NONDESTRUCTIVE ELECTROMAGNETIC CHARACTERIZATION OF PERFECT-ELECTRIC-CONDUCTOR-BACKED UNIAXIAL MATERIALS

DISSERTATION

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Abstract

As the use of anisotropic materials in electromagnetic applications continues to proliferate, it becomes increasingly important to develop non-destructive evaluation methods for those materials in their installed configuration. In many applications, these materials are permanently affixed onto conducting bodies to reduce unwanted reflections, making it impossible to collect S_{21} or S_{12} transmission measurements as used in many techniques based on the well-known Nicolson-Ross-Weir algorithm. It also makes it impractical to reorient the sample to collect orthogonal measurements aligned with the optical axes of the anisotropic material. The goal of this research is to develop a two-reflection coefficient measurement method for extracting constitutive parameters from non-destructive interrogation of a conductor backed, non-magnetic uniaxial material using a single flanged rectangular waveguide probe.

First, this dissertation presents motivation and background on complex media and their characterization. Next, a scalar-potential formulation is presented to derive Green functions describing a parallel plate region containing two layers of uniaxial material. Two measurement techniques based on those Green functions are then developed and analyzed via uncertainty analysis. One technique is validated using laboratory measurements compared to those from a mature destructive technique. Next, the advantages and disadvantages of both proposed techniques are discussed as well as suggested areas of promising future research. Ultimately, this work demonstrates that nondestructive characterization of conductor-backed uniaxial materials is not only possible, but can be achieved in an efficient, practical manner with results on par with mature destructive techniques.

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In loving memory of my mother and grandfather.

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Adam L. Brooks

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List of Acronyms

CEM computational electromagnetic 2

CIF Cauchy's Integral Formula . . . . . ix, 39, 41, 202, 203, 277, 279, 282, 300, 301

CIT Cauchy's Integral Theorem 39, 40, 277, 300

DNG double-negative material 6–8

ENG epsilon-negative material 7, 8

FDM Fused Deposition Modeling 193

FV Frequency Varying 12

GTD Geometric Theory of Diffraction 17

LHM left-handed material 7, 9

LHPC lower half plane closure . 40–43, 180, 181, 277, 279, 281, 282, 285, 286, 288, 300, 301, 303, 309, 313, 314, 325, 326, 331

LO low-observable 1

MFIE magnetic field integral equation 19, 81, 101

MNG mu-negative material 7, 8

MoM Method of Moments 19, 20, 101, 102, 125

MUT material under test . . . . . xii, 2, 3, 11–15, 18–20, 24, 100, 116, 117, 123, 124, 189–191, 193, 196, 198, 199

NDE non-destructive evaluation 2, 3, 11, 14, 15, 27

NRW Nicolson-Ross-Weir 193, 196, 197

PEC perfect electric conductor . . . . xi, 3, 12–14, 38, 96, 100, 103, 197, 198

PTD Physical Theory of Diffraction 18

RARWG Reduced Aperture Rectangular Waveguide . . . . . viii, ix, xii, xiii, 27, 125, 186–188, 194

S/FS Short/Free-Space 12

SRR split ring resonator 8, 9

TL transmission line 9

TLM Two Layer Method 13, 190

TRL Through Reflect Line 12, 192

TRR Trust Region Reflective 20

TTM Two Thickness Method 13, 14

UHPC upper half plane closure . . 40, 41, 43, 180, 277–279, 282, 286, 288, 300, 301, 305, 330

UTD Uniform Theory of Diffraction 17

VNA Vector Network Analyzer 121, 186, 190, 192, 194

NONDESTRUCTIVE ELECTROMAGNETIC CHARACTERIZATION OF PERFECT-ELECTRIC-CONDUCTOR-BACKED UNIAXIAL MATERIALS

I. Introduction and Background

1.1 Introduction

Electromagnetic characterization of a material refers to the process of determining the constitutive parameters of that material. By definition, the constitutive parameters of a material are the basis for how the material interacts with electromagnetic fields as described by Maxwell's equations. In the case of isotropic materials, those parameters can be summarized by two scalars: permittivity (ϵ) and permeability (μ). Permittivity is a measure of a material's ability to resist an electric field. Permeability is a measure of a material's magnetization in response to a magnetic field. For anisotropic materials, permittivity and permeability can change as a function of direction. Thus the constitutive parameters are each 3×3 dyads or matrices, $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$, which allow up to 18 parameters to affect the electromagnetic fields. Additionally, bianisotropic materials allow for cross-coupling terms to exist between the electric and magnetic fields, which themselves can be 3×3 dyads, $\overleftrightarrow{\xi}$ and $\overleftrightarrow{\zeta}$. This allows for up to 36 constitutive parameters to exist, depending on the class of the material.

The availability of additional constitutive parameters are attractive to engineers designing low-observable (LO) vehicles, antennas, or other electronic applications where the ability to better control the magnitude and direction of electromagnetic fields is strongly desired. Much recent work has been done designing and manufacturing anisotropic materials and how they should be incorporated into system designs.

Also, many highly-efficient computational electromagnetic (CEM) codes have been developed to simulate how incorporating such materials into applications can achieve the desired performance. Comparatively little research effort has been spent finding methods of experimentally verifying that the manufacturing processes producing these complex materials achieve the desired constitutive parameters. Without that critical piece, it becomes prohibitively expensive to iterate between design and manufacturing until the desired performance is reached. To that end, this effort seeks to help bridge a critical gap between design and final application performance by expanding the electromagnetic characterization tool set for complex media.

In most cases, electromagnetic characterization begins by measuring how a material scatters incident fields. These measurements are then analyzed by some process of inversion of Maxwell's equations to infer what the constitutive parameters must be in order for the observed fields to exist. This process is difficult enough for the isotropic case where only two parameters must be extracted. In fact, the bulk of research on measurement techniques focus on isotropic materials. Attempting to extract parameters from anisotropic materials is even more difficult due to the added complexity of the field structures produced and the added uncertainty due to more required measurements.

In general, electromagnetic characterization of materials can be divided into two classes: destructive evaluation and non-destructive evaluation (NDE). Destructive evaluation requires the precise machining of a sample to fit into a waveguide or other form of measurement apparatus. Destructive techniques can be highly accurate, but are fraught with practical issues. Close tolerances on machining may exacerbate issues such as thermal expansion of the MUT and cause air gaps to develop that drastically reduce accuracy. Additionally, for characterizing materials installed in applications *in situ*, it may be impossible to machine a sample for destructive evaluation. Therefore, there has been increased interest in NDE techniques in the research community. This effort will attempt to expand on one of the more promising NDE measurement techniques recently developed.

1.2 Problem Statement

The goal of this research effort is to find viable NDE techniques to determine constitutive parameters of complex media permanently affixed to a PEC body. Up to this point, most formulations in the literature have focused on characterization of isotropic materials or destructive evaluation of complex media. This effort will primarily focus on using a flanged waveguide probe to interrogate the MUT.

Previous formulations, including the primary research this effort is based upon [71], assumed perfect contact between the probe and the MUT and that the material could be probed from both sides. The first assumption is impractical in that it is nearly impossible to achieve perfect contact between the probe and the MUT, especially in an environment where preserving the surface quality of the material is paramount. The second assumption allows for both through (S_{12} and S_{21}) and reflection (S_{11} and S_{22}) measurement parameters to be collected. However, if a material needs to be non-destructively evaluated in an installed application, it is unreasonable to assume that access to both sides of the material under test would be available. In many applications, the MUT is permanently affixed to a conducting body to reduce unwanted reflections.

This formulation allows for either an air gap between the probe and the material under test or another uniaxial material with known constitutive parameters. In the future, it would be desirable to extend this analysis to a biaxial material case. This formulation is limitied to the uniaxial case because it allows for a faster scalar potential-based approach.

1.3 Metamaterials Definitions and History

It is useful to give some background on metamaterials and their applications. In order to understand the concept of metamaterials, one must first understand a few concepts about electromagnetic material characterization in general. *Constitutive parameters* are the set of parameters that characterize the electrical properties of a material. In the most general case, these are the dyadic parameters permittivity[1] ($\overleftrightarrow{\epsilon}$), permeability ($\overleftrightarrow{\mu}$), and magnetoelectric parameters ($\overleftrightarrow{\xi}$ and $\overleftrightarrow{\zeta}$), which satisfy the *constitutive relations*

$$\vec{B} = \overleftrightarrow{\mu} \cdot \vec{H} + \overleftrightarrow{\zeta} \cdot \vec{E} \tag{1}$$

$$\vec{D} = \overleftrightarrow{\xi} \cdot \vec{H} + \overleftrightarrow{\mu} \cdot \vec{E} \tag{2}$$

For brevity, when referring to any non-specific constitutive parameter, it will be referred to as $\overleftrightarrow{\sigma} \in \{\overleftrightarrow{\epsilon}, \overleftrightarrow{\mu}, \overleftrightarrow{\xi}, \overleftrightarrow{\zeta}\}$.

The bulk of introductory electromagnetic textbooks focus on analyzing the behavior of electromagnetic waves in the presence of simple media. The Balanis [6] textbook definitions are considered for this work. *Simple media* are defined to be linear, homogeneous, and isotropic. A medium is said to be *linear* if none of its constitutive parameters are functions of the applied field strength $\left(\forall \vec{E}_0, \vec{H}_0 \in \mathbb{C}^3, \overleftrightarrow{\sigma}\left(\vec{E}_0, \vec{H}_0\right) = \overleftrightarrow{\sigma}\right)$; otherwise, the medium is considered to be *nonlinear*. *Homogeneous media* are defined to be those whose constitutive parameters are not functions of position $\forall \vec{r} \in \mathbb{R}^3, (\overleftrightarrow{\sigma}(\vec{r}) = \overleftrightarrow{\sigma})$; otherwise, the media are considered to be *nonhomogeneous* or *inhomogeneous*. *Isotropic media* are defined as media whose constitutive parameters are not functions of direction of the applied field ($\overleftrightarrow{\sigma} = \sigma \overleftrightarrow{I}$) and there is no magnetoelectric

[1] Often conductivity is combined with permittivity to yield an effective permittivity. For this effort, any effects due to conductivity are assumed to be contained within the effective permittivity, abbreviated from ϵ_{eff} to ϵ in this work.

coupling ($\xi = \zeta = 0$); otherwise, the media are considered to be either bi-isotropic, anisotropic or bianisotropic. *Bi-isotropic media* have constitutive parameters that are not functions of direction of the applied field ($\overleftrightarrow{\sigma} = \sigma \overleftrightarrow{I}$), however they also have non-zero magnetoelectric coupling parameters ($\xi \neq 0$ and $\zeta \neq 0$). *Anisotropic* media have at least one constitutive parameter that is a function of direction of the applied field ($\exists m_1, m_2, n_1, n_2 \in \{x, y, z\} \mid \overleftrightarrow{\sigma}_{m_1 n_1} \neq \overleftrightarrow{\sigma}_{m_2 n_2}$), but there is no coupling between the magnetic and electric fields in the constitutive relations ($\overleftrightarrow{\xi} = \overleftrightarrow{\zeta} = 0$). SImilarly, *Bianisotropic* media have at least one constitutive parameter that is a function of direction of the applied field ($\exists m_1, m_2, n_1, n_2 \in \{x, y, z\} \mid \overleftrightarrow{\sigma}_{m_1 n_1} \neq \overleftrightarrow{\sigma}_{m_2 n_2}$), but they also have non-zero magnetoelectric coupling parameters ($\overleftrightarrow{\xi} \neq \overleftrightarrow{0}$ and $\overleftrightarrow{\zeta} \neq \overleftrightarrow{0}$).

Complex media are defined as any media that do not meet all three criteria for simple media. Both simple and complex media can be either dispersive or non-dispersive. A medium is said to be *non-dispersive* if none of its constitutive parameters are functions of frequency ($\forall \omega \in \mathbb{R}, \overleftrightarrow{\sigma}(\omega) = \overleftrightarrow{\sigma}$); otherwise the medium is considered to be *dispersive*. This research effort focuses on two sub-classes of complex media, uniaxial materials and biaxial materials, both of which are linear, homogeneous, and anisotropic. *Uniaxial materials* have diagonal constitutive parameter dyads where two elements of the diagonal are equal. *Biaxial materials* also have diagonal constitutive parameter dyads, but all diagonal elements are unique. Therefore,

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \quad \overleftrightarrow{\sigma}^u = \begin{bmatrix} \sigma_t & 0 & 0 \\ 0 & \sigma_t & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}, \quad \overleftrightarrow{\sigma}^b = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \tag{3}$$

where $\overleftrightarrow{\sigma}$ refers to the generic, bianisotropic constitutive parameters, $\overleftrightarrow{\sigma}^u$ refers to the uniaxial constitutive parameters, $\overleftrightarrow{\sigma}^b$ refers to the biaxial constitutive parameters.

Note from the general bianisotropic case, up to 36 constitutive parameters[2] could exist. While designing materials with this many parameters is attractive for flexibility, it comes at the cost of added analytic complexity. Utilizing uniaxial and biaxial materials allow four and six constitutive parameters as opposed to only two with simple media, greatly increasing design flexibility. Moreover, due to recent advances in manufacturing techniques such as 3-D printing, it is becoming increasingly easy to physically produce both uniaxial and biaxial materials with great precision. Some examples of naturally-occurring uniaxial materials are calcite, barium borate, ruby, and even ice. Examples of natural biaxial materials include borax, topaz, and epsom salt. These crystals exhibit interesting properties in the optical region of the electromagnetic spectrum, including birefringence [39]. *Birefringence* is an effect where horizontal and vertical polarizations of electromagnetic waves travel at different speeds through the medium. As it turns out, crystallographic structure is very closely related to the constitutive parameter dyads for a given material [need ref]. Thus, the idea of constructing metamaterials to have desired electromagnetic properties frequently begins with a crystallographic approach.

There is no formal, agreed-upon definition for metamaterials due to their diverse range of applications. One of the most inclusive fundamental definitions of metamaterials comes from Cui [25], who states that "a metamaterial is a macroscopic composite of periodic or non-periodic structure whose function is due to both the cellular architecture and the chemical composition." Direct research into metamaterials has been going on since at least the 1960s, when Veselago published his seminal work on the subject [83]. In it, Veselago postulated the existence of double-negative materials (DNGs), having both negative permittivity and negative permeability, and the properties such materials would have. His conclusions were based solely on anal-

[2] Arguably, the Post constraint may limit this number to 35 [53, 67].

ysis of Maxwell's equations without supporting them with experimental observations. Through his analysis, he postulated the existence of materials with negative group velocity and positive phase velocity, but was not the first to make that discovery. For example, negative group velocity had been observed in some crystals by Mandel'shtam [57] decades earlier.

Uniquely, Veselago further predicted the properties of such so-called left-handed materials (LHMs) who have negative group velocity and positive phase velocity. Effects such as negative index of refraction; light "tension" or attraction, as opposed to light pressure; a reverse Doppler effect; and a reverse Vavilov-Cerenkov radiation were predicted. Veselago also noted that it was possible for certain gyrotropic media to be epsilon-negative materials (ENGs) (e.g. a plasma in a magnetic field), mu-negative materials (MNGs), or DNGs (e.g. pure ferromagnetic materials and some semiconductors), depending on the permittivity and permeability dyads. *Gytropic media* are similar to biaxial and uniaxial media, but have at least one set of complex, non-zero off-diagonal elements in at least one constitutive parameter dyad. This can cause left-handed and right-handed elliptically-polarized electromagnetic waves to travel at different speeds through the medium (similar to birefringence). Neglecting absorption losses, the dyads take on a Hermitian matrix form such that

$$\overleftrightarrow{\sigma}^{g} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} + jg_z & \sigma'_{xz} - jg_y \\ \sigma'_{xy} - jg_z & \sigma'_{yy} & \sigma'_{yz} + jg_x \\ \sigma'_{xz} + jg_y & \sigma'_{yz} - jg_x & \sigma'_{zz} \end{bmatrix} \tag{4}$$

where σ' is a real, symmetric matrix and $\vec{g} = \hat{x}g_x + \hat{y}g_y + \hat{z}g_z$ is a real pseudovector known as the *gyration vector*. In the simplest gyrotropic case where the gyration vector is in the same direction as the principal axis (the $\hat{z}$-direction for simplicity here), the dyad looks similar to the uniaxial case with the first off-diagonal elements being imaginary and non-zero such that

$$\overleftrightarrow{\sigma}^g = \begin{bmatrix} \sigma_t & -j\sigma_{gz} & 0 \\ j\sigma_{gz} & \sigma_t & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \tag{5}$$

where $\sigma_{gz} = j\left(\sigma_{xy} + jg_z\right)$. Some natural examples of gyrotropic materials include ferrites and plasmas [25, 99].

While Veselago's work is now considered to be the foundation of metamaterials research, his initial paper was largely unregarded until Pendry published several observations of changes in material properties based on structure in the 1990s [62–64,66]. In [64], Pendry noted that a structure composed of very thin metallic wires exhibited electromagnetic properties in the gigahertz region similar to the properties of a bulk slab of the same metal in the ultraviolet region. Thus he demonstrated that changing the physical structure of a material could drastically alter it's electromagnetic properties. In [63], Pendry began to understand that using periodic structures of split ring resonators (SRRs) or magnetic cylinders, one could tune constitutive parameters to "values not accessible in naturally occurring materials." This meant that, through careful design of material structures, DNGs, ENGs, and MNGs could potentially be fabricated for real-world applications. This effectively sparked a metamaterial "gold rush" in the research community that continues to this day. Additionally, one can see that Pendry's work began to bridge Veselago's analytic predictions with Mandel'shtam's observations. For these reasons, Pendry is largely considered to be the father of modern metamaterial research.

1.4 Metamaterials Applications

Metamaterials have applications in numerous varied fields. Thousands of articles have been published on the subject. The two most promising military applications are controlling scattering of eletromagnetic radiation and improved antennas. Pendry illustrated how a slab of negative-index material could act as a perfect lens in both the microwave and optical regimes [62]. Shamonina detailed numerous applications including perfect lenses, super-directivity, super-wavelength focusing and imaging, photonic band-gap materials, nanoparticles, and super-resolution [76]. The first apparent experimental verification of negative refraction in the microwave region was demonstrated by Shelby, et al. [77]. To accomplish this, they utilized a two-dimensional unit cell array, with cells containing copper strips and SRRs. A good review of work being done on electrically-small and highly-directive antennas was provided by Ziolkowski [99]. Cloaking technology has became extremely popular in the research community since Pendry discussed the ability to control electromagnetic fields "at will" [65].

This effort takes advantage of a volumetric approach to metamaterials design where unit cells are embedded throughout a bulk material. However, in practice, such structures tend to be narrow-banded and extremely lossy due to their leveraging of resonant modes. In [14], Caloz provides an extensive look at a transmission line (TL) theory of metamaterials, which allows for one- and two-dimensional structures to be created that exhibit the properties of LHMs with wide bandwidth and low loss.

1.5 Material Characterization Techniques

Numerous techniques exist to characterize materials in the microwave region. Many factors influence how well each technique works including the type of material being tested, the analytic model being used, the underlying assumptions for any models used, and any numerical techniques that are applied. Numerical techniques in particular have many factors to consider including convergence criteria, minimization of error (including global versus local error minimums), and computational efficiency. In this section, a review of some of the most notable research examples in each of these areas are provided.

Isotropic Material Characterization Techniques.

The bulk of material characterization research efforts have been focused on isotropic materials. There are many reasons for this, but likely the most consequential reason is because only in the last decade or two have manufacturing processes become sophisticated enough to produce anisotropic materials on any meaningful scale. To this day, isotropic materials dominate in many applications due to their simplicity, both for production and for analysis. Due to the relative maturity of isotropic material characterization techniques, it is useful to attempt to modify one of them to characterize anisotropic materials. By doing so, there will be some intuition of expected advantages and limitations of the technique chosen. That is the approach this research effort takes.

It is important to note that for a linear system to be well-posed and directly-solvable, the number of independent measurements must be equal to the number of unknowns. For isotropic media, there are only two unknowns (ϵ and μ) and thus only two measurements are required. For uniaxial materials, there are four unknowns ($\epsilon_t, \epsilon_z, \mu_t$, and μ_z), requiring four independent measurements. Even worse, there are six unknown in biaxial materials ($\epsilon_x, \epsilon_y, \epsilon_z, \mu_x, \mu_y$, and μ_z), which in turn requires six independent measurements to have a well-posed solution. It is possible to reduce the number of measurements needed by half if the material is known to be non-dielectric ($\epsilon_x = \epsilon_y = \epsilon_z = \epsilon_0$) or non-magnetic ($\mu_x = \mu_y = \mu_z = \mu_0$), but the initial analytic formulations (provided in the proceeding chapters) will avoid making such assumptions.

There are predominantly two classes of NDE techniques for isotropic media in the microwave regime: far-field (free space) methods, near-field methods. This review will break the near-field methods down into two subclasses: single probe methods and dual probe methods. Since the goal of this effort will be X-band characterization of uniaxial and biaxial materials, this review will focus on research that is applicable between 8-12 gigahertz.

Far-Field (Free Space) Methods.

Free space methods employ either a monostatic configuration using a single transmit/receive horn antenna or a bistatic configuration using two separate transmit and receive horn antennas. It is typically assumed that the incident wavefront is a far-field plane wave and that the transverse dimensions of the MUT are infinite in extent. One notable advantage of free space methods is that no contact with the MUT is required. This is particularly useful when testing materials in extreme temperatures or other conditions that may damage measurement hardware in close proximity. Additionally, both horizontal and vertical polarizations may be characterized in both the transmit and receive planes. Yet another advantage is the ability to test over a wider bandwidth than waveguide methods as well as allowing a multitude of incident and observation angles. Unfortunately, there are several drawbacks to free space methods which may result in significant sources of error, as noted by Stewart [78]. Edge diffraction, iteractions with the sample holder, and wavefront variations can render either the plane-wave or infinite transverse extent assumptions invalid. Distance and antenna illumination patterns affect this error source greatly, so it is often beneficial to measure as large a sample as possible. This can be problematic as it may be impractical or impossible to fabricate a sample large enough to mitigate these error sources. Due to the distance required to satisfy far-field assumption, these measurement setups tend to require large amounts of space, large focusing lenses, and precise positioning of the material in relation to the antennas and lenses.

In [2], a monostatic configuration is demonstrated. Using two thicknesses of material, two independent measurements are taken with up to 10 percent error. In [34], a biastatic configuration is demonstrated. A horn antenna is placed on either side of the MUT allowing for transmission and reflection coefficients to be measured. The system is calibrated using the Through Reflect Line (TRL) measurement calibration technique [72]. This technique demonstrates a reduction in edge diffraction error by using spot focusing lenses that are 30 centimeters in diameter.

Near-Field Single Probe Methods.

Single probe near-field methods rely entirely on reflection parameter measurements. In order to get the two independent measurements required for a well-posed system, different configurations of the measurement setup are required. Often, the different configurations consist of different backing materials, different thicknesses of the MUT, or different materials applied to the front surface of the MUT. One popular method of obtaining two independent measurements is the Short/Free-Space (S/FS) method [3, 55, 81], where the first measurement is taken using a PEC backing behind the MUT and the second measurement is taken using a free space backing. Hyde noted [43] that the two measurements are complements to one another in that the PEC-backed measurement interrogates the MUT with a strong magnetic field and the free space-backed measurement interrogates the MUT with a strong electric field. Thus the results using this technique tend to be very accurate.

A Frequency Varying (FV) method is presented by Maode in [58], but obtaining the best accuracy depends on having *a priori* knowledge of the frequency-dependent behavior of the constitutive parameters. Techniques requiring *a priori* knowledge are undesirable because the required knowledge may either be flawed or unavailable. Maode also presented a Two Thickness Method (TTM) in [58] where, using an open-ended waveguide probe, the first measurement is taken with a PEC-backed MUT. The second measurement is taken using a PEC-backed MUT of different thickness than the first measurement. Maode utilized an approximate form of input admittance that, due to readily available computational resources, is not necessary today. A sensitivity analysis of the TTM shows that variations in the relative thicknesses of the two samples tested can result in large errors [17].

Dester, et al., [29] propose a Two Layer Method (TLM) as an alternative to the TTM again utilizing an open-ended waveguide for the first measurement. The second measurement adds a layer of material with known constitutive parameters on top of the MUT. Applying the two-layer parallel plate Green function for the second set of equations, extraction of the MUT constitutive parameters is possible. However, the TLM results in increased error compared to the TTM, so it should only be applied when two samples of the MUT are not available. A good example of this case would be for *in situ* measurements in an installed application, as this research intends to focus.

For the TLM, the known material should have a little loss as possible. This allows as much of the electric field as possible to interact with and interrogate the MUT. Hyde suggested an alternative approach to the TLM [43] by placing the known material behind the MUT for the second measurement which results in improved accuracy. The improvement is due to stronger penetration of the electric field into the MUT for increased interaction. While the accuracy is improved over the initially-proposed TLM, Hyde's technique is not well-suited for *in situ* measurements where the material behind the MUT is not accessible for change. Dester presents a two-iris method [ref 35] that obtains similar performance to the method proposed by Hyde in [ref 52]. In this two-iris method, a reduced aperture is used to collect the second measurement. This technique has the added advantage that it is suitable for *in situ* measurements.

Near-Field Dual Probe Methods.

While this research will focus on single-probe techniques due to the assumed lack of access to the back of the MUT, it is important to consider dual probe methods for completeness. One main advantage of using dual probe methods for characterizing isotropic media is that only one measurement configuration is needed in order to obtain two independent measurements: a reflection measurement and a transmission measurement. Hyde presents a novel dual-probe method [44] for NDE of a PEC-backed MUT. Using dominant-mode analysis only, errors are less than ten percent. It is suggested that full-wave modal analysis would improve the accuracy considerably. Compared with the TTM, this method requires only one measurement configuration for isotropic materials.

Several papers have been written demonstrating simple, accurate, and precise measurements taken with dual flanged waveguides [42–47,75]. This technique is shown to be immune to some of the errors inherent in traditional (destructive) rectangular waveguide techniques, most notably the precise machining of the MUT required to prevent air gaps around the material inside the sample holder. Additionally, this technique is relatively forgiving of minor misalignments in the transverse directions of the transmit and receive apertures [44, 47]. While the method generally requires large flanges to prevent detection of flange-edge reflections, time-gating can be utilized to mitigate such reflections and reduce the required flange size [46]. This is discussed further in the next section.

Types of Near-Field Probes.

Having discussed both single- and dual-probe near field techniques, it is important to consider the different types of near-field probes in common use and their relative advantages and limitations. Typically, near-field measurements are taken with either an open-ended coaxial waveguide probe or an open-end rectangular waveguide probe. Coaxial waveguide probes have significantly wider bandwidth compared to rectangular waveguide probes. They also provide good accuracy and are well-represented in texts and the literature [18, 31, 97, 100]. A broad review of various coaxial waveguide probe configurations and their relative errors was provided by [80]. These configurations tend to be delicate and require very precise measurement procedures. The most notorious sources of errors in coaxial waveguide probes are air gaps that develop between the MUT and the center conductor. Several air-gap mitigation techniques were provided in [74], but each adds further complexity to an already-troublesome measurement apparatus. Another issue is that measurement accuracy is frequency-dependent [24]. One successful demonstration of NDE with a coaxial waveguide probe is provided by Pournarpoulos [68], characterizing several materials up to 40 gigahertz.

Rectangular waveguide probes, which narrower in bandwidth compared to coaxial waveguide probes, offer several advantages. They are physically rugged, provide excellent accuracy, and offer improved matching with free space impedance [18]. Also, radiating field from a rectangular waveguide probe penetrate deeper into the MUT than those of coaxial waveguide probes. Most notably, the linear polarization of the fields in a rectangular waveguide allow for measurement of anisotropic materials.

Much success has been reported using variations on an open-ended flanged rectangular waveguide probe [10, 16, 27, 29, 58, 79, 82]. Inhomogeneous materials (in the form of layered, continuously-varying dielectrics) have also been successfully characterized with a similar configuration [61, 73]. One disadvantage of flanged rectangular waveguide probes is that steps must be taken to mitigate reflections from the flange edges. Hyde [45] presents a dominant-mode technique where time gating can be applied to filter out these ujnwanted reflections. This technique allows for reasonably-sized flanges to be used when measuring low-loss materials. Unfortunately, this method is shown to error that increases with frequency. It is proposed that including higher-order modes in the analysis may provide improved accuracy.

Anisotropic Material Characterization Techniques.

Far less research has been devoted to characterization of anisotropic materials than that of isotropic materials. Due to recent improvements in the production of anisotropic materials and bevy of theorized applications for such materials [9, 30, 40], interest in characterizing those materials has waxed significantly. Uniaxial materials, the simplest of anisotropic media, have become relatively easy to manufacture [21]. As indicated in an earlier section, uniaxial materials require four independent measurements in order to have a well-posed solution for the constitutive parameters.

Resonator methods have been used for the accurate characterization of both isotropic and uniaxial materials [52]. The limitations of such methods are that they are extremely narrow in bandwidth and they are destructive. Free space methods have been attempted to characterize biaxial materials [98], but are unable to reliably extract longitudinal constitutive parameters due to extreme sensitivity to measurement errors. Complex permittivity of sapphire and uniaxial alumina were successfully measured using a coaxial line probe [7], however this technique is also destructive and has difficulty with low-loss samples.

Using an open-ended waveguide, Chang [15] measures the permittivity of a dielectric fiber composite material at 30-, 60-, and 90-degree angles with respect to the longitudinal axis to obtain independent measurements. However, due to the high conductivity of the sample and its thinness, the extraction of the longitudinal permittivity component was unstable.

Rogers [71] presents a two flanged waveguide method to characterize uniaxial media with relative success. However, this method initially assumes a dominant-mode analysis and may be improved by including higher order modes. Knisely presents a square waveguide method for destructively characterizing biaxial media [50] which only requires a single cubic sample of biaxial material to be produced as opposed to three rectangular samples as used in previous methods. Knisely also presents a non-destructive single probe method of characterizing uniaxaial media [51].

Analytical Models.

The methods above describe only the data collection portion of the material characterization process. Once the measurements are physically recorded, an analytical model must be used to interpret those data. This is an inverse problem where one must infer (extract) the constitutive parameters of the material that give rise to those measurements. Thus, the more accurate the analytical model used, the more accurate the extracted constitutive parameters should be. There are two major categories of analytical models: asymptotic methods and full wave methods.

Asymptotic Methods.

Asymptotic methods are high-frequency models which assume that the smallest features of a scattering object are large compared to the wavelength of the incoming electromagnetic waves. The most frequently used asymptotic methods are the Geometric Theory of Diffraction (GTD), the Uniform Theory of Diffraction (UTD), and the Physical Theory of Diffraction (PTD) [59], which have been applied to parallel plate geometries as a special case of the canonical wedge [26, 54]. Unfortunately, for the rectangular waveguide collection methods, the electrically-large assumption does not hold up. For example, the largest feature of an X-band (8.2 to 12.4 gigahertz) waveguide is approximately 2.2 centimeters. However, the wavelength of 10 gigahertz electromagnetic waves is approximately 3 centimeters. Thus, asymptotic methods are not appropriate for this research effort.

Full Wave Methods.

There are two subcategories of full wave methods: approximate full wave methods and rigorous full wave methods. In approximate full wave methods, the principle of least action is used to avoid the differential equations required in rigorous full wave methods [5, 23, 33, 58, 100]. By approximating the admittance at the waveguide aperture, extraction of constitutive parameters is comparably straightforward. However, since sufficient computational power is readily available, this research seeks to avoid as many errors introduced by approximations and assumptions as possible.

Rigorous full wave solutions begin with the fundamental Maxwell's equations and account for all scattering properties associated with the MUT, a process well-described by Balanis in [6]. A version of this process used by Stewart [78, 79], Rogers [71], Knisely [50, 51], and numerous others is employed in this research. First, Maxwell's equations are used in conjunction with boundary conditions describing the problem-space geometry to formulate an integral equation solution. That solution is then reduced to a Green function kernel format which gives physical insight into the field structures produced by currents that exists in the measurement apparatus geometry. Next, Love's equivalence principle is used to determine the currents that exist within the waveguide aperture. Finally, a field expansion of the reflected modes is obtained using the Method of Moments (MoM), resulting in a set of magnetic field integral equations (MFIEs). These MFIEs are then used to model the forward solution predicting a theoretical set of measurements expected given a guessed set of constitutive parameters. An iterative approach (root search algorithm) is employed to estimate the values of the constitutive parameters of the MUT by comparing the MFIE predictions to the actual measurements. The accuracy of the MFIEs, and thus the estimated parameters, depends on the number of modes used in the expansion portion of the MoM solution.

Often the first 20 modes are used to expand the MoM solution due to an assumption that solution convergence typically occurs within the first 20 modes included [10]. Since infinite reflection modes exist, it stands to reason that accuracy of the model improves the more modes are included in the expansion. However, computation time increases on the order of N^2, where N is the number of modes included, so it is desirable to include only the number of modes necessary to reach solution convergence. Dester [28] notes that including 20 modes may not be the most efficient or accurate approach to obtaining true convergence, so he proposes a hybrid technique where the first 20 modes are used in conjunction with an extrapolation method. This hybrid technique is shown to produce results similar to those provided by including the first 160 modes. This research will take advantage of such computational improvements, while additionally seeking as many closed-form solutions as possible.

Numerical Solution Methods.

Unfortunately, it is not possible to obtain a fully closed-form solution to the MFIEs discussed in the previous section. Therefore, there is no choice but to resort to numerical techniques to obtain solutions to both the analytical model providing theoretical predictions (the forward problem) and the error minimization process used to estimate the constitutive parameters of the MUT (the inverse problem).

The MoM is the preferred technique for solving the forward problem [19,20,36,60]. Careful choice of basis and testing functions can significantly reduce the complexity of the MoM solutions. Here, the infinite number of waveguide modes are a natural choice to use for these expansion and testing functions. It is necessary to truncate the number of modes used, otherwise a system of infinite equations with infinite unknowns would result. As mentioned previously, the time required increases on the order of N^2 where N is the number of waveguide modes chosen for the expansion, so computational budget must be taken into consideration when choosing the number of modes to use.

Measurements are taken at discrete frequencies within the band of interest, thus the reverse problem can be accomplished at each individual frequency. This allows for the characterization of dispersive media. The Newton-Raphson method has both one-dimensional and two-dimensional variations that are simple to implement and work well for solving the reverse problem [44, 71, 78]. Additionally, there are several non-linear least squares approaches [56] that have recently come into favor [4, 43, 47, 82] since they better characterize uncertainties and frequency dependence of the constitutive parameters being estimated. A subset of these approaches, including the Trust Region Reflective (TRR) method, the Gauss-Newton method, and the Levenberg-Marquardt method, are straightforward to implement in MATLAB® [32, 56].

Green Functions.

Due to the effort required in solving Maxwell's equations, formulating the solution in terms of Green function kernels are extremely useful for capturing the results of the analysis and drastically reducing effort in characterizing other problems of similar geometry. This is shown quite clearly by Havrilla in [37]. Frequently, vector potentials are used to help find those solutions [6,19,22,35]. Scalar potentials have been growing in popularity recently for analyzing gyrotropic, chiral, and uniaxial materials [38,69, 70,84–96] due to their ability to greatly simplify analysis and outstanding physical insight they provide. This insight typically stems from how most methods decompose of the electromagnetic fields into longitudinal and transverse components.

Despite their utility, Green function kernels for solving electromagnetic analytic models are not trivially obtained. Additionally, while much progress has been made by Weiglhofer and his colleagues in this area, it has not been possible to demonstrate how the more general cases developed (for example, gyrotropic bianisotropic) reduce to simpler subclasses of those cases (uniaxial anisotropic or isotropic). However, the methods used in [38] and subsequently in [50,51,71] do show consistency between the uniaxial case and simpler subclasses. Thus, the methods used in [38] are replicated in this research to extend the work of [71] to a structure where two layers of uniaxial material sandwiched in a parallel plate geometry.

Direct Field Formulation.

The direct field approach to solving Maxwell's equations is extremely laborious in comparison to potential-based methods. The bulk of the difference is due to the need to invert numerous 3-by-3 matrices for the direct field approach. Additionally, if a vectorized form cannot be found, each term must be derived separately. As an example, note that for homogeneous, bianisotropic gyrotropic media, Maxwell's equations reduce to

$$(\nabla \times \overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta}) \cdot \vec{E} = -\vec{J}_h - j\omega\overleftrightarrow{\mu} \cdot \vec{H} \tag{6}$$

$$(\nabla \times \overleftrightarrow{I} - j\omega\overleftrightarrow{\xi}) \cdot \vec{H} = \vec{J}_e + j\omega\overleftrightarrow{\epsilon} \cdot \vec{E} \tag{7}$$

where the constitutive parameter dyads are

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_{xx} & -j\sigma_{xy} & 0 \\ j\sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}, \qquad\qquad \sigma \in \{\epsilon, \mu, \zeta, \xi\} \tag{8}$$

Through careful manipulation of (6) and (7), it can be shown that

$$\begin{aligned} &\overbrace{\left[\overleftrightarrow{\epsilon} \cdot (\nabla \times \overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta}) \cdot \overleftrightarrow{\epsilon}^{-1} \cdot (\nabla \times \overleftrightarrow{I} - j\omega\overleftrightarrow{\xi}) - \overleftrightarrow{k}^2\right]}^{\overleftrightarrow{W}_h} \cdot \vec{H} \\ &\qquad = \underbrace{-j\omega\overleftrightarrow{\epsilon} \cdot \vec{J}_h + \overleftrightarrow{\epsilon} \cdot (\nabla \times \overleftrightarrow{I} + j\omega\overleftrightarrow{\zeta}) \cdot \overleftrightarrow{\epsilon}^{-1} \cdot \vec{J}_e}_{\vec{S}_1} \end{aligned} \tag{9}$$

$$\Rightarrow \vec{H} = \overleftrightarrow{W}_h^{-1} \cdot \vec{S}_1 \tag{10}$$

where $\overleftrightarrow{W}_h$ represents the eigenvector matrix, $\vec{S}_1$ is the source term, and $\overleftrightarrow{k}^2 = \omega^2\overleftrightarrow{\epsilon} \cdot \overleftrightarrow{\mu}$. Note that when the constitutive parameter dyads are of full rank, $\overleftrightarrow{W}_h$ is also of full rank. Thus, the inversion of the 3-by-3 matrix $\overleftrightarrow{W}_h$ is necessary to recover the magnetic field, which is a very tedious process.

Scalar Potential Formulation.

The scalar potential approach is a method whereby decomposing the electromagnetic fields into subcomponents, an effective dimensionality reduction is realized for the inversion of $\overleftrightarrow{W}_h$. By breaking the electric and magnetic fields into longitudinal ($\hat{z}E_z$ and $\hat{z}H_z$) and transverse ($\vec{E}_t$ and $\vec{H}_t$) components in terms of scalar potentials

ψ, θ, Φ, and Π, it can be shown that

$$\vec{E}_t = \nabla_t \Phi - \hat{z} \times \nabla_t \theta \tag{11}$$

$$E_z = -\frac{1}{j\omega\epsilon_z}\left(\nabla_t^2 \psi + J_{ez}\right) \tag{12}$$

$$\vec{H}_t = \nabla_t \Pi - \hat{z} \times \nabla_t \psi \tag{13}$$

$$H_z = \frac{1}{j\omega\mu_z}\left(\nabla_t^2 \theta - J_{hz}\right) \tag{14}$$

where Φ and Π are directly related to ψ and θ. Therefore, the scalar potentials ψ and θ are solutions to the equations

$$\mathcal{L}_1 \psi + \mathcal{L}_2 \theta = S_1 \tag{15}$$

$$\mathcal{L}_3 \psi + \mathcal{L}_4 \theta = S_2 \tag{16}$$

where $\mathcal{L}_n$ are scalar differential operators and S_n are source terms. Using Fourier transforms and linear algebra to invert the 2-by-2 $\overleftrightarrow{\mathcal{L}}$ matrix, field recovery becomes a simple process of differentiation. This dramatically reduces the complexity of solving for the electromagnetic fields than the direct field approach. It should be noted that the most general material that can be represented in this scalar potential decomposition is the gyrotropic material [93].

1.6 Scope

Veselago and others have stressed that metamaterials can be realized using gyrotropic media. However, much attention has been given to uniaxial materials for this purpose as well [8, 9, 14, 96]. Uniaxial materials are, in fact, a simplified subclass of gyrotropic media and are easier to both analyze and manufacture. For those reasons, this research limits the focus to uniaxial materials.

1.7 Research Goals and Contribution to Science

The goals of this research were to analytically derive a meaningful set of Green functions for characterizing uniaxial materials, develop at least one constitutive parameter extraction algorithm, and validate that algorithm with experimental results and uncertainty analyses.

This research provides three major contributions to science. First, the total Green functions for the electric and magnetic fields in transverse spectral domain and longitudinal spatial domain using scalar potentials are derived. This enables many future avenues of research beyond the scope of this work. This portion of the research was presented to the community at two conferences [11, 12] .

The second major contribution is the derivation of a method to extract constitutive parameters via a flanged rectangular waveguide probe with a layer of known material applied to the MUT. A feasibility study for this technique is also provided should further research into this technique be desired. This portion of the research was presented to the community at a conference and published [12].

The third major contribution is the derivation of a method to extract constitutive parameters via a flanged rectangular waveguide probe with a reduced aperture in the flange plate. A feasibility study is provided along with experimental results. This portion of the research has been accepted for presentation at a conference and publication later in the year of this writing [13].

1.8 Assumptions

The following assumptions are applied for this analysis:

- The parallel-plate waveguide section is of infinite extent in the transverse directions and of finite extend in the longitudinal direction.

- The materials in the parallel-plate waveguide section are linear, anisotropic, dielectric, magnetic, homogeneous and of uniform thickness.

- Rectangular waveguide sections contain only free space, and thus are linear, isotropic, homogeneous, non-dielectric ($\overleftrightarrow{\epsilon} = \epsilon_0 \overleftrightarrow{I}$), non-magnetic ($\overleftrightarrow{\mu} = \mu_0 \overleftrightarrow{I}$) and of uniform thickness

- The time dependence, $e^{j\omega t}$, is assumed and therefore suppressed throughout this effort.

1.9 Notation

Arrow notation is used to signify a variable is a vector or a dyad (matrix). A single arrow over a variable indicates the variable is a vector. For example, $\vec{E}$ refers to the electric field vector. A double arrow over a variable indicates the variable is a dyad. For example, $\overleftrightarrow{\mu}$ refers to the permeability dyad.

Tilde notation is used to signify a Fourier-transformed variable. A single tilde over a variable indicates it has been transformed to the transverse spectral domain. For the purposes of this effort, the transverse spectral domain indicates the x- and y-directed components of the variable in question have been Fourier-transformed. A double tilde over a variable indicates it has been transformed to the full spectral domain, which now includes the z-directed component of the variable in question. For example, $\vec{\tilde{\tilde{H}}}$ refers to the full-spectral-domain magnetic field vector, while $\vec{\tilde{E}}$ refers to the transverse-spectral-domain electric field vector.

Subscript notation is used to describe observation and source parameters for fields, scalar potentials, Green function kernels, and measurements with the following convention:

$$X_{\text{(observation parameters)(optional source parameters)}} \tag{17}$$

For example, one would read $\vec{E}_{2h1}$ as "the electric field vector observed in region 2 resulting from magnetic currents in region 1." For Green functions, G_{e1e2} would be interpreted as "the Green function kernel that produces an electric field observed in region 1 from electric currents in region 2." Finally, S_{21} would be interpreted as "the measurement taken at port 2 resulting from excitation at port 1." Note that since source parameters are optional, E_{x2} would be interpreted as "the x-component of the electric field observed in region 2."

P notation is a special exception of subscript notation. It is used as shorthand to replace exponential functions to save space, to allow for easy reconfiguration of multiplied exponentials, and to make patterns of exponentials easier to visually recognize in equations. In this work, $P_{\text{(region parameters)(variable)}} = e^{-jk_{z\text{(region parameters)}}\text{(variable)}}$. For example, $P_{\theta 1d} = e^{-jk_{z\theta 1}d}$.

Superscripts are reserved for discriminating between different uses of the same variable. For example, $\vec{E}^p$ and $\vec{E}^s$ are used to differentiate the principal and scattered solutions for the electric field, $\vec{E}$. Set notation is used when a certain equation applies to multiple subscripted or superscripted values. For example, $\vec{E}_{\{1,2\}} = \vec{E}_{t\{1,2\}} + \hat{z}E_{z\{1,2\}}$ is a more compact way of representing the following two equations

$$\vec{E}_1 = \vec{E}_{t1} + \hat{z}E_{z1} \tag{18}$$

$$\vec{E}_2 = \vec{E}_{t2} + \hat{z}E_{z2} \tag{19}$$

Variables are also used for further compactness. In the above example, $\vec{E}_\alpha = \vec{E}_{t\alpha} + \hat{z}E_{z\alpha}, \alpha \in \{1,2\}$ may be more compact, but in some situations less readable.

1.10 Overview and Organization

This chapter provides motivation and background for NDE of complex media, focusing in particular on uniaxial and gyrotropic media. The chapter explores multiple measurement techniques, defines the scope, presents assumptions and defines notation used in this research effort. Chapter II presents a scalar-potential formulation of the transverse-spectral-domain Green functions describing the electromagnetic fields in a parallel plate region filled with two layers of uniaxial material. Chapter III presents a theory of constitutive parameter extraction using a two-layer method. Chapter IV presents the results and analyses of the two-layer method. Chapter V presents a theory of constitutive parameter extraction using a RARWG probe. Chapter VI presents the results and analyses of the RARWG probe technique. Chapter VII presents conclusions and suggested future work.

II. Potential-Based Formulation and Total Parallel-Plate Green Function for Bi-Layered Uniaxial Media

This chapter discusses analysis of a parallel plate waveguide section as depicted in Fig. 1. The analyses here follow very closely with analyses outlined in [71] and are intended to extend that work using bi-layered materials.

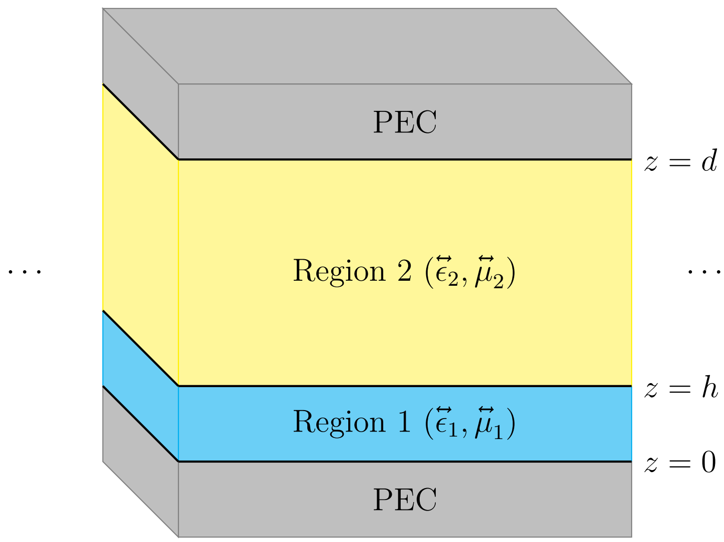

Figure 1. Cross section of parallel plate region under analysis in this chapter.

2.1 Scalar Potential Formulation for Uniaxial Material

This section will develop the scalar potential formulation needed for deriving the Green functions in later sections. Beginning with Maxwell's equations for a generic, infinite-space uniaxial material,

$$\nabla \times \vec{E} = -\vec{J}_h - j\omega\overleftrightarrow{\mu} \cdot \vec{H}, \qquad \overleftrightarrow{\mu} = \begin{bmatrix} \mu_t & 0 & 0 \\ 0 & \mu_t & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \tag{20}$$

$$\nabla \times \vec{H} = \vec{J}_e + j\omega \overleftrightarrow{\epsilon} \cdot \vec{E}, \qquad \overleftrightarrow{\epsilon} = \begin{bmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \tag{21}$$

Since the constitutive dyads are composted of one longitudinal and one transverse component each, finding a method of analyzing each component separately would be convenient. If one defines a transverse differential operator $\nabla_t := \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$, (20) can be decomposed into longitudinal and transverse components such that

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{E}_t + \hat{z}E_z\right) = -\vec{J}_{ht} - \hat{z}J_{hz} - j\omega\mu_t\vec{H}_t - \hat{z}j\omega\mu_z H_z \tag{22}$$

The longitudinal component is orthogonal to the transverse components because $\hat{z} \perp \hat{x}$ and $\hat{z} \perp \hat{y}$. Therefore, the longitudinal and transverse components are linearly independent and can be analyzed by separate equations. Thus, from (22) one can infer that

$$\nabla_t \times \vec{E}_t = -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z \tag{23}$$

$$\nabla_t \times \hat{z}E_z + \hat{z}\frac{\partial}{\partial z} \times \vec{E}_t = -\vec{J}_{ht} - j\omega\mu_t\vec{H}_t \tag{24}$$

Similarly, from (21) it can be shown that

$$\nabla_t \times \vec{H}_t = \hat{z}J_{ez} + \hat{z}j\omega\epsilon_z E_z \tag{25}$$

$$\nabla_t \times \hat{z}H_z + \hat{z}\frac{\partial}{\partial z} \times \vec{H}_t = \vec{J}_{et} + j\omega\epsilon_t\vec{E}_t \tag{26}$$

By Helmholtz theorem (also referred to as the fundamental theorem of vector calculus), any sufficiently smooth, rapidly decaying vector field $\vec{V} : \mathbb{R}^3 \to \mathbb{C}^3$ can be decomposed into a superposition of a divergence-free and a curl-free component. Thus,

$$\vec{V} = \nabla w + \nabla \times \vec{A} \tag{27}$$

where the scalar field $w : \mathbb{R}^3 \to \mathbb{C}$ and vector field $\vec{A} : \mathbb{R}^3 \to \mathbb{C}^3$. Analyzing only the transverse component of (27),

$$\begin{aligned} \vec{V}_t &= \nabla_t w + \nabla_t \times \vec{A} \\ \Rightarrow \hat{x}V_x + \hat{y}V_y &= \nabla_t w + \nabla_t \times (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z) \\ \Rightarrow \vec{V}_t &= \nabla_t w + \nabla_t \times (\hat{z}A_z) \end{aligned} \tag{28}$$

where the scalar field $A_z : \mathbb{R}^3 \to \mathbb{C}$. Therefore, the transverse components of the electric and magnetic fields and currents can be decomposed into scalar potentials such that,

$$\vec{E}_t = \nabla_t \Phi + \nabla_t \times \hat{z}\theta \tag{29}$$

$$\vec{H}_t = \nabla_t \Pi + \nabla_t \times \hat{z}\psi \tag{30}$$

$$\vec{J}_{et} = \nabla_t u_e + \nabla_t \times \hat{z}v_e \tag{31}$$

$$\vec{J}_{ht} = \nabla_t u_h + \nabla_t \times \hat{z}v_h \tag{32}$$

where scalar potentials $\Phi, \theta, \Pi, \psi, u_e, v_e, u_h, v_h : \mathbb{R}^3 \to \mathbb{C}$. Substituting (29) into (23) reveals that

$$\begin{aligned} \nabla_t \times (\nabla_t \Phi + \nabla_t \times \hat{z}\theta) &= -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z \\ \Rightarrow \cancelto{0}{\nabla_t \times \nabla_t \Phi} + \nabla_t \times \nabla_t \times \hat{z}\theta &= -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \nabla_t \cancelto{0}{(\nabla_t \cdot \hat{z}\theta)} - \nabla_t^2 \hat{z}\theta &= -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z \\
-\hat{z}\cdot\{\cdot\} \Rightarrow \nabla_t^2\theta &= J_{hz} + j\omega\mu_z H_z \\
\Rightarrow H_z &= \frac{\nabla_t^2\theta - J_{hz}}{j\omega\mu_z} \qquad (33)
\end{aligned}
$$

Similarly, substituting (30) into (25) implies that

$$
\begin{aligned}
\nabla_t \times (\nabla_t \Pi + \nabla_t \times \hat{z}\psi) &= \hat{z}J_{ez} + \hat{z}j\omega\epsilon_z E_z \\
\Rightarrow -\nabla_t^2\psi &= J_{ez} + j\omega\epsilon_z E_z \\
\Rightarrow E_z &= -\frac{\nabla_t^2\psi + J_{ez}}{j\omega\epsilon_z} \qquad (34)
\end{aligned}
$$

Substituting (29), (30) and (32) into (24) implies that

$$
\begin{aligned}
\nabla_t \times \hat{z}E_z + \hat{z}\frac{\partial}{\partial z} \times (\nabla_t \Phi + \nabla_t \times \hat{z}\theta) &= -\left(\nabla_t u_h + \nabla_t \times \hat{z}v_h\right) \\
&\quad - j\omega\mu_t \left(\nabla_t \Pi + \nabla_t \times \hat{z}\psi\right) \\
\Rightarrow -\hat{z}\times\nabla_t E_z + \hat{z}\times\nabla_t\frac{\partial\Phi}{\partial z} \overbrace{-\hat{z}\times\hat{z}\times\nabla_t\frac{\partial\theta}{\partial z}}^{\hat{z}\left(-\cancel{\hat{z}\cdot\nabla_t\frac{\partial\theta}{\partial z}}\right)+\nabla_t\frac{\partial\theta}{\partial z}\cancelto{1}{(\hat{z}\cdot\hat{z})}} &= -\nabla_t u_h - \nabla_t \times \hat{z}v_h - j\omega\mu_t\nabla_t\Pi \\
&\quad - j\omega\mu_t\nabla_t \times \hat{z}\psi \\
\Rightarrow -\hat{z}\times\nabla_t E_z + \hat{z}\times\nabla_t\frac{\partial\Phi}{\partial z} + \nabla_t\frac{\partial\theta}{\partial z} &= -\nabla_t u_h + \hat{z}\times\nabla_t v_h - j\omega\mu_t\nabla_t\Pi \\
&\quad + j\omega\mu_t\hat{z}\times\nabla_t\psi \qquad (35)
\end{aligned}
$$

Note that by vector identity, $\nabla_t \cdot \left(\nabla_t \times \vec{V}\right) = 0 \forall \vec{V} \in \mathbb{C}^3$. This implies that the $\hat{z}\times\nabla_t$ and ∇_t components of (35) are linearly independent of one another. Thus, separating (35) into $\hat{z}\times\nabla_t$ and ∇_t terms implies that

$$
\hat{z}\times\nabla_t\left(-E_z + \frac{\partial\Phi}{\partial z} - v_h - j\omega\mu_t\psi\right) = 0
$$

$$\Rightarrow -E_z + \frac{\partial \Phi}{\partial z} - v_h - j\omega\mu_t\psi = 0 \tag{36}$$

and

$$\begin{aligned}
\nabla_t \left(\frac{\partial \theta}{\partial z} + u_h + j\omega\mu_t\Pi \right) &= 0 \\
\Rightarrow \frac{\partial \theta}{\partial z} + u_h + j\omega\mu_t\Pi &= 0 \\
\Rightarrow \Pi &= -\frac{\frac{\partial \theta}{\partial z} + u_h}{j\omega\mu_t} \qquad (37)
\end{aligned}$$

By duality, (36) and (37) imply that

$$-H_z + \frac{\partial \Pi}{\partial z} + v_e + j\omega\epsilon_t\theta = 0 \text{ and} \tag{38}$$

$$\Phi = \frac{\frac{\partial \psi}{\partial z} - u_e}{j\omega\epsilon_t} \tag{39}$$

Substituting (33) and (37) into (38) implies that

$$-\frac{\nabla_t^2\theta - J_{hz}}{j\omega\mu_z} - \frac{\frac{\partial^2\theta}{\partial z^2} + \frac{\partial u_h}{\partial z}}{j\omega\mu_t} + v_e + j\omega\epsilon_t\theta = 0$$

$$j\omega\mu_t \cdot \{\cdot\} \Rightarrow -\frac{\mu_t}{\mu_z}\nabla_t^2\theta - \frac{\partial^2\theta}{\partial z^2} - \underbrace{\omega^2\epsilon_t\mu_t}_{k_t^2}\theta = -\frac{\mu_t}{\mu_z}J_{hz} + \frac{\partial u_h}{\partial z} - j\omega\mu_t v_e \tag{40}$$

By duality, (40) implies that

$$-\frac{\epsilon_t}{\epsilon_z}\nabla_t^2\psi - \frac{\partial^2\psi}{\partial z^2} - k_t^2\psi = \frac{\epsilon_t}{\epsilon_z}J_{ez} - \frac{\partial u_e}{\partial z} - j\omega\epsilon_t v_h \tag{41}$$

It is useful to consider an operator and source notation approach for the scalar potential wave equations for differential equation analysis, such that principal and scattered solutions take the form

$$\mathcal{L}_\theta \theta^p = S_\theta \tag{42}$$

$$\mathcal{L}_\psi \psi^p = S_\psi \tag{43}$$

$$\mathcal{L}_\theta \theta^s = 0 \tag{44}$$

$$\mathcal{L}_\psi \psi^s = 0 \tag{45}$$

where

$$\mathcal{L}_\theta = -\frac{\mu_t}{\mu_z}\nabla_t^2 - \frac{\partial^2}{\partial z^2} - k_t^2 \tag{46}$$

$$\mathcal{L}_\psi = -\frac{\epsilon_t}{\epsilon_z}\nabla_t^2 - \frac{\partial^2}{\partial z^2} - k_t^2 \tag{47}$$

$$S_\theta = -\frac{\mu_t}{\mu_z} J_{hz} + \frac{\partial u_h}{\partial z} - j\omega\mu_t v_e \tag{48}$$

$$S_\psi = \frac{\epsilon_t}{\epsilon_z} J_{ez} - \frac{\partial u_e}{\partial z} - j\omega\epsilon_t v_h \tag{49}$$

It is also useful to determine auxiliary current density functions for future analyses. Thus,

$$\begin{aligned}
\nabla_t \cdot (31) \Rightarrow \nabla_t \cdot \vec{J}_{et} &= \nabla_t \cdot \nabla_t u_e + \cancelto{0}{\nabla_t \cdot \nabla_t \times \hat{z} v_e} \\
\Rightarrow \nabla_t \cdot \vec{J}_{et} &= \nabla_t^2 u_e && (50) \\
\nabla_t \cdot (32) \Rightarrow \nabla_t \cdot \vec{J}_{ht} &= \nabla_t^2 u_h && (51) \\
\nabla_t \times (31) \Rightarrow \nabla_t \times \vec{J}_{et} &= \cancelto{0}{\nabla_t \times \nabla_t u_e} + \nabla_t \times \nabla_t \times \hat{z} v_e \\
\Rightarrow \nabla_t \times \vec{J}_{et} &= \nabla_t \left(\cancelto{0}{\nabla_t \cdot \hat{z} v_e}\right) - \nabla_t^2 \hat{z} v_e \\
\Rightarrow \nabla_t \times \vec{J}_{et} &= -\hat{z}\nabla_t^2 v_e && (52) \\
\nabla_t \times (32) \Rightarrow \nabla_t \times \vec{J}_{ht} &= -\hat{z}\nabla_t^2 v_h && (53)
\end{aligned}$$

Summary of Scalar Potential Functions.

Fields:

$$\vec{E} = \vec{E}_t + \hat{z}E_z \qquad \vec{H} = \vec{H}_t + \hat{z}H_z$$

$$\vec{E}_t = \nabla_t\Phi + \nabla_t \times \hat{z}\theta \qquad \vec{H}_t = \nabla_t\Pi + \nabla_t \times \hat{z}\psi$$

$$E_z = -\frac{1}{j\omega\epsilon_z}\left(\nabla_t^2\psi + J_{ez}\right) \qquad H_z = \frac{1}{j\omega\mu_z}\left(\nabla_t^2\theta - J_{hz}\right)$$

Scalar Potentials:

$$\Phi = \frac{1}{j\omega\epsilon_t}\left(\frac{\partial\psi}{\partial z} - u_e\right) \qquad \Pi = -\frac{1}{j\omega\mu_t}\left(\frac{\partial\theta}{\partial z} - u_h\right)$$

where ψ and θ must satisfy

$$\mathcal{L}_\psi\psi^p = S_\psi \qquad \mathcal{L}_\theta\theta^p = S_\theta$$

$$\mathcal{L}_\psi\psi^s = 0 \qquad \mathcal{L}_\theta\theta_s = 0$$

$$\mathcal{L}_\psi = -\frac{\epsilon_t}{\epsilon_z}\nabla_t^2 - \frac{\partial^2}{\partial z^2} - k_t^2 \qquad \mathcal{L}_\theta = -\frac{\mu_t}{\mu_z}\nabla_t^2 - \frac{\partial^2}{\partial z^2} - k_t^2$$

$$S_\psi = \frac{\epsilon_t}{\epsilon_z}J_{ez} - \frac{\partial u_e}{\partial z} - j\omega\epsilon_t v_h \qquad S_\theta = -\frac{\mu_t}{\mu_z}J_{hz} + \frac{\partial u_h}{\partial z} - j\omega\mu_t v_e$$

Auxiliary Relations:

$$\nabla_t \cdot \vec{J}_{et} = \nabla_t^2 u_e \qquad \nabla_t \cdot \vec{J}_{ht} = \nabla_t^2 u_h$$

$$\nabla_t \times \vec{J}_{et} = -\hat{z}\nabla_t^2 v_e \qquad \nabla_t \times \vec{J}_{ht} = -\hat{z}\nabla_t^2 v_h$$

$$k_t^2 = \omega^2\epsilon_t\mu_t$$

2.2 Spectral Domain Analysis

Due to the infinite extent of the transverse directions of the parallel plate waveguide, it is natural to spatially employ the Fourier transform to aid in analysis due to its infinite limits of integration and the inherent traveling wave nature built into the transform itself.

Transform Definitions and Properties.

This analysis will consider a Fourier transform in the transverse plane and in the longitudinal direction for simplicity.

$$\mathcal{F}_\rho \left\{ f\left(\vec{\rho}, z\right) \right\} := \tilde{f}\left(\vec{\lambda}_\rho, z\right) = \iint\limits_{-\infty}^{\infty} f\left(\vec{\rho}, z\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\rho \tag{54}$$

$$\mathcal{F}_\rho^{-1} \left\{ \tilde{f}\left(\vec{\lambda}_\rho, z\right) \right\} := f\left(\vec{\rho}, z\right) = \frac{1}{\left(2\pi\right)^2} \iint\limits_{-\infty}^{\infty} \tilde{f}\left(\vec{\lambda}_\rho, z\right) e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho \tag{55}$$

$$\mathcal{F}_z \left\{ \tilde{f}\left(\vec{\lambda}_\rho, z\right) \right\} := \tilde{\tilde{f}}\left(\vec{\lambda}_\rho, \lambda_z\right) = \int\limits_{-\infty}^{\infty} \tilde{f}\left(\vec{\lambda}_\rho, z\right) e^{-j\lambda_z z} dz \tag{56}$$

$$\mathcal{F}_z^{-1} \left\{ \tilde{\tilde{f}}\left(\vec{\lambda}_\rho, \lambda_z\right) \right\} := \tilde{f}\left(\vec{\lambda}_\rho, z\right) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \tilde{\tilde{f}}\left(\vec{\lambda}_\rho, \lambda_z\right) e^{j\lambda_z z} d\lambda_z \tag{57}$$

where $\vec{\rho} = \hat{x}x + \hat{y}y$, $\vec{\lambda}_\rho = \hat{x}\lambda_x + \hat{y}\lambda_y$, $d^2\rho = dxdy$, and $d^2\lambda_\rho = d\lambda_x d\lambda_y$. These transforms lead to some very useful properties for simplifying equations from the previous section.

$$\mathcal{F}_\rho \left\{ \nabla_t f \right\} = j\vec{\lambda}_\rho \tilde{f}, \qquad \mathcal{F}_\rho \left\{ \nabla_t^2 f \right\} = -\lambda_\rho^2 \tilde{f}$$

$$\mathcal{F}_z \left\{ \frac{\partial}{\partial z} \tilde{f} \right\} = j\lambda_z \tilde{\tilde{f}}, \qquad \mathcal{F}_z \left\{ \frac{\partial^2}{\partial z^2} \tilde{f} \right\} = -\lambda_z^2 \tilde{\tilde{f}}$$

2.3 Principal Solutions

This analysis begins by finding principal solutions to the scalar potential differential equations. Due to the complexity of these analyses, it is desirable to retain as much analytical work as possible that can be generalized to numerous situations going forward. To that end, Green functions are developed to minimize duplication of analytic effort. Additionally, Green function solutions will provide physical insight into the structure of the fields.

Full Spectral Domain Principal Solutions.

Begin by transforming the particular solution to ψ to the full spectral domain. $\mathcal{F}_z\left\{\mathcal{F}_\rho\left\{(43)\right\}\right\}$ implies that

$$\tilde{\tilde{\mathcal{L}}}_\psi \tilde{\tilde{\psi}}^p = \tilde{\tilde{S}}_\psi \tag{58}$$

$$\tilde{\tilde{\mathcal{L}}}_\psi = \frac{\epsilon_t}{\epsilon_z}\lambda_{\rho\psi}^2 + \lambda_z^2 - k_t^2 \tag{59}$$

$$\tilde{\tilde{S}}_\psi = \frac{\epsilon_t}{\epsilon_z}\tilde{\tilde{J}}_{ez} - j\lambda_z\tilde{\tilde{u}}_e - j\omega\epsilon_t\tilde{\tilde{v}}_h \tag{60}$$

Substituting (59) and (60) into (58) implies that

$$\left[\lambda_z^2 - \underbrace{\left(k_t^2 - \frac{\epsilon_t}{\epsilon_z}\lambda_{\rho\psi}^2\right)}_{\lambda_{z\psi}^2}\right]\tilde{\tilde{\psi}}^p = \frac{\epsilon_t}{\epsilon_z}\tilde{\tilde{J}}_{ez} - j\lambda_z\tilde{\tilde{u}}_e - j\omega\epsilon_t\tilde{\tilde{v}}_h \tag{61}$$

$$\Rightarrow \tilde{\tilde{\psi}}^p = \frac{\frac{\epsilon_t}{\epsilon_z}\tilde{\tilde{J}}_{ez} - j\lambda_z\tilde{\tilde{u}}_e - j\omega\epsilon_t\tilde{\tilde{v}}_h}{\left(\lambda_z - \lambda_{z\psi}\right)\left(\lambda_z + \lambda_{z\psi}\right)} \tag{62}$$

Utilizing the auxiliary relation, $\mathcal{F}_z\left\{\mathcal{F}_\rho\left\{(50)\right\}\right\}$ implies that

$$j\vec{\lambda}_\rho \cdot \tilde{\tilde{\vec{J}}}_{et} = -\lambda_\rho^2\tilde{\tilde{u}}_e$$

$$\Rightarrow \tilde{\tilde{u}}_e = -\frac{j\vec{\lambda}_\rho \cdot \tilde{\tilde{\vec{J}}}_{et}}{\lambda_\rho^2} \tag{63}$$

Note that $\forall \vec{V} \in \mathbb{C}^3, \vec{\lambda}_\rho \cdot \vec{V} = \vec{\lambda}_\rho \cdot \vec{V}_t$ since $\vec{\lambda}_\rho$ has no $\hat{z}$ component. Thus the dot product destroys the $\hat{z}$ component of the vector $\vec{V}$. Therefore, (63) implies that

$$\tilde{\tilde{u}}_e = -\frac{j\vec{\lambda}_\rho}{\lambda_\rho^2} \cdot \tilde{\tilde{\vec{J}}}_e \tag{64}$$

Utilizing another auxiliary relation, $\mathcal{F}_z \left\{\mathcal{F}_\rho \left\{(53)\right\}\right\}$ implies that

$$\begin{aligned}
j\vec{\lambda}_\rho \times \tilde{\tilde{\vec{J}}}_{ht} &= \hat{z}\lambda_\rho^2 \tilde{\tilde{v}}_h \\
\hat{z} \cdot \{\cdot\} \Rightarrow j\hat{z} \cdot \vec{\lambda}_\rho \times \tilde{\tilde{\vec{J}}}_{ht} &= \lambda_\rho^2 \tilde{\tilde{v}}_h \\
\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c} \Rightarrow j\hat{z} \times \vec{\lambda}_\rho \cdot \tilde{\tilde{\vec{J}}}_{ht} &= \lambda_\rho^2 \tilde{\tilde{v}}_h \\
\Rightarrow \tilde{\tilde{v}}_h &= \frac{j\hat{z} \times \vec{\lambda}_\rho \cdot \tilde{\tilde{\vec{J}}}_{ht}}{\lambda_\rho^2} \\
\Rightarrow \tilde{\tilde{v}}_h &= \frac{j\hat{z} \times \vec{\lambda}_\rho}{\lambda_\rho^2} \cdot \tilde{\tilde{\vec{J}}}_h
\end{aligned} \tag{65}$$

Substituting (64) and (65) into (62) implies that

$$\begin{aligned}
\tilde{\tilde{\psi}}^p &= \frac{\frac{\epsilon_t}{\epsilon_z}\tilde{\tilde{J}}_{ez} - j\lambda_z(-\frac{j\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2} \cdot \tilde{\tilde{\vec{J}}}_e) - j\omega\epsilon_t(\frac{j\hat{z}\times\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2} \cdot \tilde{\tilde{\vec{J}}}_h)}{(\lambda_z - \lambda_{z\psi})(\lambda_z + \lambda_{z\psi})} \\
\tilde{\tilde{\psi}}^p &= \underbrace{\left[\frac{-\vec{\lambda}_\rho \frac{\lambda_z}{\lambda_{\rho\psi}^2} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{(\lambda_z - \lambda_{z\psi})(\lambda_z + \lambda_{z\psi})}\right]}_{\tilde{\tilde{\vec{G}}}_{\psi e}^p} \cdot \tilde{\tilde{\vec{J}}}_e + \underbrace{\left[\frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}}{(\lambda_z - \lambda_{z\psi})(\lambda_z + \lambda_{z\psi})}\right]}_{\tilde{\tilde{\vec{G}}}_{\psi h}^p} \cdot \tilde{\tilde{\vec{J}}}_h
\end{aligned} \tag{66}$$

By duality, (66) implies that

$$\tilde{\tilde{\theta}}^p = \underbrace{\left[\frac{\vec{\lambda}_\rho \frac{\lambda_z}{\lambda_{\rho\theta}^2} - \hat{z}\frac{\mu_t}{\mu_z}}{(\lambda_z - \lambda_{z\theta})(\lambda_z + \lambda_{z\theta})}\right]}_{\tilde{\tilde{\vec{G}}}_{\theta h}^p} \cdot \tilde{\tilde{\vec{J}}}_h + \underbrace{\left[\frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega\mu_t}{\lambda_{\rho\theta}^2}}{(\lambda_z - \lambda_{z\theta})(\lambda_z + \lambda_{z\theta})}\right]}_{\tilde{\tilde{\vec{G}}}_{\theta e}^p} \cdot \tilde{\tilde{\vec{J}}}_e, \quad \lambda_{z\theta}^2 = k_t^2 - \frac{\mu_t}{\mu_z}\lambda_{\rho\theta}^2 \tag{67}$$

Transverse Spectral Domain Principal Solutions: $\tilde{\psi}^p$ and $\tilde{\theta}^p$.

Now that the full spectral-domain particular solutions have been obtained for $\tilde{\tilde{\psi}}^p$ and $\tilde{\tilde{\theta}}^p$, the process of returning to the spatial domain can begin. First, return to the transverse spectral domain by applying the inverse longitudinal Fourier transform. $\mathcal{F}_z^{-1}\{(66)\}$ implies that

$$\begin{aligned}\tilde{\psi}^p &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{\psi}}^p e^{j\lambda_z z} d\lambda_z \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{\vec{G}}}^p_{\psi e} \cdot \tilde{\tilde{\vec{J}}}_e e^{j\lambda_z z} d\lambda_z + \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{\vec{G}}}^p_{\psi h} \cdot \tilde{\tilde{\vec{J}}}_h e^{j\lambda_z z} d\lambda_z \end{aligned} \tag{68}$$

Assuming $\tilde{\vec{J}}_e$ and $\tilde{\vec{J}}_h$ are continuous over the finite interval $a < z' < b$ and zero every where else (i.e. the current density only exists in a bounded region with respect to z', in this case between two parallel PEC plates) implies that

$$\tilde{\tilde{\vec{J}}}_\alpha = \int_{-\infty}^{\infty} \tilde{\vec{J}}_\alpha e^{-j\lambda_z z'} dz' = \int_a^b \tilde{\vec{J}}_\alpha e^{-j\lambda_z z'} dz' \tag{69}$$

where $\alpha \in \{e, h\}$. Substituting (69) into (68) and noting that $\tilde{\tilde{\vec{G}}}^p_{\psi e}$ stays constant with respect to z' implies that

$$\begin{aligned}\tilde{\psi}^p &= \int_{-\infty}^{\infty} \frac{\tilde{\tilde{\vec{G}}}^p_{\psi e}}{2\pi} \cdot \left[\int_a^b \tilde{\vec{J}}_e e^{-j\lambda_z z'} dz'\right] e^{j\lambda_z z} d\lambda_z + \int_{-\infty}^{\infty} \frac{\tilde{\tilde{\vec{G}}}^p_{\psi h}}{2\pi} \cdot \left[\int_a^b \tilde{\vec{J}}_h e^{-j\lambda_z z'} dz'\right] e^{j\lambda_z z} d\lambda_z \\ &= \int_a^b \left[\int_{-\infty}^{\infty} \frac{1}{2\pi} \tilde{\tilde{\vec{G}}}^p_{\psi e} e^{j\lambda_z (z-z')} d\lambda_z\right] \cdot \tilde{\vec{J}}_e dz' + \int_a^b \left[\int_{-\infty}^{\infty} \frac{1}{2\pi} \tilde{\tilde{\vec{G}}}^p_{\psi h} e^{j\lambda_z (z-z')} d\lambda_z\right] \cdot \tilde{\vec{J}}_h dz' \\ &= \int_a^b \tilde{\vec{G}}^p_{\psi e} \cdot \tilde{\vec{J}}_e dz' + \int_a^b \tilde{\vec{G}}^p_{\psi h} \cdot \tilde{\vec{J}}_h dz' \end{aligned} \tag{70}$$

Similarly, it can be shown that

$$\tilde{\theta}^p = \int_a^b \vec{\vec{\tilde{G}}}{}^p_{\theta e} \cdot \vec{\tilde{J}}_e dz' + \int_a^b \vec{\vec{\tilde{G}}}{}^p_{\theta h} \cdot \vec{\tilde{J}}_h dz' \tag{71}$$

Determination of Transverse Spectral Domain Principal Green Functions.

Next, the transverse spectral domain Green functions from (70) and (71) are determined by complex plane analysis. Complex plane analysis leverages Cauchy's Integral Theorem (CIT) and Cauchy's Integral Formula (CIF) to find a finite solution to an integral with infinite limits of integration. Many situations regarding the functional form of the integrand are explored in Appendix A.

First, it can be shown that two poles exist in the complex λ_z-plane for all Green functions used in this analysis thus far. These poles are indicated by the red x's at locations $\pm\lambda_{z\alpha}$ in Fig. 2, where $\alpha \in \{\psi, \theta\}$.

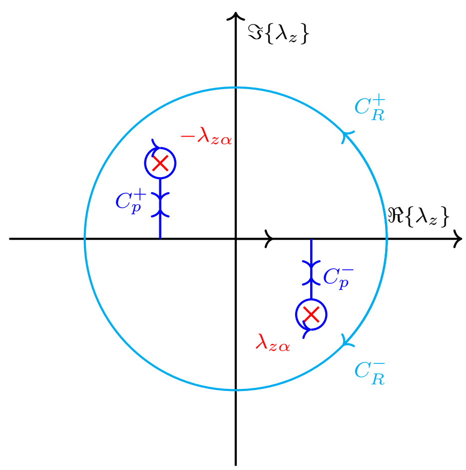

Figure 2. Complex poles (red) of the transverse spatial frequency domain principal scalar potential Green functions, deformation contours around those poles (blue) and closure contours as $R \to \infty$ (cyan) in the complex λ_z-plane.

CIT states that for any closed path of integration in the complex plane,

$$\oint f(\lambda_z) d\lambda_z = 0 \tag{72}$$

as long as $f(\lambda_z)$ is analytic inside and on the simple closed contour. In order for a real integral with infinite limits of integration to converge, a closed path of integration must be defined such that as $R \to \infty$, any terms containing R decay to zero faster than $\frac{1}{R}$. Careful choice of upper half plane closure (UHPC) and lower half plane closure (LHPC) contours (indicated in cyan as C_R^+ and C_R^- in Fig. 2) as $R \to \infty$ causes the infinite closure contour integral to decay to zero. However, the poles must also be accounted for by defining a deformation contour that circumvents the poles. Such deformation contours are indicated in blue as C_p^+ and C_p^- in Fig. 2 respectively. These circular contours are exaggerated for visibility in the figure, but in reality they are of radius $\varepsilon \to 0$. The linear "stem" components of the deformation contours are colocated in opposite directions, thus their contributions cancel completely.

The piecewise summation of the contours discussed above with the infinite real integral define closed contours. With closed contours defined, CIT becomes

$$0 = \begin{cases} \lim_{R\to\infty} \left[\int\limits_{-R}^{R} f(\lambda_z)\, d\lambda_z + \oint\limits_{C_p^+} f(\lambda_z)\, d\lambda_z + \cancelto{0}{\oint\limits_{C_R^+} f(\lambda_z)\, d\lambda_z} \right] & \text{...UHPC} \\ \lim_{R\to\infty} \left[\int\limits_{-R}^{R} f(\lambda_z)\, d\lambda_z + \oint\limits_{C_p^-} f(\lambda_z)\, d\lambda_z + \cancelto{0}{\oint\limits_{C_R^-} f(\lambda_z)\, d\lambda_z} \right] & \text{...LHPC} \end{cases}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} f(\lambda_z)\, d\lambda_z = \begin{cases} \oint\limits_{C_p^+} f(\lambda_z)\, d\lambda_z & \text{...UHPC} \\ -\oint\limits_{C_p^-} f(\lambda_z)\, d\lambda_z & \text{...LHPC} \end{cases} \tag{73}$$

CIF states that

$$\oint \frac{F(\lambda_z)}{\lambda_z - \lambda_{z0}} d\lambda_z = j2\pi F(\lambda_{z0}) \tag{74}$$

By rewriting $f(\lambda_z)$ to be of the form $\frac{F(\lambda_z)}{\lambda_z - \lambda_{z0}}$ for the respective poles that exist in the upper and lower half planes, (74) can be substituted into (73) to evaluate the infinite integrals. Note that a more generalized form of (74) is derived in Appendix A. Thus

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{F(\lambda_z)}{\lambda_z - \lambda_{z0}} d\lambda_z = \begin{cases} j2\pi F(\lambda_{z0}) & \text{...UHPC} \\ -j2\pi F(\lambda_{z0}) & \text{...LHPC} \end{cases} \tag{75}$$

Applying the above to the Green functions,

$$\begin{aligned} \vec{\vec{G}}^{p}_{\psi e} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{\vec{\tilde{G}}}^{p}_{\psi e} e^{j\lambda_z(z-z')} d\lambda_z \\ &= \int_{-\infty}^{\infty} \frac{-\vec{\lambda}_\rho \frac{\lambda_z}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\pi(\lambda_z - \lambda_{z\psi})(\lambda_z + \lambda_{z\psi})} e^{j\lambda_z(z-z')} d\lambda_z \end{aligned} \tag{76}$$

Note from Fig. 2 that under UHPC, $\lambda_{z0} = -\lambda_{z\alpha}$ and that under LHPC, $\lambda_{z0} = \lambda_{z\alpha}$. There are two cases that must be explored: the case when $z > z'$ and the case when $z < z'$. First evaluate $\vec{\vec{G}}^{p}_{\psi e}$.

When $z > z'$, that implies $z - z' > 0$. This further implies that the imaginary part of $\lambda_z, \Im\{\lambda_z\} > 0$ in order for the exponential term to decay as $\lambda_z \to \infty$, implying that UHPC is required. UHPC implies that

$$\begin{aligned} \vec{\vec{G}}^{pz^+}_{\psi e} &= \int_{-\infty}^{\infty} \frac{F(\lambda_z)}{(\lambda_z - (-\lambda_{z\psi}))} d\lambda_z \\ \Rightarrow F(\lambda_z) &= \frac{-\vec{\lambda}_\rho \frac{\lambda_z}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\pi(\lambda_z - \lambda_{z\psi})} e^{j\lambda_z(z-z')} \end{aligned} \tag{77}$$

Substituting (77) into (75) implies that

$$\vec{\vec{G}}^{pz^+}_{\psi e} = \left. \frac{j\cancel{2\pi}\left(-\vec{\lambda}_\rho \frac{\lambda_z}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}\right)}{\cancel{2\pi}\left(\lambda_z - \lambda_{z\psi}\right)} e^{j\lambda_z(z-z')} \right|_{\lambda_z = -\lambda_{z\psi}}$$

$$\Rightarrow \vec{\vec{G}}^{pz^+}_{\psi e} = -j\frac{\vec{\lambda}_\rho \frac{\lambda_{z\psi}}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}} e^{-j\lambda_{z\psi}(z-z')} \tag{78}$$

When $z < z'$, that implies $z - z' < 0$. This further implies that $\Im\{\lambda_z\} < 0$, implying that LHPC is required. LHPC implies that

$$\vec{\vec{G}}^{pz^-}_{\psi e} = \int_{-\infty}^{\infty} \frac{F(\lambda_z)}{(\lambda_z - \lambda_{z\psi})} d\lambda_z$$

$$\Rightarrow F(\lambda_z) = \frac{-\vec{\lambda}_\rho \frac{\lambda_z}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\pi\left(\lambda_z + \lambda_{z\psi}\right)} e^{j\lambda_z(z-z')} \tag{79}$$

Substituting (79) into (75) implies that

$$\vec{\vec{G}}^{pz^-}_{\psi e} = -\left. \frac{j\cancel{2\pi}\left(-\vec{\lambda}_\rho \frac{\lambda_z}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}\right)}{\cancel{2\pi}\left(\lambda_z + \lambda_{z\psi}\right)} e^{j\lambda_z(z-z')} \right|_{\lambda_z = \lambda_{z\psi}}$$

$$\Rightarrow \vec{\vec{G}}^{pz^-}_{\psi e} = -j\frac{-\vec{\lambda}_\rho \frac{\lambda_{z\psi}}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}} e^{j\lambda_{z\psi}(z-z')} \tag{80}$$

Examining (78) and (80) implies that

$$\boxed{\vec{\vec{G}}^{p}_{\psi e} = \underbrace{-j\frac{\vec{\lambda}_\rho \operatorname{sgn}(z-z') \frac{\lambda_{z\psi}}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}}_{\vec{g}^{p}_{\psi e}} e^{-j\lambda_{z\psi}|z-z'|}} \tag{81}$$

Next, evaluate $\vec{\vec{G}}^{p}_{\psi h}$.

When $z > z'$, that implies $z - z' > 0$. This further implies that $\Im\{\lambda_z\} > 0$, implying that UHPC is required. UHPC implies that

$$\vec{\vec{G}}_{\psi h}^{pz^+} = \int_{-\infty}^{\infty} \frac{F(\lambda_z)}{(\lambda_z - (-\lambda_{z\psi}))} d\lambda_z$$

$$\Rightarrow F(\lambda_z) = \frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}}{2\pi(\lambda_z - \lambda_{z\psi})} e^{j\lambda_z(z-z')} \tag{82}$$

Substituting (82) into (75) implies that

$$\vec{\vec{G}}_{\psi h}^{pz^+} = \left. \frac{j\cancel{2\pi}\left(\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}\right)}{\cancel{2\pi}(\lambda_z - \lambda_{z\psi})} e^{j\lambda_z(z-z')} \right|_{\lambda_z = -\lambda_{z\psi}}$$

$$\Rightarrow \vec{\vec{G}}_{\psi h}^{pz^+} = -j \frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}}{2\lambda_{z\psi}} e^{-j\lambda_{z\psi}(z-z')} \tag{83}$$

When $z < z'$, that implies $z - z' < 0$. This further implies that $\Im\{\lambda_z\} < 0$, implying that LHPC is required. LHPC implies that

$$\vec{\vec{G}}_{\psi h}^{pz^-} = \int_{-\infty}^{\infty} \frac{F(\lambda_z)}{(\lambda_z - \lambda_{z\psi})} d\lambda_z$$

$$\Rightarrow F(\lambda_z) = \frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}}{2\pi(\lambda_z + \lambda_{z\psi})} e^{j\lambda_z(z-z')} \tag{84}$$

Substituting (84) into (75) implies that

$$\vec{\vec{G}}_{\psi h}^{pz^-} = \left. -\frac{j\cancel{2\pi}\left(\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}\right)}{\cancel{2\pi}(\lambda_z + \lambda_{z\psi})} e^{j\lambda_z(z-z')} \right|_{\lambda_z = \lambda_{z\psi}}$$

$$\Rightarrow \vec{\vec{G}}_{\psi h}^{pz^-} = -j \frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda_{\rho\psi}^2}}{2\lambda_{z\psi}} e^{j\lambda_{z\psi}(z-z')} \tag{85}$$

Examining (83) and (85) reveals that

$$\boxed{\vec{\tilde{G}}^p_{\psi h} = \underbrace{-j\frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega \epsilon_t}{\lambda^2_{\rho\psi}}}{2\lambda_{z\psi}}}_{\vec{\tilde{g}}^p_{\psi h}} e^{-j\lambda_{z\psi}|z-z'|}} \tag{86}$$

Noting that $\tilde{\bar{\bar{G}}}^p_{\theta e}$ is the negative dual of $\tilde{\bar{\bar{G}}}^p_{\psi h}$ and that $\tilde{\bar{\bar{G}}}^p_{\theta h}$ is the negative dual of $\tilde{\bar{\bar{G}}}^p_{\psi e}$, it can be inferred that $\vec{\tilde{G}}^p_{\theta e}$ is the negative dual of $\vec{\tilde{G}}^p_{\psi h}$ and that $\vec{\tilde{G}}^p_{\theta h}$ is the negative dual of $\vec{\tilde{G}}^p_{\psi e}$. This implies that

$$\boxed{\begin{aligned} \vec{\tilde{G}}^p_{\theta e} &= \underbrace{-j\frac{\hat{z} \times \vec{\lambda}_\rho \frac{\omega \mu_t}{\lambda^2_{\rho\theta}}}{2\lambda_{z\theta}}}_{\vec{\tilde{g}}^p_{\theta e}} e^{-j\lambda_{z\theta}|z-z'|} && (87) \\ \vec{\tilde{G}}^p_{\theta h} &= \underbrace{j\frac{\vec{\lambda}_\rho \operatorname{sgn}(z-z') \frac{\lambda_{z\theta}}{\lambda^2_{\rho\theta}} + \hat{z}\frac{\mu_t}{\mu_z}}{2\lambda_{z\theta}}}_{\vec{\tilde{g}}^p_{\theta h}} e^{-j\lambda_{z\theta}|z-z'|} && (88) \end{aligned}}$$

Determination of Principal Solutions: $\tilde{\Phi}^p$ and $\tilde{\Pi}^p$.

Now that the particular solutions for $\tilde{\psi}^p$ and $\tilde{\theta}^p$ have been obtained, particular solutions to $\tilde{\Phi}^p$ and $\tilde{\Pi}^p$ must be derived in order to proceed. Begin by transforming Φ^p to the transverse spectral domain. $\mathcal{F}_\rho\{(39)\}$ implies that

$$\begin{aligned} \tilde{\Phi}^p &= \frac{\frac{\partial}{\partial z}\tilde{\psi}^p - \tilde{u}_e}{j\omega\epsilon_t} \\ &= \frac{1}{j\omega\epsilon_t}\left(\frac{\partial}{\partial z}\left[\int_a^b \vec{\tilde{G}}^p_{\psi e} \cdot \vec{\tilde{J}}_e dz' + \int_a^b \vec{\tilde{G}}^p_{\psi h} \cdot \vec{\tilde{J}}_h dz'\right] - \tilde{u}_e\right) \end{aligned} \tag{89}$$

While the constant fractional term of (89) can be brought inside the integrals with no issues, the partial derivative operator can only be brought inside under the following conditions:

- The integrand must be continuous over the interval of integration.

- The derivative of the integrand must be continuous over the interval of the integration.

These conditions pose a problem because, while the current densities $\vec{\tilde{J}}_{(e,h)}$ are assumed to be continuous over the interval $a < z' < b$, $\vec{\tilde{G}}^{p}_{\psi e}$ is discontinuous at $z' = z$ due to the signum function. Further, the derivatives of both $\vec{\tilde{G}}^{p}_{\psi e}$ and $\vec{\tilde{G}}^{p}_{\psi h}$ are discontinuous at $z' = z$ due to the absolute value term in the exponent. To mitigate these issues, it is necessary to divide each integral into two subregions where the integrands and their derivatives adhere to the continuity condition. Using principal value integration,

$$PV \int_{a}^{b} f\left(z, z'\right) dz' = \lim_{\delta \to 0} \left[\int_{a}^{z-\delta} f\left(z, z'\right) dz' + \int_{z+\delta}^{b} f\left(z, z'\right) dz' \right] \tag{90}$$

Note that when $f\left(z, z'\right) = \vec{\tilde{\tilde{G}}}^{p}_{\alpha(e,h)}$, the choice to divide the integral at $z = z'$ causes any signum terms in $\vec{\tilde{\tilde{g}}}^{p}_{\alpha(e,h)}$ to evaluate to a constant ± 1, thus making $\vec{\tilde{\tilde{g}}}^{p}_{\alpha(e,h)}$ constant with respect to both z and z'. Additionally, the problem term in the exponent is now no longer an absolute value and thus the derivative is continuous. Due to variable limits of integration in (90), Leibniz rule will be required. Leibniz integral rule states that

$$\begin{aligned} \frac{\partial}{\partial z} \left[\int_{a(z)}^{b(z)} f\left(z, z'\right) dz' \right] &= f\left(z, z' = b(z)\right) \cdot \frac{\partial b(z)}{\partial z} - f\left(z, z' = a(z)\right) \cdot \frac{\partial a(z)}{\partial z} \\ &+ \int_{a(z)}^{b(z)} \frac{\partial f\left(z, z'\right)}{\partial z} dz' \end{aligned} \tag{91}$$

where $-\infty < a(z), b(z) < \infty$, and inside the integral only the variation of $f\left(z, z'\right)$

with respect to z is considered. Applying these principles to the electric integral term of $\frac{\partial}{\partial z}\tilde{\psi}^p$ implies that

$$\begin{aligned}
\frac{\partial}{\partial z}\tilde{\psi}_e^p &= \frac{\partial}{\partial z}\int\limits_a^b \vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}|z-z'|}\cdot\vec{\tilde{J}}_e dz' \\
&= \lim_{\delta\to 0}\left[\underbrace{\frac{\partial}{\partial z}\int\limits_a^{z-\delta}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'}_{z>z'} + \underbrace{\frac{\partial}{\partial z}\int\limits_{z+\delta}^{b}\vec{\vec{g}}_{\psi e}^p e^{j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'}_{z<z'}\right]
\end{aligned}\tag{92}$$

Evaluating the portion of (92) where $z > z'$, letting $z^- = z-\delta$ and noting that $\lim_{\delta\to 0} z^- = z$ implies that

$$\begin{aligned}
\lim_{z^-\to z}\left[\frac{\partial}{\partial z}\int\limits_a^{z^-}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'\right] &= \lim_{z^-\to z}\Bigg[\cancelto{1}{\frac{\partial z^-}{\partial z}}\vec{\vec{g}}_{\psi e}^p \cancelto{1}{e^{-j\lambda_{z\psi}(z-z^-)}}\cdot\vec{\tilde{J}}_e\left(z'=z^-\right) \\
&\qquad -\cancelto{0}{\frac{\partial a}{\partial z}}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-a)}\cdot\vec{\tilde{J}}_e\left(z'=a\right) + \int\limits_a^{z^-}\frac{\partial}{\partial z}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'\Bigg] \\
&= \vec{\vec{g}}_{\psi e}^p\cdot\vec{\tilde{J}}_e(z) - j\lambda_{z\psi}\int\limits_a^{z^-}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz' \\
&= \left(-j\frac{\vec{\lambda}_\rho \cancelto{1}{\mathrm{sgn}\left(z-z'\right)}\frac{\lambda_{z\psi}}{\lambda_\rho^2} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}\right)\cdot\vec{\tilde{J}}_e(z) - j\lambda_{z\psi}\int\limits_a^{z^-}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz' \\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_\rho^2} - j\frac{\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}\right)\cdot\vec{\tilde{J}}_e(z) - j\lambda_{z\psi}\int\limits_a^{z^-}\vec{\vec{g}}_{\psi e}^p e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'
\end{aligned}\tag{93}$$

Evaluating the portion of (92) where $z < z'$, letting $z^+ = z+\delta$ and noting that

$\lim_{\delta\to 0} z^+ = z$ implies that

$$\lim_{z^+\to z}\left[\frac{\partial}{\partial z}\int\limits_{z^+}^{b}\vec{\tilde{g}}^p_{\psi e}e^{j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'\right]=\lim_{z^+\to z}\left[\cancelto{0}{\frac{\partial b}{\partial z}}\vec{\tilde{g}}^p_{\psi e}e^{j\lambda_{z\psi}(z-b)}\cdot\vec{\tilde{J}}_e\left(z'=b\right)\right.$$

$$\left.-\cancelto{1}{\frac{\partial z^+}{\partial z}}\vec{\tilde{g}}^p_{\psi e}\cancelto{1}{e^{j\lambda_{z\psi}\left(z-z^+\right)}}\cdot\vec{\tilde{J}}_e\left(z'=z^+\right)+\int\limits_{z^+}^{b}\frac{\partial}{\partial z}\vec{\tilde{g}}^p_{\psi e}e^{j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'\right]$$

$$=-\vec{\tilde{g}}^p_{\psi e}\cdot\vec{\tilde{J}}_e\left(z\right)+j\lambda_{z\psi}\int\limits_{a}^{z^-}\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'$$

$$=-\left(-j\frac{\vec{\lambda}_\rho\cancelto{-1}{\mathrm{sgn}\left(z-z'\right)}\frac{\lambda_{z\psi}}{\lambda^2_{\rho\psi}}+\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}\right)\cdot\vec{\tilde{J}}_e\left(z\right)+j\lambda_{z\psi}\int\limits_{a}^{z^-}\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'$$

$$=\left(-j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}}+j\frac{\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}\right)\cdot\vec{\tilde{J}}_e\left(z\right)+j\lambda_{z\psi}\int\limits_{a}^{z^-}\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz' \tag{94}$$

Substituting (93) and (94) into (92) implies that

$$\frac{\partial}{\partial z}\tilde{\psi}^p_e=\left(-j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}}-j\frac{\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}\right)\cdot\vec{\tilde{J}}_e\left(z\right)-j\lambda_{z\psi}\int\limits_{a}^{z^-}\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'$$

$$+\left[\left(-j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}}+j\frac{\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}\right)\cdot\vec{\tilde{J}}_e\left(z\right)dz'-j\lambda_{z\psi}\int\limits_{a}^{z^-}\left(-\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}(z-z')}\cdot\vec{\tilde{J}}_e dz'\right)\right]$$

$$=\left(-j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}}-\cancel{j\frac{\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}}-j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}}+\cancel{j\frac{\hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}}\right)\cdot\vec{\tilde{J}}_e\left(z\right)$$

$$-j\lambda_{z\psi}\int\limits_{a}^{b}\mathrm{sgn}\left(z-z'\right)\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}|z-z'|}\cdot\vec{\tilde{J}}_e dz'$$

$$=-j\frac{\vec{\lambda}_\rho}{\lambda^2_{\rho\psi}}\cdot\vec{\tilde{J}}_e-j\lambda_{z\psi}\int\limits_{a}^{b}\mathrm{sgn}\left(z-z'\right)\vec{\tilde{g}}^p_{\psi e}e^{-j\lambda_{z\psi}|z-z'|}\cdot\vec{\tilde{J}}_e dz' \tag{95}$$

$\mathcal{F}_z^{-1}\left\{(64)\right\}$ implies that

$$\tilde{u}_e = -\frac{j\vec{\lambda}_\rho}{\lambda_\rho^2} \cdot \vec{\tilde{J}}_e \tag{96}$$

Substituting (96) into (95) implies that

$$\frac{\partial}{\partial z}\tilde{\psi}_e^p = \tilde{u}_e - j\lambda_{z\psi} \int_a^b \operatorname{sgn}\left(z - z'\right) \vec{\tilde{g}}_{\psi e}^p e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_e dz' \tag{97}$$

Applying these same techniques to the magnetic integral term of $\frac{\partial}{\partial z}\tilde{\psi}^p$, it can be shown that

$$\frac{\partial}{\partial z}\tilde{\psi}_h^p = -j\lambda_{z\psi} \int_a^b \operatorname{sgn}\left(z - z'\right) \vec{\tilde{g}}_{\psi h}^p e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_h dz' \tag{98}$$

Evaluating (97) and (98) reveals that

$$\begin{aligned}
\frac{\partial}{\partial z}\tilde{\psi}^p &= \frac{\partial}{\partial z}\tilde{\psi}_e^p + \frac{\partial}{\partial z}\tilde{\psi}_h^p \\
&= \tilde{u}_e - j\lambda_{z\psi}\left[\int_a^b \operatorname{sgn}\left(z - z'\right) \vec{\tilde{g}}_{\psi e}^p e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_e dz'\right. \\
&\quad \left. + \int_a^b \operatorname{sgn}\left(z - z'\right) \vec{\tilde{g}}_{\psi h}^p e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_h dz'\right]
\end{aligned} \tag{99}$$

Substituting (99) into (89) implies that

$$\begin{aligned}
\tilde{\Phi}^p = \frac{1}{j\omega\epsilon_t}\Bigg(-j\lambda_{z\psi}\Bigg[&\int_a^b \operatorname{sgn}\left(z - z'\right) \vec{\tilde{g}}_{\psi e}^p e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_e dz' \\
&+ \int_a^b \operatorname{sgn}\left(z - z'\right) \vec{\tilde{g}}_{\psi h}^p e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_h dz'\Bigg] + \cancel{\tilde{u}_e} - \cancel{\tilde{u}_e}\Bigg)
\end{aligned}$$

$$= \int_a^b \left(-\frac{\lambda_{z\psi}}{\omega\epsilon_t}\right) \operatorname{sgn}(z-z')\, \vec{\tilde{g}}^p_{\psi e} e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_e dz'$$

$$+ \int_a^b \left(-\frac{\lambda_{z\psi}}{\omega\epsilon_t}\right) \operatorname{sgn}(z-z')\, \vec{\tilde{g}}^p_{\psi h} e^{-j\lambda_{z\psi}|z-z'|} \cdot \vec{\tilde{J}}_h dz'$$

$$= \int_a^b \underbrace{\left(-\frac{\lambda_{z\psi}}{\omega\epsilon_t}\right) \operatorname{sgn}(z-z')\, \vec{\tilde{G}}^p_{\psi e}}_{\vec{\tilde{G}}^p_{\Phi e}} \cdot \vec{\tilde{J}}_e dz' + \int_a^b \underbrace{\left(-\frac{\lambda_{z\psi}}{\omega\epsilon_t}\right) \operatorname{sgn}(z-z')\, \vec{\tilde{G}}^p_{\psi h}}_{\vec{\tilde{G}}^p_{\Phi h}} \cdot \vec{\tilde{J}}_h dz' \quad (100)$$

By duality, (100) implies that

$$\tilde{\Pi}^p = \int_a^b \underbrace{\frac{\lambda_{z\theta}}{\omega\mu_t} \operatorname{sgn}(z-z')\, \vec{\tilde{G}}^p_{\theta e}}_{\vec{\tilde{G}}^p_{\Pi e}} \cdot \vec{\tilde{J}}_e dz' + \int_a^b \underbrace{\frac{\lambda_{z\theta}}{\omega\mu_t} \operatorname{sgn}(z-z')\, \vec{\tilde{G}}^p_{\theta h}}_{\vec{\tilde{G}}^p_{\Pi h}} \cdot \vec{\tilde{J}}_h dz' \quad (101)$$

Summary of Principal Solutions.

$\tilde{\psi}^p$ Solution:

$$\boxed{\begin{aligned}
\tilde{\psi}^p &= \int_a^b \vec{\tilde{G}}^p_{\psi e} \cdot \vec{\tilde{J}}_e dz' + \int_a^b \vec{\tilde{G}}^p_{\psi h} \cdot \vec{\tilde{J}}_h dz' \\
\vec{\tilde{G}}^p_{\psi e} &= \underbrace{-j\frac{\vec{\lambda}_\rho \operatorname{sgn}(z-z')\frac{\lambda_{z\psi}}{\lambda^2_{\rho\psi}} + \hat{z}\frac{\epsilon_t}{\epsilon_z}}{2\lambda_{z\psi}}}_{\vec{\tilde{g}}^p_{\psi e}} e^{-j\lambda_{z\psi}|z-z'|} \\
\vec{\tilde{G}}^p_{\psi h} &= \underbrace{-j\frac{\hat{z}\times\vec{\lambda}_\rho \frac{\omega\epsilon_t}{\lambda^2_{\rho\psi}}}{2\lambda_{z\psi}}}_{\vec{\tilde{g}}^p_{\psi h}} e^{-j\lambda_{z\psi}|z-z'|} \\
\lambda^2_{z\psi} &= \omega^2\epsilon_t\mu_t - \frac{\epsilon_t}{\epsilon_z}\lambda^2_{\rho\psi}
\end{aligned}}$$

$\tilde{\Phi}^p$ Solution:

$$\boxed{\tilde{\Phi}^p = \int\limits_a^b \underbrace{\left(-\frac{\lambda_{z\psi}}{\omega\epsilon_t}\right)\operatorname{sgn}(z-z')\,\vec{\tilde{G}}^p_{\psi e}}_{\vec{\tilde{G}}^p_{\Phi e}} \cdot \vec{\tilde{J}}_e dz' + \int\limits_a^b \underbrace{\left(-\frac{\lambda_{z\psi}}{\omega\epsilon_t}\right)\operatorname{sgn}(z-z')\,\vec{\tilde{G}}^p_{\psi h}}_{\vec{\tilde{G}}^p_{\Phi h}} \cdot \vec{\tilde{J}}_h dz'} \tag{102}$$

$\tilde{\theta}^p$ Solution:

$$\boxed{\begin{aligned}
\tilde{\theta}^p &= \int\limits_a^b \vec{\tilde{G}}^p_{\theta e} \cdot \vec{\tilde{J}}_e dz' + \int\limits_a^b \vec{\tilde{G}}^p_{\theta h} \cdot \vec{\tilde{J}}_h dz' \\
\vec{\tilde{G}}^p_{\theta e} &= \underbrace{-j\frac{\hat{z}\times\vec{\lambda}_\rho \frac{\omega\mu_t}{\lambda^2_{\rho\theta}}}{2\lambda_{z\theta}}}_{\vec{\tilde{g}}^p_{\theta e}} e^{-j\lambda_{z\theta}|z-z'|} \\
\vec{\tilde{G}}^p_{\theta h} &= \underbrace{j\frac{\vec{\lambda}_\rho \operatorname{sgn}(z-z')\frac{\lambda_{z\theta}}{\lambda^2_{\rho\theta}} + \hat{z}\frac{\mu_t}{\mu_z}}{2\lambda_{z\theta}}}_{\vec{\tilde{g}}^p_{\theta h}} e^{-j\lambda_{z\theta}|z-z'|} \\
\lambda^2_{z\theta} &= \omega^2\epsilon_t\mu_t - \frac{\mu_t}{\mu_z}\lambda^2_{\rho\theta}
\end{aligned}}$$

$\tilde{\Pi}^p$ Solution:

$$\boxed{\tilde{\Pi}^p = \int\limits_a^b \underbrace{\frac{\lambda_{z\theta}}{\omega\mu_t}\operatorname{sgn}(z-z')\,\vec{\tilde{G}}^p_{\theta e}}_{\vec{\tilde{G}}^p_{\Pi e}} \cdot \vec{\tilde{J}}_e dz' + \int\limits_a^b \underbrace{\frac{\lambda_{z\theta}}{\omega\mu_t}\operatorname{sgn}(z-z')\,\vec{\tilde{G}}^p_{\theta h}}_{\vec{\tilde{G}}^p_{\Pi h}} \cdot \vec{\tilde{J}}_h dz'} \tag{103}$$

2.4 Scattered Solutions

Now that the principal solutions have been determined, the scattered solutions must be derived in order to find total solutions for the scalar potentials. Taking the forward transverse Fourier transform of the differential equation for Ψ^s, $\mathcal{F}_\rho\{(45)\}$

implies that

$$\left(\frac{\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}}\lambda_{\rho\psi}^2 - \frac{\partial^2}{\partial z^2} - k_{t\{1,2\}}^2\right)\tilde{\psi}_{\{1,2\}}^s = 0$$

$$\Rightarrow -\frac{\partial^2 \tilde{\psi}_{\{1,2\}}^s}{\partial z^2} - \lambda_{z\psi\{1,2\}}^2 \tilde{\psi}_{\{1,2\}}^s = 0$$

$$\Rightarrow \tilde{\psi}_{\{1,2\}}^s = \tilde{\psi}_{\{1,2\}}^+ e^{-j\lambda_{z\psi\{1,2\}}z} + \tilde{\psi}_{\{1,2\}}^- e^{j\lambda_{z\psi\{1,2\}}z} \tag{104}$$

Similarly, it can be shown that

$$\tilde{\theta}_{\{1,2\}}^s = \tilde{\theta}_{\{1,2\}}^+ e^{-j\lambda_{z\theta\{1,2\}}z} + \tilde{\theta}_{\{1,2\}}^- e^{j\lambda_{z\theta\{1,2\}}z} \tag{105}$$

Thus, the total solutions for $\tilde{\psi}_{\{1,2\}}$ and $\tilde{\theta}_{\{1,2\}}$ are

$$\tilde{\psi}_{\{1,2\}} = \tilde{\psi}_{\{1,2\}}^p + \tilde{\psi}_{\{1,2\}}^s = \tilde{\psi}_{\{1,2\}}^p + \tilde{\psi}_{\{1,2\}}^+ \underbrace{e^{-j\lambda_{z\psi\{1,2\}}z}}_{P_{\psi\{1,2\}z}} + \tilde{\psi}_{\{1,2\}}^- e^{j\lambda_{z\psi\{1,2\}}z} \tag{106}$$

$$\tilde{\theta}_{\{1,2\}} = \tilde{\theta}_{\{1,2\}}^p + \tilde{\theta}_{\{1,2\}}^s = \tilde{\theta}_{\{1,2\}}^p + \tilde{\theta}_{\{1,2\}}^+ \underbrace{e^{-j\lambda_{z\theta\{1,2\}}z}}_{P_{\theta\{1,2\}z}} + \tilde{\theta}_{\{1,2\}}^- e^{j\lambda_{z\theta\{1,2\}}z} \tag{107}$$

From this point forward, exponentials may be substituted with P notation using the convention

$$\boxed{P_{\alpha\{1,2\}\beta} = e^{-j\lambda_{z\alpha\{1,2\}}\beta}} \tag{108}$$

where $\alpha \in \{\psi, \theta\}$ and $\beta \in \mathbb{R}$. This makes it easier split up exponentials with multiple terms, makes it easier to visually identify patterns of exponential terms and in many cases makes the notation more compact. Noting that a boundary condition exists at $z = d$, it is useful to define $\tilde{\psi}_2$ and $\tilde{\theta}_2$ at that boundary. Thus by shifting the scattered solution at $z = d$ by d,

$$\tilde{\psi}_1 = \tilde{\psi}_1^p + \tilde{\psi}_1^+ P_{\psi 1z} + \tilde{\psi}_1^- P_{\psi 1z}^{-1} \tag{109}$$

$$\tilde{\psi}_2 = \tilde{\psi}_2^p + \tilde{\psi}_2^+ P_{\psi 2z} P_{\psi 2d}^{-1} + \tilde{\psi}_2^- P_{\psi 2z}^{-1} P_{\psi 2d} \tag{110}$$

$$\tilde{\theta}_1 = \tilde{\theta}_1^p + \tilde{\theta}_1^+ P_{\theta 1z} + \tilde{\theta}_1^- P_{\theta 1z}^{-1} \tag{111}$$

$$\tilde{\theta}_2 = \tilde{\theta}_2^p + \tilde{\theta}_2^+ P_{\theta 2z} P_{\theta 2d}^{-1} + \tilde{\theta}_2^- P_{\theta 2z}^{-1} P_{\theta 2d} \tag{112}$$

To find the unknown coefficients in (109) - (112), boundary conditions must be applied. Since there are eight unknown scattering coefficients, eight boundary conditions are needed to have a well-posed solution. At $z = 0$ and $z = d$, PEC boundary conditions exist (i.e. $\vec{E}_t = 0$).

Evaluating (29) at $z = 0$ and $z = d$ implies that

$$\left.\vec{E}_{t\{1,2\}}\right|_{z\in\{0,d\}} = \nabla_t \Phi_{\{1,2\}} + \nabla_t \times \hat{z}\theta_{\{1,2\}} = 0$$

$$\Rightarrow \left.\tilde{\theta}_{\{1,2\}}\right|_{z\in\{0,d\}} = \left.\tilde{\Phi}_{\{1,2\}}\right|_{z\in\{0,d\}} = 0 \tag{113}$$

$$\Rightarrow \left.\tilde{\Phi}_{\{1,2\}}\right|_{z\in\{0,d\}} = \frac{1}{j\omega\epsilon_{t\{1,2\}}} \left.\left(\frac{\partial}{\partial z}\tilde{\psi}_{\{1,2\}} - \tilde{u}_e\right)\right|_{z\in\{0,d\}} = 0 \tag{114}$$

This accounts for four of the needed equations. Next, the boundary at $z = h$ requires continuity of $\tilde{\vec{E}}_t$ and $\tilde{\vec{H}}_t$. Evaluating (29) at $z = h$ implies that

$$\left.\vec{E}_{t1}\right|_{z=h} = \left.\vec{E}_{t2}\right|_{z=h}$$

$$\Rightarrow \left. j\vec{\lambda}_\rho \tilde{\Phi}_1 - j\hat{z} \times \vec{\lambda}_\rho \tilde{\theta}_1 \right|_{z=h} = \left. j\vec{\lambda}_\rho \tilde{\Phi}_2 - j\hat{z} \times \vec{\lambda}_\rho \tilde{\theta}_2 \right|_{z=h}$$

$$\Rightarrow \left.\tilde{\theta}_1\right|_{z=h} = \left.\tilde{\theta}_2\right|_{z=h}, \left.\tilde{\Phi}_1\right|_{z=h} = \left.\tilde{\Phi}_2\right|_{z=h} \tag{115}$$

$$\Rightarrow \frac{1}{j\omega\epsilon_{t1}} \left.\left(\frac{\partial}{\partial z}\tilde{\psi}_1 - \tilde{u}_e\right)\right|_{z=h} = \frac{1}{j\omega\epsilon_{t2}} \left.\left(\frac{\partial}{\partial z}\tilde{\psi}_2 - \tilde{u}_e\right)\right|_{z=h} \tag{116}$$

Evaluating (30) at $z = h$ implies that

$$\left.\vec{H}_{t1}\right|_{z=h} = \left.\vec{H}_{t2}\right|_{z=h}$$

$$\Rightarrow j\vec{\lambda}_\rho\tilde{\Pi}_1 - j\hat{z}\times\vec{\lambda}_\rho\tilde{\psi}_1\Big|_{z=h} = j\vec{\lambda}_\rho\tilde{\Pi}_2 - j\hat{z}\times\vec{\lambda}_\rho\tilde{\psi}_2\Big|_{z=h}$$

$$\Rightarrow \tilde{\psi}_1\Big|_{z=h} = \tilde{\psi}_2\Big|_{z=h},\ \tilde{\Pi}_1\Big|_{z=h} = \tilde{\Pi}_2\Big|_{z=h} \tag{117}$$

$$\Rightarrow -\frac{1}{j\omega\mu_{t1}}\left(\frac{\partial}{\partial z}\tilde{\theta}_1 + \tilde{u}_h\right)\Bigg|_{z=h} = -\frac{1}{j\omega\mu_{t2}}\left(\frac{\partial}{\partial z}\tilde{\theta}_2 + \tilde{u}_h\right)\Bigg|_{z=h} \tag{118}$$

(115) - (118) account for the remaining four equations needed to form a well-posed system (i.e. a unique solution exists that varies continuously based on initial conditions).

Boundary Condition 1: PEC boundary condition at $z = 0$.

Evaluating (113) at $z = 0$ implies that

$$\begin{aligned} 0 = \tilde{\theta}_1\Big|_{z=0} &= \left[\tilde{\theta}_1^p + \tilde{\theta}_1^+ e^{-j\lambda_{z\theta 1}z} + \tilde{\theta}_1^- e^{j\lambda_{z\theta 1}z}\right]\Big|_{z=0} \\ &= \underbrace{\int_0^h \vec{\tilde{G}}^p_{\theta 1e}(z=0)\cdot\vec{\tilde{J}}_e dz' + \int_0^h \vec{\tilde{G}}^p_{\theta 1h}(z=0)\cdot\vec{\tilde{J}}_h dz'}_{V^-_{\theta 1}} + \tilde{\theta}_1^+ + \tilde{\theta}_1^- \end{aligned}$$

$$\Rightarrow \tilde{\theta}_1^+ + \tilde{\theta}_1^- = -V^-_{\theta 1} \tag{119}$$

Evaluating (114) at $z = 0$ implies that

$$\begin{aligned} 0 &= \frac{1}{j\omega\epsilon_{t1}}\left(\frac{\partial}{\partial z}\tilde{\psi}_1 - \tilde{u}_{e1}\right)\Bigg|_{z=0} \\ &= \left(\frac{\partial}{\partial z}\tilde{\psi}_1^p - \tilde{u}_{e1} + \frac{\partial}{\partial z}\tilde{\psi}_1^s\right)\Bigg|_{z=0} \\ &= \Bigg(\cancel{\tilde{u}_{e1}} + \int_0^h \frac{\partial}{\partial z}\vec{\tilde{G}}^p_{\psi 1e}\cdot\vec{\tilde{J}}_e dz' + \int_0^h \frac{\partial}{\partial z}\vec{\tilde{G}}^p_{\psi 1h}\cdot\vec{\tilde{J}}_h dz' - \cancel{\tilde{u}_{e1}} \\ &\qquad + \frac{\partial}{\partial z}\left(\tilde{\psi}_1^+ e^{-j\lambda_{z\psi 1}z} + \tilde{\psi}_1^- e^{j\lambda_{z\psi 1}z}\right)\Bigg)\Bigg|_{z=0} \end{aligned}$$

$$
\begin{aligned}
0 = &-j\lambda_{z\psi 1}\left(\int_0^h \operatorname{sgn}(z-z')\,\tilde{\vec{\vec{G}}}^{p}_{\psi 1e}\cdot\tilde{\vec{J}}_e dz' \right.\\
&\left.\left.+\int_0^h \operatorname{sgn}(z-z')\,\tilde{\vec{\vec{G}}}^{p}_{\psi 1h}\cdot\tilde{\vec{J}}_h dz' + \tilde{\psi}_1^{+}e^{-j\lambda_{z\psi 1}z} - \tilde{\psi}_1^{-}e^{j\lambda_{z\psi 1}z}\right)\right|_{z=0}\\
= &\int_0^h \cancelto{-1}{\operatorname{sgn}(-z')}\tilde{\vec{\vec{G}}}^{p}_{\psi 1e}(z=0)\cdot\tilde{\vec{J}}_e dz' + \int_0^h \cancelto{-1}{\operatorname{sgn}(-z')}\tilde{\vec{\vec{G}}}^{p}_{\psi 1h}(z=0)\cdot\tilde{\vec{J}}_h dz' + \tilde{\psi}_1^{+} - \tilde{\psi}_1^{-}\\
= &\underbrace{\int_0^h \tilde{\vec{\vec{G}}}^{p}_{\psi 1e}(z=0)\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\vec{\vec{G}}}^{p}_{\psi 1h}(z=0)\cdot\tilde{\vec{J}}_h dz'}_{V^{-}_{\psi 1}} - \tilde{\psi}_1^{+} + \tilde{\psi}_1^{-}\\
\Rightarrow &\ \tilde{\psi}_1^{+} - \tilde{\psi}_1^{-} = V^{-}_{\psi 1}
\end{aligned}
\tag{120}
$$

Boundary Condition 2: PEC boundary condition at $z = d$.

Evaluating (113) at $z = d$ implies that

$$
\begin{aligned}
0 = \tilde{\theta}_2\Big|_{z=d} &= \left[\tilde{\theta}_2^{p} + \tilde{\theta}_2^{+}P_{\theta 2z}P^{-1}_{\theta 2d} + \tilde{\theta}_2^{-}P^{-1}_{\theta 2z}P_{\theta 2d}\right]\Big|_{z=d}\\
&= \underbrace{\int_h^d \tilde{\vec{\vec{G}}}^{p}_{\theta 2e}(z=d)\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\vec{\vec{G}}}^{p}_{\theta 2h}(z=d)\cdot\tilde{\vec{J}}_h dz'}_{V^{+}_{\theta 2}} + \tilde{\theta}_2^{+} + \tilde{\theta}_2^{-}\\
\Rightarrow \tilde{\theta}_2^{+} + \tilde{\theta}_2^{-} &= -V^{+}_{\theta 2}
\end{aligned}
\tag{121}
$$

Evaluating (114) at $z = d$ implies that

$$
\begin{aligned}
0 = &\frac{1}{j\omega\epsilon_{t2}}\left(\frac{\partial}{\partial z}\tilde{\psi}_2 - \tilde{u}_{e2}\right)\Big|_{z=d} = \left(\frac{\partial}{\partial z}\tilde{\psi}_2^{p} - \tilde{u}_{e2} + \frac{\partial}{\partial z}\tilde{\psi}_2^{s}\right)\Big|_{z=d}\\
= &\left(\cancel{\tilde{u}_{e2}} - j\lambda_{z\psi 2}\int_h^d \operatorname{sgn}(z-z')\,\tilde{\vec{\vec{G}}}^{p}_{\psi 2e}\cdot\tilde{\vec{J}}_e dz' - j\lambda_{z\psi 2}\int_h^d \operatorname{sgn}(z-z')\,\tilde{\vec{\vec{G}}}^{p}_{\psi 2h}\cdot\tilde{\vec{J}}_h dz'\right.\\
&\left.\left. - \cancel{\tilde{u}_{e2}} + \frac{\partial}{\partial z}\left(\tilde{\psi}_2^{+}P_{\psi 2z}P^{-1}_{\psi 2d} + \tilde{\psi}_2^{-}P^{-1}_{\psi 2z}P_{\psi 2d}\right)\right)\right|_{z=d}
\end{aligned}
$$

$$
\begin{aligned}
0 = & -j\lambda_{z\psi 2}\left(\int_h^d \mathrm{sgn}\left(z-z'\right)\vec{\tilde{\vec{G}}}^{p}_{\psi 2e}\cdot\vec{\tilde{J}}_e dz' + \int_h^d \mathrm{sgn}\left(z-z'\right)\vec{\tilde{\vec{G}}}^{p}_{\psi 2h}\cdot\vec{\tilde{J}}_h dz'\right.\\
&\qquad\qquad\left.+\tilde{\psi}_2^{+}P_{\psi 2z}P_{\psi 2d}^{-1}-\tilde{\psi}_2^{-}P_{\psi 2z}^{-1}P_{\psi 2d}\right)\Big|_{z=d}\\
= & -j\lambda_{z\psi 2}\left(\int_h^d \cancelto{1}{\mathrm{sgn}\left(d-z'\right)}\vec{\tilde{\vec{G}}}^{p}_{\psi 2e}\left(z=d\right)\cdot\vec{\tilde{J}}_e dz' + \int_h^d \cancelto{1}{\mathrm{sgn}\left(d-z'\right)}\vec{\tilde{\vec{G}}}^{p}_{\psi 2h}\left(z=d\right)\cdot\vec{\tilde{J}}_h dz'\right.\\
&\qquad\qquad\left.+\tilde{\psi}_2^{+}-\tilde{\psi}_2^{-}\right)\\
= & \underbrace{\int_h^d \vec{\tilde{\vec{G}}}^{p}_{\psi 2e}\left(z=d\right)\cdot\vec{\tilde{J}}_e dz' + \int_h^d \vec{\tilde{\vec{G}}}^{p}_{\psi 2h}\left(z=d\right)\cdot\vec{\tilde{J}}_h dz'}_{V^{+}_{\psi 2}}+\tilde{\psi}_2^{+}-\tilde{\psi}_2^{-}\\
\Rightarrow & \;\tilde{\psi}_2^{+}-\tilde{\psi}_2^{-}=-V^{+}_{\psi 2}
\end{aligned}
\tag{122}
$$

Boundary Condition 3: Continuity of $\vec{E}_t$ at $z=h$.

Evaluating (115) implies that

$$
\begin{aligned}
&\tilde{\theta}_1\Big|_{z=h}=\tilde{\theta}_2\Big|_{z=h}\Rightarrow\left(\tilde{\theta}_1^{p}+\tilde{\theta}_1^{s}\right)\Big|_{z=h}=\left(\tilde{\theta}_2^{p}+\tilde{\theta}_2^{s}\right)\Big|_{z=h}\\
&\Rightarrow\left(\int_0^h \vec{\tilde{\vec{G}}}^{p}_{\theta 1e}\cdot\vec{\tilde{J}}_e dz' + \int_0^h \vec{\tilde{\vec{G}}}^{p}_{\theta 1h}\cdot\vec{\tilde{J}}_h dz' + \tilde{\theta}_1^{+}P_{\theta 1z}+\tilde{\theta}_1^{-}P_{\theta 1z}^{-1}\right)\Bigg|_{z=h}=\\
&\quad\left(\int_h^d \vec{\tilde{\vec{G}}}^{p}_{\theta 2e}\cdot\vec{\tilde{J}}_e dz' + \int_h^d \vec{\tilde{\vec{G}}}^{p}_{\theta 2h}\cdot\vec{\tilde{J}}_h dz' + \tilde{\theta}_2^{+}P_{\theta 2z}P_{\theta 2d}^{-1}+\tilde{\theta}_2^{-}P_{\theta 2z}^{-1}P_{\theta 2d}\right)\Bigg|_{z=h}\\
&\Rightarrow\underbrace{\int_0^h \vec{\tilde{\vec{G}}}^{p}_{\theta 1e}\left(z=h\right)\cdot\vec{\tilde{J}}_e dz' + \int_0^h \vec{\tilde{\vec{G}}}^{p}_{\theta 1h}\left(z=h\right)\cdot\vec{\tilde{J}}_h dz'}_{V^{+}_{\theta 1}}+\tilde{\theta}_1^{+}P_{\theta 1h}+\tilde{\theta}_1^{-}P_{\theta 1h}^{-1}=\\
&\quad\underbrace{\int_h^d \vec{\tilde{\vec{G}}}^{p}_{\theta 2e}\left(z=h\right)\cdot\vec{\tilde{J}}_e dz' + \int_h^d \vec{\tilde{\vec{G}}}^{p}_{\theta 2h}\left(z=h\right)\cdot\vec{\tilde{J}}_h dz'}_{V^{-}_{\theta 2}}+\tilde{\theta}_2^{+}P_{\theta 2h}P_{\theta 2d}^{-1}+\tilde{\theta}_2^{-}P_{\theta 2h}^{-1}P_{\theta 2d}
\end{aligned}
$$

$$\Rightarrow P_{\theta 2h}P_{\theta 2d}P_{\theta 1h}^2\tilde{\theta}_1^+ + P_{\theta 2h}P_{\theta 2d}\tilde{\theta}_1^- - P_{\theta 1h}P_{\theta 2h}^2\tilde{\theta}_2^+ - P_{\theta 1h}P_{\theta 2d}^2\tilde{\theta}_2^- = P_{\theta 1h}P_{\theta 2h}P_{\theta 2d}\left(V_{\theta 2}^- - V_{\theta 1}^+\right) \tag{123}$$

Evaluating (116) implies that

$$\begin{aligned}
&\frac{1}{j\omega\epsilon_{t1}}\left(\frac{\partial}{\partial z}\tilde{\psi}_1 - \tilde{u}_{e1}\right)\Bigg|_{z=h} = \frac{1}{j\omega\epsilon_{t2}}\left(\frac{\partial}{\partial z}\tilde{\psi}_2 - \tilde{u}_{e2}\right)\Bigg|_{z=h} \\
&\frac{1}{\epsilon_{t1}}\left(\frac{\partial}{\partial z}\tilde{\psi}_1^p - \tilde{u}_{e1} + \frac{\partial}{\partial z}\tilde{\psi}_1^s\right)\Bigg|_{z=h} = \frac{1}{\epsilon_{t2}}\left(\frac{\partial}{\partial z}\tilde{\psi}_2^p - \tilde{u}_{e2} + \frac{\partial}{\partial z}\tilde{\psi}_2^s\right)\Bigg|_{z=h} \\
&\Rightarrow \frac{1}{\epsilon_{t1}}\Bigg(\cancel{\tilde{u}_{e1}} - j\lambda_{z\psi 1}\left[\int_0^h \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 1e}^p\cdot\tilde{\vec{J}}_e dz' + \int_0^h \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 1h}^p\cdot\tilde{\vec{J}}_h dz'\right] \\
&\quad -\cancel{\tilde{u}_{e1}} + \frac{\partial}{\partial z}\left(\tilde{\psi}_1^+ P_{\psi 1z} + \tilde{\psi}_1^- P_{\psi 1z}^{-1}\right)\Bigg)\Bigg|_{z=h} = \frac{1}{\epsilon_{t2}}\Bigg\{\cancel{\tilde{u}_{e2}} - j\lambda_{z\psi 2}\Bigg[\int_h^d \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 2e}^p \\
&\qquad \cdot\tilde{\vec{J}}_e dz' + \int_h^d \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 2h}^p\cdot\tilde{\vec{J}}_h dz'\Bigg] - \cancel{\tilde{u}_{e2}} + \frac{\partial}{\partial z}\Big(\tilde{\psi}_2^+ P_{\psi 2z}P_{\psi 2d}^{-1} \\
&\qquad\qquad + \tilde{\psi}_2^- P_{\psi 2z}^{-1}P_{\psi 2d}\Big)\Bigg\}\Bigg|_{z=h} \\
&\Rightarrow -\frac{j\lambda_{z\psi 1}}{\epsilon_{t1}}\left[\int_0^h \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 1e}^p\cdot\tilde{\vec{J}}_e dz' + \int_0^h \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 1h}^p\cdot\tilde{\vec{J}}_h dz' + \tilde{\psi}_1^+ P_{\psi 1z}\right. \\
&\left. -\tilde{\psi}_1^- P_{\psi 1z}^{-1}\right]\Bigg|_{z=h} = -\frac{j\lambda_{z\psi 2}}{\epsilon_{t2}}\left[\int_h^d \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 2e}^p\cdot\tilde{\vec{J}}_e dz' + \int_h^d \operatorname{sgn}(z-z')\,\tilde{\vec{G}}_{\psi 2h}^p\cdot\tilde{\vec{J}}_h dz'\right. \\
&\qquad\qquad \left. + \tilde{\psi}_2^+ P_{\psi 2z}P_{\psi 2d}^{-1} - \tilde{\psi}_2^- P_{\psi 2z}^{-1}P_{\psi 2d}\right]\Bigg|_{z=h} \\
&\Rightarrow \frac{\lambda_{z\psi 1}}{\epsilon_{t1}}\left[\int_0^h \cancelto{1}{\operatorname{sgn}(h-z')}\tilde{\vec{G}}_{\psi 1e}^p(z=h)\cdot\tilde{\vec{J}}_e dz' + \int_0^h \cancelto{1}{\operatorname{sgn}(h-z')}\tilde{\vec{G}}_{\psi 1h}^p(z=h)\cdot\tilde{\vec{J}}_h dz'\right. \\
&\qquad \left. + \tilde{\psi}_1^+ P_{\psi 1h} - \tilde{\psi}_1^- P_{\psi 1h}^{-1}\right] = \frac{\lambda_{z\psi 2}}{\epsilon_{t2}}\left[\int_h^d \cancelto{-1}{\operatorname{sgn}(h-z')}\tilde{\vec{G}}_{\psi 2e}^p(z=h)\cdot\tilde{\vec{J}}_e dz'\right. \\
&\qquad\qquad \left. + \int_h^d \cancelto{-1}{\operatorname{sgn}(h-z')}\tilde{\vec{G}}_{\psi 2h}^p(z=h)\cdot\tilde{\vec{J}}_h dz' + \tilde{\psi}_2^+ P_{\psi 2h}P_{\psi 2d}^{-1} - \tilde{\psi}_2^- P_{\psi 2h}^{-1}P_{\psi 2d}\right]
\end{aligned}$$

$$\Rightarrow \underbrace{\int_0^h \tilde{\vec{G}}^p_{\psi 1e}\left(z=h\right)\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\vec{G}}^p_{\psi 1h}\left(z=h\right)\cdot\tilde{\vec{J}}_h dz'}_{V^+_{\psi 1}} + \tilde{\psi}^+_1 P_{\psi 1h} - \tilde{\psi}^-_1 P^{-1}_{\psi 1h} =$$

$$\underbrace{\frac{\lambda_{z\psi 2}\epsilon_{t1}}{\lambda_{z\psi 1}\epsilon_{t2}}}_{C_\psi=\frac{Z_{\psi 2}}{Z_{\psi 1}}}\left(-\underbrace{\left(\int_h^d \tilde{\vec{G}}^p_{\psi 2e}\left(z=h\right)\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\vec{G}}^p_{\psi 2h}\left(z=h\right)\cdot\tilde{\vec{J}}_h dz'\right)}_{V^-_{\psi 2}} + \tilde{\psi}^+_2 P_{\psi 2h}P^{-1}_{\psi 2d}\right.$$

$$\left. - \tilde{\psi}^-_2 P^{-1}_{\psi 2h}P_{\psi 2d}\right)$$

$$\begin{aligned}\Rightarrow P^2_{\psi 1h}P_{\psi 2h}P_{\psi 2d}\tilde{\psi}^+_1 - P_{\psi 2h}P_{\psi 2d}\tilde{\psi}^-_1 - C_\psi P_{\psi 1h}P^2_{\psi 2h}\tilde{\psi}^+_2 + C_\psi P_{\psi 1h}P^2_{\psi 2d}\tilde{\psi}^-_2 = \\ P_{\psi 1h}P_{\psi 2h}P_{\psi 2d}\left(-V^+_{\psi 1} - V^-_{\psi 2}C_\psi\right)\end{aligned} \tag{124}$$

Boundary Condition 4: Continuity of $\vec{H}_t$ at $z = h$.

Evaluating (117) implies that

$$\tilde{\psi}_1\Big|_{z=h} = \tilde{\psi}_2\Big|_{z=h} \Rightarrow \left(\tilde{\psi}^p_1 + \tilde{\psi}^s_1\right)\Big|_{z=h} = \left(\tilde{\psi}^p_2 + \tilde{\psi}^s_2\right)\Big|_{z=h}$$

$$\Rightarrow \left(\int_0^h \tilde{\vec{G}}^p_{\psi 1e}\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\vec{G}}^p_{\psi 1h}\cdot\tilde{\vec{J}}_h dz' + \tilde{\psi}^+_1 P_{\psi 1z} + \tilde{\psi}^-_1 P^{-1}_{\psi 1z}\right)\Bigg|_{z=h} =$$

$$\left(\int_h^d \tilde{\vec{G}}^p_{\psi 2e}\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\vec{G}}^p_{\psi 2h}\cdot\tilde{\vec{J}}_h dz' + \tilde{\psi}^+_2 P_{\psi 2z}P^{-1}_{\psi 2d} + \tilde{\psi}^-_2 P^{-1}_{\psi 2z}P_{\psi 2d}\right)\Bigg|_{z=h}$$

$$\Rightarrow \underbrace{\int_0^h \tilde{\vec{G}}^p_{\psi 1e}\left(z=h\right)\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\vec{G}}^p_{\psi 1h}\left(z=h\right)\cdot\tilde{\vec{J}}_h dz'}_{V^+_{\psi 1}} + \tilde{\psi}^+_1 P_{\psi 1h} + \tilde{\psi}^-_1 P^{-1}_{\psi 1h} =$$

$$\underbrace{\int_h^d \tilde{\vec{G}}^p_{\psi 2e}\left(z=h\right)\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\vec{G}}^p_{\psi 2h}\left(z=h\right)\cdot\tilde{\vec{J}}_h dz'}_{V^-_{\psi 2}} + \tilde{\psi}^+_2 P_{\psi 2h}P^{-1}_{\psi 2d} + \tilde{\psi}^-_2 P^{-1}_{\psi 2h}P_{\psi 2d}$$

$$
\begin{aligned}
\Rightarrow P_{\psi 1h}^2 P_{\psi 2h} P_{\psi 2d} \tilde{\psi}_1^+ + P_{\psi 2h} P_{\psi 2d} \tilde{\psi}_1^- - P_{\psi 1h} P_{\psi 2h}^2 \tilde{\psi}_2^+ - P_{\psi 1h} P_{\psi 2d}^2 \tilde{\psi}_2^- = \\
P_{\psi 1h} P_{\psi 2h} P_{\psi 2d} \left(-V_{\psi 1}^+ + V_{\psi 2}^-\right)
\end{aligned}
\tag{125}
$$

Evaluating (118) implies that

$$
\begin{aligned}
&-\frac{1}{j\omega\mu_{t1}}\left(\frac{\partial}{\partial z}\tilde{\theta}_1 + \tilde{u}_{h1}\right)\Bigg|_{z=h} = -\frac{1}{j\omega\mu_{t2}}\left(\frac{\partial}{\partial z}\tilde{\theta}_2 + \tilde{u}_{h2}\right)\Bigg|_{z=h} \\
&\Rightarrow \frac{1}{\mu_{t1}}\left(\frac{\partial}{\partial z}\tilde{\theta}_1^p + \tilde{u}_{h1} + \frac{\partial}{\partial z}\tilde{\theta}_1^s\right)\Bigg|_{z=h} = \frac{1}{\mu_{t2}}\left(\frac{\partial}{\partial z}\tilde{\theta}_2^p + \tilde{u}_{h2} + \frac{\partial}{\partial z}\tilde{\theta}_2^s\right)\Bigg|_{z=h} \\
&\Rightarrow \frac{1}{\mu_{t1}}\left(-\cancel{\tilde{u}_{h1}} - j\lambda_{z\theta 1}\left[\int\limits_0^h \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 1e}^p \cdot \vec{\tilde{J}}_e dz' + \int\limits_0^h \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 1h}^p \cdot \vec{\tilde{J}}_h dz'\right]\right. \\
&\quad \left. + \cancel{\tilde{u}_{h1}} + \frac{\partial}{\partial z}\left(\tilde{\theta}_1^+ P_{\theta 1z} + \tilde{\theta}_1^- P_{\theta 1z}^{-1}\right)\right)\Bigg|_{z=h} = \frac{1}{\mu_{t2}}\left(-j\lambda_{z\theta 2}\left[\int\limits_h^d \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 2e}^p \cdot \vec{\tilde{J}}_e dz'\right.\right. \\
&\quad \left.\left. + \int\limits_h^d \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 2h}^p \cdot \vec{\tilde{J}}_h dz'\right] + \cancel{\tilde{u}_{h2}} + \frac{\partial}{\partial z}\left(\tilde{\theta}_2^+ P_{\theta 2z} P_{\theta 2d}^{-1} + \tilde{\theta}_2^- P_{\theta 2z}^{-1} P_{\theta 2d}\right) - \cancel{\tilde{u}_{h2}}\right)\Bigg|_{z=h} \\
&\Rightarrow \frac{-j\lambda_{z\theta 1}}{\mu_{t1}}\left[\int\limits_0^h \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 1e}^p \cdot \vec{\tilde{J}}_e dz' + \int\limits_0^h \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 1h}^p \cdot \vec{\tilde{J}}_h dz' + \tilde{\theta}_1^+ P_{\theta 1z}\right. \\
&\left. -\tilde{\theta}_1^- P_{\theta 1z}^{-1}\right]\Bigg|_{z=h} = \frac{-j\lambda_{z\theta 2}}{\mu_{t2}}\left(\int\limits_h^d \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 2e}^p \cdot \vec{\tilde{J}}_e dz' + \int\limits_h^d \operatorname{sgn}(z-z')\,\vec{\tilde{G}}_{\theta 2h}^p \cdot \vec{\tilde{J}}_h dz'\right. \\
&\qquad\qquad \left. + \tilde{\theta}_2^+ P_{\theta 2z} P_{\theta 2d}^{-1} - \tilde{\theta}_2^- P_{\theta 2z}^{-1} P_{\theta 2d}\right)\Bigg|_{z=h} \\
&\Rightarrow \frac{-j\lambda_{z\theta 1}}{\mu_{t1}}\left[\int\limits_0^h \cancelto{1}{\operatorname{sgn}(h-z')}\vec{\tilde{G}}_{\theta 1e}^p(z=h) \cdot \vec{\tilde{J}}_e dz' + \int\limits_0^h \cancelto{1}{\operatorname{sgn}(h-z')}\vec{\tilde{G}}_{\theta 1h}^p(z=h) \cdot \vec{\tilde{J}}_h dz'\right. \\
&\qquad \left. + \tilde{\theta}_1^+ P_{\theta 1h} - \tilde{\theta}_1^- P_{\theta 1h}^{-1}\right] = \frac{-j\lambda_{z\theta 2}}{\mu_{t2}}\left[\int\limits_h^d \cancelto{-1}{\operatorname{sgn}(h-z')}\vec{\tilde{G}}_{\theta 2e}^p(z=h) \cdot \vec{\tilde{J}}_e dz'\right. \\
&\qquad\qquad \left. + \int\limits_h^d \cancelto{-1}{\operatorname{sgn}(h-z')}\vec{\tilde{G}}_{\theta 2h}^p(z=h) \cdot \vec{\tilde{J}}_h dz' + \tilde{\theta}_2^+ P_{\theta 2h} P_{\theta 2d}^{-1} - \tilde{\theta}_2^- P_{\theta 2h}^{-1} P_{\theta 2d}\right]
\end{aligned}
$$

$$\Rightarrow \underbrace{\int_0^h \vec{\tilde{G}}^p_{\theta 1e}(z=h)\cdot\vec{\tilde{J}}_e dz' + \int_0^h \vec{\tilde{G}}^p_{\theta 1h}(z=h)\cdot\vec{\tilde{J}}_h dz'}_{V^+_{\theta 1}} + \tilde{\theta}^+_1 P_{\theta 1h} - \tilde{\theta}^-_1 P^{-1}_{\theta 1h} =$$

$$\underbrace{\frac{\lambda_{z\theta 2}\mu_{t1}}{\lambda_{z\theta 1}\mu_{t2}}}_{C_\theta=\frac{Z_{\theta 1}}{Z_{\theta 2}}}\left[-\underbrace{\left(\int_h^d \vec{\tilde{G}}^p_{\theta 2e}(z=h)\cdot\vec{\tilde{J}}_e dz' + \int_h^d \vec{\tilde{G}}^p_{\theta 2h}(z=h)\cdot\vec{\tilde{J}}_h dz'\right)}_{V^-_{\theta 2}} + \tilde{\theta}^+_2 P_{\theta 2h} P^{-1}_{\theta 2d} \right.$$

$$\left. - \tilde{\theta}^-_2 P^{-1}_{\theta 2h} P_{\theta 2d}\right]$$

$$\Rightarrow P^2_{\theta 1h} P_{\theta 2h} P_{\theta 2d}\tilde{\theta}^+_1 - P_{\theta 2h} P_{\theta 2d}\tilde{\theta}^-_1 - C_\theta P_{\theta 1h} P^2_{\theta 2h}\tilde{\theta}^+_2 + C_\theta P_{\theta 1h} P^2_{\theta 2d}\tilde{\theta}^-_2 = P_{\theta 1h} P_{\theta 2h} P_{\theta 2d}\left(-V^+_{\theta 1} - V^-_{\theta 2} C_\theta\right) \tag{126}$$

Computation of Scattering Coefficients.

Now solve (119)-(126) for $\left\{\tilde{\psi},\tilde{\theta}\right\}^{\{+,-\}}_{\{1,2\}}$. Also, note that (119),(121),(123) and (126) are linearly independent of (120),(122),(124) and (125), making it possible to solve two sets of four equations using Gauss-Jordan elimination for $\overleftrightarrow{A}\vec{x}=\vec{b}$ where

$$\vec{x} = \begin{bmatrix} \alpha^+_1 \\ \alpha^-_1 \\ \alpha^+_2 \\ \alpha^-_2 \end{bmatrix}, \alpha \in \left\{\tilde{\psi},\tilde{\theta}\right\} \tag{127}$$

Solving (119), (121), (123), and (126) for $\tilde{\theta}^{\{+,-\}}_{\{1,2\}}$ via (127) implies that

$$\tilde{\theta}^+_1 = \frac{-V^+_{\theta 1} P_{\theta 1h}\left[\left(P^2_{\theta 2h} - P^2_{\theta 2d}\right) - C_\theta\left(P^2_{\theta 2h} + P^2_{\theta 2d}\right)\right] - 2V^-_{\theta 2} C_\theta P_{\theta 1h} P^2_{\theta 2h}}{\left[\left(P^2_{\theta 2h} - P^2_{\theta 2d}\right)\left(1 + P^2_{\theta 1h}\right) + C_\theta\left(P^2_{\theta 2h} + P^2_{\theta 2d}\right)\left(1 - P^2_{\theta 1h}\right)\right]} + \frac{-V^-_{\theta 1}\left[\left(P^2_{\theta 2h} - P^2_{\theta 2d}\right) + C_\theta\left(P^2_{\theta 2h} + P^2_{\theta 2d}\right)\right] + 2V^+_{\theta 2} C_\theta P_{\theta 1h} P_{\theta 2d} P_{\theta 2h}}{\left[\left(P^2_{\theta 2h} - P^2_{\theta 2d}\right)\left(1 + P^2_{\theta 1h}\right) + C_\theta\left(P^2_{\theta 2h} + P^2_{\theta 2d}\right)\left(1 - P^2_{\theta 1h}\right)\right]} \tag{128}$$

$$\begin{aligned}\tilde{\theta}_1^- &= \frac{V_{\theta 1}^+ P_{\theta 1h}\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right) - C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\right] + 2V_{\theta 2}^- C_\theta P_{\theta 1h} P_{\theta 2h}^2}{\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right)\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\left(1 - P_{\theta 1h}^2\right)\right]} \\ &+ \frac{-V_{\theta 1}^- P_{\theta 1h}^2\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right) - C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\right] - 2V_{\theta 2}^+ C_\theta P_{\theta 1h} P_{\theta 2d} P_{\theta 2h}}{\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right)\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\left(1 - P_{\theta 1h}^2\right)\right]}\end{aligned} \tag{129}$$

$$\begin{aligned}\tilde{\theta}_2^+ &= \frac{2P_{\theta 2d} P_{\theta 2h}\left(V_{\theta 1}^+ - V_{\theta 1}^- P_{\theta 1h}\right) + V_{\theta 2}^+ P_{\theta 2d}^2\left[\left(1 + P_{\theta 1h}^2\right) - C_\theta\left(1 - P_{\theta 1h}^2\right)\right]}{\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right)\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\left(1 - P_{\theta 1h}^2\right)\right]} \\ &+ \frac{-V_{\theta 2}^- P_{\theta 2d} P_{\theta 2h}\left[\left(1 + P_{\theta 1h}^2\right) - C_\theta\left(1 - P_{\theta 1h}^2\right)\right]}{\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right)\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\left(1 - P_{\theta 1h}^2\right)\right]}\end{aligned} \tag{130}$$

$$\begin{aligned}\tilde{\theta}_2^- &= \frac{2P_{\theta 2d} P_{\theta 2h}\left(-V_{\theta 1}^+ + V_{\theta 1}^- P_{\theta 1h}\right) - V_{\theta 2}^+ P_{\theta 2h}^2\left[\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(1 - P_{\theta 1h}^2\right)\right]}{\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right)\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\left(1 - P_{\theta 1h}^2\right)\right]} \\ &+ \frac{V_{\theta 2}^- P_{\theta 2d} P_{\theta 2h}\left[\left(1 + P_{\theta 1h}^2\right) - C_\theta\left(1 - P_{\theta 1h}^2\right)\right]}{\left[\left(P_{\theta 2h}^2 - P_{\theta 2d}^2\right)\left(1 + P_{\theta 1h}^2\right) + C_\theta\left(P_{\theta 2h}^2 + P_{\theta 2d}^2\right)\left(1 - P_{\theta 1h}^2\right)\right]}\end{aligned} \tag{131}$$

For the sake of brevity, $\tilde{\psi}_{\{1,2\}}^{\{+,-\}}$ coefficients are derived in Appendix B.

Transverse Spectral Domain Scattered Solutions.

Now that the scattering coefficients have been determined, the scattered solutions $\tilde{\theta}_{\{1,2\}}^s$ and $\tilde{\psi}_{\{1,2\}}^s$ can be determined. Since $\tilde{\Pi}_{\{1,2\}}^s$ and $\tilde{\Phi}_{\{1,2\}}^s$ can be computed directly from $\tilde{\theta}_{\{1,2\}}^s$ and $\tilde{\psi}_{\{1,2\}}^s$, their derivations will be omitted from this section. Starting with $\tilde{\theta}_1^s$, substituting (128) and (129) into (111), it can be shown that

$$\begin{aligned}\tilde{\theta}_1^s &= \frac{V_{\theta 1}^+\left(P_{\theta 1z}^{-1} - P_{\theta 1z}\right)\left[\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right) - C_\theta\left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\right]}{\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\ &+ \frac{-V_{\theta 1}^-\left[\left(P_{\theta 1h}^{-1} P_{\theta 1z} + P_{\theta 1h} P_{\theta 1z}^{-1}\right)\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right)\right]}{\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\ &+ \frac{-V_{\theta 1}^-\left[C_\theta\left(P_{\theta 1h}^{-1} P_{\theta 1z} - P_{\theta 1h} P_{\theta 1z}^{-1}\right)\left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\right]}{\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\ &+ \frac{2V_{\theta 2}^+ C_\theta\left(P_{\theta 1z} - P_{\theta 1z}^{-1}\right) + 2V_{\theta 2}^- C_\theta P_{\theta 2d}^{-1} P_{\theta 2h}\left(P_{\theta 1z}^{-1} - P_{\theta 1z}\right)}{\underbrace{\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)}_{D_\theta}}\end{aligned} \tag{132}$$

Breaking (132) into electric and magnetic components and then substituting (119), (121), and (123) into (132) implies that

$$\tilde{\theta}^s_{1\{e,h\}} = \int\limits_0^h D_\theta^{-1} \Big\{ \vec{\tilde{g}}^p_{\theta 1\{e,h\}} (z=h) \, P_{\theta 1h} P^{-1}_{\theta 1z'} \left(P^{-1}_{\theta 1z} - P_{\theta 1z} \right) \left[\left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right.$$

$$\left. - C_\theta \left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right]$$

$$- \vec{\tilde{g}}^p_{\theta 1\{e,h\}} (z=0) \, P_{\theta 1z'} \left[\left(P^{-1}_{\theta 1h} P_{\theta 1z} + P_{\theta 1h} P^{-1}_{\theta 1z} \right) \left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right.$$

$$\left. + C_\theta \left(P^{-1}_{\theta 1h} P_{\theta 1z} - P_{\theta 1h} P^{-1}_{\theta 1z} \right) \left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right] \Big\} \cdot \vec{\tilde{J}}_{\{e,h\}} dz'$$

$$+ \int\limits_h^d D_\theta^{-1} \Big[2 \vec{\tilde{g}}^p_{\theta 2\{e,h\}} (z=d) \, C_\theta P_{\theta 2d} P^{-1}_{\theta 2z'} \left(P_{\theta 1z} - P^{-1}_{\theta 1z} \right)$$

$$+ 2 \vec{\tilde{g}}^p_{\theta 2\{e,h\}} (z=h) \, C_\theta P^{-1}_{\theta 2d} P_{\theta 2z'} \left(P^{-1}_{\theta 1z} - P_{\theta 1z} \right) \Big] \cdot \vec{\tilde{J}}_{\{e,h\}} dz'$$

$$= \int\limits_0^h D_\theta^{-1} \Big\{ \vec{\tilde{g}}^p_{\theta 1\{e,h\}} (z=h) \left(P_{\theta 1h} P^{-1}_{\theta 1z} P^{-1}_{\theta 1z'} - P_{\theta 1h} P_{\theta 1z} P^{-1}_{\theta 1z'} \right) \left[\left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right.$$

$$\left. - C_\theta \left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right]$$

$$- \vec{\tilde{g}}^p_{\theta 1\{e,h\}} (z=0) \left[\left(P^{-1}_{\theta 1h} P_{\theta 1z} P_{\theta 1z'} + P_{\theta 1h} P^{-1}_{\theta 1z} P_{\theta 1z'} \right) \left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right.$$

$$\left. + C_\theta \left(P^{-1}_{\theta 1h} P_{\theta 1z} P_{\theta 1z'} - P_{\theta 1h} P^{-1}_{\theta 1z} P_{\theta 1z'} \right) \left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h} \right) \right] \Big\} \cdot \vec{\tilde{J}}_{\{e,h\}} dz'$$

$$+ \int\limits_h^d \left[2 C_\theta D_\theta^{-1} \left(\vec{\tilde{g}}^p_{\theta 2\{e,h\}} (z=h) \, P^{-1}_{\theta 2d} P_{\theta 2z'} - \vec{\tilde{g}}^p_{\theta 2\{e,h\}} (z=d) \, P_{\theta 2d} P^{-1}_{\theta 2z'} \right) \left(P^{-1}_{\theta 1z} - P_{\theta 1z} \right) \right]$$

$$\cdot \vec{\tilde{J}}_{\{e,h\}} dz'$$

$$= \int\limits_0^h \vec{\tilde{G}}^s_{\theta 1\{e,h\}1} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' + \int\limits_h^d \vec{\tilde{G}}^s_{\theta 1\{e,h\}2} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \tag{133}$$

For $\tilde{\theta}_2^s$, by substituting (130) and (131) into (112), it can be shown that

$$\begin{aligned}
\tilde{\theta}_2^s = & \frac{2V_{\theta 1}^{+}P_{\theta 1h}^{-1}\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right) - 2V_{\theta 1}^{-}\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right)}{\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\
& + \frac{V_{\theta 2}^{+}\left[\left(P_{\theta 2h}^{-1}P_{\theta 2z} - P_{\theta 2h}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right]}{\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\
& + \frac{V_{\theta 2}^{+}\left[-C_\theta\left(P_{\theta 2h}^{-1}P_{\theta 2z} + P_{\theta 2h}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]}{\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\
& + \frac{V_{\theta 2}^{-}\left[\left(P_{\theta 2d}P_{\theta 2z}^{-1} - P_{\theta 2d}^{-1}P_{\theta 2z}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right]}{\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)} \\
& + \frac{V_{\theta 2}^{-}\left[C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]}{\underbrace{\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)}_{D_\theta}}
\end{aligned} \tag{134}$$

Breaking (134) into electric and magnetic components and substituting (119), (121), and (123) into (134) implies that

$$\begin{aligned}
\tilde{\theta}_{2\{e,h\}}^s = & \int_0^h D_\theta^{-1}\Big[2\vec{\tilde{g}}_{\theta 1\{e,h\}}^p\left(z=h\right)P_{\theta 1z'}^{-1}\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right) \\
& -2\vec{\tilde{g}}_{\theta 1\{e,h\}}^p\left(z=0\right)P_{\theta 1z'}\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right)\Big]\cdot\vec{\tilde{J}}_{\{e,h\}}dz' \\
& + \int_h^d D_\theta^{-1}\Big\{\vec{\tilde{g}}_{\theta 2\{e,h\}}^p\left(z=d\right)P_{\theta 2d}P_{\theta 2z'}^{-1}\left[\left(P_{\theta 2h}^{-1}P_{\theta 2z} - P_{\theta 2h}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right. \\
& \left. -C_\theta\left(P_{\theta 2h}^{-1}P_{\theta 2z} + P_{\theta 2h}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right] \\
& +\vec{\tilde{g}}_{\theta 2\{e,h\}}^p\left(z=h\right)P_{\theta 2h}^{-1}P_{\theta 2z'}\left[\left(P_{\theta 2d}P_{\theta 2z}^{-1} - P_{\theta 2d}^{-1}P_{\theta 2z}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right. \\
& \left. +C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]\Big\}\cdot\vec{\tilde{J}}_{\{e,h\}}dz' \\
= & \int_0^h \vec{\tilde{G}}_{\theta 2\{e,h\}1}^s\cdot\vec{\tilde{J}}_{\{e,h\}}dz' + \int_h^d \vec{\tilde{G}}_{\theta 2\{e,h\}2}^s\cdot\vec{\tilde{J}}_{\{e,h\}}dz'
\end{aligned} \tag{135}$$

For brevity, the solutions to $\tilde{\psi}_{\{1,2\}}^s$ are derived in Apppendix B.

2.5 Transverse Spectral Domain Total Scalar Potential Green Functions

Now that the principal and scattered Green functions have been determined, they can be combined to find the total scalar potential Green functions. Begin with $\tilde{\theta}_1$. Substituting (71) and (133) into (111) implies that

$$\begin{aligned}
\tilde{\theta}_{1\{e,h\}} &= \int_0^h \underbrace{\left[\tilde{\vec{G}}^p_{\theta 1\{e,h\}} + \tilde{\vec{G}}^s_{\theta 1\{e,h\}1}\right]}_{\tilde{\vec{G}}_{\theta 1\{e,h\}1}} \cdot \tilde{\vec{J}}_{\{e,h\}} dz' + \int_h^d \underbrace{\tilde{\vec{G}}^s_{\theta 1\{e,h\}2}}_{\tilde{\vec{G}}_{\theta 1\{e,h\}2}} \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
&= \int_0^h D_\theta^{-1} \Big\{ \tilde{\vec{g}}^p_{\theta 1\{e,h\}} e^{-j\lambda_{z\theta 1}|z-z'|} \left[\left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h} + P_{\theta 1h}\right)\right. \\
&\qquad \left. + C_\theta \left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h} - P_{\theta 1h}\right)\right] \\
&\qquad + \tilde{\vec{g}}^p_{\theta 1\{e,h\}}(z=h)\left(P_{\theta 1h} P^{-1}_{\theta 1z} P^{-1}_{\theta 1z'} - P_{\theta 1h} P_{\theta 1z} P^{-1}_{\theta 1z'}\right)\left[\left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h}\right)\right. \\
&\qquad \left. - C_\theta \left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h}\right)\right] \\
&\qquad - \tilde{\vec{g}}^p_{\theta 1\{e,h\}}(z=0)\left[\left(P^{-1}_{\theta 1h} P_{\theta 1z} P_{\theta 1z'} + P_{\theta 1h} P^{-1}_{\theta 1z} P_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d} P_{\theta 2h} - P_{\theta 2d} P^{-1}_{\theta 2h}\right)\right] \\
&\qquad - \tilde{\vec{g}}^p_{\theta 1\{e,h\}}(z=0)\left[C_\theta \left(P^{-1}_{\theta 1h} P_{\theta 1z} P_{\theta 1z'} - P_{\theta 1h} P^{-1}_{\theta 1z} P_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d} P_{\theta 2h} + P_{\theta 2d} P^{-1}_{\theta 2h}\right)\right] \Big\} \\
&\qquad \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
&+ \int_h^d D_\theta^{-1} \left[2C_\theta \left(\tilde{\vec{g}}^p_{\theta 2\{e,h\}}(z=h) P^{-1}_{\theta 2d} P_{\theta 2z'} - \tilde{\vec{g}}^p_{\theta 2\{e,h\}}(z=d) P_{\theta 2d} P^{-1}_{\theta 2z'}\right)\left(P^{-1}_{\theta 1z} - P_{\theta 1z}\right)\right] \\
&\qquad \cdot \tilde{\vec{J}}_{\{e,h\}} dz'
\end{aligned} \tag{136}$$

Note that D_θ can be rewritten in terms of sine and cosine functions. Defining the thickness of region 2 as

$$\boxed{T = d - h} \tag{137}$$

implies that

$$
\begin{aligned}
D_\theta &= \left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_\theta \left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right) \\
&= j4\left[\sin\left(\lambda_{z\theta 2}\left(d-h\right)\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2}\left(d-h\right)\right)\sin\left(\lambda_{z\theta 1} h\right)\right] \\
&= j4\left[\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)\right] \qquad (138)
\end{aligned}
$$

In Chapter III, it is shown that only the potentials contributing to the magnetic field that are observed due to magnetic currents are necessary. Additionally, the observation region is coincident with the excitation region in the final extraction algorithm. Thus, this section only focuses on development of potentials resulting from magnetic currents in the observation region. The full development of potentials resulting from electric currents and magnetic currents outside the observation region is presented in Appendix C.

First, analyze the magnetic component $\tilde{\theta}_{1h}$. From (88), there are both longitudinal and transverse components. Since these components are linearly independent, they can be analyzed separately. Thus,

$$
\vec{\tilde{g}}_{\theta\{1,2\}ht}^{p} = j\,\mathrm{sgn}\left(z - z'\right)\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \qquad (139)
$$

$$
\vec{\tilde{g}}_{\theta\{1,2\}hz}^{p} = j\frac{\hat{z}\mu_{t\{1,2\}}}{2\lambda_{z\theta\{1,2\}}\mu_{z\{1,2\}}} \qquad (140)
$$

Begin by analyzing the component observed in region 1 resulting from transverse magnetic currents in region 1, $\tilde{\theta}_{1ht1}$. Substituting (139) and (138) into (136) implies

that

$$
\begin{aligned}
\vec{\vec{G}}_{\theta 1ht1} = {} & \left(\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right)\Big\{\operatorname{sgn}(z-z')\, e^{-j\lambda_{z\theta 1}|z-z'|}\Big[\big(P_{\theta 2d}^{-1}P_{\theta 2h}-P_{\theta 2d}P_{\theta 2h}^{-1}\big)\big(P_{\theta 1h}^{-1}+P_{\theta 1h}\big) \\
& +C_\theta\big(P_{\theta 2d}^{-1}P_{\theta 2h}+P_{\theta 2d}P_{\theta 2h}^{-1}\big)\big(P_{\theta 1h}^{-1}-P_{\theta 1h}\big)\Big] \\
& +\cancelto{1}{\operatorname{sgn}(h-z')}\big(P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}^{-1}-P_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}^{-1}\big)\Big[\big(P_{\theta 2d}^{-1}P_{\theta 2h}-P_{\theta 2d}P_{\theta 2h}^{-1}\big)\Big] \\
& +\cancelto{1}{\operatorname{sgn}(h-z')}\big(P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}^{-1}-P_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}^{-1}\big)\Big[-C_\theta\big(P_{\theta 2d}^{-1}P_{\theta 2h}+P_{\theta 2d}P_{\theta 2h}^{-1}\big)\Big] \\
& -\cancelto{-1}{\operatorname{sgn}(-z')}\Big[\big(P_{\theta 1h}^{-1}P_{\theta 1z}P_{\theta 1z'}+P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}\big)\big(P_{\theta 2d}^{-1}P_{\theta 2h}-P_{\theta 2d}P_{\theta 2h}^{-1}\big)\Big] \\
& -\cancelto{-1}{\operatorname{sgn}(-z')}\Big[C_\theta\big(P_{\theta 1h}^{-1}P_{\theta 1z}P_{\theta 1z'}-P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}\big)\big(P_{\theta 2d}^{-1}P_{\theta 2h}+P_{\theta 2d}P_{\theta 2h}^{-1}\big)\Big]\Big\} \\
= {} & \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\Bigg[\frac{\operatorname{sgn}(z-z')\, e^{-j\lambda_{z\theta 1}|z-z'|}\Big[\big(P_{\theta 2d}^{-1}P_{\theta 2h}-P_{\theta 2d}P_{\theta 2h}^{-1}\big)\big(P_{\theta 1h}^{-1}+P_{\theta 1h}\big)\Big]}{j4\left[\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)\right]} \\
& +\frac{\operatorname{sgn}(z-z')\, e^{-j\lambda_{z\theta 1}|z-z'|}\Big[C_\theta\big(P_{\theta 2d}^{-1}P_{\theta 2h}+P_{\theta 2d}P_{\theta 2h}^{-1}\big)\big(P_{\theta 1h}^{-1}-P_{\theta 1h}\big)\Big]}{j4\left[\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)\right]} \\
& +\frac{\big(P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}^{-1}-P_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}^{-1}\big)\big(P_{\theta 2d}^{-1}P_{\theta 2h}-P_{\theta 2d}P_{\theta 2h}^{-1}\big)}{j4\left[\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)\right]} \\
& +\frac{C_\theta\big(-P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}^{-1}+P_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}^{-1}\big)\big(P_{\theta 2d}^{-1}P_{\theta 2h}+P_{\theta 2d}P_{\theta 2h}^{-1}\big)}{j4\left[\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)\right]} \\
& +\frac{\big(P_{\theta 1h}^{-1}P_{\theta 1z}P_{\theta 1z'}+P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}\big)\big(P_{\theta 2d}^{-1}P_{\theta 2h}-P_{\theta 2d}P_{\theta 2h}^{-1}\big)}{j4\left[\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)\right]} \\
& +\frac{C_\theta\big(P_{\theta 1h}^{-1}P_{\theta 1z}P_{\theta 1z'}-P_{\theta 1h}P_{\theta 1z}^{-1}P_{\theta 1z'}\big)\big(P_{\theta 2d}^{-1}P_{\theta 2h}+P_{\theta 2d}P_{\theta 2h}^{-1}\big)}{j4\left[\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)\right]}\Bigg]
\end{aligned}
\tag{141}
$$

Due to the $\operatorname{sgn}(z-z')$ and $|z-z'|$ terms in (141), two cases must be considered.

When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}^{z+}_{\theta 1ht1} = \left(j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\theta}D_\theta}\right)\Big\{ & \cancelto{1}{\mathrm{sgn}\left(z-z'\right)} P_{\theta1z}P^{-1}_{\theta1z'}\left[\left(P^{-1}_{\theta2d}P_{\theta2h}-P_{\theta2d}P^{-1}_{\theta2h}\right)\left(P^{-1}_{\theta1h}+P_{\theta1h}\right)\right] \\
& +\cancelto{1}{\mathrm{sgn}\left(z-z'\right)} P_{\theta1z}P^{-1}_{\theta1z'}\left[C_\theta\left(P^{-1}_{\theta2d}P_{\theta2h}+P_{\theta2d}P^{-1}_{\theta2h}\right)\left(P^{-1}_{\theta1h}-P_{\theta1h}\right)\right] \\
& +\left(P_{\theta1h}P^{-1}_{\theta1z}P^{-1}_{\theta1z'}-P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h}-P_{\theta2d}P^{-1}_{\theta2h}\right) \\
& +C_\theta\left(-P_{\theta1h}P^{-1}_{\theta1z}P^{-1}_{\theta1z'}+P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h}+P_{\theta2d}P^{-1}_{\theta2h}\right) \\
& +\left(P^{-1}_{\theta1h}P_{\theta1z}P_{\theta1z'}+P_{\theta1h}P^{-1}_{\theta1z}P_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h}-P_{\theta2d}P^{-1}_{\theta2h}\right) \\
& +C_\theta\left(P^{-1}_{\theta1h}P_{\theta1z}P_{\theta1z'}-P_{\theta1h}P^{-1}_{\theta1z}P_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h}+P_{\theta2d}P^{-1}_{\theta2h}\right)\Big\} \\
= \left(j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\theta}D_\theta}\right) & \Big[\left(P^{-1}_{\theta2d}P_{\theta2h}-P_{\theta2d}P^{-1}_{\theta2h}\right)\left(P^{-1}_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}+P_{\theta1h}P^{-1}_{\theta1z}P^{-1}_{\theta1z'}\right) \\
& +C_\theta\left(P^{-1}_{\theta2d}P_{\theta2h}+P_{\theta2d}P^{-1}_{\theta2h}\right)\left(-P_{\theta1h}P^{-1}_{\theta1z}P^{-1}_{\theta1z'}+\cancel{P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}}+P^{-1}_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}\right) \\
& +C_\theta\left(P^{-1}_{\theta2d}P_{\theta2h}+P_{\theta2d}P^{-1}_{\theta2h}\right)\left(-\cancel{P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}}+P^{-1}_{\theta1h}P_{\theta1z}P_{\theta1z'}-P_{\theta1h}P^{-1}_{\theta1z}P_{\theta1z'}\right) \\
& +\left(\cancel{P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}}-\cancel{P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}}\right)\left(P^{-1}_{\theta2d}P_{\theta2h}-P_{\theta2d}P^{-1}_{\theta2h}\right) \\
& +\left(P^{-1}_{\theta1h}P_{\theta1z}P_{\theta1z'}+P_{\theta1h}P^{-1}_{\theta1z}P_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h}-P_{\theta2d}P^{-1}_{\theta2h}\right)\Big] \\
= \left(j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\theta}}\right) & \left[\frac{\sin\left(\lambda_{z\theta2}T\right)\left[\cos\left(\lambda_{z\theta1}\left(h-\left(z-z'\right)\right)\right)+\cos\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right. \\
& \left.+\frac{C_\theta\cos\left(\lambda_{z\theta2}T\right)\left[\sin\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)+\sin\left(\lambda_{z\theta1}\left(h-\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right]
\end{aligned}
\tag{142}
$$

When $z < z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}^{z-}_{\theta 1ht1} = {} & \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right)\Big\{\cancelto{-1}{\operatorname{sgn}(z-z')}P^{-1}_{\theta 1z}P_{\theta 1z'}\left[\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right)\right] \\
& +\cancelto{-1}{\operatorname{sgn}(z-z')}P^{-1}_{\theta 1z}P_{\theta 1z'}\left[C_\theta\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right)\right] \\
& +\left(P_{\theta 1h}P^{-1}_{\theta 1z}P^{-1}_{\theta 1z'}-P_{\theta 1h}P_{\theta 1z}P^{-1}_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right) \\
& +C_\theta\left(-P_{\theta 1h}P^{-1}_{\theta 1z}P^{-1}_{\theta 1z'}+P_{\theta 1h}P_{\theta 1z}P^{-1}_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right) \\
& +\left(P^{-1}_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}+P_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right) \\
& +C_\theta\left(P^{-1}_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}-P_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right)\Big\} \\
= {} & \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right)\Big[\left(-P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P_{\theta 1h}P^{-1}_{\theta 1z}P^{-1}_{\theta 1z'}-P_{\theta 1h}P_{\theta 1z}P^{-1}_{\theta 1z'}-P^{-1}_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}\right) \\
& +C_\theta\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}-\cancel{P_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}}-P^{-1}_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}\right) \\
& +C_\theta\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(\cancel{P_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}}-P_{\theta 1h}P^{-1}_{\theta 1z}P^{-1}_{\theta 1z'}+P_{\theta 1h}P_{\theta 1z}P^{-1}_{\theta 1z'}\right) \\
& +\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P^{-1}_{\theta 1h}P_{\theta 1z}P_{\theta 1z'}\right)\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right) \\
& +\left(-\cancel{P_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}}+\cancel{P_{\theta 1h}P^{-1}_{\theta 1z}P_{\theta 1z'}}\right)\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right)\Big] \\
= {} & \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\Bigg[\frac{\sin\left(\lambda_{z\theta 2}T\right)\left[\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)-\cos\left(\lambda_{z\theta 1}\left(h+\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \\
& +\frac{C_\theta\cos\left(\lambda_{z\theta 2}T\right)\left[\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)-\sin\left(\lambda_{z\theta 1}\left(h+\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\Bigg]
\end{aligned}
\tag{143}
$$

Analyzing (142) and (143) reveals that

$$
\begin{aligned}
\vec{\vec{G}}_{\theta 1ht1} = {} & \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\Bigg[\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \\
& +\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \\
& +\frac{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\left[\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)+\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\Bigg]
\end{aligned}
\tag{144}
$$

Next, analyze the component observed in region 1 due to longitudinal magnetic currents in region 1, $\tilde{\theta}_{1hz1}$. From the analysis of $\tilde{\vec{\vec{G}}}_{\theta 1e1}$ presented in Appendix C, it can be shown that substituting (140) and (138) into (136) implies that

$$\tilde{\vec{\vec{G}}}_{\theta 1hz1} = \left[\frac{\hat{z}Z_{\theta 1}}{2\omega\mu_{z1}}\right]\left[\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\left[\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)-\sin\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.$$
$$\left.+\frac{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\left[\cos\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \tag{145}$$

Now that $\tilde{\theta}_1$ has been found, determine $\tilde{\theta}_2$. Substituting (71) and (134) into (112) implies that

$$\tilde{\theta}_{2\{e,h\}} = \int_0^h \underbrace{\tilde{\vec{\vec{G}}}{}^{s}_{\theta 2\{e,h\}1}}_{\tilde{\vec{\vec{G}}}_{\theta 2\{e,h\}1}} \cdot \tilde{\vec{J}}_{\{e,h\}}dz' + \int_h^d \underbrace{\left[\tilde{\vec{\vec{G}}}{}^{p}_{\theta 2\{e,h\}} + \tilde{\vec{\vec{G}}}{}^{s}_{\theta 2\{e,h\}2}\right]}_{\tilde{\vec{\vec{G}}}_{\theta 2\{e,h\}2}} \cdot \tilde{\vec{J}}_{\{e,h\}}dz'$$
$$= \int_0^h D_\theta^{-1}\left[2\left(\tilde{\vec{\vec{g}}}{}^{p}_{\theta 1\{e,h\}}\left(z=h\right)P_{\theta 1z'}^{-1} - \tilde{\vec{\vec{g}}}{}^{p}_{\theta 1\{e,h\}}\left(z=0\right)P_{\theta 1z'}\right)\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right)\right]$$
$$\cdot\tilde{\vec{J}}_{\{e,h\}}dz' + \int_h^d D_\theta^{-1}\left\{\tilde{\vec{\vec{g}}}{}^{p}_{\theta 2\{e,h\}}e^{-j\lambda z\theta 2|z-z'|}\left[\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right.\right.$$
$$\left.+C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]$$
$$+\tilde{\vec{\vec{g}}}{}^{p}_{\theta 2\{e,h\}}\left(z=d\right)P_{\theta 2d}P_{\theta 2z'}^{-1}\left[\left(P_{\theta 2h}^{-1}P_{\theta 2z} - P_{\theta 2h}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right.$$
$$\left.-C_\theta\left(P_{\theta 2h}^{-1}P_{\theta 2z} + P_{\theta 2h}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]$$
$$+\tilde{\vec{\vec{g}}}{}^{p}_{\theta 2\{e,h\}}\left(z=h\right)P_{\theta 2h}^{-1}P_{\theta 2z'}\left[\left(P_{\theta 2d}P_{\theta 2z}^{-1} - P_{\theta 2d}^{-1}P_{\theta 2z}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right.$$
$$\left.\left.+C_\theta\left(P_{\theta 2d}^{-1}P_{\theta 2z} - P_{\theta 2d}P_{\theta 2z}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]\right\}\cdot\tilde{\vec{J}}_{\{e,h\}}dz' \tag{146}$$

Using (146) with similar substitution techniques as those used to find the components for $\tilde{\theta}_1$, the $\tilde{\vec{\vec{G}}}_{\theta 2\{e,h\}\{1,2\}}$ terms can be found in a reasonably straightforward manner. For brevity, they are derived in Appendix C. Additionally, determination of the $\tilde{\psi}_1$ and $\tilde{\psi}_2$ components proceeds in a similar manner to those of $\tilde{\theta}_1$ and $\tilde{\theta}_2$ above. Thus, the full derivation of the $\tilde{\psi}_1$ and $\tilde{\psi}_2$ components is presented in Appendix C. Now that the total Green functions have been determined for $\tilde{\theta}$ and $\tilde{\psi}$ in regions 1 and 2, determine the total Green functions for $\tilde{\Pi}$ in regions 1 and 2. Since $\tilde{\Phi}$ does not contribute to the magnetic field, its development is presented in Appendix C. Now that the total Green functions have been determined for $\tilde{\theta}$ and $\tilde{\psi}$ in regions 1 and 2, determine the total Green functions for $\tilde{\Pi}$ in regions 1 and 2. Since $\tilde{\Phi}$ does not contribute to the magnetic field, its development is presented in Appendix C. (37) and analysis from (101) imply that

$$\begin{aligned}
\tilde{\Pi}_{\{1,2\}} &= -\frac{1}{j\omega\mu_{t\{1,2\}}}\left(\int_0^h \frac{\partial}{\partial z}\left[\tilde{\vec{\vec{G}}}_{\theta\{1,2\}e1}+\tilde{\vec{\vec{G}}}_{\theta\{1,2\}ht1}+\tilde{\vec{\vec{G}}}_{\theta\{1,2\}hz1}\right]\cdot\tilde{\vec{J}}_h dz'\right.\\
&\qquad\left.+\int_h^d \frac{\partial}{\partial z}\left[\tilde{\vec{\vec{G}}}_{\theta\{1,2\}e2}+\tilde{\vec{\vec{G}}}_{\theta\{1,2\}ht2}+\tilde{\vec{\vec{G}}}_{\theta\{1,2\}hz2}\right]\cdot\tilde{\vec{J}}_h dz'\right)\\
\Rightarrow \tilde{\vec{\vec{G}}}_{\Pi\{1,2\}\{e,ht,hz\}\{1,2\}} &= -\frac{1}{j\omega\mu_{t\{1,2\}}}\frac{\partial}{\partial z}\tilde{\vec{\vec{G}}}_{\theta\{1,2\}\{e,ht,hz\}\{1,2\}}
\end{aligned}\tag{147}$$

First, analyze the components observed in region 1, $\tilde{\Pi}_1$. Begin by analyzing the component observed in region 1 resulting from transverse magnetic currents in region 1, $\tilde{\Pi}_{1ht1}$. Substituting (144) into (147) implies that

$$\begin{aligned}
\tilde{\vec{\vec{G}}}_{\Pi1ht1} = -\frac{1}{j\omega\mu_{t1}}\frac{\partial}{\partial z}&\left\{\left[\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right]\left[\frac{\sin\left(\lambda_{z\theta2}T\right)\left[\cos\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right.\right.\\
&+\frac{\sin\left(\lambda_{z\theta2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\theta1}\left(h-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\\
&\left.\left.+\frac{C_\theta\cos\left(\lambda_{z\theta2}T\right)\left[\sin\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)+\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta1}\left(h-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 \omega \mu_{t1}}\right) \left[\frac{\sin\left(\lambda_{z\theta 2} T\right)\left[\lambda_{z\theta 1} \sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\theta 2} T\right)\left[\frac{\partial}{\partial z} \operatorname{sgn}\left(z-z'\right) \cos\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad +\frac{C_\theta \cos\left(\lambda_{z\theta 2} T\right)\left[-\lambda_{z\theta 1} \cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad \left.+\frac{C_\theta \cos\left(\lambda_{z\theta 2} T\right)\left[\frac{\partial}{\partial z} \operatorname{sgn}\left(z-z'\right) \sin\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)}\right]
\end{aligned}
\tag{148}
$$

Due to the $\operatorname{sgn}(z-z')$ in (148), two cases must be analyzed. When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}_{\Pi 1 h t 1}^{z^+} &= \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 \omega \mu_{t1}}\right) \left[\frac{\sin\left(\lambda_{z\theta 2} T\right)\left[\lambda_{z\theta 1} \sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\theta 2} T\right)\left[\frac{\partial}{\partial z} \cancelto{1}{\operatorname{sgn}\left(z-z'\right)} \cos\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad +\frac{C_\theta \cos\left(\lambda_{z\theta 2} T\right)\left[-\lambda_{z\theta 1} \cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad \left.+\frac{C_\theta \cos\left(\lambda_{z\theta 2} T\right)\left[\frac{\partial}{\partial z} \cancelto{1}{\operatorname{sgn}\left(z-z'\right)} \sin\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)}\right] \\
&= \left(-\frac{\vec{\lambda}_\rho \lambda_{z\theta 1}}{2\lambda_{\rho\theta}^2 \omega \mu_{t1}}\right) \left[\frac{\sin\left(\lambda_{z\theta 2} T\right)\left[\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)+\sin\left(\lambda_{z\theta 1}\left(h-\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)}\right. \\
&\quad \left.+\frac{C_\theta \cos\left(\lambda_{z\theta 2} T\right)\left[-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)-\cos\left(\lambda_{z\theta 1}\left(h-\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right)+C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)}\right]
\end{aligned}
\tag{149}
$$

When $z < z'$, that implies that

$$\vec{\tilde{G}}_{\Pi 1ht1}^{z^-} = \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 \omega \mu_{t1}}\right) \left[\frac{\sin(\lambda_{z\theta 2}T)\left[\lambda_{z\theta 1}\sin(\lambda_{z\theta 1}(h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.$$
$$+\frac{\sin(\lambda_{z\theta 2}T)\left[\cancelto{-1}{\frac{\partial}{\partial z}\operatorname{sgn}(z-z')}\cos(\lambda_{z\theta 1}(h+(z-z')))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}$$
$$+\frac{C_\theta\cos(\lambda_{z\theta 2}T)\left[-\lambda_{z\theta 1}\cos(\lambda_{z\theta 1}(h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}$$
$$\left.+\frac{C_\theta\cos(\lambda_{z\theta 2}T)\left[\cancelto{-1}{\frac{\partial}{\partial z}\operatorname{sgn}(z-z')}\sin(\lambda_{z\theta 1}(h+(z-z')))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right]$$
$$= \left(-\frac{\vec{\lambda}_\rho \lambda_{z\theta 1}}{2\lambda_{\rho\theta}^2 \omega \mu_{t1}}\right) \left[\frac{\sin(\lambda_{z\theta 2}T)\left[\sin(\lambda_{z\theta 1}(h-z-z')) + \sin(\lambda_{z\theta 1}(h+(z-z')))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.$$
$$\left.+\frac{C_\theta\cos(\lambda_{z\theta 2}T)\left[-\cos(\lambda_{z\theta 1}(h-z-z')) - \cos(\lambda_{z\theta 1}(h+(z-z')))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right] \tag{150}$$

Analyzing (149) and (150) reveals that

$$\vec{\tilde{G}}_{\Pi 1ht1} = \left(-\frac{\vec{\lambda}_\rho \lambda_{z\theta 1}}{2\lambda_{\rho\theta}^2 \omega \mu_{t1}}\right) \left[\frac{\sin(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}(h-z-z'))}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.$$
$$+\frac{\sin(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}(h-|z-z'|))}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}$$
$$\left.+\frac{C_\theta\cos(\lambda_{z\theta 2}T)\left[-\cos(\lambda_{z\theta 1}(h-z-z')) - \cos(\lambda_{z\theta 1}(h-|z-z'|))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta \cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right]$$
$$= \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 Z_{\theta 1}}\right) \left[\frac{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\left[\sin(\lambda_{z\theta 1}(h-z-z')) + \sin(\lambda_{z\theta 1}(h-|z-z'|))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.$$
$$\left.+\frac{Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\left[-\cos(\lambda_{z\theta 1}(h-z-z')) - \cos(\lambda_{z\theta 1}(h-|z-z'|))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right] \tag{151}$$

Continuing in this manner, it is straightforward to derive the remaining $\tilde{\Pi}$ Green functions. For brevity, the remaining derivations are presented in Appendix C.

Transverse Spectral Domain Total Scalar Potential Grand Summary.

$$
\begin{aligned}
\tilde{\theta}_{\{1,2\}} = & \int_0^h \vec{\tilde{G}}_{\theta\{1,2\}e1} \cdot \vec{\tilde{J}}_e dz' + \int_0^h \left(\vec{\tilde{G}}_{\theta\{1,2\}ht1} + \vec{\tilde{G}}_{\theta\{1,2\}hz1} \right) \cdot \vec{\tilde{J}}_h dz' \\
& + \int_h^d \vec{\tilde{G}}_{\theta\{1,2\}e2} \cdot \vec{\tilde{J}}_e dz' + \int_h^d \left(\vec{\tilde{G}}_{\theta\{1,2\}ht2} + \vec{\tilde{G}}_{\theta\{1,2\}hz2} \right) \cdot \vec{\tilde{J}}_h dz'
\end{aligned}
$$

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 1e1} &= \left(\tfrac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}}{2\lambda_{\rho\theta}^2} \right) \Upsilon_1^\theta & \vec{\tilde{G}}_{\theta 1e2} &= \left(\tfrac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2} \right) \Upsilon_2^\theta \\
\vec{\tilde{G}}_{\theta 1ht1} &= \left(j\tfrac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \right) \Upsilon_3^\theta & \vec{\tilde{G}}_{\theta 1ht2} &= \left(-j\tfrac{\vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2} \right) \Upsilon_4^\theta \\
\vec{\tilde{G}}_{\theta 1hz1} &= \left(-\tfrac{\hat{z} Z_{\theta 1}}{2\omega\mu_{z1}} \right) \Upsilon_1^\theta & \vec{\tilde{G}}_{\theta 1hz2} &= \left(\tfrac{\hat{z} Z_{\theta 1}^2}{\omega\mu_{z1}} \right) \Upsilon_2^\theta \\
\vec{\tilde{G}}_{\theta 2e1} &= \left(\tfrac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}^2}{\lambda_{\rho\theta}^2} \right) \Upsilon_5^\theta & \vec{\tilde{G}}_{\theta 2e2} &= \left(\tfrac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{2\lambda_{\rho\theta}^2} \right) \Upsilon_6^\theta \\
\vec{\tilde{G}}_{\theta 2ht1} &= \left(j\tfrac{\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2} \right) \Upsilon_7^\theta & \vec{\tilde{G}}_{\theta 2ht2} &= \left(j\tfrac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \right) \Upsilon_8^\theta \\
\vec{\tilde{G}}_{\theta 2hz1} &= \left(-\tfrac{\hat{z} Z_{\theta 2}^2}{\omega\mu_{z2}} \right) \Upsilon_5^\theta & \vec{\tilde{G}}_{\theta 2hz2} &= \left(-\tfrac{\hat{z} Z_{\theta 2}}{2\omega\mu_{z2}} \right) \Upsilon_6^\theta
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Pi}_{\{1,2\}} = & \int_0^h \vec{\tilde{G}}_{\Pi\{1,2\}e1} \cdot \vec{\tilde{J}}_e dz' + \int_0^h \left(\vec{\tilde{G}}_{\Pi\{1,2\}ht1} + \vec{\tilde{G}}_{\Pi\{1,2\}hz1} \right) \cdot \vec{\tilde{J}}_h dz' \\
& + \int_h^d \vec{\tilde{G}}_{\Pi\{1,2\}e2} \cdot \vec{\tilde{J}}_e dz' + \int_h^d \left(\vec{\tilde{G}}_{\Pi\{1,2\}ht2} + \vec{\tilde{G}}_{\Pi\{1,2\}hz2} \right) \cdot \vec{\tilde{J}}_h dz'
\end{aligned}
$$

$$
\begin{aligned}
\vec{\tilde{G}}_{\Pi 1e1} &= \left(-j\tfrac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \right) \Upsilon_9^\theta & \vec{\tilde{G}}_{\Pi 1e2} &= \left(j\tfrac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2} \right) \Upsilon_{10}^\theta \\
\vec{\tilde{G}}_{\Pi 1ht1} &= \left(\tfrac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^\theta & \vec{\tilde{G}}_{\Pi 1ht2} &= \left(\tfrac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2} \right) \Upsilon_{12}^\theta \\
\vec{\tilde{G}}_{\Pi 1hz1} &= \left(j\tfrac{\hat{z}}{2\omega\mu_{z1}} \right) \Upsilon_9^\theta & \vec{\tilde{G}}_{\Pi 1hz2} &= \left(-j\tfrac{\hat{z} Z_{\theta 1}}{\omega\mu_{z1}} \right) \Upsilon_{10}^\theta \\
\vec{\tilde{G}}_{\Pi 2e1} &= \left(-j\tfrac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2} \right) \Upsilon_{13}^\theta & \vec{\tilde{G}}_{\Pi 2e2} &= \left(-j\tfrac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \right) \Upsilon_{14}^\theta \\
\vec{\tilde{G}}_{\Pi 2ht1} &= \left(\tfrac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2} \right) \Upsilon_{15}^\theta & \vec{\tilde{G}}_{\Pi 2ht2} &= \left(\tfrac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 Z_{\theta 2}} \right) \Upsilon_{16}^\theta \\
\vec{\tilde{G}}_{\Pi 2hz1} &= \left(j\tfrac{\hat{z} Z_{\theta 2}}{\omega\mu_{z2}} \right) \Upsilon_{13}^\theta & \vec{\tilde{G}}_{\Pi 2hz2} &= \left(j\tfrac{\hat{z}}{2\omega\mu_{z2}} \right) \Upsilon_{14}^\theta
\end{aligned}
$$

$$\tilde{\psi}_{\{1,2\}} = \int\limits_0^h \left(\vec{\tilde{G}}_{\psi\{1,2\}et1} + \vec{\tilde{G}}_{\psi\{1,2\}ez1}\right) \cdot \vec{\tilde{J}}_e dz' + \int\limits_0^h \vec{\tilde{G}}_{\psi\{1,2\}h1} \cdot \vec{\tilde{J}}_h dz' + \int\limits_h^d \left(\vec{\tilde{G}}_{\psi\{1,2\}et2} + \vec{\tilde{G}}_{\psi\{1,2\}ez2}\right) \cdot \vec{\tilde{J}}_e dz' + \int\limits_h^d \vec{\tilde{G}}_{\psi\{1,2\}h2} \cdot \vec{\tilde{J}}_h dz'$$

$$\vec{\tilde{G}}_{\psi 1et1} = \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_9^\psi \qquad \vec{\tilde{G}}_{\psi 1et2} = \left(j\frac{\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^\psi$$

$$\vec{\tilde{G}}_{\psi 1ez1} = \left(-\frac{\hat{z}}{2Z_{\psi 1}\omega\epsilon_{z1}}\right)\Upsilon_{11}^\psi \qquad \vec{\tilde{G}}_{\psi 1ez2} = \left(-\frac{\hat{z}Z_{\psi 2}}{Z_{\psi 1}\omega\epsilon_{z1}}\right)\Upsilon_{12}^\psi$$

$$\vec{\tilde{G}}_{\psi 1h1} = \left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{11}^\psi \qquad \vec{\tilde{G}}_{\psi 1h2} = \left(-\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi$$

$$\vec{\tilde{G}}_{\psi 2et1} = \left(-j\frac{\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^\psi \qquad \vec{\tilde{G}}_{\psi 2et2} = \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^\psi$$

$$\vec{\tilde{G}}_{\psi 2ez1} = \left(-\frac{\hat{z}Z_{\psi 1}}{Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{15}^\psi \qquad \vec{\tilde{G}}_{\psi 2ez2} = \left(-\frac{\hat{z}}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^\psi$$

$$\vec{\tilde{G}}_{\psi 2h1} = \left(-\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{15}^\psi \qquad \vec{\tilde{G}}_{\psi 2h2} = \left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{16}^\psi$$

$$\tilde{\Phi}_{\{1,2\}} = \int\limits_0^h \left(\vec{\tilde{G}}_{\Phi\{1,2\}et1} + \vec{\tilde{G}}_{\Phi\{1,2\}ez1}\right) \cdot \vec{\tilde{J}}_e dz' + \int\limits_0^h \vec{\tilde{G}}_{\Phi\{1,2\}h1} \cdot \vec{\tilde{J}}_h dz' + \int\limits_h^d \left(\vec{\tilde{G}}_{\Phi\{1,2\}et2} + \vec{\tilde{G}}_{\Phi\{1,2\}ez2}\right) \cdot \vec{\tilde{J}}_e dz' + \int\limits_h^d \vec{\tilde{G}}_{\Phi\{1,2\}h2} \cdot \vec{\tilde{J}}_h dz'$$

$$\vec{\tilde{G}}_{\Phi 1et1} = \left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi \qquad \vec{\tilde{G}}_{\Phi 1et2} = \left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi$$

$$\vec{\tilde{G}}_{\Phi 1ez1} = \left(j\frac{\hat{z}}{2\omega\epsilon_{z1}}\right)\Upsilon_3^\psi \qquad \vec{\tilde{G}}_{\Phi 1ez2} = \left(-j\frac{\hat{z}Z_{\psi 2}}{\omega\epsilon_{z1}}\right)\Upsilon_4^\psi$$

$$\vec{\tilde{G}}_{\Phi 1h1} = \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_3^\psi \qquad \vec{\tilde{G}}_{\Phi 1h2} = \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_4^\psi$$

$$\vec{\tilde{G}}_{\Phi 2et1} = \left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_5^\psi \qquad \vec{\tilde{G}}_{\Phi 2et2} = \left(-\frac{\vec{\lambda}_\rho Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi$$

$$\vec{\tilde{G}}_{\Phi 2ez1} = \left(j\frac{\hat{z}Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\Upsilon_7^\psi \qquad \vec{\tilde{G}}_{\Phi 2ez2} = \left(j\frac{\hat{z}}{2\omega\epsilon_{z2}}\right)\Upsilon_8^\psi$$

$$\vec{\tilde{G}}_{\Phi 2h1} = \left(j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_7^\psi \qquad \vec{\tilde{G}}_{\Phi 2h2} = \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_8^\psi$$

Supplemental Relations:

$$\Upsilon_1^\alpha = \left[\frac{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\left[\cos\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)-\cos\left(\lambda_{z\alpha 1}\left(h-|z-z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\left[\sin\left(\lambda_{z\alpha 1}\left(h-|z-z'|\right)\right)-\sin\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_2^\alpha = \left[\frac{\sin\left(\lambda_{z\alpha 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\alpha 1}z\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_3^\alpha = \left[\frac{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\alpha 1}\left(h-|z-z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$+\frac{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\left[\sin\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$+\frac{Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\alpha 1}\left(h-|z-z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$\left.+\frac{Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\left[\cos\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_4^\alpha = \left[\frac{\cos\left(\lambda_{z\alpha 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\alpha 1}z\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_5^\alpha = \left[\frac{\sin\left(\lambda_{z\alpha 1}z'\right)\sin\left(\lambda_{z\alpha 2}\left(d-z\right)\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_6^\alpha = \left[\frac{Z_{\alpha 1}\sin\left(\lambda_{z\alpha 1}h\right)\left[\sin\left(\lambda_{z\alpha 2}\left(T-|z-z'|\right)\right)+\sin\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\cos\left(\lambda_{z\alpha 1}h\right)\left[\cos\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)-\cos\left(\lambda_{z\alpha 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_7^\alpha = \left[\frac{\cos\left(\lambda_{z\alpha 1}z'\right)\sin\left(\lambda_{z\alpha 2}\left(d-z\right)\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_8^\alpha = \left[\frac{Z_{\alpha 1}\sin\left(\lambda_{z\alpha 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\alpha 2}\left(d-h-|z-z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$+\frac{Z_{\alpha 1}\sin\left(\lambda_{z\alpha 1}h\right)\left[-\cos\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$+\frac{Z_{\alpha 2}\cos\left(\lambda_{z\alpha 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\alpha 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$\left.+\frac{Z_{\alpha 2}\cos\left(\lambda_{z\alpha 1}h\right)\left[\sin\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$\alpha \in \{\theta, \psi\}$ $\qquad$ $T = d - h$

$Z_{\theta\{1,2\}} = \frac{\omega\mu_{t\{1,2\}}}{\lambda_{z\theta\{1,2\}}}$ $\qquad$ $Z_{\psi\{1,2\}} = \frac{\lambda_{z\psi\{1,2\}}}{\omega\epsilon_{t\{1,2\}}}$

$\lambda_{z\theta\{1,2\}}^2 = \omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\lambda_{\rho\theta}^2$ $\qquad$ $\lambda_{z\psi\{1,2\}}^2 = \omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}}\lambda_{\rho\psi}^2$

$$\Upsilon_9^\alpha = \left[\frac{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\left[\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 1}(h-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$+\frac{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\left[-\sin(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$+\frac{Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\left[\operatorname{sgn}(z-z')\cos(\lambda_{z\alpha 1}(h-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$\left.+\frac{Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\left[-\cos(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{10}^\alpha = \left[\frac{\sin(\lambda_{z\alpha 2}(d-z'))\cos(\lambda_{z\alpha 1}z)}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{11}^\alpha = \left[\frac{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\left[\cos(\lambda_{z\alpha 1}(h-|z-z'|))+\cos(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\left[-\sin(\lambda_{z\alpha 1}(h-|z-z'|))-\sin(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{12}^\alpha = \left[\frac{\cos(\lambda_{z\alpha 2}(d-z'))\cos(\lambda_{z\alpha 1}z)}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{13}^\alpha = \left[\frac{\sin(\lambda_{z\alpha 1}z')\cos(\lambda_{z\alpha 2}(d-z))}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{14}^\alpha = \left[\frac{Z_{\alpha 1}\sin(\lambda_{z\alpha 1}h)\left[\operatorname{sgn}(z-z')\cos(\lambda_{z\alpha 2}(T-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$+\frac{Z_{\alpha 1}\sin(\lambda_{z\alpha 1}h)\left[\cos(\lambda_{z\alpha 2}(d+h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$+\frac{Z_{\alpha 2}\cos(\lambda_{z\alpha 1}h)\left[\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 2}(T-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$\left.+\frac{Z_{\alpha 2}\cos(\lambda_{z\alpha 1}h)\left[-\sin(\lambda_{z\alpha 2}(d+h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{15}^\alpha = \left[\frac{\cos(\lambda_{z\alpha 1}z')\cos(\lambda_{z\alpha 2}(d-z))}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{16}^\alpha = \left[\frac{Z_{\alpha 1}\sin(\lambda_{z\alpha 1}h)\left[\sin(\lambda_{z\alpha 2}(d+h-z-z'))-\sin(\lambda_{z\alpha 2}(T-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\cos(\lambda_{z\alpha 1}h)\left[\cos(\lambda_{z\alpha 2}(T-|z-z'|))+\cos(\lambda_{z\alpha 2}(d+h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$

$\alpha \in \{\theta, \psi\}$ $\qquad$ $T = d - h$

$Z_{\theta\{1,2\}} = \frac{\omega\mu_{t(1,2)}}{\lambda_{z\theta\{1,2\}}}$ $\qquad$ $Z_{\psi\{1,2\}} = \frac{\lambda_{z\psi\{1,2\}}}{\omega\epsilon_{t\{1,2\}}}$

$\lambda_{z\theta\{1,2\}}^2 = \omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\lambda_{\rho\theta}^2$ $\qquad$ $\lambda_{z\psi\{1,2\}}^2 = \omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}}\lambda_{\rho\psi}^2$

2.6 Transverse Spectral Domain Total Field Recovery

Now that the transverse spatial frequency domain potentials have been determined, they can be used to recover the electric and magnetic fields.

Begin by taking the forward transverse Fourier transform of the total field equations, $\mathcal{F}_\rho\{(29),(30),(34),(33)\}$, which implies that

$$\vec{\tilde{E}}_{t\{1,2\}} = j\vec{\lambda}_\rho\tilde{\Phi}_{\{1,2\}} - j\hat{z}\times\vec{\lambda}_\rho\tilde{\theta}_{\{1,2\}} \tag{152}$$

$$\vec{\tilde{H}}_{t\{1,2\}} = j\vec{\lambda}_\rho\tilde{\Pi}_{\{1,2\}} - j\hat{z}\times\vec{\lambda}_\rho\tilde{\psi}_{\{1,2\}} \tag{153}$$

$$\tilde{E}_{z\{1,2\}} = -\frac{1}{j\omega\epsilon_{z\{1,2\}}}\left(-\lambda_{\rho\psi}^2\tilde{\psi}_{\{1,2\}} + \tilde{J}_{ez}\right) \tag{154}$$

$$\tilde{H}_{z\{1,2\}} = \frac{1}{j\omega\mu_{z\{1,2\}}}\left(-\lambda_{\rho\theta}^2\tilde{\theta}_{\{1,2\}} - \tilde{J}_{hz}\right) \tag{155}$$

To obtain the total electric field, (152) and (154) imply that

$$\begin{aligned}\vec{\tilde{E}}_{\{1,2\}} &= \vec{\tilde{E}}_{t\{1,2\}} + \hat{z}\tilde{E}_{z\{1,2\}} \\ &= j\vec{\lambda}_\rho\tilde{\Phi}_{\{1,2\}} - j\hat{z}\times\vec{\lambda}_\rho\tilde{\theta}_{\{1,2\}} + \hat{z}\left(-\frac{1}{j\omega\epsilon_{z\{1,2\}}}\left(-\lambda_{\rho\psi}^2\tilde{\psi}_{\{1,2\}} + \hat{z}\cdot\vec{\tilde{J}}_e\right)\right) \\ &= j\vec{\lambda}_\rho\tilde{\Phi}_{\{1,2\}} - j\hat{z}\times\vec{\lambda}_\rho\tilde{\theta}_{\{1,2\}} + \hat{z}\frac{\lambda_{\rho\psi}^2}{j\omega\epsilon_{z\{1,2\}}}\tilde{\psi}_{\{1,2\}} - \hat{z}\hat{z}\cdot\frac{\vec{\tilde{J}}_e}{j\omega\epsilon_{z\{1,2\}}}\end{aligned} \tag{156}$$

Substituting the scalar potentials $\tilde{\Phi}, \tilde{\theta}$, and $\tilde{\psi}$ into (156) implies that

$$
\begin{aligned}
\vec{\tilde{E}}_{\{1,2\}} = j\vec{\lambda}_\rho \Bigg(\int_0^h \Big[\vec{\tilde{G}}_{\Phi\{1,2\}h1} \cdot \vec{\tilde{J}}_h + \Big(\vec{\tilde{G}}_{\Phi\{1,2\}et1} + \vec{\tilde{G}}_{\Phi\{1,2\}ez1} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \\
+ \int_h^d \Big[\vec{\tilde{G}}_{\Phi\{1,2\}h2} \cdot \vec{\tilde{J}}_h + \Big(\vec{\tilde{G}}_{\Phi\{1,2\}et2} + \vec{\tilde{G}}_{\Phi\{1,2\}ez2} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \Bigg) \\
- j\hat{z} \times \vec{\lambda}_\rho \Bigg(\int_0^h \Big[\vec{\tilde{G}}_{\theta\{1,2\}e1} \cdot \vec{\tilde{J}}_e + \Big(\vec{\tilde{G}}_{\theta\{1,2\}ht1} + \vec{\tilde{G}}_{\theta\{1,2\}hz1} \Big) \cdot \vec{\tilde{J}}_h \Big] dz' \\
+ \int_h^d \Big[\vec{\tilde{G}}_{\theta\{1,2\}e2} \cdot \vec{\tilde{J}}_e + \Big(\vec{\tilde{G}}_{\theta\{1,2\}ht2} + \vec{\tilde{G}}_{\theta\{1,2\}hz2} \Big) \cdot \vec{\tilde{J}}_h \Big] dz' \Bigg) \\
+ \hat{z} \frac{\lambda_{\rho\psi}^2}{j\omega\epsilon_{z\{1,2\}}} \Bigg(\int_0^h \Big[\vec{\tilde{G}}_{\psi\{1,2\}h1} \cdot \vec{\tilde{J}}_h + \Big(\vec{\tilde{G}}_{\psi\{1,2\}et1} + \vec{\tilde{G}}_{\psi\{1,2\}ez1} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \\
+ \int_h^d \Big[\vec{\tilde{G}}_{\psi\{1,2\}h2} \cdot \vec{\tilde{J}}_h + \Big(\vec{\tilde{G}}_{\psi\{1,2\}et2} + \vec{\tilde{G}}_{\psi\{1,2\}ez2} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \Bigg) - \hat{z}\hat{z} \cdot \frac{\vec{\tilde{J}}_e}{j\omega\epsilon_{z\{1,2\}}} \\
= \int_0^h \Big[j\vec{\lambda}_\rho \vec{\tilde{G}}_{\Phi\{1,2\}h1} \cdot \vec{\tilde{J}}_h + j\vec{\lambda}_\rho \Big(\vec{\tilde{G}}_{\Phi\{1,2\}et1} + \vec{\tilde{G}}_{\Phi\{1,2\}ez1} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \\
+ \int_h^d \Big[j\vec{\lambda}_\rho \vec{\tilde{G}}_{\Phi\{1,2\}h2} \cdot \vec{\tilde{J}}_h + j\vec{\lambda}_\rho \Big(\vec{\tilde{G}}_{\Phi\{1,2\}et2} + \vec{\tilde{G}}_{\Phi\{1,2\}ez2} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \\
+ \int_0^h \Big[-j\hat{z} \times \vec{\lambda}_\rho \vec{\tilde{G}}_{\theta\{1,2\}e1} \cdot \vec{\tilde{J}}_e - j\hat{z} \times \vec{\lambda}_\rho \Big(\vec{\tilde{G}}_{\theta\{1,2\}ht1} + \vec{\tilde{G}}_{\theta\{1,2\}hz1} \Big) \cdot \vec{\tilde{J}}_h \Big] dz' \\
+ \int_h^d \Big[-j\hat{z} \times \vec{\lambda}_\rho \vec{\tilde{G}}_{\theta\{1,2\}e2} \cdot \vec{\tilde{J}}_e - j\hat{z} \times \vec{\lambda}_\rho \Big(\vec{\tilde{G}}_{\theta\{1,2\}ht2} + \vec{\tilde{G}}_{\theta\{1,2\}hz2} \Big) \cdot \vec{\tilde{J}}_h \Big] dz' \\
+ \int_0^h \Big[\hat{z} \frac{\lambda_{\rho\psi}^2}{j\omega\epsilon_{z\{1,2\}}} \vec{\tilde{G}}_{\psi\{1,2\}h1} \cdot \vec{\tilde{J}}_h + \hat{z} \frac{\lambda_{\rho\psi}^2}{j\omega\epsilon_{z\{1,2\}}} \Big(\vec{\tilde{G}}_{\psi\{1,2\}et1} + \vec{\tilde{G}}_{\psi\{1,2\}ez1} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \\
+ \int_h^d \Big[\hat{z} \frac{\lambda_{\rho\psi}^2}{j\omega\epsilon_{z\{1,2\}}} \vec{\tilde{G}}_{\psi\{1,2\}h2} \cdot \vec{\tilde{J}}_h + \hat{z} \frac{\lambda_{\rho\psi}^2}{j\omega\epsilon_{z\{1,2\}}} \Big(\vec{\tilde{G}}_{\psi\{1,2\}et2} + \vec{\tilde{G}}_{\psi\{1,2\}ez2} \Big) \cdot \vec{\tilde{J}}_e \Big] dz' \\
- \hat{z}\hat{z} \cdot \frac{\vec{\tilde{J}}_e}{j\omega\epsilon_{z\{1,2\}}}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^h \Big[j\vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\Phi\{1,2\}et1} + \tilde{\vec{G}}_{\Phi\{1,2\}ez1} \right) - j\hat{z}\times\vec{\lambda}_\rho \tilde{\vec{G}}_{\theta\{1,2\}e1} \\
&\qquad - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z\{1,2\}}} \left(\tilde{\vec{G}}_{\psi\{1,2\}et1} + \tilde{\vec{G}}_{\psi\{1,2\}ez1} \right) + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z\{1,2\}}}\delta\left(z-z'\right) \Big] \cdot \tilde{\vec{J}}_e dz' \\
&+ \int_0^h \left[j\vec{\lambda}_\rho \tilde{\vec{G}}_{\Phi\{1,2\}h1} - j\hat{z}\times\vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\theta\{1,2\}ht1} + \tilde{\vec{G}}_{\theta\{1,2\}hz1} \right) - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z\{1,2\}}} \tilde{\vec{G}}_{\psi\{1,2\}h1} \right] \cdot \tilde{\vec{J}}_h dz' \\
&\qquad + \int_h^d \Big[j\vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\Phi\{1,2\}et2} + \tilde{\vec{G}}_{\Phi\{1,2\}ez2} \right) - j\hat{z}\times\vec{\lambda}_\rho \tilde{\vec{G}}_{\theta\{1,2\}e2} \\
&\qquad - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z\{1,2\}}} \left(\tilde{\vec{G}}_{\psi\{1,2\}et2} + \tilde{\vec{G}}_{\psi\{1,2\}ez2} \right) + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z\{1,2\}}}\delta\left(z-z'\right) \Big] \cdot \tilde{\vec{J}}_e dz' \\
&+ \int_h^d \left[j\vec{\lambda}_\rho \tilde{\vec{G}}_{\Phi\{1,2\}h2} - j\hat{z}\times\vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\theta\{1,2\}ht2} + \tilde{\vec{G}}_{\theta\{1,2\}hz2} \right) - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z\{1,2\}}} \tilde{\vec{G}}_{\psi\{1,2\}h2} \right] \cdot \tilde{\vec{J}}_h dz' \\
&= \int_0^h \tilde{\overleftrightarrow{G}}_{e\{1,2\}e1} \cdot \tilde{\vec{J}}_e dz' + \int_0^h \tilde{\overleftrightarrow{G}}_{e\{1,2\}h1} \cdot \tilde{\vec{J}}_h dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{e\{1,2\}e2} \cdot \tilde{\vec{J}}_e dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{e\{1,2\}h2} \cdot \tilde{\vec{J}}_h dz'
\end{aligned}
\tag{157}
$$

Next, to obtain the total magnetic field, (153) and (155) imply that

$$
\begin{aligned}
\tilde{\vec{H}}_{\{1,2\}} &= \tilde{\vec{H}}_{t\{1,2\}} + \hat{z}\tilde{H}_{z\{1,2\}} \\
&= j\vec{\lambda}_\rho \tilde{\Pi}_{\{1,2\}} - j\hat{z}\times\vec{\lambda}_\rho\tilde{\psi}_{\{1,2\}} + \hat{z}\left(\frac{1}{j\omega\mu_{z\{1,2\}}} \left(-\lambda_{\rho\theta}^2\tilde{\theta}_{\{1,2\}} - \hat{z}\cdot\tilde{\vec{J}}_h \right) \right) \\
&= j\vec{\lambda}_\rho \tilde{\Pi}_{\{1,2\}} - j\hat{z}\times\vec{\lambda}_\rho\tilde{\psi}_{\{1,2\}} - \hat{z}\frac{\lambda_{\rho\theta}^2}{j\omega\mu_{z\{1,2\}}}\tilde{\theta}_{\{1,2\}} - \hat{z}\hat{z}\cdot\frac{\tilde{\vec{J}}_h}{j\omega\mu_{z\{1,2\}}}
\end{aligned}
\tag{158}
$$

By substituting the scalar potentials $\tilde{\Pi}, \tilde{\psi}$, and $\tilde{\theta}$ into (158), it can be shown that

$$\begin{aligned}
\vec{\tilde{H}}_{\{1,2\}} = \int_0^h \Bigg[& j\vec{\lambda}_\rho \vec{\tilde{G}}_{\Pi\{1,2\}e1} - j\hat{z} \times \vec{\lambda}_\rho \left(\vec{\tilde{G}}_{\psi\{1,2\}et1} + \vec{\tilde{G}}_{\psi\{1,2\}ez1} \right) + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z\{1,2\}}} \vec{\tilde{G}}_{\theta\{1,2\}e1} \Bigg] \\
& \cdot \vec{\tilde{J}}_e dz' + \int_0^h \Bigg[j\vec{\lambda}_\rho \left(\vec{\tilde{G}}_{\Pi\{1,2\}ht1} + \vec{\tilde{G}}_{\Pi\{1,2\}hz1} \right) - j\hat{z} \times \vec{\lambda}_\rho \vec{\tilde{G}}_{\psi\{1,2\}h1} \\
& + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z\{1,2\}}} \left(\vec{\tilde{G}}_{\theta\{1,2\}ht1} + \vec{\tilde{G}}_{\theta\{1,2\}hz1} \right) + j\frac{\hat{z}\hat{z}}{\omega\mu_{z\{1,2\}}} \delta(z - z') \Bigg] \cdot \vec{\tilde{J}}_h dz' + \int_h^d \Bigg[j\vec{\lambda}_\rho \vec{\tilde{G}}_{\Pi\{1,2\}e2} \\
& - j\hat{z} \times \vec{\lambda}_\rho \left(\vec{\tilde{G}}_{\psi\{1,2\}et2} + \vec{\tilde{G}}_{\psi\{1,2\}ez2} \right) + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z\{1,2\}}} \vec{\tilde{G}}_{\theta\{1,2\}e2} \Bigg] \cdot \vec{\tilde{J}}_e dz' \\
& + \int_h^d \Bigg[j\vec{\lambda}_\rho \left(\vec{\tilde{G}}_{\Pi\{1,2\}ht2} + \vec{\tilde{G}}_{\Pi\{1,2\}hz2} \right) - j\hat{z} \times \vec{\lambda}_\rho \vec{\tilde{G}}_{\psi\{1,2\}h2} \\
& + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z\{1,2\}}} \left(\vec{\tilde{G}}_{\theta\{1,2\}ht2} + \vec{\tilde{G}}_{\theta\{1,2\}hz2} \right) + j\frac{\hat{z}\hat{z}}{\omega\mu_{z\{1,2\}}} \delta(z - z') \Bigg] \cdot \vec{\tilde{J}}_h dz' \\
= & \int_0^h \overleftrightarrow{\tilde{G}}_{h\{1,2\}e1} \cdot \vec{\tilde{J}}_e dz' + \int_0^h \overleftrightarrow{\tilde{G}}_{h\{1,2\}h1} \cdot \vec{\tilde{J}}_h dz' + \int_h^d \overleftrightarrow{\tilde{G}}_{h\{1,2\}e2} \cdot \vec{\tilde{J}}_e dz' + \int_h^d \overleftrightarrow{\tilde{G}}_{h\{1,2\}h2} \cdot \vec{\tilde{J}}_h dz'
\end{aligned} \tag{159}$$

Determination of Transverse Spectral Domain Total Field Green Functions.

Now that the transverse spectral domain total fields equations have been determined, the Green functions contributing to those fields must be analyzed. Noting that the Green functions are dyadic in nature, it is useful to analyze some key dyads that will be used in the analyses going forward. Note that $\vec{a}\vec{b} = \left(\vec{b}\vec{a}\right)^T$ where $\vec{a}, \vec{b} \in \left\{\vec{\lambda}_\rho, \hat{z}\right\}$. Therefore, terms where the order of the vectors are reversed are omitted from this analysis. Thus,

$$\vec{\lambda}_\rho \vec{\lambda}_\rho = (\hat{x}\lambda_x + \hat{y}\lambda_y)(\hat{x}\lambda_x + \hat{y}\lambda_y) = \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & 0 \\ \lambda_x\lambda_y & \lambda_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{160}$$

$$\hat{z}\hat{z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{161}$$

$$\vec{\lambda}_\rho \hat{z} = (\hat{x}\lambda_x + \hat{y}\lambda_y)\hat{z} = \begin{bmatrix} 0 & 0 & \lambda_x \\ 0 & 0 & \lambda_y \\ 0 & 0 & 0 \end{bmatrix} \tag{162}$$

$$\left(\hat{z} \times \vec{\lambda}_\rho\right)\hat{z} = [\hat{z} \times (\hat{x}\lambda_x + \hat{y}\lambda_y)]\,\hat{z} = (\hat{y}\lambda_x - \hat{x}\lambda_y)\hat{z} = \begin{bmatrix} 0 & 0 & -\lambda_y \\ 0 & 0 & \lambda_x \\ 0 & 0 & 0 \end{bmatrix} \tag{163}$$

$$\left(\hat{z} \times \vec{\lambda}_\rho\right)\vec{\lambda}_\rho = (\hat{y}\lambda_x - \hat{x}\lambda_y)(\hat{x}\lambda_x + \hat{y}\lambda_y) = \begin{bmatrix} -\lambda_x\lambda_y & -\lambda_y^2 & 0 \\ \lambda_x^2 & \lambda_x\lambda_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{164}$$

$$\left(\hat{z} \times \vec{\lambda}_\rho\right)\left(\hat{z} \times \vec{\lambda}_\rho\right) = (\hat{y}\lambda_x - \hat{x}\lambda_y)(\hat{y}\lambda_x - \hat{x}\lambda_y) = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{165}$$

It is important to note from (29) and (34) that θ only contributes to transverse components of the electric field. Thus, components that are functions of $\lambda_{\rho\theta}$ do not contribute to E_z and are therefore TE^z components. Similarly, (30) and (33) imply that ψ only contributes to transverse components of the magnetic field. Thus, components that are functions of $\lambda_{\rho\psi}$ are TM^z components. As will be shown in

Chapter III, only the magnetic field Green function will be needed to develop the MFIEs used in the first proposed measurement technique. Thus, full development of the electric field Green functions is presented in Appendix D. Additionally, magnetic field Green functions that arise due to electric currents as well as from magnetic currents outside the observation region are developed in Appendix D.

Begin by analyzing the magnetic field component observed in region 1 resulting from magnetic currents in region 1, $\tilde{\vec{H}}_{1h1}$. Substituting (144), (145), (C.46), (151), and (C.56) into (159) implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h1h1} &= j\vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\Pi 1ht1} + \tilde{\vec{G}}_{\Pi 1hz1}\right) - j\hat{z} \times \vec{\lambda}_\rho \tilde{\vec{G}}_{\psi 1h1} + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}} \left(\tilde{\vec{G}}_{\theta 1ht1} + \tilde{\vec{G}}_{\theta 1hz1}\right) \\
&\qquad + j\frac{\hat{z}\hat{z}}{\omega\mu_{z1}}\delta\left(z - z'\right) \\
&= j\vec{\lambda}_\rho \left[\left(\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 Z_{\theta 1}}\right)\Upsilon_{11}^\theta + \left(j\frac{\hat{z}}{2\omega\mu_{z1}}\right)\Upsilon_9^\theta\right] - j\hat{z} \times \vec{\lambda}_\rho \left(-\frac{\hat{z} \times \vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{11}^\psi \\
&\qquad + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}}\left[\left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\Upsilon_3^\theta + \left(-\frac{\hat{z}Z_{\theta 1}}{2\omega\mu_{z1}}\right)\Upsilon_1^\theta\right] + j\frac{\hat{z}\hat{z}}{\omega\mu_{z1}}\delta\left(z - z'\right) \\
&= \left(\hat{z} \times \vec{\lambda}_\rho\right)\left(\hat{z} \times \vec{\lambda}_\rho\right)\left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{11}^\psi + \vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}}\right)\Upsilon_{11}^\theta \\
&\quad + \vec{\lambda}_\rho\hat{z}\left(-\frac{1}{2\omega\mu_{z1}}\right)\Upsilon_9^\theta + \hat{z}\vec{\lambda}_\rho\left(-\frac{1}{2\omega\mu_{z1}}\right)\Upsilon_3^\theta + \hat{z}\hat{z}\left(-j\frac{Z_{\theta 1}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z1}^2}\right)\Upsilon_1^\theta \\
&\qquad + \hat{z}\hat{z}\left(j\frac{\delta\left(z - z'\right)}{\omega\mu_{z1}}\right)
\end{aligned} \tag{166}$$

Breaking (166) into TM^z, TE^z, and depolarizing components ($\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h1}, \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1h1}$ and $\tilde{\overleftrightarrow{G}}^{d}_{h1h1}$ respectively where $\tilde{\overleftrightarrow{G}}_{h1h1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1h1} + \tilde{\overleftrightarrow{G}}^{d}_{h1h1}$) implies that

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h1} = \left(\hat{z} \times \vec{\lambda}_\rho\right)\left(\hat{z} \times \vec{\lambda}_\rho\right)\left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{11}^\psi$$

$$= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \Upsilon_{11}^{\psi} \tag{167}$$

$$\tilde{\overleftrightarrow{G}}_{h1h1}^{\mathrm{TE}^z} = \vec{\lambda}_\rho \vec{k}_\rho \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} + \vec{\lambda}_\rho \hat{z} \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_9^{\theta} + \hat{z}\vec{\lambda}_\rho \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_3^{\theta} + \hat{z}\hat{z} \left(-j\frac{Z_{\theta 1}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z1}^2} \right) \Upsilon_1^{\theta}$$

$$= \begin{bmatrix} \lambda_x^2 \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_x\lambda_y \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_x \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_9^{\theta} \\ \lambda_x\lambda_y \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_y^2 \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_y \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_9^{\theta} \\ \lambda_x \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_3^{\theta} & \lambda_y \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_3^{\theta} & \left(-j\frac{Z_{\theta 1}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z1}^2} \right) \Upsilon_1^{\theta} \end{bmatrix} \tag{168}$$

$$\tilde{\overleftrightarrow{G}}_{h1h1}^{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta(z-z')}{\omega\mu_{z1}} \right) \tag{169}$$

Next, analyze the magnetic field component observed in region 2 resulting from magnetic currents in region 2, $\tilde{\vec{H}}_{2h2}$. Substituting (C.18), (C.20), (C.50), (C.62), and (C.64) into (159) implies that

$$\tilde{\overleftrightarrow{G}}_{h2h2} = j\vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\Pi 2ht2} + \tilde{\vec{G}}_{\Pi 2hz2} \right) - j\hat{z} \times \vec{\lambda}_\rho \tilde{\vec{G}}_{\psi 2h2} + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}} \left(\tilde{\vec{G}}_{\theta 2ht2} + \tilde{\vec{G}}_{\theta 2hz2} \right) + j\frac{\hat{z}\hat{z}}{\omega\mu_{z2}} \delta(z-z')$$

$$= j\vec{\lambda}_\rho \left[\left(\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 Z_{\theta 2}} \right) \Upsilon_{16}^{\theta} + \left(j\frac{\hat{z}}{2\omega\mu_{z2}} \right) \Upsilon_{14}^{\theta} \right] - j\hat{z} \times \vec{\lambda}_\rho \left(-\frac{\hat{z} \times \vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 Z_{\psi 2}} \right) \Upsilon_{16}^{\psi} + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}} \left[\left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \right) \Upsilon_8^{\theta} + \left(-\frac{\hat{z} Z_{\theta 2}}{2\omega\mu_{z2}} \right) \Upsilon_6^{\theta} \right] + j\frac{\hat{z}\hat{z}}{\omega\mu_{z2}} \delta(z-z')$$

$$= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{16}^{\psi} + \vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right)\Upsilon_{16}^{\theta}$$
$$+ \vec{\lambda}_\rho\hat{z}\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{14}^{\theta} + \hat{z}\vec{\lambda}_\rho\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{8}^{\theta} + \hat{z}\hat{z}\left(-j\frac{Z_{\theta 2}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z2}^2}\right)\Upsilon_{6}^{\theta}$$
$$+ \hat{z}\hat{z}\left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z2}}\right) \tag{170}$$

Breaking (170) into TM^z, TE^z, and depolarizing components implies that

$$\tilde{\overleftrightarrow{G}}_{h2h2}^{\mathrm{TM}^z} = \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{16}^{\psi}$$
$$= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(j\frac{1}{2k_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{16}^{\psi} \tag{171}$$

$$\tilde{\overleftrightarrow{G}}_{h2h2}^{\mathrm{TE}^z} = \vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right)\Upsilon_{16}^{\theta} + \vec{\lambda}_\rho\hat{z}\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{14}^{\theta} + \hat{z}\vec{\lambda}_\rho\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{8}^{\theta}$$
$$+ \hat{z}\hat{z}\left(-j\frac{Z_{\theta 2}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z2}^2}\right)\Upsilon_{6}^{\theta}$$
$$= \begin{bmatrix} \lambda_x^2\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right)\Upsilon_{16}^{\theta} & \lambda_x\lambda_y\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right)\Upsilon_{16}^{\theta} & \lambda_x\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{14}^{\theta} \\ \lambda_x\lambda_y\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right)\Upsilon_{16}^{\theta} & \lambda_y^2\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right)\Upsilon_{16}^{\theta} & \lambda_y\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{14}^{\theta} \\ \lambda_x\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{8}^{\theta} & \lambda_y\left(-\frac{1}{2\omega\mu_{z2}}\right)\Upsilon_{8}^{\theta} & \left(-j\frac{Z_{\theta 2}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z2}^2}\right)\Upsilon_{6}^{\theta} \end{bmatrix} \tag{172}$$

$$\tilde{\overleftrightarrow{G}}_{h2h2}^{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z2}}\right) \tag{173}$$

Transverse Spectral Domain Total Electromagnetic Field Grand Summary.

Fields:

$$\boxed{\begin{aligned}
\tilde{\vec{E}}_1 &= \int_0^h \tilde{\overleftrightarrow{G}}_{e1e1}\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\overleftrightarrow{G}}_{e1h1}\cdot\tilde{\vec{J}}_h dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{e1e2}\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{e1h2}\cdot\tilde{\vec{J}}_h dz'\\
\tilde{\vec{E}}_2 &= \int_0^h \tilde{\overleftrightarrow{G}}_{e2e1}\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\overleftrightarrow{G}}_{e2h1}\cdot\tilde{\vec{J}}_h dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{e2e2}\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{e2h2}\cdot\tilde{\vec{J}}_h dz'\\
\tilde{\vec{H}}_1 &= \int_0^h \tilde{\overleftrightarrow{G}}_{h1e1}\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\overleftrightarrow{G}}_{h1h1}\cdot\tilde{\vec{J}}_h dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{h1e2}\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{h1h2}\cdot\tilde{\vec{J}}_h dz'\\
\tilde{\vec{H}}_2 &= \int_0^h \tilde{\overleftrightarrow{G}}_{h2e1}\cdot\tilde{\vec{J}}_e dz' + \int_0^h \tilde{\overleftrightarrow{G}}_{h2h1}\cdot\tilde{\vec{J}}_h dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{h2e2}\cdot\tilde{\vec{J}}_e dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{h2h2}\cdot\tilde{\vec{J}}_h dz'
\end{aligned}}$$

Electric (ee) Green Functions:

$$\boxed{\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1e1} &= \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1e1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1e1} + \tilde{\overleftrightarrow{G}}^{d}_{e1e1}\\
\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1e1} &= \begin{bmatrix} k_x^2\left(-j\frac{Z_{\psi1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_x\lambda_y\left(-j\frac{Z_{\psi1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_x\left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_3^\psi\\ \lambda_x\lambda_y\left(-j\frac{Z_{\psi1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_y^2\left(-j\frac{Z_{\psi1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_y\left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_3^\psi\\ \lambda_x\left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_9^\psi & \lambda_y\left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_9^\psi & \left(j\frac{\lambda_{\rho\psi}^2}{2Z_{\psi1}\omega^2\epsilon_{z1}^2}\right)\Upsilon_{11}^\psi \end{bmatrix}\\
\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1e1} &= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0\\ -\lambda_x\lambda_y & \lambda_x^2 & 0\\ 0 & 0 & 0 \end{bmatrix}\left(-j\frac{Z_{\theta1}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_1^\theta\\
\tilde{\overleftrightarrow{G}}^{d}_{e1e1} &= \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix}\left(j\frac{\delta(z-z')}{\omega\epsilon_{z1}}\right)
\end{aligned}}$$

$$\tilde{\overleftrightarrow{G}}_{e1e2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1e2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1e2} + \tilde{\overleftrightarrow{G}}^{d}_{e1e2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1e2} = Z_{\psi 2} \begin{bmatrix} \lambda_x^2 \left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^{\psi} & \lambda_x\lambda_y \left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^{\psi} & \lambda_x \left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_4^{\psi} \\ \lambda_x\lambda_y \left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^{\psi} & \lambda_y^2 \left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^{\psi} & \lambda_y \left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_4^{\psi} \\ \lambda_x \left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_{10}^{\psi} & \lambda_y \left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_{10}^{\psi} & \left(j\frac{\lambda_{\rho\psi}^2}{\omega^2\epsilon_{z1}^2 Z_{\psi 1}}\right)\Upsilon_{12}^{\psi} \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1e2} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(-j\frac{Z_{\theta 1}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_2^{\theta}$$

$$\tilde{\overleftrightarrow{G}}^{d}_{e1e2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta(z-z')}{\omega\epsilon_{z1}}\right)$$

$$\tilde{\overleftrightarrow{G}}_{e2e1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2e1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2e1} + \tilde{\overleftrightarrow{G}}^{d}_{e2e1}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2e1} = Z_{\psi 1} \begin{bmatrix} \lambda_x^2 \left(-j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_5^{\psi} & \lambda_x\lambda_y \left(-j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_5^{\psi} & \lambda_x \left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon_7^{\psi} \\ \lambda_x\lambda_y \left(-j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_5^{\psi} & \lambda_y^2 \left(-j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_5^{\psi} & \lambda_y \left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon_7^{\psi} \\ \lambda_x \left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon_{13}^{\psi} & \lambda_y \left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon_{13}^{\psi} & \left(j\frac{\lambda_{\rho\psi}^2}{\omega^2\epsilon_{z2}^2 Z_{\psi 2}}\right)\Upsilon_{15}^{\psi} \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2e1} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(-j\frac{Z_{\theta 2}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_5^{\theta}$$

$$\tilde{\overleftrightarrow{G}}^{d}_{e2e1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta(z-z')}{\omega\epsilon_{z2}}\right)$$

$$\tilde{\overleftrightarrow{G}}_{e2e2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2e2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2e2} + \tilde{\overleftrightarrow{G}}^{d}_{e2e2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2e2} = \begin{bmatrix} \lambda_x^2\left(-j\frac{Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi & \lambda_x\lambda_y\left(-j\frac{Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi & \lambda_x\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon_8^\psi \\ \lambda_x\lambda_y\left(-j\frac{Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi & \lambda_y^2\left(-j\frac{Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi & \lambda_y\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon_8^\psi \\ \lambda_x\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon_{14}^\psi & \lambda_y\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon_{14}^\psi & \left(j\frac{\lambda_{\rho\psi}^2}{2Z_{\psi 2}\omega^2\epsilon_{z2}^2}\right)\Upsilon_{16}^\psi \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2e2} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(-j\frac{Z_{\theta 2}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_6^\theta$$

$$\tilde{\overleftrightarrow{G}}^{d}_{e2e2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z2}}\right)$$

Magnetoelectric (eh) Green Functions:

$$\tilde{\overleftrightarrow{G}}_{e1h1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1h1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1h1}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1h1} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_3^\psi & \lambda_x^2\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_3^\psi & 0 \\ -\lambda_y^2\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_3^\psi & \lambda_x\lambda_y\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_3^\psi & 0 \\ -\lambda_y\left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right)\Upsilon_{11}^\psi & \lambda_x\left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right)\Upsilon_{11}^\psi & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1h1} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_3^\theta & -\lambda_y^2\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_3^\theta & -\lambda_y\left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right)\Upsilon_1^\theta \\ \lambda_x^2\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_3^\theta & \lambda_x\lambda_y\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_3^\theta & \lambda_x\left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right)\Upsilon_1^\theta \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}_{e1h2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1h2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1h2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1h2} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{4} & \lambda_x^2\left(\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{4} & 0 \\ -\lambda_y^2\left(\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{4} & \lambda_x\lambda_y\left(\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{4} & 0 \\ -\lambda_y\left(j\frac{Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon^{\psi}_{12} & \lambda_x\left(j\frac{Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon^{\psi}_{12} & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e1h2} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{Z_{\theta 1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{4} & -\lambda_y^2\left(-\frac{Z_{\theta 1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{4} & -\lambda_y\left(j\frac{Z^2_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon^{\theta}_{2} \\ \lambda_x^2\left(-\frac{Z_{\theta 1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{4} & \lambda_x\lambda_y\left(-\frac{Z_{\theta 1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{4} & \lambda_x\left(j\frac{Z^2_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon^{\theta}_{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}_{e2h1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2h1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2h1}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2h1} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{Z_{\psi 1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{7} & \lambda_x^2\left(-\frac{Z_{\psi 1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{7} & 0 \\ -\lambda_y^2\left(-\frac{Z_{\psi 1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{7} & \lambda_x\lambda_y\left(-\frac{Z_{\psi 1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{7} & 0 \\ -\lambda_y\left(j\frac{Z_{\psi 1}}{\omega\epsilon_{z2}Z_{\psi 2}}\right)\Upsilon^{\psi}_{15} & \lambda_x\left(j\frac{Z_{\psi 1}}{\omega\epsilon_{z2}Z_{\psi 2}}\right)\Upsilon^{\psi}_{15} & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2h1} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{Z_{\theta 2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{7} & -\lambda_y^2\left(\frac{Z_{\theta 2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{7} & -\lambda_y\left(j\frac{Z^2_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon^{\theta}_{5} \\ \lambda_x^2\left(\frac{Z_{\theta 2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{7} & \lambda_x\lambda_y\left(\frac{Z_{\theta 2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{7} & \lambda_x\left(j\frac{Z^2_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon^{\theta}_{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}_{e2h2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2h2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2h2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2h2} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{8} & \lambda_x^2\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{8} & 0 \\ -\lambda_y^2\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{8} & \lambda_x\lambda_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{8} & 0 \\ -\lambda_y\left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon^{\psi}_{16} & \lambda_x\left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon^{\psi}_{16} & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2h2} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{8} & -\lambda_y^2\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{8} & -\lambda_y\left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon^{\theta}_{6} \\ \lambda_x^2\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{8} & \lambda_x\lambda_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{8} & \lambda_x\left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon^{\theta}_{6} \\ 0 & 0 & 0 \end{bmatrix}$$

Magnetoelectric (he) Green Functions:

$$\tilde{\overleftrightarrow{G}}_{h1e1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1e1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1e1}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1e1} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_9 & -\lambda^2_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_9 & -\lambda_y\left(j\frac{1}{2Z_{\psi1}\omega\epsilon_{z1}}\right)\Upsilon^\psi_{11} \\ \lambda^2_x\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_9 & \lambda_x\lambda_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_9 & \lambda_x\left(j\frac{1}{2Z_{\psi1}\omega\epsilon_{z1}}\right)\Upsilon^\psi_{11} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1e1} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_9 & \lambda^2_x\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_9 & 0 \\ -\lambda^2_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_9 & \lambda_x\lambda_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_9 & 0 \\ -\lambda_y\left(j\frac{Z_{\theta1}}{2\omega\mu_{z1}}\right)\Upsilon^\theta_1 & \lambda_x\left(j\frac{Z_{\theta1}}{2\omega\mu_{z1}}\right)\Upsilon^\theta_1 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}_{h1e2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1e2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1e2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1e2} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{Z_{\psi2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_{10} & -\lambda^2_y\left(\frac{Z_{\psi2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_{10} & -\lambda_y\left(j\frac{Z_{\psi2}}{\omega\epsilon_{z1}Z_{\psi1}}\right)\Upsilon^\psi_{12} \\ \lambda^2_x\left(\frac{Z_{\psi2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_{10} & \lambda_x\lambda_y\left(\frac{Z_{\psi2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_{10} & \lambda_x\left(j\frac{Z_{\psi2}}{\omega\epsilon_{z1}Z_{\psi1}}\right)\Upsilon^\psi_{12} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1e2} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{Z_{\theta1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_{10} & \lambda^2_x\left(-\frac{Z_{\theta1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_{10} & 0 \\ -\lambda^2_y\left(-\frac{Z_{\theta1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_{10} & \lambda_x\lambda_y\left(-\frac{Z_{\theta1}}{\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_{10} & 0 \\ -\lambda_y\left(j\frac{Z^2_{\theta1}}{\omega\mu_{z1}}\right)\Upsilon^\theta_2 & \lambda_x\left(j\frac{Z^2_{\theta1}}{\omega\mu_{z1}}\right)\Upsilon^\theta_2 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}_{h2e1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h2e1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h2e1}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h2e1} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{Z_{\psi1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{13} & -\lambda^2_y\left(-\frac{Z_{\psi1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{13} & -\lambda_y\left(j\frac{Z_{\psi1}}{\omega\epsilon_{z2}Z_{\psi2}}\right)\Upsilon^{\psi}_{15} \\ \lambda^2_x\left(-\frac{Z_{\psi1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{13} & \lambda^2_x\left(-\frac{Z_{\psi1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{13} & \lambda_x\left(j\frac{Z_{\psi1}}{\omega\epsilon_{z2}Z_{\psi2}}\right)\Upsilon^{\psi}_{15} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h2e1} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{Z_{\theta2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{13} & \lambda^2_x\left(\frac{Z_{\theta2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{13} & 0 \\ -\lambda^2_y\left(\frac{Z_{\theta2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{13} & \lambda_x\lambda_y\left(\frac{Z_{\theta2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{13} & 0 \\ -\lambda_y\left(j\frac{Z^2_{\theta2}}{\omega\mu_{z2}}\right)\Upsilon^{\theta}_{5} & \lambda_x\left(j\frac{Z^2_{\theta2}}{\omega\mu_{z2}}\right)\Upsilon^{\theta}_{5} & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}_{h2e2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h2e2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h2e2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h2e2} = \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{14} & -\lambda^2_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{14} & -\lambda_y\left(j\frac{1}{2Z_{\psi2}\omega\epsilon_{z2}}\right)\Upsilon^{\psi}_{16} \\ \lambda^2_x\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{14} & \lambda_x\lambda_y\left(-\frac{1}{2\lambda^2_{\rho\psi}}\right)\Upsilon^{\psi}_{14} & \lambda_x\left(j\frac{1}{2Z_{\psi2}\omega\epsilon_{z2}}\right)\Upsilon^{\psi}_{16} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h2e2} = \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{14} & \lambda^2_x\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{14} & 0 \\ -\lambda^2_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{14} & \lambda_x\lambda_y\left(\frac{1}{2\lambda^2_{\rho\theta}}\right)\Upsilon^{\theta}_{14} & 0 \\ -\lambda_y\left(j\frac{Z_{\theta2}}{2\omega\mu_{z2}}\right)\Upsilon^{\theta}_{6} & \lambda_x\left(j\frac{Z_{\theta2}}{2\omega\mu_{z2}}\right)\Upsilon^{\theta}_{6} & 0 \end{bmatrix}$$

Magnetic (hh) Green Functions:

$$\tilde{\overleftrightarrow{G}}_{h1h1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h1} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1h1} + \tilde{\overleftrightarrow{G}}^{d}_{h1h1}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h1} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \Upsilon_{11}^{\psi}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1h1} = \begin{bmatrix} \lambda_x^2 \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_x\lambda_y \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_x \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_{9}^{\theta} \\ \lambda_x\lambda_y \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_y^2 \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}} \right) \Upsilon_{11}^{\theta} & \lambda_y \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_{9}^{\theta} \\ \lambda_x \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_{3}^{\theta} & \lambda_y \left(-\frac{1}{2\omega\mu_{z1}} \right) \Upsilon_{3}^{\theta} & \left(-j\frac{Z_{\theta 1}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z1}^2} \right) \Upsilon_{1}^{\theta} \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{d}_{h1h1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta\left(z - z'\right)}{\omega\mu_{z1}} \right)$$

$$\tilde{\overleftrightarrow{G}}_{h1h2} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h2} + \tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1h2} + \tilde{\overleftrightarrow{G}}^{d}_{h1h2}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h2} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \Upsilon_{12}^{\psi}$$

$$\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{h1h2} = \begin{bmatrix} \lambda_x^2 \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{12}^{\theta} & \lambda_x\lambda_y \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{12}^{\theta} & \lambda_x \left(\frac{Z_{\theta 1}}{\omega\mu_{z1}} \right) \Upsilon_{10}^{\theta} \\ \lambda_x\lambda_y \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{12}^{\theta} & \lambda_y^2 \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{12}^{\theta} & \lambda_y \left(\frac{Z_{\theta 1}}{\omega\mu_{z1}} \right) \Upsilon_{10}^{\theta} \\ \lambda_x \left(\frac{Z_{\theta 1}}{\omega\mu_{z1}} \right) \Upsilon_{4}^{\theta} & \lambda_y \left(\frac{Z_{\theta 1}}{\omega\mu_{z1}} \right) \Upsilon_{4}^{\theta} & \left(-j\frac{Z_{\theta 1}^2 k_{\rho\theta}^2}{\omega^2\mu_{z1}^2} \right) \Upsilon_{2}^{\theta} \end{bmatrix}$$

$$\tilde{\overleftrightarrow{G}}^{d}_{h1h2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta\left(z - z'\right)}{\omega\mu_{z1}} \right)$$

$$\overleftrightarrow{\tilde{G}}_{h2h1} = \overleftrightarrow{\tilde{G}}^{\mathrm{TM}^z}_{h2h1} + \overleftrightarrow{\tilde{G}}^{\mathrm{TE}^z}_{h2h1} + \overleftrightarrow{\tilde{G}}^{d}_{h2h1}$$

$$\overleftrightarrow{\tilde{G}}^{\mathrm{TM}^z}_{h2h1} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(j\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2 Z_{\psi2}} \right) \Upsilon_{15}^{\psi}$$

$$\overleftrightarrow{\tilde{G}}^{\mathrm{TE}^z}_{h2h1} = \begin{bmatrix} \lambda_x^2 \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{15}^{\theta} & \lambda_x\lambda_y \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{15}^{\theta} & \lambda_x \left(-\frac{Z_{\theta2}}{\omega\mu_{z2}} \right) \Upsilon_{13}^{\theta} \\ \lambda_x\lambda_y \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{15}^{\theta} & \lambda_y^2 \left(j\frac{1}{\lambda_{\rho\theta}^2} \right) \Upsilon_{15}^{\theta} & \lambda_y \left(-\frac{Z_{\theta2}}{\omega\mu_{z2}} \right) \Upsilon_{13}^{\theta} \\ \lambda_x \left(-\frac{Z_{\theta2}}{\omega\mu_{z2}} \right) \Upsilon_{7}^{\theta} & \lambda_y \left(-\frac{Z_{\theta2}}{\omega\mu_{z2}} \right) \Upsilon_{7}^{\theta} & \left(-j\frac{Z_{\theta2}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z2}^2} \right) \Upsilon_{5}^{\theta} \end{bmatrix}$$

$$\overleftrightarrow{\tilde{G}}^{d}_{h2h1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z2}} \right)$$

$$\overleftrightarrow{\tilde{G}}_{h2h2} = \overleftrightarrow{\tilde{G}}^{\mathrm{TM}^z}_{h2h2} + \overleftrightarrow{\tilde{G}}^{\mathrm{TE}^z}_{h2h2} + \overleftrightarrow{\tilde{G}}^{d}_{h2h2}$$

$$\overleftrightarrow{\tilde{G}}^{\mathrm{TM}^z}_{h2h2} = \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(j\frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi2}} \right) \Upsilon_{16}^{\psi}$$

$$\overleftrightarrow{\tilde{G}}^{\mathrm{TE}^z}_{h2h2} = \begin{bmatrix} \lambda_x^2 \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta2}} \right) \Upsilon_{16}^{\theta} & \lambda_x\lambda_y \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta2}} \right) \Upsilon_{16}^{\theta} & \lambda_x \left(-\frac{1}{2\omega\mu_{z2}} \right) \Upsilon_{14}^{\theta} \\ \lambda_x\lambda_y \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta2}} \right) \Upsilon_{16}^{\theta} & \lambda_y^2 \left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta2}} \right) \Upsilon_{16}^{\theta} & \lambda_y \left(-\frac{1}{2\omega\mu_{z2}} \right) \Upsilon_{14}^{\theta} \\ \lambda_x \left(-\frac{1}{2\omega\mu_{z2}} \right) \Upsilon_{8}^{\theta} & \lambda_y \left(-\frac{1}{2\omega\mu_{z2}} \right) \Upsilon_{8}^{\theta} & \left(-j\frac{Z_{\theta2}\lambda_{\rho\theta}^2}{2\omega^2\mu_{z2}^2} \right) \Upsilon_{6}^{\theta} \end{bmatrix}$$

$$\overleftrightarrow{\tilde{G}}^{d}_{h2h2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z2}} \right)$$

Supplemental Relations:

$$\Upsilon_1^\alpha = \left[\frac{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\left[\cos\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)-\cos\left(\lambda_{z\alpha 1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\left[\sin\left(\lambda_{z\alpha 1}\left(h-\left|z-z'\right|\right)\right)-\sin\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_2^\alpha = \left[\frac{\sin\left(\lambda_{z\alpha 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\alpha 1}z\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_3^\alpha = \left[\frac{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\alpha 1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$+\frac{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\left[\sin\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$+\frac{Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\alpha 1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$\left.+\frac{Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\left[\cos\left(\lambda_{z\alpha 1}\left(h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_4^\alpha = \left[\frac{\cos\left(\lambda_{z\alpha 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\alpha 1}z\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_5^\alpha = \left[\frac{\sin\left(\lambda_{z\alpha 1}z'\right)\sin\left(\lambda_{z\alpha 2}\left(d-z\right)\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_6^\alpha = \left[\frac{Z_{\alpha 1}\sin\left(\lambda_{z\alpha 1}h\right)\left[\sin\left(\lambda_{z\alpha 2}\left(T-\left|z-z'\right|\right)\right)+\sin\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\cos\left(\lambda_{z\alpha 1}h\right)\left[\cos\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)-\cos\left(\lambda_{z\alpha 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_7^\alpha = \left[\frac{\cos\left(\lambda_{z\alpha 1}z'\right)\sin\left(\lambda_{z\alpha 2}\left(d-z\right)\right)}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$$\Upsilon_8^\alpha = \left[\frac{Z_{\alpha 1}\sin\left(\lambda_{z\alpha 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\alpha 2}\left(d-h-\left|z-z'\right|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right.$$
$$+\frac{Z_{\alpha 1}\sin\left(\lambda_{z\alpha 1}h\right)\left[-\cos\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$+\frac{Z_{\alpha 2}\cos\left(\lambda_{z\alpha 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\alpha 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}$$
$$\left.+\frac{Z_{\alpha 2}\cos\left(\lambda_{z\alpha 1}h\right)\left[\sin\left(\lambda_{z\alpha 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2}T\right)\sin\left(\lambda_{z\alpha 1}h\right)+Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2}T\right)\cos\left(\lambda_{z\alpha 1}h\right)}\right]$$

$\alpha \in \{\theta, \psi\}$ $\qquad T = d - h$

$Z_{\theta\{1,2\}} = \frac{\omega\mu_{t\{1,2\}}}{\lambda_{z\theta\{1,2\}}}$ $\qquad Z_{\psi\{1,2\}} = \frac{\lambda_{z\psi\{1,2\}}}{\omega\epsilon_{t\{1,2\}}}$

$\lambda_{z\theta\{1,2\}}^2 = \omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\lambda_{\rho\theta}^2$ $\qquad \lambda_{z\psi\{1,2\}}^2 = \omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}}\lambda_{\rho\psi}^2$

$$\Upsilon_9^\alpha = \left[\frac{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\left[\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 1}(h-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$+\frac{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\left[-\sin(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$+\frac{Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\left[\operatorname{sgn}(z-z')\cos(\lambda_{z\alpha 1}(h-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$\left.+\frac{Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\left[-\cos(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{10}^\alpha = \left[\frac{\sin(\lambda_{z\alpha 2}(d-z'))\cos(\lambda_{z\alpha 1}z)}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{11}^\alpha = \left[\frac{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\left[\cos(\lambda_{z\alpha 1}(h-|z-z'|))+\cos(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\left[-\sin(\lambda_{z\alpha 1}(h-|z-z'|))-\sin(\lambda_{z\alpha 1}(h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{12}^\alpha = \left[\frac{\cos(\lambda_{z\alpha 2}(d-z'))\cos(\lambda_{z\alpha 1}z)}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{13}^\alpha = \left[\frac{\sin(\lambda_{z\alpha 1}z')\cos(\lambda_{z\alpha 2}(d-z))}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{14}^\alpha = \left[\frac{Z_{\alpha 1}\sin(\lambda_{z\alpha 1}h)\left[\operatorname{sgn}(z-z')\cos(\lambda_{z\alpha 2}(T-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$+\frac{Z_{\alpha 1}\sin(\lambda_{z\alpha 1}h)\left[\cos(\lambda_{z\alpha 2}(d+h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$+\frac{Z_{\alpha 2}\cos(\lambda_{z\alpha 1}h)\left[\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 2}(T-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}$$
$$\left.+\frac{Z_{\alpha 2}\cos(\lambda_{z\alpha 1}h)\left[-\sin(\lambda_{z\alpha 2}(d+h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{15}^\alpha = \left[\frac{\cos(\lambda_{z\alpha 1}z')\cos(\lambda_{z\alpha 2}(d-z))}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$
$$\Upsilon_{16}^\alpha = \left[\frac{Z_{\alpha 1}\sin(\lambda_{z\alpha 1}h)\left[\sin(\lambda_{z\alpha 2}(d+h-z-z'))-\sin(\lambda_{z\alpha 2}(T-|z-z'|))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right.$$
$$\left.+\frac{Z_{\alpha 2}\cos(\lambda_{z\alpha 1}h)\left[\cos(\lambda_{z\alpha 2}(T-|z-z'|))+\cos(\lambda_{z\alpha 2}(d+h-z-z'))\right]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}T)\sin(\lambda_{z\alpha 1}h)+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}T)\cos(\lambda_{z\alpha 1}h)}\right]$$

$\alpha \in \{\theta, \psi\}$ $\quad$ $T = d - h$

$Z_{\theta\{1,2\}} = \frac{\omega \mu_{t(1,2)}}{\lambda_{z\theta\{1,2\}}}$ $\quad$ $Z_{\psi\{1,2\}} = \frac{\lambda_{z\psi\{1,2\}}}{\omega \epsilon_{t\{1,2\}}}$

$\lambda_{z\theta\{1,2\}}^2 = \omega^2 \epsilon_{t\{1,2\}} \mu_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}} \lambda_{\rho\theta}^2$ $\quad$ $\lambda_{z\psi\{1,2\}}^2 = \omega^2 \epsilon_{t\{1,2\}} \mu_{t\{1,2\}} - \frac{\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}} \lambda_{\rho\psi}^2$

2.7 Sanity Check of Model

As a basic cross-check of this model, it is important to verify that when $h \to 0$, the model agrees with the model developed by Rogers in [71]. First note, that the only intrinsic difference between this model and the one developed in [71] are the Υ terms. Next, noting that as $h \to 0$, region 1 ceases to exist. Therefore, there is no value in cross-checking equations that have sources and/or observations in region 1. Thus, only $\Upsilon^{\alpha}_{\{6,8,14,16\}}$ need to be checked.

$$
\begin{aligned}
\lim_{h\to 0} \Upsilon^{\alpha}_{6} = & \left[\frac{Z_{\alpha 1} \cancelto{0}{\sin(0)} \left[\sin\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right) + \sin\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2} d\right) \cancelto{0}{\sin(0)} + Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2} d\right) \cancelto{1}{\cos(0)}} \right. \\
& \left. + \frac{Z_{\alpha 2} \cancelto{1}{\cos(0)} \left[\cos\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right) - \cos\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2} d\right) \cancelto{0}{\sin(0)} + Z_{\alpha 2}\sin\left(k_{z\alpha 2} d\right) \cancelto{1}{\cos(0)}} \right] \\
= & \frac{\cancel{Z_{\alpha 2}} \left[\cos\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right) - \cos\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right)\right]}{\cancel{Z_{\alpha 2}} \sin\left(\lambda_{z\alpha 2} d\right)} \\
= & \frac{\cos\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right) - \cos\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right)}{\sin\left(\lambda_{z\alpha 2} d\right)}
\end{aligned} \tag{174}
$$

$$
\begin{aligned}
\lim_{h\to 0} \Upsilon^{\alpha}_{8} = & \left[\frac{Z_{\alpha 1} \cancelto{0}{\sin(0)} \left[\mathrm{sgn}\left(z - z'\right)\cos\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2} d\right) \cancelto{0}{\sin(0)} + Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2} d\right) \cancelto{1}{\cos(0)}} \right. \\
& + \frac{Z_{\alpha 1} \cancelto{0}{\sin(0)} \left[-\cos\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2} d\right) \cancelto{0}{\sin(0)} + Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2} d\right) \cancelto{1}{\cos(0)}} \\
& + \frac{Z_{\alpha 2} \cancelto{1}{\cos(0)} \left[\mathrm{sgn}\left(z - z'\right)\sin\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2} d\right) \cancelto{0}{\sin(0)} + Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2} d\right) \cancelto{1}{\cos(0)}} \\
& \left. + \frac{Z_{\alpha 2} \cancelto{1}{\cos(0)} \left[\sin\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right)\right]}{Z_{\alpha 1}\cos\left(\lambda_{z\alpha 2} d\right) \cancelto{0}{\sin(0)} + Z_{\alpha 2}\sin\left(\lambda_{z\alpha 2} d\right) \cancelto{1}{\cos(0)}} \right] \\
= & \frac{\cancel{Z_{\alpha 2}} \left[\mathrm{sgn}\left(z - z'\right)\sin\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right) + \sin\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right)\right]}{\cancel{Z_{\alpha 2}} \sin\left(\lambda_{z\alpha 2} d\right)} \\
= & \frac{\mathrm{sgn}\left(z - z'\right)\sin\left(\lambda_{z\alpha 2}\left(d - |z - z'|\right)\right) + \sin\left(\lambda_{z\alpha 2}\left(d - z - z'\right)\right)}{\sin\left(\lambda_{z\alpha 2} d\right)}
\end{aligned} \tag{175}
$$

$$
\begin{aligned}
\lim_{h\to 0}\Upsilon_{14}^{\alpha} = &\left[\frac{Z_{\alpha 1}\cancelto{0}{\sin(\theta)}[\operatorname{sgn}(z-z')\cos(\lambda_{z\alpha 2}(d-|z-z'|))]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}d)\cancelto{0}{\sin(\theta)}+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}d)\cancelto{1}{\cos(\theta)}}\right.\\
&+\frac{Z_{\alpha 1}\cancelto{0}{\sin(\theta)}[\cos(\lambda_{z\alpha 2}(d-z-z'))]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}d)\cancelto{0}{\sin(\theta)}+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}d)\cancelto{1}{\cos(\theta)}}\\
&+\frac{Z_{\alpha 2}\cancelto{1}{\cos(\theta)}[\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 2}(d-|z-z'|))]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}d)\cancelto{0}{\sin(\theta)}+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}d)\cancelto{1}{\cos(\theta)}}\\
&\left.+\frac{Z_{\alpha 2}\cancelto{1}{\cos(\theta)}[-\sin(\lambda_{z\alpha 2}(d-z-z'))]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}d)\cancelto{0}{\sin(\theta)}+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}d)\cancelto{1}{\cos(\theta)}}\right]\\
=&\frac{\cancel{Z_{\alpha 2}}[\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 2}(d-|z-z'|))-\sin(\lambda_{z\alpha 2}(d-z-z'))]}{\cancel{Z_{\alpha 2}}\sin(\lambda_{z\alpha 2}d)}\\
=&\frac{\operatorname{sgn}(z-z')\sin(\lambda_{z\alpha 2}(d-|z-z'|))-\sin(\lambda_{z\alpha 2}(d-z-z'))}{\sin(\lambda_{z\alpha 2}d)}
\end{aligned}
\tag{176}
$$

$$
\begin{aligned}
\lim_{h\to 0}\Upsilon_{16}^{\alpha} = &\left[\frac{Z_{\alpha 1}\cancelto{0}{\sin(\theta)}[\sin(\lambda_{z\alpha 2}(d-z-z'))-\sin(\lambda_{z\alpha 2}(d-|z-z'|))]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}d)\cancelto{0}{\sin(\theta)}+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}d)\cancelto{1}{\cos(\theta)}}\right.\\
&\left.+\frac{Z_{\alpha 2}\cancelto{1}{\cos(\theta)}[\cos(\lambda_{z\alpha 2}(d-|z-z'|))+\cos(\lambda_{z\alpha 2}(d-z-z'))]}{Z_{\alpha 1}\cos(\lambda_{z\alpha 2}d)\cancelto{0}{\sin(\theta)}+Z_{\alpha 2}\sin(\lambda_{z\alpha 2}d)\cancelto{1}{\cos(\theta)}}\right]\\
=&\frac{\cancel{Z_{\alpha 2}}[\cos(\lambda_{z\alpha 2}(d-|z-z'|))+\cos(\lambda_{z\alpha 2}(d-z-z'))]}{\cancel{Z_{\alpha 2}}\sin(\lambda_{z\alpha 2}d)}\\
=&\frac{\cos(\lambda_{z\alpha 2}(d-|z-z'|))+\cos(\lambda_{z\alpha 2}(d-z-z'))}{\sin(\lambda_{z\alpha 2}d)}
\end{aligned}
\tag{177}
$$

The results of these limit calculations correspond exactly with the functions derived by Rogers in [71].

2.8 Physical Interpretation of Results

It is useful analyze how the resulting components manifest themselves physically. From (37), we see that a transverse lamellar magnetic current ($\vec{J}_{ht_l} = \nabla_t u_h$) supports a transverse lamellar magnetic field ($\vec{H}_{t_l} = \nabla_t \Pi$), a transverse rotational electric field

($\vec{E}_{t_r} = \nabla_t \times \hat{z}\theta$) and, consequently through (33), a longitudinal magnetic field ($\hat{z}H_z$). From (36), we see that a transverse rotational magnetic current ($\vec{J}_{ht_r} = \nabla_t \times \hat{z}v_h$) supports a transverse rotational magnetic field ($\vec{H}_{t_r} = \nabla_t \times \hat{z}\psi$), a transverse lamellar electric field ($\vec{E}_{t_l} = \nabla_t\Phi$), and a longitudinal electric field ($\hat{z}E_z$). Continuing the analysis in a similar fashion, we see that the fields supported by the various current density types can be summarized by

$$\exists\left(\vec{J}_{et_l} \text{ or } \vec{J}_{ht_r} \text{ or } \hat{z}J_{ez}\right) \Rightarrow \exists\left(\vec{E}_{t_l} \text{ and } \hat{z}E_z \text{ and } \vec{H}_{t_r}\right) \Rightarrow \mathrm{TM}^z \tag{178}$$

$$\exists\left(\vec{J}_{et_r} \text{ or } \vec{J}_{ht_l} \text{ or } \hat{z}J_{hz}\right) \Rightarrow \exists\left(\vec{E}_{t_r} \text{ and } \hat{z}H_z \text{ and } \vec{H}_{t_l}\right) \Rightarrow \mathrm{TE}^z. \tag{179}$$

These results make general physical sense. First, each type of current density supports an equivalent, opposite-directed field consistent across all supporting current densities. Due to Love's equivalence principle, we can replace an electric field at a waveguide aperture with an equivalent magnetic current on a PEC surface. Therefore, let us focus on the transverse magnetic currents as they are directly applicable in this research.

A transverse rotational magnetic current supports a transverse rotational magnetic field in the opposite direction, as depicted in Fig. 3. Second, each directly-supported field generates a complementary field of the opposing type. For example, a transverse rotational magnetic field (supported by a transverse rotational magnetic current) generates both transverse lamellar and longitudinal electric field components as depicted in Fig. 3. Since there is no longitudinal magnetic field component, a TM^z field structure results.

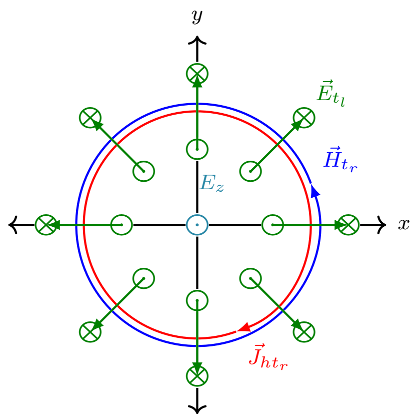

Figure 3. Fields supported by a transverse rotational magnetic current, $\vec{J}_{ht_r}$ as viewed from above with $z > 0$.

However, when the supporting current is transverse lamellar in nature, no complementary longitudinal field is generated. For example, no longitudinal electric field is generated by a lamellar magnetic current, as depicted in Fig. 4. This is because any positive-z longitudinal component resulting from the electric field rotating around a particular radial magnetic current is immediately canceled by a negative-z longitudinal component in the same position from the electric field rotating around an adjacent radial of the magnetic current. The resulting transverse rotational electric field does, however, support a longitudinal magnetic field, thus resulting in a TE^z field structure.

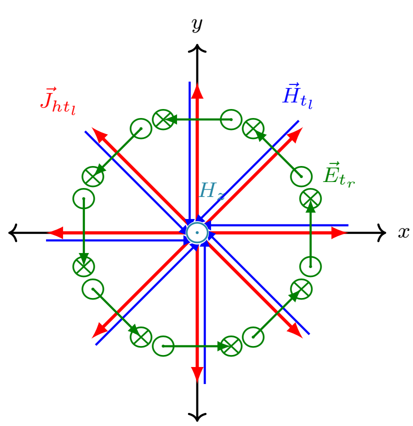

Figure 4. Fields supported by a transverse lamellar magnetic current, $\vec{J}_{ht_l}$ as viewed from above with $z > 0$.

These results correlate well with the Green functions developed earlier. For example, analyze the TE^z component of the electric and magnetic fields generated by magnetic current densities. We begin by looking at the scalar potential origins of each of the electric and magnetic field components observed in region 1 supported by magnetic current densities in region 1.

$$\tilde{\overleftrightarrow{G}}_{e1h1}^{\mathrm{TM}^z} \propto \begin{bmatrix} \tilde{\Phi}_{1h1} & \tilde{\Phi}_{1h1} & 0 \\ \tilde{\Phi}_{1h1} & \tilde{\Phi}_{1h1} & 0 \\ \tilde{\psi}_{1h1} & \tilde{\psi}_{1h1} & 0 \end{bmatrix} \tag{180}$$

$$\tilde{\overleftrightarrow{G}}_{e1h1}^{\mathrm{TE}^z} \propto \begin{bmatrix} \tilde{\theta}_{1h1} & \tilde{\theta}_{1h1} & \tilde{\theta}_{1h1} \\ \tilde{\theta}_{1h1} & \tilde{\theta}_{1h1} & \tilde{\theta}_{1h1} \\ 0 & 0 & 0 \end{bmatrix} \tag{181}$$

$$\tilde{\overleftrightarrow{G}}_{e1h1}^{d} = \overleftrightarrow{0} \tag{182}$$

$$\tilde{\overleftrightarrow{G}}_{h1h1}^{\mathrm{TM}^z} \propto \begin{bmatrix} \tilde{\psi}_{1h1} & \tilde{\psi}_{1h1} & 0 \\ \tilde{\psi}_{1h1} & \tilde{\psi}_{1h1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{183}$$

$$\tilde{\overleftrightarrow{G}}_{h1h1}^{\mathrm{TE}^z} \propto \begin{bmatrix} \tilde{\Pi}_{1h1} & \tilde{\Pi}_{1h1} & \tilde{\Pi}_{1h1} \\ \tilde{\Pi}_{1h1} & \tilde{\Pi}_{1h1} & \tilde{\Pi}_{1h1} \\ \tilde{\theta}_{1h1} & \tilde{\theta}_{1h1} & \tilde{\theta}_{1h1} \end{bmatrix} \tag{184}$$

$$\tilde{\overleftrightarrow{G}}_{h1h1}^{d} \propto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{185}$$

It can be shown that, through $\nabla_t \cdot \nabla_t \times \{\cdot\} = \nabla_t \times \nabla_t \cdot \{\cdot\} = 0$ cancellations from the underlying scalar potential Green functions,

$$\tilde{\Phi}_{1h1} \cdot \tilde{\vec{J}}_{ht_l} = 0, \tilde{\psi}_{1h1} \cdot \tilde{\vec{J}}_{ht_l} = 0 \tag{186}$$

$$\tilde{\theta}_{h1} \cdot \tilde{\vec{J}}_{ht_r} = 0, \tilde{\Pi}_{h1} \cdot \tilde{\vec{J}}_{ht_r} = 0. \tag{187}$$

Thus, we find that $\tilde{\vec{J}}_{ht_l}$ only supports $\tilde{\theta}$ and $\tilde{\Pi}$, and therefore $\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e1h1} = \tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{h1h1} = 0$. Depolarizing terms are killed off in the dot product of (159). Thus, $\tilde{\vec{J}}_{ht_l}$ only supports a TE^z field structure. Similarly, we find that $\tilde{\vec{J}}_{ht_r}$ only supports $\tilde{\Phi}$ and $\tilde{\psi}$, and therefore only supports a TM^z field structure. These results perfectly agree with the earlier predictions from the Maxwell equations derivations laid out in (178) and (179).

III. Theory of the Extraction of Uniaxial Material Parameters Using Two-Layer Method

Using the total Green functions for uniaxial materials derived in Chapter II, an extraction theory for uniaxial materials can now be developed. For this effort, a single flanged rectangular waveguide probe will be used to interrogate a MUT that is permanently affixed to a PEC surface as depicted in fig 5. The waveguide aperture will be sized appropriately for the bandwidth of interest. The flange will be sized appropriately to allow time gating of flange edge reflections in the measurements [45]. This derivation will follow similar principles to those used in [71], substituting the dual-layer uniaxial material theory developed in Chapter II for the single-layer material theory presented in [71].

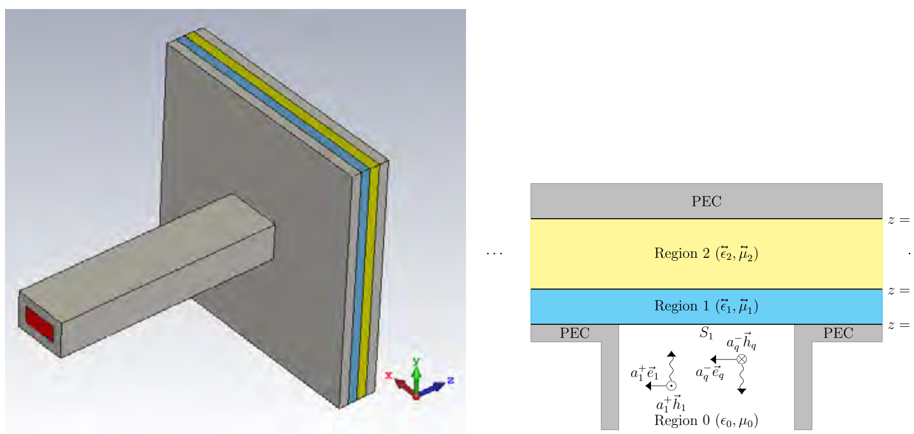

Figure 5. Perspective view (left) and cross section (right) of parallel plate and rectangular waveguide regions under analysis in this chapter. Region 0 (white) is the rectangular waveguide region filled with free space, Region 1 (cyan) is a material with known constitutive parameters, and Region 2 (yellow) is the material under test (yellow).

The amplitude of an incoming wave in the rectangular waveguide is a_1^+. Since this wave is propagating in a rectangular waveguide, the excitation frequency can be chosen such that only the TE_{10}^z mode propagates in the forward direction. When the incoming wave encounters the discontinuity at the probe aperture, infinitely many reflection modes $(q \to \infty)$ are produced in the reverse direction with amplitude a_q^-.

Therefore, the reflection coefficient can be defined as

$$\Gamma_q = \frac{a_q^-}{a_1^+} \tag{188}$$

$$\Rightarrow \Gamma_1 = \frac{a_1^-}{a_1^+} = S_{11}^{\text{thy}} \tag{189}$$

Using a combination of continuity of tangential fields, Love's equivalence, and the MoM, a coupled set of MFIEs will be developed. From those MFIEs, a technique for deriving $\vec{\epsilon}$ and $\vec{\mu}$ will be shown.

3.1 Rectangular Waveguide Analysis

Begin by analyzing the transverse electric field in the waveguide region (region 0). The total transverse electric field is the sum of the dominant mode excitation wave in the forward direction and the infinite reflected modes in the reverse direction. Since it is impossible to analyze infinite modes in a computational environment, truncate the infinite modes to some large finite number of modes Q. Thus,

$$\vec{E}_{t0} = a_1^+ \vec{e}_1 e^{-jk_z z} + \sum_{q=1}^{Q} a_q^- \vec{e}_q e^{jk_z z} \tag{190}$$

where q is an index that selects from all possible reflected modes ($\vec{e}_q$), including TE_{mn}^z and TM_{mn}^z modes, in increasing order by cutoff frequency. For reference, the first 20 modes are tabulated in Appendix E. Note from fig 5 that the waveguide region meets the parallel plate region in the form of an aperture (S_1) at $z = 0$. Therefore, at the aperture (190) implies that

$$\vec{e}_a := \vec{E}_{t0}\,(z = 0) = a_1^+ \vec{e}_1 + \sum_{q=1}^{Q} a_q^- \vec{e}_q \tag{191}$$

As part of the MoM, the unknowns must first be expanded. Note that (191) already has the unknowns expanded. Next, testing both sides by the p^{th} mode ($\vec{e}_p$) implies that

$$\begin{aligned}
\int\limits_{S_1} \vec{e}_p \cdot \vec{e}_a dS &= \int\limits_{S_1} \vec{e}_p \cdot a_1^+ \vec{e}_1 dS + \int\limits_{S_1} \vec{e}_p \cdot \sum_{q=1}^{Q} a_q^- \vec{e}_q dS \\
&= a_1^+ \int\limits_{S_1} \vec{e}_p \cdot \vec{e}_1 dS + \sum_{q=1}^{Q} a_q^- \int\limits_{S_1} \vec{e}_p \cdot \vec{e}_q dS \\
&= a_1^+ \delta_{p1} + \sum_{q=1}^{Q} a_q^- \delta_{pq}, \\
\Rightarrow a_p^- &= \int\limits_{S_1} \vec{e}_p \cdot \vec{e}_a dS - a_1^+ \delta_{p1}
\end{aligned} \tag{192}$$

Noting that p is just a dummy index variable, substituting $p = q$ into (192) implies that

$$a_q^- = \int\limits_{S_1} \vec{e}_q \cdot \vec{e}_a dS - a_1^+ \delta_{q1} \tag{193}$$

Having analyzed the transverse electric field in the rectangular waveguide, now analyze the transverse magnetic field. From fig 5, it can be seen that the total transverse magnetic field is

$$\vec{H}_{t0} = a_1^+ \vec{h}_1 e^{-jk_z z} - \sum_{q=1}^{Q} a_q^- \vec{h}_q e^{jk_z z} \tag{194}$$

Again, analyzing the magnetic field at the aperture implies that

$$\vec{H}_{t0}\left(z = 0\right) = a_1^+ \vec{h}_1 - \sum_{q=1}^{Q} a_q^- \vec{h}_q \tag{195}$$

Substituting (193) into (195) implies that

$$\begin{aligned}\vec{H}_{t0}(z=0) &= a_1^+\vec{h}_1 - \sum_{q=1}^{Q}\left[\int_{S_1}\vec{e}_q\cdot\vec{e}_a dS - a_1^+\delta_{q1}\right]\vec{h}_q \\ &= a_1^+\vec{h}_1 - \sum_{q=1}^{Q}\vec{h}_q\int_{S_1}\vec{e}_q\cdot\vec{e}_a dS + \sum_{q=1}^{Q}a_1^+\vec{h}_q\delta_{q1} \\ &= 2a_1^+\vec{h}_1 - \sum_{q=1}^{Q}\vec{h}_q\int_{S_1}\vec{e}_q\cdot\vec{e}_a dS \qquad (196)\end{aligned}$$

3.2 Parallel Plate Region Analysis

To analyze the fields in the parallel plate region, Love's equivalence principle is used to replace the transverse electric field across the aperture with a magnetic current across a virtual PEC surface. The equivalent structure is illustrated in fig 6.

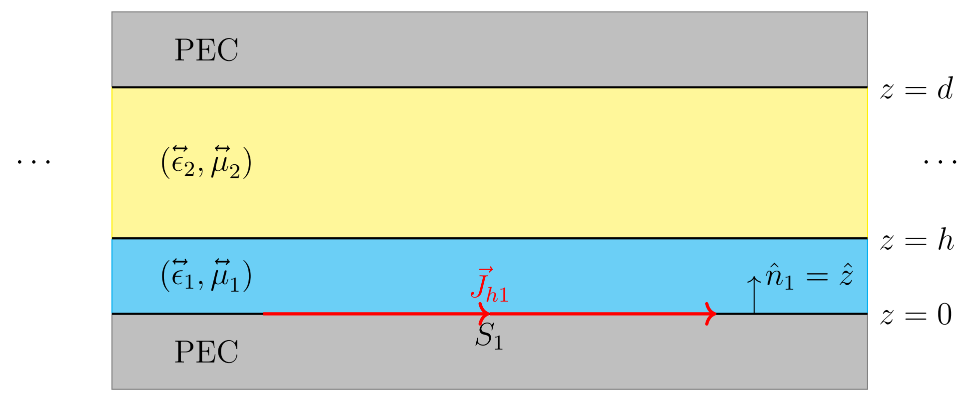

Figure 6. Cross section of Love's equivalent parallel plate region under analysis in this section. Region 1 (cyan) is a material with known constitutive parameters and Region 2 (yellow) is the material under test (yellow). The equivalent PEC surface magnetic current replacing the aperture S_1 is $\vec{J}_{h1}$ (red).

Love's equivalence principle states that

$$\vec{J}_{ht1} = -\hat{n}_1 \times \vec{E}_{t1} = -\hat{z}\times\vec{E}_{t1} \qquad (197)$$

Noting that continuity of tangential fields is required at the aperture boundary implies

that

$$\vec{J}_{ht1} = -\hat{z} \times \vec{E}_{t1} = -\hat{z} \times \vec{e}_a \tag{198}$$

Utilizing the Green functions derived in Chapter II, this equivalent magnetic current can be used to predict the magnetic field observed in the parallel plate region. Note, however, that since the Green functions were determined in the transverse-spectral domain $\left(\vec{\lambda}_\rho, z\right)$, they must be converted back to the spatial domain to be of use here. Employing the forward and reverse transverse Fourier transforms implies that

$$\begin{aligned}
\vec{H}_{\{1,2\}}(\hat{\rho}, z) &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \tilde{\vec{H}}_{\{1,2\}} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_0^h \tilde{\overleftrightarrow{G}}_{h\{1,2\}h1} \cdot \tilde{\vec{J}}_{h1} dz' + \int_h^d \tilde{\overleftrightarrow{G}}_{h\{1,2\}h2} \cdot \tilde{\vec{J}}_{h2} dz' \right] e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_0^h \tilde{\overleftrightarrow{G}}_{h\{1,2\}h1} \cdot \left(\iint_{-\infty}^{\infty} \vec{J}_{h1} e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} d^2\rho' \right) dz' \right. \\
&\quad \left. + \int_h^d \tilde{\overleftrightarrow{G}}_{h\{1,2\}h2} \cdot \left(\iint_{-\infty}^{\infty} \vec{J}_{h2} e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} d^2\rho' \right) dz' \right] e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho \\
&= \iint_{-\infty}^{\infty} \frac{1}{4\pi^2} \left[\int_{V_1'} \tilde{\overleftrightarrow{G}}_{h\{1,2\}h1} \cdot \vec{J}_{h1} e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} dV_1' + \int_{V_2'} \tilde{\overleftrightarrow{G}}_{h\{1,2\}h2} \cdot \vec{J}_{h2} e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} dV_2' \right] d^2\lambda_\rho \\
&= \int_{V_1'} \underbrace{\left[\iint_{-\infty}^{\infty} \frac{1}{4\pi^2} \tilde{\overleftrightarrow{G}}_{h\{1,2\}h1} e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} d^2\lambda_\rho \right]}_{\overleftrightarrow{G}_{h\{1,2\}h1}} \cdot \vec{J}_{h1} dV_1' \\
&\quad + \int_{V_2'} \underbrace{\left[\iint_{-\infty}^{\infty} \frac{1}{4\pi^2} \tilde{\overleftrightarrow{G}}_{h\{1,2\}h2} e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} d^2\lambda_\rho \right]}_{\overleftrightarrow{G}_{h\{1,2\}h2}} \cdot \vec{J}_{h2} dV_2'
\end{aligned} \tag{199}$$

The magnetic field observed just above the aperture at $\vec{r} = \vec{r}_1^{\,+} := (x, y, z^+)$ may either be from region 1 or region 2, depending on whether $h > 0$ or $h = 0$ respectively. If $h > 0$, the observation and equivalent magnetic current only occur in region 1. If $h = 0$, the observation and equivalent magnetic current only occur in region 2. Thus, substituting the equivalent magnetic current into (199) implies that

$$\begin{aligned}
\vec{H}_t\left(\vec{r}_1^{\,+}\right) &= \begin{cases} \iint\limits_{-\infty}^{\infty} \left[\frac{1}{4\pi^2} \int\limits_{V_1'} \tilde{\overleftrightarrow{G}}_{h1h1} \cdot \vec{J}_{ht1} \delta\left(z' - z^+\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} dV_1' \right] d^2\lambda_\rho, & h > 0 \\ \iint\limits_{-\infty}^{\infty} \left[\frac{1}{4\pi^2} \int\limits_{V_2'} \tilde{\overleftrightarrow{G}}_{h2h2} \cdot \vec{J}_{ht2} \delta\left(z' - z^+\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} dV_2' \right] d^2\lambda_\rho, & h = 0 \end{cases} \\
&= \begin{cases} \iint\limits_{-\infty}^{\infty} \left[\frac{1}{4\pi^2} \int\limits_0^b \int\limits_0^a \tilde{\overleftrightarrow{G}}_{h1h1} \cdot \left(-\hat{z} \times \vec{e}_{a1}\left(\vec{r}_1^{\,\prime}\right)\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} dx' dy' \right] d^2\lambda_\rho, & h > 0 \\ \iint\limits_{-\infty}^{\infty} \left[\frac{1}{4\pi^2} \int\limits_0^b \int\limits_0^a \tilde{\overleftrightarrow{G}}_{h2h2} \cdot \left(-\hat{z} \times \vec{e}_{a1}\left(\vec{r}_1^{\,\prime}\right)\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho} - \vec{\rho}')} dx' dy' \right] d^2\lambda_\rho, & h = 0 \end{cases}
\end{aligned} \tag{200}$$

Since the $h = 0$ case is analyzed by Rogers in [71], that derivation is not repeated here. Hence, this section only focuses on the case where $h > 0$.

Case I: $h > 0$**.**

First, expand the unknown electric field in the aperture such that,

$$\vec{e}_a = \sum_{n=1}^{N} a_1^+ C_n \vec{e}_n \tag{201}$$

where C_n are unknown constants to be determined. By enforcing continuity of tangential magnetic fields across the aperture at S_1, (196), (200), and (201) imply that

$$\vec{H}_{t0}\left(\vec{r}_1^{\,-}\right) = \vec{H}_t\left(\vec{r}_1^{\,+}\right)$$

$$
\begin{aligned}
&(196),(200) \Rightarrow 2a_1^{+}\vec{h}_1\left(\overleftarrow{r_1}\right) - \sum_{q=1}^{Q}\vec{h}_q\left(\overleftarrow{r_1}\right)\int_{S_1}\vec{e}_q\left(\overleftarrow{r_1}\right)\cdot\vec{e}_a\left(\overleftarrow{r_1}\right)dS \\
&\quad = \iint_{-\infty}^{\infty}\left[\frac{1}{4\pi^2}\int_0^b\int_0^a \tilde{\overleftrightarrow{G}}_{h1h1}\cdot\left(-\hat{z}\times\vec{e}_a\left(\vec{r}_1'\right)\right)e^{j\vec{\lambda}_\rho\cdot(\vec{\rho}-\vec{\rho}')}dx'dy'\right]d^2\lambda_\rho \\
&(201) \Rightarrow 2a_1^{+}\vec{h}_1\left(\overleftarrow{r_1}\right) - \sum_{q=1}^{Q}\vec{h}_q\left(\overleftarrow{r_1}\right)\int_{S_1}\vec{e}_q\left(\overleftarrow{r_1}\right)\cdot\left(\sum_{n=1}^{N}a_1^{+}C_n\vec{e}_n\left(\overleftarrow{r_1}\right)\right)dS \\
&\quad = \iint_{-\infty}^{\infty}\left[\frac{1}{4\pi^2}\int_0^b\int_0^a \tilde{\overleftrightarrow{G}}_{h1h1}\cdot\left(-\hat{z}\times\sum_{n=1}^{N}a_1^{+}C_n\vec{e}_n\left(\vec{r}_1'\right)\right)e^{j\vec{\lambda}_\rho\cdot(\vec{\rho}-\vec{\rho}')}dx'dy'\right]d^2\lambda_\rho \\
&\Rightarrow 2\vec{h}_1\left(\overleftarrow{r_1}\right) - \sum_{q=1}^{Q}\sum_{n=1}^{N}C_n\vec{h}_q\left(\overleftarrow{r_1}\right)\underbrace{\int_{S_1}\vec{e}_q\left(\overleftarrow{r_1}\right)\cdot\vec{e}_n\left(\overleftarrow{r_1}\right)dS}_{\delta_{qn}} \\
&\quad = \sum_{n=1}^{N}C_n\iint_{-\infty}^{\infty}\left[\frac{1}{4\pi^2}\int_0^b\int_0^a \tilde{\overleftrightarrow{G}}_{h1h1}\cdot\left(-\hat{z}\times\vec{e}_n\left(\vec{r}_1'\right)\right)e^{j\vec{\lambda}_\rho\cdot(\vec{\rho}-\vec{\rho}')}dx'dy'\right]d^2\lambda_\rho \\
&\Rightarrow 2\vec{h}_1\left(\overleftarrow{r_1}\right) - \sum_{n=1}^{N}C_n\vec{h}_n\left(\overleftarrow{r_1}\right) = \sum_{n=1}^{N}C_n\iint_{-\infty}^{\infty}\left[\frac{1}{4\pi^2}\int_0^b\int_0^a \tilde{\overleftrightarrow{G}}_{h1h1}\right. \\
&\qquad\qquad \left.\cdot\left(-\hat{z}\times\vec{e}_n\left(\vec{r}_1'\right)\right)e^{j\vec{\lambda}_\rho\cdot(\vec{\rho}-\vec{\rho}')}dx'dy'\right]d^2\lambda_\rho
\end{aligned}
\tag{202}
$$

Substituting $Z_n\vec{h}_n = \hat{z}\times\vec{e}_n$ into (202) implies that

$$
2\vec{h}_1\left(\overleftarrow{r_1}\right) = \sum_{n=1}^{N}C_n\left\{\vec{h}_n\left(\overleftarrow{r_1}\right) - \iint_{-\infty}^{\infty}\left[\frac{Z_n}{4\pi^2}\int_0^b\int_0^a \tilde{\overleftrightarrow{G}}_{h1h1}\cdot\vec{h}_n\left(\vec{r}_1'\right)e^{j\vec{\lambda}_\rho\cdot(\vec{\rho}-\vec{\rho}')}dx'dy'\right]d^2\lambda_\rho\right\}
\tag{203}
$$

Testing (203) with the operator $\int_{S_1}\vec{h}_m\left(\vec{r}_1\right)\cdot\{\cdot\}\,dS, m = 1,\ldots,N$ and noting that in the tangential field continuity limit, $\vec{r}_1^{\,+} = \vec{r}_1^{\,-} = \vec{r}_1$ implies that

$$2\sum_{m=1}^{N}\int\limits_{S_1}\vec{h}_m\left(\vec{r}_1\right)\cdot\vec{h}_1\left(\vec{r}_1\right)dS = \sum_{m=1}^{N}\sum_{n=1}^{N}C_n\left\{\int\limits_{S_1}\vec{h}_m\left(\vec{r}_1\right)\cdot\vec{h}_n\left(\vec{r}_1\right)dS\right.$$
$$-\frac{Z_n}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left(\left[\int\limits_0^b\int\limits_0^a\vec{h}_m\left(\vec{r}_1\right)e^{j\vec{\lambda}_\rho\cdot\vec{\rho}}dxdy\right]\cdot\tilde{\overleftrightarrow{G}}_{h1h1}\right. \tag{204}$$
$$\left.\left.\cdot\left[\int\limits_0^b\int\limits_0^a\vec{h}_n\left(\vec{r}_1^{\,\prime}\right)e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}^{\,\prime}}dx'dy'\right]\right)d^2\lambda_\rho\right\}$$

Note that (204) can be separated into a system of N linearly-independent equations that can be rewritten in the form

$$\underbrace{\begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \cdots & A_{N,N}\end{bmatrix}}_{N\times N}\underbrace{\begin{bmatrix} C_1 \\ \vdots \\ C_N\end{bmatrix}}_{N\times 1} = \underbrace{\begin{bmatrix} B_1 \\ \vdots \\ B_N\end{bmatrix}}_{N\times 1} \tag{205}$$

where

$$A_{m,n} = \int\limits_{S_1}\vec{h}_m\left(\vec{r}_1\right)\cdot\vec{h}_n\left(\vec{r}_1\right)dS - \frac{Z_n}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left[\left(\int\limits_0^b\int\limits_0^a\vec{h}_m\left(\vec{r}_1\right)e^{j\vec{\lambda}_\rho\cdot\vec{\rho}}dxdy\right)\right.$$
$$\left.\cdot\tilde{\overleftrightarrow{G}}_{h1h1}\cdot\left(\int\limits_0^b\int\limits_0^a\vec{h}_n\left(\vec{r}_1^{\,\prime}\right)e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}^{\,\prime}}dx'dy'\right)\right]d^2\lambda_\rho \tag{206}$$

$$B_m = 2\int\limits_{S_1}\vec{h}_m\left(\vec{r}_1\right)\cdot\vec{h}_1\left(\vec{r}_1\right)dS \tag{207}$$

From (205), it can be seen that there are N equations for finding N unknown coefficients C_n. Thus the system is well-posed and can be solved through traditional linear algebra techniques. In order to simplify the equations, begin by evaluating as many of the integrals as possible. Beginning with the source integral,

$$\int_0^b \int_0^a \vec{h}_n\left(\vec{r}_1^{\,\prime}\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}^{\,\prime}} dx'dy' = \int_0^b \int_0^a \vec{h}_n\left(x',y',0\right) e^{-j\lambda_x x'} e^{-j\lambda_y y'} dx'dy' \tag{208}$$

Rewriting the transverse magnetic field in the aperture as $\vec{h}_n = \hat{x}h_{nx} + \hat{y}h_{ny}$ implies that

$$\int_0^b \int_0^a \vec{h}_n\left(\vec{r}_1^{\,\prime}\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}^{\,\prime}} dx'dy' = \int_0^b \int_0^a \left(\hat{x}h_{nx} + \hat{y}h_{ny}\right) e^{-j\lambda_x x'} e^{-j\lambda_y y'} dx'dy' \tag{209}$$

Incorporating the TE^z and TM^z modal field representations for a rectangular waveguide, provided by [6], implies that

$$\begin{aligned}\int_0^b \int_0^a \vec{h}_n\left(\vec{r}_1^{\,\prime}\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}^{\,\prime}} dx'dy' = \hat{x}M_{xn}^h &\int_0^b \int_0^a \sin\left(k_{xn}x'\right)\cos\left(k_{yn}y'\right) e^{-j\lambda_x x'} e^{-j\lambda_y y'} dx'dy' \\ + \hat{y}M_{yn}^h &\int_0^b \int_0^a \cos\left(k_{xn}x'\right)\sin\left(k_{yn}y'\right) e^{-j\lambda_x x'} e^{-j\lambda_y y'} dx'dy'\end{aligned} \tag{210}$$

where $M^h_{\{x,y\}\{m,n\}}$ is an amplitude coefficient that depends on the type of mode as indicated in table 1.

Table 1. Modal Amplitude Coefficients

	$M^h_{x\{m,n\}}$	$M^h_{y\{m,n\}}$	$Z_{\{m,n\}}$
$\mathrm{TE}^z_{v_{\{m,n\}}w_{\{m,n\}}}$	$\beta\frac{k_{x\{m,n\}}}{Z_{\{m,n\}}}$	$\beta\frac{k_{y\{m,n\}}}{Z_{\{m,n\}}}$	$\frac{\omega\mu_0}{k_{z\{m,n\}}}$
$\mathrm{TM}^z_{v_{\{m,n\}}w_{\{m,n\}}}$	$\beta k_{y\{m,n\}}$	$-\beta k_{x\{m,n\}}$	$\frac{k_{z\{m,n\}}}{\omega\epsilon_0}$

$k_{x\{m,n\}}$	$k_{y\{m,n\}}$	$k_{z\{m,n\}}$	$k_{c\{m,n\}}$
$\frac{v_{\{m,n\}}\pi}{a}$	$\frac{w_{\{m,n\}}\pi}{b}$	$\sqrt{k_0^2 - k_{c\{m,n\}}^2}$	$\sqrt{k_{x\{m,n\}}^2 + k_{y\{m,n\}}^2}$

Applying separation of variables to (210) implies that

$$\int_0^b \int_0^a \vec{h}_n\left(\vec{r}_1'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dx'dy' = \hat{x}M_{xn}^h \int_0^a \sin\left(k_{xn}x'\right) e^{-j\lambda_x x'} dx' \int_0^b \cos\left(k_{yn}y'\right) e^{-j\lambda_y y'} dy' + \hat{y}M_{yn}^h \int_0^a \cos\left(k_{xn}x'\right) e^{-j\lambda_x x'} dx' \int_0^b \sin\left(k_{yn}y'\right) e^{-j\lambda_y y'} dy' \tag{211}$$

A closed-form solution exists for (211). From Appendix B of [78], it can be shown that in general

$$\int_0^X \sin\left(\frac{u\pi}{X}x\right) e^{\pm j\lambda_x x} dx = -\frac{u\pi}{X}\left[\frac{\left(1-(-1)^u e^{\pm j\lambda_x X}\right)}{\left(\lambda_x+\frac{u\pi}{X}\right)\left(\lambda_x-\frac{u\pi}{X}\right)}\right] \tag{212}$$

$$\int_0^X \cos\left(\frac{u\pi}{X}x\right) e^{\pm j\lambda_x x} dx = \pm j\lambda_x\left[\frac{\left(1-(-1)^u e^{\pm j\lambda_x X}\right)}{\left(\lambda_x+\frac{u\pi}{X}\right)\left(\lambda_x-\frac{u\pi}{X}\right)}\right] \tag{213}$$

where $u \in \left\{v_{\{m,n\}}, w_{\{m,n\}}\right\}$. Due to symmetry of the waveguide, $v_{\{m,n\}}$ must be odd and $w_{\{m,n\}}$ must be even when following the solution presented in [78]. Substituting (212) and (213) into (211) implies that

$$\begin{aligned}
&\int_0^b \int_0^a \vec{h}_n\left(\vec{r}_1'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dx'dy' \\
&= \hat{x}M_{xn}^h\left(-k_{xn}\left[\frac{\left(1-(-1)^{v_n} e^{-j\lambda_x a}\right)}{\left(\lambda_x+k_{xn}\right)\left(\lambda_x-k_{xn}\right)}\right]\right)\left(-j\lambda_y\left[\frac{\left(1-(-1)^{w_n} e^{-j\lambda_y b}\right)}{\left(\lambda_y+k_{yn}\right)\left(\lambda_y-k_{yn}\right)}\right]\right) \\
&\quad + \hat{y}M_{yn}^h\left(-j\lambda_x\left[\frac{\left(1-(-1)^{v_n} e^{-j\lambda_x a}\right)}{\left(\lambda_x+k_{xn}\right)\left(\lambda_x-k_{xn}\right)}\right]\right)\left(-k_{yn}\left[\frac{\left(1-(-1)^{w_n} e^{-j\lambda_y b}\right)}{\left(\lambda_y+k_{yn}\right)\left(\lambda_y-k_{yn}\right)}\right]\right) \\
&= \left[\frac{\left(1-(-1)^{v_n} e^{-j\lambda_x a}\right)\left(1-(-1)^{w_n} e^{-j\lambda_y b}\right)}{\left(\lambda_x+k_{xn}\right)\left(\lambda_x-k_{xn}\right)\left(\lambda_y+k_{yn}\right)\left(\lambda_y-k_{yn}\right)}\right]\left[\hat{x}jM_{xn}^h k_{xn}\lambda_y + \hat{y}jM_{yn}^h \lambda_x k_{yn}\right]
\end{aligned} \tag{214}$$

Next, analyzing the testing functions in a similar manner implies that

$$\int_0^b \int_0^a \vec{h}_m(\vec{r}_1)\, e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dx dy = \hat{x} M_{xm}^h \int_0^a \sin(k_{xm} x)\, e^{j\lambda_x x} dx \int_0^b \cos(k_{ym} y)\, e^{j\lambda_y y} dy$$

$$+ \hat{y} M_{ym}^h \int_0^a \cos(k_{xm} x)\, e^{j\lambda_x x} dx \int_0^b \sin(k_{ym} y)\, e^{j\lambda_y y} dy$$

$$= \hat{x} M_{xm}^h \left(-k_{xm} \left[\frac{\left(1 - (-1)^{v_m} e^{j\lambda_x a}\right)}{(\lambda_x + k_{xm})(\lambda_x - k_{xm})} \right] \right) \left(j\lambda_y \left[\frac{\left(1 - (-1)^{w_m} e^{j\lambda_y b}\right)}{(\lambda_y + k_{ym})(\lambda_y - k_{ym})} \right] \right)$$

$$+ \hat{y} M_{ym}^h \left(j\lambda_x \left[\frac{\left(1 - (-1)^{v_m} e^{j\lambda_x a}\right)}{(\lambda_x + k_{xm})(\lambda_x - k_{xm})} \right] \right) \left(-k_{ym} \left[\frac{\left(1 - (-1)^{w_m} e^{j\lambda_y b}\right)}{(\lambda_y + k_{ym})(\lambda_y - k_{ym})} \right] \right)$$

$$= - \left[\frac{\left(1 - (-1)^{v_m} e^{j\lambda_x a}\right)\left(1 - (-1)^{w_m} e^{j\lambda_y b}\right)}{(\lambda_x + k_{xm})(\lambda_x - k_{xm})(\lambda_y + k_{ym})(\lambda_y - k_{ym})} \right] \left[\hat{x} j M_{xm}^h k_{xm} \lambda_y + \hat{y} j M_{ym}^h \lambda_x k_{ym} \right] \tag{215}$$

Analyzing the excitation integrals implies that

$$\int_{S_1} \vec{h}_m(\vec{r}_1) \cdot \vec{h}_n(\vec{r}_1)\, dS = \int_{S_1} \left[\hat{x} M_{xm}^h \sin(k_{xm} x) \cos(k_{ym} y) \right.$$

$$\left. + \hat{y} M_{ym}^h \cos(k_{xm} x) \sin(k_{ym} y) \right] \cdot \left[\hat{x} M_{xn}^h \sin(k_{xn} x) \cos(k_{yn} y) \right.$$

$$\left. + \hat{y} M_{yn}^h \cos(k_{xn} x) \sin(k_{yn} y) \right] dS$$

$$= \int_{S_1} M_{xm}^h M_{xn}^h \sin(k_{xm} x) \sin(k_{xn} x) \cos(k_{ym} y) \cos(k_{yn} y) \tag{216}$$

$$+ M_{ym}^h M_{yn}^h \cos(k_{xm} x) \cos(k_{xn} x) \sin(k_{ym} y) \sin(k_{yn} y)\, dS$$

Due to mode orthogonality, when $m \neq n$ (216) evaluates to 0. Therefore, with the added assumption of separation of variables, (216) implies that

$$
\begin{aligned}
\int_{S_1} \vec{h}_m(\vec{r}_1) \cdot \vec{h}_n(\vec{r}_1)\, dS = \delta_{m,n} \Bigg[& \left(M_{xm}^h\right)^2 \int_0^a \sin^2(k_{xm}x)\, dx \int_0^b \cos^2(k_{ym}y)\, dy \\
& + \left(M_{ym}^h\right)^2 \int_0^a \cos^2(k_{xm}x)\, dx \int_0^b \sin^2(k_{ym}y)\, dy \Bigg] \\
= \delta_{m,n} \Bigg[& \left(M_{xm}^h\right)^2 \left(\frac{2k_{xm}a - \sin(2k_{xm}a)}{4k_{xm}}\right) \left(\frac{2k_{ym}b + \sin(2k_{ym}b)}{4k_{ym}}\right) \\
& + \left(M_{ym}^h\right)^2 \left(\frac{2k_{xm}a + \sin(2k_{xm}a)}{4k_{xm}}\right) \left(\frac{2k_{ym}b - \sin(2k_{ym}b)}{4k_{ym}}\right) \Bigg] \\
= \delta_{m,n} \Bigg[& \left(M_{xm}^h\right)^2 \left(\frac{2k_{xm}a - \cancelto{0}{\sin(2v_m\pi)}}{4k_{xm}}\right) \left(\frac{2k_{ym}b + \cancelto{0}{\sin(2w_m\pi)}}{4k_{ym}}\right) \\
& + \left(M_{ym}^h\right)^2 \left(\frac{2k_{xm}a + \cancelto{0}{\sin(2v_m\pi)}}{4k_{xm}}\right) \left(\frac{2k_{ym}b - \cancelto{0}{\sin(2w_m\pi)}}{4k_{ym}}\right) \Bigg] \\
= \delta_{m,n} & \left(\frac{ab}{4}\right) \left[\left(M_{xm}^h\right)^2 + \left(M_{ym}^h\right)^2\right] \left(1 + \delta_{w_m,0}\right)
\end{aligned}
\tag{217}
$$

Consider the special case when $w_m = 0$. From table 1, that implies $k_{ym} = 0$, which further implies that

$$
\int_0^b \cos^2(k_{ym}y)\, dy = \int_0^b \cancelto{1}{\cos^2(0)} dy = b \tag{218}
$$

as opposed to $\frac{b}{2}$ for all nonzero values of w_m. Therefore, the $(1 + \delta_{w_m,0})$ term in (217) neatly accounts for this discrepancy. Recall that by careful selection of operating frequency, $\vec{h}_1$ is constrained to the TE_{10}^z mode. Thus for B_m, $w_m = w_n = 0$, which implies that

$$
B_m = 2 \int_{S_1} \vec{h}_m(\vec{r}_1) \cdot \vec{h}_1(\vec{r}_1)\, dS
$$

$$
\begin{aligned}
&= 2\delta_{m,1}\left(\frac{ab}{4}\right)\left[\left(M_{xm}^{h}\right)^{2}+\cancelto{0}{\left(M_{ym}^{h}\right)^{2}}\right]\left(1+\cancelto{1}{\delta_{w_m,0}}\right) \\
&= \delta_{m,1}ab\left(\beta\frac{k_{x1}}{Z_1}\right)^{2}
\end{aligned}
\tag{219}
$$

Now that the excitation matrix B has been simplified, use the source, test and excitation integrals determined above to simplify $A_{m,n}$. Substituting (214), (215), and (217) into (206) implies that

$$
\begin{aligned}
A_{m,n} &= \delta_{m,n}\left(\frac{ab}{4}\right)\left[\left(M_{xm}^{h}\right)^{2}+\left(M_{ym}^{h}\right)^{2}\right]\left(1+\delta_{w_m,0}\right) \\
&- \frac{Z_n}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left[\left(-\left[\frac{\left(1-(-1)^{v_m}e^{j\lambda_x a}\right)\left(1-(-1)^{w_m}e^{j\lambda_y b}\right)}{\left(\lambda_x+k_{xm}\right)\left(\lambda_x-k_{xm}\right)\left(\lambda_y+k_{ym}\right)\left(\lambda_y-k_{ym}\right)}\right]\left[\hat{x}jM_{xm}^{h}k_{xm}\lambda_y\right.\right.\right. \\
&\quad \left.\left.+\hat{y}jM_{ym}^{h}\lambda_x k_{ym}\right]\right)\cdot\overleftrightarrow{\tilde{G}}_{h1h1}\cdot\left(\left[\frac{\left(1-(-1)^{v_n}e^{-j\lambda_x a}\right)\left(1-(-1)^{w_n}e^{-j\lambda_y b}\right)}{\left(\lambda_x+k_{xn}\right)\left(\lambda_x-k_{xn}\right)\left(\lambda_y+k_{yn}\right)\left(\lambda_y-k_{yn}\right)}\right]\right. \\
&\quad \left.\left.\cdot\left[\hat{x}jM_{xn}^{h}k_{xn}\lambda_y+\hat{y}jM_{yn}^{h}\lambda_x k_{yn}\right]\right)\right]d^2\lambda_\rho \\
&= \delta_{m,n}\left(\frac{ab}{4}\right)\left[\left(M_{xm}^{h}\right)^{2}+\left(M_{ym}^{h}\right)^{2}\right]\left(1+\delta_{w_m,0}\right) \\
&\quad - \frac{Z_n}{4\pi^2}\int\limits_{-\infty}^{\infty}\left\{\left[\frac{\left(1-(-1)^{v_m}e^{j\lambda_x a}\right)\left(1-(-1)^{v_n}e^{-j\lambda_x a}\right)}{\left(\lambda_x+k_{xm}\right)\left(\lambda_x-k_{xm}\right)\left(\lambda_x+k_{xn}\right)\left(\lambda_x-k_{xn}\right)}\right]\right. \\
&\int\limits_{-\infty}^{\infty}\left[\frac{\left(1-(-1)^{w_m}e^{j\lambda_y b}\right)\left(1-(-1)^{w_n}e^{-j\lambda_y b}\right)}{\left(\lambda_y+k_{ym}\right)\left(\lambda_y-k_{ym}\right)\left(\lambda_y+k_{yn}\right)\left(\lambda_y-k_{yn}\right)}\right]\left[M_{xm}^{h}M_{xn}^{h}k_{xm}k_{xn}\lambda_y^2\tilde{G}_{h1h1,xx}\right. \\
&\quad +M_{xm}^{h}M_{yn}^{h}k_{xm}k_{yn}\lambda_x\lambda_y\tilde{G}_{h1h1,xy}+M_{ym}^{h}M_{xn}^{h}k_{ym}k_{xn}\lambda_x\lambda_y\tilde{G}_{h1h1,yx} \\
&\quad \left.\left.+M_{ym}^{h}M_{yn}^{h}k_{ym}k_{yn}\lambda_x^2\tilde{G}_{h1h1,yy}\right]d\lambda_y\right\}d\lambda_x
\end{aligned}
\tag{220}
$$

where $\tilde{G}_{h1h1,xx}$ refers to the $\hat{x}\hat{x}$ element of $\overleftrightarrow{\tilde{G}}_{h1h1}$.

3.3 Dominant Mode Summary

The λ_y integral portion of (220) can be evaluated analytically in the complex λ_y plane. With respect to $k_{y(m,n)}$, there are five possible cases:

- Case I: $w_m = w_n = 0$;
- Case II: $w_m \neq 0, w_n = 0$;
- Case III: $w_m = 0, w_n \neq 0$;
- Case IV: $w_m = w_n \neq 0$;
- Case V: $w_m \neq w_n \neq 0$.

Case I implies only the dominant mode is present. While this is the least accurate case, it is also the easiest case to analyze. Therefore, this effort will focus solely on Case I.

Case I: $w_m = w_n = 0$**.**

If $w_m = w_n = 0$, that implies that $k_{ym} = k_{yn} = 0$. This simplifies (220) such that

$$
\begin{aligned}
A_{m,n} &= \delta_{m,n}\left(\frac{ab}{2}\right)\left[\left(M^h_{xm}\right)^2 + \cancelto{0}{\left(M^h_{ym}\right)^2}\right] \\
&- \frac{Z_n M^h_{xm} M^h_{xn} k_{xm} k_{xn}}{4\pi^2} \int_{-\infty}^{\infty} \left\{ \left[\frac{\left(1-(-1)^{v_m} e^{j\lambda_x a}\right)\left(1-(-1)^{v_n} e^{-j\lambda_x a}\right)}{(\lambda_x + k_{xm})(\lambda_x - k_{xm})(\lambda_x + k_{xn})(\lambda_x - k_{xn})} \right] \right. \\
&\qquad \left. \int_{-\infty}^{\infty} \left[\frac{\left(1-e^{j\lambda_y b}\right)\left(1-e^{-j\lambda_y b}\right)}{\lambda_y^2} \right] \tilde{G}_{h1h1,xx} d\lambda_y \right\} d\lambda_x
\end{aligned}
\tag{221}
$$

For brevity, the details of evaluating the integral with respect to λ_y are presented in Appendix F.

$$\boxed{\begin{aligned}
&\vec{C} = \overleftrightarrow{A}^{-1}\vec{B}\\
&A_{m,n} = \delta_{m,n}\frac{1}{Z_m Z_n} - \frac{Z_n M^h_{xm} M^h_{xn} k_{xm} k_{xn}}{2\pi}\int_{-\infty}^{\infty} C^{\lambda_x}\left(\Omega^{\mathrm{TE}^z} + \Omega^{\mathrm{TM}^z}\right) d\lambda_x\\
&C^{\lambda_x} = \left[\frac{\left(1-(-1)^{v_m}\, e^{j\lambda_x a}\right)\left(1-(-1)^{v_n}\, e^{-j\lambda_x a}\right)}{\left(\lambda_x + k_{xm}\right)\left(\lambda_x - k_{xm}\right)\left(\lambda_x + k_{xn}\right)\left(\lambda_x - k_{xn}\right)}\right]\\
&\Omega^{\mathrm{TE}^z} = j\left[\frac{b}{Z^*_{\theta 1}}\right]\left[\frac{Z^*_{\theta 1} - Z^*_{\theta 2}\tan\left(k^*_{z\theta 2}T\right)\tan\left(\lambda^*_{z\theta 1}h\right)}{Z^*_{\theta 1}\tan\left(\lambda^*_{z\theta 1}h\right) + Z^*_{\theta 2}\tan\left(\lambda^*_{z\theta 2}T\right)}\right]\\
&\qquad - \sum_{\ell=0}^{L}\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2\left(1-e^{j\lambda_y b}\right)}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TE}^z}}\right]\left[1 - \frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 1}h\right)\tan\left(\lambda_{z\theta 2}T\right)\right]\Bigg|_{\lambda_y = -\lambda_{y\theta\ell}}\\
&\Omega^{\mathrm{TM}^z} = -\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TM}^z}}\left[1 - \frac{Z_{\psi 2}}{Z_{\psi 1}}\tan\left(\lambda_{z\psi 1}h\right)\tan\left(\lambda_{z\psi 2}T\right)\right]\Bigg|_{\lambda_y = -\lambda_{y\psi\ell}}\\
&Z^*_{\theta(1,2)} = \frac{\omega\mu_{t(1,2)}}{\lambda^*_{z\theta(1,2)}},\ \lambda^*_{z\theta(1,2)} = \sqrt{k^2_{t(1,2)} - \frac{\mu_{t(1,2)}}{\mu_{z(1,2)}}\lambda_x^2}\\
&D^{\mathrm{TE}^z} = Z^2_{\theta 1}\mu_{z2}\left[\frac{\tan\left(\lambda_{z\theta 1}h\right)}{\lambda_{z\theta 1}} - h\right] + Z^2_{\theta 2}\mu_{z1}\left[\frac{\tan\left(\lambda_{z\theta 2}T\right)}{\lambda_{z\theta 2}} - T\right]\\
&\qquad\qquad + Z_{\theta 1}Z_{\theta 2}\tan\left(\lambda_{z\theta 1}h\right)\tan\left(\lambda_{z\theta 2}T\right)\left[h\mu_{z2} + T\mu_{z1}\right]\\
&D^{\mathrm{TM}^z} = \frac{1}{Z_{\psi 1}Z_{\psi 2}}\left[h\epsilon_{z2}Z^2_{\psi 2} + T\epsilon_{z1}Z^2_{\psi 1}\right]\tan\left(\lambda_{z\psi 1}h\right)\tan\left(\lambda_{z\psi 2}T\right)\\
&\qquad\qquad - \epsilon_{z2}\left[\frac{\tan\left(\lambda_{z\psi 1}h\right)}{\lambda_{z\psi 1}} + h\right] - \epsilon_{z1}\left[\frac{\tan\left(\lambda_{z\psi 2}T\right)}{\lambda_{z\psi 2}} + T\right]\\
&B_m = \frac{2}{Z_m^2}\delta_{m,1}
\end{aligned}}$$

where the $Z_{(m,n)}$ and $M^h_{x(m,n)}$ terms are determined via Table 1 and the $\lambda_{y(\theta,\psi)\ell}$ terms are determined numerically via the techniques described in [41].

Special Case: $h \to 0$**.**

In Appendix F, it is shown that as $h \to 0$

$$\lim_{h\to 0} \Omega^{\mathrm{TE}^z} = \frac{jb\lambda^*_{z\theta 2}\cos\left(\lambda^*_{z\theta 2}T\right)}{\omega\mu_{t2}\sin\left(\lambda^*_{z\theta 2}T\right)} - \frac{2\mu_{z2}\lambda_x^2}{\omega\mu_{t2}^2 T^3}\sum_{\ell=0}^{L}\frac{\left(1-e^{-j\lambda_{y\theta\ell}b}\right)(\pi\ell)^2}{\lambda^3_{y\theta\ell}\left(\lambda_x^2+\lambda^2_{y\theta\ell}\right)} \tag{222}$$

$$\lim_{h\to 0} \Omega^{\mathrm{TM}^z} = -\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z2}\left(1-e^{-j\lambda_{y\psi\ell}b}\right)}{\lambda_{y\psi\ell}\left(\lambda_x^2+\lambda^2_{y\psi\ell}\right)T\left(\delta_{0,l}+1\right)} \tag{223}$$

These results are in perfect agreement with the single-layer model derived in [71].

Extraction Algorithm.

Once the $A_{m,n}$ and B_m terms are determined, the unknown C_n terms can be solved. Those C_n terms will then be used to find theoretical reflection coefficients. Substituting (201) into (193) implies that

$$\begin{aligned}
S_{11}^{\mathrm{thy}} &= \frac{a_1^-}{a_1^+} \\
&= \frac{1}{a_1^+}\left(a_q^-\right)\Big|_{q=1} \\
&= \frac{1}{a_1^+}\left(\int_{S_1}\vec{e}_q\cdot\vec{e}_{a1}dS - a_1^+\delta_{q1}\right)\Bigg|_{q=1} \\
&= \frac{1}{a_1^+}\left[\int_{S_1}\vec{e}_1\cdot\left(\sum_{n=1}^{N}a_1^+C_n\vec{e}_n\right)dS - a_1^+\right] \\
&= \int_{S_1}\left(\sum_{n=1}^{N}C_n\vec{e}_1\cdot\vec{e}_n\right)dS - 1 \\
&= C_1 - 1
\end{aligned} \tag{224}$$

Now that the process of determining the theoretical reflection coefficient has been established, it can be used to extract constitutive parameters from measurements by finding the arguments that minimize the difference between the theory and measured data. Namely, in the case of a nonmagnetic material, two least-squares objective functions are simultaneously minimized to find the corresponding constitutive parameters. Thus,

$$\underset{\epsilon_{rt},\epsilon_{rz}\in\mathbb{C}}{\operatorname{argmin}}\left\{\begin{array}{c}\left(S_{11}^{\text{thy}}\left(h\neq 0\right)-S_{11}^{\text{m1}}\right)^{2}\\ \left(S_{11}^{\text{thy}}\left(h=0\right)-S_{11}^{\text{m2}}\right)^{2}\end{array}\right\} \tag{225}$$

If the nonmagnetic assumption does not hold, additional measurements may be taken by varying the size of region 1 (i.e. taking each measurement at a different distance from the MUT) and/or varying known material parameters in region 1.

IV. Results of The Two-Layer Method

In an attempt to validate the forward model developed in Chapter III, the model is implemented in MATLAB® and compared with results simulated using the commercial software CST Microwave Studio®. First, it is assumed that the MUT is lossy, nonmagnetic (i.e. $\mu_{rt2} = \mu_{rz2} = 1$), and that the known uniaxial layer of material is air (i.e. $\epsilon_{rt1} = \epsilon_{rz1} = \mu_{rt1} = \mu_{rz1} = 1$). It is also assumed for comparison that the CST Microwave Studio® data are "true," given that the commercial software is relatively mature. Various heights (h) of the known material, MUT thicknesses ($T = d - h$), and MUT permittivity parameter sets are tested in an attempt to get a broad range of comparison results. The results of these comparisons are mixed and accuracy of the results is dependent on two main factors: whether or not $h > 0$ and the specific combination of ϵ_{rt2} and ϵ_{rz2} chosen. Figure 7 illustrates how changing h affects accuracy. Note that each case plotted in Fig 7 has a subplot for the magnitude component $\left|S_{11}^{\mathrm{thy}}\right|$ and the phase component $\angle S_{11}^{\mathrm{thy}}$. The phase component is constrained to the range $-\pi < \angle S_{11}^{\mathrm{thy}} < \pi$, which accounts for large jumps in the phase plots as phase wrapping occurs.

It is intuitive that when $h = 0$, the accuracy of the comparisons with CST Microwave Studio® solutions is on par with those shown in [71]. This is because the model presented in this research collapses to the identical model presented in [71], as shown in Chapter III. It is interesting to note in the CST Microwave Studio® results that the magnitude of the reflection parameter drops dramatically after a critical frequency. This critical frequency decreases as h increases. This is intuitive because as the parallel plates get further apart, their cutoff frequency lowers, allowing more energy to propagate in the transverse directions in the parallel plate region. When h is the dominant source of model inaccuracy, it is typically the region of frequencies above this critical frequency that demonstrates the largest error (as demonstrated in fig. 7). This suggests that the MATLAB® implementation of the model has issues coping with transitions from evanescent to propagating parallel plate waveguide modes. Next, fig. 8 illustrates how the choice of permittivity values dramatically affects accuracy.

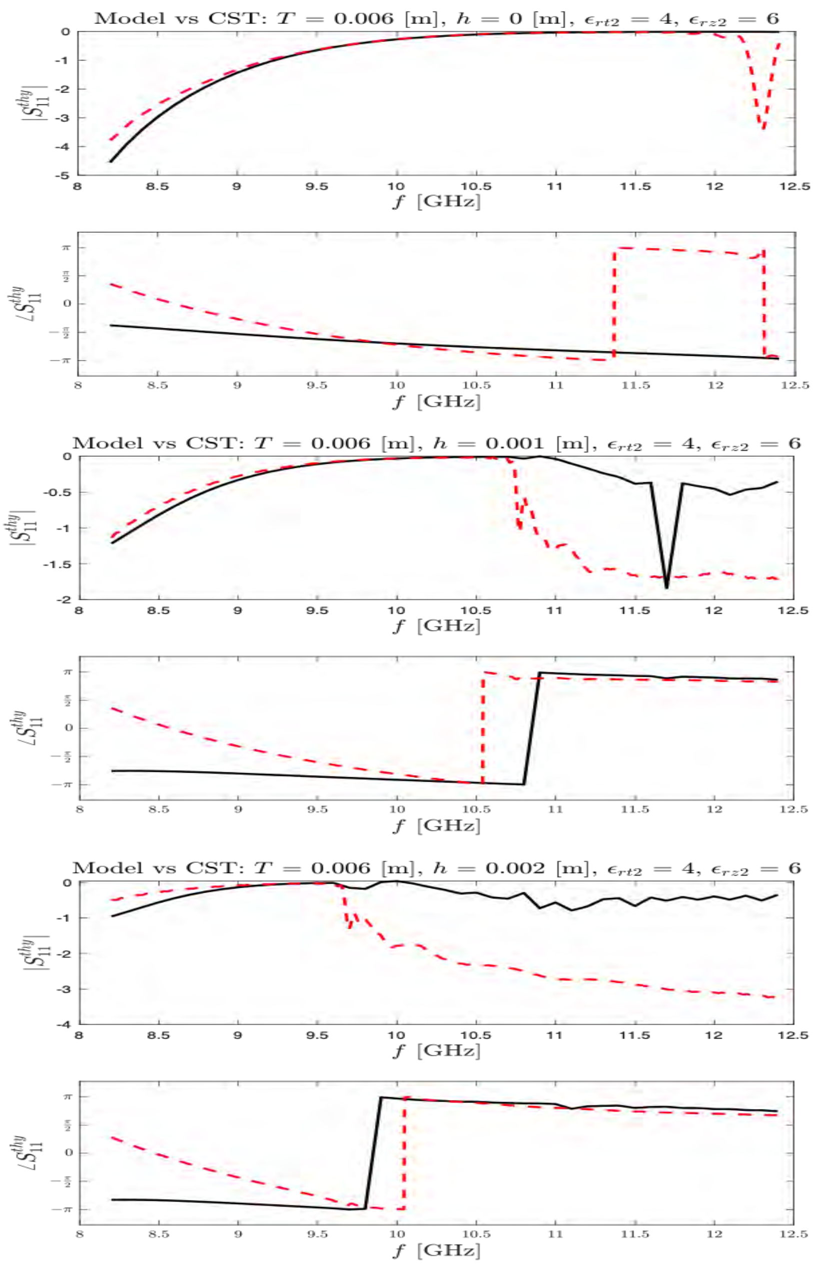

Figure 7. Comparison of two-layer model (solid black) with CST Microwave Studio® (dashed red) with varying h values (top: $h = 0$ mm, middle: $h = 1$ mm, bottom: $h = 2$ mm). All other values are constant ($T = 6$ mm, $\epsilon_{rt2} = 4$, $\epsilon_{rz2} = 6$, $\tan\delta = 0.001$).

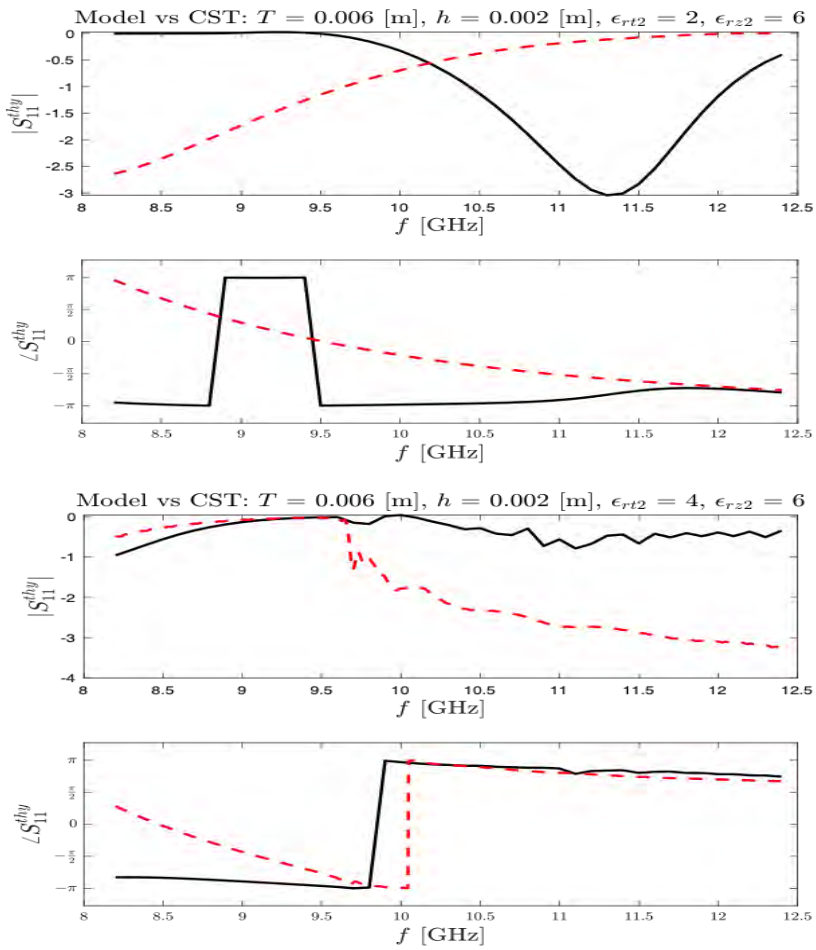

Figure 8. Comparison of two-layer model (solid black) with CST Microwave Studio® (dashed red) with varying ϵ_{rt2} values (top: $\epsilon_{rt2} = 2$, bottom: $\epsilon_{rt2} = 4$). All other values are constant ($T = 6$ mm, $h = 2$ mm, $\epsilon_{rz2} = 6$, $\tan\delta = 0.001$).

It is not immediately apparent why different combinations of permittivity parameters show radically-different accuracy when compared with CST Microwave Studio® results. It is suspected that the infinite numerical λ_x integral may be the source of these errors. In an attempt to trace the problem, several example integrand functions are examined with respect to λ_x. In all cases, functions that integrate to reasonably-accurate results (barring the h considerations previously mentioned) show smooth, monotonic or weakly monotonic (usually increasing) behavior that decays to zero as λ_x gets large. The functions that do not integrate to relatively accurate results exhibit asymptotic discontinuities reminiscent of the tangent function at one or more λ_x values before decaying to zero as λ_x gets large. This is mildly intuitive upon examination of the complicated trigonometric form of the model's kernel functions.

Considerable time and effort is taken to ascertain and mitigate both of these suspected sources of instability in the MATLAB® implementation. Rather than continuing to troubleshoot the issue, it is decided that the effort is better spent exploring a technique that is both computationally simpler and ultimately more stable.

In an effort to characterize the likely effectiveness of the two-layer technique as described above, CST Microwave Studio® simulations are used to qualitatively assess how well the technique would likely perform in practice, assuming a stable MATLAB® realization could be constructed. To accomplish this, two families of curves are produced. In one set, depicted in fig. 9, ϵ_{z2} is kept constant, while a broad range of ϵ_{t2} values are explored. In the other set, depicted in fig. 10, ϵ_{t2} is kept constant, while a broad range of ϵ_{z2} values are explored. To illustrate the role measurement uncertainty plays, a Monte Carlo simulation is performed with 1000 samples taken. The total reflection parameter uncertainty is estimated by

$$u_{S_{11}} = \sigma_{\hat{S}_{11}} \tag{226}$$

$$\hat{S}_{11} = S_{11} + \Delta S_{11} \tag{227}$$

$$\Delta S_{11} = \sqrt{\left(\Delta T \frac{\partial}{\partial T} S_{11}\right)^2 + \left(\Delta h \frac{\partial}{\partial h} S_{11}\right)^2 + (\Delta S_{11}^{\text{ms}})^2} \tag{228}$$

where $\Delta\alpha_i = \bar{\alpha} - \alpha_i$ and α_i is the i^{th} sample from either a uniform or normal random distribution α. In this case, ΔT and Δh are computed from 1000 samples each of uniform distributions around the nominal values of T and $h \pm 0.004$ inch. Simulated values of $\frac{\partial}{\partial T} S_{11}$ and $\frac{\partial}{\partial h} S_{11}$ are provided by CST Microwave Studio®'s sensitivity analysis feature. $\Delta S_{11}^{\text{ms}}$ is computed from 1000 samples of a normal distribution around the nominal value of S_{11} with $\sigma^{\text{ms}}_{S_{11}}$ values for the Agilent E8362B Vector Network Analyzer (VNA) provided by Agilent's uncertainty calculator [1].

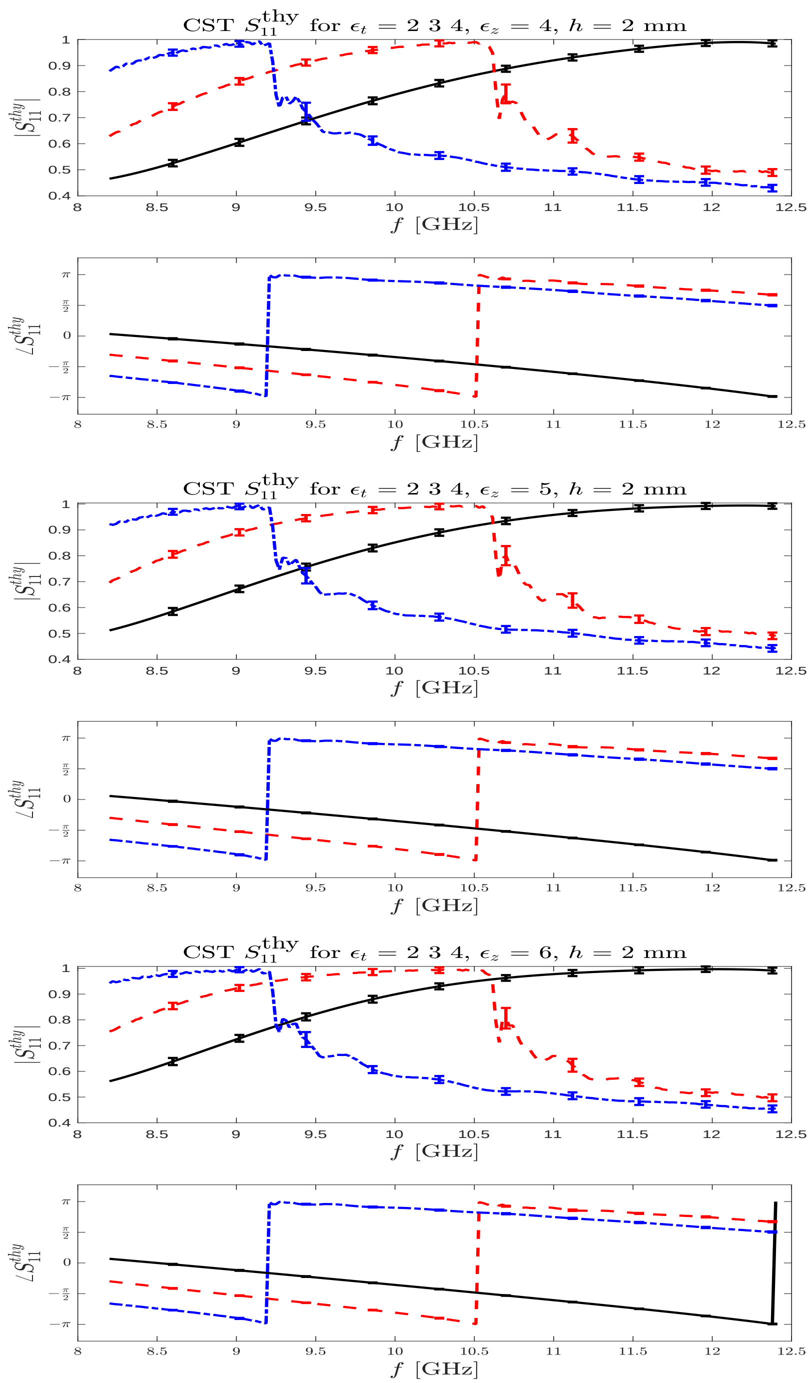

Figure 9. Comparison of two-layer CST Microwave Studio® simulated data with constant ϵ_{z2} and varying $\epsilon_{t2} \in \{2\ \text{(solid black)}, 3\ \text{(dashed red)}, 4\ \text{(dot-dashed blue)}\}$ with $h =$ 2 mm.

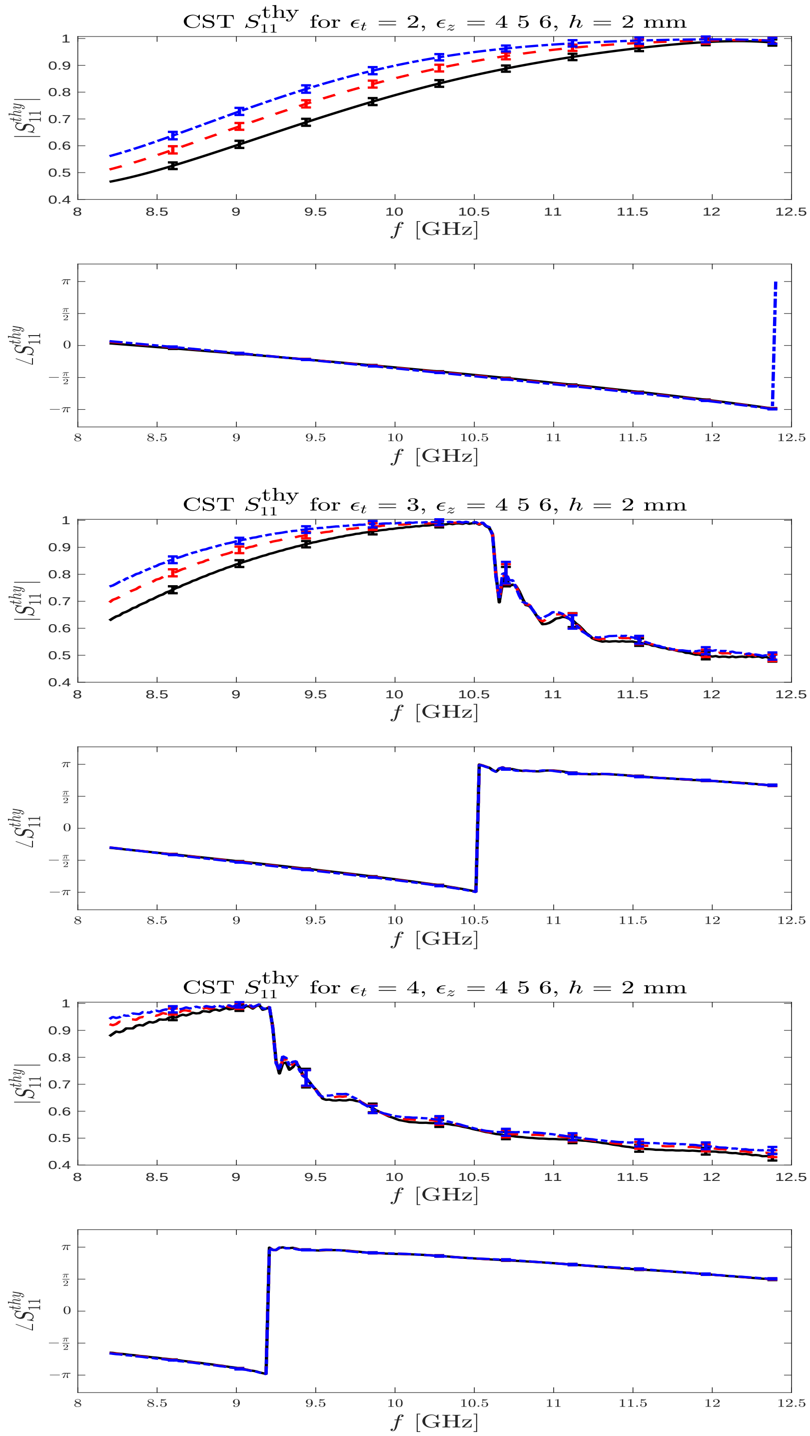

Figure 10. Comparison of two-layer CST Microwave Studio® simulated data with constant ϵ_{t2} **and varying** $\epsilon_{z2} \in \{4 \text{ (solid black)}, 5 \text{ (dashed red)}, 6 \text{ (dot-dashed blue)}\}$ **with** $h =$ **2 mm.**

Ambiguity occurs at frequencies where it would be difficult to tell which value of a given constitutive parameter would have produced the observed reflection parameter value in an inverse problem. This is relatively straightforward to determine visually, as those areas occur where plot lines are very close together or cross over. If the plot lines are close enough together, measurement uncertainty would likely make the extraction impractical to impossible.

Note that when ϵ_{z2} is kept constant, the reflection parameter changes dramatically as ϵ_{t2} changes, having only limited areas of ambiguity. However, when ϵ_{t2} is kept constant, the reflection parameter changes in only minutely-detectable ways, especially when ϵ_{t2} is large. Further, there are large regions of ambiguity, particularly at higher frequencies. To further substantiate this phenomenon, observe the field structure in the MUT at a depth of 0.1 mm below the MUT surface, as depicted in fig. 11. Note that in both cases ($h = 0$ mm and $h = 2$ mm), the maximum values for the electric field in the $\hat{y}$ direction are significantly higher than those in the $\hat{z}$ direction. Thus, $\epsilon_{y2} = \epsilon_{t2}$ is much more strongly implicated in the resulting reflection parameter measurements than ϵ_{z2}. Therefore, it can be concluded that this technique would likely do reasonably well at extracting ϵ_{t2} but would do a comparatively poor job of extracting ϵ_{z2} in an inverse problem, especially for large values of ϵ_{t2}.

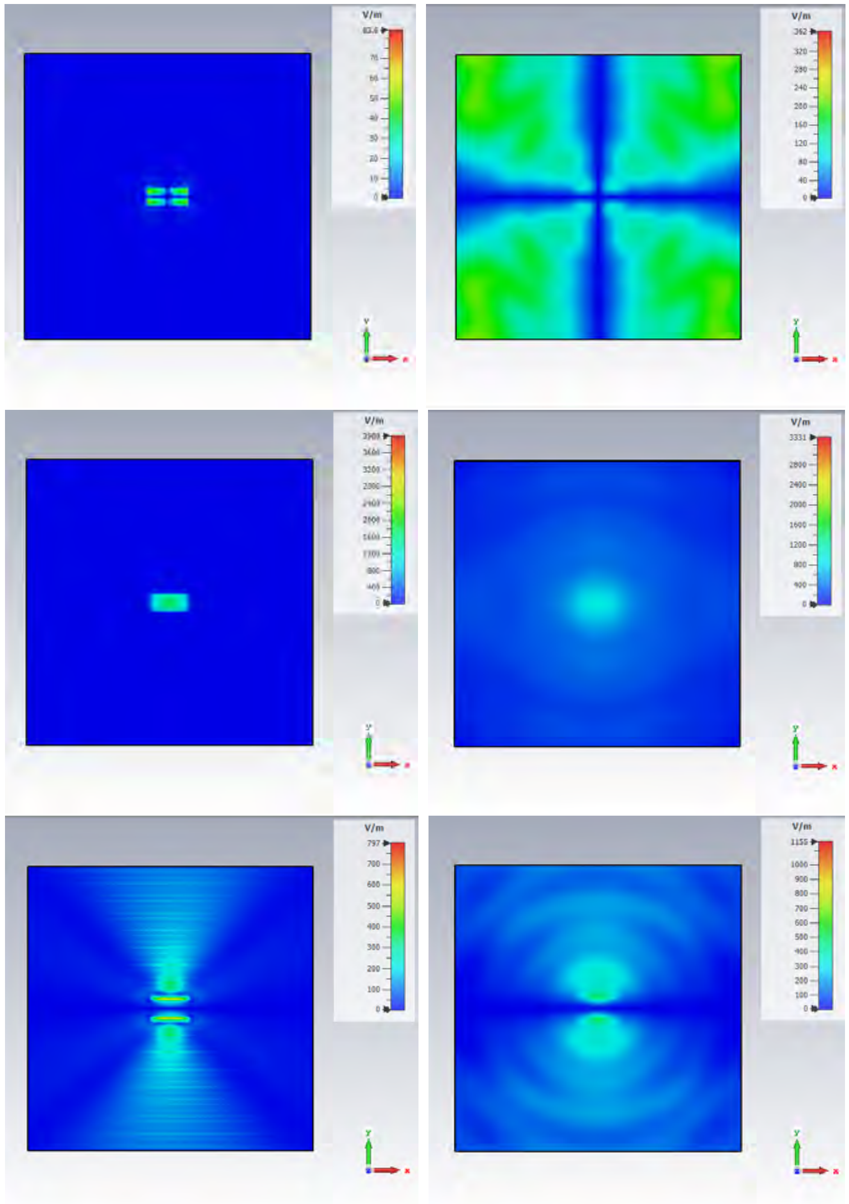

Figure 11. Comparison of two-layer CST Microwave Studio® electric fields maximum values at 0.1 mm below MUT surface. Rows: E_x (top), E_y (middle), E_z (bottom). Columns: $h = 0$ mm (left), $h = 2$ mm (right).

V. Theory of Extraction of Uniaxial Material Parameters Using RARWG Probe Method

In another attempt at exciting an electric field in the $\hat{z}$ direction, this chapter explores using a flanged rectangular waveguide probe that has a reduced aperture region at the flange plate. The aperture is symmetrically reduced in only the y dimension, as depicted in fig. 12. It is hypothesized that the jump discontinuity in the rectangular waveguide structure could excite TM^z modes in the parallel plate region, and thus implicate ϵ_z for extraction. As with the previous technique, a MoM approach is used in concert with the Green functions derived in Chapter II. In order to apply the MoM, there are several steps that must occur including field expansion, applying boundary conditions, applying testing functions, solving the resultant system of equations for the unknown variables.

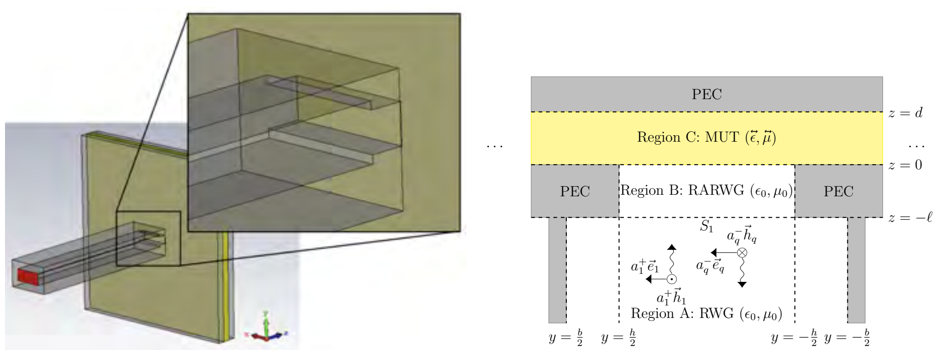

Figure 12. Perspective view (left) and cross section (right) of parallel plate and rectangular waveguide regions. Region A (white) is the rectangular waveguide region of height b filled with free space, Region B (white) is a rectangular waveguide region of height h filled with free space, and Region C (yellow) is the material under test. It is assumed that the flanged region $-\ell < z < d$ extends infinetely in the $\hat{x}$ and $\hat{y}$ directions.

5.1 Field Expansion

Due to excitation symmetry and scattering geometry, the first index (x-variation) is forced to be odd (1, 3, ...) and the second index (y-variation) is forced to be even (0, 2, ...). Furthermore, the second index of TM^z modes is forced to be non-zero.

Thus, it can be shown that fields in regions A and B can be summarized by

$$\left.\begin{aligned}
\vec{E}_t^A &= A_{1,0}^{+,\mathrm{TE}^z}\,\vec{e}_{1,0}^{A,\mathrm{TE}^z}\,e^{-\gamma_{z1,0}^A(z+\ell)} \\
&\quad + \sum_{\substack{n=1\\q=1}}^{N,Q} A_{v_n,w_q}^{-,\mathrm{TE}^z}\,\vec{e}_{v_n,w_q}^{A,\mathrm{TE}^z}\,e^{\gamma_{zv_n,w_q}^A(z+\ell)} + \sum_{\substack{n=1\\q=2}}^{N,Q} A_{v_n,w_q}^{-,\mathrm{TM}^z}\,\vec{e}_{v_n,w_q}^{A,\mathrm{TM}^z}\,e^{\gamma_{zv_n,w_q}^A(z+\ell)} \\
\vec{H}_t^A &= A_{1,0}^{+,\mathrm{TE}^z}\,\vec{h}_{1,0}^{A,\mathrm{TE}^z}\,e^{-\gamma_{z1,0}^A(z+\ell)} \\
&\quad - \sum_{\substack{n=1\\q=1}}^{N,Q} A_{v_n,w_q}^{-,\mathrm{TE}^z}\,\vec{h}_{v_n,w_q}^{A,\mathrm{TE}^z}\,e^{\gamma_{zv_n,w_q}^A(z+\ell)} - \sum_{\substack{n=1\\q=2}}^{N,Q} A_{v_n,w_q}^{-,\mathrm{TM}^z}\,\vec{h}_{v_n,w_q}^{A,\mathrm{TM}^z}\,e^{\gamma_{zv_n,w_q}^A(z+\ell)}
\end{aligned}\right\} z < -\ell \tag{229}$$

$$\left.\begin{aligned}
\vec{E}_t^B &= \sum_{\substack{n=1\\q=1}}^{N,Q} B_{v_n,w_q}^{+,\mathrm{TE}^z}\,\vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}\,e^{-\gamma_{zv_n,w_q}^B(z+\ell)} + \sum_{\substack{n=1\\q=1}}^{N,Q} B_{v_n,w_q}^{-,\mathrm{TE}^z}\,\vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}\,e^{\gamma_{zv_n,w_q}^B(z+\ell)} \\
&\quad + \sum_{\substack{n=1\\q=2}}^{N,Q} B_{v_n,w_q}^{+,\mathrm{TM}^z}\,\vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}\,e^{-\gamma_{zv_n,w_q}^B(z+\ell)} + \sum_{\substack{n=1\\q=2}}^{N,Q} B_{v_n,w_q}^{-,\mathrm{TM}^z}\,\vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}\,e^{\gamma_{zv_n,w_q}^B(z+\ell)} \\
\vec{H}_t^B &= \sum_{\substack{n=1\\q=1}}^{N,Q} B_{v_n,w_q}^{+,\mathrm{TE}^z}\,\vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\,e^{-\gamma_{zv_n,w_q}^B(z+\ell)} - \sum_{\substack{n=1\\q=1}}^{N,Q} B_{v_n,w_q}^{-,\mathrm{TE}^z}\,\vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\,e^{\gamma_{zv_n,w_q}^B(z+\ell)} \\
&\quad + \sum_{\substack{n=1\\q=2}}^{N,Q} B_{v_n,w_q}^{+,\mathrm{TM}^z}\,\vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\,e^{-\gamma_{zv_n,w_q}^B(z+\ell)} - \sum_{\substack{n=1\\q=2}}^{N,Q} B_{v_n,w_q}^{-,\mathrm{TM}^z}\,\vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\,e^{\gamma_{zv_n,w_q}^B(z+\ell)}
\end{aligned}\right\} -\ell < z < 0 \tag{230}$$

where $v_\alpha = 2\alpha - 1$, $w_\alpha = 2\left(\alpha - 1\right)$, and $\alpha \in \mathbb{N}$.

Since the fields in the parallel-plate region are fully described by Green functions, there is no need to account for amplitudes on forward- and reverse-traveling waves. However, unknown amplitudes on the TE^z and TM^z components of those fields must be found. Therefore, the fields in region C can be described by

$$\left.\begin{aligned}\vec{E}^{C} &= \sum_{\substack{n=1\\q=1}}^{N,Q} C^{\mathrm{TE}^z}_{v_n,w_q}\vec{e}^{C,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} C^{\mathrm{TM}^z}_{v_n,w_q}\vec{e}^{C,\mathrm{TM}^z}_{v_n,w_q}\\ \vec{H}^{C} &= \sum_{\substack{n=1\\q=1}}^{N,Q} C^{\mathrm{TE}^z}_{v_n,w_q}\vec{h}^{C,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} C^{\mathrm{TM}^z}_{v_n,w_q}\vec{h}^{C,\mathrm{TM}^z}_{v_n,w_q}\end{aligned}\right\} 0 < z < d \tag{231}$$

Balanis shows in [6] that

$$\begin{aligned}\vec{e}^{A,\mathrm{TE}^z}_{v_n,w_q} = \hat{x}k^{A}_{yw_q}\cos\left(k_{xv_n}\left(x+\frac{a}{2}\right)\right)\sin\left(k_{yw_q}\left(y+\frac{b}{2}\right)\right)\\ -\hat{y}k^{A}_{xv_n}\sin\left(k_{xv_n}\left(x+\frac{a}{2}\right)\right)\cos\left(k_{yw_q}\left(y+\frac{b}{2}\right)\right)\end{aligned} \tag{232}$$

$$k_{xv_n} = \frac{(2n-1)\pi}{a}, k^{A}_{yw_q} = \frac{2(q-1)\pi}{b} \tag{233}$$

$$\Rightarrow \cos\left(k_{xv_n}\left(x+\frac{a}{2}\right)\right) = \cos\left(k_{xv_n}x + n\pi - \frac{\pi}{2}\right)$$

$$= \sin\left(k_{xv_n}x + n\pi\right)$$

$$= (-1)^{n}\sin\left(k_{xv_n}x\right) \tag{234}$$

$$\sin\left(k^{A}_{yw_q}\left(y+\frac{b}{2}\right)\right) = \sin\left(k^{A}_{yw_q}y + (q-1)\pi\right)$$

$$= (-1)^{q-1}\sin\left(k^{A}_{yw_q}y\right) \tag{235}$$

$$\sin\left(k_{xv_n}\left(x+\frac{a}{2}\right)\right) = \sin\left(k_{xv_n}x + \frac{(2n-1)\pi}{2}\right)$$

$$= \sin\left(k_{xv_n}x + n\pi - \frac{\pi}{2}\right)$$

$$= (-1)^{n-1}\cos\left(k_{xv_n}x\right) \tag{236}$$

$$\cos\left(k^{A}_{yw_q}\left(y+\frac{b}{2}\right)\right) = \cos\left(k^{A}_{yw_q}y + (q-1)\pi\right)$$

$$= (-1)^{q-1}\cos\left(k^{A}_{yw_q}y\right) \tag{237}$$

$$\begin{aligned}\Rightarrow \vec{e}^{A,\mathrm{TE}^z}_{v_n,w_q} = \hat{x}k^{A}_{yw_q}(-1)^{n}\sin\left(k_{xv_n}x\right)(-1)^{q-1}\sin\left(k^{A}_{yw_q}y\right)\\ -\hat{y}k_{xv_n}(-1)^{n-1}\cos\left(k_{xv_n}x\right)(-1)^{q-1}\cos\left(k^{A}_{yw_q}y\right)\end{aligned}$$

$$
\begin{aligned}
&= \hat{x}\left(-1\right)^{n+q+1} k^A_{yw_q} \sin\left(k_{xv_n} x\right) \sin\left(k^A_{yw_q} y\right) \\
&\quad + \hat{y}\left(-1\right)^{n+q+1} k_{xv_n} \cos\left(k_{xv_n} x\right) \cos\left(k^A_{yw_q} y\right)
\end{aligned}
\tag{238}
$$

Again, Balanis shows in [6] that

$$
\begin{aligned}
\vec{e}^{A,\mathrm{TM}^z}_{v_n,w_q} &= \hat{x} k_{xv_n} \cos\left(k_{xv_n}\left(x+\frac{a}{2}\right)\right) \sin\left(k^A_{yw_q}\left(y+\frac{b}{2}\right)\right) \\
&\quad + \hat{y} k^A_{yw_q} \sin\left(k_{xv_n}\left(x+\frac{a}{2}\right)\right) \cos\left(k^A_{yw_q}\left(y+\frac{b}{2}\right)\right) \\
&= \hat{x} k_{xv_n}\left(-1\right)^{n+q+1} \sin\left(k_{xv_n} x\right) \sin\left(k^A_{yw_q} y\right) \\
&\quad + \hat{y} k^A_{yw_q}\left(-1\right)^{n+q} \cos\left(k_{xv_n} x\right) \cos\left(k^A_{yw_q} y\right)
\end{aligned}
\tag{239}
$$

Summary Region A

$$
\boxed{
\begin{aligned}
&\vec{e}^{A,\mathrm{TE}^z}_{v_n,w_q} = \hat{x}\left(-1\right)^{n+q+1} k^A_{yw_q} \sin\left(k_{xv_n} x\right) \sin\left(k^A_{yw_q} y\right) \\
&\qquad + \hat{y}\left(-1\right)^{n+q+1} k_{xv_n} \cos\left(k_{xv_n} x\right) \cos\left(k^A_{yw_q} y\right) \\
&\vec{e}^{A,\mathrm{TM}^z}_{v_n,w_q} = \hat{x} k_{xv_n}\left(-1\right)^{n+q+1} \sin\left(k_{xv_n} x\right) \sin\left(k^A_{yw_q} y\right) \\
&\qquad + \hat{y} k^A_{yw_q}\left(-1\right)^{n+q} \cos\left(k_{xv_n} x\right) \cos\left(k^A_{yw_q} y\right) \\
&\vec{h}^{A,\mathrm{TE}^z}_{v_n,w_q} = \frac{\hat{z}\times\vec{e}^{A,\mathrm{TE}^z}_{v_n,w_q}}{Z^{A,\mathrm{TE}^z}_{v_n,w_q}}, \vec{h}^{A,\mathrm{TM}^z}_{v_n,w_q} = \frac{\hat{z}\times\vec{e}^{A,\mathrm{TM}^z}_{v_n,w_q}}{Z^{A,\mathrm{TM}^z}_{v_n,w_q}} \\
&Z^{A,\mathrm{TE}^z}_{v_n,w_q} = \frac{j\omega\mu_0}{\gamma^A_{zv_n,w_q}}, Z^{A,\mathrm{TM}^z}_{v_n,w_q} = \frac{\gamma^A_{zv_n,w_q}}{j\omega\epsilon_0} \\
&k_{xv_n} = \frac{\left(2n-1\right)\pi}{a}, k^A_{yw_q} = \frac{2\left(q-1\right)\pi}{b}, \gamma^A_{zv_n,w_q} = \sqrt{k^2_{xv_n}+k^{A2}_{yw_q}-k^2_0} \\
&v_\alpha = 2\alpha-1, w_\alpha = 2\left(\alpha-1\right), \alpha\in\mathbb{N}
\end{aligned}
}
\tag{240}
$$

Summary Region B

$$\boxed{\begin{aligned}
\vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q} &= \hat{x}\left(-1\right)^{n+q+1} k^B_{yw_q} \sin\left(k_{xv_n} x\right) \sin\left(k^B_{yw_q} y\right) \\
&\quad + \hat{y}\left(-1\right)^{n+q+1} k_{xv_n} \cos\left(k_{xv_n} x\right) \cos\left(k^B_{yw_q} y\right) \\
\vec{e}^{B,\mathrm{TM}^z}_{v_n,w_q} &= \hat{x} k_{xv_n} \left(-1\right)^{n+q+1} \sin\left(k_{xv_n} x\right) \sin\left(k^B_{yw_q} y\right) \\
&\quad + \hat{y} k^B_{yw_q} \left(-1\right)^{n+q} \cos\left(k_{xv_n} x\right) \cos\left(k^B_{yw_q} y\right) \\
\vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} &= \frac{\hat{z} \times \vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q}}{Z^{B,\mathrm{TE}^z}_{v_n,w_q}}, \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} = \frac{\hat{z} \times \vec{e}^{B,\mathrm{TM}^z}_{v_n,w_q}}{Z^{B,\mathrm{TM}^z}_{v_n,w_q}} \\
Z^{B,\mathrm{TE}^z}_{v_n,w_q} &= \frac{j\omega\mu_0}{\gamma^B_{zv_n,w_q}}, Z^{B,\mathrm{TM}^z}_{v_n,w_q} = \frac{\gamma^B_{zv_n,w_q}}{j\omega\epsilon_0} \\
k_{xv_n} &= \frac{\left(2n-1\right)\pi}{a}, k^B_{yw_q} = \frac{2\left(q-1\right)\pi}{h}, \gamma^B_{zv_n,w_q} = \sqrt{k^2_{xv_n} + k^{B2}_{yw_q} - k^2_B} \\
v_\alpha &= 2\alpha - 1, w_\alpha = 2\left(\alpha - 1\right), \alpha \in \mathbb{N}
\end{aligned}} \tag{241}$$

The fields in region C are sustained by the Love's-equivalent transverse magnetic current in the aperture between regions B and C and are determined via the Green functions provided by Rogers in [71] and verified in this work. Thus,

$$\begin{aligned}
\vec{J}_{ht} &= -\hat{n} \times \vec{E}^C_t\left(z = 0^+\right) = -\hat{z} \times \left(\vec{E}^{C,\mathrm{TE}^z}_t\left(z = 0^+\right) + \vec{E}^{C,\mathrm{TM}^z}_t\left(z = 0^+\right)\right) \\
&= -\hat{z} \times \vec{e}_a
\end{aligned} \tag{242}$$

where $\vec{e}_a$ refers to the electric field in the aperture. Recalling that the Green functions developed earlier are in the transverse-spectral domain $\left(\vec{\lambda}_\rho, z\right)$ implies that

$$\begin{aligned}
\vec{H}^C\left(\vec{\rho}, z\right) &= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \tilde{\vec{H}}^C e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_0^d \tilde{\overleftrightarrow{G}}_{hh} \cdot \tilde{\vec{J}}_h dz'\right] e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_0^d \tilde{\overleftrightarrow{G}}_{hh} \cdot \left(\iint\limits_S \vec{J}_h e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} d{\rho'}^2 \right) dz' \right] e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{V'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \vec{J}_h e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dV' \right] d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{V'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \vec{J}_{ht} \delta\left(z' - z\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dV' \right] d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \vec{J}_{ht} e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho \\
&= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \left(-\hat{z} \times \vec{e}_a\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho \qquad (243)
\end{aligned}
$$

where S' refers to the closed aperture surface defined by $-\frac{a}{2} < x' < \frac{a}{2}, -\frac{h}{2} < y' < \frac{h}{2}$.

Note that in the aperture, the fields must behave as they would in a rectangular waveguide filled with the same material as region B. Thus the expanded field relations $\vec{e}_t^{B,\mathrm{TE}^z} = \vec{e}_t^{C,\mathrm{TE}^z}$, $\vec{e}_t^{B,\mathrm{TM}^z} = \vec{e}_t^{C,\mathrm{TM}^z}$, $\vec{h}_t^{B,\mathrm{TE}^z} = \vec{h}_t^{C,\mathrm{TE}^z}$, and $\vec{h}_t^{B,\mathrm{TM}^z} = \vec{h}_t^{C,\mathrm{TM}^z}$ must hold true. Therefore, the expanded aperture electric field can be written as

$$
\vec{e}_a = \sum_{\substack{n=1 \\ q=1}}^{N,Q} C_{v_n,w_q}^{\mathrm{TE}^z} \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} + \sum_{\substack{n=1 \\ q=2}}^{N,Q} C_{v_n,w_q}^{\mathrm{TM}^z} \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} \qquad (244)
$$

Thus,

$$
\vec{H}^C\left(\vec{\rho}, z\right) = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \left(-\hat{z} \times \vec{e}_a\right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho
$$

$$
\begin{aligned}
&= -\sum_{\substack{n=1\\q=1}}^{N,Q} C_{v_n,w_q}^{\mathrm{TE}^z} \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \left(\hat{z} \times \vec{e}_{v_n,w_q}^{\,B,\mathrm{TE}^z} \right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho \\
&\quad - \sum_{\substack{n=1\\q=2}}^{N,Q} C_{v_n,w_q}^{\mathrm{TM}^z} \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \left(\hat{z} \times \vec{e}_{v_n,w_q}^{\,B,\mathrm{TM}^z} \right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho \\
&= -\sum_{\substack{n=1\\q=1}}^{N,Q} C_{v_n,w_q}^{\mathrm{TE}^z} \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \vec{h}_{v_n,w_q}^{\,B,\mathrm{TE}^z} \left(\vec{\rho}' \right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho \\
&\quad - \sum_{\substack{n=1\\q=2}}^{N,Q} C_{v_n,w_q}^{\mathrm{TM}^z} \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_{S'} \tilde{\overleftrightarrow{G}}_{hh} \cdot \vec{h}_{v_n,w_q}^{\,B,\mathrm{TM}^z} \left(\vec{\rho}' \right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho
\end{aligned} \tag{245}
$$

In similar fashion, it can be shown that

$$
\begin{aligned}
\vec{E}^C \left(\vec{\rho}, z \right) = &-\sum_{\substack{n=1\\q=1}}^{N,Q} C_{v_n,w_q}^{\mathrm{TE}^z} \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_{S'} \tilde{\overleftrightarrow{G}}_{eh} \cdot \vec{h}_{v_n,w_q}^{\,B,\mathrm{TE}^z} \left(\vec{\rho}' \right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho \\
&- \sum_{\substack{n=1\\q=2}}^{N,Q} C_{v_n,w_q}^{\mathrm{TM}^z} \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_{S'} \tilde{\overleftrightarrow{G}}_{eh} \cdot \vec{h}_{v_n,w_q}^{\,B,\mathrm{TM}^z} \left(\vec{\rho}' \right) e^{j\vec{\lambda}_\rho \cdot (\vec{\rho}-\vec{\rho}')} dS' \right] d^2\lambda_\rho
\end{aligned} \tag{246}
$$

5.2 Application of Boundary Conditions

Boundary Condition 1: $\vec{E}_t^A \left(z = -\ell^- \right) = \begin{cases} 0 \ldots \, |y| > \frac{h}{2} \\ \vec{E}_t^B \left(z = -\ell^+ \right) \ldots \, |y| < \frac{h}{2} \end{cases}$

$$
\begin{aligned}
&\Rightarrow A_{1,0}^{+,\mathrm{TE}^z} \vec{e}_{1,0}^{\,A,\mathrm{TE}^z} + \sum_{\substack{n=1\\q=1}}^{N,Q} A_{v_n,w_q}^{-,\mathrm{TE}^z} \vec{e}_{v_n,w_q}^{\,A,\mathrm{TE}^z} + \sum_{\substack{n=1\\q=2}}^{N,Q} A_{v_n,w_q}^{-,\mathrm{TM}^z} \vec{e}_{v_n,w_q}^{\,A,\mathrm{TM}^z} \\
&= \begin{cases} 0 \ldots \, |y| > \frac{h}{2} \\ \sum_{\substack{n=1\\q=1}}^{N,Q} B_{v_n,w_q}^{+,\mathrm{TE}^z} \vec{e}_{v_n,w_q}^{\,B,\mathrm{TE}^z} + \sum_{\substack{n=1\\q=1}}^{N,Q} B_{v_n,w_q}^{-,\mathrm{TE}^z} \vec{e}_{v_n,w_q}^{\,B,\mathrm{TE}^z} \\ + \sum_{\substack{n=1\\q=2}}^{N,Q} B_{v_n,w_q}^{+,\mathrm{TM}^z} \vec{e}_{v_n,w_q}^{\,B,\mathrm{TM}^z} + \sum_{\substack{n=1\\q=2}}^{N,Q} B_{v_n,w_q}^{-,\mathrm{TM}^z} \vec{e}_{v_n,w_q}^{\,B,\mathrm{TM}^z} \ldots \, |y| < \frac{h}{2} \end{cases}
\end{aligned} \tag{247}
$$

To aid readability, define reflection and through parameters in relation to the initial excitation wave such that

$$\begin{aligned}
R^{A,\mathrm{TE}^z}_{v_n,w_q} &= \frac{A^{-,\mathrm{TE}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} & R^{A,\mathrm{TM}^z}_{v_n,w_q} &= \frac{A^{-,\mathrm{TM}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} \\
T^{B,\mathrm{TE}^z}_{v_n,w_q} &= \frac{B^{+,\mathrm{TE}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} & T^{B,\mathrm{TM}^z}_{v_n,w_q} &= \frac{B^{+,\mathrm{TM}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} \\
R^{B,\mathrm{TE}^z}_{v_n,w_q} &= \frac{B^{-,\mathrm{TE}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} & R^{B,\mathrm{TM}^z}_{v_n,w_q} &= \frac{B^{-,\mathrm{TM}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} \\
T^{C,\mathrm{TE}^z}_{v_n,w_q} &= \frac{C^{\mathrm{TE}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}} & T^{C,\mathrm{TM}^z}_{v_n,w_q} &= \frac{C^{\mathrm{TM}^z}_{v_n,w_q}}{A^{+,\mathrm{TE}^z}_{1,0}}
\end{aligned} \tag{248}$$

This implies that

$$\begin{aligned}
&\vec{e}^{A,\mathrm{TE}^z}_{1,0} + \sum_{\substack{n=1\\q=1}}^{N,Q} R^{A,\mathrm{TE}^z}_{v_n,w_q}\vec{e}^{A,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} R^{A,\mathrm{TM}^z}_{v_n,w_q}\vec{e}^{A,\mathrm{TM}^z}_{v_n,w_q} \\
&= \begin{cases} 0 \ldots |y| > \frac{h}{2} \\ \sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q}\vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q}\vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q} \\ + \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q}\vec{e}^{B,\mathrm{TM}^z}_{v_n,w_q} + \sum_{\substack{n=1\\p=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q}\vec{e}^{B,\mathrm{TM}^z}_{v_n,w_q} \ldots |y| < \frac{h}{2} \end{cases}
\end{aligned} \tag{249}$$

Boundary Condition 2: $\vec{H}^A_t\left(z=-\ell^-\right) = \vec{H}^B_t\left(z=-\ell^+\right) \ldots |y| < \frac{h}{2}$

$$\begin{aligned}
\Rightarrow \vec{h}^{A,\mathrm{TE}^z}_{1,0} - \sum_{\substack{n=1\\q=1}}^{N,Q} R^{A,\mathrm{TE}^z}_{v_n,w_q}\vec{h}^{A,\mathrm{TE}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=2}}^{N,Q} R^{A,\mathrm{TM}^z}_{v_n,w_q}\vec{h}^{A,\mathrm{TM}^z}_{v_n,w_q} \\
= \sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q}\vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q}\vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} \\
+ \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q}\vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q}\vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} \ldots |y| < \frac{h}{2}
\end{aligned} \tag{250}$$

Boundary Condition 3: $\vec{H}_t^C\left(z=0^+\right)=\vec{H}_t^B\left(z=0^-\right)...|y|<\frac{h}{2}$

$\overleftrightarrow{I}_t\vec{H}^C$ is used in place of $\vec{H}_t^C$ for application of this boundary condition. When the testing operator is applied later, it will absorb the $\overleftrightarrow{I}_t$ component. Therefore, $\vec{H}_t^B\left(z=0^-\right)=\overleftrightarrow{I}_t\vec{H}^C\left(z=0^+\right)$

$$\begin{aligned}
&\Rightarrow \sum_{\substack{n=1\\q=1}}^{N,Q} T_{v_n,w_q}^{B,\mathrm{TE}^z}\vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}e^{-\gamma_{zv_n,w_q}^B\ell}-\sum_{\substack{n=1\\q=1}}^{N,Q} R_{v_n,w_q}^{B,\mathrm{TE}^z}\vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}e^{\gamma_{zv_n,w_q}^B\ell}\\
&\qquad+\sum_{\substack{n=1\\q=2}}^{N,Q} T_{v_n,w_q}^{B,\mathrm{TM}^z}\vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}e^{-\gamma_{zv_n,w_q}^B\ell}-\sum_{\substack{n=1\\q=2}}^{N,Q} R_{v_n,w_q}^{B,\mathrm{TM}^z}\vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}e^{\gamma_{zv_n,w_q}^B\ell}\\
&=-\sum_{\substack{n=1\\q=1}}^{N,Q} T_{v_n,w_q}^{C,\mathrm{TE}^z}\frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left[\int\limits_{S'}\overleftrightarrow{I}_t\tilde{\overleftrightarrow{G}}_{hh}\left(z=z'=0\right)\cdot\vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\left(\vec{\rho}'\right)e^{j\vec{\lambda}_\rho\cdot\left(\vec{\rho}-\vec{\rho}'\right)}dS'\right]d^2\lambda_\rho\\
&\quad-\sum_{\substack{n=1\\q=2}}^{N,Q} T_{v_n,w_q}^{C,\mathrm{TM}^z}\frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left[\int\limits_{S'}\overleftrightarrow{I}_t\tilde{\overleftrightarrow{G}}_{hh}\left(z=z'=0\right)\cdot\vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\left(\vec{\rho}'\right)e^{j\vec{\lambda}_\rho\cdot\left(\vec{\rho}-\vec{\rho}'\right)}dS'\right]d^2\lambda_\rho
\end{aligned}\tag{251}$$

Boundary Condition 4: $\vec{E}_t^C\left(z=0^+\right)=\begin{cases}0...|y|>\frac{h}{2}\\ \vec{E}_t^B\left(z=0^-\right)...|y|<\frac{h}{2}\end{cases}$

In a similar fashion to Boundary Condition 3, it can be shown that

$$\begin{aligned}
&\sum\nolimits_{\substack{n=1\\q=1}}^{N,Q} T_{v_n,w_q}^{B,\mathrm{TE}^z}\vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}e^{-\gamma_{zv_n,w_q}^B\ell}+\sum\nolimits_{\substack{n=1\\q=1}}^{N,Q} R_{v_n,w_q}^{B,\mathrm{TE}^z}\vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}e^{\gamma_{zv_n,w_q}^B\ell}\\
&\qquad+\sum\nolimits_{\substack{n=1\\q=2}}^{N,Q} T_{v_n,w_q}^{B,\mathrm{TM}^z}\vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}e^{-\gamma_{zv_n,w_q}^B\ell}+\sum\nolimits_{\substack{n=1\\q=2}}^{N,Q} R_{v_n,w_q}^{B,\mathrm{TM}^z}\vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}e^{\gamma_{zv_n,w_q}^B\ell}\\
&=-\sum_{\substack{n=1\\q=1}}^{N,Q} T_{v_n,w_q}^{C,\mathrm{TE}^z}\frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left[\int\limits_{S'}\overleftrightarrow{I}_t\tilde{\overleftrightarrow{G}}_{eh}\left(z=z'=0\right)\cdot\vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\left(\vec{\rho}'\right)e^{j\vec{\lambda}_\rho\cdot\left(\vec{\rho}-\vec{\rho}'\right)}dS'\right]d^2\lambda_\rho\\
&\quad-\sum_{\substack{n=1\\q=2}}^{N,Q} T_{v_n,w_q}^{C,\mathrm{TM}^z}\frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left[\int\limits_{S'}\overleftrightarrow{I}_t\tilde{\overleftrightarrow{G}}_{eh}\left(z=z'=0\right)\cdot\vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\left(\vec{\rho}'\right)e^{j\vec{\lambda}_\rho\cdot\left(\vec{\rho}-\vec{\rho}'\right)}dS'\right]d^2\lambda_\rho
\end{aligned}\tag{252}$$

5.3 Application of Testing Operators

Applying the testing operator $\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \{\}\, dS$ to (249) implies that

$$
\begin{aligned}
&\overbrace{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TE}^z}_{1,0}\, dS}^{\bar{A}^{(m1)}_{p1}} + \sum_{\substack{n=1\\q=1}}^{N,Q} R^{A,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{A}^{(mn)}_{pq}} \\
&+ \sum_{\substack{n=1\\q=2}}^{N,Q} R^{A,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{B}^{(mn)}_{pq}} = \sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{C}^{(mn)}_{pq}} \\
&+ \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{C}^{(mn)}_{pq}} + \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TM}^z}_{v_n,w_q}\, dS} \\
&\qquad + \sum_{\substack{n=1\\p=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{e}^{\,B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TM}^z}_{v_n,w_q}\, dS}
\end{aligned}
\tag{253}
$$

which implies that

$$
\boxed{
\begin{aligned}
&\sum_{\substack{n=1\\q=1}}^{N,Q} \bar{A}^{(mn)}_{pq} R^{A,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{B}^{(mn)}_{pq} R^{A,\mathrm{TM}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{C}^{(mn)}_{pq} T^{B,\mathrm{TE}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{C}^{(mn)}_{pq} R^{B,\mathrm{TE}^z}_{v_n,w_q} \\
&= -\bar{A}^{(m1)}_{p1} ... m = 1, ..., N; p = 1, ..., Q
\end{aligned}
}
\tag{254}
$$

Applying the testing operator $\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \{\}\, dS$ to (249) implies that

$$
\begin{aligned}
&\overbrace{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TE}^z}_{1,0}\, dS}^{\bar{D}^{(m1)}_{p1}} + \sum_{\substack{n=1\\q=1}}^{N,Q} R^{A,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{D}^{(mn)}_{pq}} \\
&+ \sum_{\substack{n=1\\q=2}}^{N,Q} R^{A,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{E}^{(mn)}_{pq}} = \sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \cancelto{0}{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TE}^z}_{v_n,w_q}\, dS} \\
&+ \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \cancelto{0}{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TE}^z}_{v_n,w_q}\, dS} + \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{F}^{(mn)}_{pq}} \\
&\qquad\qquad + \sum_{\substack{n=1\\p=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{F}^{(mn)}_{pq}}
\end{aligned}
\tag{255}
$$

which implies that

$$
\boxed{
\begin{aligned}
&\sum_{\substack{n=1\\q=1}}^{N,Q} \bar{D}^{(mn)}_{pq} R^{A,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{E}^{(mn)}_{pq} R^{A,\mathrm{TM}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{F}^{(mn)}_{pq} T^{B,\mathrm{TM}^z}_{v_n,w_q} - \sum_{\substack{n=1\\p=2}}^{N,Q} \bar{F}^{(mn)}_{pq} R^{B,\mathrm{TM}^z}_{v_n,w_q} \\
&= -\bar{D}^{(m1)}_{p1} ... m = 1, ..., N; p = 2, ..., Q
\end{aligned}
}
\tag{256}
$$

Applying the testing operator $\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \{\}\, dS$ to (250) implies that

$$
\begin{aligned}
&\overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TE}^z}_{1,0}\, dS}^{\bar{G}^{(m1)}_{p1}} - \sum_{\substack{n=1\\q=1}}^{N,Q} R^{A,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{G}^{(mn)}_{pq}} \\
&- \sum_{\substack{n=1\\q=2}}^{N,Q} R^{A,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{H}^{(mn)}_{pq}} = \sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{I}^{(mn)}_{pq}} \\
&- \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{I}^{(mn)}_{pq}} + \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q}\, dS} \\
&\qquad\qquad - \sum_{\substack{n=1\\q=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q}\, dS}
\end{aligned}
\tag{257}
$$

which implies that

$$
\boxed{
\begin{aligned}
&\sum_{\substack{n=1\\q=1}}^{N,Q} \bar{G}^{(mn)}_{pq} R^{A,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{H}^{(mn)}_{pq} R^{A,\mathrm{TM}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{I}^{(mn)}_{pq} T^{B,\mathrm{TE}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{I}^{(mn)}_{pq} R^{B,\mathrm{TE}^z}_{v_n,w_q} \\
&= \bar{G}^{(m1)}_{p1} \ldots m = 1, \ldots, N; p = 1, \ldots, Q
\end{aligned}
}
\tag{258}
$$

Applying the testing operator $\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \{\}\, dS$ to (250) implies that

$$\begin{aligned}
&\overbrace{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TE}^z}_{1,0}\, dS}^{\bar{J}^{(m1)}_{p1}} - \sum_{\substack{n=1\\q=1}}^{N,Q} R^{A,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{J}^{(mn)}_{pq}} \\
&- \sum_{\substack{n=1\\q=2}}^{N,Q} R^{A,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{K}^{(mn)}_{pq}} = \sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\, dS} \\
&- \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\, dS} + \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{L}^{(mn)}_{pq}} \\
&\qquad\qquad - \sum_{\substack{n=1\\q=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{L}^{(mn)}_{pq}}
\end{aligned} \tag{259}$$

which implies that

$$\boxed{\begin{aligned}
&\sum_{\substack{n=1\\q=1}}^{N,Q} \bar{J}^{(mn)}_{pq} R^{A,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{K}^{(mn)}_{pq} R^{A,\mathrm{TM}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{L}^{(mn)}_{pq} T^{B,\mathrm{TM}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{L}^{(mn)}_{pq} R^{B,\mathrm{TM}^z}_{v_n,w_q} \\
&= \bar{J}^{(m1)}_{p1} ... m = 1, ..., N; p = 2, ..., Q
\end{aligned}} \tag{260}$$

Applying the testing operator $\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \{\}\, dS$ to (251) implies that

$$\sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{-\gamma^B_{zv_n,w_q}\ell} dS}^{\bar{M}^{(mn)}_{pq}} - \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{\gamma^B_{zv_n,w_q}\ell} dS}^{\bar{N}^{(mn)}_{pq}}$$

$$+ \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} e^{-\gamma^B_{zv_n,w_q}\ell} dS}$$

$$- \sum_{\substack{n=1\\q=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \cancelto{0}{\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} e^{\gamma^B_{zv_n,w_q}\ell} dS} = -\sum_{\substack{n=1\\q=1}}^{N,Q} T^{C,\mathrm{TE}^z}_{v_n,w_q} \bar{O}^{(mn)}_{pq} - \sum_{\substack{n=1\\q=2}}^{N,Q} T^{C,\mathrm{TM}^z}_{v_n,w_q} \bar{P}^{(mn)}_{pq} \tag{261}$$

$$\bar{O}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TE}^z}_{v_n,w_q}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\substack{z=0\\z'=0}\right) \cdot \int\limits_{S'} \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{262}$$

$$\bar{P}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TM}^z}_{v_n,w_q}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\substack{z=0\\z'=0}\right) \cdot \int\limits_{S'} \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{263}$$

which implies that

$$\boxed{\begin{aligned} &\sum_{\substack{n=1\\q=1}}^{N,Q} \bar{M}^{(mn)}_{pq} T^{B,\mathrm{TE}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{N}^{(mn)}_{pq} R^{B,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{O}^{(mn)}_{pq} T^{C,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{P}^{(mn)}_{pq} T^{C,\mathrm{TM}^z}_{v_n,w_q} \\ &= 0 ... m = 1, ..., N; p = 1, ..., Q \end{aligned}} \tag{264}$$

Applying the testing operator $\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \{\}\, dS$ to (251) implies that

$$\begin{aligned}
&\sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \cancelto{0}{\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{-\gamma^B_{zv_n,w_q}\ell} dS} - \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \cancelto{0}{\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{\gamma^B_{zv_n,w_q}\ell} dS} \\
&+ \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} e^{-\gamma^B_{zv_n,w_q}\ell} dS}^{\bar{Q}^{(mn)}_{pq}} \\
&- \sum_{\substack{n=1\\q=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} e^{\gamma^B_{zv_n,w_q}\ell} dS}^{\bar{R}^{(mn)}_{pq}} \\
&= -\sum_{\substack{n=1\\q=1}}^{N,Q} T^{C,\mathrm{TE}^z}_{v_n,w_q} \bar{S}^{(mn)}_{pq} - \sum_{\substack{n=1\\q=2}}^{N,Q} T^{C,\mathrm{TM}^z}_{v_n,w_q} \bar{T}^{(mn)}_{pq}
\end{aligned} \tag{265}$$

$$\bar{S}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TE}^z}_{v_n,w_q}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\z'=0}\right) \cdot \int_{S'} \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{266}$$

$$\bar{T}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TM}^z}_{v_n,w_q}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}^{B,\mathrm{TM}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\z'=0}\right) \cdot \int_{S'} \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q}\left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{267}$$

which implies that

$$\boxed{\begin{aligned}
&\sum_{\substack{n=1\\q=2}}^{N,Q} \bar{Q}^{(mn)}_{pq} T^{B,\mathrm{TM}^z}_{v_n,w_q} - \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{R}^{(mn)}_{pq} R^{B,\mathrm{TM}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{S}^{(mn)}_{pq} T^{C,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{T}^{(mn)}_{pq} T^{C,\mathrm{TM}^z}_{v_n,w_q} \\
&= 0 ... m = 1, ..., N; p = 2, ..., Q
\end{aligned}} \tag{268}$$

Applying the testing operator $\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \{\}\, dS$ to (252) implies that

$$\sum_{\substack{n=1\\q=1}}^{N,Q} T_{v_n,w_q}^{B,\mathrm{TE}^z} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} e^{-\gamma_{zv_n,w_q}^B \ell} dS}^{\bar{U}_{pq}^{(mn)}} + \sum_{\substack{n=1\\q=1}}^{N,Q} R_{v_n,w_q}^{B,\mathrm{TE}^z} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} e^{\gamma_{zv_n,w_q}^B \ell} dS}^{\bar{V}_{pq}^{(mn)}}$$
$$+ \sum_{\substack{n=1\\q=2}}^{N,Q} T_{v_n,w_q}^{B,\mathrm{TM}^z} \cancelto{0}{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} e^{-\gamma_{zv_n,w_q}^B \ell} dS} + \sum_{\substack{n=1\\q=2}}^{N,Q} R_{v_n,w_q}^{B,\mathrm{TM}^z} \cancelto{0}{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} e^{\gamma_{zv_n,w_q}^B \ell} dS}$$
$$= -\sum_{\substack{n=1\\q=1}}^{N,Q} T_{v_n,w_q}^{C,\mathrm{TE}^z} \bar{W}_{pq}^{(mn)} - \sum_{\substack{n=1\\q=2}}^{N,Q} T_{v_n,w_q}^{C,\mathrm{TM}^z} \bar{X}_{pq}^{(mn)} \tag{269}$$

$$\bar{W}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0\\ z'=0 \end{smallmatrix}\right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{270}$$

$$\bar{X}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0\\ z'=0 \end{smallmatrix}\right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{271}$$

which implies that

$$\boxed{\begin{aligned} &\sum_{\substack{n=1\\q=1}}^{N,Q} \bar{U}_{pq}^{(mn)} T_{v_n,w_q}^{B,\mathrm{TE}^z} + \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{V}_{pq}^{(mn)} R_{v_n,w_q}^{B,\mathrm{TE}^z} + \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{W}_{pq}^{(mn)} T_{v_n,w_q}^{C,\mathrm{TE}^z} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{X}_{pq}^{(mn)} T_{v_n,w_q}^{C,\mathrm{TM}^z} \\ &= 0 ... m = 1, ..., N; p = 1, ..., Q \end{aligned}} \tag{272}$$

Finally, applying the testing operator $\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \{\} \, dS$ to (252) implies that

$$\sum_{\substack{n=1\\q=1}}^{N,Q} T^{B,\mathrm{TE}^z}_{v_n,w_q} \xcancel{\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{-\gamma^B_{zv_n,w_q}\ell} dS}^{\,0} + \sum_{\substack{n=1\\q=1}}^{N,Q} R^{B,\mathrm{TE}^z}_{v_n,w_q} \xcancel{\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{\gamma^B_{zv_n,w_q}\ell} dS}^{\,0}$$

$$+ \sum_{\substack{n=1\\q=2}}^{N,Q} T^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{B,\mathrm{TM}^z}_{v_n,w_q} e^{-\gamma^B_{zv_n,w_q}\ell} dS}^{\bar{Y}^{(mn)}_{pq}}$$

$$+ \sum_{\substack{n=1\\q=2}}^{N,Q} R^{B,\mathrm{TM}^z}_{v_n,w_q} \overbrace{\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{B,\mathrm{TM}^z}_{v_n,w_q} e^{\gamma^B_{zv_n,w_q}\ell} dS}^{\bar{Z}^{(mn)}_{pq}} = -\sum_{\substack{n=1\\q=1}}^{N,Q} T^{C,\mathrm{TE}^z}_{v_n,w_q} \bar{\Gamma}^{(mn)}_{pq} - \sum_{\substack{n=1\\q=2}}^{N,Q} T^{C,\mathrm{TM}^z}_{v_n,w_q} \bar{\Delta}^{(mn)}_{pq} \tag{273}$$

$$\bar{\Gamma}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TE}^z}_{v_n,w_q}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\substack{z=0\\z'=0} \right) \cdot \int\limits_{S'} \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q} (\vec{\rho}\,') \, e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}\,'} dS' \right] d^2\lambda_\rho \tag{274}$$

$$\bar{\Delta}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TM}^z}_{v_n,w_q}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}^{B,\mathrm{TM}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\substack{z=0\\z'=0} \right) \cdot \int\limits_{S'} \vec{h}^{B,\mathrm{TM}^z}_{v_n,w_q} (\vec{\rho}\,') \, e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}\,'} dS' \right] d^2\lambda_\rho \tag{275}$$

which implies that

$$\boxed{\begin{array}{l} \displaystyle\sum_{\substack{n=1\\q=2}}^{N,Q} \bar{Y}^{(mn)}_{pq} T^{B,\mathrm{TM}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{Z}^{(mn)}_{pq} R^{B,\mathrm{TM}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=1}}^{N,Q} \bar{\Gamma}^{(mn)}_{pq} T^{C,\mathrm{TE}^z}_{v_n,w_q} + \sum_{\substack{n=1\\q=2}}^{N,Q} \bar{\Delta}^{(mn)}_{pq} T^{C,\mathrm{TM}^z}_{v_n,w_q} \\ = 0 ... m = 1, ..., N; p = 2, ..., Q \end{array}} \tag{276}$$

$$\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{A}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{B}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TM}^z}-\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{C}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TE}^z}-\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{C}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TE}^z}$$

$$=-\bar{A}_{p1}^{(m1)}...m=1,...,N;p=1,...,Q \tag{277}$$

$$\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{D}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{E}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TM}^z}-\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{F}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TM}^z}-\sum_{\substack{n=1\\p=2}}^{N,Q}\bar{F}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TM}^z}$$

$$=-\bar{D}_{p1}^{(m1)}...m=1,...,N;p=2,...,Q \tag{278}$$

$$\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{G}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{H}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TM}^z}+\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{I}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TE}^z}-\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{I}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TE}^z}$$

$$=\bar{G}_{p1}^{(m1)}...m=1,...,N;p=1,...,Q \tag{279}$$

$$\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{J}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{K}_{pq}^{(mn)}R_{v_n,w_q}^{A,\mathrm{TM}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{L}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TM}^z}-\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{L}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TM}^z}$$

$$=\bar{J}_{p1}^{(m1)}...m=1,...,N;p=2,...,Q \tag{280}$$

$$\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{M}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TE}^z}-\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{N}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{O}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{P}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TM}^z}$$

$$=0...m=1,...,N;p=1,...,Q \tag{281}$$

$$\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{Q}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TM}^z}-\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{R}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TM}^z}+\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{S}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{T}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TM}^z}$$

$$=0...m=1,...,N;p=2,...,Q \tag{282}$$

$$\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{U}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{V}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{W}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{X}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TM}^z}$$

$$=0...m=1,...,N;p=1,...,Q \tag{283}$$

$$\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{Y}_{pq}^{(mn)}T_{v_n,w_q}^{B,\mathrm{TM}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{Z}_{pq}^{(mn)}R_{v_n,w_q}^{B,\mathrm{TM}^z}+\sum_{\substack{n=1\\q=1}}^{N,Q}\bar{\Gamma}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TE}^z}+\sum_{\substack{n=1\\q=2}}^{N,Q}\bar{\Delta}_{pq}^{(mn)}T_{v_n,w_q}^{C,\mathrm{TM}^z}$$

$$=0...m=1,...,N;p=2,...,Q \tag{284}$$

$$\bar{A}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TE}^z}\, dS, \qquad \bar{B}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TM}^z}\, dS$$

$$\bar{C}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS, \qquad \bar{D}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TE}^z}\, dS$$

$$\bar{E}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TM}^z}\, dS, \qquad \bar{F}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS$$

$$\bar{G}_{pq}^{(mn)} = \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{A,\mathrm{TE}^z}\, dS, \qquad \bar{H}_{pq}^{(mn)} = \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{A,\mathrm{TM}^z}\, dS$$

$$\bar{I}_{pq}^{(mn)} = \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS, \qquad \bar{J}_{pq}^{(mn)} = \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{A,\mathrm{TE}^z}\, dS$$

$$\bar{K}_{pq}^{(mn)} = \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{A,\mathrm{TM}^z}\, dS, \qquad \bar{L}_{pq}^{(mn)} = \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS$$

$$\bar{M}_{pq}^{(mn)} = P_B \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS, \quad \bar{N}_{pq}^{(mn)} = P_B^{-1} \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS$$

$$\bar{O}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{P}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{Q}_{pq}^{(mn)} = P_B \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS, \quad \bar{R}_{pq}^{(mn)} = P_B^{-1} \int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS$$

$$\bar{S}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{T}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{U}_{pq}^{(mn)} = P_B \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS, \quad \bar{V}_{pq}^{(mn)} = P_B^{-1} \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS$$

$$\bar{W}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{X}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{Y}_{pq}^{(mn)} = P_B \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS, \quad \bar{Z}_{pq}^{(mn)} = P_B^{-1} \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS$$

$$\bar{\Gamma}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{S} \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left({}^{z=0}_{z'=0} \right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} \left(\vec{\rho}' \right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$\bar{\Delta}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{S} \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left({}^{z=0}_{z'=0} \right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}' \right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \tag{285}$$

where $P_B = e^{-\gamma_{zv_n,w_q}^B \ell}$.

$$\overbrace{\begin{bmatrix} \bar{A} & \bar{B} & -\bar{C} & -\bar{C} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{D} & \bar{E} & \bar{0} & \bar{0} & -\bar{F} & -\bar{F} & \bar{0} & \bar{0} \\ \bar{G} & \bar{H} & \bar{I} & -\bar{I} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{J} & \bar{K} & \bar{0} & \bar{0} & \bar{L} & -\bar{L} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{M} & -\bar{N} & \bar{0} & \bar{0} & \bar{O} & \bar{P} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{Q} & -\bar{R} & \bar{S} & \bar{T} \\ \bar{0} & \bar{0} & \bar{U} & \bar{V} & \bar{0} & \bar{0} & \bar{W} & \bar{X} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{Y} & \bar{Z} & \bar{\Gamma} & \bar{\Delta} \end{bmatrix}}^{\overleftrightarrow{\mathbf{A}}} \overbrace{\begin{bmatrix} \vec{R}^{A,\mathrm{TE}^z} \\ \vec{R}^{A,\mathrm{TM}^z} \\ \vec{T}^{B,\mathrm{TE}^z} \\ \vec{R}^{B,\mathrm{TE}^z} \\ \vec{T}^{B,\mathrm{TM}^z} \\ \vec{R}^{B,\mathrm{TM}^z} \\ \vec{T}^{C,\mathrm{TE}^z} \\ \vec{T}^{C,\mathrm{TM}^z} \end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix} -\vec{A} \\ -\vec{D} \\ \vec{G} \\ \vec{J} \\ \vec{0}_1 \\ \vec{0}_2 \\ \vec{0}_3 \\ \vec{0}_4 \end{bmatrix}}^{\vec{b}} \tag{286}$$

where all submatrices of $\overleftrightarrow{\mathbf{A}}$ are of the general pattern

$$\bar{A} = \begin{bmatrix} \bar{A}_{1,1}^{(1,1)} & \bar{A}_{1,2}^{(1,1)} & \cdots & \bar{A}_{1,Q}^{(1,1)} & \bar{A}_{1,1}^{(1,2)} & \cdots & \bar{A}_{1,Q}^{(1,N)} \\ \bar{A}_{2,1}^{(1,1)} & \bar{A}_{2,2}^{(1,1)} & \cdots & \bar{A}_{2,Q}^{(1,1)} & \bar{A}_{2,1}^{(1,2)} & \cdots & \bar{A}_{2,Q}^{(1,N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{A}_{Q,1}^{(1,1)} & \bar{A}_{Q,2}^{(1,1)} & \cdots & \bar{A}_{Q,Q}^{(1,1)} & \bar{A}_{Q,1}^{(1,2)} & \cdots & \bar{A}_{Q,Q}^{(1,N)} \\ \bar{A}_{1,1}^{(2,1)} & \bar{A}_{1,2}^{(2,1)} & \cdots & \bar{A}_{1,Q}^{(2,1)} & \bar{A}_{1,1}^{(2,2)} & \cdots & \bar{A}_{1,Q}^{(2,N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{A}_{Q,1}^{(N,1)} & \bar{A}_{Q,2}^{(N,1)} & \cdots & \bar{A}_{Q,Q}^{(N,1)} & \bar{A}_{Q,1}^{(N,2)} & \cdots & \bar{A}_{Q,Q}^{(N,N)} \end{bmatrix} \tag{287}$$

except that any submatrices related to TM^z components or testing operators will have the corresponding index beginning at 2, thus reducing the dimension of the submatrix accordingly. It can be shown that the submatrices have the following dimensions:

$$\bar{A}, \bar{C}, \bar{G}, \bar{I}, \bar{M}, \bar{N}, \bar{O}, \bar{U}, \bar{V}, \bar{W} \in \mathbb{C}^{NQ \times NQ} \tag{288}$$

$$\bar{B}, \bar{H}, \bar{P}, \bar{X} \in \mathbb{C}^{NQ \times N(Q-1)} \tag{289}$$

$$\bar{D}, \bar{J}, \bar{S}, \bar{\Gamma} \in \mathbb{C}^{N(Q-1) \times NQ} \tag{290}$$

$$\bar{E}, \bar{F}, \bar{K}, \bar{L}, \bar{Q}, \bar{R}, \bar{T}, \bar{Y}, \bar{Z}, \bar{\Delta} \in \mathbb{C}^{N(Q-1) \times N(Q-1)} \tag{291}$$

This implies that the grand matrix $\overleftrightarrow{\mathbf{A}} \in \mathbb{C}^{[4N(Q-1)+4NQ] \times [4N(Q-1)+4NQ]}$. Next, the subvectors of $\vec{x}$ are of the general pattern

$$\vec{R}^{A,\mathrm{TE}^z} = \begin{bmatrix} R_{1,1}^{A,\mathrm{TE}^z} & R_{1,2}^{A,\mathrm{TE}^z} & \cdots & R_{1,Q}^{A,\mathrm{TE}^z} & R_{2,1}^{A,\mathrm{TE}^z} & \cdots & R_{N,Q}^{A,\mathrm{TE}^z} \end{bmatrix}^T \tag{292}$$

with the same caveat that any unknown TM^z amplitude will begin the second index at 2 instead of 1. Thus

$$\begin{aligned} &\vec{R}^{A,\mathrm{TE}^z} \in \mathbb{C}^{NQ}, && \vec{R}^{A,\mathrm{TM}^z} \in \mathbb{C}^{N(Q-1)}, && \vec{T}^{B,\mathrm{TE}^z} \in \mathbb{C}^{NQ}, && \vec{R}^{B,\mathrm{TE}^z} \in \mathbb{C}^{NQ} \\ &\vec{T}^{B,\mathrm{TM}^z} \in \mathbb{C}^{N(Q-1)}, && \vec{R}^{B,\mathrm{TM}^z} \in \mathbb{C}^{N(Q-1)}, && \vec{T}^{C,\mathrm{TE}^z} \in \mathbb{C}^{NQ}, && \vec{T}^{C,\mathrm{TM}^z} \in \mathbb{C}^{N(Q-1)} \end{aligned} \tag{293}$$

This implies that the grand unknown vector $\vec{x} \in \mathbb{C}^{4N(Q-1)+4NQ}$, which is in second-dimension agreement with $\overleftrightarrow{\mathbf{A}}$ as is required. Finally, the subvectors of the excitation vector $\vec{b}$ are of the form

$$\vec{A} = \begin{bmatrix} \bar{A}_{1,1}^{(1,1)} & \bar{A}_{2,1}^{(1,1)} & \cdots & \bar{A}_{Q,1}^{(1,1)} & \bar{A}_{1,1}^{(2,1)} & \cdots & \bar{A}_{Q,1}^{(N,1)} \end{bmatrix}^T \tag{294}$$

with the same caveat that any excitation vector associated with a TM^z testing operator will begin at subscripted index 2 instead of 1. Therefore,

$$\begin{aligned} &\vec{A} \in \mathbb{C}^{NQ}, \quad \vec{D} \in \mathbb{C}^{N(Q-1)}, \quad \vec{G} \in \mathbb{C}^{NQ}, \quad \vec{J} \in \mathbb{C}^{N(Q-1)} \\ &\vec{0}_1 \in \mathbb{C}^{NQ}, \quad \vec{0}_2 \in \mathbb{C}^{N(Q-1)}, \quad \vec{0}_3 \in \mathbb{C}^{NQ}, \quad \vec{0}_4 \in \mathbb{C}^{N(Q-1)} \end{aligned} \tag{295}$$

This implies that the grand excitation vector $\vec{b} \in \mathbb{C}^{4N(Q-1)+4NQ}$, which is in first-dimension agreement with $\overleftrightarrow{\mathbf{A}}$ as is required.

5.4 Submatrix Equations

It is important to note the following identities prior to proceeding to evaluate the integrals from the previous sections.

$$\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(k_{xv_n}x\right) dx = 2\int\limits_{0}^{\frac{a}{2}} \cos^2\left(k_{xv_n}x\right) dx = 2\left(\frac{x}{2} + \frac{\sin\left(2\frac{(2n-1)\pi}{a}x\right)}{4k_{xv_n}}\right)\Bigg|_0^{\frac{a}{2}} = \frac{a}{2} \tag{296}$$

$$\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \sin^2\left(k_{xv_n}x\right) dx = 2\int\limits_{0}^{\frac{a}{2}} \sin^2\left(k_{xv_n}x\right) dx = 2\left(\frac{x}{2} - \frac{\sin\left(2\frac{(2n-1)\pi}{a}x\right)}{4k_{xv_n}}\right)\Bigg|_0^{\frac{a}{2}} = \frac{a}{2} \tag{297}$$

$$\int \sin\left(my\right)\sin\left(ny\right) dy = \frac{\sin\left[\left(m-n\right)y\right]}{2\left(m-n\right)} - \frac{\sin\left[\left(m+n\right)y\right]}{2\left(m+n\right)} ...m^2 \neq n^2 \tag{298}$$

$$\int \cos\left(my\right)\cos\left(ny\right) dy = \frac{\sin\left[\left(m-n\right)y\right]}{2\left(m-n\right)} + \frac{\sin\left[\left(m+n\right)y\right]}{2\left(m+n\right)} ...m^2 \neq n^2 \tag{299}$$

$$\begin{aligned} \int\limits_{-r}^{r} \sin\left(\frac{u\pi}{2r}x\right) e^{\pm j\lambda_x x} dx &= j\left[\frac{\left(\frac{u\pi}{r}\right)\cos\left(\frac{u\pi}{2}\right)\sin\left(\lambda_x r\right) - 2\lambda_x \sin\left(\frac{u\pi}{2}\right)\cos\left(\lambda_x r\right)}{\left(\lambda_x + \frac{u\pi}{2r}\right)\left(\lambda_x - \frac{u\pi}{2r}\right)}\right] \\ &= \begin{cases} -j2\lambda_x \left[\frac{\sin\left(\frac{u\pi}{2}\right)\cos(\lambda_x r)}{\left(\lambda_x+\frac{u\pi}{2r}\right)\left(\lambda_x-\frac{u\pi}{2r}\right)}\right] ...u \in 2\mathbb{N}+1 \\ j2\left(\frac{u\pi}{2r}\right)\left[\frac{\cos\left(\frac{u\pi}{2}\right)\sin(\lambda_x r)}{\left(\lambda_x+\frac{u\pi}{2r}\right)\left(\lambda_x-\frac{u\pi}{2r}\right)}\right] ...u \in 2\mathbb{N} \end{cases} \end{aligned} \tag{300}$$

$$\int_{-r}^{r} \cos\left(\frac{u\pi}{2r}x\right) e^{\pm j\lambda_x x} dx = -\frac{\left(\frac{u\pi}{r}\right)\sin\left(\frac{u\pi}{2}\right)\cos\left(\lambda_x r\right) - 2\lambda_x \cos\left(\frac{u\pi}{2}\right)\sin\left(\lambda_x r\right)}{\left(\lambda_x + \frac{u\pi}{2r}\right)\left(\lambda_x - \frac{u\pi}{2r}\right)}$$

$$= \begin{cases} -\frac{\left(\frac{u\pi}{r}\right)\sin\left(\frac{u\pi}{2}\right)\cos(\lambda_x r)}{\left(\lambda_x+\frac{u\pi}{2r}\right)\left(\lambda_x-\frac{u\pi}{2r}\right)} & ...u \in 2\mathbb{N}+1 \\ \frac{2\lambda_x \cos\left(\frac{u\pi}{2}\right)\sin(\lambda_x r)}{\left(\lambda_x+\frac{u\pi}{2r}\right)\left(\lambda_x-\frac{u\pi}{2r}\right)} & ...u \in 2\mathbb{N} \end{cases} \tag{301}$$

$$\bar{A}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TE}^z} dS$$

$$= (-1)^{m+n+p+q}\, k_{yw_q}^A k_{yw_p}^B \overbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(k_{xv_m}x\right)\sin\left(k_{xv_n}x\right)dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k_{yw_p}^B y\right)\sin\left(k_{yw_q}^A y\right)dy}^{\Theta_1}$$

$$+ (-1)^{m+n+p+q}\, k_{xv_m}k_{xv_n} \overbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_m}x\right)\cos\left(k_{xv_n}x\right)dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k_{yw_p}^B y\right)\cos\left(k_{yw_q}^A y\right)dy}^{\Theta_2}$$

$$\boxed{\begin{aligned} &\bar{A}_{pq}^{(mn)} = (-1)^{m+n+p+q}\,\frac{a}{2}\delta_{mn}\left(k_{yw_q}^A k_{yw_p}^B \Theta_1 + k_{xv_m}k_{xv_n}\Theta_2\right) \\ &\quad\Theta_1 = \begin{cases} \frac{h}{2}...k_{yw_q}^A = k_{yw_p}^B \\ \frac{\sin\left[\left(k_{yw_p}^B - k_{yw_q}^A\right)\frac{h}{2}\right]}{k_{yw_p}^B - k_{yw_q}^A} - \frac{\sin\left[\left(k_{yw_p}^B + k_{yw_q}^A\right)\frac{h}{2}\right]}{k_{yw_p}^B + k_{yw_q}^A}...k_{yw_q}^A \neq k_{yw_p}^B \end{cases} \\ &\quad\Theta_2 = \begin{cases} \frac{h}{2}...k_{yw_q}^A = k_{yw_p}^B \\ \frac{\sin\left[\left(k_{yw_p}^B - k_{yw_q}^A\right)\frac{h}{2}\right]}{k_{yw_p}^B - k_{yw_q}^A} + \frac{\sin\left[\left(k_{yw_p}^B + k_{yw_q}^A\right)\frac{h}{2}\right]}{k_{yw_p}^B + k_{yw_q}^A}...k_{yw_q}^A \neq k_{yw_p}^B \end{cases} \end{aligned}} \tag{302}$$

$$\bar{B}_{pq}^{(mn)} = \int_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TM}^z} dS$$

$$= (-1)^{m+n+p+q}\, k_{yw_p}^{B} k_{xv_n} \overbrace{\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(k_{xv_m} x\right) \sin\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k_{yw_p}^{B} y\right) \sin\left(k_{yw_q}^{A} y\right) dy}^{\Theta_1}$$

$$+ (-1)^{m+n+p+q}\, k_{xv_m} k_{yw_q}^{A} \overbrace{\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_m} x\right) \cos\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k_{yw_p}^{B} y\right) \cos\left(k_{yw_q}^{A} y\right) dy}^{\Theta_2}$$

$$\boxed{\bar{B}_{pq}^{(mn)} = (-1)^{m+n+p+q}\, \frac{a}{2}\delta_{mn} \left(k_{yw_p}^{B} k_{xv_n} \Theta_1 + k_{xv_m} k_{yw_q}^{A} \Theta_2\right)} \tag{303}$$

$$\bar{C}_{pq}^{(mn)} = \int\limits_{S} \vec{e}_{v_m,w_p}^{\,B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{\,B,\mathrm{TE}^z}\, dS$$

$$= (-1)^{m+n+p+q}\, k_{yw_p}^{B} k_{yw_q}^{B} \overbrace{\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(k_{xv_m} x\right) \sin\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k_{yw_p}^{B} y\right) \sin\left(k_{yw_q}^{B} y\right) dy}^{\frac{h}{2}\delta_{pq}}$$

$$+ (-1)^{m+n+p+q}\, k_{xv_m} k_{xv_n} \overbrace{\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_m} x\right) \cos\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k_{yw_p}^{B} y\right) \cos\left(k_{yw_q}^{B} y\right) dy}^{\frac{h}{2}\delta_{pq}}$$

$$\boxed{\bar{C}_{pq}^{(mn)} = (-1)^{m+n+p+q}\, \frac{ah}{4}\delta_{mn}\delta_{pq} \left(k_{yw_p}^{B} k_{yw_q}^{B} + k_{xv_m} k_{xv_n}\right)} \tag{304}$$

$$\bar{D}_{pq}^{(mn)} = \int\limits_{S} \vec{e}_{v_m,w_p}^{\,B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{\,A,\mathrm{TE}^z}\, dS$$

$$= (-1)^{m+n+p+q}\, k_{xv_m} k^A_{yw_q} \overbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(k_{xv_m} x\right) \sin\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k^B_{yw_p} y\right) \sin\left(k^A_{yw_q} y\right) dy}^{\Theta_1}$$

$$- (-1)^{m+n+p+q}\, k^B_{yw_p} k_{xv_n} \overbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_m} x\right) \cos\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k^B_{yw_p} y\right) \cos\left(k^A_{yw_q} y\right) dy}^{\Theta_2}$$

$$\boxed{\bar{D}^{(mn)}_{pq} = (-1)^{m+n+p+q}\, \frac{a}{2}\delta_{mn} \left(k_{xv_m} k^A_{yw_q} \Theta_1 - k^B_{yw_p} k_{xv_n} \Theta_2\right)} \tag{305}$$

$$\bar{E}^{(mn)}_{pq} = \int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,A,\mathrm{TM}^z}_{v_n,w_q}\, dS$$

$$= (-1)^{m+n+p+q}\, k_{xv_m} k_{xv_n} \overbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(k_{xv_m} x\right) \sin\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k^B_{yw_p} y\right) \sin\left(k^A_{yw_q} y\right) dy}^{\Theta_1}$$

$$+ (-1)^{m+n+p+q}\, k^B_{yw_p} k^A_{yw_q} \overbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_m} x\right) \cos\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k^B_{yw_p} y\right) \cos\left(k^A_{yw_q} y\right) dy}^{\Theta_2}$$

$$\boxed{\bar{E}^{(mn)}_{pq} = (-1)^{m+n+p+q}\, \frac{a}{2}\delta_{mn} \left(k_{xv_m} k_{xv_n} \Theta_1 + k^B_{yw_p} k^A_{yw_q} \Theta_2\right)} \tag{306}$$

$$\bar{F}^{(mn)}_{pq} = \int_S \vec{e}^{\,B,\mathrm{TM}^z}_{v_m,w_p} \cdot \vec{e}^{\,B,\mathrm{TM}^z}_{v_n,w_q}\, dS$$

$$= (-1)^{m+n+p+q}\, k_{xv_m} k_{xv_n} \overbrace{\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(k_{xv_m} x\right) \sin\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k^B_{yw_p} y\right) \sin\left(k^B_{yw_q} y\right) dy}^{\frac{h}{2}\delta_{pq}}$$

$$+ (-1)^{m+n+p+q}\, k^B_{yw_p} k^B_{yw_q} \overbrace{\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_m} x\right) \cos\left(k_{xv_n} x\right) dx}^{\frac{a}{2}\delta_{mn}} \overbrace{\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k^B_{yw_p} y\right) \cos\left(k^B_{yw_q} y\right) dy}^{\frac{h}{2}\delta_{pq}}$$

$$\boxed{\bar{F}^{(mn)}_{pq} = (-1)^{m+n+p+q} \frac{ah}{4} \delta_{mn}\delta_{pq} \left(k_{xv_m} k_{xv_n} + k^B_{yw_p} k^B_{yw_q}\right) = \bar{C}^{(mn)}_{pq}} \tag{307}$$

$$\bar{G}^{(mn)}_{pq} = \int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TE}^z}_{v_n,w_q}\, dS = \frac{1}{Z^{B,\mathrm{TE}^z}_{v_m,w_p} Z^{A,\mathrm{TE}^z}_{v_n,w_q}} \overbrace{\int\limits_S \vec{e}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{A,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{A}^{(mn)}_{pq}}$$

$$\boxed{\bar{G}^{(mn)}_{pq} = \frac{\bar{A}^{(mn)}_{pq}}{Z^{B,\mathrm{TE}^z}_{v_m,w_p} Z^{A,\mathrm{TE}^z}_{v_n,w_q}}} \tag{308}$$

$$\bar{H}^{(mn)}_{pq} = \int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{A,\mathrm{TM}^z}_{v_n,w_q}\, dS = \frac{1}{Z^{B,\mathrm{TE}^z}_{v_m,w_p} Z^{A,\mathrm{TM}^z}_{v_n,w_q}} \overbrace{\int\limits_S \vec{e}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{A,\mathrm{TM}^z}_{v_n,w_q}\, dS}^{\bar{B}^{(mn)}_{pq}}$$

$$\boxed{\bar{H}^{(mn)}_{pq} = \frac{\bar{B}^{(mn)}_{pq}}{Z^{B,\mathrm{TE}^z}_{v_m,w_p} Z^{A,\mathrm{TM}^z}_{v_n,w_q}}} \tag{309}$$

$$\bar{I}^{(mn)}_{pq} = \int\limits_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\, dS = \frac{1}{Z^{B,\mathrm{TE}^z}_{v_m,w_p} Z^{B,\mathrm{TE}^z}_{v_n,w_q}} \overbrace{\int\limits_S \vec{e}^{B,\mathrm{TE}^z}_{v_m,w_p} \cdot \vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q}\, dS}^{\bar{C}^{(mn)}_{pq}}$$

$$\boxed{\bar{I}_{pq}^{(mn)} = \frac{\bar{C}_{pq}^{(mn)}}{Z_{v_m,w_p}^{B,\mathrm{TE}^z}\, Z_{v_n,w_q}^{B,\mathrm{TE}^z}}} \tag{310}$$

$$\bar{J}_{pq}^{(mn)} = \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{A,\mathrm{TE}^z}\, dS = \frac{1}{Z_{v_m,w_p}^{B,\mathrm{TM}^z}\, Z_{v_n,w_q}^{A,\mathrm{TE}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TE}^z}\, dS}^{\bar{D}_{pq}^{(mn)}}$$

$$\boxed{\bar{J}_{pq}^{(mn)} = \frac{\bar{D}_{pq}^{(mn)}}{Z_{v_m,w_p}^{B,\mathrm{TM}^z}\, Z_{v_n,w_q}^{A,\mathrm{TE}^z}}} \tag{311}$$

$$\bar{K}_{pq}^{(mn)} = \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{A,\mathrm{TM}^z}\, dS = \frac{1}{Z_{v_m,w_p}^{B,\mathrm{TM}^z}\, Z_{v_n,w_q}^{A,\mathrm{TM}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{A,\mathrm{TM}^z}\, dS}^{\bar{E}_{pq}^{(mn)}}$$

$$\boxed{\bar{K}_{pq}^{(mn)} = \frac{\bar{E}_{pq}^{(mn)}}{Z_{v_m,w_p}^{B,\mathrm{TM}^z}\, Z_{v_n,w_q}^{A,\mathrm{TM}^z}}} \tag{312}$$

$$\bar{L}_{pq}^{(mn)} = \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS = \frac{1}{Z_{v_m,w_p}^{B,\mathrm{TM}^z}\, Z_{v_n,w_q}^{B,\mathrm{TM}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z}\, dS}^{\bar{F}_{pq}^{(mn)}}$$

$$\boxed{\bar{L}_{pq}^{(mn)} = \frac{\bar{F}_{pq}^{(mn)}}{Z_{v_m,w_p}^{B,\mathrm{TM}^z}\, Z_{v_n,w_q}^{B,\mathrm{TM}^z}}} \tag{313}$$

$$\bar{M}_{pq}^{(mn)} = P_B \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS = \frac{P_B}{Z_{v_m,w_p}^{B,\mathrm{TE}^z}\, Z_{v_n,w_q}^{B,\mathrm{TE}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z}\, dS}^{\bar{C}_{pq}^{(mn)}}$$

$$\boxed{\bar{M}_{pq}^{(mn)} = \frac{P_B \bar{C}_{pq}^{(mn)}}{Z_{v_m,w_p}^{B,\mathrm{TE}^z} Z_{v_n,w_q}^{B,\mathrm{TE}^z}}} \tag{314}$$

$$\bar{N}_{pq}^{(mn)} = P_B^{-1} \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} dS = \frac{P_B^{-1}}{Z_{v_m,w_p}^{B,\mathrm{TE}^z} Z_{v_n,w_q}^{B,\mathrm{TE}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} dS}^{\bar{C}_{pq}^{(mn)}}$$

$$\boxed{\bar{N}_{pq}^{(mn)} = \frac{P_B^{-1} \bar{C}_{pq}^{(mn)}}{Z_{v_m,w_p}^{B,\mathrm{TE}^z} Z_{v_n,w_q}^{B,\mathrm{TE}^z}}} \tag{315}$$

$$\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(\overbrace{\frac{v_n \pi}{2\frac{a}{2}}}^{k_{xv_n}} x\right) e^{\pm j\lambda_x x} dx = -j2\lambda_x \left[\frac{\sin\left(\frac{v_n\pi}{2}\right)\cos\left(\lambda_x \frac{a}{2}\right)}{\left(\lambda_x + \frac{v_n\pi}{2\frac{a}{2}}\right)\left(\lambda_x - \frac{v_n\pi}{2\frac{a}{2}}\right)}\right]$$

$$= j2\lambda_x \left[\frac{(-1)^n \cos\left(\lambda_x \frac{a}{2}\right)}{(\lambda_x + k_{xv_n})(\lambda_x - k_{xv_n})}\right] \tag{316}$$

$$\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(\overbrace{\frac{w_q \pi}{2\frac{h}{2}}}^{k_{yw_q}^B} y\right) e^{\pm j\lambda_y y} dy = j2\left(\frac{w_q\pi}{2\frac{h}{2}}\right) \left[\frac{\cos\left(\frac{w_q\pi}{2}\right)\sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + \frac{w_q\pi}{2\frac{h}{2}}\right)\left(\lambda_y - \frac{w_q\pi}{2\frac{h}{2}}\right)}\right]$$

$$= -j2k_{yw_q}^B \left[\frac{(-1)^q \sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)}\right] \tag{317}$$

$$\Rightarrow \int\limits_{-\frac{a}{2}}^{\frac{a}{2}} \sin(k_{xv_n} x) e^{\pm j\lambda_x x} dx \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sin\left(k_{yw_q}^B y\right) e^{\pm j\lambda_y y} dy$$

$$= \left(j2\lambda_x \left[\frac{(-1)^n \cos\left(\lambda_x \frac{a}{2}\right)}{(\lambda_x + k_{xv_n})(\lambda_x - k_{xv_n})}\right]\right) \left(-j2k_{yw_q}^B \left[\frac{(-1)^q \sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)}\right]\right)$$

$$= (-1)^{n+q}\, k^B_{yw_q} \lambda_x \overbrace{\left[\frac{4\cos\left(\lambda_x \frac{a}{2}\right)\sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_x + k_{xv_n}\right)\left(\lambda_x - k_{xv_n}\right)\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)} \right]}^{\Theta_3^{nq}} \tag{318}$$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(\overbrace{\frac{v_n\pi}{2\frac{a}{2}}}^{k_{xv_n}} x\right) e^{\pm j\lambda_x x} dx = -\frac{\left(\frac{v_n\pi}{\frac{a}{2}}\right)\sin\left(\frac{v_n\pi}{2}\right)\cos\left(\lambda_x \frac{a}{2}\right)}{\left(\lambda_x + \frac{v_n\pi}{2\frac{a}{2}}\right)\left(\lambda_x - \frac{v_n\pi}{2\frac{a}{2}}\right)}$$

$$= 2k_{xv_n} \frac{(-1)^n \cos\left(\lambda_x \frac{a}{2}\right)}{\left(\lambda_x + k_{xv_n}\right)\left(\lambda_x - k_{xv_n}\right)} \tag{319}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(\overbrace{\frac{w_q\pi}{2\frac{h}{2}}}^{k^B_{yw_q}} y\right) e^{\pm j\lambda_y y} dy = \frac{2\lambda_y \cos\left(\frac{w_q\pi}{2}\right)\sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + \frac{w_q\pi}{2\frac{h}{2}}\right)\left(\lambda_y - \frac{w_q\pi}{2\frac{h}{2}}\right)}$$

$$= -\frac{2\lambda_y (-1)^q \sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)} \tag{320}$$

$$\Rightarrow \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(k_{xv_n} x\right) e^{\pm j\lambda_x x} dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos\left(k^B_{yw_q} y\right) e^{\pm j\lambda_y y} dy$$

$$= \left(2k_{xv_n} \frac{(-1)^n \cos\left(\lambda_x \frac{a}{2}\right)}{\left(\lambda_x + k_{xv_n}\right)\left(\lambda_x - k_{xv_n}\right)}\right)\left(-\frac{2\lambda_y (-1)^q \sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)}\right)$$

$$= (-1)^{n+q+1}\, k_{xv_n} \lambda_y \overbrace{\left[\frac{4\cos\left(\lambda_x \frac{a}{2}\right)\sin\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_x + k_{xv_n}\right)\left(\lambda_x - k_{xv_n}\right)\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)} \right]}^{\Theta_3^{nq}} \tag{321}$$

$$\begin{aligned} \Theta_3 &= \Theta_3^{mp}\Theta_3^{nq} \\ &= \left\{ \left[\frac{4\cos^2\left(\lambda_x \frac{a}{2}\right)}{\left(\lambda_x + k_{xv_m}\right)\left(\lambda_x - k_{xv_m}\right)\left(\lambda_x + k_{xv_n}\right)\left(\lambda_x - k_{xv_n}\right)} \right] \right. \\ &\quad \left. \left[\frac{4\sin^2\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k^B_{yw_p}\right)\left(\lambda_y - k^B_{yw_p}\right)\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)} \right] \right\} \end{aligned} \tag{322}$$

$$\bar{O}^{(mn)}_{pq} = \frac{Z^{B,\mathrm{TE}^z}_{v_n,w_q}}{4\pi^2} \iint_{-\infty}^{\infty} \left[\int_S \vec{h}^{B,\mathrm{TE}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \overleftrightarrow{\tilde{G}}_{hh}\left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int_{S'} \vec{h}^{B,\mathrm{TE}^z}_{v_n,w_q}\left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \hat{z}\times \vec{e}^{B,\mathrm{TE}^z}_{v_m,w_p} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\ z'=0}\right) \cdot \int\limits_{S'} \hat{z}\times\vec{e}^{B,\mathrm{TE}^z}_{v_n,w_q} e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS'\right] d^2\lambda_\rho \\
&= \frac{1}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}} \iint\limits_{-\infty}^{\infty} \Bigg\{ \hat{z}\times \Bigg[\hat{x}\left(-1\right)^{m+p+1} k^B_{yw_p} \int\limits_S \sin\left(k_{xv_m}x\right)\sin\left(k^B_{yw_p}y\right) e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \\
&\qquad + \hat{y}\left(-1\right)^{m+p+1} k_{xv_m} \int\limits_S \cos\left(k_{xv_m}x\right)\cos\left(k^B_{yw_p}y\right) e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \Bigg] \cdot \tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\ z'=0}\right) \\
&\qquad\quad \cdot \hat{z}\times \Bigg[\hat{x}\left(-1\right)^{n+q+1} k^B_{yw_q} \int\limits_{S'} \sin\left(k_{xv_n}x'\right)\sin\left(k^B_{yw_q}y'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \\
&\qquad\qquad + \hat{y}\left(-1\right)^{n+q+1} k_{xv_n} \int\limits_{S'} \cos\left(k_{xv_n}x'\right)\cos\left(k^B_{yw_q}y'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}'} dS' \Bigg] \Bigg\} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}} \iint\limits_{-\infty}^{\infty} \Big\{ \Big[\hat{y}\left(-1\right)^{m+p+1} k^B_{yw_p} \left(\left(-1\right)^{m+p} k^B_{yw_q}\lambda_x\Theta_3^{mp}\right) \\
&\quad - \hat{x}\left(-1\right)^{m+p+1} k_{xv_m}\left(\left(-1\right)^{m+p+1} k_{xv_m}\lambda_y\Theta_3^{mp}\right)\Big] \cdot \tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\ z'=0}\right) \\
&\qquad \cdot \Big[\hat{y}\left(-1\right)^{n+q+1} k^B_{yw_q} \left(\left(-1\right)^{n+q} k^B_{yw_q}\lambda_x\Theta_3^{nq}\right) \\
&\qquad\quad - \hat{x}\left(-1\right)^{n+q+1} k_{xv_n}\left(\left(-1\right)^{n+q+1} k_{xv_n}\lambda_y\Theta_3^{nq}\right)\Big] \Big\} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}} \iint\limits_{-\infty}^{\infty} \Bigg\{ \Bigg[\hat{y}\cancelto{-1}{\left(-1\right)^{2m+2p+1}} k^{B2}_{yw_p}\lambda_x\Theta_3^{mp} - \hat{x}\cancelto{1}{\left(-1\right)^{2m+2p+2}} k^2_{xv_m}\lambda_y\Theta_3^{mp}\Bigg] \\
&\quad \cdot\tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\ z'=0}\right) \cdot \Bigg[\hat{y}\cancelto{-1}{\left(-1\right)^{2n+2q+1}} k^{B2}_{yw_q}\lambda_x\Theta_3^{nq} - \hat{x}\cancelto{1}{\left(-1\right)^{2n+2q+2}} k^2_{xv_n}\lambda_y\Theta_3^{nq}\Bigg] \Bigg\} d^2\lambda_\rho \\
&= \frac{1}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}} \iint\limits_{-\infty}^{\infty} \Bigg\{ \overbrace{\Theta_3^{mp}\left(-\hat{x}k^2_{xv_m}\lambda_y - \hat{y}k^{B2}_{yw_p}\lambda_x\right)}^{\vec{I}^{\mathrm{TE}^z}_{mp}} \cdot\tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\ z'=0}\right) \\
&\qquad\qquad \cdot \overbrace{\Theta_3^{nq}\left(-\hat{x}k^2_{xv_n}\lambda_y - \hat{y}k^{B2}_{yw_q}\lambda_x\right)}^{\vec{I}^{\mathrm{TE}^z}_{nq}} \Bigg\} d^2\lambda_\rho
\end{aligned}
$$

$$\boxed{\begin{aligned}\bar{O}_{pq}^{(mn)} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \left\{ \Theta_3 \left(\hat{x} k_{xv_m}^2 \lambda_y + \hat{y} k_{yw_p}^{B2} \lambda_x \right) \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \right. \\ &\qquad \left. \cdot \left(\hat{x} k_{xv_n}^2 \lambda_y + \hat{y} k_{yw_q}^{B2} \lambda_x \right) \right\} d^2\lambda_\rho \\ \Theta_3 &= \left\{ \overbrace{\left[\frac{4\cos^2\left(\lambda_x \frac{a}{2}\right)}{(\lambda_x + k_{xv_m})(\lambda_x - k_{xv_m})(\lambda_x + k_{xv_n})(\lambda_x - k_{xv_n})} \right]}^{\Theta_3^{\lambda_x}} \right. \\ &\qquad \left. \cdot \underbrace{\left[\frac{4\sin^2\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)} \right]}_{\Theta_3^{\lambda_y}} \right\}\end{aligned}} \tag{323}$$

$$\begin{aligned}\bar{P}_{pq}^{(mn)} &= \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \\ &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_S \left(\hat{z} \times \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \right) e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TE}^z}} \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \right. \\ &\qquad \left. \cdot \int\limits_{S'} \left(\hat{z} \times \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}'\right) \right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \\ &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \left\{ \vec{I}_{mp}^{\mathrm{TE}^z} \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \overbrace{\left[k_{xv_n} k_{yw_q}^B \Theta_3^{nq} \left(\hat{x}\lambda_y - \hat{y}\lambda_x \right) \right]}^{\vec{I}_{nq}^{\mathrm{TM}^z}} \right\} d^2\lambda_\rho\end{aligned}$$

$$\boxed{\begin{aligned}\bar{P}_{pq}^{(mn)} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \left\{ k_{xv_n} k_{yw_q}^B \Theta_3 \left(-\hat{x} k_{xv_m}^2 \lambda_y - \hat{y} k_{yw_p}^{B2} \lambda_x \right) \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \right. \\ &\qquad \left. \cdot \left(\hat{x}\lambda_y - \hat{y}\lambda_x \right) \right\} d^2\lambda_\rho\end{aligned}} \tag{324}$$

$$\bar{Q}_{pq}^{(mn)} = P_B \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} dS = \frac{P_B}{Z_{v_m,w_p}^{B,\mathrm{TM}^z} Z_{v_n,w_q}^{B,\mathrm{TM}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} dS}^{\bar{F}_{pq}^{(mn)}}$$

$$\boxed{\bar{Q}_{pq}^{(mn)} = \frac{P_B}{Z_{v_m,w_p}^{B,\mathrm{TM}^z} Z_{v_n,w_q}^{B,\mathrm{TM}^z}} \bar{F}_{pq}^{(mn)}} \tag{325}$$

$$\bar{R}_{pq}^{(mn)} = P_B^{-1} \int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} dS = \frac{P_B^{-1}}{Z_{v_m,w_p}^{B,\mathrm{TM}^z} Z_{v_n,w_q}^{B,\mathrm{TM}^z}} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} dS}^{\bar{F}_{pq}^{(mn)}}$$

$$\boxed{\bar{R}_{pq}^{(mn)} = \frac{P_B^{-1}}{Z_{v_m,w_p}^{B,\mathrm{TM}^z} Z_{v_n,w_q}^{B,\mathrm{TM}^z}} \bar{F}_{pq}^{(mn)}} \tag{326}$$

$$\begin{aligned}
\bar{S}_{pq}^{(mn)} &= \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} \left(\vec{\rho}' \right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho \\
&= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_S \hat{z} \times \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TM}^z}} \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \overbrace{\int\limits_{S'} \hat{z} \times \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS'}^{\vec{I}_{nq}^{\mathrm{TE}^z}} \right] d^2\lambda_\rho
\end{aligned}$$

$$\boxed{\begin{aligned}\bar{S}_{pq}^{(mn)} = \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \iint\limits_{-\infty}^{\infty} \Big\{ & k_{xv_m} k_{yw_p}^B \Theta_3 \left(\hat{x}\lambda_y - \hat{y}\lambda_x \right) \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \\ & \cdot \left(-\hat{x} k_{xv_n}^2 \lambda_y - \hat{y} k_{yw_q}^{B2} \lambda_x \right) \Big\} d^2\lambda_\rho \end{aligned}} \tag{327}$$

$$\bar{T}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_{S} \vec{h}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} \left(\vec{\rho}\,'\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}\,'} dS' \right] d^2\lambda_\rho$$

$$= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_{S} \hat{z} \times \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TM}^z}} \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \overbrace{\int\limits_{S'} \hat{z} \times \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}\,'} dS'}^{\vec{I}_{nq}^{\mathrm{TM}^z}} \right] d^2\lambda_\rho$$

$$\boxed{\begin{aligned} \bar{T}_{pq}^{(mn)} = \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \iint\limits_{-\infty}^{\infty} \Big\{ k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B \Theta_3 \left(\hat{x}\lambda_y - \hat{y}\lambda_x\right) \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \\ \cdot \left(\hat{x}\lambda_y - \hat{y}\lambda_x\right) \Big\} d^2\lambda_\rho \end{aligned}} \tag{328}$$

$$\bar{U}_{pq}^{(mn)} = P_B \overbrace{\int\limits_{S} \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} dS}^{\bar{C}_{pq}^{(mn)}}$$

$$\boxed{\bar{U}_{pq}^{(mn)} = P_B \bar{C}_{pq}^{(mn)}} \tag{329}$$

$$\bar{V}_{pq}^{(mn)} = P_B^{-1} \overbrace{\int\limits_{S} \vec{e}_{v_m,w_p}^{B,\mathrm{TE}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} dS}^{\bar{C}_{pq}^{(mn)}}$$

$$\boxed{\bar{V}_{pq}^{(mn)} = P_B^{-1} \bar{C}_{pq}^{(mn)}} \tag{330}$$

$$\bar{W}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}_{v_m,w_p}^{\,B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{\,B,\mathrm{TE}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{\,B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TE}^z} \times \hat{z}} \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \overbrace{\int\limits_{S'} \hat{z} \times \vec{e}_{v_n,w_q}^{\,B,\mathrm{TE}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS'}^{\vec{I}_{nq}^{\mathrm{TE}^z}} \right] d^2\lambda_\rho$$

$$\boxed{\begin{aligned} \bar{W}_{pq}^{(mn)} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Big\{ \Theta_3 \left(\hat{x} k_{yw_p}^{B2} \lambda_x - \hat{y} k_{xv_m}^2 \lambda_y \right) \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \\ \cdot \left(\hat{x} k_{xv_n}^2 \lambda_y + \hat{y} k_{yw_q}^{B2} \lambda_x \right) \Big\} d^2\lambda_\rho \end{aligned}} \tag{331}$$

$$\bar{X}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}_{v_m,w_p}^{\,B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{\,B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{\,B,\mathrm{TE}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TE}^z} \times \hat{z}} \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \overbrace{\int\limits_{S'} \hat{z} \times \vec{e}_{v_n,w_q}^{\,B,\mathrm{TM}^z} \left(\vec{\rho}'\right) e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS'}^{\vec{I}_{nq}^{\mathrm{TM}^z}} \right] d^2\lambda_\rho$$

$$\boxed{\begin{aligned} \bar{X}_{pq}^{(mn)} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Big\{ k_{xv_n} k_{yw_q}^B \Theta_3 \left(-\hat{x} k_{yw_p}^{B2} \lambda_x + \hat{y} k_{xv_m}^2 \lambda_y \right) \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \\ \cdot \left(\hat{x} \lambda_y - \hat{y} \lambda_x \right) \Big\} d^2\lambda_\rho \end{aligned}} \tag{332}$$

$$\bar{Y}_{pq}^{(mn)} = P_B \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{\,B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{\,B,\mathrm{TM}^z} dS}^{\bar{F}_{pq}^{(mn)}}$$

$$\boxed{\bar{Y}_{pq}^{(mn)} = P_B \bar{F}_{pq}^{(mn)}} \tag{333}$$

$$\bar{Z}_{pq}^{(mn)} = P_B^{-1} \overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} \cdot \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} dS}^{\bar{F}_{pq}^{(mn)}}$$

$$\boxed{\bar{Z}_{pq}^{(mn)} = P_B^{-1} \bar{F}_{pq}^{(mn)}} \tag{334}$$

$$\bar{\Gamma}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TE}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TE}^z} (\vec{\rho}') \, e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TM}^z} \times \hat{z}} \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \overbrace{\int\limits_{S'} \hat{z} \times \vec{e}_{v_n,w_q}^{B,\mathrm{TE}^z} (\vec{\rho}') \, e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS'}^{\vec{I}_{nq}^{\mathrm{TE}^z}} \right] d^2\lambda_\rho$$

$$\boxed{\begin{aligned} \bar{\Gamma}_{pq}^{(mn)} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Big\{ & k_{xv_m} k_{yw_p}^B \Theta_3 \left(\hat{x}\lambda_x + \hat{y}\lambda_y \right) \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \\ & \cdot \Big(\hat{x} k_{xv_n}^2 \lambda_y + \hat{y} k_{yw_q}^{B2} \lambda_x \Big) \Big\} \, d^2\lambda_\rho \end{aligned}} \tag{335}$$

$$\bar{\Delta}_{pq}^{(mn)} = \frac{Z_{v_n,w_q}^{B,\mathrm{TM}^z}}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \int\limits_{S'} \vec{h}_{v_n,w_q}^{B,\mathrm{TM}^z} (\vec{\rho}') \, e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS' \right] d^2\lambda_\rho$$

$$= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left[\overbrace{\int\limits_S \vec{e}_{v_m,w_p}^{B,\mathrm{TM}^z} e^{j\vec{\lambda}_\rho \cdot \vec{\rho}} dS}^{\vec{I}_{mp}^{\mathrm{TM}^z} \times \hat{z}} \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) \cdot \overbrace{\int\limits_{S'} \hat{z} \times \vec{e}_{v_n,w_q}^{B,\mathrm{TM}^z} (\vec{\rho}') \, e^{-j\vec{\lambda}_\rho \cdot \vec{\rho}'} dS'}^{\vec{I}_{nq}^{\mathrm{TM}^z}} \right] d^2\lambda_\rho$$

$$\boxed{\bar{\Delta}_{pq}^{(mn)} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \left\{ k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B \Theta_3 \left(-\hat{x}\lambda_x - \hat{y}\lambda_y \right) \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\substack{z=0 \\ z'=0} \right) \cdot \left(\hat{x}\lambda_y - \hat{y}\lambda_x \right) \right\} d^2\lambda_\rho} \tag{336}$$

Solving for Green Function-supported Equations.

Using the general solutions to the λ_y integrals from Appendix F, we can begin solving the Green function-supported submatrices.

$$\begin{aligned} \bar{O}_{pq}^{(mn)} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint_{-\infty}^{\infty} \left\{ \Theta_3 \left(\hat{x} k_{xv_m}^2 \lambda_y + \hat{y} k_{yw_p}^{B2} \lambda_x \right) \cdot \tilde{\overleftrightarrow{G}}_{hh} \left(\substack{z=0 \\ z'=0} \right) \right. \\ &\qquad \left. \cdot \left(\hat{x} k_{xv_n}^2 \lambda_y + \hat{y} k_{yw_q}^{B2} \lambda_x \right) \right\} d^2\lambda_\rho \\ &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint_{-\infty}^{\infty} \left\{ \Theta_3 \left(k_{xv_m}^2 k_{xv_n}^2 \lambda_y^2 \tilde{G}_{hh,xx} \left(\substack{z=0 \\ z'=0} \right) + k_{xv_m}^2 k_{yw_q}^{B2} \lambda_x \lambda_y \tilde{G}_{hh,xy} \left(\substack{z=0 \\ z'=0} \right) \right. \right. \\ &\qquad \left. \left. + k_{yw_p}^{B2} k_{xv_n}^2 \lambda_x \lambda_y \tilde{G}_{hh,yx} \left(\substack{z=0 \\ z'=0} \right) + k_{yw_p}^{B2} k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{hh,yy} \left(\substack{z=0 \\ z'=0} \right) \right) \right\} d^2\lambda_\rho \end{aligned} \tag{337}$$

The above can be broken up into TE^z and TM^z components. Additionally, it can be broken into components based on matrix position of the element taken from $\tilde{\overleftrightarrow{G}}_{hh}$. Therefore, beginning with the xx term

$$\begin{aligned} \bar{O}_{pq,xx}^{(mn)\mathrm{TE}^z} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint_{-\infty}^{\infty} \Theta_3 k_{xv_m}^2 k_{xv_n}^2 \lambda_y^2 \tilde{G}_{hh,xx}^{\mathrm{TE}^z} \left(\substack{z=0 \\ z'=0} \right) d^2\lambda_\rho \\ &= \frac{j k_{xv_m}^2 k_{xv_n}^2}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \overbrace{\int_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \lambda_{z\theta} \Upsilon_{16}^{\theta}}{2\lambda_\rho^2} d\lambda_y d\lambda_x}^{\Theta_4} \end{aligned}$$

$$
\begin{aligned}
&= \frac{jk_{xv_m}^2 k_{xv_n}^2}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}\,\omega\mu_t} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \left\{ 2\pi\delta_{p,q}\delta_{p,1} \left[\frac{h\lambda_{z\theta}^* \cos\left(\lambda_{z\theta}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\theta}^* d\right)} \right] \right. \\
&\quad + \pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right) \left[\frac{h\lambda_{z\theta p}\cos\left(\lambda_{z\theta p} d\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\sin\left(\lambda_{z\theta p} d\right)} \right] \\
&\quad - 2\pi \left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})} k_t \lambda_x \left(1 - e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\left(\lambda_x^2 + k_{yw_q}^{B2}\right)\sin\left(k_t d\right)} \right] \\
&\quad \left. + j\frac{4\pi}{d} \sum_{i=0}^{\infty} \frac{\left(\frac{i\pi}{d}\right)^2 \lambda_{y\theta i}\left(1 - e^{-j\lambda_{y\theta i} h}\right)}{\left(\lambda_{y\theta i}^2 - k_{yw_p}^{B2}\right)\left(\lambda_{y\theta i}^2 - k_{yw_q}^{B2}\right)\left(\lambda_x^2 + \lambda_{y\theta i}^2\right)\tau_\theta} \right\} d\lambda_x
\end{aligned} \tag{338}
$$

$$
\begin{aligned}
\bar{O}_{pq,xx}^{(mn)\mathrm{TM}^z} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{xv_m}^2 k_{xv_n}^2 \lambda_y^2 \tilde{G}_{hh,xx}^{\mathrm{TM}^z}\left({z=0 \atop z'=0}\right) d^2\lambda_\rho \\
&= \frac{j\omega\epsilon_t k_{xv_m}^2 k_{xv_n}^2}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \overbrace{\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y}\lambda_y^4 \lambda_{z\psi}^{-1}\Upsilon_{16}^{\psi}}{2\lambda_\rho^2} d\lambda_y d\lambda_x}^{\Theta_5} \\
&= \frac{j\omega\epsilon_t k_{xv_m}^2 k_{xv_n}^2}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \left\{ \pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right) \left[\frac{hk_{yw_p}^{B2}\cos\left(\lambda_{z\psi p} d\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\lambda_{z\psi p}\sin\left(\lambda_{z\psi p} d\right)} \right] \right. \\
&\qquad + 2\pi \left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})} \lambda_x^3 \left(1 - e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{k_t\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\left(\lambda_x^2 + k_{yw_q}^{B2}\right)\sin\left(k_t d\right)} \right] \\
&\qquad \left. + j\frac{4\pi}{d} \sum_{i=0}^{\infty} \frac{\lambda_{y\psi i}^3 \left(1 - e^{-j\lambda_{y\psi i} h}\right)}{\left(\lambda_{y\psi i}^2 - k_{yw_p}^{B2}\right)\left(\lambda_{y\psi i}^2 - k_{yw_q}^{B2}\right)\left(\lambda_x^2 + \lambda_{y\psi i}^2\right)\left(1+\delta_{i,0}\right)\tau_\psi} \right\} d\lambda_x
\end{aligned} \tag{339}
$$

It is important to note that $\frac{k_t}{\omega\mu_t} = \frac{\omega\epsilon_t}{k_t}$. Thus, on close inspection, it can be seen that the terms containing k_t from both the TE^z and TM^z components of the xx term cancel when ultimately added together. Thus,

$$\boxed{\begin{aligned}
\bar{O}_{pq,xx}^{(mn)\mathrm{TE}^z} &= \frac{jk_{xv_m}^2 k_{xv_n}^2}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}\omega\mu_t}\Theta_4 \\
\Theta_4 &= \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\lambda_x^2 \left\{ 2\pi\delta_{p,q}\delta_{p,1}\left[\frac{h\lambda_{z\theta}^*\cos\left(\lambda_{z\theta}^* d\right)}{\lambda_x^2\sin\left(\lambda_{z\theta}^* d\right)}\right]\right. \\
&\quad +\pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)\left[\frac{h\lambda_{z\theta p}\cos\left(\lambda_{z\theta p}d\right)}{\left(\lambda_x^2+k_{yw_p}^{B2}\right)\sin\left(\lambda_{z\theta p}d\right)}\right] \\
&\quad \left. +j\frac{4\pi}{d}\sum_{i=0}^{\infty}\frac{\left(\frac{i\pi}{d}\right)^2\lambda_{y\theta i}\left(1-e^{-j\lambda_{y\theta i}h}\right)}{\left(\lambda_{y\theta i}^2-k_{yw_p}^{B2}\right)\left(\lambda_{y\theta i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\theta i}^2\right)\tau_\theta}\right\} d\lambda_x
\end{aligned}} \tag{340}$$

$$\boxed{\begin{aligned}
\bar{O}_{pq,xx}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m}^2 k_{xv_n}^2}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}}\Theta_5 \\
\Theta_5 &= \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\left\{\left[\frac{\pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)hk_{yw_p}^{B2}\cos\left(\lambda_{z\psi p}d\right)}{\left(\lambda_x^2+k_{yw_p}^{B2}\right)\lambda_{z\psi p}\sin\left(\lambda_{z\psi p}d\right)}\right]\right. \\
&\quad \left. +j\frac{4\pi}{d}\sum_{i=0}^{\infty}\frac{\lambda_{y\psi i}^3\left(1-e^{-j\lambda_{y\psi i}h}\right)}{\left(\lambda_{y\psi i}^2-k_{yw_p}^{B2}\right)\left(\lambda_{y\psi i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\psi i}^2\right)\left(1+\delta_{i,0}\right)\tau_\psi}\right\} d\lambda_x
\end{aligned}} \tag{341}$$

Next, analyze the xy terms

$$\begin{aligned}
\bar{O}_{pq,xy}^{(mn)\mathrm{TE}^z} &= \frac{k_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}}\iint_{-\infty}^{\infty}\Theta_3\lambda_x\lambda_y\tilde{G}_{hh,xy}^{\mathrm{TE}^z}\left(\substack{z=0\\z'=0}\right)d^2\lambda_\rho \\
&= \frac{jk_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}\omega\mu_t}\int_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^2\int_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^2\lambda_{z\theta}\Upsilon_{16}^{\theta}}{2\lambda_\rho^2}d\lambda_y d\lambda_x
\end{aligned}$$

$$\boxed{\bar{O}_{pq,xy}^{(mn)\mathrm{TE}^z} = \frac{jk_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}\omega\mu_t}\Theta_4} \tag{342}$$

$$
\begin{aligned}
\bar{O}_{pq,xy}^{(mn)\mathrm{TM}^z} &= \frac{k_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 \lambda_x \lambda_y \tilde{G}_{hh,xy}^{\mathrm{TM}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) d^2\lambda_\rho \\
&= \frac{-j\omega\epsilon_t k_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \overbrace{\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \lambda_{z\psi}^{-1} \Upsilon_{16}^{\psi}}{2\lambda_\rho^2} d\lambda_y d\lambda_x}^{\Theta_6} \\
&= \frac{-j\omega\epsilon_t k_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \left\{ 2\pi \delta_{p,q} \delta_{p,1} \left[\frac{h \cos\left(\lambda_{z\psi}^* d\right)}{\lambda_x^2 \lambda_{z\psi}^* \sin\left(\lambda_{z\psi}^* d\right)} \right] \right. \\
&\qquad + \pi \delta_{p,q} \left(1 - \delta_{p,1}\right) \left(1 - \delta_{q,1}\right) \left[\frac{h \cos\left(\lambda_{z\psi p} d\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right) \lambda_{z\psi p} \sin\left(\lambda_{z\psi p} d\right)} \right] \\
&\qquad - 2\pi \left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})} \lambda_x \left(1 - e^{-\lambda_x h}\right) \cos\left(k_t d\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right) \left(\lambda_x^2 + k_{yw_q}^{B2}\right) k_t \sin\left(k_t d\right)} \right] \\
&\qquad \left. + j \frac{4\pi}{d} \sum_{i=0}^{\infty} \frac{\lambda_{y\psi i} \left(1 - e^{-j\lambda_{y\psi i} h}\right)}{\left(\lambda_{y\psi i}^2 - k_{yw_p}^{B2}\right) \left(\lambda_{y\psi i}^2 - k_{yw_q}^{B2}\right) \left(\lambda_x^2 + \lambda_{y\psi i}^2\right) \left(1 + \delta_{i,0}\right) \tau_\psi} \right\} d\lambda_x
\end{aligned}
\tag{343}
$$

Again, the k_t components of $\bar{O}_{pq,xy}^{(mn)\mathrm{TE}^z}$ and $\bar{O}_{pq,xy}^{(mn)\mathrm{TM}^z}$ perfectly cancel when added together. Furthermore, everywhere Θ_6 appears in this work, it is multiplied by either a $k_{yw_p}^B$ or a $k_{yw_q}^B$ or both. Thus, any term in Θ_6 requiring $p = q = 1$ is guaranteed to be destroyed. Therefore,

$$
\boxed{
\begin{aligned}
\bar{O}_{pq,xy}^{(mn)\mathrm{TM}^z} &= \frac{-j\omega\epsilon_t k_{xv_m}^2 k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \Theta_6 \\
\Theta_6 &= \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \left\{ \pi \delta_{p,q} \left(1 - \delta_{p,1}\right) \left(1 - \delta_{q,1}\right) \left[\frac{h \cos\left(\lambda_{z\psi p} d\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right) \lambda_{z\psi p} \sin\left(\lambda_{z\psi p} d\right)} \right] \right. \\
&\qquad \left. + j \frac{4\pi}{d} \sum_{i=0}^{\infty} \frac{\lambda_{y\psi i} \left(1 - e^{-j\lambda_{y\psi i} h}\right)}{\left(\lambda_{y\psi i}^2 - k_{yw_p}^{B2}\right) \left(\lambda_{y\psi i}^2 - k_{yw_q}^{B2}\right) \left(\lambda_x^2 + \lambda_{y\psi i}^2\right) \left(1 + \delta_{i,0}\right) \tau_\psi} \right\} d\lambda_x
\end{aligned}
}
\tag{344}
$$

Next, analyze the yx terms.

$$\bar{O}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{yw_p}^{B2} k_{xv_n}^2 \lambda_x \lambda_y \tilde{G}_{hh,yx}^{\mathrm{TE}^z} \left(\substack{z=0 \\ z'=0} \right) d^2\lambda_\rho$$

$$= \frac{j k_{xv_n}^2 k_{yw_p}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \lambda_{z\theta} \Upsilon_{16}^{\theta}}{2\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{O}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{j k_{xv_n}^2 k_{yw_p}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \Theta_4} \tag{345}$$

$$\bar{O}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{yw_p}^{B2} k_{xv_n}^2 \lambda_x \lambda_y \tilde{G}_{hh,yx}^{\mathrm{TM}^z} \left(\substack{z=0 \\ z'=0} \right) d^2\lambda_\rho$$

$$= \frac{-j\omega\epsilon_t k_{xv_n}^2 k_{yw_p}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \lambda_{z\psi}^{-1} \Upsilon_{16}^{\psi}}{2\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{O}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{-j\omega\epsilon_t k_{xv_n}^2 k_{yw_p}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \Theta_6} \tag{346}$$

Again, it can be shown that the k_t components of the yx terms perfectly cancel when added together. Finally, analyze the yy terms.

$$\bar{O}_{pq,yy}^{(mn)\mathrm{TE}^z} = \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{yw_p}^{B2} k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{hh,yy}^{\mathrm{TE}^z} \left(\substack{z=0 \\ z'=0} \right) d^2\lambda_\rho$$

$$= \frac{j k_{yw_p}^{B2} k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \lambda_{z\theta} \Upsilon_{16}^{\theta}}{2\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{O}_{pq,yy}^{(mn)\mathrm{TE}^z} = \frac{j k_{yw_p}^{B2} k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \Theta_4} \tag{347}$$

$$
\begin{aligned}
\bar{O}_{pq,yy}^{(mn)\mathrm{TM}^z} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{yw_p}^{B2} k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{hh,yy}^{\mathrm{TM}^z}\left(\substack{z=0\\z'=0}\right) d^2\lambda_\rho \\
&= \frac{j\omega\epsilon_t k_{yw_p}^{B2} k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \overbrace{\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^4 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^0 \lambda_{z\psi}^{-1} \Upsilon_{16}^{\psi}}{2\lambda_\rho^2} d\lambda_y d\lambda_x}^{\Theta_7} \\
&= \frac{j\omega\epsilon_t k_{yw_p}^{B2} k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^4 \left\{ -2\pi\delta_{p,q}\delta_{p,1} \left\{ \frac{h^3 \cos\left(\lambda_{z\psi}^* d\right)}{\lambda_x^2 \lambda_{z\psi}^* \sin\left(\lambda_{z\psi}^* d\right)} \right.\right. \\
&\quad \left. + \frac{12h}{\lambda_x^2 \sin\left(\lambda_{z\psi}^* d\right)} \left(\frac{2\cos\left(\lambda_{z\psi}^* d\right)}{\lambda_x^2 \lambda_{z\psi}^*} + \tau_\psi \left[-\frac{\cos\left(\lambda_{z\psi}^* d\right)}{\lambda_{z\psi}^{*3}} - \frac{d}{\lambda_{z\psi}^{*2} \sin\left(\lambda_{z\psi}^*\right)} \right] \right) \right\} \\
&\quad -2\pi \left[\frac{h\cos\left(\lambda_{z\psi}^* d\right)}{\lambda_x^2 \lambda_{z\psi}^* \sin\left(\lambda_{z\psi}^* d\right)} \right] \left[\frac{\delta_{p,1}\left(1-\delta_{q,1}\right)}{k_{yw_q}^{B2}} + \frac{\delta_{q,1}\left(1-\delta_{p,1}\right)}{k_{yw_p}^{B2}} \right] \\
&\quad +\pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right) \left[\frac{h\cos\left(\lambda_{z\psi p} d\right)}{k_{yw_p}^{B2}\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\lambda_{z\psi p}\sin\left(\lambda_{z\psi p} d\right)} \right] \\
&\quad +2\pi \left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})}\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\left(\lambda_x^2 + k_{yw_q}^{B2}\right) k_t \sin\left(k_t d\right)} \right] \\
&\quad \left. +j\frac{4\pi}{d}\sum_{i=0}^{\infty} \frac{\left(1-e^{-j\lambda_{y\psi i} h}\right)}{\lambda_{y\psi i}\left(\lambda_{y\psi i}^2 - k_{yw_p}^{B2}\right)\left(\lambda_{y\psi i}^2 - k_{yw_q}^{B2}\right)\left(\lambda_x^2 + \lambda_{y\psi i}^2\right)\left(1+\delta_{i,0}\right)\tau_\psi} \right\} d\lambda_x
\end{aligned}
\tag{348}
$$

Once again, the k_t components from both yy terms perfectly cancel out. Furthermore, it can be shown that anywhere Θ_7 appears in this work it is multiplied by $k_{yw_p}^B$ and $k_{yw_q}^B$. Thus any terms that require $p = 1$ or $q = 1$ are guaranteed to be destroyed.

Therefore,

$$\boxed{\begin{aligned}\bar{O}^{(mn)\mathrm{TM}^z}_{pq,yy} &= \frac{j\omega\epsilon_t k^{B2}_{yw_p} k^{B2}_{yw_q}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}}\Theta_7 \\ \Theta_7 = \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\lambda_x^4 &\left\{ \left[\frac{\pi h \cos(\lambda_{z\psi p} d)\,\delta_{p,q}\,(1-\delta_{p,1})\,(1-\delta_{q,1})}{k^{B2}_{yw_p}\left(\lambda_x^2 + k^{B2}_{yw_p}\right)\lambda_{z\psi p}\sin(\lambda_{z\psi p} d)} \right] \right. \\ &\left. + j\frac{4\pi}{d}\sum_{i=0}^{\infty} \frac{\left(1 - e^{-j\lambda_{y\psi i} h}\right)}{\lambda_{y\psi i}\left(\lambda^2_{y\psi i} - k^{B2}_{yw_p}\right)\left(\lambda^2_{y\psi i} - k^{B2}_{yw_q}\right)\left(\lambda_x^2 + \lambda^2_{y\psi i}\right)(1+\delta_{i,0})\,\tau_\psi} \right\} d\lambda_x\end{aligned}} \tag{349}$$

In summary,

$$\boxed{\begin{aligned}\bar{O}^{(mn)\mathrm{TE}^z}_{pq,xx} &= \frac{jk^2_{xv_m}k^2_{xv_n}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}\,\omega\mu_t}\Theta_4 & \bar{O}^{(mn)\mathrm{TE}^z}_{pq,xy} &= \frac{jk^2_{xv_m}k^{B2}_{yw_q}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}\,\omega\mu_t}\Theta_4 \\ \bar{O}^{(mn)\mathrm{TE}^z}_{pq,yx} &= \frac{jk^2_{xv_n}k^{B2}_{yw_p}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}\,\omega\mu_t}\Theta_4 & \bar{O}^{(mn)\mathrm{TE}^z}_{pq,yy} &= \frac{jk^{B2}_{yw_p}k^{B2}_{yw_q}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}\,\omega\mu_t}\Theta_4\end{aligned}} \tag{350}$$

$$\boxed{\begin{aligned}\bar{O}^{(mn)\mathrm{TM}^z}_{pq,xx} &= \frac{j\omega\epsilon_t k^2_{xv_m}k^2_{xv_n}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}}\Theta_5 & \bar{O}^{(mn)\mathrm{TM}^z}_{pq,xy} &= \frac{-j\omega\epsilon_t k^2_{xv_m}k^{B2}_{yw_q}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}}\Theta_6 \\ \bar{O}^{(mn)\mathrm{TM}^z}_{pq,yx} &= \frac{-j\omega\epsilon_t k^2_{xv_n}k^{B2}_{yw_p}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}}\Theta_6 & \bar{O}^{(mn)\mathrm{TM}^z}_{pq,yy} &= \frac{j\omega\epsilon_t k^{B2}_{yw_p}k^{B2}_{yw_q}}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}}\Theta_7\end{aligned}} \tag{351}$$

When all of these components are added together, it can be shown that

$$\boxed{\begin{aligned}\bar{O}^{(mn)}_{pq} &= \frac{j}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}\,\omega\mu_t}\left[k^2_{xv_m}\left(k^2_{xv_n} + k^{B2}_{yw_q}\right) + k^{B2}_{yw_p}\left(k^2_{xv_n} + k^{B2}_{yw_q}\right)\right]\Theta_4 \\ &+ \frac{j\omega\epsilon_t}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}}\left[k^2_{xv_m}\left(k^2_{xv_n}\Theta_5 - k^{B2}_{yw_q}\Theta_6\right) + k^{B2}_{yw_p}\left(k^{B2}_{yw_q}\Theta_7 - k^2_{xv_n}\Theta_6\right)\right]\end{aligned}} \tag{352}$$

$$\bar{P}^{(mn)}_{pq} = \frac{1}{4\pi^2 Z^{B,\mathrm{TE}^z}_{v_m,w_p}} \iint_{-\infty}^{\infty} \left\{ k_{xv_n}k^B_{yw_q}\Theta_3\left(-\hat{x}k^2_{xv_m}\lambda_y - \hat{y}k^{B2}_{yw_p}\lambda_x\right)\cdot \overset{\leftrightarrow}{\tilde{G}}_{hh}\left(\begin{smallmatrix} z=0 \\ z'=0\end{smallmatrix}\right)\cdot(\hat{x}\lambda_y \right.$$

$$\left. -\hat{y}\lambda_x)\right\} d^2\lambda_\rho$$

$$= \frac{k_{xv_n} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 \Big\{ -k_{xv_m}^2 \lambda_y^2 \tilde{G}_{hh,xx} \left(\substack{z=0\\z'=0}\right) + k_{xv_m}^2 \lambda_x \lambda_y \tilde{G}_{hh,xy} \left(\substack{z=0\\z'=0}\right) - k_{yw_p}^{B2} \lambda_x \lambda_y \tilde{G}_{hh,yx} \left(\substack{z=0\\z'=0}\right) + k_{yw_p}^{B2} \lambda_x^2 \tilde{G}_{hh,yy} \left(\substack{z=0\\z'=0}\right) \Big\} d^2\lambda_\rho \tag{353}$$

By careful inspection, it can be shown that

$$\boxed{\begin{aligned} \bar{P}_{pq,xx}^{(mn)\mathrm{TE}^z} &= -\frac{j k_{xv_m}^2 k_{xv_n} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \Theta_4 & \bar{P}_{pq,xy}^{(mn)\mathrm{TE}^z} &= \frac{j k_{xv_m}^2 k_{xv_n} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \Theta_4 \\ \bar{P}_{pq,yx}^{(mn)\mathrm{TE}^z} &= -\frac{j k_{xv_n} k_{yw_p}^{B2} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \Theta_4 & \bar{P}_{pq,yy}^{(mn)\mathrm{TE}^z} &= \frac{j k_{xv_n} k_{yw_p}^{B2} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z} \omega\mu_t} \Theta_4 \end{aligned}} \tag{354}$$

$$\boxed{\begin{aligned} \bar{P}_{pq,xx}^{(mn)\mathrm{TM}^z} &= -\frac{j\omega\epsilon_t k_{xv_m}^2 k_{xv_n} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \Theta_5 & \bar{P}_{pq,xy}^{(mn)\mathrm{TM}^z} &= -\frac{j\omega\epsilon_t k_{xv_m}^2 k_{xv_n} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \Theta_6 \\ \bar{P}_{pq,yx}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_n} k_{yw_p}^{B2} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \Theta_6 & \bar{P}_{pq,yy}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_n} k_{yw_p}^{B2} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \Theta_7 \end{aligned}} \tag{355}$$

When all of these components are added together, the TE^z components completely cancel. Therefore, it can be shown that

$$\boxed{\bar{P}_{pq}^{(mn)} = \frac{j\omega\epsilon_t k_{xv_n} k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TE}^z}} \left[k_{yw_p}^{B2} \left(\Theta_6 + \Theta_7\right) - k_{xv_m}^2 \left(\Theta_5 + \Theta_6\right) \right]} \tag{356}$$

$$\begin{aligned} \bar{S}_{pq}^{(mn)} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \iint\limits_{-\infty}^{\infty} \Big\{ k_{xv_m} k_{yw_p}^B \Theta_3 \left(\hat{x}\lambda_y - \hat{y}\lambda_x\right) \cdot \overleftrightarrow{\tilde{G}}_{hh} \left(\substack{z=0\\z'=0}\right) \cdot \Big(-\hat{x} k_{xv_n}^2 \lambda_y \\ &\qquad - \hat{y} k_{yw_q}^{B2} \lambda_x \Big) \Big\} d^2\lambda_\rho \\ &= \frac{k_{xv_m} k_{yw_p}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \iint\limits_{-\infty}^{\infty} \Theta_3 \Big\{ -k_{xv_n}^2 \lambda_y^2 \tilde{G}_{hh,xx} \left(\substack{z=0\\z'=0}\right) - k_{yw_q}^{B2} \lambda_x \lambda_y \tilde{G}_{hh,xy} \left(\substack{z=0\\z'=0}\right) \\ &\qquad + k_{xv_n}^2 \lambda_x \lambda_y \tilde{G}_{hh,yx} \left(\substack{z=0\\z'=0}\right) + k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{hh,yy} \left(\substack{z=0\\z'=0}\right) \Big\} d^2\lambda_\rho \end{aligned} \tag{357}$$

By careful inspection, it can be shown that

$$\boxed{\begin{aligned} \bar{S}_{pq,xx}^{(mn)\mathrm{TE}^z} &= -\frac{jk_{xv_m}k_{xv_n}^2 k_{yw_p}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 & \bar{S}_{pq,xy}^{(mn)\mathrm{TE}^z} &= -\frac{jk_{xv_m}k_{yw_p}^B k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 \\ \bar{S}_{pq,yx}^{(mn)\mathrm{TE}^z} &= \frac{jk_{xv_m}k_{xv_n}^2 k_{yw_p}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 & \bar{S}_{pq,yy}^{(mn)\mathrm{TE}^z} &= \frac{jk_{xv_m}k_{yw_p}^B k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 \end{aligned}} \tag{358}$$

$$\boxed{\begin{aligned} \bar{S}_{pq,xx}^{(mn)\mathrm{TM}^z} &= -\frac{j\omega\epsilon_t k_{xv_m}k_{xv_n}^2 k_{yw_p}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_5 & \bar{S}_{pq,xy}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m}k_{yw_p}^B k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_6 \\ \bar{S}_{pq,yx}^{(mn)\mathrm{TM}^z} &= -\frac{j\omega\epsilon_t k_{xv_m}k_{xv_n}^2 k_{yw_p}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_6 & \bar{S}_{pq,yy}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m}k_{yw_p}^B k_{yw_q}^{B2}}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_7 \end{aligned}} \tag{359}$$

When all of these components are added together, the TE^z components completely cancel. Thus,

$$\boxed{\bar{S}_{pq}^{(mn)} = \frac{j\omega\epsilon_t k_{xv_m}k_{yw_p}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\left[k_{yw_q}^{B2}\left(\Theta_6+\Theta_7\right) - k_{xv_n}^2\left(\Theta_5+\Theta_6\right)\right]} \tag{360}$$

$$\begin{aligned} \bar{T}_{pq}^{(mn)} &= \frac{1}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\iint\limits_{-\infty}^{\infty}\Big\{k_{xv_m}k_{xv_n}k_{yw_p}^B k_{yw_q}^B\Theta_3\left(\hat{x}\lambda_y-\hat{y}\lambda_x\right)\cdot\tilde{\overleftrightarrow{G}}_{hh}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)\cdot\left(\hat{x}\lambda_y\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.-\hat{y}\lambda_x\right)\Big\}\,d^2\lambda_\rho \\ &= \frac{k_{xv_m}k_{xv_n}k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\iint_{-\infty}^{\infty}\Theta_3\Big\{\lambda_y^2\tilde{G}_{hh,xx}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)-\lambda_x\lambda_y\tilde{G}_{hh,xy}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right) \\ &\qquad\qquad -\lambda_x\lambda_y\tilde{G}_{hh,yx}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)+\lambda_x^2\tilde{G}_{hh,yy}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)\Big\}\,d^2\lambda_\rho \end{aligned} \tag{361}$$

By careful inspection, it can be shown that

$$\boxed{\begin{aligned} \bar{T}_{pq,xx}^{(mn)\mathrm{TE}^z} &= \frac{jk_{xv_m}k_{xv_n}k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 & \bar{T}_{pq,xy}^{(mn)\mathrm{TE}^z} &= -\frac{jk_{xv_m}k_{xv_n}k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 \\ \bar{T}_{pq,yx}^{(mn)\mathrm{TE}^z} &= -\frac{jk_{xv_m}k_{xv_n}k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 & \bar{T}_{pq,yy}^{(mn)\mathrm{TE}^z} &= \frac{jk_{xv_m}k_{xv_n}k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}\,\omega\mu_t}\Theta_4 \end{aligned}} \tag{362}$$

$$\boxed{\begin{aligned} \bar{T}_{pq,xx}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_5 & \bar{T}_{pq,xy}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_6 \\ \bar{T}_{pq,yx}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_6 & \bar{T}_{pq,yy}^{(mn)\mathrm{TM}^z} &= \frac{j\omega\epsilon_t k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}}\Theta_7 \end{aligned}} \tag{363}$$

When all of these components are added together, the TE^z components completely cancel. Thus,

$$\boxed{\bar{T}_{pq}^{(mn)} = \frac{j\omega\epsilon_t k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2 Z_{v_m,w_p}^{B,\mathrm{TM}^z}} \left(\Theta_5 + 2\Theta_6 + \Theta_7\right)} \tag{364}$$

$$\begin{aligned} \bar{W}_{pq}^{(mn)} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Big\{ \Theta_3 \left(\hat{x} k_{yw_p}^{B2} \lambda_x - \hat{y} k_{xv_m}^2 \lambda_y\right) \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(z = z' = 0\right) \cdot \Big(\hat{x} k_{xv_n}^2 \lambda_y \\ + \hat{y} k_{yw_q}^{B2} \lambda_x\Big) \Big\} \, d^2\lambda_\rho \\ = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \Big\{ k_{xv_n}^2 k_{yw_p}^{B2} \lambda_x \lambda_y \tilde{G}_{eh,xx} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) + k_{yw_p}^{B2} k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{eh,xy} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \\ - k_{xv_m}^2 k_{xv_n}^2 \lambda_y^2 \tilde{G}_{eh,yx} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) - k_{xv_m}^2 k_{yw_q}^{B2} \lambda_x \lambda_y \tilde{G}_{eh,yy} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \Big\} \, d^2\lambda_\rho \end{aligned} \tag{365}$$

The above can be broken into components based on their matrix position in $\tilde{\overleftrightarrow{G}}_{eh}$ and by TE^z and TM^z components. Beginning with the xx components,

$$\begin{aligned} \bar{W}_{pq,xx}^{(mn)\mathrm{TE}^z} &= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{xv_n}^2 k_{yw_p}^{B2} \lambda_x \lambda_y \tilde{G}_{eh,xx}^{\mathrm{TE}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) d^2\lambda_\rho \\ &= -\frac{k_{xv_n}^2 k_{yw_p}^{B2}}{8\pi^2} \overbrace{\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \Upsilon_8^\theta}{\lambda_\rho^2} d\lambda_y d\lambda_x}^{\Theta_8} \end{aligned}$$

$$= -\frac{k_{xv_n}^2 k_{yw_p}^{B2}}{8\pi^2} \int_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \left\{ 2\pi\delta_{p,q}\delta_{p,1} \left[\frac{h}{\lambda_x^2}\right] + \frac{\pi h \delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right)} \right.$$
$$\left. -2\pi \left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})}\lambda_x\left(1-e^{-\lambda_x h}\right)}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\left(\lambda_x^2 + k_{yw_q}^{B2}\right)}\right]\right\} d\lambda_x \tag{366}$$

In a later section, it is shown that the last term in Θ_8 will always cancel with the last term from another added term. Therefore

$$\boxed{\begin{aligned} &\bar{W}_{pq,xx}^{(mn)\mathrm{TE}^z} = -\frac{k_{xv_n}^2 k_{yw_p}^{B2}}{8\pi^2}\Theta_8 \\ &\Theta_8 = \int_{-\infty}^{\infty} \Theta_3^{\lambda_x} \left\{ 2\pi\delta_{p,q}\delta_{p,1} h + \pi\left[\frac{\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right) h \lambda_x^2}{\left(\lambda_x^2 + k_{yw_p}^{B2}\right)}\right]\right\} d\lambda_x \end{aligned}} \tag{367}$$

$$\begin{aligned} \bar{W}_{pq,xx}^{(mn)\mathrm{TM}^z} &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \Theta_3 k_{xv_n}^2 k_{yw_p}^{B2} \lambda_x \lambda_y \tilde{G}_{eh,xx}^{\mathrm{TM}^z} \left(\substack{z=0 \\ z'=0}\right) d^2\lambda_\rho \\ &= \frac{k_{xv_n}^2 k_{yw_p}^{B2}}{4\pi^2} \iint_{-\infty}^{\infty} \Theta_3 \lambda_x \lambda_y \left(\frac{\lambda_x \lambda_y \Upsilon_8^{\psi}}{2\lambda_\rho^2}\right) d^2\lambda_\rho \\ &= \frac{k_{xv_n}^2 k_{yw_p}^{B2}}{8\pi^2}\Theta_8 \end{aligned}$$

$$\boxed{\bar{W}_{pq,xx}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_n}^2 k_{yw_p}^{B2}}{8\pi^2}\Theta_8} \tag{368}$$

Note that when added together, these two components entirely cancel. Thus, there is no $\bar{W}_{pq,xx}^{(mn)}$ contribution. In similar fashion, it can be shown that there is also no $\bar{W}_{pq,yy}^{(mn)}$ contribution due to similar cancellation of terms. Next, analyze the xy terms

$$\bar{W}_{pq,xy}^{(mn)\mathrm{TE}^z} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \Theta_3 k_{yw_p}^{B2} k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{eh,xy}^{\mathrm{TE}^z} \left(\substack{z=0 \\ z'=0}\right) d^2\lambda_\rho$$

$$= -\frac{k_{yw_p}^{B2} k_{yw_q}^{B2}}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \Upsilon_8^{\theta}}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{W}_{pq,xy}^{(mn)\mathrm{TE}^z} = -\frac{k_{yw_p}^{B2} k_{yw_q}^{B2}}{8\pi^2} \Theta_8} \tag{369}$$

$$\bar{W}_{pq,xy}^{(mn)\mathrm{TM}^z} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{yw_p}^{B2} k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{eh,xy}^{\mathrm{TM}^z} \left(\substack{z=0 \\ z'=0}\right) d^2\lambda_\rho$$

$$\begin{aligned}
&= -\frac{k_{yw_p}^{B2} k_{yw_q}^{B2}}{8\pi^2} \overbrace{\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^4 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^0 \Upsilon_8^{\theta}}{\lambda_\rho^2} d\lambda_y d\lambda_x}^{\Theta_9} \\
&= -\frac{k_{yw_p}^{B2} k_{yw_q}^{B2}}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^4 \left\{ -2\pi\delta_{p,q}\delta_{p,1} \left[\frac{h^3}{\lambda_x^2} + \frac{12h}{\lambda_x^4}\right] \right. \\
&-2\pi\delta_{p,1}\left(1-\delta_{q,1}\right)\left[\frac{h}{k_{yw_q}^{B2}\lambda_x^2}\right] - 2\pi\delta_{q,1}\left(1-\delta_{p,1}\right)\left[\frac{h}{k_{yw_p}^{B2}\lambda_x^2}\right] \\
&+\pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)\left[\frac{h}{k_{yw_p}^{B2}\left(\lambda_x^2 + k_{yw_p}^{B2}\right)}\right] \\
&\left. +2\pi\left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})}\left(1-e^{-\lambda_x h}\right)}{\lambda_x\left(\lambda_x^2 + k_{yw_p}^{B2}\right)\left(\lambda_x^2 + k_{yw_q}^{B2}\right)}\right]\right\} d\lambda_x
\end{aligned} \tag{370}$$

Note that when Θ_8 and Θ_9 are added together, the last term cancels completely. Furthermore, anywhere Θ_9 appears in this work it is multiplied by both $k_{yw_p}^{B}$ and $k_{yw_q}^{B}$. Thus, any term in Θ_9 that requires either $p = 1$ or $q = 1$ is guaranteed to be destroyed. Therefore

$$\boxed{\begin{aligned}\bar{W}^{(mn)\mathrm{TM}^z}_{pq,xy} &= -\frac{k^{B2}_{yw_p}k^{B2}_{yw_q}}{8\pi^2}\Theta_9\\ \Theta_9 &= \int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\left\{\pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)\left[\frac{h\lambda_x^4}{k^{B2}_{yw_p}\left(\lambda_x^2+k^{B2}_{yw_p}\right)}\right]\right\}d\lambda_x\end{aligned}} \tag{371}$$

Finally analyze the yx components.

$$\begin{aligned}\bar{W}^{(mn)\mathrm{TE}^z}_{pq,yx} &= \frac{1}{4\pi^2}\iint\limits_{-\infty}^{\infty}\Theta_3\left\{-k^2_{xv_m}k^2_{xv_n}\lambda_y^2\tilde{G}^{\mathrm{TE}^z}_{eh,yx}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)\right\}d^2\lambda_\rho\\ &= -\frac{k^2_{xv_m}k^2_{xv_n}}{8\pi^2}\int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^2\int\limits_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^2\Upsilon_8^\theta}{\lambda_\rho^2}d\lambda_y d\lambda_x\end{aligned}$$

$$\boxed{\bar{W}^{(mn)\mathrm{TE}^z}_{pq,yx} = -\frac{k^2_{xv_m}k^2_{xv_n}}{8\pi^2}\Theta_8} \tag{372}$$

$$\begin{aligned}\bar{W}^{(mn)\mathrm{TM}^z}_{pq,yx} &= \frac{1}{4\pi^2}\iint\limits_{-\infty}^{\infty}\Theta_3\left\{-k^2_{xv_m}k^2_{xv_n}\lambda_y^2\tilde{G}^{\mathrm{TM}^z}_{eh,yx}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)\right\}d^2\lambda_\rho\\ &= -\frac{k^2_{xv_m}k^2_{xv_n}}{8\pi^2}\overbrace{\int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\int\limits_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^4\Upsilon_8^\theta}{\lambda_\rho^2}d\lambda_y d\lambda_x}^{\Theta_{10}}\\ &= -\frac{k^2_{xv_m}k^2_{xv_n}}{8\pi^2}\int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\left\{\pi\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)\left[\frac{hk^{B2}_{yw_p}}{\left(\lambda_x^2+k^{B2}_{yw_p}\right)}\right]\right.\\ &\qquad\left.+2\pi\left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})}\lambda_x^3\left(1-e^{-\lambda_x h}\right)}{\left(\lambda_x^2+k^{B2}_{yw_p}\right)\left(\lambda_x^2+k^{B2}_{yw_q}\right)}\right]\right\}d\lambda_x\end{aligned} \tag{373}$$

It is important to note that when Θ_8 and Θ_{10} are added together, the last term cancels completely. Therefore

$$\boxed{\begin{aligned}\bar{W}_{pq,yx}^{(mn)\mathrm{TM}^z} &= -\frac{k_{xv_m}^2 k_{xv_n}^2}{8\pi^2}\Theta_{10} \\ \Theta_{10} &= \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\pi\left[\frac{\delta_{p,q}\left(1-\delta_{p,1}\right)\left(1-\delta_{q,1}\right)hk_{yw_p}^{B2}}{\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right]d\lambda_x\end{aligned}} \tag{374}$$

In summary

$$\boxed{\begin{aligned}\bar{W}_{pq,xx}^{(mn)\mathrm{TE}^z} &= -\bar{W}_{pq,xx}^{(mn)\mathrm{TM}^z} & \bar{W}_{pq,xy}^{(mn)\mathrm{TE}^z} &= -\frac{k_{yw_p}^{B2}k_{yw_q}^{B2}}{8\pi^2}\Theta_8 \\ \bar{W}_{pq,yx}^{(mn)\mathrm{TE}^z} &= -\frac{k_{xv_m}^2 k_{xv_n}^2}{8\pi^2}\Theta_8 & \bar{W}_{pq,yy}^{(mn)\mathrm{TE}^z} &= -\bar{W}_{pq,yy}^{(mn)\mathrm{TM}^z} \\ \bar{W}_{pq,xy}^{(mn)\mathrm{TM}^z} &= -\frac{k_{yw_p}^{B2}k_{yw_q}^{B2}}{8\pi^2}\Theta_9 & \bar{W}_{pq,yx}^{(mn)\mathrm{TM}^z} &= -\frac{k_{xv_m}^2 k_{xv_n}^2}{8\pi^2}\Theta_{10}\end{aligned}} \tag{375}$$

When all these components are added together, the TE^z and TM^z components components completely cancel for the xx and yy matrix position components. Thus

$$\boxed{\bar{W}_{pq}^{(mn)} = -\frac{1}{8\pi^2}\left[k_{xv_m}^2 k_{xv_n}^2\left(\Theta_8+\Theta_{10}\right)+k_{yw_p}^{B2}k_{yw_q}^{B2}\left(\Theta_8+\Theta_9\right)\right]} \tag{376}$$

$$\begin{aligned}\bar{X}_{pq}^{(mn)} &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\left\{k_{xv_n}k_{yw_q}^{B}\Theta_3\left(-\hat{x}k_{yw_p}^{B2}\lambda_x+\hat{y}k_{xv_m}^2\lambda_y\right)\cdot\overleftrightarrow{\tilde{G}}_{eh}\left(\substack{z=0\\z'=0}\right)\cdot\left(\hat{x}\lambda_y-\hat{y}\lambda_x\right)\right\}d^2\lambda_\rho \\ &= \frac{k_{xv_n}k_{yw_q}^{B}}{4\pi^2}\iint_{-\infty}^{\infty}\Theta_3\left\{-k_{yw_p}^{B2}\lambda_x\lambda_y\tilde{G}_{eh,xx}\left(\substack{z=0\\z'=0}\right)+k_{yw_p}^{B2}\lambda_x^2\tilde{G}_{eh,xy}\left(\substack{z=0\\z'=0}\right)\right. \\ &\qquad \left.+k_{xv_m}^2\lambda_y^2\tilde{G}_{eh,yx}\left(\substack{z=0\\z'=0}\right)-k_{xv_m}^2\lambda_x\lambda_y\tilde{G}_{eh,yy}\left(\substack{z=0\\z'=0}\right)\right\}d^2\lambda_\rho\end{aligned} \tag{377}$$

In a similar fashion to the previous section, it can be shown that xx and yy TE^z components cancel with their respective TM^z components. Therefore, begin by analyzing the xy components.

$$\bar{X}_{pq,xy}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_n}k_{yw_q}^B}{4\pi^2}\iint\limits_{-\infty}^{\infty}\Theta_3 k_{yw_p}^{B2}\lambda_x^2\tilde{G}_{eh,xy}^{\mathrm{TE}^z}\left(\substack{z=0\\z'=0}\right)d^2\lambda_\rho$$

$$= -\frac{k_{xv_n}k_{yw_p}^{B2}k_{yw_q}^B}{8\pi^2}\int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^2\int\limits_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^2\Upsilon_8^\theta}{\lambda_\rho^2}d\lambda_y d\lambda_x$$

$$\boxed{\bar{X}_{pq,xy}^{(mn)\mathrm{TE}^z} = -\frac{k_{xv_n}k_{yw_p}^{B2}k_{yw_q}^B}{8\pi^2}\Theta_8} \tag{378}$$

$$\bar{X}_{pq,xy}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_n}k_{yw_q}^B}{4\pi^2}\iint\limits_{-\infty}^{\infty}\Theta_3 k_{yw_p}^{B2}\lambda_x^2\tilde{G}_{eh,xy}^{\mathrm{TM}^z}\left(\substack{z=0\\z'=0}\right)d^2\lambda_\rho$$

$$= -\frac{k_{xv_n}k_{yw_p}^{B2}k_{yw_q}^B}{8\pi^2}\int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^4\int\limits_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^0\Upsilon_8^\psi}{\lambda_\rho^2}d\lambda_y d\lambda_x$$

$$\boxed{\bar{X}_{pq,xy}^{(mn)\mathrm{TM}^z} = -\frac{k_{xv_n}k_{yw_p}^{B2}k_{yw_q}^B}{8\pi^2}\Theta_9} \tag{379}$$

Next, analyze the yx components.

$$\bar{X}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_n}k_{yw_q}^B}{4\pi^2}\iint\limits_{-\infty}^{\infty}\Theta_3 k_{xv_m}^2\lambda_y^2\tilde{G}_{eh,yx}^{\mathrm{TE}^z}\left(\substack{z=0\\z'=0}\right)d^2\lambda_\rho$$

$$= \frac{k_{xv_m}^2 k_{xv_n}k_{yw_q}^B}{8\pi^2}\int\limits_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^2\int\limits_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_x}\lambda_y^2\Upsilon_8^\theta}{\lambda_\rho^2}d\lambda_y d\lambda_x$$

$$\boxed{\bar{X}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_m}^2 k_{xv_n}k_{yw_q}^B}{8\pi^2}\Theta_8} \tag{380}$$

$$\bar{X}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_n}k_{yw_q}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{xv_m}^2 \lambda_y^2 \tilde{G}_{eh,yx}^{\mathrm{TM}^z} \left(\substack{z=0\\z'=0}\right) d^2\lambda_\rho$$

$$= \frac{k_{xv_m}^2 k_{xv_n} k_{yw_q}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^4 \Upsilon_8^\psi}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{X}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_m}^2 k_{xv_n} k_{yw_q}^B}{8\pi^2} \Theta_{10}} \tag{381}$$

Adding all the components together implies that

$$\boxed{\bar{X}_{pq}^{(mn)} = \frac{k_{xv_n}k_{yw_q}^B}{8\pi^2} \left[k_{xv_m}^2 \left(\Theta_8 + \Theta_{10}\right) - k_{yw_p}^{B2} \left(\Theta_8 + \Theta_9\right)\right]} \tag{382}$$

$$\begin{aligned} \bar{\Gamma}_{pq}^{(mn)} &= \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left\{ k_{xv_m}k_{yw_p}^B \Theta_3 \left(\hat{x}\lambda_x + \hat{y}\lambda_y\right) \cdot \overleftrightarrow{\tilde{G}}_{eh} \left(\substack{z=0\\z'=0}\right) \cdot \left(\hat{x}k_{xv_n}^2 \lambda_y + \hat{y} k_{yw_q}^{B2} \lambda_x\right) \right\} d^2\lambda_\rho \\ &= \frac{k_{xv_m}k_{yw_p}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \left\{ k_{xv_n}^2 \lambda_x \lambda_y \tilde{G}_{eh,xx} \left(\substack{z=0\\z'=0}\right) + k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{eh,xy} \left(\substack{z=0\\z'=0}\right) \right. \\ &\qquad \left. + k_{xv_n}^2 \lambda_y^2 \tilde{G}_{eh,yx} \left(\substack{z=0\\z'=0}\right) + k_{yw_q}^{B2} \lambda_x \lambda_y \tilde{G}_{eh,yy} \left(\substack{z=0\\z'=0}\right) \right\} d^2\lambda_\rho \end{aligned} \tag{383}$$

It can be shown that xx and yy TE^z components cancel with their respective TM^z components. Therefore, begin by analyzing the xy components.

$$\bar{\Gamma}_{pq,xy}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_m}k_{yw_p}^B}{4\pi^2} \iint_{-\infty}^{\infty} \Theta_3 k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{eh,xy}^{\mathrm{TE}^z} \left(\substack{z=0\\z'=0}\right) d^2\lambda_\rho$$

$$= -\frac{k_{xv_m}k_{yw_p}^B k_{yw_q}^{B2}}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \Upsilon_8^\theta}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Gamma}_{pq,xy}^{(mn)\mathrm{TE}^z} = -\frac{k_{xv_m}k_{yw_p}^B k_{yw_q}^{B2}}{8\pi^2} \Theta_8} \tag{384}$$

$$\bar{\Gamma}_{pq,xy}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_m} k_{yw_p}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{yw_q}^{B2} \lambda_x^2 \tilde{G}_{eh,xy}^{\mathrm{TM}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) d^2\lambda_\rho$$

$$= -\frac{k_{xv_m} k_{yw_p}^B k_{yw_q}^{B2}}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^4 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^0 \Upsilon_8^\psi}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Gamma}_{pq,xy}^{(mn)\mathrm{TM}^z} = -\frac{k_{xv_m} k_{yw_p}^B k_{yw_q}^{B2}}{8\pi^2} \Theta_9} \tag{385}$$

Next, analyze the yx components.

$$\bar{\Gamma}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_m} k_{yw_p}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{xv_n}^2 \lambda_y^2 \tilde{G}_{eh,yx}^{\mathrm{TE}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) d^2\lambda_\rho$$

$$= \frac{k_{xv_m} k_{xv_n}^2 k_{yw_p}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \Upsilon_8^\theta}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Gamma}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_m} k_{xv_n}^2 k_{yw_p}^B}{8\pi^2} \Theta_8} \tag{386}$$

$$\bar{\Gamma}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_m} k_{yw_p}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 k_{xv_n}^2 \lambda_y^2 \tilde{G}_{eh,yx}^{\mathrm{TM}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix} \right) d^2\lambda_\rho$$

$$= \frac{k_{xv_m} k_{xv_n}^2 k_{yw_p}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^4 \Upsilon_8^\psi}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Gamma}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_m} k_{xv_n}^2 k_{yw_p}^B}{8\pi^2} \Theta_{10}} \tag{387}$$

Adding all of these components together implies that

$$\boxed{\bar{\Gamma}_{pq}^{(mn)} = \frac{k_{xv_m} k_{yw_p}^B}{8\pi^2} \left[k_{xv_n}^2 \left(\Theta_8 + \Theta_{10} \right) - k_{yw_q}^{B2} \left(\Theta_8 + \Theta_9 \right) \right]} \tag{388}$$

$$\bar{\Delta}_{pq}^{(mn)} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \left\{ k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B \Theta_3 \left(-\hat{x}\lambda_x - \hat{y}\lambda_y\right) \cdot \tilde{\overleftrightarrow{G}}_{eh} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \cdot \left(\hat{x}\lambda_y - \hat{y}\lambda_x\right) \right\} d^2\lambda_\rho$$

$$= \frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \left\{ -\lambda_x\lambda_y \tilde{G}_{eh,xx} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) + \lambda_x^2 \tilde{G}_{eh,xy} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \right.$$
$$\left. -\lambda_y^2 \tilde{G}_{eh,yx} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) + \lambda_x\lambda_y \tilde{G}_{eh,yy} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) \right\} d^2\lambda_\rho \tag{389}$$

It can be shown that xx and yy TE^z components cancel with their respective TM^z components. Therefore, begin by analyzing the xy components.

$$\bar{\Delta}_{pq,xy}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \lambda_x^2 \tilde{G}_{eh,xy}^{\mathrm{TE}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) d^2\lambda_\rho$$
$$= -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \Upsilon_8^\theta}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Delta}_{pq,xy}^{(mn)\mathrm{TE}^z} = -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \Theta_8} \tag{390}$$

$$\bar{\Delta}_{pq,xy}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \lambda_x^2 \tilde{G}_{eh,xy}^{\mathrm{TM}^z} \left(\begin{smallmatrix} z=0 \\ z'=0 \end{smallmatrix}\right) d^2\lambda_\rho$$
$$= -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^4 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^0 \Upsilon_8^\psi}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Delta}_{pq,xy}^{(mn)\mathrm{TM}^z} = -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \Theta_9} \tag{391}$$

$$\bar{\Delta}_{pq,yx}^{(mn)\mathrm{TE}^z} = \frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \left\{ -\lambda_y^2 \tilde{G}_{eh,yx}^{\mathrm{TE}^z} \left(\substack{z=0 \\ z'=0} \right) \right\} d^2\lambda_\rho$$

$$= -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^2 \Upsilon_8^\theta}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Delta}_{pq,yx}^{(mn)\mathrm{TE}^z} = -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \Theta_8} \tag{392}$$

$$\bar{\Delta}_{pq,yx}^{(mn)\mathrm{TM}^z} = \frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{4\pi^2} \iint\limits_{-\infty}^{\infty} \Theta_3 \left\{ -\lambda_y^2 \tilde{G}_{eh,yx}^{\mathrm{TM}^z} \left(\substack{z=0 \\ z'=0} \right) \right\} d^2\lambda_\rho$$

$$= -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x} \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y} \lambda_y^4 \Upsilon_8^\psi}{\lambda_\rho^2} d\lambda_y d\lambda_x$$

$$\boxed{\bar{\Delta}_{pq,yx}^{(mn)\mathrm{TM}^z} = -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \Theta_{10}} \tag{393}$$

Adding all these components together implies that

$$\boxed{\bar{\Delta}_{pq}^{(mn)} = -\frac{k_{xv_m} k_{xv_n} k_{yw_p}^B k_{yw_q}^B}{8\pi^2} \left(2\Theta_8 + \Theta_9 + \Theta_{10} \right)} \tag{394}$$

5.5 Dominant Mode Analysis

If it is assumed that only the dominant TE_{10}^z mode is present in the analysis, that implies that all submatrices of $\overleftrightarrow{A}$ containing TM^z observations or excitations no longer exist. Therefore, $\bar{B}$, $\bar{D}$, $\bar{E}$, $\bar{F}$, $\bar{H}$, $\bar{J}$, $\bar{K}$, $\bar{L}$, $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\bar{T}$, $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{\Gamma}$, and $\bar{\Delta}$

all cease to exist. That implies that the matrix equations become

$$\overbrace{\begin{bmatrix} \bar{A}_{1,1}^{(1,1)} & -\bar{C}_{1,1}^{(1,1)} & -\bar{C}_{1,1}^{(1,1)} & 0 \\ \bar{G}_{1,1}^{(1,1)} & \bar{I}_{1,1}^{(1,1)} & -\bar{I}_{1,1}^{(1,1)} & 0 \\ 0 & \bar{M}_{1,1}^{(1,1)} & -\bar{N}_{1,1}^{(1,1)} & \bar{O}_{1,1}^{(1,1)} \\ 0 & \bar{U}_{1,1}^{(1,1)} & \bar{V}_{1,1}^{(1,1)} & \bar{W}_{1,1}^{(1,1)} \end{bmatrix}}^{\overleftrightarrow{\mathbf{A}}} \overbrace{\begin{bmatrix} R^{A,\mathrm{TE}^z} \\ T^{B,\mathrm{TE}^z} \\ R^{B,\mathrm{TE}^z} \\ T^{C,\mathrm{TE}^z} \end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix} -\bar{A}_{1,1}^{(1,1)} \\ \bar{G}_{1,1}^{(1,1)} \\ 0 \\ 0 \end{bmatrix}}^{\vec{b}} \tag{395}$$

$$\begin{aligned} \bar{A}_{1,1}^{(1,1)} &= \tfrac{h\pi^2}{4a}, & \bar{C}_{1,1}^{(1,1)} &= \tfrac{h\pi^2}{4a} = \bar{A}_{1,1}^{(1,1)} \\ \bar{G}_{1,1}^{(1,1)} &= \tfrac{\bar{A}_{1,1}^{(1,1)}}{Z_{v_1,w_1}^{B,\mathrm{TE}^z} Z_{v_1,w_1}^{A,\mathrm{TE}^z}}, & \bar{I}_{1,1}^{(1,1)} &= \tfrac{\bar{C}_{1,1}^{(1,1)}}{Z_{v_1,w_1}^{B,\mathrm{TE}^z} Z_{v_1,w_1}^{B,\mathrm{TE}^z}} \\ \bar{M}_{1,1}^{(1,1)} &= \tfrac{P_B\bar{C}_{1,1}^{(1,1)}}{Z_{v_1,w_1}^{B,\mathrm{TE}^z} Z_{v_1,w_1}^{B,\mathrm{TE}^z}} = P_B\bar{I}_{1,1}^{(1,1)}, & \bar{N}_{1,1}^{(1,1)} &= \tfrac{P_B^{-1}\bar{C}_{1,1}^{(1,1)}}{Z_{v_1,w_1}^{B,\mathrm{TE}^z} Z_{v_1,w_1}^{B,\mathrm{TE}^z}} = P_B^{-1}\bar{I}_{1,1}^{(1,1)} \\ \bar{U}_{1,1}^{(1,1)} &= P_B\bar{C}_{1,1}^{(1,1)} = P_B\bar{A}_{1,1}^{(1,1)}, & \bar{V}_{1,1}^{(1,1)} &= P_B^{-1}\bar{C}_{1,1}^{(1,1)} = P_B^{-1}\bar{A}_{1,1}^{(1,1)} \end{aligned} \tag{396}$$

$$\bar{O}_{1,1}^{(1,1)} = \frac{jk_{xv_1}^4}{4\pi^2 Z_{v_1,w_1}^{B,\mathrm{TE}^z}\,\omega\mu_t}\Theta_4 + \frac{jk_{xv_1}^4\omega\epsilon_t}{4\pi^2 Z_{v_1,w_1}^{B,\mathrm{TE}^z}}\Theta_5 = \frac{j\pi^2}{4a^4 Z_{v_1,w_1}^{B,\mathrm{TE}^z}}\left[\frac{1}{\omega\mu_t}\Theta_4 + \omega\epsilon_t\Theta_5\right] \tag{397}$$

$$\Theta_4 = \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\lambda_x^2 \left\{ 2\pi\left[\frac{h\lambda_{z\theta}^*\cos\left(\lambda_{z\theta}^* d\right)}{\lambda_x^2\sin\left(\lambda_{z\theta}^* d\right)}\right] + j\frac{4\pi}{d}\sum_{i=0}^{\infty}\frac{\left(\frac{i\pi}{d}\right)^2\left(1-e^{-j\lambda_{y\theta i}h}\right)}{\lambda_{y\theta i}^3\left(\lambda_x^2+\lambda_{y\theta i}^2\right)\tau_\theta}\right\} d\lambda_x \tag{398}$$

$$\Theta_5 = \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\left\{ j\frac{4\pi}{d}\sum_{i=0}^{\infty}\frac{\left(1-e^{-j\lambda_{y\psi i}h}\right)}{\lambda_{y\psi i}\left(\lambda_x^2+\lambda_{y\psi i}^2\right)\left(1+\delta_{i,0}\right)\tau_\psi}\right\} d\lambda_x \tag{399}$$

$$\Theta_3^{\lambda_x} = \frac{4\cos^2\left(\lambda_x\frac{a}{2}\right)}{\left(\lambda_x^2-\frac{\pi^2}{a^2}\right)^2} \tag{400}$$

$$\bar{W}_{1,1}^{(1,1)} = -\frac{k_{xv_1}^4}{8\pi^2}\left(\Theta_8+\Theta_{10}\right) \tag{401}$$

$$\begin{aligned} \Theta_8 &= \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\left\{2\pi\delta_{1,1}\delta_{1,1}h + \pi\left[\frac{\delta_{1,1}\left(1-\delta_{1,1}\right)\left(1-\delta_{1,1}\right)h\lambda_x^2}{\left(\lambda_x^2+k_{yw_1}^{B2}\right)}\right]\right\} d\lambda_x \\ &= 2\pi h\int_{-\infty}^{\infty}\frac{4\cos^2\left(\lambda_x\frac{a}{2}\right)}{\left(\lambda_x+k_{xv_1}\right)\left(\lambda_x-k_{xv_1}\right)\left(\lambda_x+k_{xv_1}\right)\left(\lambda_x-k_{xv_1}\right)}d\lambda_x \end{aligned} \tag{402}$$

Noting that $\cos^2\left(\lambda_x \frac{a}{2}\right) = 1 - \sin^2\left(\lambda_x \frac{a}{2}\right)$ implies that

$$\cos^2\left(\lambda_x \frac{a}{2}\right) = 1 - \frac{1 - e^{j\lambda_x a}}{4} - \frac{1 - e^{-j\lambda_x a}}{4} = \frac{1 + e^{j\lambda_x a}}{4} + \frac{1 + e^{-j\lambda_x a}}{4} \tag{403}$$

$$\Rightarrow \Theta_8 = 2\pi h \left[\int_{-\infty}^{\infty} \frac{\left(1 + e^{j\lambda_x a}\right)}{\left(\lambda_x + k_{xv_1}\right)^2 \left(\lambda_x - k_{xv_1}\right)^2} d\lambda_x + \int_{-\infty}^{\infty} \frac{\left(1 + e^{-j\lambda_x a}\right)}{\left(\lambda_x + k_{xv_1}\right)^2 \left(\lambda_x - k_{xv_1}\right)^2} d\lambda_x \right] \tag{404}$$

The first integral can be evaluated under UHPC, while the second integral can be evaluated under LHPC. It can be seen that there are order-2 poles at $\lambda_x = \pm k_{xv_1}$, as depicted in the complex plane in fig. 13.

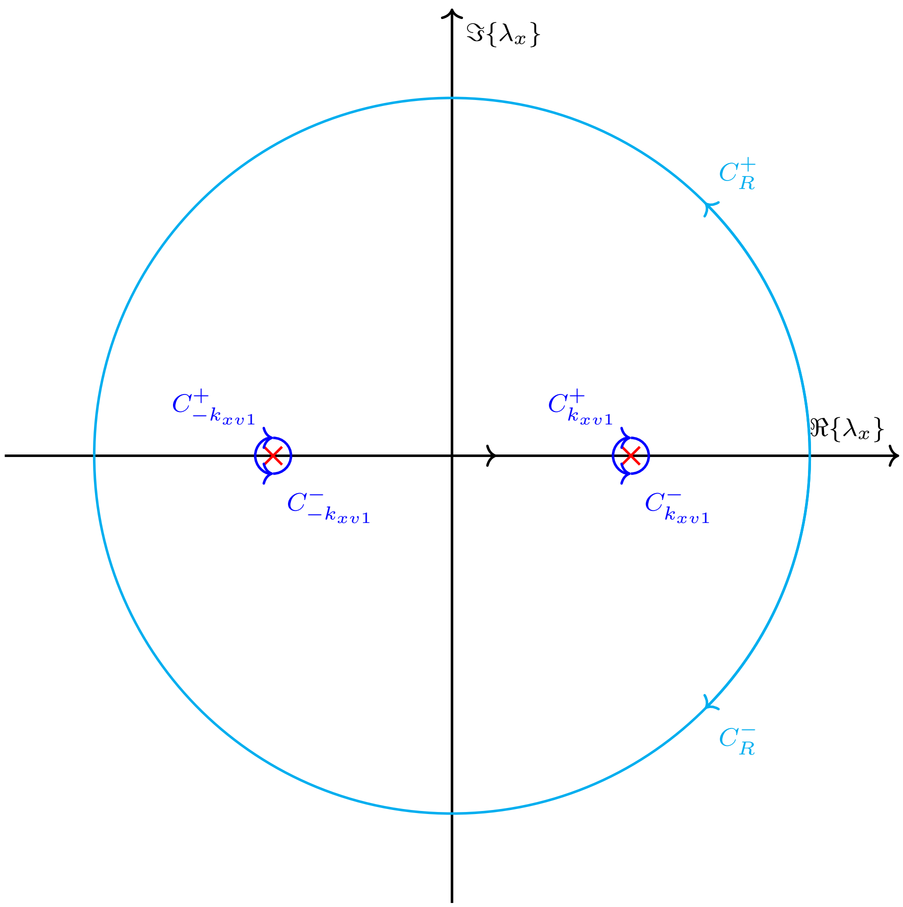

Figure 13. Complex poles (red) of Θ_8, deformation contours around those poles (blue) and closure contours as $R \to \infty$ (cyan) in the complex λ_x-plane.

Begin by analyzing the $C^{+}_{k_{xv_1}}$ contribution, where the integrand is suppressed for notational convenience.

$$
\begin{aligned}
\oint_{C^{+}_{k_{xv_1}}} &= j\pi \frac{\partial}{\partial \lambda_x} \left[\frac{\left(1 + e^{j\lambda_x a}\right)}{\left(\lambda_x + k_{xv_1}\right)^2} \right] \Bigg|_{\lambda_x = k_{xv_1}} \\
&= j\pi \left[\frac{-ja}{\left(k_{xv_1} + k_{xv_1}\right)^2} \right] \\
&= \pi \left[\frac{a}{4\frac{\pi^2}{a^2}} \right] \\
&= \frac{a^3}{4\pi}
\end{aligned}
\tag{405}
$$

It can be shown that under LHPC

$$
\oint_{C^{-}_{k_{xv_1}}} = -\frac{a^3}{4\pi} \tag{406}
$$

Next, analyze the $C^{+}_{-k_{xv_1}}$ contribution.

$$
\begin{aligned}
\oint_{C^{+}_{-k_{xv_1}}} &= j\pi \frac{\partial}{\partial \lambda_x} \left[\frac{\left(1 + e^{j\lambda_x a}\right)}{\left(\lambda_x - k_{xv_1}\right)^2} \right] \Bigg|_{\lambda_x = -k_{xv_1}} \\
&= j\pi \left[\frac{-ja}{\left(-k_{xv_1} - k_{xv_1}\right)^2} \right] \\
&= \pi \left[\frac{a}{4\frac{\pi^2}{a^2}} \right] \\
&= \frac{a^3}{4\pi}
\end{aligned}
\tag{407}
$$

Finally, it can be shown that under LHPC

$$
\oint_{C^{-}_{-k_{xv_1}}} = -\frac{a^3}{4\pi} \tag{408}
$$

Combining all these residue contributions together implies that

$$\Theta_8 = 2\pi h \left[\frac{a^3}{\pi}\right] = 2a^3 h \tag{409}$$

$$\Theta_{10} = \int_{-\infty}^{\infty} \Theta_3^{\lambda_x} \pi \left[\frac{\delta_{1,1}\left(1-\delta_{1,1}\right)\left(1-\delta_{1,1}\right) h k_{yw_1}^{B2}}{\left(\lambda_x^2 + k_{yw_1}^{B2}\right)}\right] d\lambda_x = 0 \tag{410}$$

Therefore

$$\begin{aligned}
\bar{W}_{1,1}^{(1,1)} &= -\frac{k_{xv_1}^4}{8\pi^2}\left(2a^3 h\right) \\
&= -\frac{2a^3 h}{8\pi^2}\left(\frac{\pi}{a}\right)^4 \\
&= -\frac{h\pi^2}{4a} = -\bar{A}_{1,1}^{(1,1)}
\end{aligned} \tag{411}$$

Note that if region B is filled with air

$$\bar{G}_{1,1}^{(1,1)} = \frac{\bar{A}_{1,1}^{(1,1)}}{\left(Z_{v_1,w_1}^{A,\mathrm{TE}^z}\right)^2} \tag{412}$$

$$\bar{I}_{1,1}^{(1,1)} = \bar{G}_{1,1}^{(1,1)} \tag{413}$$

$$\bar{M}_{1,1}^{(1,1)} = P_B \bar{G}_{1,1}^{(1,1)} \tag{414}$$

$$\bar{N}_{1,1}^{(1,1)} = P_B^{-1} \bar{G}_{1,1}^{(1,1)} \tag{415}$$

Now, the matrix equation can be further simplified. If region B is filled with air, the first and fourth rows can be divided by $\bar{A}_{1,1}^{(1,1)}$. The second and third rows can be divided by $\bar{G}_{1,1}^{(1,1)}$. This simplifies the matrix equations to

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & P_B & -P_B^{-1} & \frac{\bar{O}_{1,1}^{(1,1)}}{\bar{G}_{1,1}^{(1,1)}} \\ 0 & P_B & P_B^{-1} & -1 \end{bmatrix} \begin{bmatrix} R^{A,\mathrm{TE}^z} \\ T^{B,\mathrm{TE}^z} \\ R^{B,\mathrm{TE}^z} \\ T^{C,\mathrm{TE}^z} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{416}$$

By Gauss-Jordan elimination, it can be shown that

$$\begin{bmatrix} R^{A,\mathrm{TE}^z} \\ T^{B,\mathrm{TE}^z} \\ R^{B,\mathrm{TE}^z} \\ T^{C,\mathrm{TE}^z} \end{bmatrix} = \begin{bmatrix} P_B^2 \left[\frac{\bar{G}_{1,1}^{(1,1)}+\bar{O}_{1,1}^{(1,1)}}{\bar{G}_{1,1}^{(1,1)}-\bar{O}_{1,1}^{(1,1)}}\right] \\ 1 \\ P_B^2 \left[\frac{\bar{G}_{1,1}^{(1,1)}+\bar{O}_{1,1}^{(1,1)}}{\bar{G}_{1,1}^{(1,1)}-\bar{O}_{1,1}^{(1,1)}}\right] \\ -2P_B\left(\frac{\bar{G}_{1,1}^{(1,1)}}{\bar{O}_{1,1}^{(1,1)}-\bar{G}_{1,1}^{(1,1)}}\right) \end{bmatrix} \tag{417}$$

$$\begin{aligned}
\Rightarrow S_{11}^{(1)} &= R^{A,\mathrm{TE}^z} = P_B^2\left[\frac{\bar{G}_{1,1}^{(1,1)}+\bar{O}_{1,1}^{(1,1)}}{\bar{G}_{1,1}^{(1,1)}-\bar{O}_{1,1}^{(1,1)}}\right] \\
&= P_B^2\left[\frac{\left(\frac{h\pi^2}{4a\left(Z_{v_1,w_1}^{B,\mathrm{TE}^z}\right)^2}\right)+\left(\frac{j\pi^2}{8a^4 Z_{v_1,w_1}^{B,\mathrm{TE}^z}}\left[\frac{1}{\omega\mu_t}\Theta_4+\omega\epsilon_t\Theta_5\right]\right)}{\left(\frac{h\pi^2}{4a\left(Z_{v_1,w_1}^{B,\mathrm{TE}^z}\right)^2}\right)-\left(\frac{j\pi^2}{8a^4 Z_{v_1,w_1}^{B,\mathrm{TE}^z}}\left[\frac{1}{\omega\mu_t}\Theta_4+\omega\epsilon_t\Theta_5\right]\right)}\right] \\
&= P_B^2\left[\frac{2a^3h\omega\mu_t + jZ_{v_1,w_1}^{B,\mathrm{TE}^z}\left(\Theta_4+k_t^2\Theta_5\right)}{2a^3h\omega\mu_t - \underbrace{jZ_{v_1,w_1}^{B,\mathrm{TE}^z}\left(\Theta_4+k_t^2\Theta_5\right)}_{\Theta_{11}}}\right]
\end{aligned} \tag{418}$$

$$\begin{aligned}
\Theta_{11} &= jZ_{v_1,w_1}^{B,\mathrm{TE}^z}\left(\Theta_4+k_t^2\Theta_5\right) \\
&= 2\pi Z_{v_1,w_1}^{B,\mathrm{TE}^z}\int_{-\infty}^{\infty}\Theta_3^{\lambda_x}\left\{\frac{jh\lambda_{z\theta}^*\cos\left(\lambda_{z\theta}^* d\right)}{\sin\left(\lambda_{z\theta}^* d\right)}\right. \\
&\quad\left.-\frac{2}{d}\sum_{i=0}^{\infty}\left[\frac{\lambda_x^2\left(\frac{i\pi}{d}\right)^2\left(1-e^{-j\lambda_{y\theta i}h}\right)}{\lambda_{y\theta i}^3\left(\lambda_x^2+\lambda_{y\theta i}^2\right)\tau_\theta}+\frac{k_t^2\left(1-e^{-j\lambda_{y\psi i}h}\right)}{\lambda_{y\psi i}\left(\lambda_x^2+\lambda_{y\psi i}^2\right)\left(1+\delta_{i,0}\right)\tau_\psi}\right]\right\}d\lambda_x
\end{aligned} \tag{419}$$

For a second measurement, it is impractical to reliably calibrate the system while making $\ell = 0$. Therefore, the reduced aperture section is replaced with a full aperture section of length ℓ. In this manner, the calibrated measurement location is the same. Thus, the measurements are of identical form except the second measurement substitutes $h = b$. A close inspection of the functional form shows the two measurements are likely linearly independent due to the exponentials in the summation terms. The measurements are linearly dependent only when $\lambda_{y\theta i} = \lambda_{y\psi i} = \lambda_{y\alpha i} \iff \tau_\theta = \tau_\psi$ while $h = b + \frac{2\pi m}{\lambda_{y\alpha i}} ... m \in \mathbb{Z}$. For a non-magnetic, uniaxial material, $\tau_\theta = 1 \neq \tau_\psi$, thus linear independence is guaranteed.

5.6 Dominant Mode Analysis Grand Summary

$$
\boxed{
\begin{array}{lll}
\multicolumn{3}{l}{S_{11}^{(1)} = P_B^2 \left[\frac{2a^3 h\omega\mu_t + \Theta_{11}}{2a^3 h\omega\mu_t - \Theta_{11}}\right], \quad S_{11}^{(2)} = S_{11}^{(1)}\Big|_{h=b}} \\
\multicolumn{3}{l}{\Theta_{11} = 8\pi Z_{v_1,w_1}^{A,\mathrm{TE}^z} \int\limits_{-\infty}^{\infty} \frac{\cos^2\left(\lambda_x \frac{a}{2}\right)}{\left(\lambda_x^2 - \left(\frac{\pi}{a}\right)^2\right)^2} \left\{ \frac{jh\lambda_{z\theta}^* \cos\left(\lambda_{z\theta}^* d\right)}{\sin\left(\lambda_{z\theta}^* d\right)} \right.} \\
\multicolumn{3}{l}{\left. \quad - \frac{2}{d} \sum_{i=0}^{\infty} \left[\frac{\lambda_x^2 \left(\frac{i\pi}{d}\right)^2 \left(1 - e^{-j\lambda_{y\theta i} h}\right)}{\lambda_{y\theta i}^3 \left(\lambda_x^2 + \lambda_{y\theta i}^2\right) \tau_\theta} + \frac{k_t^2 \left(1 - e^{-j\lambda_{y\psi i} h}\right)}{\lambda_{y\psi i} \left(\lambda_x^2 + \lambda_{y\psi i}^2\right) \left(1 + \delta_{i,0}\right) \tau_\psi} \right] \right\} d\lambda_x} \\
P_B = e^{-\gamma_{zv_1,w_1}^A \ell}, & \gamma_{zv_1,w_1}^A = \sqrt{\left(\frac{\pi}{a}\right)^2 - k_0^2}, & Z_{v_1,w_1}^{A,\mathrm{TE}^z} = \frac{j\omega\mu_0}{\gamma_{zv_1,w_1}^A} \\
\lambda_{z\theta}^* = \sqrt{k_t^2 - \tau_\theta \lambda_x^2}, & \tau_\theta = \frac{\mu_t}{\mu_z}, & \tau_\psi = \frac{\epsilon_t}{\epsilon_z} \\
k_t^2 = \omega^2 \epsilon_t \mu_t, & \multicolumn{2}{l}{\lambda_{y\alpha i} = \pm\sqrt{\tau_\alpha^{-1}\left[k_t^2 - \left(\frac{i\pi}{d}\right)^2\right] - \lambda_x^2}, \quad \alpha \in \{\theta, \psi\}} \\
k_B = k_0, & \multicolumn{2}{l}{\delta_{i,0} = \begin{cases} 1 ... & i = 0 \\ 0 ... & i \neq 0 \end{cases}}
\end{array}
}
\tag{420}
$$

Now that the process of determining the theoretical reflection coefficient has been established, it can be used to extract constitutive parameters from measurements by finding the arguments that minimize the difference between the theory and measured data. Namely, in the case of a nonmagnetic material, two least-squares objective functions are simultaneously minimized to find the corresponding constitutive parameters. Thus,

$$\underset{\epsilon_{rt},\epsilon_{rz}\in\mathbb{C}}{\operatorname{argmin}} \left\{ \begin{array}{c} \left(S_{11}^{\text{thy}}\left(h \neq b\right) - S_{11}^{\text{m1}}\right)^2 \\ \left(S_{11}^{\text{thy}}\left(h = b\right) - S_{11}^{\text{m2}}\right)^2 \end{array} \right\} \tag{421}$$

If the nonmagnetic assumption does not hold, additional measurements can be added by using apertures of different height h.

VI. RARWG Probe Technique Results

In an effort to characterize the likely effectiveness of the RARWG technique as described above, CST Microwave Studio® simulations are used to qualitatively assess the predicted performance of the technique in practice, assuming a stable MATLAB® realization could be constructed. To accomplish this, similar to analysis performed in Chapter IV, two families of curves are produced. In one set, depicted in fig. 14, ϵ_z is kept constant, while a broad range of ϵ_t values are explored. In the other set, depicted in fig. 15, ϵ_t is kept constant, while a broad range of ϵ_z values are explored. To illustrate the role measurement uncertainty plays, a Monte Carlo simulation is performed with 1000 samples taken. The total reflection parameter uncertainty is estimated by

$$u_{S_{11}} = \sigma_{\hat{S}_{11}} \tag{422}$$

$$\hat{S}_{11} = S_{11} + \Delta S_{11} \tag{423}$$

$$\Delta S_{11} = \sqrt{\left(\Delta T \frac{\partial}{\partial T} S_{11}\right)^2 + (\Delta S_{11}^{\mathrm{ms}})^2} \tag{424}$$

where $\Delta\alpha_i = \bar{\alpha} - \alpha_i$ and α_i is the i^{th} sample from a random distribution α. In this case, ΔT is computed from 1000 samples of uniform distributions around the nominal value of $T \pm 0.004$ inch. Simulated values of $\frac{\partial}{\partial T} S_{11}$ are provided by CST Microwave Studio®'s sensitivity analysis feature. $\Delta S_{11}^{\mathrm{ms}}$ is computed from 1000 samples of a normal distribution around the nominal value of S_{11} with $\sigma_{S_{11}}^{\mathrm{ms}}$ values for the Agilent E8362B VNA provided by Agilent's uncertainty calculator [1].

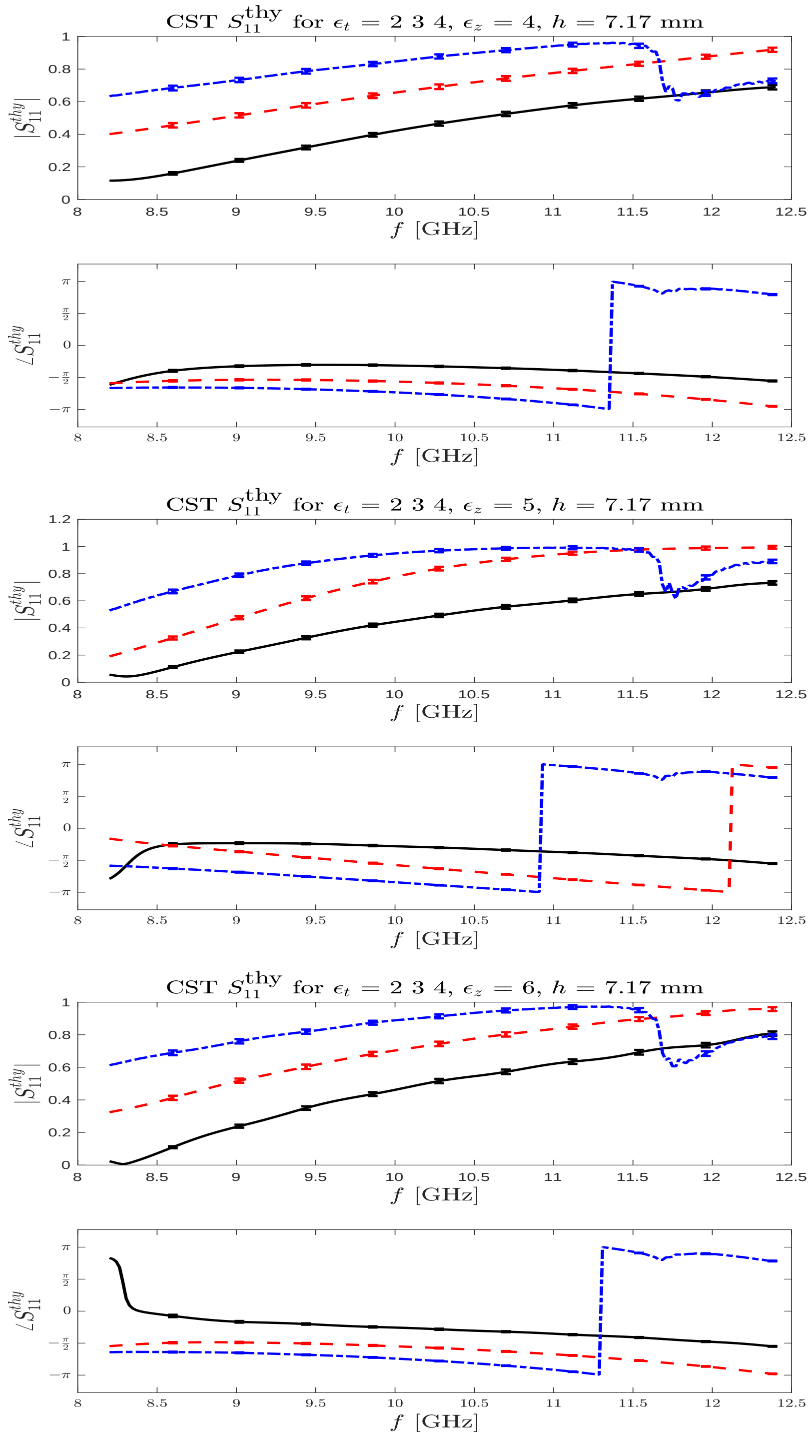

Figure 14. Comparison of RARWG CST Microwave Studio® simulated data with constant ϵ_z and varying ϵ_t values with $h = 7.17$ mm.

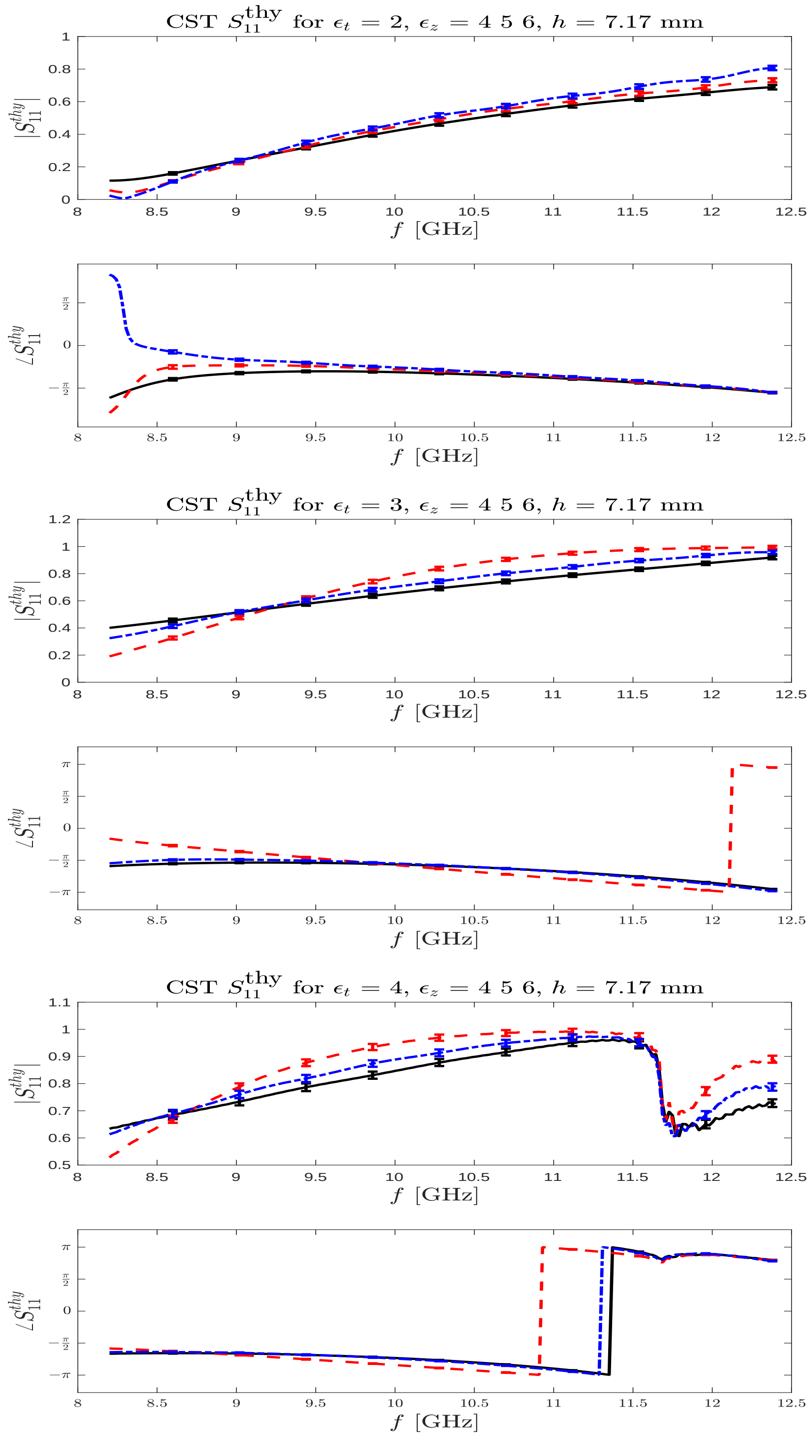

Figure 15. Comparison of RARWG CST Microwave Studio® simulated data with constant ϵ_t and varying ϵ_z values with $h = 7.17$ mm.

Ambiguity occurs at frequencies where it would be difficult to tell which value of a given constitutive parameter would have produced the observed reflection parameter value in an inverse problem. This is relatively straightforward to determine visually, as those areas occur where plot lines are very close together or cross over. If the plot lines are close enough together, measurement uncertainty would likely make the extraction impossible.

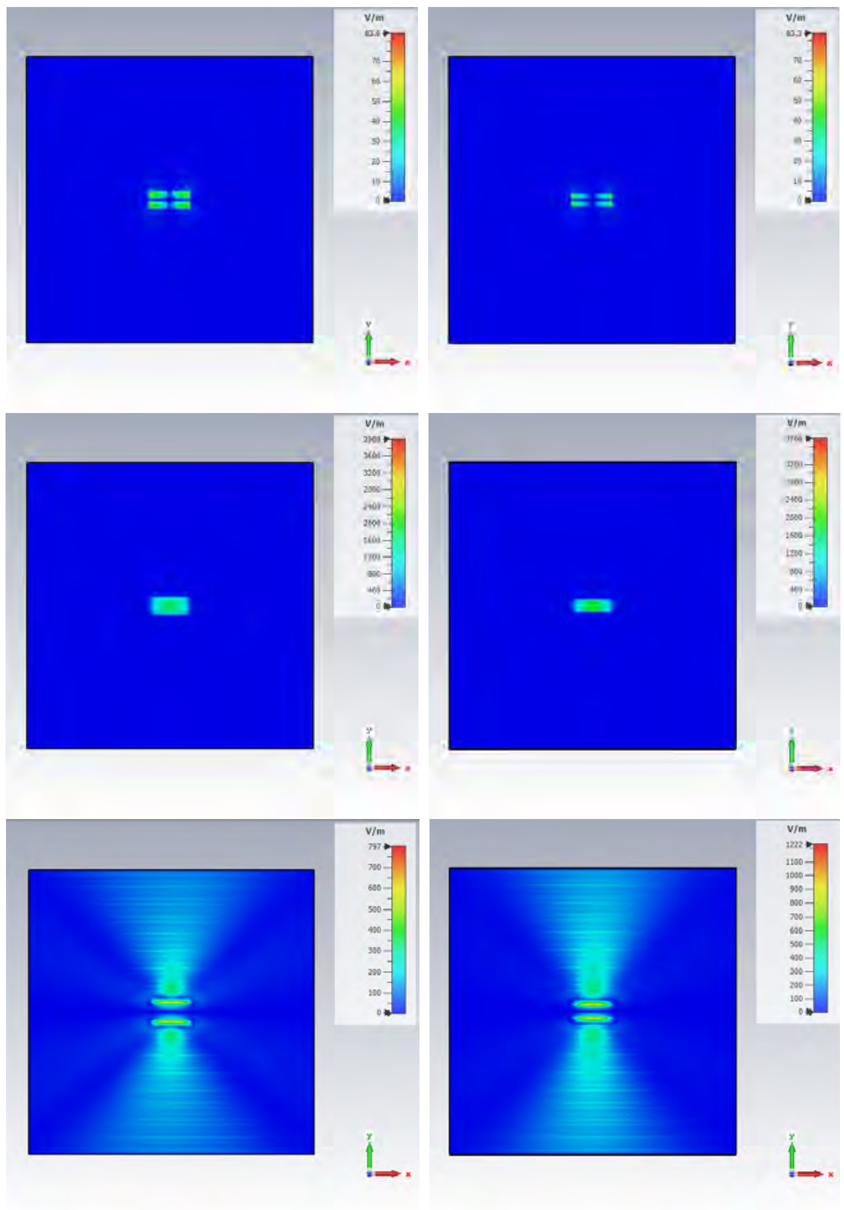

Figure 16. Comparison of CST Microwave Studio® electric field maximum values at 0.1 mm below MUT surface. Rows: E_x (top), E_y (middle), E_z (bottom). Columns: Full aperture (left), reduced aperture (right).

Note that when ϵ_z is kept constant, the reflection parameter again changes dramatically as ϵ_t changes, having limited areas of ambiguity mostly at higher frequencies. However, when ϵ_t is kept constant, again the reflection parameter changes in only minutely-detectable ways, especially when ϵ_t is small. Further, there are large regions of potential ambiguity, though in slightly different areas than those occurring using the TLM. To further substantiate this phenomenon, observe the field structure in the MUT at a depth of 0.1 mm below the MUT surface, as depicted in fig. 16. Note that in both cases (full aperture and reduced aperture), the maximum values for the electric field in the $\hat{y}$ direction are significantly higher than those in the $\hat{z}$ direction. Thus, $\epsilon_y = \epsilon_t$ is much more strongly implicated in the resulting reflection parameter measurements than ϵ_z. However, also observe that reducing the aperture size greatly increases the maximum value of E_z within the MUT. Thus, it can be concluded that this technique would likely do reasonably well at extracting ϵ_t but would do a comparatively poor job of extracting ϵ_z in an inverse problem, especially for small values of ϵ_t. However, this technique would likely outperform the TLM at extracting ϵ_z. To empirically validate these performance predictions, laboratory measurements are taken via the procedures outlined in the next section.

6.1 Experimental Setup and Results

The Agilent E8362B VNA is used to take two S_{11} measurements of a non-magnetic uniaxial material: one with a full-aperture flange plate, the second with a reduced-aperture flange plate. These measurements are then run through the algorithm described in Chapter V to extract ϵ_t and ϵ_z. The setup is designed to characterize the material in the X-band (8.2–12.4 GHz) at 201 frequencies. Thus the waveguide transverse dimensions are the standard $a = 0.9$ inch and $b = 0.4$ inch respectively. The flange plate thicknesses are $\ell = 0.25$ inch. Finally, the reduced-aperture dimension is $h = 7.17$ mm. Figure 17 depicts how a full-aperture and reduced-aperture flanged rectangular waveguide probe are used to measure a MUT clamped in place with a highly-conductive metal backing.

Figure 17. Full-aperture plate (top). Reduced-aperture plate (middle). Experimental measurement setup with conductor-backed MUT clamped in place (bottom).

The system is calibrated to the plane where the flange plate meets the rectangular waveguide using the TRL calibration algorithm built into the VNA. To eliminate reflection from the edges of the flange plate, thus approximating the infinite parallel-plate measurement, a time gate window is applied using the function built into the VNA. The cutoff for this window is 600 ps after the calibration plane. This cutoff is determined empirically by taking the widest window possible where ringing effects are no longer observed in the S_{11} measurement.

The uniaxial material used for experimental measurements is designed using crystalographic symmetry techniques using periodic tetragonal dielectric inclusions embedded in a bulk dielectric medium, as described in [48]. In [48], Knisely shows that

$$\epsilon_t = \begin{cases} \frac{\epsilon_m(w-w_c x)}{w} + \frac{\epsilon_a \epsilon_m l_c w_c x h}{lw[\epsilon_a h + h_c y(\epsilon_m - \epsilon_a)]} & ...\epsilon_a < \epsilon_m \\ \frac{h}{\frac{h-h_c y}{\epsilon_m} + \frac{lwh_c y}{l_c(\epsilon_m w + w_c x(\epsilon_a - \epsilon_m))}} & ...\epsilon_a > \epsilon_m \end{cases} \tag{425}$$

$$\epsilon_z = \frac{\epsilon_m l}{l_c} + \frac{h_c w_c l x y \left(\epsilon_a - \epsilon_m\right)}{l_c h w} \tag{426}$$

where x is the number of columns of inclusions in the $\hat{x}$ direction, y is the number rows of inclusions in the $\hat{y}$ direction, ϵ_m is the relative permittivity of the bulk material, ϵ_a is the relative permittivity of the inclusion material, $\{w, h, l\}$ are the dimensions of the overall material in $\{\hat{x}, \hat{y}, \hat{z}\}$ directions respectively, and $\{w_c, h_c, l_c\}$ are dimensions of the individual cell inclusions in $\{\hat{x}, \hat{y}, \hat{z}\}$ directions respectively. In order for the material to be uniaxial, $w_c = l_c$ with the same density and distribution of cells in the $\hat{x}$ and $\hat{y}$ directions.

Two material samples are designed for measurement in this effort: a low-contrast sample and a high-contrast sample. The samples are dimensionally identical. The low-contrast material has $\epsilon_a = 1$, i.e. cells filled with air. The high-contrast material has cells filled with ESD-PETG filament material, $\epsilon_a \approx 9$ at 10 GHz. For both materials, the bulk material is a nylon filament material, $\epsilon_m \approx 2.8$ at 10 GHz. These materials are characterized isotropically in bulk by Knisely in [49]. The sample dimensions are $w = h = 6$ inches, $w_c = h_c = 0.0625$ inch, $l = l_c = 0.25$ inch, $x = y = 48$. With these parameters, the low-contrast sample is predicted to have $\epsilon_t \approx 2.1$ and $\epsilon_z \approx 2.4$. The high-contrast material is predicted to have $\epsilon_t \approx 3.8$ and $\epsilon_z \approx 4.3$.

The samples are fabricated using a dual-nozzle Fused Deposition Modeling (FDM) 3-D printer. Due to imperfections inherent in the printing process itself, minuscule air gaps are unintentionally introduced between printed layers, between parallel lines in the materials, and at transitions from one material to another (i.e. around the inclusions). Thus the resulting parameters are expected not to perfectly match the theoretical effective performance predicted. This is shown by Knisely et al in [49], who produced a sample material with the same crystallographic structure but smaller bulk dimensions ($w = h = l = 0.9$ inch) in order to characterize the cubic sample in a rectangular-to-square waveguide structure. This allows a Nicolson-Ross-Weir (NRW)-based extraction algorithm that provides a direct inverse solution without requiring a root search algorithm. The extracted results from [49] for the high-contrast material are presented in fig. 18 for reference.

The algorithm in the previous chapter is validated by taking measurements of a $6 \times 6 \times 0.25$ inch slab using the same crystallographic structure as the cube measured in [49] with error bars based solely on the gauge error associated with measuring the MUT thickness $d \pm 0.004$ inch. The inverse solutions are found using MATLAB®'s `lsqnonlin` function with the `trust-region-reflective` algorithm and root search boundary constraints. The results are depicted in fig. 18. These results are in very close agreement with the results obtained using the cube sample in [49]. It is important to note that the bounding constraints for initial guesses needed to be within

≈ ±0.5 of the actual value in order to achieve adequate convergence of an inverse solution. This is likely due to the ambiguities noted in the previous section. By narrowing down the search area, it is more likely that the correct forward model parameters are found via the root search algorithm. Additionally, observe that there are several locations (especially at higher frequencies) where ϵ_z solutions hit the root search boundaries (denoted by dotted lines in fig. 18). These areas correlate well with likely areas of ambiguity predicted in the previous section (i.e. at frequencies near and above 11 GHz). Finally, there are large errors at the lowest few frequencies and highest few frequencies. These are associated with artifacts introduced into the S_{11} measurements due to the time gating algorithm built into the VNA.

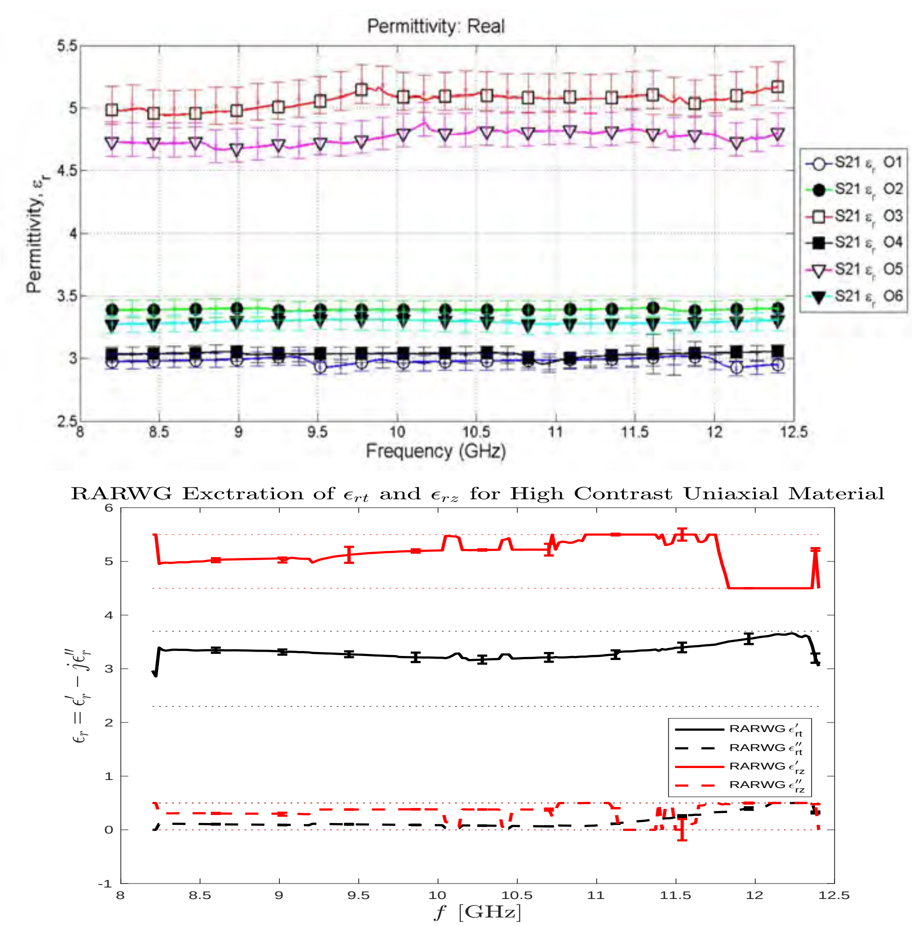

Figure 18. Cube sample measurement results for high contrast material from [49] (top). Extracted ϵ_t and ϵ_z using the RARWG method (bottom).

It is possible that another root search algorithm or set of objective functions may perform better at the inverse problem than `lsqnonlin` with the proposed set of objective functions. Also, applying some assumptions about continuity and smoothness of solutions with respect to frequency may help mitigate divergent results. None of these avenues of possible improvement are explored in this effort due to time constraints.

VII. Conclusions

7.1 Conclusions

A bi-layered uniaxial parallel plate waveguide Green function has been developed to support two proposed non-destructive measurement techniques and to gain physical insight so that the most promising avenues of future research can be identified. The utility of both measurement techniques is initially explored based on sensitivity analysis.

In the first technique, measurement diversity is achieved by varying the properties of a known uniaxial layer applied to the MUT. It is determined that this technique would likely have a difficult time extracting ϵ_z from the MUT due to ambiguity, especially at high frequencies. Additionally, the technique is difficult implement due to the complexity of the forward model. The potential benefits of this technique are that it allows air gap analysis and may allow for non-destructive measurements using a robotic probe that has highly-accurate positioning information.

In the second technique, measurement diversity is achieved by varying the size of the rectangular waveguide aperture region leading into the parallel plate waveguide. Laboratory measurments are taken to validate a dominant mode model for the second technique, demonstrating its relative utility. It is determined that this technique has a less difficult time extracting ϵ_z due to less ambiguity than the first technique. This is likely a result of stronger E_z fields induced in the MUT. However, ambiguity is still a large factor impacting the ability to extract ϵ_z. For the sample tested in this effort, ϵ_z extraction performance is poor above 11 GHz. This technique performs well in comparison to a destructive, NRW-based technique and can be used to non-destructively characterize uniaxial materials that are permanently affixed to highly-conductive surfaces. Also, this technique benefits from a relatively simple forward model. Furthermore, a dominant-mode analysis is sufficient to provide results comparable to the NRW-based technique, which saves considerable computational resources. Ultimately, the forward model in this technique is significantly more computationally efficient than using commercial software such as CST Microwave Studio®. For example, in the models tested, the proposed technique running on a consumer-class laptop only require a few minutes to produce what take over 12 hours on a professional desktop-class machine running CST Microwave Studio®.

Ultimately, this research effort provides three main contributions to the scientific community. The first contribution is a unique bi-layered uniaxial Green function that is applicable far beyond the scope of this research. The second contribution predicts challenges in performance of the first proposed measurement technique. It is recommended that continued effort to produce a stable implementation of the extraction algorithm would yield a practical, valuable technique because there is no need to change the field applicator between measurements. The third contribution demonstrates the simplicity, speed, and accuracy of the second proposed measurement technique. The technique is viable for immediate real-world application, but would likely benefit further from additional refinement of the root search algorithm. Finally, this work demonstrates that non-destructive characterization of PEC-backed uniaxial materials is not only possible, but can be achieved in an efficient, practical manner with results on par with mature destructive techniques.

7.2 Future Work

Based on this research, in particular the Green functions developed, there are multiple promising avenues for future research to improve the extraction performance of ϵ_t and ϵ_z from non-magnetic, conductor backed uniaxial materials. The most critical factor for improving ϵ_z extraction is to find a method that implicates an even stronger E_z field within the MUT. As is noted in the physical interpretations listed in Chapter II, the best method for obtaining a strong E_z in the parallel-plate region is to have a rotating transverse magnetic current present. This can be achieved by Love's equivalence using either a flanged coaxial or circular waveguide probe. The purely lamellar electric fields associated with the TM^z modes in the waveguide aperture can be replaced with purely rotational magnetic surface currents on a PEC surface in the parallel plate waveguide structure.

There are many challenges associated with this approach. One challenge is the comparably difficult problem of analyzing the closed-form theory of the circular/coaxial waveguide aperture-to-parallel plate region solutions compared with the rectangular waveguide solutions. Though complicated, this challenge is not insurmountable. Another challenge, particularly with the circular waveguide, is isolating a TM^z mode for measurement. The dominant mode in a circular waveguide is the TE^z_{11} mode. Balanis suggests a method of coupling from the dominant TE^z_{10} mode in a rectangular waveguide to the TM^z_{01} mode in a circular waveguide, depicted in fig. 19 [6]. While this approach may ultimately prove of value, one of the main advantages of using the circular waveguide compared to the rectangular waveguide, increased bandwidth, would be lost. There is also the question whether, upon interacting with the MUT at the circular waveguide aperture, would the dominant TE^z_{11} mode be excited in the circular waveguide? If so, would the coupling interface back to the rectangular waveguide effectively filter those spurious modal excitations out?

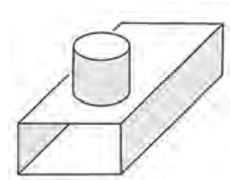

Figure 19. Suggested coupler to excite the TM^z_{01} mode in circular waveguide from the TE^z_{10} mode in rectangular waveguide and vice versa [6].

Another potential approach would be to use a free-space measurement system (possibly a focused-beam setup) to bistatically interrogate the MUT from an angle in order to induce both E_z and $\vec{E}_t$ components within the MUT. There are numerous prcatical challenges with this approach, including elliptical beam errors. One major challenge comes into play when measuring materials that have high ϵ_z. Due to Snell's Law, these materials have a tendency to "straighten" the interrogating wave toward normal to the material surface. In effect, this could greatly reduce the E_z component, reducing the ability to extract ϵ_z.

Finally, another approach would be to explore a two port near-field measurement apparatus where two rectangular waveguides meet the MUT at an angle. There are numerous challenges to this approach, however. First, the optimum angle for such an apparatus would likely be dependent on both the constitutive parameters and physical dimensions of the MUT. It would likely be challenging to devise an apparatus that would allow the angle of this interface to be arbitrarily changed. Furthermore, it would be very challenging to develop a closed-form mathematical theory to describe such a structure. However, by adjusting the size of an air-gap layer between a fixed-angle apparatus, similar to the one depicted in fig. 20, the system may be "tuned" to maximize energy coupling from port 1 to port 2. Then, measuring the size of this air gap along with the S_{11} and S_{21} parameters could provide the necessary model parameters to extract the MUT constitutive parameters. Realistically, however, there would be little energy expected to be returned to port 1 after tuning in this angled configuration. Likely, an additional measurement would be necessary for measurement diversity, either by tuning the system so that all the energy returns to port 1 or by taking another measurement with a standard flanged rectangular waveguide probe like the one used in this effort.

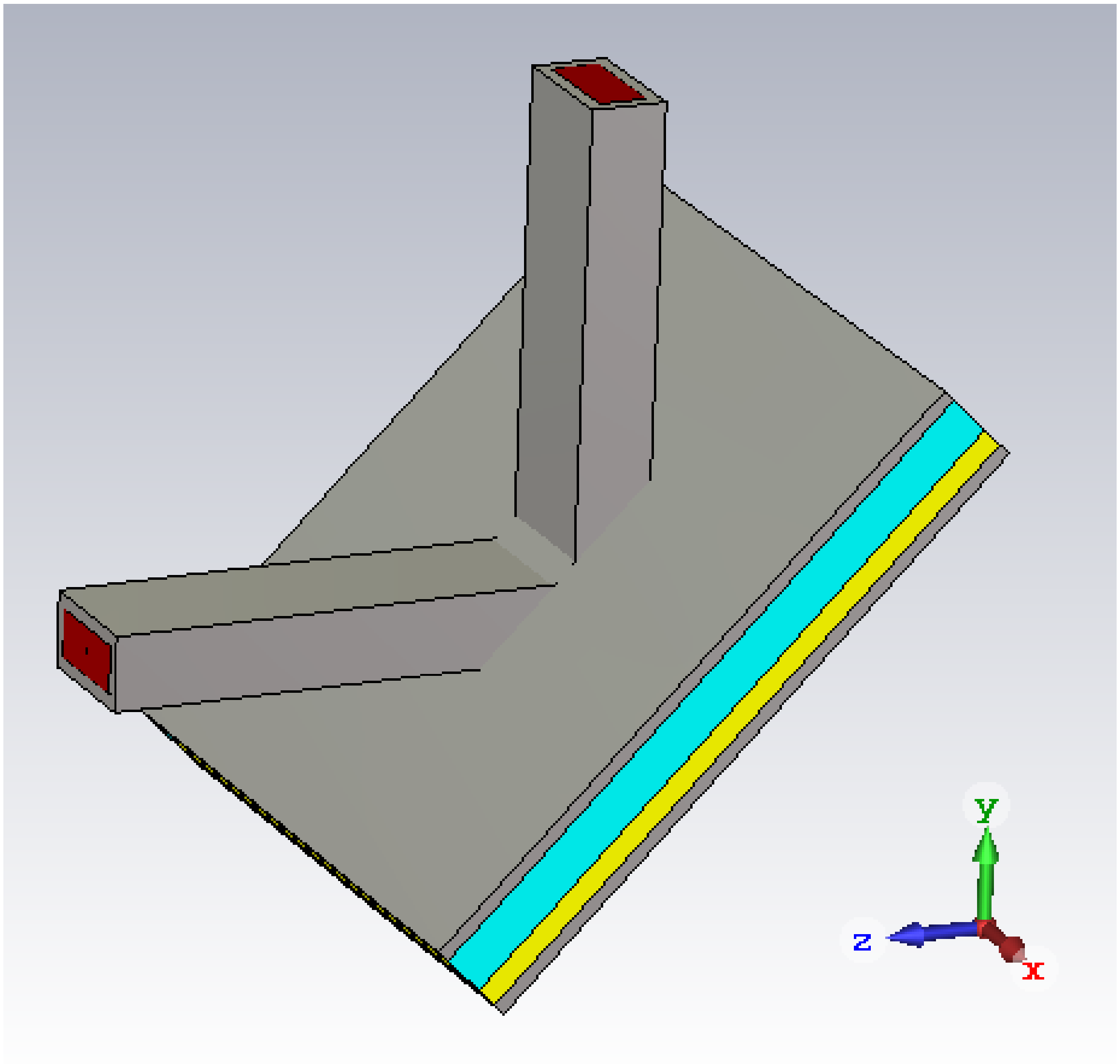

Figure 20. Two-port angled rectangular waveguide probe.

Appendix

A. Generalized Cauchy's Integral Formula (CIF)

Since it is possible for multiple poles to exist at the same location and not every pole is of the simple form $(z - z_0)$, it is useful to develop a generalized version of CIF. First, recall the standard form of CIF

$$\oint_C \frac{f(z)}{z - z_0} dz = j2\pi f(z_0) \tag{A.1}$$

From here, derive the formula for two simple poles at $z = z_0$. First, note that

$$\frac{\partial}{\partial z_0} \frac{1}{z - z_0} = \frac{1}{(z - z_0)^2} \tag{A.2}$$

which implies that

$$\begin{aligned} \oint_C \frac{f(z)}{(z - z_0)^2} dz &= \frac{\partial}{\partial z_0} \oint_C \frac{f(z)}{(z - z_0)} dz = j2\pi \frac{\partial}{\partial z_0} f(z_0) = j2\pi \frac{\partial}{\partial z} f(z) \bigg|_{z=z_0} \\ &= \frac{j2\pi}{1} \frac{\partial}{\partial z} f(z) \bigg|_{z=z_0} \end{aligned} \tag{A.3}$$

Next, look at the formula for three simple poles at $z = z_0$. Note that

$$\frac{\partial}{\partial z_0} \frac{1}{(z - z_0)^2} = \frac{2}{(z - z_0)^3} \tag{A.4}$$

which implies that

$$\oint\limits_C \frac{f(z)}{(z-z_0)^3}dz = \frac{1}{2}\frac{\partial^2}{\partial {z_0}^2}\oint\limits_C \frac{f(z)}{(z-z_0)}dz = \frac{j2\pi}{2}\frac{\partial^2}{\partial {z_0}^2}f(z_0) = \frac{j2\pi}{1\cdot 2}\frac{\partial^2}{\partial z^2}f(z)\bigg|_{z=z_0} \tag{A.5}$$

Next, look at the formula for four simple poles at $z = z_0$. Note that

$$\frac{\partial}{\partial z_0}\frac{1}{(z-z_0)^3} = \frac{3}{(z-z_0)^4} \tag{A.6}$$

which implies that

$$\oint\limits_C \frac{f(z)}{(z-z_0)^4}dz = \frac{1}{2\cdot 3}\frac{\partial^3}{\partial {z_0}^3}\oint\limits_C \frac{f(z)}{(z-z_0)}dz = \frac{j2\pi}{2\cdot 3}\frac{\partial^3}{\partial {z_0}^3}f(z_0) = \frac{j2\pi}{1\cdot 2\cdot 3}\frac{\partial^3}{\partial z^3}f(z)\bigg|_{z=z_0} \tag{A.7}$$

Now a clear pattern can be seen with respect to simple poles of degree n, implying by induction that

$$\boxed{\oint\limits_C \frac{f(z)}{(z-z_0)^n}dz = \frac{j2\pi}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}f(z)\bigg|_{z=z_0}} \tag{A.8}$$

This standard CIF generalized to simple poles of degree n is really only useful if the denominator is easily factored into simple poles. However, it is now of interest to determine a more generalized CIF that can handle an arbitrary rational function that is not necessarily able to be easily factored, $F(z) = \frac{N(z)}{D(z)}$. Begin with a Taylor-series expansion of $D(z)$ at a point $z = z_0$ where $D(z_0) = 0$

$$\begin{aligned} D(z) = D(z_0) + \frac{(z-z_0)^1}{1!}\frac{\partial}{\partial z}D(z)\bigg|_{z=z_0} + \frac{(z-z_0)^2}{2!}\frac{\partial^2}{\partial z^2}D(z)\bigg|_{z=z_0} + \cdots \\ + \frac{(z-z_0)^n}{n!}\left[\frac{\partial^n}{\partial z^n}D(z) + \frac{(z-z_0)}{n+1}\frac{\partial^{n+1}}{\partial z^{n+1}}D(z) + \cdots\right]\bigg|_{z=z_0} \end{aligned} \tag{A.9}$$

Note that, since $D(z)$ is of degree n, all derivatives of $D(z)$ greater than order n will be zero. If all the derivatives of $D(z_0)$ below degree n are coincidentally zero,

$$\oint_C \frac{N(z)}{D(z)}dz = \frac{n!}{\frac{\partial^n}{\partial {z_0}^n}D(z_0)} \oint_C \frac{N(z)}{(z-z_0)^n}dz = \frac{n!}{\frac{\partial^n}{\partial {z_0}^n}D(z_0)} \frac{j2\pi}{(n-1)!} \frac{\partial^{n-1}}{\partial {z_0}^{n-1}} N(z_0) \quad \text{(A.10)}$$

$$\boxed{\oint_C \frac{N(z)}{D(z)}dz = j2\pi n \left. \frac{\frac{\partial^{n-1}}{\partial z^{n-1}}N(z)}{\frac{\partial^n}{\partial z^n}D(z)} \right|_{z=z_0} \ldots \text{if } \left. \frac{\partial^k}{\partial z^k} D(z) \right|_{z=z_0} = 0, k \in \{0,\ldots,n-1\}} \quad \text{(A.11)}$$

$$\oint_C \frac{N(z)}{D(z)}dz = j2\pi \left. \frac{N(z)}{\frac{\partial}{\partial z}D(z)} \right|_{z=z_0} \ldots \text{if} D(z_0) = 0, \left. \frac{\partial D(z)}{\partial z} \right|_{z=z_0} \neq 0, n = 1 \quad \text{(A.12)}$$

B. Full $\tilde{\psi}^s$ Development

B.1 Computation of $\tilde{\psi}_{\{1,2\}}^{\{+,-\}}$ Scattering Coefficients

Solving (120), (122), (124), and (125) for $\tilde{\psi}_{\{1,2\}}^{\{+,-\}}$ via (127) implies that

$$\begin{aligned}\tilde{\psi}_1^+ = &\frac{V_{\psi 1}^+ P_{\psi 1h}\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right) - C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\ &\frac{V_{\psi 1}^- \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\ &+ \frac{2V_{\psi 2}^+ C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 2}^- C_\psi P_{\psi 1h} P_{\psi 2h}^2}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]}\end{aligned} \tag{B.1}$$

$$\begin{aligned}\tilde{\psi}_1^- = &\frac{V_{\psi 1}^+ P_{\psi 1h}\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right) - C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\ &\frac{V_{\psi 1}^- P_{\psi 1h}^2\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right) - C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\ &+ \frac{2V_{\psi 2}^+ C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 2}^- C_\psi P_{\psi 1h} P_{\psi 2h}^2}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]}\end{aligned} \tag{B.2}$$

$$\begin{aligned}\tilde{\psi}_2^+ = &\frac{2V_{\psi 1}^+ P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 1}^- P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} - V_{\psi 2}^+ P_{\psi 2d}^2\left[\left(1 - P_{\psi 1h}^2\right) - C_\psi\left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\ &+ \frac{-V_{\psi 2}^- P_{\psi 2d} P_{\psi 2h}\left[\left(1 - P_{\psi 1h}^2\right) - C_\psi\left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]}\end{aligned} \tag{B.3}$$

$$\begin{aligned}\tilde{\psi}_2^- = &\frac{2V_{\psi 1}^+ P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 1}^- P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} + V_{\psi 2}^+ P_{\psi 2h}^2\left[\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\ &+ \frac{-V_{\psi 2}^- P_{\psi 2d} P_{\psi 2h}\left[\left(1 - P_{\psi 1h}^2\right) - C_\psi\left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi\left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]}\end{aligned} \tag{B.4}$$

B.2 Transverse Spectral Domain $\tilde{\psi}^s$ Scattered Solutions

First, determine $\tilde{\psi}_1^s$. Substituting (B.1) and (B.2) into (109) implies that

$$\begin{aligned}
\tilde{\psi}_1^s = &\left(\frac{V_{\psi 1}^{+} P_{\psi 1h} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \right. \\
&+ \frac{V_{\psi 1}^{-} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&\left. + \frac{2V_{\psi 2}^{+} C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 2}^{-} C_\psi P_{\psi 1h} P_{\psi 2h}^2}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \right) P_{\psi 1z} \\
&+ \left(\frac{V_{\psi 1}^{+} P_{\psi 1h} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \right. \\
&+ \frac{V_{\psi 1}^{-} P_{\psi 1h}^2 \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&\left. + \frac{2V_{\psi 2}^{+} C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 2}^{-} C_\psi P_{\psi 1h} P_{\psi 2h}^2}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \right) P_{\psi 1z}^{-1} \\
= &\frac{V_{\psi 1}^{+} P_{\psi 1h} P_{\psi 1z} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&+ \frac{V_{\psi 1}^{+} P_{\psi 1h} P_{\psi 1z}^{-1} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&+ \frac{V_{\psi 1}^{-} P_{\psi 1z} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&+ \frac{V_{\psi 1}^{-} P_{\psi 1h}^2 P_{\psi 1z}^{-1} \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&+ \frac{2V_{\psi 2}^{+} C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} P_{\psi 1z} + 2V_{\psi 2}^{+} C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} P_{\psi 1z}^{-1}}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&+ \frac{2V_{\psi 2}^{-} C_\psi P_{\psi 1h} P_{\psi 2h}^2 P_{\psi 1z} + 2V_{\psi 2}^{-} C_\psi P_{\psi 1h} P_{\psi 2h}^2 P_{\psi 1z}^{-1}}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
= &\frac{P_{\psi 1h}^{-1} P_{\psi 2d}^{-1} P_{\psi 2h}^{-1}}{P_{\psi 1h}^{-1} P_{\psi 2d}^{-1} P_{\psi 2h}^{-1}} \left[\frac{V_{\psi 1}^{+} P_{\psi 1h} \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) - C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \right. \\
&+ \frac{V_{\psi 1}^{-} \left[\left(P_{\psi 1z} + P_{\psi 1h}^2 P_{\psi 1z}^{-1} \right) \left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) + C_\psi \left(P_{\psi 1z} - P_{\psi 1h}^2 P_{\psi 1z}^{-1} \right) \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \\
&\left. + \frac{2V_{\psi 2}^{+} C_\psi P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) + 2V_{\psi 2}^{-} C_\psi P_{\psi 1h} P_{\psi 2h}^2 \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right)}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2 \right) \left(1 - P_{\psi 1h}^2 \right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2 \right) \left(1 + P_{\psi 1h}^2 \right) \right]} \right]
\end{aligned}$$

$$
\begin{aligned}
&= D_\psi^{-1} \left\{ V_{\psi 1}^{+} \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \left[\left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) - C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \right] \right. \\
&\qquad + V_{\psi 1}^{-} P_{\psi 1h}^{-1} \left[\left(P_{\psi 1z} + P_{\psi 1h}^{2} P_{\psi 1z}^{-1} \right) \left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \right] \\
&\qquad + V_{\psi 1}^{-} P_{\psi 1h}^{-1} \left[C_\psi \left(P_{\psi 1z} - P_{\psi 1h}^{2} P_{\psi 1z}^{-1} \right) \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \right] \\
&\qquad \left. + 2 V_{\psi 2}^{+} C_\psi \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) + 2 V_{\psi 2}^{-} C_\psi P_{\psi 2d}^{-1} P_{\psi 2h} \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \right\}
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
D_\psi &= \left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \\
&\quad + C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right)
\end{aligned} \tag{B.6}
$$

Breaking (B.5) into electric and magnetic components and substituting (120), (122), and (124) into (B.5) implies that

$$
\begin{aligned}
\tilde{\psi}_{1\{e,h\}}^{s} &= \int_0^h \tilde{\vec{G}}_{\psi 1\{e,h\}}^{p} (z = h) \, D_\psi^{-1} \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \left[\left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \right. \\
&\qquad \left. - C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \right] \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
&+ \int_0^h \tilde{\vec{G}}_{\psi 1\{e,h\}}^{p} (z = 0) \, P_{\psi 1h}^{-1} D_\psi^{-1} \left[\left(P_{\psi 1z} + P_{\psi 1h}^{2} P_{\psi 1z}^{-1} \right) \left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \right. \\
&\qquad \left. + C_\psi \left(P_{\psi 1z} - P_{\psi 1h}^{2} P_{\psi 1z}^{-1} \right) \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \right] \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
&\qquad + \int_h^d \tilde{\vec{G}}_{\psi 2\{e,h\}}^{p} (z = d) \, D_\psi^{-1} \left(2 C_\psi \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \right) \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
&\qquad + \int_h^d \tilde{\vec{G}}_{\psi 2(e,h)}^{p} (z = h) \, D_\psi^{-1} \left(2 C_\psi P_{\psi 2d}^{-1} P_{\psi 2h} \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \right) \cdot \tilde{\vec{J}}_{\{e,h\}} dz'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^h \vec{\tilde{g}}^{p}_{\psi 1\{e,h\}}\left(z=h\right) P_{\psi 1h} P^{-1}_{\psi 1z'} D^{-1}_{\psi} \left(P_{\psi 1z} + P^{-1}_{\psi 1z}\right) \left[\left(P_{\psi 2d} P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d} P_{\psi 2h}\right)\right. \\
&\qquad\qquad \left. - C_{\psi}\left(P^{-1}_{\psi 2d} P_{\psi 2h} - P_{\psi 2d} P^{-1}_{\psi 2h}\right)\right] \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&+ \int_0^h \vec{\tilde{g}}^{p}_{\psi 1\{e,h\}}\left(z=0\right) P_{\psi 1z'} D^{-1}_{\psi} P^{-1}_{\psi 1h} \left[\left(P_{\psi 1z} + P^{2}_{\psi 1h} P^{-1}_{\psi 1z}\right) \left(P_{\psi 2d} P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d} P_{\psi 2h}\right)\right. \\
&\qquad\qquad \left. + C_{\psi}\left(P_{\psi 1z} - P^{2}_{\psi 1h} P^{-1}_{\psi 1z}\right)\left(P^{-1}_{\psi 2d} P_{\psi 2h} - P_{\psi 2d} P^{-1}_{\psi 2h}\right)\right] \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&\qquad + \int_h^d \vec{\tilde{g}}^{p}_{\psi 2\{e,h\}}\left(z=d\right) P_{\psi 2d} P^{-1}_{\psi 2z'} D^{-1}_{\psi} 2C_{\psi}\left(P_{\psi 1z} + P^{-1}_{\psi 1z}\right) \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&\qquad + \int_h^d \vec{\tilde{g}}^{p}_{\psi 2\{e,h\}}\left(z=h\right) P^{-1}_{\psi 2h} P_{\psi 2z'} D^{-1}_{\psi} 2C_{\psi} P^{-1}_{\psi 2d} P_{\psi 2h}\left(P_{\psi 1z} + P^{-1}_{\psi 1z}\right) \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&= \int_0^h D^{-1}_{\psi} \left\{\vec{\tilde{g}}^{p}_{\psi 1\{e,h\}}\left(z=h\right)\left(P_{\psi 1h} P_{\psi 1z} P^{-1}_{\psi 1z'} + P_{\psi 1h} P^{-1}_{\psi 1z} P^{-1}_{\psi 1z'}\right)\left[\left(P_{\psi 2d} P^{-1}_{\psi 2h}\right.\right.\right. \\
&\quad \left. + P^{-1}_{\psi 2d} P_{\psi 2h}\right) - C_{\psi}\left(P^{-1}_{\psi 2d} P_{\psi 2h} - P_{\psi 2d} P^{-1}_{\psi 2h}\right)\Big] + \vec{\tilde{g}}^{p}_{\psi 1\{e,h\}}\left(z=0\right)\Big[\left(P^{-1}_{\psi 1h} P_{\psi 1z} P_{\psi 1z'}\right. \\
&\quad \left. + P_{\psi 1h} P^{-1}_{\psi 1z} P_{\psi 1z'}\right)\left(P_{\psi 2d} P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d} P_{\psi 2h}\right)\Big] + \vec{\tilde{g}}^{p}_{\psi 1\{e,h\}}\left(z=0\right)\Big[C_{\psi}\left(P^{-1}_{\psi 1h} P_{\psi 1z} P_{\psi 1z'}\right. \\
&\qquad\qquad \left.\left.\left. - P_{\psi 1h} P^{-1}_{\psi 1z} P_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d} P_{\psi 2h} - P_{\psi 2d} P^{-1}_{\psi 2h}\right)\right]\right\} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&\quad + \int_h^d D^{-1}_{\psi}\left[2C_{\psi}\left(\vec{\tilde{g}}^{p}_{\psi 2\{e,h\}}\left(z=d\right) P_{\psi 2d} P^{-1}_{\psi 2z'} + \vec{\tilde{g}}^{p}_{\psi 2\{e,h\}}\left(z=h\right) P^{-1}_{\psi 2d} P_{\psi 2z'}\right)\left(P_{\psi 1z}\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\left. + P^{-1}_{\psi 1z}\right)\right] \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&= \int_0^h \vec{\tilde{G}}^{s}_{\psi 1\{e,h\}1} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' + \int_h^d \vec{\tilde{G}}^{s}_{\psi 1\{e,h\}2} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \qquad \text{(B.7)}
\end{aligned}
$$

Next, determine $\tilde{\psi}_2^s$. Substituting (B.3) and (B.4) into (110) implies that

$$
\begin{aligned}
\tilde{\psi}_2^s = &\left(\frac{2V_{\psi 1}^{+} P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 1}^{-} P_{\psi 1h} P_{\psi 2d} P_{\psi 2h}}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \right. \\
&+ \frac{-V_{\psi 2}^{+} P_{\psi 2d}^2 \left[\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
&\left. + \frac{-V_{\psi 2}^{-} P_{\psi 2d} P_{\psi 2h} \left[\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \right) P_{\psi 2z} P_{\psi 2d}^{-1} \\
&+ \left(\frac{2V_{\psi 1}^{+} P_{\psi 2d} P_{\psi 2h} + 2V_{\psi 1}^{-} P_{\psi 1h} P_{\psi 2d} P_{\psi 2h} + V_{\psi 2}^{+} P_{\psi 2h}^2 \left[\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \right. \\
&\left. + \frac{-V_{\psi 2}^{-} P_{\psi 2d} P_{\psi 2h} \left[\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \right) P_{\psi 2z}^{-1} P_{\psi 2d} \\
= &\frac{2V_{\psi 1}^{+} P_{\psi 2h} P_{\psi 2z} + 2V_{\psi 1}^{+} P_{\psi 2d}^2 P_{\psi 2h} P_{\psi 2z}^{-1} + 2V_{\psi 1}^{-} P_{\psi 1h} P_{\psi 2d}^2 P_{\psi 2h} P_{\psi 2z}^{-1}}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
&+ \frac{2V_{\psi 1}^{-} P_{\psi 1h} P_{\psi 2h} P_{\psi 2z} + V_{\psi 2}^{+} P_{\psi 2d} P_{\psi 2h}^2 P_{\psi 2z}^{-1} \left[\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
&+ \frac{-V_{\psi 2}^{+} P_{\psi 2d} P_{\psi 2z} \left[\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
&+ \frac{-V_{\psi 2}^{-} P_{\psi 2h} P_{\psi 2z} \left[\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
&+ \frac{-V_{\psi 2}^{-} P_{\psi 2d}^2 P_{\psi 2h} P_{\psi 2z}^{-1} \left[\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
= &\frac{P_{\psi 1h}^{-1} P_{\psi 2d}^{-1} P_{\psi 2h}^{-1}}{P_{\psi 1h}^{-1} P_{\psi 2d}^{-1} P_{\psi 2h}^{-1}} \left[\frac{2V_{\psi 1}^{+} P_{\psi 2h} \left(P_{\psi 2z} + P_{\psi 2d}^2 P_{\psi 2z}^{-1}\right) + 2V_{\psi 1}^{-} P_{\psi 1h} P_{\psi 2h} \left(P_{\psi 2z} + P_{\psi 2d}^2 P_{\psi 2z}^{-1}\right)}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \right. \\
&+ \frac{V_{\psi 2}^{+} P_{\psi 2d} \left[\left(P_{\psi 2h}^2 P_{\psi 2z}^{-1} - P_{\psi 2z}\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 P_{\psi 2z}^{-1} + P_{\psi 2z}\right)\left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \\
&\left. + \frac{-V_{\psi 2}^{-} P_{\psi 2h} \left[\left(P_{\psi 2z} + P_{\psi 2d}^2 P_{\psi 2z}^{-1}\right)\left(1 - P_{\psi 1h}^2\right) - C_\psi \left(P_{\psi 2z} + P_{\psi 2d}^2 P_{\psi 2z}^{-1}\right)\left(1 + P_{\psi 1h}^2\right)\right]}{\left[\left(P_{\psi 2d}^2 + P_{\psi 2h}^2\right)\left(1 - P_{\psi 1h}^2\right) + C_\psi \left(P_{\psi 2h}^2 - P_{\psi 2d}^2\right)\left(1 + P_{\psi 1h}^2\right)\right]} \right]
\end{aligned}
$$

$$
\begin{aligned}
= D_\psi^{-1} \{ & 2V_{\psi 1}^{+} P_{\psi 1h}^{-1} \left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) + 2V_{\psi 1}^{-} \left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \\
& + V_{\psi 2}^{+} \left[\left(P_{\psi 2h} P_{\psi 2z}^{-1} - P_{\psi 2h}^{-1} P_{\psi 2z} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \right] \\
& + V_{\psi 2}^{+} \left[C_\psi \left(P_{\psi 2h} P_{\psi 2z}^{-1} + P_{\psi 2h}^{-1} P_{\psi 2z} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right) \right] \\
& - V_{\psi 2}^{-} \left[\left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \right] \\
& - V_{\psi 2}^{-} \left[- C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right) \right] \}
\end{aligned} \tag{B.8}
$$

Breaking (B.8) into electric and magnetic components and substituting (120), (122),and (124) into (B.8) implies that

$$
\begin{aligned}
\tilde{\psi}_{2\{e,h\}}^{s} = & \int_0^h \tilde{\vec{G}}_{\psi 1\{e,h\}}^{p} (z = h) \, D_\psi^{-1} 2 P_{\psi 1h}^{-1} \left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
& + \int_0^h \tilde{\vec{G}}_{\psi 1\{e,h\}}^{p} (z = 0) \, D_\psi^{-1} 2 \left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
& + \int_h^d \tilde{\vec{G}}_{\psi 2\{e,h\}}^{p} (z = d) \, D_\psi^{-1} \left[\left(P_{\psi 2h} P_{\psi 2z}^{-1} - P_{\psi 2h}^{-1} P_{\psi 2z} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \right. \\
& \qquad \left. + C_\psi \left(P_{\psi 2h} P_{\psi 2z}^{-1} + P_{\psi 2h}^{-1} P_{\psi 2z} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right) \right] \cdot \tilde{\vec{J}}_{\{e,h\}} dz' \\
& + \int_h^d \tilde{\vec{G}}_{\psi 2\{e,h\}}^{p} (z = h) \, D_\psi^{-1} \left(- \left[\left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \right. \right. \\
& \qquad \left. \left. - C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2z} + P_{\psi 2d} P_{\psi 2z}^{-1} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right) \right] \right) \cdot \tilde{\vec{J}}_{\{e,h\}} dz'
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_0^h \vec{\tilde{g}}^p_{\psi 1\{e,h\}}\left(z=h\right) P_{\psi 1h} P^{-1}_{\psi 1z'} D^{-1}_{\psi} 2 P^{-1}_{\psi 1h}\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right) \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&\qquad + \int\limits_0^h \vec{\tilde{g}}^p_{\psi 1\{e,h\}}\left(z=0\right) P_{\psi 1z'} D^{-1}_{\psi} 2\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right) \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&\quad + \int\limits_h^d \vec{\tilde{g}}^p_{\psi 2\{e,h\}}\left(z=d\right) P_{\psi 2d} P^{-1}_{\psi 2z'} D^{-1}_{\psi}\left[\left(P_{\psi 2h} P^{-1}_{\psi 2z} - P^{-1}_{\psi 2h} P_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right. \\
&\qquad\qquad \left. + C_{\psi}\left(P_{\psi 2h} P^{-1}_{\psi 2z} + P^{-1}_{\psi 2h} P_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right] \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&+ \int\limits_h^d \vec{\tilde{g}}^p_{\psi 2\{e,h\}}\left(z=h\right) P^{-1}_{\psi 2h} P_{\psi 2z'} D^{-1}_{\psi}\left(-\left[\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right.\right. \\
&\qquad\qquad\qquad \left.\left. - C_{\psi}\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right]\right) \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&= \int_0^h D^{-1}_{\psi}\left[2\vec{\tilde{g}}^p_{\psi 1\{e,h\}}\left(z=h\right) P^{-1}_{\psi 1z'}\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\right. \\
&\qquad \left. + 2\vec{\tilde{g}}^p_{\psi 1\{e,h\}}\left(z=0\right) P_{\psi 1z'}\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\right] \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&+ \int_h^d D^{-1}_{\psi}\left\{\vec{\tilde{g}}^p_{\psi 2\{e,h\}}\left(z=d\right) P_{\psi 2d} P^{-1}_{\psi 2z'}\left[\left(P_{\psi 2h} P^{-1}_{\psi 2z} - P^{-1}_{\psi 2h} P_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right.\right. \\
&\qquad\qquad \left. + C_{\psi}\left(P_{\psi 2h} P^{-1}_{\psi 2z} + P^{-1}_{\psi 2h} P_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right] \\
&\qquad - \vec{\tilde{g}}^p_{\psi 2\{e,h\}}\left(z=h\right) P^{-1}_{\psi 2h} P_{\psi 2z'}\left[\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right. \\
&\qquad\qquad\qquad \left.\left. - C_{\psi}\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right]\right\} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&= \int_0^h D^{-1}_{\psi}\left\{2\left(\vec{\tilde{g}}^p_{\psi 1\{e,h\}}\left(z=h\right) P^{-1}_{\psi 1z'} + \vec{\tilde{g}}^p_{\psi 1\{e,h\}}\left(z=0\right) P_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d} P_{\psi 2z}\right.\right. \\
&\left.\left. + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\right\} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' + \int_h^d D^{-1}_{\psi}\left[\vec{\tilde{g}}^p_{\psi 2\{e,h\}}\left(z=d\right) P_{\psi 2d} P^{-1}_{\psi 2z'}\left[\left(P_{\psi 2h} P^{-1}_{\psi 2z}\right.\right.\right. \\
&\left. - P^{-1}_{\psi 2h} P_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right) + C_{\psi}\left(P_{\psi 2h} P^{-1}_{\psi 2z} + P^{-1}_{\psi 2h} P_{\psi 2z}\right)\left.\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right] \\
&\quad - \vec{\tilde{g}}^p_{\psi 2\{e,h\}}\left(z=h\right) P^{-1}_{\psi 2h} P_{\psi 2z'}\left[\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right. \\
&\qquad\qquad\qquad \left.\left. - C_{\psi}\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right]\right] \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
&= \int\limits_0^h \vec{\tilde{G}}^s_{\psi 2\{e,h\}1} \cdot \vec{\tilde{J}}_{\{e,h\}} dz' + \int\limits_h^d \vec{\tilde{G}}^s_{\psi 2\{e,h\}2} \cdot \vec{\tilde{J}}_{\{e,h\}} dz'
\end{aligned}
\tag{B.9}
$$

C. Full Development of Remaining Total Scalar Potential Green Functions

Since only the magnetic field components due to magnetic currents in region 1 and region 2 are needed for the proposed measurement technique, the full development of electric field components, magnetic field components that arise from electric currents, and magnetic field components that arise from magnetic currents outside the observation region is unnecessary in the main body of this research. For completeness, this appendix presents their full development.

C.1 $\tilde{\theta}_e$ Development

Begin by analyzing the electric component $\tilde{\theta}_{1e}$ observed in region 1. The electric component can be divided into two elements, one resulting from electric currents in region 1 and the other resulting from electric currents in region 2. Begin by analyzing the component observed in region 1 resulting from electric currents in region 1, $\tilde{\theta}_{1e1}$. From (87), there is no longitudinal component, implying that

$$\vec{\tilde{g}}^{p}_{\theta\{1,2\}e} = -j\frac{\hat{z} \times \vec{\lambda}_{\rho}\omega\mu_{t\{1,2\}}}{2\lambda^2_{\rho\theta}\lambda_{z\theta\{1,2\}}} \tag{C.1}$$

Substituting (C.1) and (138) into (136) implies that

$$\begin{aligned}\vec{\tilde{G}}_{\theta1e1} = \left(-j\frac{\hat{z} \times \vec{\lambda}_{\rho}\omega\mu_{t1}}{2\lambda^2_{\rho\theta}\lambda_{z\theta1}}\right) D_{\theta}^{-1} \Big\{ & e^{-j\lambda_{z\theta1}|z-z'|} \left[\left(P^{-1}_{\theta2d}P_{\theta2h} - P_{\theta2d}P^{-1}_{\theta2h}\right)\left(P^{-1}_{\theta1h} + P_{\theta1h}\right)\right. \\ & \left. + C_{\theta}\left(P^{-1}_{\theta2d}P_{\theta2h} + P_{\theta2d}P^{-1}_{\theta2h}\right)\left(P^{-1}_{\theta1h} - P_{\theta1h}\right)\right] \\ & + \left(P_{\theta1h}P^{-1}_{\theta1z}P^{-1}_{\theta1z'} - P_{\theta1h}P_{\theta1z}P^{-1}_{\theta1z'}\right)\left[\left(P^{-1}_{\theta2d}P_{\theta2h} - P_{\theta2d}P^{-1}_{\theta2h}\right)\right. \\ & \left. - C_{\theta}\left(P^{-1}_{\theta2d}P_{\theta2h} + P_{\theta2d}P^{-1}_{\theta2h}\right)\right] - \left(P^{-1}_{\theta1h}P_{\theta1z}P_{\theta1z'} + P_{\theta1h}P^{-1}_{\theta1z}P_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h} - P_{\theta2d}P^{-1}_{\theta2h}\right) \\ & - C_{\theta}\left(P^{-1}_{\theta1h}P_{\theta1z}P_{\theta1z'} - P_{\theta1h}P^{-1}_{\theta1z}P_{\theta1z'}\right)\left(P^{-1}_{\theta2d}P_{\theta2h} + P_{\theta2d}P^{-1}_{\theta2h}\right)\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right) D_\theta^{-1}\Big\{e^{-j\lambda_{z\theta1}|z-z'|}\left[\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\right] \\
&\quad +e^{-j\lambda_{z\theta1}|z-z'|}\left[C_\theta\left(P_{\theta2d}^{-1}P_{\theta2h}+P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\right]+\big(P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1} \\
&\quad -P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\big)\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right) \\
&\quad +C_\theta\left(-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}+P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}+P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\right)\big(P_{\theta2d}^{-1}P_{\theta2h} \\
&\qquad +P_{\theta2d}P_{\theta2h}^{-1}\big)\Big\}
\end{aligned}
\tag{C.2}
$$

Due to the $|z-z'|$ term in the exponent of (C.2), two cases must be analyzed. When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}_{\theta1e1}^{z+} &= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right) D_\theta^{-1}\Big\{P_{\theta1z}P_{\theta1z'}^{-1}\left[\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\right] \\
&\quad +P_{\theta1z}P_{\theta1z'}^{-1}\left[C_\theta\left(P_{\theta2d}^{-1}P_{\theta2h}+P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\right]+\big(P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1} \\
&\quad -P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\big)\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right) \\
&\quad +C_\theta\left(-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}+P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}+P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\right)\big(P_{\theta2d}^{-1}P_{\theta2h} \\
&\qquad +P_{\theta2d}P_{\theta2h}^{-1}\big)\Big\} \\
&= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right) D_\theta^{-1}\Big[\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)\Big(P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}^{-1}+\cancel{P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}} \\
&\quad +P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}\Big)+\Big(-\cancel{P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\Big)\big(P_{\theta2d}^{-1}P_{\theta2h} \\
&\quad -P_{\theta2d}P_{\theta2h}^{-1}\big)+C_\theta\Big(P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}^{-1}-\cancel{P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}}-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}\Big)\big(P_{\theta2d}^{-1}P_{\theta2h} \\
&\quad +P_{\theta2d}P_{\theta2h}^{-1}\big)+C_\theta\Big(\cancel{P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}+P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\Big)\big(P_{\theta2d}^{-1}P_{\theta2h} \\
&\qquad +P_{\theta2d}P_{\theta2h}^{-1}\big)\Big]
\end{aligned}
$$

$$= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right)\left[\frac{\sin\left(\lambda_{z\theta2}T\right)\left[\sin\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)-\sin\left(\lambda_{z\theta1}\left(h-\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right.$$
$$\left.+\frac{C_\theta\cos\left(\lambda_{z\theta2}T\right)\left[\cos\left(\lambda_{z\theta1}\left(h-\left(z-z'\right)\right)\right)-\cos\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right] \tag{C.3}$$

When $z < z'$, that implies that

$$\vec{\vec{G}}^{z-}_{\theta1e1} = \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right)D_\theta^{-1}\left[P_{\theta1z}^{-1}P_{\theta1z'}\left[\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\right]\right.$$
$$+P_{\theta1z}^{-1}P_{\theta1z'}\left[C_\theta\left(P_{\theta2d}^{-1}P_{\theta2h}+P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\right]+\left(P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}-P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}\right.$$
$$\left.-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\right)\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)+C_\theta\left(-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}\right.$$
$$\left.\left.+P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}+P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}\right)\left(P_{\theta2d}^{-1}P_{\theta2h}+P_{\theta2d}P_{\theta2h}^{-1}\right)\right]$$
$$= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right)D_\theta^{-1}\left[\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}P_{\theta1z}^{-1}P_{\theta1z'}+\cancel{P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}}\right.\right.$$
$$\left.+P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}\right)+\left(-P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}-\cancel{P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}}\right)\left(P_{\theta2d}^{-1}P_{\theta2h}\right.$$
$$\left.-P_{\theta2d}P_{\theta2h}^{-1}\right)+C_\theta\left(P_{\theta1h}^{-1}P_{\theta1z}^{-1}P_{\theta1z'}-\cancel{P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}}-P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}^{-1}\right)\left(P_{\theta2d}^{-1}P_{\theta2h}\right.$$
$$\left.+P_{\theta2d}P_{\theta2h}^{-1}\right)+C_\theta\left(P_{\theta1h}P_{\theta1z}P_{\theta1z'}^{-1}-P_{\theta1h}^{-1}P_{\theta1z}P_{\theta1z'}+\cancel{P_{\theta1h}P_{\theta1z}^{-1}P_{\theta1z'}}\right)\left(P_{\theta2d}^{-1}P_{\theta2h}\right.$$
$$\left.\left.+P_{\theta2d}P_{\theta2h}^{-1}\right)\right]$$
$$= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}}\right)\left[\frac{\sin\left(\lambda_{z\theta2}T\right)\left[\sin\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)-\sin\left(\lambda_{z\theta1}\left(h+\left(z-z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right.$$
$$\left.+\frac{C_\theta\cos\left(\lambda_{z\theta2}T\right)\left[\cos\left(\lambda_{z\theta1}\left(h+\left(z-z'\right)\right)\right)-\cos\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_\theta\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right] \tag{C.4}$$

Analyzing (C.3) and (C.4) reveals that

$$\vec{\vec{G}}_{\theta1e1} = \left[\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta1}D_\theta}\right]\left[\sin\left(\lambda_{z\theta1}\left(h-|z-z'|\right)\right)-\sin\left(\lambda_{z\theta2}T\right)\left[\sin\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)\right]\right.$$
$$\left.-C_\theta\cos\left(\lambda_{z\theta2}T\right)\left[\cos\left(\lambda_{z\theta1}\left(h-|z-z'|\right)\right)-\cos\left(\lambda_{z\theta1}\left(h-z-z'\right)\right)\right]\right]$$

$$
= \left[-\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}}{2\lambda_{\rho\theta}^2} \right] \left[\frac{Z_{\theta 2} \sin\left(\lambda_{z\theta 2} T\right) \left[\sin\left(\lambda_{z\theta 1}\left(h - z - z'\right)\right) - \sin\left(\lambda_{z\theta 1}\left(h - |z - z'|\right)\right)\right]}{Z_{\theta 2} \sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \right.
$$

$$
\left. + \frac{Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \left[Z_{\theta 2} \cos\left(\lambda_{z\theta 1}\left(h - |z - z'|\right)\right) - \cos\left(\lambda_{z\theta 1}\left(h - z - z'\right)\right)\right]}{Z_{\theta 2} \sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \right] \tag{C.5}
$$

Next, analyze the component observed in region 1 resulting from electric currents in region 2, $\tilde{\theta}_{1e2}$. Substituting (C.1) and (138) into (136) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 1e2} &= \left(-j \frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}}{2\lambda_{\rho\theta}^2} \right) \left[\frac{2C_\theta \left(P_{\theta 2d}^{-1} P_{\theta 2z'} - P_{\theta 2d} P_{\theta 2z'}^{-1}\right) \left(P_{\theta 1z}^{-1} - P_{\theta 1z}\right)}{j4 \left[\sin\left(\lambda_{z\theta 2} T\right) \cos\left(k_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)\right]} \right] \\
&= \left(\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2 Z_{\theta 2}} \right) \left[\frac{\sin\left(\lambda_{z\theta 2}\left(d - z'\right)\right) \sin\left(\lambda_{z\theta 1} z\right)}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \right] \\
&= \left(\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2} \right) \left[\frac{\sin\left(\lambda_{z\theta 2}\left(d - z'\right)\right) \sin\left(\lambda_{z\theta 1} z\right)}{Z_{\theta 2} \sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \right]
\end{aligned} \tag{C.6}
$$

Now analyze the component observed from region 2 resulting from electric currents in region 1, $\tilde{\theta}_{2e1}$. Substituting (C.1) and (138) into (146) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 2e1} &= \left(-j \frac{\hat{z} \times \vec{\lambda}_\rho \omega \mu_{t2}}{2\lambda_{\rho\theta}^2 \lambda_{z\theta 2}} \right) \left[\frac{2 \left(P_{\theta 1z'}^{-1} - P_{\theta 1z'}\right) \left(P_{\theta 2d}^{-1} P_{\theta 2z} - P_{\theta 2d} P_{\theta 2z}^{-1}\right)}{j4 \left[\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)\right]} \right] \\
&= \left(\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2} \right) \left[\frac{\sin\left(\lambda_{z\theta 1} z'\right) \sin\left(\lambda_{z\theta 2}\left(d - z\right)\right)}{\sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \right] \\
&= \left(\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 2}^2}{\lambda_{\rho\theta}^2} \right) \left[\frac{\sin\left(\lambda_{z\theta 1} z'\right) \sin\left(\lambda_{z\theta 2}\left(d - z\right)\right)}{Z_{\theta 2} \sin\left(\lambda_{z\theta 2} T\right) \cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \sin\left(\lambda_{z\theta 1} h\right)} \right]
\end{aligned} \tag{C.7}
$$

Next, analyze the component observed in region 2 resulting from electric currents in region 2, $\tilde{\theta}_{2e2}$. Substituting (C.1) and (138) into (146) implies that

$$\begin{aligned}\tilde{\vec{G}}_{\theta 2e2} = \left(-j\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta 2}}\right) D_{\theta}^{-1} \Big[& e^{-j\lambda_{z\theta 2}|z-z'|}\left[\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right] \\ & +e^{-j\lambda_{z\theta 2}|z-z'|}\left[C_{\theta}\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right] \\ & + \left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1} - P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) \\ & -C_{\theta}\left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1} + P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right) \\ & + \left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'} - P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) \\ & +C_{\theta}\left(P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'} - P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 1z'}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\Big]\end{aligned} \tag{C.8}$$

Due to the $|z-z'|$ term in the exponent of (C.8), two cases must be considered. When $z > z'$, that implies that

$$\begin{aligned}\tilde{\vec{G}}_{\theta 2e2}^{z+} = & \left(-j\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta 2}}\right) D_{\theta}^{-1} \Big\{ P_{\theta 2z}P_{\theta 2z'}^{-1}\left[\left(P_{\theta 2d}^{-1}P_{\theta 2h} - P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right] \\ & +P_{\theta 2z}P_{\theta 2z'}^{-1}\left[C_{\theta}\left(P_{\theta 2d}^{-1}P_{\theta 2h} + P_{\theta 2d}P_{\theta 2h}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right] \\ & + \left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1} - P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) \\ & -C_{\theta}\left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1} + P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right) \\ & + \left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'} - P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) \\ & +C_{\theta}\left(P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'} - P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\Big\} \\ = & \left(-j\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta 2}}\right) D_{\theta}^{-1} \Big[\left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'} - P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) \\ & + \Big(P_{\theta 2d}^{-1}P_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}^{-1} + \cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1}} - \cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1}} \\ & -P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\Big)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right) + C_{\theta}\Big(P_{\theta 2d}^{-1}P_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}^{-1} - \cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1}} \\ & -P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\Big)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right) \\ & +C_{\theta}\left(\cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1}} + P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'} - P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \left[-\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta2}}\right]\left[\frac{\cos\left(\lambda_{z\theta1}h\right)\cos\left(\lambda_{z\theta2}\left(T-\left(z-z'\right)\right)\right)}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_{\theta}\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right.\\
&\quad +\frac{\cos\left(\lambda_{z\theta1}h\right)\left[-\cos\left(\lambda_{z\theta2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_{\theta}\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\\
&\quad \left.+\frac{C_{\theta}\sin\left(\lambda_{z\theta1}h\right)\left[-\sin\left(\lambda_{z\theta2}\left(T-\left(z-z'\right)\right)\right)-\sin\left(\lambda_{z\theta2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_{\theta}\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right]
\end{aligned}
\tag{C.9}
$$

When $z < z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}_{\theta2e2}^{z-} &= \left(-j\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta2}}\right)D_{\theta}^{-1}\left\{P_{\theta2z}^{-1}P_{\theta2z'}\left[\left(P_{\theta2d}^{-1}P_{\theta2h}-P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\right]\right.\\
&\quad +P_{\theta2z}^{-1}P_{\theta2z'}\left[C_{\theta}\left(P_{\theta2d}^{-1}P_{\theta2h}+P_{\theta2d}P_{\theta2h}^{-1}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\right]\\
&\quad +\left(P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}^{-1}-P_{\theta2d}P_{\theta2h}P_{\theta2z}^{-1}P_{\theta2z'}^{-1}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\\
&\quad -C_{\theta}\left(P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}^{-1}+P_{\theta2d}P_{\theta2h}P_{\theta2z}^{-1}P_{\theta2z'}^{-1}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\\
&\quad +\left(P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}^{-1}P_{\theta2z'}-P_{\theta2d}^{-1}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\\
&\quad \left.+C_{\theta}\left(P_{\theta2d}^{-1}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}-P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}^{-1}P_{\theta2z'}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\right\}\\
&= \left(-j\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta2}}\right)D_{\theta}^{-1}\left[\left(P_{\theta2d}^{-1}P_{\theta2h}P_{\theta2z}^{-1}P_{\theta2z'}-\cancel{P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}^{-1}P_{\theta2z'}}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)\right.\\
&\quad +\left(\cancel{P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}^{-1}P_{\theta2z'}}-P_{\theta2d}^{-1}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}+P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}^{-1}\right.\\
&\quad \left.-P_{\theta2d}P_{\theta2h}P_{\theta2z}^{-1}P_{\theta2z'}^{-1}\right)\left(P_{\theta1h}^{-1}+P_{\theta1h}\right)+C_{\theta}\left(-P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}^{-1}-P_{\theta2d}P_{\theta2h}P_{\theta2z}^{-1}P_{\theta2z'}^{-1}\right.\\
&\quad \left.+P_{\theta2d}^{-1}P_{\theta2h}P_{\theta2z}^{-1}P_{\theta2z'}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\\
&\quad \left.+C_{\theta}\left(\cancel{P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}^{-1}P_{\theta2z'}}+P_{\theta2d}^{-1}P_{\theta2h}^{-1}P_{\theta2z}P_{\theta2z'}-\cancel{P_{\theta2d}P_{\theta2h}^{-1}P_{\theta2z}^{-1}P_{\theta2z'}}\right)\left(P_{\theta1h}^{-1}-P_{\theta1h}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[-\frac{\hat{z}\times\vec{\lambda}_{\rho}\omega\mu_{t2}}{2\lambda_{\rho\theta}^{2}\lambda_{z\theta2}}\right]\left[\frac{\cos\left(\lambda_{z\theta1}h\right)\cos\left(\lambda_{z\theta2}\left(T+\left(z-z'\right)\right)\right)}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_{\theta}\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right.\\
&\quad +\frac{\cos\left(\lambda_{z\theta1}h\right)\left[-\cos\left(\lambda_{z\theta2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_{\theta}\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\\
&\quad \left.+\frac{C_{\theta}\sin\left(\lambda_{z\theta1}h\right)\left[-\sin\left(\lambda_{z\theta2}\left(T+\left(z-z'\right)\right)\right)-\sin\left(\lambda_{z\theta2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta2}T\right)\cos\left(\lambda_{z\theta1}h\right)+C_{\theta}\cos\left(\lambda_{z\theta2}T\right)\sin\left(\lambda_{z\theta1}h\right)}\right]
\end{aligned}
\tag{C.10}
$$

Analyzing (C.9) and (C.10) reveals that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 2e2} = &\left[\frac{\hat{z}\times\vec{\lambda}_\rho \omega\mu_{t2}}{2\lambda_{\rho\theta}^2\lambda_{z\theta 2}}\right]\left[\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.\\
&+\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\
&\left.+\frac{C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\\
= &\left[\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{2\lambda_{\rho\theta}^2}\right]\left[\frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1}h\right)\left[\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.\\
&+\frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1}h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\
&\left.+\frac{Z_{\theta 1}\sin\left(\lambda_{z\theta 1}h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned}
\tag{C.11}
$$

C.2 Remaining $\tilde{\theta}_h$ Development

First, analyze the component observed in region 1 due to transverse magnetic currents in region 2, $\tilde{\theta}_{1ht2}$. Substituting (139) and (138) into (136) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 1ht2} = &\left[\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right]\left[\frac{2C_\theta\left(\cancelto{-1}{\mathrm{sgn}\left(h-z'\right)}P_{\theta 2d}^{-1}P_{\theta 2z'}-\cancelto{1}{\mathrm{sgn}\left(d-z'\right)}P_{\theta 2d}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1z}^{-1}-P_{\theta 1z}\right)}{j4\left[\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)\right]}\right]\\
= &\left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{-j8C_\theta\cos\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\theta 1}z\right)}{j4\left[\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)\right]}\right]\\
= &\left(-j\frac{\vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\left[\frac{\cos\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\theta 1}z\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned}
\tag{C.12}
$$

Next, analyze the component observed in region 1 resulting from longitudinal magnetic currents in region 2, $\tilde{\theta}_{1hz2}$. From analysis of $\vec{\tilde{G}}_{\theta 1e2}$, it can be shown that substituting (140) and (138) into (136) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 1hz2} &= \left(j \frac{\hat{z}\mu_{t1}}{2\lambda_{z\theta 1}\mu_{z1}} \right) \left[\frac{-8C_\theta \sin\left(\lambda_{z\theta 2}\left(d - z'\right)\right) \sin\left(\lambda_{z\theta 1} z\right)}{j4\left[\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right) + C_\theta \cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)\right]} \right] \\
&= \left(-\frac{\hat{z}Z_{\theta 1}^2}{\omega\mu_{z1}} \right) \left[\frac{\sin\left(\lambda_{z\theta 2}\left(d - z'\right)\right) \sin\left(\lambda_{z\theta 1} z\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right) + Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \right]
\end{aligned}
\tag{C.13}
$$

Next, analyze the component observed in region 2 resulting from transverse magnetic currents in region 1, $\tilde{\theta}_{2ht1}$. Substituting (139) and (138) into (146) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 2ht1} &= \left(j \frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2} \right) \left[\frac{2\left(\cancelto{1}{\operatorname{sgn}\left(h - z'\right)} P_{\theta 1z'}^{-1} - \cancelto{-1}{\operatorname{sgn}\left(-z'\right)} P_{\theta 1z'} \right) \left(P_{\theta 2d}^{-1} P_{\theta 2z} - P_{\theta 2d} P_{\theta 2z}^{-1} \right)}{j4\left[\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right) + C_\theta \cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)\right]} \right] \\
&= \left(j \frac{\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2} \right) \left[\frac{\cos\left(\lambda_{z\theta 1} z'\right) \sin\left(\lambda_{z\theta 2}\left(d - z\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right) + Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \right]
\end{aligned}
\tag{C.14}
$$

Next, analyze the component observed in region 2 resulting from transverse magnetic currents in region 2, $\tilde{\theta}_{2ht2}$. By substituting (139) and (138) into (146), it can be shown that

$$
\begin{aligned}
\vec{\tilde{G}}_{\theta 2ht2} = \left[\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta} \right] \Big\{ & \operatorname{sgn}\left(z - z'\right) e^{-j\lambda_{z\theta 2}|z-z'|} \left[\left(P_{\theta 2d}^{-1} P_{\theta 2h} - P_{\theta 2d} P_{\theta 2h}^{-1} \right) \left(P_{\theta 1h}^{-1} + P_{\theta 1h} \right) \right] \\
& + \operatorname{sgn}\left(z - z'\right) e^{-j\lambda_{z\theta 2}|z-z'|} \left[C_\theta \left(P_{\theta 2d}^{-1} P_{\theta 2h} + P_{\theta 2d} P_{\theta 2h}^{-1} \right) \left(P_{\theta 1h}^{-1} - P_{\theta 1h} \right) \right] \\
& + \left(P_{\theta 2d} P_{\theta 2h}^{-1} P_{\theta 2z} P_{\theta 2z'}^{-1} - P_{\theta 2d} P_{\theta 2h} P_{\theta 2z}^{-1} P_{\theta 2z'}^{-1} - P_{\theta 2d} P_{\theta 2h}^{-1} P_{\theta 2z}^{-1} P_{\theta 2z'} \right) \left(P_{\theta 1h}^{-1} + P_{\theta 1h} \right) \\
& + \left(P_{\theta 2d}^{-1} P_{\theta 2h}^{-1} P_{\theta 2z} P_{\theta 2z'} \right) \left(P_{\theta 1h}^{-1} + P_{\theta 1h} \right) + C_\theta \left(-P_{\theta 2d} P_{\theta 2h}^{-1} P_{\theta 2z} P_{\theta 2z'}^{-1} \right) \left(P_{\theta 1h}^{-1} - P_{\theta 1h} \right) \\
& + C_\theta \left(-P_{\theta 2d} P_{\theta 2h} P_{\theta 2z}^{-1} P_{\theta 2z'}^{-1} - P_{\theta 2d}^{-1} P_{\theta 2h}^{-1} P_{\theta 2z} P_{\theta 2z'} + P_{\theta 2d} P_{\theta 2h}^{-1} P_{\theta 2z}^{-1} P_{\theta 2z'} \right) \left(P_{\theta 1h}^{-1} - P_{\theta 1h} \right) \Big\}
\end{aligned}
\tag{C.15}
$$

Due to the $\operatorname{sgn}\left(z - z'\right)$ and $|z - z'|$ terms in (C.15), two cases must be investigated.

When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\tilde{G}}^{z+}_{\theta 2ht2} = & \left[\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right]\left\{\cancelto{1}{\operatorname{sgn}\left(z-z'\right)} P_{\theta 2z}P^{-1}_{\theta 2z'}\left[\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right)\right]\right. \\
& +\cancelto{1}{\operatorname{sgn}\left(z-z'\right)} P_{\theta 2z}P^{-1}_{\theta 2z'}\left[C_\theta\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right)\right] \\
& +\left(P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}-P_{\theta 2d}P_{\theta 2h}P^{-1}_{\theta 2z}P^{-1}_{\theta 2z'}-P_{\theta 2d}P^{-1}_{\theta 2h}P^{-1}_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right) \\
& +\left(P^{-1}_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right)+C_\theta\left(-P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right) \\
& \left.+C_\theta\left(-P_{\theta 2d}P_{\theta 2h}P^{-1}_{\theta 2z}P^{-1}_{\theta 2z'}-P^{-1}_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}+P_{\theta 2d}P^{-1}_{\theta 2h}P^{-1}_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right)\right\} \\
= & \left[\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right]\left[\left(\cancel{P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}}-P_{\theta 2d}P_{\theta 2h}P^{-1}_{\theta 2z}P^{-1}_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right)\right. \\
& +\left(P^{-1}_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}-P_{\theta 2d}P^{-1}_{\theta 2h}P^{-1}_{\theta 2z}P_{\theta 2z'}-\cancel{P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right) \\
& +\left(P^{-1}_{\theta 2d}P_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right) \\
& +C_\theta\left(P^{-1}_{\theta 2d}P_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}+\cancel{P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}}-\cancel{P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right) \\
& \left.+C_\theta\left(-P_{\theta 2d}P_{\theta 2h}P^{-1}_{\theta 2z}P^{-1}_{\theta 2z'}-P^{-1}_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}+P_{\theta 2d}P^{-1}_{\theta 2h}P^{-1}_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right)\right] \\
= & \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[\sin\left(k_{z\theta 2}\left(T-\left(z-z'\right)\right)\right)+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right. \\
& \left.+\frac{C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\cos\left(\lambda_{z\theta 2}\left(T-\left(z-z'\right)\right)\right)-\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned}
\tag{C.16}
$$

When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\tilde{G}}^{z-}_{\theta 2ht2} = & \left(\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right)\left\{\cancelto{-1}{\operatorname{sgn}\left(z-z'\right)} P^{-1}_{\theta 2z}P_{\theta 2z'}\left[\left(P^{-1}_{\theta 2d}P_{\theta 2h}-P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right)\right]\right. \\
& +\cancelto{-1}{\operatorname{sgn}\left(z-z'\right)} P^{-1}_{\theta 2z}P_{\theta 2z'}\left[C_\theta\left(P^{-1}_{\theta 2d}P_{\theta 2h}+P_{\theta 2d}P^{-1}_{\theta 2h}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right)\right] \\
& +\left(P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}-P_{\theta 2d}P_{\theta 2h}P^{-1}_{\theta 2z}P^{-1}_{\theta 2z'}-P_{\theta 2d}P^{-1}_{\theta 2h}P^{-1}_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right) \\
& +\left(P^{-1}_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}+P_{\theta 1h}\right)+C_\theta\left(-P_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P^{-1}_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right) \\
& \left.+C_\theta\left(-P_{\theta 2d}P_{\theta 2h}P^{-1}_{\theta 2z}P^{-1}_{\theta 2z'}-P^{-1}_{\theta 2d}P^{-1}_{\theta 2h}P_{\theta 2z}P_{\theta 2z'}+P_{\theta 2d}P^{-1}_{\theta 2h}P^{-1}_{\theta 2z}P_{\theta 2z'}\right)\left(P^{-1}_{\theta 1h}-P_{\theta 1h}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right]\left[\left(P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1} - P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\right.\\
&\quad + \left(P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'} - P_{\theta 2d}^{-1}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'} - \cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'}}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\\
&\quad + \left(\cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'}}\right)\left(P_{\theta 1h}^{-1} + P_{\theta 1h}\right)\\
&\quad + C_\theta\left(-P_{\theta 2d}^{-1}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'} - \cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'}} - P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'}^{-1}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\\
&\quad \left. + C_\theta\left(-P_{\theta 2d}P_{\theta 2h}P_{\theta 2z}^{-1}P_{\theta 2z'}^{-1} - P_{\theta 2d}^{-1}P_{\theta 2h}^{-1}P_{\theta 2z}P_{\theta 2z'} + \cancel{P_{\theta 2d}P_{\theta 2h}^{-1}P_{\theta 2z}^{-1}P_{\theta 2z'}}\right)\left(P_{\theta 1h}^{-1} - P_{\theta 1h}\right)\right]\\
&= \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{\cos(\lambda_{z\theta 1}h)\left[-\sin(\lambda_{z\theta 2}(T+(z-z'))) + \sin(\lambda_{z\theta 2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.\\
&\quad \left. + \frac{C_\theta\sin(\lambda_{z\theta 1}h)\left[-\cos(\lambda_{z\theta 2}(T+(z-z'))) - \cos(\lambda_{z\theta 2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(k_{z\theta 1}h) + C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right]
\end{aligned}
\tag{C.17}
$$

Analyzing (C.16) and (C.17) reveals that

$$
\begin{aligned}
\vec{\vec{G}}_{\theta 2ht2} &= \left(\frac{j\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{\cos(\lambda_{z\theta 1}h)\left[\operatorname{sgn}(z-z')\sin(\lambda_{z\theta 2}(T-|z-z'|))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.\\
&\quad + \frac{\cos(\lambda_{z\theta 1}h)\left[\sin(\lambda_{z\theta 2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\\
&\quad \left. + \frac{C_\theta\sin(\lambda_{z\theta 1}h)\left[\operatorname{sgn}(z-z')\cos(\lambda_{z\theta 2}(T-|z-z'|)) - \cos(\lambda_{z\theta 2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right]\\
&= \left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{Z_{\theta 2}\cos(\lambda_{z\theta 1}h)\left[\operatorname{sgn}(z-z')\sin(\lambda_{z\theta 2}(T-|z-z'|))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right.\\
&\quad + \frac{Z_{\theta 2}\cos(\lambda_{z\theta 1}h)\left[\sin(\lambda_{z\theta 2}(d+h-z-z'))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\\
&\quad \left. + \frac{Z_{\theta 1}\sin(\lambda_{z\theta 1}h)\left[\operatorname{sgn}(z-z')\cos(\lambda_{z\theta 2}(T-|z-z'|)) - \cos(k_{z\theta 2}(d+h-z-z'))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h) + Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}\right]
\end{aligned}
\tag{C.18}
$$

Now analyze the component observed in region 2 resulting from longitudinal magnetic currents in region 1, $\tilde{\theta}_{2hz1}$. Similar to the analysis for $\vec{\vec{G}}_{\theta 2e1}$, it can be shown that substituting (140) and (138) into (146) implies that

$$\begin{aligned}\tilde{\vec{\vec{G}}}_{\theta 2hz1} &= \left(j\frac{\hat{z}\mu_{t2}}{2\lambda_{z\theta 2}\mu_{z2}}\right)\left[\frac{-8\sin\left(\lambda_{z\theta 1}z'\right)\sin\left(\lambda_{z\theta 2}\left(d-z\right)\right)}{j4\left[\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)\right]}\right]\\ &= \left(-\frac{\hat{z}Z_{\theta 2}^2}{\omega\mu_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\theta 1}z'\right)\sin\left(\lambda_{z\theta 2}\left(d-z\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\end{aligned} \tag{C.19}$$

Finally, analyze the component observed in region 2 resulting from longitudinal magnetic currents in region 2, $\tilde{\theta}_{2hz2}$. Similar to the analysis for $\tilde{\vec{\vec{G}}}_{\theta 2e2}$, it can be shown that substituting (140) and (138) into (146) implies that

$$\begin{aligned}\tilde{\vec{\vec{G}}}_{\theta 2hz2} = &\left(-\frac{\hat{z}\mu_{t2}}{2\lambda_{z\theta 2}\mu_{z2}}\right)\left[\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.\\ &+\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\ &\left.+\frac{C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\\ = &\left(-\frac{\hat{z}Z_{\theta 2}}{2\omega\mu_{z2}}\right)\left[\frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1}h\right)\left[\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.\\ &+\frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1}h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\ &\left.+\frac{Z_{\theta 1}\sin\left(\lambda_{z\theta 1}h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\end{aligned} \tag{C.20}$$

C.3 $\tilde{\psi}$ Development

Determine $\tilde{\psi}_1$ and $\tilde{\psi}_2$. Substituting (70) and (B.5) into (109) implies that

$$\tilde{\psi}_{1\{e,h\}} = \int_0^h \left[\tilde{\vec{\vec{G}}}^p_{\psi 1\{e,h\}} + \tilde{\vec{\vec{G}}}^s_{\psi 1\{e,h\}1}\right]\cdot\tilde{\vec{J}}_{\{e,h\}}dz' + \int_h^d \tilde{\vec{\vec{G}}}^s_{\psi 1\{e,h\}2}\cdot\tilde{\vec{J}}_{\{e,h\}}dz'$$

$$
\begin{aligned}
= \int_0^h D_\psi^{-1} \Big\{ & \vec{\tilde{g}}_{\psi 1\{e,h\}} e^{-jk_{z\psi 1}|z-z'|} \left[\left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \right. \\
& \left. + C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right) \right] \\
& + \vec{\tilde{g}}^p_{\psi 1\{e,h\}} (z=h) \left(P_{\psi 1h} P_{\psi 1z} P_{\psi 1z'}^{-1} + P_{\psi 1h} P_{\psi 1z}^{-1} P_{\psi 1z'}^{-1} \right) \left[\left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \right. \\
& \left. - C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \right] \\
& + \vec{\tilde{g}}^p_{\psi 1\{e,h\}} (z=0) \left[\left(P_{\psi 1h}^{-1} P_{\psi 1z} P_{\psi 1z'} + P_{\psi 1h} P_{\psi 1z}^{-1} P_{\psi 1z'} \right) \left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \right] \\
& + \vec{\tilde{g}}^p_{\psi 1\{e,h\}} (z=0) \left[C_\psi \left(P_{\psi 1h}^{-1} P_{\psi 1z} P_{\psi 1z'} - P_{\psi 1h} P_{\psi 1z}^{-1} P_{\psi 1z'} \right) \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \right] \Big\} \\
& \cdot \vec{\tilde{J}}_{\{e,h\}} dz' \\
+ \int_h^d & D_\psi^{-1} 2 C_\psi \left(\vec{\tilde{g}}^p_{\psi 2\{e,h\}} (z=d) P_{\psi 2d} P_{\psi 2z'}^{-1} + \vec{\tilde{g}}^p_{\psi 2(e,h)} (z=h) P_{\psi 2d}^{-1} P_{\psi 2z'} \right) \left(P_{\psi 1z} + P_{\psi 1z}^{-1} \right) \\
& \cdot \vec{\tilde{J}}_{\{e,h\}} dz'
\end{aligned}
\tag{C.21}
$$

Note that D_ψ can be rewritten in terms of sine and cosine functions such that

$$
\begin{aligned}
D_\psi &= \left(P_{\psi 2d} P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1} P_{\psi 2h} \right) \left(P_{\psi 1h}^{-1} - P_{\psi 1h} \right) \\
&\quad + C_\psi \left(P_{\psi 2d}^{-1} P_{\psi 2h} - P_{\psi 2d} P_{\psi 2h}^{-1} \right) \left(P_{\psi 1h}^{-1} + P_{\psi 1h} \right) \\
&= j4 \left[\cos\left(\lambda_{z\psi 2} T\right) \sin\left(\lambda_{z\psi 1} h\right) + C_\psi \sin\left(\lambda_{z\psi 2} T\right) \cos\left(\lambda_{z\psi 1} h\right) \right]
\end{aligned}
\tag{C.22}
$$

Beginning with the electric component of $\tilde{\psi}_1$, note that from (81) there are both transverse and longitudinal components of $\vec{\tilde{g}}^p_{\psi e1}$ such that

$$
\tilde{g}^p_{\psi\{1,2\}et} = -j \frac{\vec{\lambda}_\rho \operatorname{sgn}(z-z')}{2\lambda_{\rho\psi}^2} \tag{C.23}
$$

$$
\tilde{g}^p_{\psi\{1,2\}ez} = -j \frac{\hat{z}\epsilon_{t\{1,2\}}}{2\lambda_{z\psi\{1,2\}}\epsilon_{z\{1,2\}}} \tag{C.24}
$$

First, determine the component observed in region 1 resulting from transverse electric currents in region 1, $\tilde{\psi}_{1et1}$. Substituting (C.23) and (C.22) into (C.21) implies that

$$\begin{aligned}
\tilde{\vec{G}}_{\psi 1et1} = {} & \left[-\frac{j\vec{\lambda}_{\rho}}{2\lambda_{\rho\psi}^{2}D_{\psi}}\right]\Big\{\operatorname{sgn}(z-z')\,e^{-j\lambda_{z\psi1}|z-z'|}\left[\left(P_{\psi2d}P_{\psi2h}^{-1}+P_{\psi2d}^{-1}P_{\psi2h}\right)\left(P_{\psi1h}^{-1}\right.\right. \\
& \left.\left. -P_{\psi1h}\right)\right]+\operatorname{sgn}(z-z')\,e^{-j\lambda_{z\psi1}|z-z'|}\left[C_{\psi}\left(P_{\psi2d}^{-1}P_{\psi2h}-P_{\psi2d}P_{\psi2h}^{-1}\right)\left(P_{\psi1h}^{-1}+P_{\psi1h}\right)\right] \\
& +\cancelto{1}{\operatorname{sgn}(h-z')}\left(P_{\psi1h}P_{\psi1z}P_{\psi1z'}^{-1}+P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}^{-1}\right)\left[\left(P_{\psi2d}P_{\psi2h}^{-1}+P_{\psi2d}^{-1}P_{\psi2h}\right)\right] \\
& +\cancelto{1}{\operatorname{sgn}(h-z')}\left(P_{\psi1h}P_{\psi1z}P_{\psi1z'}^{-1}+P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}^{-1}\right)\left[-C_{\psi}\left(P_{\psi2d}^{-1}P_{\psi2h}-P_{\psi2d}P_{\psi2h}^{-1}\right)\right] \\
& +\cancelto{-1}{\operatorname{sgn}(-z')}\left[\left(P_{\psi1h}^{-1}P_{\psi1z}P_{\psi1z'}+P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}\right)\left(P_{\psi2d}P_{\psi2h}^{-1}+P_{\psi2d}^{-1}P_{\psi2h}\right)\right] \\
& +\cancelto{-1}{\operatorname{sgn}(-z')}\left[C_{\psi}\left(P_{\psi1h}^{-1}P_{\psi1z}P_{\psi1z'}-P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}\right)\left(P_{\psi2d}^{-1}P_{\psi2h}-P_{\psi2d}P_{\psi2h}^{-1}\right)\right]\Big\} \\
= {} & \left(-j\frac{\vec{\lambda}_{\rho}}{2\lambda_{\rho\psi}^{2}D_{\psi}}\right)\Big\{\operatorname{sgn}(z-z')\,e^{-j\lambda_{z\psi1}|z-z'|}\left[\left(P_{\psi2d}P_{\psi2h}^{-1}+P_{\psi2d}^{-1}P_{\psi2h}\right)\left(P_{\psi1h}^{-1}-P_{\psi1h}\right)\right] \\
& +\operatorname{sgn}(z-z')\,e^{-j\lambda_{z\psi1}|z-z'|}\left[C_{\psi}\left(P_{\psi2d}^{-1}P_{\psi2h}-P_{\psi2d}P_{\psi2h}^{-1}\right)\left(P_{\psi1h}^{-1}+P_{\psi1h}\right)\right] \\
& +\left(P_{\psi1h}P_{\psi1z}P_{\psi1z'}^{-1}+P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}^{-1}-P_{\psi1h}^{-1}P_{\psi1z}P_{\psi1z'}\right)\left(P_{\psi2d}P_{\psi2h}^{-1}+P_{\psi2d}^{-1}P_{\psi2h}\right) \\
& +\left(-P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}\right)\left(P_{\psi2d}P_{\psi2h}^{-1}+P_{\psi2d}^{-1}P_{\psi2h}\right) \\
& +C_{\psi}\left(-P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}^{-1}-P_{\psi1h}^{-1}P_{\psi1z}P_{\psi1z'}+P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}\right)\left(P_{\psi2d}^{-1}P_{\psi2h}-P_{\psi2d}P_{\psi2h}^{-1}\right) \\
& +C_{\psi}\left(-P_{\psi1h}P_{\psi1z}P_{\psi1z'}^{-1}\right)\left(P_{\psi2d}^{-1}P_{\psi2h}-P_{\psi2d}P_{\psi2h}^{-1}\right)\Big\}
\end{aligned} \tag{C.25}$$

Due to the $\operatorname{sgn}(z-z')$ and $|z-z'|$ terms in (C.25), two cases must be analyzed.

When $z > z'$, that implies that

$$
\begin{aligned}
\tilde{\vec{G}}^{z+}_{\psi 1et1} = {} & \left[-\frac{j\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}D_\psi}\right]\Big\{\cancelto{1}{\mathrm{sgn}\left(z-z'\right)}P_{\psi 1z}P^{-1}_{\psi 1z'}\left[\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}\right.\right.\\
& \left.\left.-P_{\psi 1h}\right)\right]+\cancelto{1}{\mathrm{sgn}\left(z-z'\right)}P_{\psi 1z}P^{-1}_{\psi 1z'}\left[C_\psi\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\right]\\
& +\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}+P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}-P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right.\\
& \left.-P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}\right)+C_\psi\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(-P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}-P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right.\\
& \left.+P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}-P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}\right)\Big\}\\
= {} & \left[-\frac{j\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}D_\psi}\right]\Big[\Big(\cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}}-P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\Big)\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\\
& +\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\Big(P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}-P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}-\cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}}\Big)\\
& +\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}\right)\\
& +C_\psi\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\Big(P^{-1}_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}+\cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}}-\cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}}\Big)\\
& +C_\psi\left(-P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}-P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}+P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\Big]\\
= {} & \left(-j\frac{\vec{\lambda}_\rho}{2k^2_{\rho\psi}}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\sin\left(\lambda_{z\psi 1}\left(h-\left(z-z'\right)\right)\right)-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
& \left.+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-\left(z-z'\right)\right)\right)-\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned}
\tag{C.26}
$$

When $z < z'$, that implies that

$$
\begin{aligned}
\tilde{\vec{G}}^{z-}_{\psi 1et1} = {} & \left(-\frac{j\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}D_\psi}\right)\Big[\cancelto{-1}{\mathrm{sgn}\left(z-z'\right)}P^{-1}_{\psi 1z}P_{\psi 1z'}\left[\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}\right.\right.\\
& \left.\left.-P_{\psi 1h}\right)\right]+\cancelto{-1}{\mathrm{sgn}\left(z-z'\right)}P^{-1}_{\psi 1z}P_{\psi 1z'}\left[C_\psi\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\right]\\
& +\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}+P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}-P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right.\\
& \left.-P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}\right)+C_\psi\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(-P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}-P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right.\\
& \left.+P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}-P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 D_\psi}\right)\Big[\left(P_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}^{-1} - P_{\psi 1h}^{-1}P_{\psi 1z}P_{\psi 1z'}\right)\left(P_{\psi 2d}P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1}P_{\psi 2h}\right)\\
&\quad + \left(P_{\psi 2d}P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1}P_{\psi 2h}\right)\left(-\cancel{P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}} - P_{\psi 1h}^{-1}P_{\psi 1z}^{-1}P_{\psi 1z'} + \cancel{P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}}\right)\\
&\qquad + \left(P_{\psi 2d}P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1}P_{\psi 2h}\right)\left(P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}^{-1}\right)\\
&\quad + C_\psi\left(P_{\psi 2d}^{-1}P_{\psi 2h} - P_{\psi 2d}P_{\psi 2h}^{-1}\right)\left(-P_{\psi 1h}^{-1}P_{\psi 1z}^{-1}P_{\psi 1z'} - \cancel{P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}} - P_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}^{-1}\right)\\
&\quad + C_\psi\left(-P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}^{-1} - P_{\psi 1h}^{-1}P_{\psi 1z}P_{\psi 1z'} + \cancel{P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}}\right)\left(P_{\psi 2d}^{-1}P_{\psi 2h} - P_{\psi 2d}P_{\psi 2h}^{-1}\right)\Big]\\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h+\left(z-z'\right)\right)\right) - \sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&\qquad\left. + \frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[-\cos\left(\lambda_{z\psi 1}\left(h+\left(z-z'\right)\right)\right) - \cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned}
\tag{C.27}
$$

Analyzing (C.26) and (C.27) reveals that

$$
\begin{aligned}
\vec{\vec{G}}_{\psi 1et1} &= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&\qquad + \frac{\cos\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&\quad\left. + \frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right) - \cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&\qquad + \frac{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&\quad\left. + \frac{Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right) - \cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned}
\tag{C.28}
$$

Next, analyze the component observed in region 1 resulting from transverse electric currents in region 2, $\tilde{\psi}_{1et2}$. Substituting (C.23) and (C.22) into (C.21) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\psi 1et2} &= \left[\frac{2C_\psi \vec{\lambda}_\rho}{j2\lambda_{\rho\psi}^2}\right] \left[\frac{\left[\cancelto{1}{\operatorname{sgn}(d-z')} P_{\psi 2d} P_{\psi 2z'}^{-1} + \cancelto{-1}{\operatorname{sgn}(h-z')} P_{\psi 2d}^{-1} P_{\psi 2z'}\right]\left[P_{\psi 1z} + P_{\psi 1z}^{-1}\right]}{j4\left[\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi \sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)\right]}\right] \\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{-j8C_\psi \sin\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1}z\right)}{j4\left[\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_\psi \sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)\right]}\right] \\
&= \left(j\frac{\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\left[\frac{\sin\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1}z\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.29}$$

Next, analyze the component observed in region 1 resulting from longitudinal electric currents in region 1, $\tilde{\psi}_{1ez1}$. Substituting (C.24) and (C.22) into (C.21) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\psi 1ez1} = \left(-j\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi 1}\epsilon_{z1}}\right) D_\psi^{-1} \Big\{ & e^{-j\lambda_{z\psi 1}|z-z'|}\Big[\left(P_{\psi 2d}P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1}P_{\psi 2h}\right)\left(P_{\psi 1h}^{-1} - P_{\psi 1h}\right) \\
& + C_\psi\left(P_{\psi 2d}^{-1}P_{\psi 2h} - P_{\psi 2d}P_{\psi 2h}^{-1}\right)\left(P_{\psi 1h}^{-1} + P_{\psi 1h}\right)\Big] \\
& + \left(P_{\psi 2d}P_{\psi 2h}^{-1} + P_{\psi 2d}^{-1}P_{\psi 2h}\right)\left(P_{\psi 1h}^{-1}P_{\psi 1z}P_{\psi 1z'} + P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'} + P_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}^{-1}\right. \\
& \left. + P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}^{-1}\right) + C_\psi\left(P_{\psi 2d}^{-1}P_{\psi 2h} - P_{\psi 2d}P_{\psi 2h}^{-1}\right)\left(-P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'} - P_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}^{-1}\right. \\
& \left. - P_{\psi 1h}P_{\psi 1z}^{-1}P_{\psi 1z'}^{-1} + P_{\psi 1h}^{-1}P_{\psi 1z}P_{\psi 1z'}\right)\Big\}
\end{aligned} \tag{C.30}$$

Due to the $|z-z'|$ term in (C.30), two cases must be analyzed. When $z > z'$, that

implies that

$$
\begin{aligned}
\vec{\bar{G}}^{z+}_{\psi 1ez1} = {} & \left(-j\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi 1}\epsilon_{z1}}\right) D_{\psi}^{-1} \left[P_{\psi 1z}P^{-1}_{\psi 1z'} \left[\left(P_{\psi 2d}P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right]\right. \\
& + P_{\psi 1z}P^{-1}_{\psi 1z'} \left[C_{\psi}\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right] \\
& + \left(P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'} + P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'} + P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}\right)\left(P_{\psi 2d}P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d}P_{\psi 2h}\right) \\
& + \left(P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}\right)\left(P_{\psi 2d}P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d}P_{\psi 2h}\right) \\
& + C_{\psi}\left(-P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'} - P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'} - P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right) \\
& \left. + C_{\psi}\left(P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right)\right] \\
= {} & \left(-\frac{j\hat{z}\epsilon_{t1}}{2\lambda_{z\psi 1}\epsilon_{z1}D_{\psi}}\right)\left[\left(P_{\psi 2d}P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'} + P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'}\right.\right. \\
& \left. + \cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}} + P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'} + P^{-1}_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'} - \cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}}\right) \\
& + C_{\psi}\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'} + \cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}} + P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right) \\
& \left. + C_{\psi}\left(-P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'} - \cancel{P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}} - P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right)\right] \\
= {} & \left(-\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi 1}\epsilon_{z1}}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-(z-z')\right)\right) + \cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
& \left. + \frac{C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h-(z-z')\right)\right) - \sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right) + C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned}
\tag{C.31}
$$

When $z < z'$, that implies that

$$
\begin{aligned}
\vec{\bar{G}}^{z-}_{\psi 1ez1} = {} & \left(-j\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi 1}\epsilon_{z1}}\right) D_{\psi}^{-1} \left\{P^{-1}_{\psi 1z}P_{\psi 1z'} \left[\left(P_{\psi 2d}P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h} - P_{\psi 1h}\right)\right]\right. \\
& + P^{-1}_{\psi 1z}P_{\psi 1z'} \left[C_{\psi}\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h} + P_{\psi 1h}\right)\right] \\
& + \left(P_{\psi 2d}P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'} + P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'} + P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'}\right. \\
& \left. + P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}\right) + C_{\psi}\left(P^{-1}_{\psi 1h}P_{\psi 1z}P_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right) \\
& \left. + C_{\psi}\left(-P_{\psi 1h}P^{-1}_{\psi 1z}P_{\psi 1z'} - P_{\psi 1h}P_{\psi 1z}P^{-1}_{\psi 1z'} - P_{\psi 1h}P^{-1}_{\psi 1z}P^{-1}_{\psi 1z'}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h} - P_{\psi 2d}P^{-1}_{\psi 2h}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-j\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi1}\epsilon_{z1}}\right) D_{\psi}^{-1} \Big[\left(P_{\psi2d}P_{\psi2h}^{-1} + P_{\psi2d}^{-1}P_{\psi2h}\right) \Big(P_{\psi1h}^{-1}P_{\psi1z}P_{\psi1z'} + \cancel{P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}} \\
&\qquad + P_{\psi1h}P_{\psi1z}P_{\psi1z'}^{-1} + P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}^{-1} + P_{\psi1h}^{-1}P_{\psi1z}^{-1}P_{\psi1z'} - \cancel{P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}} \Big) \\
&\quad + C_{\psi}\left(P_{\psi2d}^{-1}P_{\psi2h} - P_{\psi2d}P_{\psi2h}^{-1}\right) \Big(P_{\psi1h}^{-1}P_{\psi1z}^{-1}P_{\psi1z'} + \cancel{P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}} + P_{\psi1h}^{-1}P_{\psi1z}P_{\psi1z'} \Big) \\
&\quad + C_{\psi}\Big(-\cancel{P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}} - P_{\psi1h}P_{\psi1z}P_{\psi1z'}^{-1} - P_{\psi1h}P_{\psi1z}^{-1}P_{\psi1z'}^{-1} \Big) \left(P_{\psi2d}^{-1}P_{\psi2h} - P_{\psi2d}P_{\psi2h}^{-1}\right) \Big] \\
&= \left(-\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi1}\epsilon_{z1}}\right) \left[\frac{\cos(\lambda_{z\psi2}T)\left[\cos(\lambda_{z\psi1}(h+(z-z'))) + \cos(\lambda_{z\psi1}(h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_{\psi}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \right. \\
&\qquad \left. + \frac{C_{\psi}\sin(\lambda_{z\psi2}T)\left[-\sin(\lambda_{z\psi1}(h+(z-z'))) - \sin(\lambda_{z\psi1}(h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_{\psi}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \right]
\end{aligned}
\tag{C.32}
$$

Analyzing (C.31) and (C.32) reveals that

$$
\begin{aligned}
\vec{\tilde{G}}_{\psi1ez1} &= \left[\frac{-\hat{z}\epsilon_{t1}}{2\lambda_{z\psi1}\epsilon_{z1}}\right] \left[\frac{\cos(\lambda_{z\psi2}T)\left[\cos(\lambda_{z\psi1}(h-|z-z'|))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_{\psi}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \right. \\
&\qquad + \frac{\cos(\lambda_{z\psi2}T)\left[\cos(\lambda_{z\psi1}(h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_{\psi}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \\
&\qquad \left. + \frac{C_{\psi}\sin(\lambda_{z\psi2}T)\left[-\sin(\lambda_{z\psi1}(h-|z-z'|)) - \sin(\lambda_{z\psi1}(h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_{\psi}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \right] \\
&= \left[\frac{-\hat{z}}{2Z_{\psi1}\omega\epsilon_{z1}}\right] \left[\frac{Z_{\psi1}\cos(\lambda_{z\psi2}T)\left[\cos(\lambda_{z\psi1}(h-|z-z'|)) + \cos(\lambda_{z\psi1}(h-z-z'))\right]}{Z_{\psi1}\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + Z_{\psi2}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \right. \\
&\qquad \left. + \frac{Z_{\psi2}\sin(\lambda_{z\psi2}T)\left[-\sin(\lambda_{z\psi1}(h-|z-z'|)) - \sin(\lambda_{z\psi1}(h-z-z'))\right]}{Z_{\psi1}\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + Z_{\psi2}\sin(k_{z\psi2}T)\cos(\lambda_{z\psi1}h)} \right]
\end{aligned}
\tag{C.33}
$$

Next, analyze the component observed in region 1 resulting from longitudinal electric currents in region 2, $\tilde{\psi}_{1ez2}$. Substituting (C.24) and (C.22) into (C.21) implies that

$$
\vec{\tilde{G}}_{\psi1ez2} = \left(-j\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi1}\epsilon_{z1}}\right) \left[\frac{2C_{\psi}\left(P_{\psi2d}P_{\psi2z'}^{-1} + P_{\psi2d}^{-1}P_{\psi2z'}\right)\left(P_{\psi1z} + P_{\psi1z}^{-1}\right)}{j4\left[\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_{\psi}\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)\right]} \right]
$$

$$= \left(-\frac{\hat{z} Z_{\psi 2}}{Z_{\psi 1}\omega\epsilon_{z1}}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1}z\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \tag{C.34}$$

Now analyze the component observed in region 2 resulting from transverse electric currents in region 1, $\tilde{\psi}_{2et1}$. Substituting (C.23) and (C.22) into (C.48) implies that

$$\begin{aligned}
\vec{\vec{\tilde{G}}}_{\psi 2et1} &= \left(-\frac{j2\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{\left(\cancelto{1}{\operatorname{sgn}\left(h-z'\right)}P_{\psi 1z'}^{-1}+\cancelto{-1}{\operatorname{sgn}\left(-z'\right)}P_{\psi 1z'}\right)\left(P_{\psi 2d}^{-1}P_{\psi 2z}+P_{\psi 2d}P_{\psi 2z}^{-1}\right)}{j4\left[\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)\right]}\right] \\
&= \left(-j\frac{\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}z'\right)\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.35}$$

Next, analyze the component observed in region 2 resulting from transverse electric currents in region 2, $\tilde{\psi}_{2et2}$. Substituting (C.23) and (C.22) into (C.48) implies that

$$\begin{aligned}
\vec{\vec{\tilde{G}}}_{\psi 2et2} = \left[\frac{-j\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 D_\psi}\right]&\Big\{\operatorname{sgn}\left(z-z'\right)e^{-j\lambda_{z\psi 2}|z-z'|}\left[\left(P_{\psi 2d}P_{\psi 2h}^{-1}+P_{\psi 2d}^{-1}P_{\psi 2h}\right)\left(P_{\psi 1h}^{-1}\right.\right. \\
&\left.\left.-P_{\psi 1h}\right)\right]+\operatorname{sgn}\left(z-z'\right)e^{-j\lambda_{z\psi 2}|z-z'|}\left[C_\psi\left(P_{\psi 2d}^{-1}P_{\psi 2h}-P_{\psi 2d}P_{\psi 2h}^{-1}\right)\left(P_{\psi 1h}^{-1}+P_{\psi 1h}\right)\right] \\
&+\cancelto{1}{\operatorname{sgn}\left(d-z'\right)}P_{\psi 2d}P_{\psi 2z'}^{-1}\left[\left(P_{\psi 2h}P_{\psi 2z}^{-1}-P_{\psi 2h}^{-1}P_{\psi 2z}\right)\left(P_{\psi 1h}^{-1}-P_{\psi 1h}\right)\right] \\
&+\cancelto{1}{\operatorname{sgn}\left(d-z'\right)}P_{\psi 2d}P_{\psi 2z'}^{-1}\left[C_\psi\left(P_{\psi 2h}P_{\psi 2z}^{-1}+P_{\psi 2h}^{-1}P_{\psi 2z}\right)\left(P_{\psi 1h}^{-1}+P_{\psi 1h}\right)\right] \\
&-\cancelto{-1}{\operatorname{sgn}\left(h-z'\right)}P_{\psi 2h}^{-1}P_{\psi 2z'}\left[\left(P_{\psi 2d}^{-1}P_{\psi 2z}+P_{\psi 2d}P_{\psi 2z}^{-1}\right)\left(P_{\psi 1h}^{-1}-P_{\psi 1h}\right)\right] \\
&-\cancelto{-1}{\operatorname{sgn}\left(h-z'\right)}P_{\psi 2h}^{-1}P_{\psi 2z'}\left[-C_\psi\left(P_{\psi 2d}^{-1}P_{\psi 2z}+P_{\psi 2d}P_{\psi 2z}^{-1}\right)\left(P_{\psi 1h}^{-1}+P_{\psi 1h}\right)\right]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{-j\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}D_\psi}\right]\Big\{\mathrm{sgn}\left(z-z'\right)e^{-j\lambda_{z\psi2}|z-z'|}\left[\left(P_{\psi2d}P^{-1}_{\psi2h}+P^{-1}_{\psi2d}P_{\psi2h}\right)\left(P^{-1}_{\psi1h}-P_{\psi1h}\right)\right] \\
&\quad + \mathrm{sgn}\left(z-z'\right)e^{-j\lambda_{z\psi2}|z-z'|}\left[C_\psi\left(P^{-1}_{\psi2d}P_{\psi2h}-P_{\psi2d}P^{-1}_{\psi2h}\right)\left(P^{-1}_{\psi1h}+P_{\psi1h}\right)\right] \\
&\quad + \left(P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'}+P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}+P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'}\right)\left(P^{-1}_{\psi1h}-P_{\psi1h}\right) \\
&\quad + \left(-P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}\right)\left(P^{-1}_{\psi1h}-P_{\psi1h}\right)+C_\psi\left(P^{-1}_{\psi1h}+P_{\psi1h}\right)\Big(P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'} \\
&\qquad +P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}-P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'}-P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}\Big)\Big\}
\end{aligned}
\tag{C.36}
$$

Due to the $\mathrm{sgn}\left(z-z'\right)$ and $|z-z'|$ terms in (C.36), two cases must be analyzed. When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}^{z+}_{\psi2et2} &= \left[\frac{-j\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}D_\psi}\right]\Big\{\cancelto{1}{\mathrm{sgn}\left(z-z'\right)}P_{\psi2z}P^{-1}_{\psi2z'}\left[\left(P_{\psi2d}P^{-1}_{\psi2h}+P^{-1}_{\psi2d}P_{\psi2h}\right)\left(P^{-1}_{\psi1h}-P_{\psi1h}\right)\right] \\
&\quad + \cancelto{1}{\mathrm{sgn}\left(z-z'\right)}P_{\psi2z}P^{-1}_{\psi2z'}\left[C_\psi\left(P^{-1}_{\psi2d}P_{\psi2h}-P_{\psi2d}P^{-1}_{\psi2h}\right)\left(P^{-1}_{\psi1h}+P_{\psi1h}\right)\right] \\
&\quad + \left(P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'}+P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}+P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'}\right)\left(P^{-1}_{\psi1h}-P_{\psi1h}\right) \\
&\quad + \left(-P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}\right)\left(P^{-1}_{\psi1h}-P_{\psi1h}\right)+C_\psi\left(P^{-1}_{\psi1h}+P_{\psi1h}\right)\Big(P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'} \\
&\qquad +P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}-P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'}-P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}\Big)\Big\} \\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}D_\psi}\right)\Big[\left(P^{-1}_{\psi1h}-P_{\psi1h}\right)\Big(P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'}+P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'} \\
&\quad +P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'}-\cancel{P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}}+\cancel{P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}} \\
&\quad +P^{-1}_{\psi2d}P_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}\Big)+C_\psi\left(P^{-1}_{\psi1h}+P_{\psi1h}\right)\Big(P^{-1}_{\psi2d}P_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'} \\
&\quad -\cancel{P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}}+P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'}+\cancel{P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}} \\
&\qquad -P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'}-P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{\sin(\lambda_{z\psi1}h)\left[\cos(\lambda_{z\psi2}(T-(z-z'))) + \cos(\lambda_{z\psi2}(d+h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_\psi\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)}\right.\\
&\qquad\left.+\frac{C_\psi\cos(\lambda_{z\psi1}h)\left[\sin(\lambda_{z\psi2}(T-(z-z'))) - \sin(\lambda_{z\psi2}(d+h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_\psi\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)}\right]
\end{aligned}
\tag{C.37}
$$

When $z < z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}^{z-}_{\psi2et2} &= \left[\frac{-j\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 D_\psi}\right]\Big\{\cancelto{-1}{\mathrm{sgn}(z-z')}P^{-1}_{\psi2z}P_{\psi2z'}\left[\left(P_{\psi2d}P^{-1}_{\psi2h} + P^{-1}_{\psi2d}P_{\psi2h}\right)\left(P^{-1}_{\psi1h} - P_{\psi1h}\right)\right]\\
&\quad+\cancelto{-1}{\mathrm{sgn}(z-z')}P^{-1}_{\psi2z}P_{\psi2z'}\left[C_\psi\left(P^{-1}_{\psi2d}P_{\psi2h} - P_{\psi2d}P^{-1}_{\psi2h}\right)\left(P^{-1}_{\psi1h} + P_{\psi1h}\right)\right]\\
&\quad+\left(P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'} + P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'} + P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'}\right)\left(P^{-1}_{\psi1h} - P_{\psi1h}\right)\\
&\quad+\left(-P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}\right)\left(P^{-1}_{\psi1h} - P_{\psi1h}\right) + C_\psi\left(P^{-1}_{\psi1h} + P_{\psi1h}\right)\big(P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'}\\
&\qquad+P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'} - P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'} - P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}\big)\Big\}\\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 D_\psi}\right)\Big[\left(P^{-1}_{\psi1h} - P_{\psi1h}\right)\big(P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'} + \cancel{P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}}\\
&\qquad+P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'} - P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'} - \cancel{P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}}\\
&\qquad-P^{-1}_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}\big) + C_\psi\left(P^{-1}_{\psi1h} + P_{\psi1h}\right)\big(-P^{-1}_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}\\
&\qquad+\cancel{P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}} + P_{\psi2d}P_{\psi2h}P^{-1}_{\psi2z}P^{-1}_{\psi2z'} + P_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P^{-1}_{\psi2z'}\\
&\qquad\qquad-P^{-1}_{\psi2d}P^{-1}_{\psi2h}P_{\psi2z}P_{\psi2z'} - \cancel{P_{\psi2d}P^{-1}_{\psi2h}P^{-1}_{\psi2z}P_{\psi2z'}}\big)\Big]\\
&= \left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{\sin(\lambda_{z\psi1}h)\left[-\cos(\lambda_{z\psi2}(T+(z-z')))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_\psi\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)}\right.\\
&\qquad+\frac{\sin(\lambda_{z\psi1}h)\left[\cos(\lambda_{z\psi2}(d+h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_\psi\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)}\\
&\quad\left.+\frac{C_\psi\cos(\lambda_{z\psi1}h)\left[-\sin(\lambda_{z\psi2}(T+(z-z'))) - \sin(\lambda_{z\psi2}(d+h-z-z'))\right]}{\cos(\lambda_{z\psi2}T)\sin(\lambda_{z\psi1}h) + C_\psi\sin(\lambda_{z\psi2}T)\cos(\lambda_{z\psi1}h)}\right]
\end{aligned}
\tag{C.38}
$$

Analyzing (C.37) and (C.38) reveals that

$$\begin{aligned}\vec{\tilde{G}}_{\psi 2et2} = &\left[-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right]\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\ &+\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\ &\left.+\frac{C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)-\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\\ =&\left[-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right]\left[\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\ &+\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\ &+\frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\ &\left.+\frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.39}$$

Next, analyze the component observed in region 2 resulting from longitudinal electric currents in region 1, $\tilde{\psi}_{2ez1}$. Substituting (C.24) and (C.22) into (C.48) implies that

$$\begin{aligned}\vec{\tilde{G}}_{\psi 2ez1} &= \left(-j\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}}\right)\left[\frac{2\left(P_{\psi 1z'}^{-1}+P_{\psi 1z'}\right)\left(P_{\psi 2d}^{-1}P_{\psi 2z}+P_{\psi 2d}P_{\psi 2z}^{-1}\right)}{j4\left[\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)\right]}\right]\\ &=\left(-\frac{\hat{z}Z_{\psi 1}}{Z_{\psi 2}\omega\epsilon_{z2}}\right)\left[\frac{\cos\left(\lambda_{z\psi 1}z'\right)\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.40}$$

Next, analyze the component observed in region 2 resulting from longitudinal electric currents in region 2, $\tilde{\psi}_{2ez2}$. Substituting (C.24) and (C.22) into (C.48) implies

that

$$
\begin{aligned}
\vec{\tilde{G}}_{\psi 2ez2} = \left(-j\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}D_{\psi}}\right) &\Big\{ e^{-j\lambda_{z\psi 2}|z-z'|}\left[\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)\right] \\
&+e^{-j\lambda_{z\psi 2}|z-z'|}\left[C_{\psi}\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\right] \\
&+\left(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}-P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}-P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right) \\
&+\left(-P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)+C_{\psi}\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\Big(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'} \\
&+P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}+P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}+P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\Big)\Big\}
\end{aligned}
\tag{C.41}
$$

Due to the $|z-z'|$ term in (C.41), two cases must be analyzed. When $z > z'$, that implies that

$$
\begin{aligned}
\vec{\tilde{G}}^{z^+}_{\psi 2ez2} &= \left(-j\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}}\right)D^{-1}_{\psi}\Big\{P_{\psi 2z}P^{-1}_{\psi 2z'}\left[\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)\right] \\
&\quad+P_{\psi 2z}P^{-1}_{\psi 2z'}\left[C_{\psi}\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\right] \\
&\quad+\left(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}-P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}-P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right) \\
&\quad+\left(-P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)+C_{\psi}\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\Big(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'} \\
&\quad+P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}+P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}+P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\Big)\Big\} \\
&= \left(-j\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}D_{\psi}}\right)\Big[\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)\Big(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}-\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}} \\
&\quad-P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}-P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}+\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}} \\
&\quad+P^{-1}_{\psi 2d}P_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}\Big)+C_{\psi}\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\Big(P^{-1}_{\psi 2d}P_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'} \\
&\quad-\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}}+P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}+\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}} \\
&\quad+P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}+P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-(z-z')\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\quad \left.+\frac{C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-(z-z')\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned}
\tag{C.42}
$$

When $z < z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}^{z-}_{\psi 2ez2} &= \left(-j\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}D_\psi}\right)\left[P^{-1}_{\psi 2z}P_{\psi 2z'}\left[\left(P_{\psi 2d}P^{-1}_{\psi 2h}+P^{-1}_{\psi 2d}P_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)\right]\right. \\
&\quad +P^{-1}_{\psi 2z}P_{\psi 2z'}\left[C_\psi\left(P^{-1}_{\psi 2d}P_{\psi 2h}-P_{\psi 2d}P^{-1}_{\psi 2h}\right)\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\right] \\
&\quad +\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)\left(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}-P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}-P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}\right. \\
&\quad \left.-P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\right)+C_\psi\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\left(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}\right. \\
&\quad \left.\left.+P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}+P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}+P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\right)\right] \\
&= \left(-j\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}D_\psi}\right)\left[\left(P^{-1}_{\psi 1h}-P_{\psi 1h}\right)\left(P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}-P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'}\right.\right. \\
&\quad -P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}-\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}}+\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}} \\
&\quad \left.+P^{-1}_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\right)+C_\psi\left(P^{-1}_{\psi 1h}+P_{\psi 1h}\right)\left(P^{-1}_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}\right. \\
&\quad -\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}}+P_{\psi 2d}P_{\psi 2h}P^{-1}_{\psi 2z}P^{-1}_{\psi 2z'}+P_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P^{-1}_{\psi 2z'} \\
&\quad \left.\left.+P^{-1}_{\psi 2d}P^{-1}_{\psi 2h}P_{\psi 2z}P_{\psi 2z'}+\cancel{P_{\psi 2d}P^{-1}_{\psi 2h}P^{-1}_{\psi 2z}P_{\psi 2z'}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T+(z-z')\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\quad \left.+\frac{C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T+(z-z')\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned}
\tag{C.43}
$$

Analyzing (C.42) and (C.43) reveals that

$$\begin{aligned}\vec{\vec{G}}_{\psi 2ez2} &= \left(-\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\ &\quad+\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\ &\quad\left.+\frac{C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\\ &= \left(-\frac{\hat{z}}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\left[\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\ &\quad+\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\ &\quad\left.+\frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.44}$$

For the magnetic component, note that from (86) there is no longitudinal component. This implies that

$$\vec{g}^{p}_{\psi\{1,2\}h} = -j\frac{\hat{z}\times\vec{\lambda}_\rho\omega\epsilon_{t\{1,2\}}}{2\lambda_{z\psi\{1,2\}}\lambda^2_{\rho\psi}} \tag{C.45}$$

Analyze the component observed in region 1 resulting from transverse magnetic currents in region 1, $\tilde{\psi}_{1h1}$. From the analysis of $\tilde{\psi}_{1ez1}$, it can be shown that substituting (C.45) and (C.22) into (C.21) implies that

$$\begin{aligned}\vec{\vec{G}}_{\psi 1h1} &= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\epsilon_{t1}}{2\lambda_{z\psi 1}\lambda^2_{\rho\psi}}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\ &\quad+\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\ &\quad\left.+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned}$$

$$= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\left[\frac{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.$$
$$+\frac{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}$$
$$\left.+\frac{Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \tag{C.46}$$

From the analysis of $\tilde{\psi}_{1ez2}$, it can be shown that substituting (C.45) and (C.22) into (C.21) implies that

$$\tilde{\vec{G}}_{\psi 1h2} = \left(-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\epsilon_{t1}}{\lambda_{z\psi 1}\lambda_{\rho\psi}^2}\right)\left[\frac{C_\psi\cos\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1}z\right)}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]$$
$$= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1}z\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \tag{C.47}$$

Now that $\tilde{\psi}_1$ has been found, determine $\tilde{\psi}_2$. Substituting (70) and (B.8) into (110) implies that

$$\tilde{\psi}_{2\{e,h\}} = \int_0^h \tilde{\vec{G}}^s_{\psi 2\{e,h\}1}\cdot\tilde{\vec{J}}_{\{e,h\}}dz' + \int_h^d \left[\tilde{\vec{G}}^p_{\psi 2\{e,h\}}+\tilde{\vec{G}}^s_{\psi 2\{e,h\}2}\right]\cdot\tilde{\vec{J}}_{\{e,h\}}dz'$$

$$
\begin{aligned}
&= \int_0^h D_\psi^{-1} \left[2 \left(\vec{\tilde{g}}^p_{\psi 1\{e,h\}} (z=h) P^{-1}_{\psi 1 z'} + \vec{\tilde{g}}^p_{\psi 1\{e,h\}} (z=0) P_{\psi 1 z'} \right) \left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z} \right) \right] \\
&\quad \cdot \vec{\tilde{J}}_{\{e,h\}} dz' + \int_h^d D_\psi^{-1} \left\{ \vec{\tilde{g}}^p_{\psi 2\{e,h\}} e^{-jk_{z\psi 2}|z-z'|} \left[\left(P_{\psi 2d} P^{-1}_{\psi 2h} + P^{-1}_{\psi 2d} P_{\psi 2h} \right) \left(P^{-1}_{\psi 1h} - P_{\psi 1h} \right) \right. \right. \\
&\qquad \left. + C_\psi \left(P^{-1}_{\psi 2d} P_{\psi 2h} - P_{\psi 2d} P^{-1}_{\psi 2h} \right) \left(P^{-1}_{\psi 1h} + P_{\psi 1h} \right) \right] \\
&\quad + \vec{\tilde{g}}^p_{\psi 2\{e,h\}} (z=d) P_{\psi 2d} P^{-1}_{\psi 2z'} \left[\left(P_{\psi 2h} P^{-1}_{\psi 2z} - P^{-1}_{\psi 2h} P_{\psi 2z} \right) \left(P^{-1}_{\psi 1h} - P_{\psi 1h} \right) \right. \\
&\qquad \left. + C_\psi \left(P_{\psi 2h} P^{-1}_{\psi 2z} + P^{-1}_{\psi 2h} P_{\psi 2z} \right) \left(P^{-1}_{\psi 1h} + P_{\psi 1h} \right) \right] \\
&\quad - \vec{\tilde{g}}^p_{\psi 2\{e,h\}} (z=h) P^{-1}_{\psi 2h} P_{\psi 2z'} \left[\left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z} \right) \left(P^{-1}_{\psi 1h} - P_{\psi 1h} \right) \right. \\
&\qquad \left. \left. - C_\psi \left(P^{-1}_{\psi 2d} P_{\psi 2z} + P_{\psi 2d} P^{-1}_{\psi 2z} \right) \left(P^{-1}_{\psi 1h} + P_{\psi 1h} \right) \right] \right\} \cdot \vec{\tilde{J}}_{\{e,h\}} dz'
\end{aligned}
\tag{C.48}
$$

First, analyze the component observed in region 2 resulting from magnetic currents in region 1, $\tilde{\psi}_{2h1}$. Similar to the analysis of $\tilde{\psi}_{2ez1}$, it can be shown that substituting (C.45) and (C.22) into (C.48) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\psi 2h1} &= \left(-\frac{\hat{z} \times \vec{\lambda}_\rho \omega \epsilon_{t2}}{\lambda_{z\psi 2} \lambda^2_{\rho\psi}} \right) \left[\frac{\cos(\lambda_{z\psi 1} z') \cos(\lambda_{z\psi 2}(d-z))}{(\cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + C_\psi \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h))} \right] \\
&= \left(-\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\psi 1}}{\lambda^2_{\rho\psi} Z_{\psi 2}} \right) \left[\frac{\cos(\lambda_{z\psi 1} z') \cos(\lambda_{z\psi 2}(d-z))}{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h)} \right]
\end{aligned}
\tag{C.49}
$$

Next, analyze the component observed in region 2 resulting from magnetic currents in region 2, $\tilde{\psi}_{2h2}$. Similar to analysis of $\tilde{\psi}_{2ez2}$, it can be shown that substituting (C.45) and (C.22) into (C.48) implies that

$$\begin{aligned}
\tilde{\vec{G}}_{\psi 2h2} &= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho \omega \epsilon_{t2}}{2\lambda_{z\psi 2}\lambda_{\rho\psi}^2}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&\quad+\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&\quad\left.+\frac{C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\\
&= \left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\left[\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&\quad+\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&\quad\left.+\frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.50}$$

C.4 $\tilde{\Pi}_e$ Development

Begin by analyzing the component observed in region 1 resulting from electric currents in region 1, $\tilde{\Pi}_{1e1}$. Substituting (C.5) into (147) implies that

$$\begin{aligned}
\tilde{\vec{G}}_{\Pi 1e1} &= \frac{-1}{j\omega\mu_{t1}}\frac{\partial}{\partial z}\left\{\left[-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t1}}{2\lambda_{\rho\theta}^2\lambda_{z\theta 1}D_\theta}\right]\left[\sin\left(\lambda_{z\theta 2}T\right)\left[\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]\right.\right.\\
&\quad+\sin\left(\lambda_{z\theta 2}T\right)\left[-\sin\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)\right]\\
&\quad\left.\left.+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\left[\cos\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]\right]\right\}\\
&= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2\lambda_{z\theta 1}}\right)\left[\frac{\sin\left(\lambda_{z\theta 2}T\right)\left[\frac{\partial}{\partial z}\sin\left(k_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.\\
&\quad+\frac{\sin\left(\lambda_{z\theta 2}T\right)\left[-\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\
&\quad\left.+\frac{C_\theta\cos\left(\lambda_{z\theta 2}T\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)-\frac{\partial}{\partial z}\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned}$$

$$= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.$$
$$+\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\left[-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}$$
$$\left.+\frac{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta 1}\left(h-\left|z-z'\right|\right)\right)-\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \tag{C.51}$$

Next, analyze the component observed in region 1 resulting from electric currents in region 2, $\tilde{\Pi}_{1e2}$. Substituting (C.6) into (147) implies that

$$\begin{aligned}
\vec{\vec{G}}_{\Pi 1e2} &= -\frac{1}{j\omega\mu_{t1}}\frac{\partial}{\partial z}\left[\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2 Z_{\theta 2} D_\theta}\right)\left[\sin\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\theta 1}z\right)\right]\right] \\
&= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2\omega\mu_{t1}Z_{\theta 2}}\right)\left[\frac{\sin\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 1}z\right)}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \\
&= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\left[\frac{\sin\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\theta 1}z\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned} \tag{C.52}$$

Now analyze the component observed in region 2 resulting from electric currents in region 1, $\tilde{\Pi}_{2e1}$. Substituting (C.7) into (147) implies that

$$\begin{aligned}
\vec{\vec{G}}_{\Pi 2e1} &= -\frac{1}{j\omega\mu_{t2}}\frac{\partial}{\partial z}\left[\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2 D_\theta}\right)\left[\sin\left(\lambda_{z\theta 1}z'\right)\sin\left(\lambda_{z\theta 2}\left(d-z\right)\right)\right]\right] \\
&= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2\omega\mu_{t2}}\right)\left[\frac{\sin\left(\lambda_{z\theta 1}z'\right)\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 2}\left(d-z\right)\right)}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \\
&= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right)\left[\frac{\sin\left(\lambda_{z\theta 1}z'\right)\cos\left(\lambda_{z\theta 2}\left(d-z\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned} \tag{C.53}$$

Next, analyze the component observed in region 2 resulting from electric currents in region 2, $\tilde{\Pi}_{2e2}$. Substituting (C.11) into (147) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\Pi 2e2} &= -\frac{1}{j\omega\mu_{t2}}\frac{\partial}{\partial z}\left[\left(\frac{\hat{z}\times\vec{\lambda}_\rho\omega\mu_{t2}}{2\lambda_{\rho\theta}^2\lambda_{z\theta 2}D_\theta}\right)\left[\cos\left(k_{z\theta 1}h\right)\left[\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]\right.\right. \\
&\qquad + \cos\left(\lambda_{z\theta 1}h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)\right] \\
&\qquad \left.\left. + C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]\right]\right] \\
&= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2\lambda_{z\theta 2}}\right)\left[\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right. \\
&\qquad + \frac{\cos\left(\lambda_{z\theta 1}h\right)\left[-\frac{\partial}{\partial z}\cos\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \\
&\qquad \left. + \frac{C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)+\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \\
&= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\left[\frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right. \\
&\qquad + \frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1}h\right)\left[-\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)} \\
&\qquad \left. + \frac{Z_{\theta 1}\sin\left(\lambda_{z\theta 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\theta 2}\left(T-|z-z'|\right)\right)+\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned}
\tag{C.54}
$$

C.5 Remaining $\tilde{\Pi}_h$ Development

First, analyze the component observed in region 1 resulting from transverse magnetic currents in region 2, $\tilde{\Pi}_{1ht2}$. Substituting (C.12) into (147) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\Pi 1ht2} &= -\frac{1}{j\omega\mu_{t1}}\frac{\partial}{\partial z}\left[\left(-j\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2 D_\theta}\right)\left[C_\theta\cos\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\theta 1}z\right)\right]\right] \\
&= \left(\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2\omega\mu_{t1}}\right)\left[\frac{C_\theta\cos\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 1}z\right)}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \\
&= \left(\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2}\right)\left[\frac{\cos\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\theta 1}z\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]
\end{aligned}
\tag{C.55}
$$

Next, analyze the component observed in region 1 resulting from longitudinal magnetic currents in region 1, $\tilde{\Pi}_{1hz1}$. Substituting (145) into (147) implies that

$$\vec{\tilde{G}}_{\Pi 1hz1} = -\frac{1}{j\omega\mu_{t1}}\frac{\partial}{\partial z}\left[\left(\frac{\hat{z}\mu_{t1}}{2\lambda_{z\theta 1}\mu_{z1}D_\theta}\right)\left[\sin\left(\lambda_{z\theta 2}T\right)\left[\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right.\right.\right.$$
$$\left.-\sin\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)\right] + C_\theta\cos\left(\lambda_{z\theta 2}T\right)\left[\cos\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)\right.$$
$$\left.\left.\left.-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]\right]\right]$$
$$= \left[\frac{j\hat{z}}{2\lambda_{z\theta 1}\omega\mu_{z1}}\right]\left[\frac{\sin\left(\lambda_{z\theta 2}T\right)\left[\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)-\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.$$
$$\left.+\frac{C_\theta\cos\left(\lambda_{z\theta 2}T\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)-\frac{\partial}{\partial z}\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]$$
$$= \left(j\frac{\hat{z}}{2\omega\mu_{z1}}\right)\left[\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\left[-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.$$
$$+\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\left[-\cos\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}$$
$$\left.+\frac{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta 1}\left(h-|z-z'|\right)\right)-\sin\left(\lambda_{z\theta 1}\left(h-z-z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \tag{C.56}$$

Next, analyze the component observed in region 1 resulting from longitudinal magnetic currents in region 2, $\tilde{\Pi}_{1hz2}$. Substituting (C.13) into (147) implies that

$$\vec{\tilde{G}}_{\Pi 1hz2} = -\frac{1}{j\omega\mu_{t1}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}Z_{\theta 1}^2}{\omega\mu_{z1}Z_{\theta 2}D_\theta}\right)\left[\sin\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\theta 1}z\right)\right]\right]$$
$$= \left(-j\frac{\hat{z}Z_{\theta 1}^2}{\omega^2\mu_{t1}\mu_{z1}Z_{\theta 2}}\right)\left[\frac{\sin\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 1}z\right)}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]$$
$$= \left(-j\frac{\hat{z}Z_{\theta 1}}{\omega\mu_{z1}}\right)\left[\frac{\sin\left(\lambda_{z\theta 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\theta 1}z\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right] \tag{C.57}$$

Now that the components observed in region 1 have been determined, analyze the components observed in region 2, $\tilde{\Pi}_2$. First, analyze the component observed in region 2 resulting from transverse magnetic currents in region 1, $\tilde{\Pi}_{2ht1}$. Substituting

(C.14) into (147) implies that

$$\begin{aligned}\tilde{\vec{G}}_{\Pi 2ht1} &= -\frac{1}{j\omega\mu_{t2}}\frac{\partial}{\partial z}\left[\left(j\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2 D_\theta}\right)\left[\cos\left(\lambda_{z\theta 1}z'\right)\sin\left(\lambda_{z\theta 2}\left(d-z\right)\right)\right]\right]\\ &= \left(-\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2\omega\mu_{t2}}\right)\left[\frac{\cos\left(\lambda_{z\theta 1}z'\right)\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 2}\left(d-z\right)\right)}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\\ &= \left(\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2}\right)\left[\frac{\cos\left(\lambda_{z\theta 1}z'\right)\cos\left(\lambda_{z\theta 2}\left(d-z\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\end{aligned} \tag{C.58}$$

Next, analyze the component observed in region 2 resulting from transverse magnetic currents in region 2, $\tilde{\Pi}_{2ht2}$. Substituting (C.18) into (147) implies that

$$\begin{aligned}\tilde{\vec{G}}_{\Pi 2ht2} = -\frac{1}{j\omega\mu_{t2}}\frac{\partial}{\partial z}&\left[\left(j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 D_\theta}\right)\left[\cos\left(\lambda_{z\theta 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)\right.\right.\right.\\ &+\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\left.\right]+C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)\right.\\ &\left.\left.\left.-\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]\right]\right]\\ = \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2\omega\mu_{t2}}\right)&\left[\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right.\\ &+\frac{\cos\left(\lambda_{z\theta 1}h\right)\left[-\lambda_{z\theta 2}\cos\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\ &+\frac{C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\theta 2}\left(T-\left|z-z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\\ &\left.+\frac{C_\theta\sin\left(\lambda_{z\theta 1}h\right)\left[-\lambda_{z\theta 2}\sin\left(\lambda_{z\theta 2}\left(d+h-z-z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)+C_\theta\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)}\right]\end{aligned} \tag{C.59}$$

Due to the $\operatorname{sgn}\left(z-z'\right)$ in (C.59), two cases must be analyzed. When $z>z'$, that

implies that

$$\begin{aligned}
\vec{\vec{G}}^{z^+}_{\Pi 2ht2} = {} & \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2\omega\mu_{t2}}\right)\left[\frac{\cos(\lambda_{z\theta1}h)\left[\frac{\partial}{\partial z}\cancelto{1}{\mathrm{sgn}(z-z')}\sin(\lambda_{z\theta2}(T-(z-z')))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}\right. \\
& +\frac{\cos(\lambda_{z\theta1}h)\left[-\lambda_{z\theta2}\cos(\lambda_{z\theta2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)} \\
& +\frac{C_\theta\sin(\lambda_{z\theta1}h)\left[\frac{\partial}{\partial z}\cancelto{1}{\mathrm{sgn}(z-z')}\cos(\lambda_{z\theta2}(T-(z-z')))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)} \\
& \left.+\frac{C_\theta\sin(\lambda_{z\theta1}h)\left[-\lambda_{z\theta2}\sin(\lambda_{z\theta2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}\right] \\
= {} & \left(-\frac{\vec{\lambda}_\rho\lambda_{z\theta2}}{2\lambda_{\rho\theta}^2\omega\mu_{t2}}\right)\left[\frac{\cos(\lambda_{z\theta1}h)\left[-\cos(\lambda_{z\theta2}(T-(z-z')))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}\right. \\
& +\frac{\cos(\lambda_{z\theta1}h)\left[-\cos(\lambda_{z\theta2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)} \\
& \left.+\frac{C_\theta\sin(\lambda_{z\theta1}h)\left[\sin(\lambda_{z\theta2}(T-(z-z')))-\sin(\lambda_{z\theta2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}\right]
\end{aligned} \tag{C.60}$$

When $z < z'$, that implies that

$$\begin{aligned}
\vec{\vec{G}}^{z^-}_{\Pi 2ht2} = {} & \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2\omega\mu_{t2}}\right)\left[\frac{\cos(\lambda_{z\theta1}h)\left[\frac{\partial}{\partial z}\cancelto{-1}{\mathrm{sgn}(z-z')}\sin(\lambda_{z\theta2}(T+(z-z')))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}\right. \\
& +\frac{\cos(\lambda_{z\theta1}h)\left[-\lambda_{z\theta2}\cos(\lambda_{z\theta2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)} \\
& +\frac{C_\theta\sin(\lambda_{z\theta1}h)\left[\frac{\partial}{\partial z}\cancelto{-1}{\mathrm{sgn}(z-z')}\cos(\lambda_{z\theta2}(T+(z-z')))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)} \\
& \left.+\frac{C_\theta\sin(\lambda_{z\theta1}h)\left[-\lambda_{z\theta2}\sin(\lambda_{z\theta2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)+C_\theta\cos(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{\vec{\lambda}_\rho \lambda_{z\theta 2}}{2\lambda_{\rho\theta}^2 \omega \mu_{t2}}\right) \left[\frac{\cos\left(\lambda_{z\theta 1} h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T + \left(z - z'\right)\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right. \\
&\quad + \frac{\cos\left(\lambda_{z\theta 1} h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(d + h - z - z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad \left. + \frac{C_\theta \sin\left(\lambda_{z\theta 1} h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T + \left(z - z'\right)\right)\right) - \sin\left(\lambda_{z\theta 2}\left(d + h - z - z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right]
\end{aligned}
\tag{C.61}
$$

Analyzing (C.60) and (C.61) reveals that

$$
\begin{aligned}
\vec{\tilde{G}}_{\Pi 2ht2} &= \left(-\frac{\vec{\lambda}_\rho \lambda_{z\theta 2}}{2\lambda_{\rho\theta}^2 \omega \mu_{t2}}\right) \left[\frac{\cos\left(\lambda_{z\theta 1} h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T - \left|z - z'\right|\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right. \\
&\quad + \frac{\cos\left(\lambda_{z\theta 1} h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(d + h - z - z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad \left. + \frac{C_\theta \sin\left(\lambda_{z\theta 1} h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T - \left|z - z'\right|\right)\right) - \sin\left(\lambda_{z\theta 2}\left(d + h - z - z'\right)\right)\right]}{\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right] \\
&= \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2 Z_{\theta 2}}\right) \left[\frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1} h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(T - \left|z - z'\right|\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1}\cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right. \\
&\quad + \frac{Z_{\theta 2}\cos\left(\lambda_{z\theta 1} h\right)\left[-\cos\left(\lambda_{z\theta 2}\left(d + h - z - z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1}\cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)} \\
&\quad \left. + \frac{Z_{\theta 1}\sin\left(\lambda_{z\theta 1} h\right)\left[\sin\left(\lambda_{z\theta 2}\left(T - \left|z - z'\right|\right)\right) - \sin\left(\lambda_{z\theta 2}\left(d + h - z - z'\right)\right)\right]}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1}\cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right]
\end{aligned}
\tag{C.62}
$$

Now analyze the component observed in region 2 resulting from longitudinal magnetic currents in region 1, $\tilde{\Pi}_{2hz1}$. Substituting (C.19) into (147) implies that

$$
\begin{aligned}
\vec{\tilde{G}}_{\Pi 2hz1} &= -\frac{1}{j\omega\mu_{t2}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}Z_{\theta 2}}{\omega\mu_{z2}D_\theta}\right)\left[\sin\left(\lambda_{z\theta 1} z'\right)\sin\left(\lambda_{z\theta 2}\left(d - z\right)\right)\right]\right] \\
&= \left(-j\frac{\hat{z}Z_{\theta 2}}{\omega^2\mu_{t2}\mu_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\theta 1} z'\right)\frac{\partial}{\partial z}\sin\left(\lambda_{z\theta 2}\left(d - z\right)\right)}{\left[\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + C_\theta \cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)\right]}\right] \\
&= \left(j\frac{\hat{z}Z_{\theta 2}}{\omega\mu_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\theta 1} z'\right)\cos\left(\lambda_{z\theta 2}\left(d - z\right)\right)}{Z_{\theta 2}\sin\left(\lambda_{z\theta 2} T\right)\cos\left(\lambda_{z\theta 1} h\right) + Z_{\theta 1}\cos\left(\lambda_{z\theta 2} T\right)\sin\left(\lambda_{z\theta 1} h\right)}\right]
\end{aligned}
\tag{C.63}
$$

Finally, analyze the component observed in region 2 resulting from longitudinal magnetic currents in region 2, $\tilde{\Pi}_{2hz2}$. Substituting (C.20) into (147) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\Pi 2hz2} = -\frac{1}{j\omega\mu_{t2}}\frac{\partial}{\partial z}\Bigg[\left(-\frac{\hat{z}\mu_{t2}}{2\lambda_{z\theta 2}\mu_{z2}D_\theta}\right)&[\cos(\lambda_{z\theta 1}h)\left[\cos(\lambda_{z\theta 2}(d+h-z-z'))\right] \\
&+\cos(\lambda_{z\theta 1}h)\left[-\cos(\lambda_{z\theta 2}(T-|z-z'|))\right] \\
+C_\theta\sin(\lambda_{z\theta 1}h)&\left[\sin(\lambda_{z\theta 2}(T-|z-z'|))+\sin(\lambda_{z\theta 2}(d+h-z-z'))\right]]\Bigg] \\
=\left(-j\frac{\hat{z}}{2\lambda_{z\theta 2}\omega\mu_{z2}}\right)\Bigg[&\frac{\cos(\lambda_{z\theta 1}h)\left[\frac{\partial}{\partial z}\cos(\lambda_{z\theta 2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)} \\
&+\frac{\cos(\lambda_{z\theta 1}h)\left[-\frac{\partial}{\partial z}\cos(\lambda_{z\theta 2}(T-|z-z'|))\right]}{\sin(\lambda_{z\theta 2}T)\cos(k_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)} \\
+\frac{C_\theta\sin(\lambda_{z\theta 1}h)\left[\frac{\partial}{\partial z}\sin(\lambda_{z\theta 2}(T-|z-z'|))+\frac{\partial}{\partial z}\sin(\lambda_{z\theta 2}(d+h-z-z'))\right]}{\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+C_\theta\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}&\Bigg] \\
=\left(j\frac{\hat{z}}{2\omega\mu_{z2}}\right)\Bigg[&\frac{Z_{\theta 2}\cos(\lambda_{z\theta 1}h)\left[\mathrm{sgn}(z-z')\sin(\lambda_{z\theta 2}(T-|z-z'|))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)} \\
&+\frac{Z_{\theta 2}\cos(\lambda_{z\theta 1}h)\left[-\sin(\lambda_{z\theta 2}(d+h-z-z'))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)} \\
+\frac{Z_{\theta 1}\sin(\lambda_{z\theta 1}h)\left[\mathrm{sgn}(z-z')\cos(\lambda_{z\theta 2}(T-|z-z'|))+\cos(\lambda_{z\theta 2}(d+h-z-z'))\right]}{Z_{\theta 2}\sin(\lambda_{z\theta 2}T)\cos(\lambda_{z\theta 1}h)+Z_{\theta 1}\cos(\lambda_{z\theta 2}T)\sin(\lambda_{z\theta 1}h)}&\Bigg]
\end{aligned} \tag{C.64}$$

C.6 $\tilde{\Phi}$ Development

Now determine transverse spectral domain total Green functions for $\tilde{\Phi}$. (39) and the analysis from (100) imply that

$$\begin{aligned}
\tilde{\Phi}_{\{1,2\}} = \frac{1}{j\omega\epsilon_{t\{1,2\}}}\Bigg(&\int_0^h \frac{\partial}{\partial z}\left[\vec{\tilde{G}}_{\psi\{1,2\}et1}+\vec{\tilde{G}}_{\psi\{1,2\}ez1}+\vec{\tilde{G}}_{\psi\{1,2\}h1}\right]\cdot\vec{\tilde{J}}_h dz' \\
&+\int_h^d \frac{\partial}{\partial z}\left[\vec{\tilde{G}}_{\psi\{1,2\}et2}+\vec{\tilde{G}}_{\psi\{1,2\}ez2}+\vec{\tilde{G}}_{\psi\{1,2\}h2}\right]\cdot\vec{\tilde{J}}_h dz'\Bigg) \\
\Rightarrow \vec{\tilde{G}}_{\Phi\{1,2\}\{et,ez,h\}\{1,2\}} = &\frac{1}{j\omega\epsilon_{t\{1,2\}}}\frac{\partial}{\partial z}\vec{\tilde{G}}_{\psi\{1,2\}\{et,ez,h\}\{1,2\}}
\end{aligned} \tag{C.65}$$

Begin by analyzing the component observed in region 1 resulting from transverse electric currents in region 1, $\tilde{\Phi}_{1et1}$. Substituting (C.28) into (C.65) implies that

$$\vec{\tilde{G}}_{\Phi 1et1} = \frac{1}{j\omega\epsilon_{t1}}\frac{\partial}{\partial z}\left[\left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 D_\psi}\right)\left[\cos\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)\right.\right.\right.$$
$$\left.-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]$$
$$\left.\left.+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)-\cos\left(k_{z\psi 1}\left(h-z-z'\right)\right)\right]\right]\right]$$
$$\begin{aligned}= \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2\omega\epsilon_{t1}}\right)&\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&+\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\lambda_{z\psi 1}\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 1}\left(h-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&\left.+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[-\lambda_{z\psi 1}\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.66}$$

Due to the $\operatorname{sgn}\left(z-z'\right)$ in (C.66), two cases must be analyzed. When $z > z'$, that implies that

$$\begin{aligned}\vec{\tilde{G}}^{z^+}_{\Phi 1et1} = \left(-\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2\omega\epsilon_{t1}}\right)&\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\frac{\partial}{\partial z}\cancelto{1}{\operatorname{sgn}\left(z-z'\right)}\sin\left(\lambda_{z\psi 1}\left(h-\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&+\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[k_{z\psi 1}\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[\frac{\partial}{\partial z}\cancelto{1}{\operatorname{sgn}\left(z-z'\right)}\cos\left(\lambda_{z\psi 1}\left(h-\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\\
&\left.+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[-\lambda_{z\psi 1}\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\\
= \left[-\frac{\vec{\lambda}_\rho\lambda_{z\psi 1}}{2\lambda_{\rho\psi}^2\omega\epsilon_{t1}}\right]&\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[-\cos\left(\lambda_{z\psi 1}\left(h-\left(z-z'\right)\right)\right)+\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right.\\
&\left.+\frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[\sin\left(\lambda_{z\psi 1}\left(h-\left(z-z'\right)\right)\right)-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.67}$$

When $z < z'$, that implies that

$$\vec{\vec{G}}^{z^-}_{\Phi 1et1} = \left(-\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}\omega\epsilon_{t1}}\right)\left[\frac{\cos\left(\lambda_{z\psi2}T\right)\left[\frac{\partial}{\partial z}\cancel{\text{sgn}\left(z-z'\right)}\overset{-1}{\sin}\left(\lambda_{z\psi1}\left(h+\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left({}_{z\psi1}h\right)}\right.$$
$$+\frac{\cos\left(\lambda_{z\psi2}T\right)\left[\lambda_{z\psi1}\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}$$
$$+\frac{C_\psi\sin\left(\lambda_{z\psi2}T\right)\left[\frac{\partial}{\partial z}\cancel{\text{sgn}\left(z-z'\right)}\overset{-1}{\cos}\left(\lambda_{z\psi1}\left(h+\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}$$
$$\left.\frac{C_\psi\sin\left(\lambda_{z\psi2}T\right)\left[-\lambda_{z\psi1}\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]$$
$$= \left[-\frac{\vec{\lambda}_\rho\lambda_{z\psi1}}{2\lambda^2_{\rho\psi}\omega\epsilon_{t1}}\right]\left[\frac{\cos\left(\lambda_{z\psi2}T\right)\left[-\cos\left(\lambda_{z\psi1}\left(h+\left(z-z'\right)\right)\right)+\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right.$$
$$\left.+\frac{C_\psi\sin\left(\lambda_{z\psi2}T\right)\left[\sin\left(\lambda_{z\psi1}\left(h+\left(z-z'\right)\right)\right)-\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \tag{C.68}$$

Analyzing (C.67) and (C.68) reveals that

$$\vec{\vec{G}}_{\Phi 1et1} = \left(-\frac{\vec{\lambda}_\rho\lambda_{z\psi1}}{2\lambda^2_{\rho\psi}\omega\epsilon_{t1}}\right)\left[\frac{\cos\left(\lambda_{z\psi2}T\right)\left[\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right.$$
$$+\frac{\cos\left(\lambda_{z\psi2}T\right)\left[-\cos\left(\lambda_{z\psi1}\left(h-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}$$
$$\left.+\frac{C_\psi\sin\left(\lambda_{z\psi2}T\right)\left[\sin\left(\lambda_{z\psi1}\left(h-\left|z-z'\right|\right)\right)-\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]$$
$$= \left(-\frac{\vec{\lambda}_\rho Z_{\psi1}}{2\lambda^2_{\rho\psi}}\right)\left[\frac{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\left[\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)-\cos\left(\lambda_{z\psi1}\left(h-\left|z-z'\right|\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right.$$
$$\left.+\frac{Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\left[\sin\left(\lambda_{z\psi1}\left(h-\left|z-z'\right|\right)\right)-\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \tag{C.69}$$

Next, analyze the component observed in region 1 resulting from transverse electric currents in region 2, $\tilde{\Phi}_{1et2}$. Substituting (C.29) into (C.65) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\Phi 1et2} &= \frac{1}{j\omega\epsilon_{t1}}\frac{\partial}{\partial z}\left[\left(j\frac{\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2}\right)\left[\frac{C_\psi \sin\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1}z\right)}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\right] \\
&= \left(\frac{\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2\omega\epsilon_{t1}}\right)\left[\frac{\frac{\lambda_{z\psi 2}\epsilon_{t1}}{\lambda_{z\psi 1}\epsilon_{t2}}\sin\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 1}z\right)}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \\
&= \left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\left[\frac{\sin\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\psi 1}z\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.70}$$

Now analyze the component observed in region 1 resulting from longitudinal electric currents in region 1, $\tilde{\Phi}_{1ez1}$. Substituting (C.33) into (C.65) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\Phi 1ez1} &= \frac{1}{j\omega\epsilon_{t1}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}\epsilon_{t1}}{2\lambda_{z\psi 1}\epsilon_{z1}D_\psi}\right)\left[\cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)\right]\right.\right. \\
&\qquad + \cos\left(\lambda_{z\psi 2}T\right)\left[\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right] \\
&\qquad \left.\left. + C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[-\sin\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)-\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]\right]\right] \\
&= \left(j\frac{\hat{z}}{2\lambda_{z\psi 1}\omega\epsilon_{z1}}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
&\qquad + \frac{\cos\left(\lambda_{z\psi 2}T\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\qquad \left. + \frac{C_\psi\sin\left(\lambda_{z\psi 2}T\right)\left[-\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)-\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \\
&= \left(j\frac{\hat{z}}{2\omega\epsilon_{z1}}\right)\left[\frac{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
&\qquad + \frac{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\left[\sin\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\qquad \left. + \frac{Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 1}\left(h-|z-z'|\right)\right)+\cos\left(\lambda_{z\psi 1}\left(h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.71}$$

Next, analyze the component observed in region 1 resulting from longitudinal electric currents in region 2, $\tilde{\Phi}_{1ez2}$. Substituting (C.34) into (C.65) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\Phi 1ez2} &= \frac{1}{j\omega\epsilon_{t1}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}}{\omega\epsilon_{z1}Z_{\psi1}D_\psi}\right)\left[C_\psi\cos\left(\lambda_{z\psi2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi1}z\right)\right]\right] \\
&= \left(j\frac{\hat{z}}{\omega^2\epsilon_{t1}\epsilon_{z1}Z_{\psi1}}\right)\left[\frac{\frac{Z_{\psi2}}{Z_{\psi1}}\cos\left(\lambda_{z\psi2}\left(d-z'\right)\right)\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi1}z\right)}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \\
&= \left(-j\frac{\hat{z}Z_{\psi2}}{\omega\epsilon_{z1}}\right)\left[\frac{\cos\left(\lambda_{z\psi2}\left(d-z'\right)\right)\sin\left(\lambda_{z\psi1}z\right)}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]
\end{aligned} \tag{C.72}$$

Now analyze the component observed in region 1 resulting from magnetic currents in region 1, $\tilde{\Phi}_{1h1}$. Substituting (C.46) into (C.65) implies that

$$\begin{aligned}
\vec{\tilde{G}}_{\Phi 1h1} &= \frac{1}{j\omega\epsilon_{t1}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\epsilon_{t1}}{2\lambda_{z\psi1}\lambda_{\rho\psi}^2D_\psi}\right)\left[\cos\left(\lambda_{z\psi2}T\right)\left[\cos\left(\lambda_{z\psi1}\left(h-|z-z'|\right)\right)\right]\right.\right. \\
&\qquad + \cos\left(\lambda_{z\psi2}T\right)\left[\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right] \\
&\qquad \left.\left. + C_\psi\sin\left(\lambda_{z\psi2}T\right)\left[-\sin\left(\lambda_{z\psi1}\left(h-|z-z'|\right)\right)-\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]\right]\right] \\
&= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{z\psi1}\lambda_{\rho\psi}^2}\right)\left[\frac{\cos\left(\lambda_{z\psi2}T\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi1}\left(h-|z-z'|\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
&\qquad + \frac{\cos\left(\lambda_{z\psi2}T\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\qquad \left. + \frac{C_\psi\sin\left(\lambda_{z\psi2}T\right)\left[-\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi1}\left(h-|z-z'|\right)\right)-\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \\
&= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi1}\left(h-|z-z'|\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
&\qquad + \frac{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\left[\sin\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\qquad \left. + \frac{Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi1}\left(h-|z-z'|\right)\right)+\cos\left(\lambda_{z\psi1}\left(h-z-z'\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]
\end{aligned} \tag{C.73}$$

Next, analyze the component observed in region 1 resulting from magnetic currents in region 2, $\tilde{\Phi}_{1h2}$. Substituting (C.47) into (C.65) implies that

$$\begin{aligned}\vec{\tilde{G}}_{\Phi 1h2} &= \frac{1}{j\omega\epsilon_{t1}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2 Z_{\psi 1} D_\psi}\right)\left[C_\psi \cos\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\cos\left(\lambda_{z\psi 1} z\right)\right]\right] \\ &= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2\omega\epsilon_{t1} Z_{\psi 1}}\right)\left[\frac{\frac{Z_{\psi 2}}{Z_{\psi 1}}\cos\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 1} z\right)}{\cos\left(\lambda_{z\psi 2} T\right)\sin\left(\lambda_{z\psi 1} h\right)+C_\psi \sin\left(\lambda_{z\psi 2} T\right)\cos\left(\lambda_{z\psi 1} h\right)}\right] \\ &= \left(-j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\left[\frac{\cos\left(\lambda_{z\psi 2}\left(d-z'\right)\right)\sin\left(\lambda_{z\psi 1} z\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2} T\right)\sin\left(\lambda_{z\psi 1} h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2} T\right)\cos\left(\lambda_{z\psi 1} h\right)}\right]\end{aligned} \tag{C.74}$$

Now that the components observed in region 1 have been determined, analyze the components observed in region 2, $\tilde{\Phi}_2$. Begin by analyzing the component observed in region 2 resulting from transverse electric currents in region 1, $\tilde{\Phi}_{2et1}$. Substituting (C.35) into (C.65) implies that

$$\begin{aligned}\vec{\tilde{G}}_{\Phi 2et1} &= \frac{1}{j\omega\epsilon_{t2}}\frac{\partial}{\partial z}\left[\left(-j\frac{\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2 D_\psi}\right)\left[\sin\left(\lambda_{z\psi 1} z'\right)\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)\right]\right] \\ &= \left(-\frac{\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2\omega\epsilon_{t2}}\right)\left[\frac{\sin\left(\lambda_{z\psi 1} z'\right)\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{\cos\left(\lambda_{z\psi 2} T\right)\sin\left(\lambda_{z\psi 1} h\right)+C_\psi \sin\left(\lambda_{z\psi 2} T\right)\cos\left(\lambda_{z\psi 1} h\right)}\right] \\ &= \left(-\frac{\vec{\lambda}_\rho Z_{\psi 1} Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\left[\frac{\sin\left(\lambda_{z\psi 1} z'\right)\sin\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2} T\right)\sin\left(\lambda_{z\psi 1} h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2} T\right)\cos\left(\lambda_{z\psi 1} h\right)}\right]\end{aligned} \tag{C.75}$$

Next, analyze the component observed in region 2 resulting from transverse electric currents in region 2, $\tilde{\Phi}_{2et2}$. Substituting (C.39) into (C.65) implies that

$$\begin{aligned}\vec{\tilde{G}}_{\Phi 2et2} = \frac{1}{j\omega\epsilon_{t2}}\frac{\partial}{\partial z}\Bigg[\left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2 D_\psi}\right)&\left[\sin\left(\lambda_{z\psi 1} h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right.\right. \\ +\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\big] &+ C_\psi \cos\left(\lambda_{z\psi 1} h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right. \\ &\left.\left.\left.-\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]\right]\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{\vec{\lambda}_{\rho}}{2\lambda_{\rho\psi}^{2}\omega\epsilon_{t2}}\right)\left[\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi2}\left(T-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\lambda_{z\psi2}\sin\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\quad +\frac{C_{\psi}\cos\left(\lambda_{z\psi1}h\right)\left[\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi2}\left(T-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\quad \left.+\frac{C_{\psi}\cos\left(\lambda_{z\psi1}h\right)\left[\lambda_{z\psi2}\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]
\end{aligned}
\tag{C.76}
$$

Due to the $\operatorname{sgn}(z-z')$ term in (C.76), two cases must be analyzed. When $z>z'$, that implies that

$$
\begin{aligned}
\vec{\vec{G}}_{\Phi 2et2}^{z+} &= \left(-\frac{\vec{\lambda}_{\rho}}{2\lambda_{\rho\psi}^{2}\omega\epsilon_{t2}}\right)\left[\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\cancelto{1}{\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)}\cos\left(\lambda_{z\psi2}\left(T-\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\lambda_{z\psi2}\sin\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\quad +\frac{C_{\psi}\cos\left(\lambda_{z\psi1}h\right)\left[\cancelto{1}{\frac{\partial}{\partial z}\operatorname{sgn}\left(z-z'\right)}\sin\left(\lambda_{z\psi2}\left(T-\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\quad \left.+\frac{C_{\psi}\cos\left(\lambda_{z\psi1}h\right)\left[\lambda_{z\psi2}\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \\
&= \left(-\frac{\vec{\lambda}_{\rho}\lambda_{z\psi2}}{2\lambda_{\rho\psi}^{2}\omega\epsilon_{t2}}\right)\left[\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(T-\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
&\quad +\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
&\quad \left.+\frac{C_{\psi}\cos\left(\lambda_{z\psi1}h\right)\left[-\cos\left(\lambda_{z\psi2}\left(T-\left(z-z'\right)\right)\right)+\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]
\end{aligned}
\tag{C.77}
$$

When $z < z'$, that implies that

$$\begin{aligned}
\vec{\vec{G}}^{z-}_{\Phi 2et2} = \left(-\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}\omega\epsilon_{t2}}\right) & \left[\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\frac{\partial}{\partial z}\cancelto{-1}{\operatorname{sgn}\left(z-z'\right)}\cos\left(\lambda_{z\psi2}\left(T+\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
& +\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\lambda_{z\psi2}\sin\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
& +\frac{C_\psi\cos\left(\lambda_{z\psi1}h\right)\left[\frac{\partial}{\partial z}\cancelto{-1}{\operatorname{sgn}\left(z-z'\right)}\sin\left(\lambda_{z\psi2}\left(T+\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
& \left.+\frac{C_\psi\cos\left(\lambda_{z\psi1}h\right)\left[\lambda_{z\psi2}\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \\
= \left(-\frac{\vec{\lambda}_\rho\lambda_{z\psi2}}{2\lambda^2_{\rho\psi}\omega\epsilon_{t2}}\right) & \left[\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(T+\left(z-z'\right)\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
& +\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
& \left.+\frac{C_\psi\cos\left(\lambda_{z\psi1}h\right)\left[-\cos\left(\lambda_{z\psi2}\left(T+\left(z-z'\right)\right)\right)+\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]
\end{aligned} \tag{C.78}$$

Analyzing (C.77) and (C.78) reveals that

$$\begin{aligned}
\vec{\vec{G}}_{\Phi 2et2} = \left(-\frac{\vec{\lambda}_\rho\lambda_{z\psi2}}{2\lambda^2_{\rho\psi}\omega\epsilon_{t2}}\right) & \left[\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(T-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
& +\frac{\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
& \left.+\frac{C_\psi\cos\left(\lambda_{z\psi1}h\right)\left[-\cos\left(\lambda_{z\psi2}\left(T-\left|z-z'\right|\right)\right)+\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+C_\psi\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right] \\
= \left(-\frac{\vec{\lambda}_\rho Z_{\psi2}}{2\lambda^2_{\rho\psi}}\right) & \left[\frac{Z_{\psi1}\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(\lambda_{z\psi2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right. \\
& +\frac{Z_{\psi1}\sin\left(\lambda_{z\psi1}h\right)\left[\sin\left(k_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)} \\
& \left.+\frac{Z_{\psi2}\cos\left(\lambda_{z\psi1}h\right)\left[-\cos\left(\lambda_{z\psi2}\left(T-\left|z-z'\right|\right)\right)+\cos\left(\lambda_{z\psi2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi1}\cos\left(\lambda_{z\psi2}T\right)\sin\left(\lambda_{z\psi1}h\right)+Z_{\psi2}\sin\left(\lambda_{z\psi2}T\right)\cos\left(\lambda_{z\psi1}h\right)}\right]
\end{aligned} \tag{C.79}$$

Now analyze the component observed in region 2 resulting from longitudinal electric currents in region 1, $\tilde{\Phi}_{2ez1}$. Substituting (C.40) into (C.65) implies that

$$\begin{aligned}
\vec{\vec{G}}_{\Phi 2ez1} &= \frac{1}{j\omega\epsilon_{t2}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}}{\omega\epsilon_{z2}Z_{\psi 2}D_{\psi}}\right)\left[\cos\left(\lambda_{z\psi 1}z'\right)\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)\right]\right] \\
&= \left(j\frac{\hat{z}}{\omega^2\epsilon_{t2}\epsilon_{z2}Z_{\psi 2}}\right)\left[\frac{\cos\left(\lambda_{z\psi 1}z'\right)\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{\left(\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)\right)}\right] \\
&= \left(j\frac{\hat{z}Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\left[\frac{\cos\left(\lambda_{z\psi 1}z'\right)\sin\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.80}$$

Next, analyze the component observed in region 2 resulting from longitudinal electric currents in region 2, $\tilde{\Phi}_{2ez2}$. Substituting (C.44) into (C.65) implies that

$$\begin{aligned}
\vec{\vec{G}}_{\Phi 2ez2} &= \frac{1}{j\omega\epsilon_{t2}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}\epsilon_{t2}}{2\lambda_{z\psi 2}\epsilon_{z2}D_{\psi}}\right)\left[\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)\right]\right.\right. \\
&\qquad + \sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right] \\
&\qquad \left.\left. + C_{\psi}\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]\right]\right] \\
&= \left(j\frac{\hat{z}}{2\lambda_{z\psi 2}\omega\epsilon_{z2}}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[-\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
&\qquad + \frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\qquad \left. + \frac{C_{\psi}\cos\left(\lambda_{z\psi 1}h\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)+\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_{\psi}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \\
&= \left(j\frac{\hat{z}}{2\omega\epsilon_{z2}}\right)\left[\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\
&\qquad + \frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[-\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\qquad + \frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 2}\left(T-|z-z'|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\
&\qquad \left. + \frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]
\end{aligned} \tag{C.81}$$

Next, analyze the component observed in region 2 resulting from magnetic currents in region 1, $\tilde{\Phi}_{2h1}$. Substituting (C.49) into (C.65) implies that

$$\begin{aligned}\vec{\vec{G}}_{\Phi 2h1} &= \frac{1}{j\omega\epsilon_{t2}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2 Z_{\psi 2} D_\psi}\right)\left[\cos\left(\lambda_{z\psi 1}z'\right)\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)\right]\right] \\ &= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{\lambda_{\rho\psi}^2\omega\epsilon_{t2}Z_{\psi 2}}\right)\left[\frac{\cos\left(\lambda_{z\psi 1}z'\right)\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{\left(\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)\right)}\right] \\ &= \left(j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\left[\frac{\cos\left(\lambda_{z\psi 1}z'\right)\sin\left(\lambda_{z\psi 2}\left(d-z\right)\right)}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.82}$$

Finally, analyze the component observed in region 2 resulting from magnetic currents in region 2, $\tilde{\Phi}_{2h2}$. Substituting (C.50) into (C.65) implies that

$$\begin{aligned}\vec{\vec{G}}_{\Phi 2h2} = {} & \frac{1}{j\omega\epsilon_{t2}}\frac{\partial}{\partial z}\left[\left(-\frac{\hat{z}\times\vec{\lambda}_\rho\omega\epsilon_{t2}}{2\lambda_{z\psi 2}\lambda_{\rho\psi}^2}\right)\left[\sin\left(\lambda_{z\psi 1}h\right)\left[-\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]\right.\right. \\ & \qquad +\sin\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right] \\ & \left.\left. +C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)+\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]\right]\right] \\ = {} & \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2\lambda_{z\psi 2}}\right)\left[\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[-\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\ & +\frac{\sin\left(\lambda_{z\psi 1}h\right)\left[\frac{\partial}{\partial z}\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\ & \left.+\frac{C_\psi\cos\left(\lambda_{z\psi 1}h\right)\left[\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)+\frac{\partial}{\partial z}\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+C_\psi\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right] \\ = {} & \left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\left[\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\cos\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right. \\ & +\frac{Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\left[-\cos\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\ & +\frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\operatorname{sgn}\left(z-z'\right)\sin\left(\lambda_{z\psi 2}\left(T-\left|z-z'\right|\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)} \\ & \left.+\frac{Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\left[\sin\left(\lambda_{z\psi 2}\left(d+h-z-z'\right)\right)\right]}{Z_{\psi 1}\cos\left(\lambda_{z\psi 2}T\right)\sin\left(\lambda_{z\psi 1}h\right)+Z_{\psi 2}\sin\left(\lambda_{z\psi 2}T\right)\cos\left(\lambda_{z\psi 1}h\right)}\right]\end{aligned} \tag{C.83}$$

D. Full Development of Unused Electromagnetic Field Total Green Functions

Since only the magnetic field components due to magnetic currents in region 1 and region 2 are needed for the proposed measurement technique, the full development of electric field components and magnetic field components that arise from electric currents is unnecessary in the main body of this research. For completeness, this appendix presents their full development. Additionally, since this research does not leverage field observations in regions other that the source region. Therefore, magnetic observation terms resulting from currents in regions other than the observation region are presented here in case they are needed for future work.

This research does not leverage field observations in regions other that the source region. Therefore, magnetic observation terms resulting from currents in regions other than the observation region are presented here in case they are needed for future work.

D.1 $\tilde{\vec{E}}$ Development

Analyze the component observed in region 1 resulting from electric currents in region 1, $\tilde{\vec{E}}_{1e1}$. Substituting (C.5), (C.28), (C.33), (C.69), and (C.71) into (157) implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1e1} &= j\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\Phi 1et1} + \tilde{\vec{G}}_{\Phi 1ez1}\right) - j\hat{z}\times\vec{\lambda}_\rho\tilde{\vec{G}}_{\theta 1e1} - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z1}}\left(\tilde{\vec{G}}_{\psi 1et1} + \tilde{\vec{G}}_{\psi 1ez1}\right) \\
&\qquad + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z1}}\delta\left(z-z'\right) \\
&= j\vec{\lambda}_\rho\left[\left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi + \left(j\frac{\hat{z}}{2\omega\epsilon_{z1}}\right)\Upsilon_3^\psi\right] - j\hat{z}\times\vec{\lambda}_\rho\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_1^\theta \\
&\quad - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z1}}\left[\left(-j\frac{\vec{k}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_9^\psi + \left(-\frac{\hat{z}}{2Z_{\psi 1}\omega\epsilon_{z1}}\right)\Upsilon_{11}^\psi\right] + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z1}}\delta\left(z-z'\right)
\end{aligned}$$

$$
\left.\begin{aligned}
&= \vec{\lambda}_\rho \vec{\lambda}_\rho \left(-j\frac{Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right) \Upsilon_1^\psi + \vec{k}_\rho \hat{z} \left(-\frac{1}{2\omega\epsilon_{z1}}\right) \Upsilon_3^\psi \\
&\qquad + \hat{z}\vec{\lambda}_\rho \left(-\frac{1}{2\omega\epsilon_{z1}}\right) \Upsilon_9^\psi + \hat{z}\hat{z} \left(j\frac{\lambda_{\rho\psi}^2}{2Z_{\psi 1}\omega^2\epsilon_{z1}^2}\right) \Upsilon_{11}^\psi
\end{aligned}\right\} \tilde{\overleftrightarrow{G}}_{e1e1}^{\mathrm{TM}^z}
$$

$$
+ \underbrace{\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 1}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_1^\theta}_{\tilde{\overleftrightarrow{G}}_{e1e1}^{\mathrm{TE}^z}} + \underbrace{\hat{z}\hat{z}\left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z1}}\right)}_{\tilde{\overleftrightarrow{G}}_{e1e1}^{d}} \tag{D.1}
$$

Separating (D.1) into TM^z, TE^z, and depolarizing terms implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1e1}^{\mathrm{TM}^z} &= \vec{\lambda}_\rho \vec{\lambda}_\rho \left(-j\frac{Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right) \Upsilon_1^\psi + \vec{\lambda}_\rho \hat{z} \left(-\frac{1}{2\omega\epsilon_{z1}}\right) \Upsilon_3^\psi + \hat{z}\vec{\lambda}_\rho \left(-\frac{1}{2\omega\epsilon_{z1}}\right) \Upsilon_9^\psi \\
&\qquad + \hat{z}\hat{z} \left(j\frac{k_{\rho\psi}^2}{2Z_{\psi 1}\omega^2\epsilon_{z1}^2}\right) \Upsilon_{11}^\psi \\
&= \begin{bmatrix}
\lambda_x^2 \left(-j\frac{Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_x\lambda_y \left(-j\frac{Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_x \left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_3^\psi \\
\lambda_x\lambda_y \left(-j\frac{Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_y^2 \left(-j\frac{Z_{\psi 1}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_1^\psi & \lambda_y \left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_3^\psi \\
\lambda_x \left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_9^\psi & \lambda_y \left(-\frac{1}{2\omega\epsilon_{z1}}\right)\Upsilon_9^\psi & \left(j\frac{\lambda_{\rho\psi}^2}{2Z_{\psi 1}\omega^2\epsilon_{z1}^2}\right)\Upsilon_{11}^\psi
\end{bmatrix}
\end{aligned} \tag{D.2}
$$

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1e1}^{\mathrm{TE}^z} &= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 1}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_1^\theta \\
&= \begin{bmatrix}
\lambda_y^2 & -\lambda_x k_y & 0 \\
-\lambda_x\lambda_y & \lambda_x^2 & 0 \\
0 & 0 & 0
\end{bmatrix} \left(-j\frac{Z_{\theta 1}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_1^\theta
\end{aligned} \tag{D.3}
$$

$$
\tilde{\overleftrightarrow{G}}_{e1e1}^{d} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z1}}\right) \tag{D.4}
$$

Next, analyze the electric field component observed in region 1 resulting from electric currents in region 2, $\tilde{\vec{E}}_{1e2}$. Substituting (C.6),(C.29), (C.34), (C.70), and

(C.72) into (157) implies that

$$\begin{aligned}
\overleftrightarrow{\tilde{G}}_{e1e2} &= j\vec{\lambda}_\rho\left(\vec{\tilde{G}}_{\Phi 1et2}+\vec{\tilde{G}}_{\Phi 1ez2}\right) - j\hat{z}\times\vec{\lambda}_\rho\vec{\tilde{G}}_{\theta 1e2} - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z1}}\left(\vec{\tilde{G}}_{\psi 1et2}+\vec{\tilde{G}}_{\psi 1ez2}\right)\\
&\qquad + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z1}}\delta\left(z-z'\right)\\
&= j\vec{\lambda}_\rho\left[\left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi + \left(-j\frac{\hat{z}Z_{\psi 2}}{\omega\epsilon_{z1}}\right)\Upsilon_4^\psi\right] - j\hat{z}\times\vec{\lambda}_\rho\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_2^\theta\\
&\qquad - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z1}}\left[\left(j\frac{\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^\psi + \left(-\frac{\hat{z}Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon_{12}^\psi\right] + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z1}}\delta\left(z-z'\right)\\
&= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(-j\frac{Z_{\psi 1}Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi + \vec{\lambda}_\rho\hat{z}\left(\frac{Z_{\psi 2}}{\omega\epsilon_{z1}}\right)\Upsilon_4^\psi + \hat{z}\vec{\lambda}_\rho\left(\frac{Z_{\psi 2}}{\omega\epsilon_{z1}}\right)\Upsilon_{10}^\psi\\
&\quad + \hat{z}\hat{z}\left(j\frac{\lambda_{\rho\psi}^2 Z_{\psi 2}}{\omega^2\epsilon_{z1}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi + \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 1}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_2^\theta + \hat{z}\hat{z}\left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z1}}\right)
\end{aligned} \tag{D.5}$$

Breaking (D.5) into TM^z, TE^z and depolarizing components implies that

$$\begin{aligned}
\overleftrightarrow{\tilde{G}}_{e1e2}^{\mathrm{TM}^z} &= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(-j\frac{Z_{\psi 1}Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi + \vec{\lambda}_\rho\hat{z}\left(\frac{Z_{\psi 2}}{\omega\epsilon_{z1}}\right)\Upsilon_4^\psi + \hat{z}\vec{\lambda}_\rho\left(\frac{Z_{\psi 2}}{\omega\epsilon_{z1}}\right)\Upsilon_{10}^\psi\\
&\qquad + \hat{z}\hat{z}\left(j\frac{\lambda_{\rho\psi}^2 Z_{\psi 2}}{\omega^2\epsilon_{z1}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi\\
&= Z_{\psi 2}\begin{bmatrix}
\lambda_x^2\left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi & \lambda_x\lambda_y\left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi & \lambda_x\left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_4^\psi\\
\lambda_x\lambda_y\left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi & \lambda_y^2\left(-j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_2^\psi & \lambda_y\left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_4^\psi\\
\lambda_x\left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_{10}^\psi & \lambda_y\left(\frac{1}{\omega\epsilon_{z1}}\right)\Upsilon_{10}^\psi & \left(j\frac{\lambda_{\rho\psi}^2}{\omega^2\epsilon_{z1}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi
\end{bmatrix}
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
\overleftrightarrow{\tilde{G}}_{e1e2}^{\mathrm{TE}^z} &= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 1}^2}{k_{\rho\theta}^2}\right)\Upsilon_2^\theta\\
&= \begin{bmatrix}
\lambda_y^2 & -\lambda_x\lambda_y & 0\\
-\lambda_x\lambda_y & \lambda_x^2 & 0\\
0 & 0 & 0
\end{bmatrix}\left(-j\frac{Z_{\theta 1}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_2^\theta
\end{aligned} \tag{D.7}$$

$$\tilde{\overleftrightarrow{G}}^{d}_{e1e2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j\frac{\delta\left(z - z'\right)}{\omega\epsilon_{z1}} \right) \tag{D.8}$$

Now analyze the electric field component observed in region 2 resulting from electric currents in region 1, $\tilde{\vec{E}}_{2e1}$. Substituting (C.7),(C.35),(C.40),(C.75), and (C.80) into (157) implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e2e1} &= j\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\Phi 2et1} + \tilde{\vec{G}}_{\Phi 2ez1}\right) - j\hat{z}\times\vec{\lambda}_\rho\tilde{\vec{G}}_{\theta 2e1} - j\frac{\hat{z}\lambda^2_{\rho\psi}}{\omega\epsilon_{z2}}\left(\tilde{\vec{G}}_{\psi 2et1} + \tilde{\vec{G}}_{\psi 2ez1}\right) \\
&\qquad + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z2}}\delta\left(z - z'\right) \\
&= j\vec{\lambda}_\rho\left[\left(-\frac{\vec{\lambda}_\rho Z_{\psi 1}Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 + \left(j\frac{\hat{z}Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_7\right] - j\hat{z}\times\vec{\lambda}_\rho\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z^2_{\theta 2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_5 \\
&\qquad - j\frac{\hat{z}\lambda^2_{\rho\psi}}{\omega\epsilon_{z2}}\left[\left(-j\frac{\vec{\lambda}_\rho Z_{\psi 1}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_{13} + \left(-\frac{\hat{z}Z_{\psi 1}}{\omega\epsilon_{z2}Z_{\psi 2}}\right)\Upsilon^\psi_{15}\right] + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z2}}\delta\left(z - z'\right) \\
&= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(-j\frac{Z_{\psi 1}Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 + \vec{\lambda}_\rho\hat{z}\left(-\frac{Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_7 + \hat{z}\vec{\lambda}_\rho\left(-\frac{Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_{13} \\
&\quad + \hat{z}\hat{z}\left(j\frac{\lambda^2_{\rho\psi}Z_{\psi 1}}{\omega^2\epsilon^2_{z2}Z_{\psi 2}}\right)\Upsilon^\psi_{15} + \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z^2_{\theta 2}}{\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_5 + \hat{z}\hat{z}\left(j\frac{\delta\left(z - z'\right)}{\omega\epsilon_{z2}}\right)
\end{aligned} \tag{D.9}$$

Breaking (D.9) in TM^z, TE^z, and depolarizing components implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2e1} &= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(-j\frac{Z_{\psi 1}Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 + \vec{\lambda}_\rho\hat{z}\left(-\frac{Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_7 + \hat{z}\vec{\lambda}_\rho\left(-\frac{Z_{\psi 1}}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_{13} \\
&\qquad + \hat{z}\hat{z}\left(j\frac{\lambda^2_{\rho\psi}Z_{\psi 1}}{\omega^2\epsilon^2_{z2}Z_{\psi 2}}\right)\Upsilon^\psi_{15} \\
&= Z_{\psi 1}\begin{bmatrix} \lambda^2_x\left(-j\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 & \lambda_x\lambda_y\left(-j\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 & \lambda_x\left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_7 \\ \lambda_x\lambda_y\left(-j\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 & \lambda^2_y\left(-j\frac{Z_{\psi 2}}{\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_5 & \lambda_y\left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_7 \\ \lambda_x\left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_{13} & \lambda_y\left(-\frac{1}{\omega\epsilon_{z2}}\right)\Upsilon^\psi_{13} & \left(j\frac{\lambda^2_{\rho\psi}}{\omega^2\epsilon^2_{z2}Z_{\psi 2}}\right)\Upsilon^\psi_{15} \end{bmatrix}
\end{aligned} \tag{D.10}$$

$$\tilde{\overleftrightarrow{G}}_{e2e1}^{\mathrm{TE}^z} = \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 2}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_5^\theta$$

$$= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(-j\frac{Z_{\theta 2}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_5^\theta \tag{D.11}$$

$$\tilde{\overleftrightarrow{G}}_{e2e1}^{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z2}}\right) \tag{D.12}$$

Next, analyze the electric field component observed in region 2 resulting from electric currents in region 2, $\tilde{\vec{E}}_{2e2}$. Substituting (C.11),(C.39),(C.44),(C.79), and (C.81) into (157) implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e2e2} &= j\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\Phi 2et2} + \tilde{\vec{G}}_{\Phi 2ez2}\right) - j\hat{z}\times\vec{\lambda}_\rho\tilde{\vec{G}}_{\theta 2e2} - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z2}}\left(\tilde{\vec{G}}_{\psi 2et2} + \tilde{\vec{G}}_{\psi 2ez2}\right) \\
&\qquad + j\frac{\hat{z}\hat{z}}{\omega\epsilon_{z2}}\delta\left(z-z'\right) \\
&= j\vec{\lambda}_\rho\left[\left(-\frac{\vec{\lambda}_\rho Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi + \left(j\frac{\hat{z}}{2\omega\epsilon_{z2}}\right)\Upsilon_8^\psi\right] - j\hat{z}\times\vec{\lambda}_\rho\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_6^\theta \\
&\qquad - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z2}}\left[\left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^\psi + \left(-\frac{\hat{z}}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^\psi\right] + j\frac{\hat{z}\hat{z}}{\omega\epsilon_2}\delta\left(z-z'\right) \\
&= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(-j\frac{Z_{\psi 2}}{2\lambda_{\rho\psi}^2}\right)\Upsilon_6^\psi + \vec{\lambda}_\rho\hat{z}\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon_8^\psi + \hat{z}\vec{\lambda}_\rho\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon_{14}^\psi \\
&\qquad + \hat{z}\hat{z}\left(j\frac{k_{\rho\psi}^2}{2Z_{\psi 2}\omega^2\epsilon_{z2}^2}\right)\Upsilon_{16}^\psi + \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 2}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_6^\theta + \hat{z}\hat{z}\left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z2}}\right)
\end{aligned} \tag{D.13}$$

Breaking (D.13) into TM^z, TE^z, and depolarizing components implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}^{\mathrm{TM}^z}_{e2e2} &= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(-j\frac{Z_{\psi 2}}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_6 + \vec{\lambda}_\rho\hat{z}\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon^\psi_8 + \hat{z}\vec{\lambda}_\rho\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon^\psi_{14} \\
&\qquad + \hat{z}\hat{z}\left(j\frac{\lambda^2_{\rho\psi}}{2Z_{\psi 2}\omega^2\epsilon^2_{z2}}\right)\Upsilon^\psi_{16} \\
&= \begin{bmatrix} \lambda_x^2\left(-j\frac{Z_{\psi 2}}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_6 & \lambda_x\lambda_y\left(-j\frac{Z_{\psi 2}}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_6 & \lambda_x\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon^\psi_8 \\ \lambda_x\lambda_y\left(-j\frac{Z_{\psi 2}}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_6 & \lambda_y^2\left(-j\frac{Z_{\psi 2}}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_6 & \lambda_y\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon^\psi_8 \\ \lambda_x\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon^\psi_{14} & \lambda_y\left(-\frac{1}{2\omega\epsilon_{z2}}\right)\Upsilon^\psi_{14} & \left(j\frac{\lambda^2_{\rho\psi}}{2Z_{\psi 2}\omega^2\epsilon^2_{z2}}\right)\Upsilon^\psi_{16} \end{bmatrix} && \text{(D.14)} \\
\tilde{\overleftrightarrow{G}}^{\mathrm{TE}^z}_{e2e2} &= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-j\frac{Z_{\theta 2}}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_6, \\
&= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(-j\frac{Z_{\theta 2}}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_6 && \text{(D.15)} \\
\tilde{\overleftrightarrow{G}}^{d}_{e2e2} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(j\frac{\delta\left(z-z'\right)}{\omega\epsilon_{z2}}\right) && \text{(D.16)}
\end{aligned}$$

Now that the electric field resulting from electric currents has been determined, the electric field resulting from magnetic currents must be analyzed. Begin with the electric field component observed in region 1 resulting from magnetic currents in region 1, $\tilde{\vec{E}}_{1h1}$. Substituting (144),(145),(C.46), and (C.73) into (157) implies that

$$\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1h1} &= j\vec{\lambda}_\rho\tilde{\vec{G}}_{\Phi 1h1} - j\hat{z}\times\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\theta 1ht1} + \tilde{\vec{G}}_{\theta 1hz1}\right) - j\frac{\hat{z}\lambda^2_{\rho\psi}}{\omega\epsilon_{z1}}\tilde{\vec{G}}_{\psi 1h1} \\
&= j\vec{\lambda}_\rho\left(j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}}\right)\Upsilon^\psi_3 - j\hat{z}\times\vec{\lambda}_\rho\left[\left(j\frac{\vec{\lambda}_\rho}{2\lambda^2_{\rho\theta}}\right)\Upsilon^\theta_3 + \left(-\frac{\hat{z}Z_{\theta 1}}{2\omega\mu_{z1}}\right)\Upsilon^\theta_1\right] \\
&\qquad - j\frac{\hat{z}\lambda^2_{\rho\psi}}{\omega\epsilon_{z1}}\left(-\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda^2_{\rho\psi}Z_{\psi 1}}\right)\Upsilon^\psi_{11}
\end{aligned}$$

$$
\begin{aligned}
&= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_3^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi \\
&\quad + \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_3^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta
\end{aligned}
\tag{D.17}
$$

Breaking (D.17) in TM^z and TE^z terms implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1h1}^{\mathrm{TM}^z} &= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_3^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi \\
&= \begin{bmatrix}
-\lambda_x\lambda_y \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_3^\psi & \lambda_x^2 \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_3^\psi & 0 \\
-\lambda_y^2 \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_3^\psi & \lambda_x\lambda_y \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_3^\psi & 0 \\
-\lambda_y \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi & \lambda_x \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi & 0
\end{bmatrix}
\end{aligned}
\tag{D.18}
$$

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1h1}^{\mathrm{TE}^z} &= \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_3^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta \\
&= \begin{bmatrix}
-\lambda_x\lambda_y \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_3^\theta & -\lambda_y^2 \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_3^\theta & -\lambda_y \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta \\
\lambda_x^2 \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_3^\theta & \lambda_x\lambda_y \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_3^\theta & \lambda_x \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta \\
0 & 0 & 0
\end{bmatrix}
\end{aligned}
\tag{D.19}
$$

Next, analyze the electric field component observed in region 1 resulting from magnetic currents in region 2, $\tilde{\vec{E}}_{1h2}$. Substituting (C.12),(C.13),(C.47), and (C.74) into (157) implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1h2} &= j\vec{\lambda}_\rho \tilde{\vec{G}}_{\Phi 1h2} - j\hat{z} \times \vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\theta 1ht2} + \tilde{\vec{G}}_{\theta 1hz2}\right) - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z1}} \tilde{\vec{G}}_{\psi 1h2} \\
&= j\vec{\lambda}_\rho \left(-j\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi - j\hat{z} \times \vec{\lambda}_\rho \left[\left(-j\frac{\vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta + \left(-\frac{\hat{z} Z_{\theta 1}^2}{\omega\mu_{z1}}\right) \Upsilon_2^\theta\right] \\
&\qquad - j\frac{\hat{z}\lambda_{\rho\psi}^2}{\omega\epsilon_{z1}} \left(-\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right) \Upsilon_{12}^\psi
\end{aligned}
$$

$$
\begin{aligned}
&= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j \frac{Z_{\psi 2}}{\omega \epsilon_{z1} Z_{\psi 1}}\right) \Upsilon_{12}^\psi \\
&\quad + \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j \frac{Z_{\theta 1}^2}{\omega \mu_{z1}}\right) \Upsilon_2^\theta
\end{aligned}
\tag{D.20}
$$

Breaking (D.20) into TM^z and TE^z components implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1h2}^{\mathrm{TM}^z} &= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j \frac{Z_{\psi 2}}{\omega \epsilon_{z1} Z_{\psi 1}}\right) \Upsilon_{12}^\psi \\
&= \begin{bmatrix}
-\lambda_x \lambda_y \left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi & \lambda_x^2 \left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi & 0 \\
-\lambda_y^2 \left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi & \lambda_x \lambda_y \left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_4^\psi & 0 \\
-\lambda_y \left(j \frac{Z_{\psi 2}}{\omega \epsilon_{z1} Z_{\psi 1}}\right) \Upsilon_{12}^\psi & \lambda_x \left(j \frac{Z_{\psi 2}}{\omega \epsilon_{z1} Z_{\psi 1}}\right) \Upsilon_{12}^\psi & 0
\end{bmatrix}
\end{aligned}
\tag{D.21}
$$

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e1h2}^{\mathrm{TE}^z} &= \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j \frac{Z_{\theta 1}^2}{\omega \mu_{z1}}\right) \Upsilon_2^\theta \\
&= \begin{bmatrix}
-\lambda_x \lambda_y \left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta & -\lambda_y^2 \left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta & -\lambda_y \left(j \frac{Z_{\theta 1}^2}{\omega \mu_{z1}}\right) \Upsilon_2^\theta \\
\lambda_x^2 \left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta & \lambda_x \lambda_y \left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_4^\theta & \lambda_x \left(j \frac{Z_{\theta 1}^2}{\omega \mu_{z1}}\right) \Upsilon_2^\theta \\
0 & 0 & 0
\end{bmatrix}
\end{aligned}
\tag{D.22}
$$

Now analyze the electric field component observed in region 2 resulting from magnetic currents in region 1, $\tilde{\vec{E}}_{2h1}$. Substituting (C.14), (C.19), (C.49), and (C.82) into (157) implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e2h1} &= j \vec{\lambda}_\rho \tilde{\vec{G}}_{\Phi 2h1} - j \hat{z} \times \vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\theta 2ht1} + \tilde{\vec{G}}_{\theta 2hz1}\right) - j \frac{\hat{z} k_{\rho\psi}^2}{\omega \epsilon_{z2}} \tilde{\vec{G}}_{\psi 2h1} \\
&= j \vec{\lambda}_\rho \left(j \frac{\hat{z} \times \vec{\lambda}_\rho Z_{\psi 1}}{k_{\rho\psi}^2}\right) \Upsilon_7^\psi - j \hat{z} \times \vec{\lambda}_\rho \left[\left(j \frac{\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right) \Upsilon_7^\theta + \left(-\frac{\hat{z} Z_{\theta 2}^2}{\omega \mu_{z2}}\right) \Upsilon_5^\theta\right] \\
&\qquad - j \frac{\hat{z} \lambda_{\rho\psi}^2}{\omega \epsilon_{z2}} \left(-\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2 Z_{\psi 2}}\right) \Upsilon_{15}^\psi
\end{aligned}
$$

$$= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(-\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right) \Upsilon_7^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j \frac{Z_{\psi 1}}{\omega \epsilon_{z2} Z_{\psi 2}}\right) \Upsilon_{15}^\psi + \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(\frac{Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right) \Upsilon_7^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j \frac{Z_{\theta 2}^2}{\omega \mu_{z2}}\right) \Upsilon_5^\theta \tag{D.23}$$

Breaking (D.23) into TM^z and TE^z components implies that

$$\begin{aligned}\overleftrightarrow{\tilde{G}}_{e2h1}^{\text{TM}^z} &= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(-\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right) \Upsilon_7^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j \frac{Z_{\psi 1}}{\omega \epsilon_{z2} Z_{\psi 2}}\right) \Upsilon_{15}^\psi \\ &= \begin{bmatrix} -\lambda_x \lambda_y \left(-\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right) \Upsilon_7^\psi & \lambda_x^2 \left(-\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right) \Upsilon_7^\psi & 0 \\ -\lambda_y^2 \left(-\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right) \Upsilon_7^\psi & \lambda_x \lambda_y \left(-\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right) \Upsilon_7^\psi & 0 \\ -\lambda_y \left(j \frac{Z_{\psi 1}}{\omega \epsilon_{z2} Z_{\psi 2}}\right) \Upsilon_{15}^\psi & \lambda_x \left(j \frac{Z_{\psi 1}}{\omega \epsilon_{z2} Z_{\psi 2}}\right) \Upsilon_{15}^\psi & 0 \end{bmatrix}\end{aligned} \tag{D.24}$$

$$\begin{aligned}\overleftrightarrow{\tilde{G}}_{e2h1}^{\text{TE}^z} &= \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(\frac{Z_{\theta 2}}{k_{\rho\theta}^2}\right) \Upsilon_7^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j \frac{Z_{\theta 2}^2}{\omega \mu_{z2}}\right) \Upsilon_5^\theta \\ &= \begin{bmatrix} -\lambda_x \lambda_y \left(\frac{Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right) \Upsilon_7^\theta & -\lambda_y^2 \left(\frac{Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right) \Upsilon_7^\theta & -\lambda_y \left(j \frac{Z_{\theta 2}^2}{\omega \mu_{z2}}\right) \Upsilon_5^\theta \\ \lambda_x^2 \left(\frac{Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right) \Upsilon_7^\theta & \lambda_x \lambda_y \left(\frac{Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right) \Upsilon_7^\theta & \lambda_x \left(j \frac{Z_{\theta 2}^2}{\omega \mu_{z2}}\right) \Upsilon_5^\theta \\ 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{D.25}$$

Finally, analyze the electric field component observed in region 2 resulting from magnetic currents in region 2, $\vec{\tilde{E}}_{2h2}$. Substituting (C.18), (C.20), (C.50), and (C.83) into (157) implies that

$$\begin{aligned}\overleftrightarrow{\tilde{G}}_{e2h2} &= j\vec{k}_\rho \vec{\tilde{G}}_{\Phi 2h2} - j\hat{z} \times \vec{\lambda}_\rho \left(\vec{\tilde{G}}_{\theta 2ht2} + \vec{\tilde{G}}_{\theta 2hz2}\right) - j \frac{\hat{z}\lambda_{\rho\psi}^2}{\omega \epsilon_{z2}} \vec{\tilde{G}}_{\psi 2h2} \\ &= j\vec{\lambda}_\rho \left(j \frac{\hat{z} \times \vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right) \Upsilon_8^\psi - j\hat{z} \times \vec{\lambda}_\rho \left[\left(j \frac{\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right) \Upsilon_8^\theta + \left(-\frac{\hat{z} Z_{\theta 2}}{2\omega \mu_{z2}}\right) \Upsilon_6^\theta\right] \\ &\qquad - j \frac{\hat{z}\lambda_{\rho\psi}^2}{\omega \epsilon_{z2}} \left(-\frac{\hat{z} \times \vec{\lambda}_\rho}{2k_{\rho\psi}^2 Z_{\psi 2}}\right) \Upsilon_{16}^\psi\end{aligned}$$

$$
\begin{aligned}
&= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_8^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right) \Upsilon_{16}^\psi \\
&\quad + \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_8^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right) \Upsilon_6^\theta
\end{aligned}
\tag{D.26}
$$

Breaking (D.26) into TM^z and TE^z components implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e2h2}^{\mathrm{TM}^z} &= \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_8^\psi + \hat{z} \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right) \Upsilon_{16}^\psi \\
&= \begin{bmatrix} -\lambda_x\lambda_y \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_8^\psi & \lambda_x^2 \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_8^\psi & 0 \\ -\lambda_y^2 \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_8^\psi & \lambda_x\lambda_y \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_8^\psi & 0 \\ -\lambda_y \left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^\psi & \lambda_x \left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^\psi & 0 \end{bmatrix}
\end{aligned}
\tag{D.27}
$$

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{e2h2}^{\mathrm{TE}^z} &= \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_8^\theta + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right) \Upsilon_6^\theta \\
&= \begin{bmatrix} -\lambda_x\lambda_y \left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_8^\theta & -\lambda_y^2 \left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_8^\theta & -\lambda_y \left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon_6^\theta \\ \lambda_x^2 \left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_8^\theta & \lambda_x\lambda_y \left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_8^\theta & \lambda_x \left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon_6^\theta \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{D.28}
$$

D.2 $\tilde{\vec{H}}_e$ Development

Begin with the magnetic field component observed in region 1 resulting from electric currents in region 1, $\tilde{\vec{H}}_{1e1}$. Substituting (C.5), (C.28), (C.33), and (C.51) into (159) implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h1e1} &= j\vec{\lambda}_\rho \tilde{\vec{G}}_{\Pi 1e1} - j\hat{z} \times \vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\psi 1et1} + \tilde{\vec{G}}_{\psi 1ez1}\right) + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}} \tilde{\vec{G}}_{\theta 1e1} \\
&= j\vec{\lambda}_\rho \left(-j\frac{\hat{z} \times \vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta - j\hat{z} \times \vec{\lambda}_\rho \left[\left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi + \left(-\frac{\hat{z}}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi\right] \\
&\qquad + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}} \left(\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}}{2\lambda_{\rho\theta}^2}\right) \Upsilon_1^\theta
\end{aligned}
$$

$$= \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi$$
$$+ \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta + \hat{z}\left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta \tag{D.29}$$

Breaking (D.29) into TM^z and TE^z components implies that

$$\tilde{\vec{\vec{G}}}_{h1e1}^{\mathrm{TM}^z} = \left(\hat{z} \times \vec{\lambda}_\rho\right) \vec{\lambda}_\rho \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi + \left(\hat{z} \times \vec{\lambda}_\rho\right) \hat{z} \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi$$
$$= \begin{bmatrix} -\lambda_x\lambda_y \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi & -\lambda_y^2 \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi & -\lambda_y \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi \\ \lambda_x^2 \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi & \lambda_x\lambda_y \left(-\frac{1}{2\lambda_{\rho\psi}^2}\right) \Upsilon_9^\psi & \lambda_x \left(j\frac{1}{2Z_{\psi 1}\omega\epsilon_{z1}}\right) \Upsilon_{11}^\psi \\ 0 & 0 & 0 \end{bmatrix} \tag{D.30}$$

$$\tilde{\vec{\vec{G}}}_{h1e1}^{\mathrm{TE}^z} = \vec{\lambda}_\rho \left(\hat{z} \times \vec{\lambda}_\rho\right) \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta + \hat{z}\left(\hat{z} \times \vec{\lambda}_\rho\right) \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta$$
$$= \begin{bmatrix} -\lambda_x\lambda_y \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta & \lambda_x^2 \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta & 0 \\ -\lambda_y^2 \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta & \lambda_x\lambda_y \left(\frac{1}{2\lambda_{\rho\theta}^2}\right) \Upsilon_9^\theta & 0 \\ -\lambda_y \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta & \lambda_x \left(j\frac{Z_{\theta 1}}{2\omega\mu_{z1}}\right) \Upsilon_1^\theta & 0 \end{bmatrix} \tag{D.31}$$

Next, analyze the magnetic field component observed in region 1 resulting from electric currents in region 2, $\tilde{\vec{\vec{H}}}_{1e2}$. Substituting (C.6), (C.29), (C.34), and (C.52) into (159) implies that

$$\tilde{\vec{\vec{G}}}_{h1e2} = j\vec{\lambda}_\rho \tilde{\vec{G}}_{\Pi 1e2} - j\hat{z} \times \vec{\lambda}_\rho \left(\tilde{\vec{G}}_{\psi 1et2} + \tilde{\vec{G}}_{\psi 1ez2}\right) + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}} \tilde{\vec{G}}_{\theta 1e2}$$
$$= j\vec{\lambda}_\rho \left(j\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right) \Upsilon_{10}^\theta - j\hat{z} \times \vec{\lambda}_\rho \left[\left(j\frac{\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right) \Upsilon_{10}^\psi + \left(-\frac{\hat{z} Z_{\psi 2}}{\omega\epsilon_{z1} Z_{\psi 1}}\right) \Upsilon_{12}^\psi\right]$$
$$+ j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}} \left(\frac{\hat{z} \times \vec{\lambda}_\rho Z_{\theta 1}^2}{\lambda_{\rho\theta}^2}\right) \Upsilon_2^\theta$$

$$
\begin{aligned}
&= \left(\hat{z}\times\vec{\lambda}_\rho\right)\vec{\lambda}_\rho\left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^{\psi} + \left(\hat{z}\times\vec{\lambda}_\rho\right)\hat{z}\left(j\frac{Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon_{12}^{\psi} \\
&\quad + \vec{\lambda}_\rho\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{10}^{\theta} + \hat{z}\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\theta 1}^2}{\omega\mu_{z1}}\right)\Upsilon_{2}^{\theta}
\end{aligned} \tag{D.32}
$$

Breaking (D.32) into TM^z and TE^z components implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h1e2}^{\mathrm{TM}^z} &= \left(\hat{z}\times\vec{\lambda}_\rho\right)\vec{\lambda}_\rho\left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^{\psi} + \left(\hat{z}\times\vec{\lambda}_\rho\right)\hat{z}\left(j\frac{Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon_{12}^{\psi} \\
&= \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^{\psi} & -\lambda_y^2\left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^{\psi} & -\lambda_y\left(j\frac{Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon_{12}^{\psi} \\ \lambda_x^2\left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^{\psi} & \lambda_x\lambda_y\left(\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{10}^{\psi} & \lambda_x\left(j\frac{Z_{\psi 2}}{\omega\epsilon_{z1}Z_{\psi 1}}\right)\Upsilon_{12}^{\psi} \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{D.33}
$$

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h1e2}^{\mathrm{TE}^z} &= \vec{\lambda}_\rho\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{10}^{\theta} + \hat{z}\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\theta 1}^2}{\omega\mu_{z1}}\right)\Upsilon_{2}^{\theta} \\
&= \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{10}^{\theta} & \lambda_x^2\left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{10}^{\theta} & 0 \\ -\lambda_y^2\left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{10}^{\theta} & \lambda_x\lambda_y\left(-\frac{Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{10}^{\theta} & 0 \\ -\lambda_y\left(j\frac{Z_{\theta 1}^2}{\omega\mu_{z1}}\right)\Upsilon_{2}^{\theta} & \lambda_x\left(j\frac{Z_{\theta 1}^2}{\omega\mu_{z1}}\right)\Upsilon_{2}^{\theta} & 0 \end{bmatrix}
\end{aligned} \tag{D.34}
$$

Now analyze the magnetic component observed in region 2 resulting from electric currents in region 1, $\tilde{\vec{H}}_{2e1}$. Substituting (C.7), (C.35), (C.40), and (C.53) into (159) implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h2e1} &= j\vec{\lambda}_\rho\tilde{\vec{G}}_{\Pi 2e1} - j\hat{z}\times\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\psi 2et1} + \tilde{\vec{G}}_{\psi 2ez1}\right) + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}}\tilde{\vec{G}}_{\theta 2e1} \\
&= j\vec{\lambda}_\rho\left(-j\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta} - j\hat{z}\times\vec{\lambda}_\rho\left[\left(-j\frac{\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi} + \left(-\frac{\hat{z}Z_{\psi 1}}{\omega\epsilon_{z2}Z_{\psi 2}}\right)\Upsilon_{15}^{\psi}\right] \\
&\qquad + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}}\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta 2}^2}{\lambda_{\rho\theta}^2}\right)\Upsilon_{5}^{\theta}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\hat{z}\times\vec{\lambda}_\rho\right)\vec{\lambda}_\rho\left(-\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi}+\left(\hat{z}\times\vec{\lambda}_\rho\right)\hat{z}\left(j\frac{Z_{\psi1}}{\omega\epsilon_{z2}Z_{\psi2}}\right)\Upsilon_{15}^{\psi}\\
&\qquad +\vec{\lambda}_\rho\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\frac{Z_{\theta2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta}+\hat{z}\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\theta2}^2}{\omega\mu_{z2}}\right)\Upsilon_{5}^{\theta}
\end{aligned}
\tag{D.35}
$$

Breaking (D.35) into TM^z and TE^z components implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h2e1}^{\mathrm{TM}^z} &= \left(\hat{z}\times\vec{\lambda}_\rho\right)\vec{\lambda}_\rho\left(-\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi}+\left(\hat{z}\times\vec{\lambda}_\rho\right)\hat{z}\left(j\frac{Z_{\psi1}}{\omega\epsilon_{z2}Z_{\psi2}}\right)\Upsilon_{15}^{\psi}\\
&= \begin{bmatrix}
-\lambda_x\lambda_y\left(-\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi} & -\lambda_y^2\left(-\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi} & -\lambda_y\left(j\frac{Z_{\psi1}}{\omega\epsilon_{z2}Z_{\psi2}}\right)\Upsilon_{15}^{\psi}\\
\lambda_x^2\left(-\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi} & \lambda_x^2\left(-\frac{Z_{\psi1}}{\lambda_{\rho\psi}^2}\right)\Upsilon_{13}^{\psi} & \lambda_x\left(j\frac{Z_{\psi1}}{\omega\epsilon_{z2}Z_{\psi2}}\right)\Upsilon_{15}^{\psi}\\
0 & 0 & 0
\end{bmatrix}
\end{aligned}
\tag{D.36}
$$

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h2e1}^{\mathrm{TE}^z} &= \vec{\lambda}_\rho\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\frac{Z_{\theta2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta}+\hat{z}\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\theta2}^2}{\omega\mu_{z2}}\right)\Upsilon_{5}^{\theta}\\
&= \begin{bmatrix}
-\lambda_x\lambda_y\left(\frac{Z_{\theta2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta} & \lambda_x^2\left(\frac{Z_{\theta2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta} & 0\\
-\lambda_y^2\left(\frac{Z_{\theta2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta} & \lambda_x\lambda_y\left(\frac{Z_{\theta2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_{13}^{\theta} & 0\\
-\lambda_y\left(j\frac{Z_{\theta2}^2}{\omega\mu_{z2}}\right)\Upsilon_{5}^{\theta} & \lambda_x\left(j\frac{Z_{\theta2}^2}{\omega\mu_{z2}}\right)\Upsilon_{5}^{\theta} & 0
\end{bmatrix}
\end{aligned}
\tag{D.37}
$$

Next, analyze the magnetic field component observed in region 2 resulting from electric currents in region 2, $\tilde{\vec{H}}_{2e2}$. Substituting (C.11), (C.39), (C.44), and (C.54) into (159) implies that

$$
\begin{aligned}
\tilde{\overleftrightarrow{G}}_{h2e2} &= j\vec{\lambda}_\rho\tilde{\vec{G}}_{\Pi2e2}-j\hat{z}\times\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\psi2et2}+\tilde{\vec{G}}_{\psi2ez2}\right)+j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}}\tilde{\vec{G}}_{\theta2e2}\\
&= j\vec{\lambda}_\rho\left(-j\frac{\hat{z}\times\vec{\lambda}_\rho}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta}-j\hat{z}\times\vec{\lambda}_\rho\left[\left(-j\frac{\vec{\lambda}_\rho}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi}+\left(-\frac{\hat{z}}{2Z_{\psi2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^{\psi}\right]\\
&\qquad\qquad +j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}}\left(\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\theta2}}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{6}^{\theta}
\end{aligned}
$$

$$= \left(\hat{z}\times\vec{\lambda}_\rho\right)\vec{\lambda}_\rho\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi} + \left(\hat{z}\times\vec{\lambda}_\rho\right)\hat{z}\left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^{\psi}$$
$$+ \vec{\lambda}_\rho\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta} + \hat{z}\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon_{6}^{\theta} \quad \text{(D.38)}$$

Breaking (D.38) into TM^z and TE^z components implies that

$$\tilde{\overleftrightarrow{G}}_{h2e2}^{\text{TM}^z} = \left(\hat{z}\times\vec{k}_\rho\right)\vec{\lambda}_\rho\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi} + \left(\hat{z}\times\vec{\lambda}_\rho\right)\hat{z}\left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^{\psi}$$
$$= \begin{bmatrix} -\lambda_x\lambda_y\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi} & -\lambda_y^2\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi} & -\lambda_y\left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^{\psi} \\ \lambda_x^2\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi} & \lambda_x\lambda_y\left(-\frac{1}{2\lambda_{\rho\psi}^2}\right)\Upsilon_{14}^{\psi} & \lambda_x\left(j\frac{1}{2Z_{\psi 2}\omega\epsilon_{z2}}\right)\Upsilon_{16}^{\psi} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(D.39)}$$

$$\tilde{\overleftrightarrow{G}}_{h2e2}^{\text{TE}^z} = \vec{\lambda}_\rho\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta} + \hat{z}\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon_{6}^{\theta}$$
$$= \begin{bmatrix} -\lambda_x\lambda_y\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta} & \lambda_x^2\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta} & 0 \\ -\lambda_y^2\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta} & \lambda_x\lambda_y\left(\frac{1}{2\lambda_{\rho\theta}^2}\right)\Upsilon_{14}^{\theta} & 0 \\ -\lambda_y\left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon_{6}^{\theta} & \lambda_x\left(j\frac{Z_{\theta 2}}{2\omega\mu_{z2}}\right)\Upsilon_{6}^{\theta} & 0 \end{bmatrix} \quad \text{(D.40)}$$

D.3 $\tilde{\vec{H}}$ Cross-Term Development

First, analyze the magnetic field component observed in region 1 resulting from magnetic currents in region 2, $\tilde{\vec{H}}_{1h2}$. Substituting (C.12), (C.13), (C.47), (C.55), and (C.57) into (159) implies that

$$\tilde{\overleftrightarrow{G}}_{h1h2} = j\vec{\lambda}_\rho\left(\tilde{\vec{G}}_{\Pi 1ht2} + \tilde{\vec{G}}_{\Pi 1hz2}\right) - j\hat{z}\times\vec{\lambda}_\rho\tilde{\vec{G}}_{\psi 1h2} + j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}}\left(\tilde{\vec{G}}_{\theta 1ht2} + \tilde{\vec{G}}_{\theta 1hz2}\right)$$
$$+ j\frac{\hat{z}\hat{z}}{\omega\mu_{z1}}\delta\left(z-z'\right)$$

$$= j\vec{\lambda}_\rho \left[\left(\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta + \left(-j\frac{\hat{z}Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_{10}^\theta\right] - j\hat{z}\times\vec{\lambda}_\rho\left(-\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi$$
$$+ j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z1}}\left[\left(-j\frac{\vec{\lambda}_\rho Z_{\theta 1}}{\lambda_{\rho\theta}^2}\right)\Upsilon_4^\theta + \left(-\frac{\hat{z}Z_{\theta 1}^2}{\omega\mu_{z1}}\right)\Upsilon_2^\theta\right] + j\frac{\hat{z}\hat{z}}{\omega\mu_{z1}}\delta\left(z-z'\right)$$
$$= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi + \vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta + \vec{\lambda}_\rho\hat{z}\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_{10}^\theta$$
$$+ \hat{z}\vec{\lambda}_\rho\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_4^\theta + \hat{z}\hat{z}\left(-j\frac{Z_{\theta 1}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z1}^2}\right)\Upsilon_2^\theta + \hat{z}\hat{z}\left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z1}}\right) \tag{D.41}$$

Breaking (D.41) into TM^z, TE^z, and depolarizing components implies that

$$\tilde{\overleftrightarrow{G}}_{h1h2}^{\mathrm{TM}^z} = \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi$$
$$= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(j\frac{Z_{\psi 2}}{\lambda_{\rho\psi}^2 Z_{\psi 1}}\right)\Upsilon_{12}^\psi \tag{D.42}$$
$$\tilde{\overleftrightarrow{G}}_{h1h2}^{\mathrm{TE}^z} = \vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta + \vec{\lambda}_\rho\hat{z}\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_{10}^\theta + \hat{z}\vec{\lambda}_\rho\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_4^\theta$$
$$+ \hat{z}\hat{z}\left(-j\frac{Z_{\theta 1}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z1}^2}\right)\Upsilon_2^\theta$$
$$= \begin{bmatrix} \lambda_x^2\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta & \lambda_x\lambda_y\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta & \lambda_x\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_{10}^\theta \\ \lambda_x\lambda_y\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta & \lambda_y^2\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{12}^\theta & \lambda_y\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_{10}^\theta \\ \lambda_x\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_4^\theta & \lambda_y\left(\frac{Z_{\theta 1}}{\omega\mu_{z1}}\right)\Upsilon_4^\theta & \left(-j\frac{Z_{\theta 1}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z1}^2}\right)\Upsilon_2^\theta \end{bmatrix} \tag{D.43}$$
$$\tilde{\overleftrightarrow{G}}_{h1h2}^{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z1}}\right) \tag{D.44}$$

Finally, analyze the magnetic field component observed in region 2 resulting from magnetic currents in region 1, $\tilde{\vec{H}}_{2h1}$. Substituting (C.14), (C.19), (C.49), (C.58), and

(C.63) into (159) implies that

$$\begin{aligned}
\overleftrightarrow{\tilde{G}}_{h2h1} &= j\vec{\lambda}_\rho\left(\vec{\tilde{G}}_{\Pi 2ht1}+\vec{\tilde{G}}_{\Pi 2hz1}\right)-j\hat{z}\times\vec{\lambda}_\rho\vec{\tilde{G}}_{\psi 2h1}+j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}}\left(\vec{\tilde{G}}_{\theta 2ht1}+\vec{\tilde{G}}_{\theta 2hz1}\right) \\
&\qquad +j\frac{\hat{z}\hat{z}}{\omega\mu_{z2}}\delta\left(z-z'\right) \\
&= j\vec{\lambda}_\rho\left[\left(\frac{\vec{\lambda}_\rho}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta+\left(j\frac{\hat{z}Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_{13}^\theta\right]-j\hat{z}\times\vec{\lambda}_\rho\left(-\frac{\hat{z}\times\vec{\lambda}_\rho Z_{\psi 1}}{\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{15}^\psi \\
&\qquad +j\frac{\hat{z}\lambda_{\rho\theta}^2}{\omega\mu_{z2}}\left[\left(j\frac{\vec{\lambda}_\rho Z_{\theta 2}}{\lambda_{\rho\theta}^2}\right)\Upsilon_7^\theta+\left(-\frac{\hat{z}Z_{\theta 2}^2}{\omega\mu_{z2}}\right)\Upsilon_5^\theta\right]+j\frac{\hat{z}\hat{z}}{\omega\mu_{z2}}\delta\left(z-z'\right) \\
&= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{15}^\psi+\vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta+\vec{\lambda}_\rho\hat{z}\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_{13}^\theta \\
&\qquad +\hat{z}\vec{\lambda}_\rho\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_7^\theta+\hat{z}\hat{z}\left(-j\frac{Z_{\theta 2}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z2}^2}\right)\Upsilon_5^\theta+\hat{z}\hat{z}\left(j\frac{\delta\left(z-z'\right)}{\omega\mu_{z2}}\right)
\end{aligned} \tag{D.45}$$

Breaking (D.45) into TM^z, TE^z, and depolarizing components implies that

$$\begin{aligned}
\overleftrightarrow{\tilde{G}}_{h2h1}^{\mathrm{TM}^z} &= \left(\hat{z}\times\vec{\lambda}_\rho\right)\left(\hat{z}\times\vec{\lambda}_\rho\right)\left(j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{15}^\psi \\
&= \begin{bmatrix} \lambda_y^2 & -\lambda_x\lambda_y & 0 \\ -\lambda_x\lambda_y & \lambda_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(j\frac{Z_{\psi 1}}{\lambda_{\rho\psi}^2 Z_{\psi 2}}\right)\Upsilon_{15}^\psi
\end{aligned} \tag{D.46}$$

$$\begin{aligned}
\overleftrightarrow{\tilde{G}}_{h2h1}^{\mathrm{TE}^z} &= \vec{\lambda}_\rho\vec{\lambda}_\rho\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta+\vec{\lambda}_\rho\hat{z}\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_{13}^\theta+\hat{z}\vec{\lambda}_\rho\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_7^\theta \\
&\qquad +\hat{z}\hat{z}\left(-j\frac{Z_{\theta 2}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z2}^2}\right)\Upsilon_5^\theta \\
&= \begin{bmatrix} \lambda_x^2\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta & \lambda_x\lambda_y\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta & \lambda_x\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_{13}^\theta \\ \lambda_x\lambda_y\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta & \lambda_y^2\left(j\frac{1}{\lambda_{\rho\theta}^2}\right)\Upsilon_{15}^\theta & \lambda_y\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_{13}^\theta \\ \lambda_x\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_7^\theta & \lambda_y\left(-\frac{Z_{\theta 2}}{\omega\mu_{z2}}\right)\Upsilon_7^\theta & \left(-j\frac{Z_{\theta 2}^2\lambda_{\rho\theta}^2}{\omega^2\mu_{z2}^2}\right)\Upsilon_5^\theta \end{bmatrix}
\end{aligned} \tag{D.47}$$

$$\overset{\leftrightarrow}{\tilde{G}}{}^{d}_{h2h1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(j \frac{\delta\left(z - z'\right)}{\omega \mu_{z2}} \right) \tag{D.48}$$

E. TE^z and TM^z X-Band Rectangular Waveguide Modes

The index q in table 2 selects from the first 20 TE^z_{mn} and TM^z_{mn} modes in order of increasing cutoff frequency for an X-band rectangular waveguide. Note that, due to physical symmetry, only odd values of m and even values of n are allowed to be excited at the aperture.

Table 2. First 20 X-Band Rectangular Waveguide Modes

q	Mode	f_{cq} (GHz)	q	Mode	f_{cq} (GHz)
1	TE^z_{10}	6.56	11	TE^z_{72}	54.56
2	TE^z_{30}	19.67	12	TM^z_{72}	54.56
3	TE^z_{12}	30.22	13	TE^z_{90}	59.01
4	TM^z_{12}	30.22	14	TE^z_{14}	59.37
5	TE^z_{50}	32.78	15	TM^z_{14}	59.37
6	TE^z_{32}	35.46	16	TE^z_{34}	62.20
7	TM^z_{32}	35.46	17	TM^z_{34}	62.20
8	TE^z_{52}	44.10	18	TE^z_{92}	65.97
9	TM^z_{52}	44.10	19	TM^z_{92}	65.97
10	TE^z_{70}	45.89	20	TE^z_{54}	67.50

F. Evaluation of λ_y and λ_x Integrals

This appendix presents details of evaluating infinite integrals with respect to λ_y and λ_x when possible.

F.1 λ_y Integrals for Two-Layer Method

Focusing on the inner integral with respect to λ_y from (221), an attempt will be made to evaluate and simplify the following:

$$\int_{-\infty}^{\infty}\left[\frac{\left(1-e^{j\lambda_y b}\right)\left(1-e^{-j\lambda_y b}\right)}{\lambda_y^2}\right]\tilde{G}_{h1h1,xx}d\lambda_y = \int_{-\infty}^{\infty}\left[\frac{\left(1-e^{j\lambda_y b}\right)}{\lambda_y^2}\right]\tilde{G}_{h1h1,xx}d\lambda_y$$
$$+\int_{-\infty}^{\infty}\left[\frac{\left(1-e^{-j\lambda_y b}\right)}{\lambda_y^2}\right]\tilde{G}_{h1h1,xx}d\lambda_y \tag{F.1}$$

Recall that $\tilde{\overleftrightarrow{G}}_{h1h1}$ has both TE^z and TM^z components that can be analyzed separately. Taking advantage of that fact will simplify the analysis a bit.

TE^z **Contribution.**

From Chapter II, it was determined that

$$\tilde{G}^{\mathrm{TE}^z}_{h1h1,xx} = \lambda_x^2\left(j\frac{1}{2\lambda_{\rho\theta}^2 Z_{\theta 1}}\right)\Upsilon^{\theta}_{11} \tag{F.2}$$

$$\begin{aligned}\Upsilon^{\theta}_{11}\left({}^{z=0}_{z'=0}\right) &= \left[\frac{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\left[\cos\left(\lambda_{z\theta 1}h\right)+\cos\left(\lambda_{z\theta 1}h\right)\right]}{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)+Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)}\right.\\ &\qquad\left.+\frac{Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\left[-\sin\left(\lambda_{z\theta 1}h\right)-\sin\left(\lambda_{z\theta 1}h\right)\right]}{Z_{\theta 1}\cos\left(\lambda_{z\theta 2}T\right)\sin\left(\lambda_{z\theta 1}h\right)+Z_{\theta 2}\sin\left(\lambda_{z\theta 2}T\right)\cos\left(\lambda_{z\theta 1}h\right)}\right]\\ &= 2\frac{Z_{\theta 1}-Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)}{Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right)+Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)}\end{aligned} \tag{F.3}$$

Substituting (F.3) into (F.2) implies that

$$\tilde{G}^{\mathrm{TE}^z}_{h1h1,xx}\left({z=0 \atop z'=0}\right) = \lambda_x^2 \left(j\frac{1}{\lambda_{\rho\theta}^2 Z_{\theta 1}}\right)\left[\frac{Z_{\theta 1} - Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)}{Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right) + Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)}\right] \tag{F.4}$$

Substituting (F.4) into (F.1) implies that

$$\begin{aligned}
&\int\limits_{-\infty}^{\infty}\left[\frac{\left(1-e^{j\lambda_y b}\right)}{\lambda_y^2}\right]\tilde{G}_{h1h1,xx}dk_y + \int\limits_{-\infty}^{\infty}\left[\frac{\left(1-e^{-j\lambda_y b}\right)}{\lambda_y^2}\right]\tilde{G}_{h1h1,xx}d\lambda_y \\
&= \int\limits_{-\infty}^{\infty}\left[\frac{\left(1-e^{j\lambda_y b}\right)}{\lambda_y^2}\right]\left\{\lambda_x^2\left(j\frac{1}{\lambda_{\rho\theta}^2 Z_{\theta 1}}\right)\left[\frac{Z_{\theta 1} - Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)}{Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right) + Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)}\right]\right\}d\lambda_y \\
&\quad + \int\limits_{-\infty}^{\infty}\left[\frac{\left(1-e^{-j\lambda_y b}\right)}{\lambda_y^2}\right]\left\{\lambda_x^2\left(j\frac{1}{\lambda_{\rho\theta}^2 Z_{\theta 1}}\right)\left[\frac{Z_{\theta 1} - Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)}{Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right) + Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)}\right]\right\}d\lambda_y \\
&= \int\limits_{-\infty}^{\infty} j\frac{\lambda_x^2\lambda_{z\theta 1}\left(1-e^{j\lambda_y b}\right)\left[Z_{\theta 1} - Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]}{\omega\mu_{t1}\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)\left[Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right) + Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\right]}d\lambda_y \\
&\quad + \int\limits_{-\infty}^{\infty} j\frac{\lambda_x^2\lambda_{z\theta 1}\left(1-e^{-j\lambda_y b}\right)\left[Z_{\theta 1} - Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]}{\omega\mu_{t1}\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)\left[Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right) + Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right)\right]}d\lambda_y
\end{aligned} \tag{F.5}$$

In order to employ complex plane analysis, it must be shown that the integrands of the integrals in (F.5) vanishes on the λ_y controur $\to\infty$ in modulus. First, note that $\lambda_{z\theta\{1,2\}} = \sqrt{\omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2+\lambda_y^2\right)}$. That implies that

$$\begin{aligned}
\lim_{\lambda_y\to\infty}\lambda_{z\theta\{1,2\}} &= \lim_{\lambda_y\to\infty}\sqrt{\omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2+\lambda_y^2\right)} \\
&= \lim_{\lambda_y\to\infty}\sqrt{\lambda_y^2\left[\cancelto{0}{\frac{\omega^2\epsilon_{t\{1,2\}}\mu_{t\{1,2\}}}{\lambda_y^2}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\cancelto{0}{\frac{\lambda_x^2}{\lambda_y^2}}+1\right)\right]} \\
&= j\sqrt{\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}}\lim_{\lambda_y\to\infty}\lambda_y\to\infty
\end{aligned} \tag{F.6}$$

Next, note that $Z_{\theta\{1,2\}} = \frac{\omega\mu_{t\{1,2\}}}{\lambda_{z\theta\{1,2\}}}$. (F.6) implies that

$$\lim_{\lambda_y \to \infty} Z_{\theta\{1,2\}} = 0 \tag{F.7}$$

Next, the behavior of $\lambda_{z\theta 1} Z_{\theta 2}$ must be analyzed as $\lambda_y \to \infty$.

$$\begin{aligned}
\lim_{\lambda_y \to \infty} \lambda_{z\theta 1} Z_{\theta 2} &= \lim_{\lambda_y \to \infty} \frac{\omega\mu_{t2}\lambda_{z\theta 1}}{\lambda_{z\theta 2}} \\
&= \omega\mu_{t2} \lim_{\lambda_y \to \infty} \frac{j\cancel{\lambda_y}\sqrt{\frac{\mu_{t1}}{\mu_{z1}}}}{j\cancel{\lambda_y}\sqrt{\frac{\mu_{t2}}{\mu_{z2}}}} \\
&= \omega\mu_{t2}\sqrt{\frac{\mu_{t1}\mu_{z2}}{\mu_{t2}\mu_{z1}}}
\end{aligned} \tag{F.8}$$

Next, the behavior of $\lambda_y Z_{\theta\{1,2\}}$ must be analyzed as $\lambda_y \to \infty$.

$$\begin{aligned}
\lim_{\lambda_y \to \infty} \lambda_y Z_{\theta\{1,2\}} &= \lim_{\lambda_y \to \infty} \frac{\lambda_y \omega\mu_{t\{1,2\}}}{\lambda_{z\theta\{1,2\}}} \\
&= \lim_{\lambda_y \to \infty} \frac{\cancel{\lambda_y}\omega\mu_{t\{1,2\}}}{j\cancel{\lambda_y}\sqrt{\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}}} \\
&= -j\omega\sqrt{\mu_{t\{1,2\}}\mu_{z\{1,2\}}}
\end{aligned} \tag{F.9}$$

Finally, the behavior of $\tan\left(\lambda_{z\theta\{1,2\}}C\right)$ behaves as $\lambda_y \to \infty$, where $C \in \mathbb{R}$.

$$\begin{aligned}
\lim_{\lambda_y \to \infty} \tan\left(\lambda_{z\theta\{1,2\}}C\right) &= \lim_{\lambda_y \to \infty} \tan\left(j\lambda_y C\sqrt{\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}}\right) \\
&= j \lim_{\lambda_y \to \infty} \bcancel{\tanh\left(\lambda_y C\sqrt{\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}}\right)}^{\,1} \\
&= j
\end{aligned} \tag{F.10}$$

Studying the first integrand,

$$\begin{aligned}
&\lim_{\lambda_y\to\infty} \frac{j\lambda_x^2\lambda_{z\theta1}\left(1-e^{j\lambda_y b}\right)\left[Z_{\theta1}-Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\tan\left(\lambda_{z\theta1}h\right)\right]}{\omega\mu_{t1}\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)\left[Z_{\theta1}\tan\left(\lambda_{z\theta1}h\right)+Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\right]}\\
&=\lim_{\lambda_y\to\infty} \frac{j\lambda_x^2\left(1-e^{j\lambda_y b}\right)\left[\lambda_{z\theta1}Z_{\theta1}-\lambda_{z\theta1}Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\tan\left(\lambda_{z\theta1}h\right)\right]}{\omega\mu_{t1}\lambda_y\left(\lambda_x^2+\lambda_y^2\right)\left[\lambda_y Z_{\theta1}\tan\left(\lambda_{z\theta1}h\right)+\lambda_y Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\right]}\\
&=\lim_{\lambda_y\to\infty} \frac{j\lambda_x^2\left(1-e^{j\lambda_y b}\right)\left[\omega\mu_{t1}-j^2\omega\mu_{t2}\sqrt{\frac{\mu_{t1}\mu_{z2}}{\mu_{t2}\mu_{z1}}}\right]}{\omega\mu_{t1}\lambda_y\left(\lambda_x^2+\lambda_y^2\right)\left[\left(-j\omega\sqrt{\mu_{t1}\mu_{z1}}\right)j+\left(-j\omega\sqrt{\mu_{t2}\mu_{z2}}\right)j\right]}\\
&=j\frac{\lambda_x^2}{\omega\mu_{t1}}\left[\frac{\mu_{t1}+\mu_{t2}\sqrt{\frac{\mu_{t1}\mu_{z2}}{\mu_{t2}\mu_{z1}}}}{\sqrt{\mu_{t1}\mu_{z1}}+\sqrt{\mu_{t2}\mu_{z2}}}\right]\lim_{\lambda_y\to\infty}\frac{\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)}\\
&=0 \iff \Im\left\{\lambda_y\right\}>0\Rightarrow \text{UHPC}
\end{aligned} \tag{F.11}$$

Similarly for the second integrand, it can be shown that the integrand decays to zero if and only if the imaginary part of $\lambda_y < 0$. Thus LHPC is required to evaluate the second integral in (F.5). Examining (F.5), it can be seen that there are several sets of poles that must be accounted for when evaluating each integral using CIT and CIF. As depicted in fig. 21, there is a second-order pole $C_0^{\pm}$ at $\lambda_y = 0$, simple poles $C_{j\lambda_x}^{\pm}$ at $\lambda_y = \pm j\lambda_x$, and an infinite number of poles $C_\ell^{\pm}$ satisfying the transcendental equation

$$Z_{\theta1}\tan\left(\lambda_{z\theta1}h\right)+Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)=0 \tag{F.12}$$

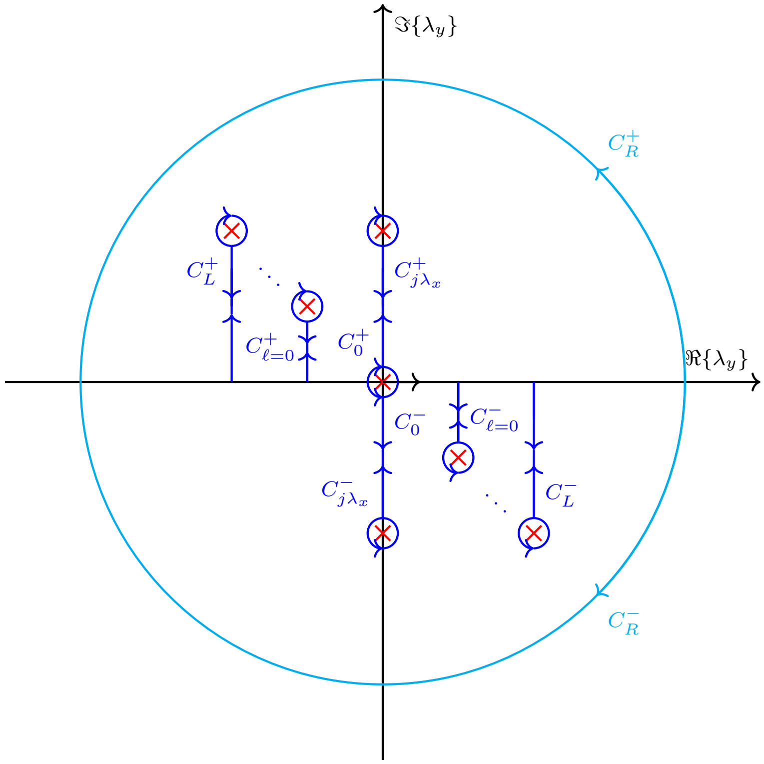

Figure 21. Complex poles (red) of the transverse spatial frequency domain principal scalar potential Green functions, deformation contours around those poles (blue) and closure contours as $R \to \infty$ (cyan) in the complex λ_y-plane.

Recall from Chapter II that for UHPC

$$
\begin{aligned}
&\lim_{R\to\infty} \int_{-R}^{R} + \oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \oint_{\sum C_\ell^+} + \cancelto{0}{\oint_{C_R^+}} = 0 \\
&\Rightarrow \int_{-\infty}^{\infty} = \oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \sum_{\ell=0}^{L} \oint_{C_\ell^+}
\end{aligned}
\tag{F.13}
$$

Also, recall from Chapter II that for LHPC

$$\lim_{R\to\infty}\int_{-R}^{R} + \oint_{C_0^-} + \oint_{C_{-j\lambda_x}^-} + \oint_{\sum C_\ell^-} + \cancelto{0}{\oint_{C_R^-}} = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} = -\oint_{C_0^-} - \oint_{C_{-j\lambda_x}^-} - \sum_{\ell=0}^{L}\oint_{C_\ell^-} \tag{F.14}$$

To evaluate the full integral with respect to λ_y represented in (F.5), add the UHPC and LHPC components together such that

$$\int_{-\infty}^{\infty} = \oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \sum_{\ell=0}^{L}\oint_{C_\ell^+} - \oint_{C_0^-} - \oint_{C_{-j\lambda_x}^-} - \sum_{\ell=0}^{L}\oint_{C_\ell^-} \tag{F.15}$$

It is important to note that the deformation contour C_0^+ around the pole at $\lambda_y = 0$ (or any purely-real pole, for that matter) only requires a semi-circular path in UHPC, thus the usual value obtained from CIF is halved. The other half of the deformation contour will be accounted for in LHPC since all poles residing on the real axis will be included in both UHPC and LHPC. Let us study each of these components' contributions, beginning with C_0^+.

The value of the second-order pole contribution at $\lambda_y = 0$ must be divided in half since the contour around the pole is semi-circular. Thus, from Appendix A,

$$\oint_{C_0^+} \frac{N(\lambda_y)}{D(\lambda_y)} d\lambda_y = j\pi \frac{\partial}{\partial \lambda_y} N(\lambda_y)\Big|_{\lambda_y=0} \tag{F.16}$$

where

$$
N\left(\lambda_y\right) = j\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\frac{\lambda_{z\theta 1}\left(1-e^{j\lambda_y b}\right)\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]}{\left(\lambda_x^2+\lambda_y^2\right)\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]} \tag{F.17}
$$

$$
D\left(\lambda_y\right) = \lambda_y^2 \tag{F.18}
$$

which implies that

$$
\begin{aligned}
\oint_{C_0^+} &= j\pi \lim_{\lambda_y\to 0}\frac{\partial}{\partial\lambda_y}\left\{j\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\frac{\lambda_{z\theta 1}\left(1-e^{j\lambda_y b}\right)\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]}{\left(\lambda_x^2+\lambda_y^2\right)\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]}\right\} \\
&= -\pi\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\lim_{\lambda_y\to 0}\frac{\frac{\partial}{\partial\lambda_y}\left\{\lambda_{z\theta 1}\left(1-e^{j\lambda_y b}\right)\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]\right\}}{\lambda_\rho^2\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]} \\
&= -\pi\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\lim_{\lambda_y\to 0}\left\{\frac{\lambda_{z\theta 1}\frac{\partial}{\partial\lambda_y}\left\{\left(1-e^{j\lambda_y b}\right)\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]\right\}}{\lambda_\rho^2\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]}\right. \\
&\qquad\left.+\frac{\cancelto{0}{\left(1-e^{j\lambda_y b}\right)}\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]\frac{\partial}{\partial\lambda_y}\lambda_{z\theta 1}}{\lambda_\rho^2\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]}\right\} \\
&= -\pi\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\lim_{\lambda_y\to 0}\left\{\frac{\lambda_{z\theta 1}\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]\frac{\partial}{\partial\lambda_y}\left(1-e^{j\lambda_y b}\right)}{\lambda_\rho^2\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]}\right. \\
&\qquad\left.+\frac{\lambda_{z\theta 1}\cancelto{0}{\left(1-e^{j\lambda_y b}\right)}\frac{\partial}{\partial\lambda_y}\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]}{\lambda_\rho^2\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]}\right\} \\
&= -\pi\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\lim_{\lambda_y\to 0}\frac{\lambda_{z\theta 1}\left[1-\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\tan\left(\lambda_{z\theta 1}h\right)\right]\left(-jbe^{j\lambda_y b}\right)}{\lambda_\rho^2\left[\tan\left(\lambda_{z\theta 1}h\right)+\frac{Z_{\theta 2}}{Z_{\theta 1}}\tan\left(\lambda_{z\theta 2}T\right)\right]} \\
&= jb\pi\left[\frac{\cancel{\lambda_x^2}}{\omega\mu_{t1}}\right]\frac{\lambda_{z\theta 1}^*\left[Z_{\theta 1}^*-Z_{\theta 2}^*\tan\left(\lambda_{z\theta 2}^*T\right)\tan\left(\lambda_{z\theta 1}^*h\right)\right]}{\cancel{\lambda_x^2}\left[Z_{\theta 1}^*\tan\left(\lambda_{z\theta 1}^*h\right)+Z_{\theta 2}^*\tan\left(\lambda_{z\theta 2}^*T\right)\right]} \\
&= j\pi\left[\frac{b}{Z_{\theta 1}^*}\right]\left[\frac{Z_{\theta 1}^*-Z_{\theta 2}^*\tan\left(\lambda_{z\theta 2}^*T\right)\tan\left(\lambda_{z\theta 1}^*h\right)}{Z_{\theta 1}^*\tan\left(\lambda_{z\theta 1}^*h\right)+Z_{\theta 2}^*\tan\left(\lambda_{z\theta 2}^*T\right)}\right]
\end{aligned} \tag{F.19}
$$

where

$$Z^*_{\theta\{1,2\}} = \frac{\omega\mu_{t\{1,2\}}}{\lambda^*_{z\theta\{1,2\}}}, \lambda^*_{z\theta\{1,2\}} = \sqrt{k^2_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\lambda_x^2} \tag{F.20}$$

Note that the only difference for LHPC around the pole at $\lambda_y = 0$ is that the exponential term becomes $\left(1 - e^{-j\lambda_y b}\right)$. Due to the derivative step, the sign change in the exponent causes a sign change in the final value, thus

$$\oint_{C_0^-} = -\oint_{C_0^+} \tag{F.21}$$

Next, analyzing the contribution of the deformation contour $C^+_{j\lambda_x}$ around the simple pole at $k_y = j\lambda_x$ implies that

$$\oint_{C^+_{j\lambda_x}} = j2\pi \lim_{k_y \to j\lambda_x} \left\{ j \left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right] \frac{\lambda_{z\theta1}\left(1 - e^{j\lambda_y b}\right)\left[Z_{\theta1} - Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\tan\left(\lambda_{z\theta1}h\right)\right]}{\lambda_y^2\left(\lambda_y + j\lambda_x\right)\left[Z_{\theta1}\tan\left(\lambda_{z\theta1}h\right) + Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\right]} \right\} \tag{F.22}$$

To aid in evaluating the limit as $\lambda_y \to j\lambda_x$, analyze how $\lambda_{z\theta\{1,2\}}$ behaves in the limit.

$$\begin{aligned}
\lim_{\lambda_y \to j\lambda_x} \lambda_{z\theta\{1,2\}} &= \sqrt{k^2_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2 + \lambda_y^2\right)} \\
&= \sqrt{k^2_{t\{1,2\}} - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\cancelto{0}{\left(\lambda_x^2 - \lambda_x^2\right)}} \\
&= \pm k_{t\{1,2\}}
\end{aligned} \tag{F.23}$$

The sign of $k_{t\{1,2\}}$ is unimportant because (F.22) is even with respect to $k_{t\{1,2\}}$.

Substituting $\lambda_{z\theta\{1,2\}} = k_{t\{1,2\}}$ into (F.22) implies that

$$\begin{aligned}\oint\limits_{C^{+}_{j\lambda_x}} &= j2\pi \left\{ j \left[\frac{\lambda_x^2}{\omega\mu_{t1}} \right] \frac{k_{t1}\left(1 - e^{-\lambda_x b}\right)\left[\frac{\omega\mu_{t1}}{k_{t1}} - \frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)\tan\left(k_{t1}h\right)\right]}{-\lambda_x^2\left(j\lambda_x + j\lambda_x\right)\left[\frac{\omega\mu_{t1}}{k_{t1}}\tan\left(k_{t1}h\right) + \frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)\right]} \right\} \\ &= -j\pi \left[\frac{k_{t1}\left(1 - e^{-\lambda_x b}\right)}{\omega\mu_{t1}\lambda_x} \right] \left[\frac{\frac{\omega\mu_{t1}}{k_{t1}} - \frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)\tan\left(k_{t1}h\right)}{\frac{\omega\mu_{t1}}{k_{t1}}\tan\left(k_{t1}h\right) + \frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)} \right]\end{aligned} \tag{F.24}$$

For LHPC around the pole at $\lambda_y = -j\lambda_x$, the sign of the exponent changes twice in comparison to the UHPC case due to the additional sign change of the pole, so the numerators are identical. Additionally, the $\lambda_{z\theta\{1,2\}}$, and subsequently $k_{t\{1,2\}}$, terms are identical. However, one denominator term changes such that

$$\left(-j\lambda_x - j\lambda_x\right) = -j2\lambda_x \tag{F.25}$$

Thus,

$$\oint\limits_{C^{-}_{-j\lambda_x}} = -\oint\limits_{C^{+}_{j\lambda_x}} \tag{F.26}$$

Finally, analyzing the contribution of the deformation contours C^{+}_{ℓ} around the $\lambda_y = -\lambda_{y\ell}$ values that satisfy $Z_{\theta 1}\tan\left(\lambda_{z\theta 1}h\right) + Z_{\theta 2}\tan\left(\lambda_{z\theta 2}T\right) = 0$. Due to the periodic nature of the tangent function, only one pole should exist at each $\lambda_{y\ell}$ value, and thus each $\lambda_{y\ell}$ pole is first-order. Since these poles are not in the familiar $(\lambda_y - \lambda_{y0})$ form, the generalized CIF derived in Appendix A will be used, which states that

$$\oint\limits_{C^{+}_{\ell}} \frac{N\left(\lambda_y\right)}{D\left(\lambda_y\right)} d\lambda_y = j2\pi \left. \frac{N\left(\lambda_y\right)}{\frac{\partial}{\partial\lambda_y} D\left(\lambda_y\right)} \right|_{\lambda_y = -\lambda_{y\theta\ell}} \tag{F.27}$$

From (F.5), it can be seen that in the neighborhood of $\lambda_y = -\lambda_{y\theta\ell}$

$$N\left(\lambda_y\right) = j\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\frac{\lambda_{z\theta1}\left(1-e^{j\lambda_y b}\right)\left[Z_{\theta1} - Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right)\tan\left(\lambda_{z\theta1}h\right)\right]}{\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)} \tag{F.28}$$

$$D\left(\lambda_y\right) = Z_{\theta1}\tan\left(\lambda_{z\theta1}h\right) + Z_{\theta2}\tan\left(\lambda_{z\theta2}T\right) \tag{F.29}$$

In order to simplify the derivative, it is helpful to express the denominator in terms sines and cosines versus tangent functions. Thus

$$\begin{aligned}
N\left(\lambda_y\right) &= j\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\frac{\lambda_{z\theta1}\left(1-e^{j\lambda_y b}\right)\left[Z_{\theta1} - Z_{\theta2}\frac{\sin(\lambda_{z\theta2}T)\sin(\lambda_{z\theta1}h)}{\cos(\lambda_{z\theta2}T)\cos(\lambda_{z\theta1}h)}\right]}{\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)}\\
D\left(\lambda_y\right) &= \frac{Z_{\theta1}\sin\left(\lambda_{z\theta1}h\right)}{\cos\left(\lambda_{z\theta1}h\right)} + \frac{Z_{\theta2}\sin\left(\lambda_{z\theta2}T\right)}{\cos\left(\lambda_{z\theta2}T\right)}\\
\Rightarrow N\left(\lambda_y\right) &= j\left[\frac{\lambda_x^2}{\omega\mu_{t1}}\right]\left[\frac{\lambda_{z\theta1}\left(1-e^{j\lambda_y b}\right)\left[Z_{\theta1}\cos\left(\lambda_{z\theta1}h\right)\cos\left(\lambda_{z\theta2}T\right)\right]}{\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)}\right.\\
&\qquad\left.+\frac{\lambda_{z\theta1}\left(1-e^{j\lambda_y b}\right)\left[-Z_{\theta2}\sin\left(\lambda_{z\theta1}h\right)\sin\left(\lambda_{z\theta2}T\right)\right]}{\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)}\right]
\end{aligned} \tag{F.30}$$

$$D\left(\lambda_y\right) = Z_{\theta1}\sin\left(\lambda_{z\theta1}h\right)\cos\left(\lambda_{z\theta2}T\right) + Z_{\theta2}\cos\left(\lambda_{z\theta1}h\right)\sin\left(\lambda_{z\theta2}T\right) \tag{F.31}$$

Recalling that

$$\begin{aligned}
\lambda_{z\theta\{1,2\}} &= \pm\sqrt{k_{t\{1,2\}}^2 - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\lambda_{\rho\theta}^2}\\
&= \pm\sqrt{k_{t\{1,2\}}^2 - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2+\lambda_y^2\right)}
\end{aligned}$$

and that

$$Z_{\theta\{1,2\}} = \frac{\omega\mu_{t\{1,2\}}}{\lambda_{z\theta\{1,2\}}}$$

implies that

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda_y} D\left(\lambda_y\right)=\frac{\partial}{\partial \lambda_y}\left[Z_{\theta 1} \sin \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right)\right]+\frac{\partial}{\partial \lambda_y}\left[Z_{\theta 2} \cos \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right)\right] \\
&=\sin \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} Z_{\theta 1}+Z_{\theta 1} \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \sin \left(\lambda_{z \theta 1} h\right) \\
&\quad+Z_{\theta 1} \sin \left(\lambda_{z \theta 1} h\right) \frac{\partial}{\partial \lambda_y} \cos \left(\lambda_{z \theta 2} T\right)+\cos \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} Z_{\theta 2} \\
&\quad+Z_{\theta 2} \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \cos \left(\lambda_{z \theta 1} h\right)+Z_{\theta 2} \cos \left(\lambda_{z \theta 1} h\right) \frac{\partial}{\partial \lambda_y} \sin \left(\lambda_{z \theta 2} T\right) \\
&=\sin \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y}\left[\frac{\omega \mu_{t 1}}{\lambda_{z \theta 1}}\right]+Z_{\theta 1} \cos \left(\lambda_{z \theta 2} T\right) \cos \left(\lambda_{z \theta 1} h\right) \frac{\partial}{\partial \lambda_y}\left[\lambda_{z \theta 1} h\right] \\
&\quad-Z_{\theta 1} \sin \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y}\left[\lambda_{z \theta 2} T\right]+\cos \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y}\left[\frac{\omega \mu_{t 2}}{\lambda_{z \theta 2}}\right] \\
&\quad-Z_{\theta 2} \sin \left(\lambda_{z \theta 2} T\right) \sin \left(\lambda_{z \theta 1} h\right) \frac{\partial}{\partial \lambda_y}\left[\lambda_{z \theta 1} h\right]+Z_{\theta 2} \cos \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y}\left[\lambda_{z \theta 2} T\right] \\
&=\left[-\frac{\omega \mu_{t 1}}{\lambda_{z \theta 1}^2}\right] \sin \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 1}+h Z_{\theta 1} \cos \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 1} \\
&\quad-T Z_{\theta 1} \sin \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 2}+\left[-\frac{\omega \mu_{t 2}}{\lambda_{z \theta 2}^2}\right] \cos \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 2} \\
&\qquad-h Z_{\theta 2} \sin \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 1}+T Z_{\theta 2} \cos \left(\lambda_{z \theta 1} h\right) \cos \left(\lambda_{z \theta 2} T\right) \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 2} \\
&=\left\{Z_{\theta 1} \cos \left(\lambda_{z \theta 2} T\right)\left[h \cos \left(\lambda_{z \theta 1} h\right)-\frac{\sin \left(\lambda_{z \theta 1} h\right)}{\lambda_{z \theta 1}}\right]\right. \\
&\left.-h Z_{\theta 2} \sin \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right)\right\} \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 1}+\left\{Z_{\theta 2} \cos \left(\lambda_{z \theta 1} h\right)\left[T \cos \left(\lambda_{z \theta 2} T\right)\right.\right. \\
&\qquad\qquad\left.\left.-\frac{\sin \left(\lambda_{z \theta 2} T\right)}{\lambda_{z \theta 2}}\right]-T Z_{\theta 1} \sin \left(\lambda_{z \theta 1} h\right) \sin \left(\lambda_{z \theta 2} T\right)\right\} \frac{\partial}{\partial \lambda_y} \lambda_{z \theta 2}
\end{aligned} \tag{F.32}
$$

In order to find $\frac{\partial}{\partial \lambda_y} D\left(\lambda_y\right)$, it will first be helpful to find $\frac{\partial}{\partial \lambda_y} \lambda_{z \theta\{1,2\}}$.

$$
\begin{aligned}
\frac{\partial}{\partial \lambda_y} \lambda_{z \theta\{1,2\}} &=\pm \frac{\partial}{\partial \lambda_y} \sqrt{k_{t\{1,2\}}^2-\frac{\mu_{t\{1,2\}}}{\mu_{z(1,2)}}\left(\lambda_x^2+\lambda_y^2\right)} \\
&=\pm \frac{1}{2 \sqrt{k_{t\{1,2\}}^2-\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2+\lambda_y^2\right)}} \frac{\partial}{\partial \lambda_y}\left[k_{t\{1,2\}}^2-\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2+\lambda_y^2\right)\right] \\
&=\pm \frac{1}{2 \sqrt{k_{t\{1,2\}}^2-\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2+\lambda_y^2\right)}}\left[-\frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(2 \lambda_y\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mp \frac{\lambda_y \mu_{t\{1,2\}}}{\mu_{z\{1,2\}} \sqrt{k_{t\{1,2\}}^2 - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\left(\lambda_x^2 + \lambda_y^2\right)}} \\
&= -\frac{\lambda_y \mu_{t\{1,2\}}}{\mu_{z\{1,2\}} \lambda_{z\theta\{1,2\}}} \\
&= -\frac{\lambda_y Z_{\theta\{1,2\}}}{\omega \mu_{z\{1,2\}}}
\end{aligned} \tag{F.33}
$$

Therefore, substituting (F.33) into (F.32) implies that

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda_y} D\left(\lambda_y\right) = \left[-\frac{\lambda_y Z_{\theta 1}}{\omega \mu_{z1}}\right] \left\{ Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \left[h \cos\left(\lambda_{z\theta 1} h\right) - \frac{\sin\left(\lambda_{z\theta 1} h\right)}{\lambda_{z\theta 1}} \right] \right. \\
&\left. - h Z_{\theta 2} \sin\left(\lambda_{z\theta 1} h\right) \sin\left(\lambda_{z\theta 2} T\right) \right\} + \left[-\frac{\lambda_y Z_{\theta 2}}{\omega \mu_{z2}}\right] \left\{ Z_{\theta 2} \cos\left(\lambda_{z\theta 1} h\right) \left[T \cos\left(\lambda_{z\theta 2} T\right) \right. \right. \\
&\qquad\qquad \left. \left. - \frac{\sin\left(\lambda_{z\theta 2} T\right)}{\lambda_{z\theta 2}} \right] - T Z_{\theta 1} \sin\left(\lambda_{z\theta 1} h\right) \sin\left(\lambda_{z\theta 2} T\right) \right\} \\
&= \frac{\lambda_y Z_{\theta 1}}{\omega \mu_{z1}} \left\{ Z_{\theta 1} \cos\left(\lambda_{z\theta 2} T\right) \left[\frac{\sin\left(\lambda_{z\theta 1} h\right)}{\lambda_{z\theta 1}} - h \cos\left(\lambda_{z\theta 1} h\right) \right] \right. \\
&\left. + h Z_{\theta 2} \sin\left(\lambda_{z\theta 1} h\right) \sin\left(\lambda_{z\theta 2} T\right) \right\} + \frac{\lambda_y Z_{\theta 2}}{\omega \mu_{z2}} \left\{ Z_{\theta 2} \cos\left(\lambda_{z\theta 1} h\right) \left[\frac{\sin\left(\lambda_{z\theta 2} T\right)}{\lambda_{z\theta 2}} \right. \right. \\
&\qquad\qquad \left. \left. - T \cos\left(\lambda_{z\theta 2} T\right) \right] + T Z_{\theta 1} \sin\left(\lambda_{z\theta 1} h\right) \sin\left(\lambda_{z\theta 2} T\right) \right\}
\end{aligned} \tag{F.34}
$$

which implies that

$$
\oint_{C_\ell^+} \frac{N}{D} d\lambda_y = - \left[\frac{2\pi \omega \mu_{z1} \mu_{z2} \lambda_x^2 \left(1 - e^{j\lambda_y b}\right)}{\lambda_y^3 \left(\lambda_x^2 + k_y^2\right) D^{\mathrm{TE}^z}} \right] \left[1 - \frac{Z_{\theta 2}}{Z_{\theta 1}} \tan\left(\lambda_{z\theta 1} h\right) \tan\left(\lambda_{z\theta 2} T\right) \right] \Bigg|_{\lambda_y = -\lambda_{y\theta\ell}} \tag{F.35}
$$

where

$$
\begin{aligned}
D^{\mathrm{TE}^z} = Z_{\theta 1}^2 \mu_{z2} \left[\frac{\tan\left(\lambda_{z\theta 1} h\right)}{\lambda_{z\theta 1}} - h \right] + Z_{\theta 2}^2 \mu_{z1} \left[\frac{\tan\left(k_{z\theta 2} T\right)}{\lambda_{z\theta 2}} - T \right] \\
+ Z_{\theta 1} Z_{\theta 2} \tan\left(\lambda_{z\theta 1} h\right) \tan\left(\lambda_{z\theta 2} T\right) \left[h \mu_{z2} + T \mu_{z1} \right]
\end{aligned} \tag{F.36}
$$

For LHPC, note that $\lambda_y \to -\lambda_y$. Recall that $\lambda_{z\theta(1,2)}$, and therefore $Z_{\theta(1,2)}$, are even with respect to λ_y. Therefore, all but the exponential term of the numerator and all but the λ_y^3 term of the denominator remain identical. The sign of the exponential term changes twice, thus the entire numerator term is identical to that in UHPC. However, due to the λ_y^3 term in the denominator, there is a sign change in LHPC versus UHPC. Thus,

$$\sum_{\ell=0}^{L} \oint_{C_\ell^+} = -\sum_{\ell=0}^{L} \oint_{C_\ell^-} \tag{F.37}$$

Thus, it has been shown that

$$\int_{-\infty}^{\infty} = 2\left(\oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \sum_{\ell=0}^{L} \oint_{C_\ell^+} \right) \tag{F.38}$$

Substituting the TE^z components above into (221) implies that

$$\begin{aligned} A_{m,n}^{\mathrm{TE}^z} &= \delta_{m,n}\left(\frac{ab}{2}\right)\left(M_{xm}^h\right)^2 \\ &- \frac{Z_n M_{xm}^h M_{xn}^h k_{xm} k_{xn}}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\left(1-(-1)^{v_m} e^{j\lambda_x a}\right)\left(1-(-1)^{v_n} e^{-j\lambda_x a}\right)}{\left(\lambda_x + k_{xm}\right)\left(\lambda_x - k_{xm}\right)\left(\lambda_x + k_{xn}\right)\left(\lambda_x - k_{xn}\right)} \right] \Omega^{\mathrm{TE}^z} d\lambda_x \end{aligned} \tag{F.39}$$

$$\begin{aligned} \Omega^{\mathrm{TE}^z} = j &\left[\frac{b}{Z_{\theta1}^*}\right]\left[\frac{Z_{\theta1}^* - Z_{\theta2}^* \tan\left(\lambda_{z\theta2}^* T\right)\tan\left(\lambda_{z\theta1}^* h\right)}{Z_{\theta1}^* \tan\left(\lambda_{z\theta1}^* h\right) + Z_{\theta2}^* \tan\left(\lambda_{z\theta2}^* T\right)}\right] \\ &- j\left[\frac{k_{t1}\left(1-e^{-\lambda_x b}\right)}{\omega\mu_{t1} k_x}\right]\left[\frac{\frac{\omega\mu_{t1}}{k_{t1}} - \frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)\tan\left(k_{t1}h\right)}{\frac{\omega\mu_{t1}}{k_{t1}}\tan\left(k_{t1}h\right) + \frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)}\right] \\ - \sum_{\ell=1}^{L} &\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right) D^{\mathrm{TE}^z}}\right]\left(1-e^{j\lambda_y b}\right)\left[1-\frac{Z_{\theta2}}{Z_{\theta1}}\tan\left(\lambda_{z\theta1}h\right)\tan\left(\lambda_{z\theta2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}} \end{aligned} \tag{F.40}$$

$$Z_{\theta\{1,2\}}^* = \frac{\omega\mu_{t\{1,2\}}}{\lambda_{z\theta\{1,2\}}^*}, \lambda_{z\theta\{1,2\}}^* = \sqrt{k_{t\{1,2\}}^2 - \frac{\mu_{t\{1,2\}}}{\mu_{z\{1,2\}}}\lambda_x^2} \tag{F.41}$$

$$D^{\mathrm{TE}^z} = Z_{\theta 1}^2 \mu_{z2} \left[\frac{\tan(\lambda_{z\theta 1} h)}{\lambda_{z\theta 1}} - h \right] + Z_{\theta 2}^2 \mu_{z1} \left[\frac{\tan(\lambda_{z\theta 2} T)}{\lambda_{z\theta 2}} - T \right] + Z_{\theta 1} Z_{\theta 2} \tan(\lambda_{z\theta 1} h) \tan(k_{z\theta 2} T) \left[h\mu_{z2} + T\mu_{z1} \right] \tag{F.42}$$

TM^z **Contribution.**

From Chapter II, it was determined that

$$\tilde{G}_{h1h1,xx}^{\mathrm{TM}^z} = \lambda_y^2 \left(j \frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \Upsilon_{11}^{\psi} \tag{F.43}$$

$$\begin{aligned}
\Upsilon_{11}^{\psi} \left({}^{z=0}_{z'=0} \right) &= \left[\frac{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \left[\cos(\lambda_{z\psi 1}(h - |z - z'|)) + \cos(\lambda_{z\psi 1}(h - z - z')) \right]}{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h)} \right. \\
&\quad \left. + \frac{Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \left[-\sin(\lambda_{z\psi 1}(h - |z - z'|)) - \sin(\lambda_{z\psi 1}(h - z - z')) \right]}{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h)} \right] \\
&= \left[\frac{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \left[\cos(\lambda_{z\psi 1} h) + \cos(\lambda_{z\psi 1} h) \right]}{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h)} \right. \\
&\quad \left. + \frac{Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \left[-\sin(\lambda_{z\psi 1} h) - \sin(\lambda_{z\psi 1} h) \right]}{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h)} \right] \\
&= 2 \frac{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h) - Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h)}{Z_{\psi 1} \cos(\lambda_{z\psi 2} T) \sin(\lambda_{z\psi 1} h) + Z_{\psi 2} \sin(\lambda_{z\psi 2} T) \cos(\lambda_{z\psi 1} h)} \\
&= 2 \frac{Z_{\psi 1} - Z_{\psi 2} \tan(\lambda_{z\psi 2} T) \tan(\lambda_{z\psi 1} h)}{Z_{\psi 1} \tan(\lambda_{z\psi 1} h) + Z_{\psi 2} \tan(\lambda_{z\psi 2} T)}
\end{aligned} \tag{F.44}$$

Substituting (F.44) into (F.43) implies that

$$\begin{aligned}
\tilde{G}_{h1h1,xx}^{\mathrm{TM}^z} \left({}^{z=0}_{z'=0} \right) &= \lambda_y^2 \left(j \frac{1}{2\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \left[2 \frac{Z_{\psi 1} - Z_{\psi 2} \tan(\lambda_{z\psi 2} T) \tan(\lambda_{z\psi 1} h)}{Z_{\psi 1} \tan(\lambda_{z\psi 1} h) + Z_{\psi 2} \tan(\lambda_{z\psi 2} T)} \right] \\
&= \lambda_y^2 \left(j \frac{1}{\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \left[\frac{Z_{\psi 1} - Z_{\psi 2} \tan(\lambda_{z\psi 2} T) \tan(\lambda_{z\psi 1} h)}{Z_{\psi 1} \tan(\lambda_{z\psi 1} h) + Z_{\psi 2} \tan(\lambda_{z\psi 2} T)} \right]
\end{aligned} \tag{F.45}$$

Substituting (F.45) into (F.1) implies that

$$\int_{-\infty}^{\infty} \left[\frac{\left(1 - e^{j\lambda_y b}\right)}{\lambda_y^2} \right] \tilde{G}_{h1h1,xx} d\lambda_y + \int_{-\infty}^{\infty} \left[\frac{\left(1 - e^{-j\lambda_y b}\right)}{\lambda_y^2} \right] \tilde{G}_{h1h1,xx} d\lambda_y$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left[\frac{\left(1 - e^{j\lambda_y b}\right)}{\lambda_y^2} \right] \left\{ \lambda_y^2 \left(j \frac{1}{\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \left[\frac{Z_{\psi 1} - Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right) \tan\left(\lambda_{z\psi 1} h\right)}{Z_{\psi 1} \tan\left(\lambda_{z\psi 1} h\right) + Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right)} \right] \right\} d\lambda_y \\
&\quad + \int_{-\infty}^{\infty} \left[\frac{\left(1 - e^{-j\lambda_y b}\right)}{\lambda_y^2} \right] \left\{ \lambda_y^2 \left(j \frac{1}{\lambda_{\rho\psi}^2 Z_{\psi 1}} \right) \left[\frac{Z_{\psi 1} - Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right) \tan\left(\lambda_{z\psi 1} h\right)}{Z_{\psi 1} \tan\left(\lambda_{z\psi 1} h\right) + Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right)} \right] \right\} d\lambda_y \\
&= \int_{-\infty}^{\infty} \frac{j\omega\epsilon_{t1}\left(1 - e^{j\lambda_y b}\right)\left[Z_{\psi 1} - Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right) \tan\left(\lambda_{z\psi 1} h\right)\right]}{\lambda_{z\psi 1}\left(\lambda_x^2 + \lambda_y^2\right)\left[Z_{\psi 1} \tan\left(\lambda_{z\psi 1} h\right) + Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right)\right]} d\lambda_y \\
&\quad + \int_{-\infty}^{\infty} \frac{j\omega\epsilon_{t1}\left(1 - e^{-j\lambda_y b}\right)\left[Z_{\psi 1} - Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right) \tan\left(\lambda_{z\psi 1} h\right)\right]}{\lambda_{z\psi 1}\left(\lambda_x^2 + \lambda_y^2\right)\left[Z_{\psi 1} \tan\left(\lambda_{z\psi 1} h\right) + Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right)\right]} d\lambda_y
\end{aligned} \tag{F.46}
$$

As with the TE^z analysis, the first integral can be evaluated via UHPC and the second integral can be evaluated via LHPC. Through similar analyses, it can be shown that

$$
\int_{-\infty}^{\infty} = 2 \left(\oint_{C_{j\lambda_x}^{+}} + \sum_{\ell=0}^{L} \oint_{C_\ell^{+}} \right) \tag{F.47}
$$

Note that since the λ_y^2 term in the denominator was canceled with a λ_y^2 term in the numerator, there is no longer a pole at $\lambda_y = 0$.

First, looking at the $C_{j\lambda_x}^{+}$ pole implies that

$$
\oint_{C_{j\lambda_x}^{+}} = j2\pi \lim_{\lambda_y \to j\lambda_x} \left\{ \frac{j\omega\epsilon_{t1}\left(1 - e^{j\lambda_y b}\right)\left[Z_{\psi 1} - Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right) \tan\left(\lambda_{z\psi 1} h\right)\right]}{\lambda_{z\psi 1}\left(j\lambda_x + \lambda_y\right)\left[Z_{\psi 1} \tan\left(\lambda_{z\psi 1} h\right) + Z_{\psi 2} \tan\left(\lambda_{z\psi 2} T\right)\right]} \right\} \tag{F.48}
$$

Analyzing how $\lambda_{z\psi\{1,2\}}$ behaves as $\lambda_y \to j\lambda_x$ implies that

$$
\begin{aligned}
\lim_{\lambda_y \to j\lambda_x} \lambda_{z\psi\{1,2\}} &= \sqrt{k_{t\{1,2\}}^2 - \frac{\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}} \cancelto{0}{\left(\lambda_x^2 - \lambda_x^2\right)}} \\
&= \pm k_{t\{1,2\}}
\end{aligned} \tag{F.49}
$$

This implies that

$$
\begin{aligned}
\oint_{C^+_{j\lambda_x}} &= j2\pi \left\{ j\omega\epsilon_{t1} \frac{\left(1 - e^{-\lambda_x b}\right) \left[\frac{k_{t1}}{\omega\epsilon_{t1}} - \frac{k_{t2}}{\omega\epsilon_{t2}} \tan\left(k_{t2}T\right) \tan\left(k_{t1}h\right)\right]}{k_{t1}\left(j\lambda_x + j\lambda_x\right) \left[\frac{k_{t1}}{\omega\epsilon_{t1}} \tan\left(k_{t1}h\right) + \frac{k_{t2}}{\omega\epsilon_{t2}} \tan\left(k_{t2}T\right)\right]} \right\} \\
&= j\pi \left\{ \frac{\omega\epsilon_{t1}\left(1 - e^{-\lambda_x b}\right) \left[\frac{k_{t1}}{\omega\epsilon_{t1}} - \frac{k_{t2}}{\omega\epsilon_{t2}} \tan\left(k_{t2}T\right) \tan\left(k_{t1}h\right)\right]}{k_{t1}\lambda_x \left[\frac{k_{t1}}{\omega\epsilon_{t1}} \tan\left(k_{t1}h\right) + \frac{k_{t2}}{\omega\epsilon_{t2}} \tan\left(k_{t2}T\right)\right]} \right\} \qquad \text{(F.50)}
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\frac{k_{t\{1,2\}}}{\omega\epsilon_{t\{1,2\}}} &= \frac{\omega\sqrt{\epsilon_{t\{1,2\}}\mu_{t\{1,2\}}}}{\omega\epsilon_{t\{1,2\}}} \\
&= \frac{\omega\cancel{\epsilon_{t\{1,2\}}}\mu_{t\{1,2\}}}{\omega\cancel{\epsilon_{t\{1,2\}}}\sqrt{\epsilon_{t\{1,2\}}\mu_{t\{1,2\}}}} \\
&= \frac{\omega\mu_{t\{1,2\}}}{k_{t\{1,2\}}} \qquad \text{(F.51)}
\end{aligned}
$$

implies that

$$
\oint_{C^+_{j\lambda_x}} = j\pi \left\{ \frac{k_{t1}\left(1 - e^{-\lambda_x b}\right) \left[\frac{\omega\mu_{t1}}{k_{t1}} - \frac{\omega\mu_{t2}}{k_{t2}} \tan\left(k_{t2}T\right) \tan\left(k_{t1}h\right)\right]}{\omega\mu_{t1}\lambda_x \left[\frac{\omega\mu_{t1}}{k_{t1}} \tan\left(k_{t1}h\right) + \frac{\omega\mu_{t2}}{k_{t2}} \tan\left(k_{t2}T\right)\right]} \right\} \qquad \text{(F.52)}
$$

Next, looking at the C^+_ℓ poles,

$$
\oint_{C^+_\ell} \frac{N}{D} d\lambda_y = j2\pi \left. \frac{N}{\frac{\partial}{\partial\lambda_y} D} \right|_{\lambda_y = -\lambda_{y\psi\ell}} \qquad \text{(F.53)}
$$

where

$$
N = \frac{j\omega\epsilon_{t1}\left(1 - e^{j\lambda_y b}\right) \left[Z_{\psi 1} - Z_{\psi 2} \tan\left(\lambda_{z\psi 2}T\right) \tan\left(\lambda_{z\psi 1}h\right)\right]}{\lambda_{z\psi 1}\left(\lambda_x^2 + \lambda_y^2\right)} \qquad \text{(F.54)}
$$

$$
D = Z_{\psi 1} \tan\left(\lambda_{z\psi 1}h\right) + Z_{\psi 2} \tan\left(\lambda_{z\psi 2}T\right) \qquad \text{(F.55)}
$$

In order to simplify the derivative, it is helpful to express the denominator in terms sines and cosines versus tangent functions. Thus

$$N = \frac{j\omega\epsilon_{t1}\left(1-e^{j\lambda_y b}\right)\left[Z_{\psi 1}\cos\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)-Z_{\psi 2}\sin\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\right]}{\lambda_{z\psi 1}\left(\lambda_x^2+\lambda_y^2\right)} \tag{F.56}$$

$$D = Z_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)+Z_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right) \tag{F.57}$$

which implies that

$$\begin{aligned}
\frac{\partial}{\partial\lambda_y}D &= \frac{\partial}{\partial\lambda_y}\left[\frac{\lambda_{z\psi 1}}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)\right]+\frac{\partial}{\partial\lambda_y}\left[\frac{\lambda_{z\psi 2}}{\omega\epsilon_{t2}}\cos\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\right]\\
&= \frac{1}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 1}+\frac{\lambda_{z\psi 1}}{\omega\epsilon_{t1}}\cos\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\sin\left(\lambda_{z\psi 1}h\right)\\
&\quad+\frac{\lambda_{z\psi 1}}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi 1}h\right)\frac{\partial}{\partial\lambda_y}\cos\left(\lambda_{z\psi 2}T\right)+\frac{1}{\omega\epsilon_{t2}}\cos\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 2}\\
&\qquad+\frac{\lambda_{z\psi 2}}{\omega\epsilon_{t2}}\sin\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\cos\left(\lambda_{z\psi 1}h\right)+\frac{\lambda_{z\psi 2}}{\omega\epsilon_{t2}}\cos\left(\lambda_{z\psi 1}h\right)\frac{\partial}{\partial\lambda_y}\sin\left(\lambda_{z\psi 2}T\right)\\
&= \frac{1}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 1}+hZ_{\psi 1}\cos\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 1}\\
&\quad-TZ_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 2}+\frac{1}{\omega\epsilon_{t2}}\cos\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 2}\\
&\quad-hZ_{\psi 2}\sin\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 1}+TZ_{\psi 2}\cos\left(\lambda_{z\psi 1}h\right)\cos\left(\lambda_{z\psi 2}T\right)\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 2}\\
&= \left[\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 1}\right]\left\{\cos\left(\lambda_{z\psi 2}T\right)\left[\frac{1}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi 1}h\right)+hZ_{\psi 1}\cos\left(\lambda_{z\psi 1}h\right)\right]\right.\\
&\left.-hZ_{\psi 2}\sin\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\right\}+\left[\frac{\partial}{\partial\lambda_y}\lambda_{z\psi 2}\right]\left\{\cos\left(\lambda_{z\psi 1}h\right)\left[\frac{1}{\omega\epsilon_{t2}}\sin\left(\lambda_{z\psi 2}T\right)\right.\right.\\
&\qquad\left.\left.+TZ_{\psi 2}\cos\left(\lambda_{z\psi 2}T\right)\right]-TZ_{\psi 1}\sin\left(\lambda_{z\psi 1}h\right)\sin\left(\lambda_{z\psi 2}T\right)\right\}
\end{aligned} \tag{F.58}$$

It can be shown that

$$\frac{\partial}{\partial\lambda_y}\lambda_{z\psi\{1,2\}} = -\frac{\lambda_y\epsilon_{t\{1,2\}}}{\epsilon_{z\{1,2\}}\lambda_{z\psi\{1,2\}}} \tag{F.59}$$

which implies that

$$\frac{\partial}{\partial\lambda_y}D = \left[\frac{\lambda_y \epsilon_{t1}}{\epsilon_{z1}\lambda_{z\psi1}}\right]\left\{hZ_{\psi2}\sin\left(\lambda_{z\psi1}h\right)\sin\left(\lambda_{z\psi2}T\right) - \cos\left(\lambda_{z\psi2}T\right)\left[\frac{1}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi1}h\right)\right.\right.$$
$$\left.\left. + hZ_{\psi1}\cos\left(\lambda_{z\psi1}h\right)\right]\right\} + \left[\frac{\lambda_y\epsilon_{t2}}{\epsilon_{z2}\lambda_{z\psi2}}\right]\left\{TZ_{\psi1}\sin\left(\lambda_{z\psi1}h\right)\sin\left(\lambda_{z\psi2}T\right)\right.$$
$$\left. - \cos\left(\lambda_{z\psi1}h\right)\left[\frac{1}{\omega\epsilon_{t2}}\sin\left(\lambda_{z\psi2}T\right) + TZ_{\psi2}\cos\left(\lambda_{z\psi2}T\right)\right]\right\}$$
$$= \frac{\lambda_y}{\omega\epsilon_{z1}Z_{\psi1}}\left\{hZ_{\psi2}\sin\left(\lambda_{z\psi1}h\right)\sin\left(\lambda_{z\psi2}T\right) - \cos\left(\lambda_{z\psi2}T\right)\left[\frac{1}{\omega\epsilon_{t1}}\sin\left(\lambda_{z\psi1}h\right)\right.\right.$$
$$\left.\left. + hZ_{\psi1}\cos\left(\lambda_{z\psi1}h\right)\right]\right\} + \frac{\lambda_y}{\omega\epsilon_{z2}Z_{\psi2}}\left\{TZ_{\psi1}\sin\left(\lambda_{z\psi1}h\right)\sin\left(\lambda_{z\psi2}T\right)\right. \tag{F.60}$$
$$\left. - \cos\left(\lambda_{z\psi1}h\right)\left[\frac{1}{\omega\epsilon_{t2}}\sin\left(\lambda_{z\psi2}T\right) + TZ_{\psi2}\cos\left(\lambda_{z\psi2}T\right)\right]\right\}$$

Therefore

$$\oint_{C_\ell^+}\frac{N}{D}d\lambda_y = -\frac{2\pi\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TM}^z}}\left[1 - \frac{Z_{\psi2}}{Z_{\psi1}}\tan\left(\lambda_{z\psi1}h\right)\tan\left(\lambda_{z\psi2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}} \tag{F.61}$$

$$D^{\mathrm{TM}^z} = \frac{1}{Z_{\psi1}Z_{\psi2}}\left[h\epsilon_{z2}Z_{\psi2}^2 + T\epsilon_{z1}Z_{\psi1}^2\right]\tan\left(\lambda_{z\psi1}h\right)\tan\left(\lambda_{z\psi2}T\right)$$
$$- \epsilon_{z2}\left[\frac{\tan\left(k_{z\psi1}h\right)}{\lambda_{z\psi1}} + h\right] - \epsilon_{z1}\left[\frac{\tan\left(\lambda_{z\psi2}T\right)}{\lambda_{z\psi2}} + T\right] \tag{F.62}$$

Substituting the TM^z components above into (221) implies that

$$A_{m,n}^{\mathrm{TM}^z} = \delta_{m,n}\left(\frac{ab}{2}\right)\left(M_{xm}^h + M_{ym}^h\right)^2$$
$$- \frac{Z_n M_{xm}^h M_{xn}^h k_{xm}k_{xn}}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\left(1-(-1)^{v_m}e^{jk_x a}\right)\left(1-(-1)^{v_n}e^{-jk_x a}\right)}{\left(k_x+k_{xm}\right)\left(k_x-k_{xm}\right)\left(k_x+k_{xn}\right)\left(k_x-k_{xn}\right)}\right]\Omega^{\mathrm{TM}^z}dk_x \tag{F.63}$$

where

$$
\begin{aligned}
\Omega^{\mathrm{TM}^z} &= j\left\{\frac{k_{t1}\left(1-e^{-\lambda_x b}\right)\left[\frac{\omega\mu_{t1}}{k_{t1}}-\frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)\tan\left(k_{t1}h\right)\right]}{\omega\mu_{t1}\lambda_x\left[\frac{\omega\mu_{t1}}{k_{t1}}\tan\left(k_{t1}h\right)+\frac{\omega\mu_{t2}}{k_{t2}}\tan\left(k_{t2}T\right)\right]}\right\} \\
&\quad -\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TM}^z}}\left[1-\frac{Z_{\psi2}}{Z_{\psi1}}\tan\left(\lambda_{z\psi1}h\right)\tan\left(\lambda_{z\psi2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}}
\end{aligned} \tag{F.64}
$$

$$
\begin{aligned}
D^{\mathrm{TM}^z} &= \frac{1}{Z_{\psi1}Z_{\psi2}}\left[h\epsilon_{z2}Z_{\psi2}^2+T\epsilon_{z1}Z_{\psi1}^2\right]\tan\left(\lambda_{z\psi1}h\right)\tan\left(\lambda_{z\psi2}T\right) \\
&\qquad -\epsilon_{z2}\left[\frac{\tan\left(\lambda_{z\psi1}h\right)}{\lambda_{z\psi1}}+h\right]-\epsilon_{z1}\left[\frac{\tan\left(\lambda_{z\psi2}T\right)}{\lambda_{z\psi2}}+T\right]
\end{aligned} \tag{F.65}
$$

It is interesting to note that when the TE^z and TM^z components are added together, the contributions due to $\lambda_y = j\lambda_x$ perfectly cancel one another. Thus,

$$
\begin{aligned}
A_{m,n} &= \delta_{m,n}\left(\frac{ab}{2}\right)\left(M_{xm}^h\right)^2 \\
&\quad -\frac{Z_n M_{xm}^h M_{xn}^h k_{xm}k_{xn}}{2\pi}\int_{-\infty}^{\infty}C^{\lambda_x}\left(\Omega^{\mathrm{TE}^z}+\Omega^{\mathrm{TM}^z}\right)d\lambda_x \\
C^{\lambda_x} &= \left[\frac{\left(1-(-1)^{v_m}e^{j\lambda_x a}\right)\left(1-(-1)^{v_n}e^{-j\lambda_x a}\right)}{\left(\lambda_x+k_{xm}\right)\left(\lambda_x-k_{xm}\right)\left(\lambda_x+k_{xn}\right)\left(\lambda_x-k_{xn}\right)}\right] \\
\Omega^{\mathrm{TE}^z} &= j\left[\frac{b}{Z_{\theta1}^*}\right]\left[\frac{Z_{\theta1}^*-Z_{\theta2}^*\tan\left(\lambda_{z\theta2}^*T\right)\tan\left(\lambda_{z\theta1}^*h\right)}{Z_{\theta1}^*\tan\left(\lambda_{z\theta1}^*h\right)+Z_{\theta2}^*\tan\left(\lambda_{z\theta2}^*T\right)}\right] \\
&\quad -\sum_{\ell=0}^{L}\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TE}^z}}\right]\left(1-e^{j\lambda_y b}\right)\left[1-\frac{Z_{\theta2}}{Z_{\theta1}}\tan\left(\lambda_{z\theta1}h\right)\tan\left(\lambda_{z\theta2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}} \\
\Omega^{\mathrm{TM}^z} &= -\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TM}^z}}\left[1-\frac{Z_{\psi2}}{Z_{\psi1}}\tan\left(\lambda_{z\psi1}h\right)\tan\left(\lambda_{z\psi2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}} \\
Z_{\theta(1,2)}^* &= \frac{\omega\mu_{t(1,2)}}{\lambda_{z\theta(1,2)}^*}, \lambda_{z\theta(1,2)}^* = \sqrt{k_{t(1,2)}^2-\frac{\mu_{t(1,2)}}{\mu_{z(1,2)}}\lambda_x^2} \\
D^{\mathrm{TE}^z} &= Z_{\theta1}^2\mu_{z2}\left[\frac{\tan\left(\lambda_{z\theta1}h\right)}{\lambda_{z\theta1}}-h\right]+Z_{\theta2}^2\mu_{z1}\left[\frac{\tan\left(\lambda_{z\theta2}T\right)}{\lambda_{z\theta2}}-T\right] \\
&\qquad +Z_{\theta1}Z_{\theta2}\tan\left(\lambda_{z\theta1}h\right)\tan\left(\lambda_{z\theta2}T\right)\left[h\mu_{z2}+T\mu_{z1}\right]
\end{aligned}
$$

$$D^{\mathrm{TM}^z} = \frac{1}{Z_{\psi 1} Z_{\psi 2}} \left[h\epsilon_{z2} Z_{\psi 2}^2 + T\epsilon_{z1} Z_{\psi 1}^2 \right] \tan\left(\lambda_{z\psi 1} h\right) \tan\left(\lambda_{z\psi 2} T\right) - \epsilon_{z2} \left[\frac{\tan\left(\lambda_{z\psi 1} h\right)}{\lambda_{z\psi 1}} + h \right] - \epsilon_{z1} \left[\frac{\tan\left(\lambda_{z\psi 2} T\right)}{\lambda_{z\psi 2}} + T \right] \tag{F.66}$$

Allowing the constant $\beta = \frac{\sqrt{2}}{k_{cm}\sqrt{ab}}$ implies that the unknown coefficients $\vec{C}$ can be found by

$$\boxed{\begin{aligned}
&\vec{C} = \overleftrightarrow{A}^{-1} \vec{B} \\
&A_{m,n} = \delta_{m,n} \frac{1}{Z_m Z_n} - \frac{Z_n M_{xm}^h M_{xn}^h k_{xm} k_{xn}}{2\pi} \int_{-\infty}^{\infty} C^{\lambda_x} \left(\Omega^{\mathrm{TE}^z} + \Omega^{\mathrm{TM}^z} \right) d\lambda_x \\
&C^{\lambda_x} = \left[\frac{\left(1 - (-1)^{v_m} e^{j\lambda_x a}\right)\left(1 - (-1)^{v_n} e^{-j\lambda_x a}\right)}{(\lambda_x + k_{xm})(\lambda_x - k_{xm})(\lambda_x + k_{xn})(\lambda_x - k_{xn})} \right] \\
&\Omega^{\mathrm{TE}^z} = j \left[\frac{b}{Z_{\theta 1}^*} \right] \left[\frac{Z_{\theta 1}^* - Z_{\theta 2}^* \tan\left(k_{z\theta 2}^* T\right) \tan\left(\lambda_{z\theta 1}^* h\right)}{Z_{\theta 1}^* \tan\left(\lambda_{z\theta 1}^* h\right) + Z_{\theta 2}^* \tan\left(\lambda_{z\theta 2}^* T\right)} \right] \\
&\quad - \sum_{\ell=0}^{L} \left[\frac{2\omega \mu_{z1} \mu_{z2} \lambda_x^2 \left(1 - e^{j\lambda_y b}\right)}{\lambda_y^3 \left(\lambda_x^2 + \lambda_y^2\right) D^{\mathrm{TE}^z}} \right] \left[1 - \frac{Z_{\theta 2}}{Z_{\theta 1}} \tan\left(\lambda_{z\theta 1} h\right) \tan\left(\lambda_{z\theta 2} T\right) \right] \Bigg|_{\lambda_y = -\lambda_{y\theta\ell}} \\
&\Omega^{\mathrm{TM}^z} = - \sum_{\ell=0}^{L} \frac{2\omega \epsilon_{z1} \epsilon_{z2} \left(1 - e^{j\lambda_y b}\right)}{\lambda_y \left(\lambda_x^2 + \lambda_y^2\right) D^{\mathrm{TM}^z}} \left[1 - \frac{Z_{\psi 2}}{Z_{\psi 1}} \tan\left(\lambda_{z\psi 1} h\right) \tan\left(\lambda_{z\psi 2} T\right) \right] \Bigg|_{\lambda_y = -\lambda_{y\psi\ell}} \\
&Z_{\theta(1,2)}^* = \frac{\omega \mu_{t(1,2)}}{\lambda_{z\theta(1,2)}^*}, \lambda_{z\theta(1,2)}^* = \sqrt{k_{t(1,2)}^2 - \frac{\mu_{t(1,2)}}{\mu_{z(1,2)}} \lambda_x^2} \\
&D^{\mathrm{TE}^z} = Z_{\theta 1}^2 \mu_{z2} \left[\frac{\tan\left(\lambda_{z\theta 1} h\right)}{\lambda_{z\theta 1}} - h \right] + Z_{\theta 2}^2 \mu_{z1} \left[\frac{\tan\left(\lambda_{z\theta 2} T\right)}{\lambda_{z\theta 2}} - T \right] \\
&\qquad + Z_{\theta 1} Z_{\theta 2} \tan\left(\lambda_{z\theta 1} h\right) \tan\left(\lambda_{z\theta 2} T\right) \left[h\mu_{z2} + T\mu_{z1} \right] \\
&D^{\mathrm{TM}^z} = \frac{1}{Z_{\psi 1} Z_{\psi 2}} \left[h\epsilon_{z2} Z_{\psi 2}^2 + T\epsilon_{z1} Z_{\psi 1}^2 \right] \tan\left(\lambda_{z\psi 1} h\right) \tan\left(\lambda_{z\psi 2} T\right) \\
&\qquad - \epsilon_{z2} \left[\frac{\tan\left(\lambda_{z\psi 1} h\right)}{\lambda_{z\psi 1}} + h \right] - \epsilon_{z1} \left[\frac{\tan\left(\lambda_{z\psi 2} T\right)}{\lambda_{z\psi 2}} + T \right] \\
&B_m = \frac{2}{Z_m^2} \delta_{m,1}
\end{aligned}}$$

where the $Z_{\{m,n\}}$ and $M_{x\{m,n\}}^h$ terms are determined via Table 1 and the $\lambda_{y\{\theta,\psi\}\ell}$ terms are determined numerically via the techniques described by Hsu in [41].

Special Case: $h \to 0$**.**

Finally, exploring the special case where $h \to 0$, the first integrand of (F.5) simplifies to

$$j \frac{\lambda_x^2 \lambda_{z\theta 1} \left(1 - e^{j\lambda_y b}\right) \left[Z_{\theta 1} - Z_{\theta 2} \tan\left(\lambda_{z\theta 2} T\right) \cancelto{0}{\tan\left(\lambda_{z\theta 1} h\right)} \right]}{\omega \mu_{t1} \lambda_y^2 \left(\lambda_x^2 + \lambda_y^2\right) \left[Z_{\theta 1} \cancelto{0}{\tan\left(\lambda_{z\theta 1} h\right)} + Z_{\theta 2} \tan\left(\lambda_{z\theta 2} T\right) \right]} \tag{F.67}$$

$$= j \frac{\lambda_x^2 \cancel{\lambda_{z\theta 1}} \left(1 - e^{j\lambda_y b}\right) \cancel{Z_{\theta 1}}}{\cancel{\omega \mu_{t1}} \lambda_y^2 \left(\lambda_x^2 + \lambda_y^2\right) Z_{\theta 2} \tan\left(\lambda_{z\theta 2} T\right)} \tag{F.68}$$

$$= j \frac{\lambda_x^2 \left(1 - e^{j\lambda_y b}\right)}{\lambda_y^2 \left(\lambda_x^2 + \lambda_y^2\right) Z_{\theta 2} \tan\left(\lambda_{z\theta 2} T\right)} \tag{F.69}$$

$$= j \frac{\lambda_x^2 \left(1 - e^{j\lambda_y b}\right) \cos\left(\lambda_{z\theta 2} T\right)}{\lambda_y^2 \left(\lambda_x^2 + \lambda_y^2\right) Z_{\theta 2} \sin\left(\lambda_{z\theta 2} T\right)} \tag{F.70}$$

Therefore, when analyzing the C_ℓ^+ component when $h = 0$, it is noted that

$$N\left(\lambda_y\right) = j \frac{\lambda_x^2 \left(1 - e^{j\lambda_y b}\right) \cos\left(\lambda_{z\theta 2} T\right)}{\lambda_y^2 \left(\lambda_x^2 + \lambda_y^2\right) Z_{\theta 2}} \tag{F.71}$$

$$D\left(\lambda_y\right) = \sin\left(\lambda_{z\theta 2} T\right) \tag{F.72}$$

It is now evident that when $h = 0$, the C_ℓ^+ poles occur everywhere that $\lambda_{z\theta 2} T = \pi \ell$. Therefore,

$$\lambda_{z\theta 2} T = \pi \ell \tag{F.73}$$

$$\lambda_{z\theta 2} = \frac{\pi \ell}{T} \tag{F.74}$$

$$\sqrt{k_{t2}^2 - \frac{\mu_{t2}}{\mu_{z2}} \lambda_x^2 - \frac{\mu_{t2}}{\mu_{z2}} \lambda_{y\theta l}^2} = \frac{\pi \ell}{T} \tag{F.75}$$

$$k_{t2}^2 - \frac{\mu_{t2}}{\mu_{z2}} \lambda_x^2 - \frac{\mu_{t2}}{\mu_{z2}} \lambda_{y\theta \ell}^2 = \left(\frac{\pi \ell}{T}\right)^2 \tag{F.76}$$

$$\frac{\mu_{t2}}{\mu_{z2}}\lambda_{y\theta\ell}^2 = k_{t2}^2 - \frac{\mu_{t2}}{\mu_{z2}}\lambda_x^2 - \left(\frac{\pi\ell}{T}\right)^2 \tag{F.77}$$

$$\lambda_{y\theta\ell}^2 = \frac{\mu_{z2}}{\mu_{t2}}k_{t2}^2 - \lambda_x^2 - \frac{\mu_{z2}}{\mu_{t2}}\left(\frac{\pi\ell}{T}\right)^2 \tag{F.78}$$

$$\lambda_{y\theta\ell} = \pm\sqrt{\frac{\mu_{z2}}{\mu_{t2}}k_{t2}^2 - \lambda_x^2 - \frac{\mu_{z2}}{\mu_{t2}}\left(\frac{\pi\ell}{T}\right)^2} \tag{F.79}$$

Similarly, it can be shown that

$$\lambda_{z\psi 2} = \frac{\pi\ell}{T} \tag{F.80}$$

$$\lambda_{y\psi\ell} = \pm\sqrt{\frac{\epsilon_{z2}}{\epsilon_{t2}}k_{t2}^2 - \lambda_x^2 - \frac{\epsilon_{z2}}{\epsilon_{t2}}\left(\frac{\pi\ell}{T}\right)^2} \tag{F.81}$$

Analyzing the TE^z results from the previous section in the limit as $h \to 0$,

$$\begin{aligned}
\lim_{h\to 0}\Omega^{\mathrm{TE}^z} &= \lim_{h\to 0} j\left[\frac{b}{Z_{\theta1}^*}\right]\left[\frac{Z_{\theta1}^* - Z_{\theta2}^*\tan\left(\lambda_{z\theta2}^* T\right)\tan\left(\lambda_{z\theta1}^* h\right)}{Z_{\theta1}^*\tan\left(\lambda_{z\theta1}^* h\right) + Z_{\theta2}^*\tan\left(\lambda_{z\theta2}^* T\right)}\right] \\
&- \sum_{\ell=0}^{L}\lim_{h\to 0}\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2\left(1-e^{j\lambda_y b}\right)}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TE}^z}}\right]\left[1-\frac{Z_{\theta2}}{Z_{\theta1}}\tan\left(\lambda_{z\theta1}h\right)\tan\left(\lambda_{z\theta2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}} \\
&= jb\left[\frac{1-\frac{Z_{\theta2}^*}{Z_{\theta1}^*}\tan\left(\lambda_{z\theta2}^* T\right)\cancelto{0}{\lim_{h\to 0}\tan\left(\lambda_{z\theta1}^* h\right)}}{Z_{\theta1}^*\cancelto{0}{\lim_{h\to 0}\tan\left(\lambda_{z\theta1}^* h\right)} + Z_{\theta2}^*\tan\left(\lambda_{z\theta2}^* T\right)}\right] \\
&- \sum_{\ell=0}^{L}\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2\left(1-e^{j\lambda_y b}\right)}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right)\lim_{h\to 0}D^{\mathrm{TE}^z}}\right]\left[1-\frac{Z_{\theta2}}{Z_{\theta1}}\cancelto{0}{\lim_{h\to 0}\tan\left(\lambda_{z\theta1}h\right)}\tan\left(\lambda_{z\theta2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}} \\
&= \frac{jb\lambda_{z\theta2}^*\cos\left(\lambda_{z\theta2}^* T\right)}{\omega\mu_{t2}\sin\left(\lambda_{z\theta2}^* T\right)} - \sum_{\ell=0}^{L}\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2\left(1-e^{j\lambda_y b}\right)}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right)\lim_{h\to 0}D^{\mathrm{TE}^z}}\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}}
\end{aligned} \tag{F.82}$$

$$\begin{aligned}
\lim_{h\to 0}D^{\mathrm{TE}^z}\Big|_{\lambda_y=-\lambda_{y\theta\ell}} &= \Bigg\{Z_{\theta1}^2\mu_{z2}\lim_{h\to 0}\left[\frac{\cancelto{0}{\tan\left(\lambda_{z\theta1}h\right)}}{\lambda_{z\theta1}} - \cancelto{0}{h}\right] + \lim_{h\to 0}Z_{\theta2}^2\mu_{z1}\left[\frac{\tan\left(\lambda_{z\theta2}T\right)}{\lambda_{z\theta2}}\right. \\
&\left. -T\right] + \lim_{h\to 0}Z_{\theta1}Z_{\theta2}\cancelto{0}{\tan\left(\lambda_{z\theta1}h\right)}\tan\left(\lambda_{z\theta2}T\right)\left[h\mu_{z2}+T\mu_{z1}\right]\Bigg\}\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}}
\end{aligned}$$

$$= \lim_{h\to 0} \frac{\left(\frac{\omega\mu_{t2}}{\lambda_{z\theta 2}}\right)^2 \mu_{z1}}{\lambda_{z\theta 2}} \left[\tan\left(\lambda_{z\theta 2}T\right) - T\lambda_{z\theta 2}\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}}$$

$$= \frac{\left(\frac{\omega\mu_{t2}T}{\pi l}\right)^2 \mu_{z1}}{\lambda_{z\theta 2}} \left[\cancelto{0}{\tan\left(\pi\ell\right)} - Tk_{z\theta 2}\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}}$$

$$= -\left(\frac{\omega\mu_{t2}}{\pi\ell}\right)^2 \mu_{z1}T^3\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}} \tag{F.83}$$

which implies that

$$\lim_{h\to 0}\Omega^{\mathrm{TE}^z} = \frac{jb\lambda^*_{z\theta 2}\cos\left(\lambda^*_{z\theta 2}T\right)}{\omega\mu_{t2}\sin\left(\lambda^*_{z\theta 2}T\right)} - \sum_{\ell=0}^{L}\left[\frac{2\omega\mu_{z1}\mu_{z2}\lambda_x^2\left(1-e^{j\lambda_y b}\right)}{\lambda_y^3\left(\lambda_x^2+\lambda_y^2\right)\left[-\left(\frac{\omega\mu_{t2}}{\pi\ell}\right)^2\mu_{z1}T^3\right]}\right]\Bigg|_{\lambda_y=-\lambda_{y\theta\ell}}$$

$$= \frac{jb\lambda^*_{z\theta 2}\cos\left(\lambda^*_{z\theta 2}T\right)}{\omega\mu_{t2}\sin\left(\lambda^*_{z\theta 2}T\right)} - \frac{2\mu_{z2}\lambda_x^2}{\omega\mu_{t2}^2T^3}\sum_{\ell=0}^{L}\frac{\left(1-e^{-j\lambda_{y\theta\ell}b}\right)\left(\pi\ell\right)^2}{\lambda^3_{y\theta\ell}\left(\lambda_x^2+\lambda^2_{y\theta\ell}\right)} \tag{F.84}$$

Similarly, analyzing the TM^z results from the previous section in the limit as $h \to 0$,

$$\lim_{h\to 0}\Omega^{\mathrm{TM}^z} = -\lim_{h\to 0}\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)D^{\mathrm{TM}^z}}\Big[1 - \frac{Z_{\psi 2}}{Z_{\psi 1}}\tan\left(\lambda_{z\psi 1}h\right)\tan\left(\lambda_{z\psi 2}T\right)\Big]\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}}$$

$$= -\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)\lim_{h\to 0}D^{\mathrm{TM}^z}}\left[1-\frac{Z_{\psi 2}}{Z_{\psi 1}}\lim_{h\to 0}\cancelto{0}{\tan\left(\lambda_{z\psi 1}h\right)}\tan\left(\lambda_{z\psi 2}T\right)\right]\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}}$$

$$= -\sum_{\ell=0}^{L}\frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1-e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2+\lambda_y^2\right)\lim_{h\to 0}D^{\mathrm{TM}^z}}\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}} \tag{F.85}$$

$$\lim_{h\to 0}D^{\mathrm{TM}^z} = \Bigg\{\frac{1}{Z_{\psi 1}Z_{\psi 2}}\left[h\epsilon_{z2}Z^2_{\psi 2}+T\epsilon_{z1}Z^2_{\psi 1}\right]\lim_{h\to 0}\cancelto{0}{\tan\left(\lambda_{z\psi 1}h\right)}\tan\left(\lambda_{z\psi 2}T\right)$$
$$-\epsilon_{z2}\lim_{h\to 0}\left[\frac{\cancelto{0}{\tan\left(\lambda_{z\psi 1}h\right)}}{\lambda_{z\psi 1}}+\cancelto{0}{h}\right] - \epsilon_{z1}\lim_{h\to 0}\left[\frac{\tan\left(\lambda_{z\psi 2}T\right)}{\lambda_{z\psi 2}}+T\right]\Bigg\}\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}}$$

$$= -\epsilon_{z1}T\left[\frac{\tan\left(\pi\ell\right)}{\pi\ell}+1\right]\Bigg|_{\lambda_y=-\lambda_{y\psi\ell}}$$

$$
\begin{aligned}
&= -\,\epsilon_{z1} T \left[\frac{\operatorname{sinc}(\pi\ell)}{\cancelto{1}{\cos(\pi\ell)}} + 1 \right] \Bigg|_{\lambda_y = -\lambda_{y\psi\ell}} \\
&= -\,\epsilon_{z1} T \left(\delta_{0,l} + 1\right) \big|_{\lambda_y = -\lambda_{y\psi\ell}} \qquad \text{(F.86)}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\lim_{h\to 0} \Omega^{\mathrm{TM}^z} &= -\sum_{\ell=0}^{L} \frac{2\omega\epsilon_{z1}\epsilon_{z2}\left(1 - e^{j\lambda_y b}\right)}{\lambda_y\left(\lambda_x^2 + \lambda_y^2\right)\left[-\epsilon_{z1} T\left(\delta_{0,l}+1\right)\right]} \Bigg|_{\lambda_y = -\lambda_{y\psi\ell}} \\
&= -\sum_{\ell=0}^{L} \frac{2\omega\epsilon_{z2}\left(1 - e^{-j\lambda_{y\psi\ell} b}\right)}{\lambda_{y\psi\ell}\left(\lambda_x^2 + \lambda_{y\psi\ell}^2\right) T\left(\delta_{0,l}+1\right)} \qquad \text{(F.87)}
\end{aligned}
$$

F.2 λ_y Integrals for Reduced Aperture Waveguide Probe Method

In an effort to reduce duplication of effort, it is helpful to develop generalized solutions to the λ_y integrals. There are two major cases that must be explored: $h = b$ and $h \neq b$.

F.3 Assuming $h = b \Rightarrow k_{yw_p}^A = k_{yw_p}^B, k_{yw_q}^A = k_{yw_q}^B$

There are two general forms of these integrals that must be explored, one stemming from $\tilde{\overleftrightarrow{G}}_{hh}$ and the other from $\tilde{\overleftrightarrow{G}}_{eh}$. Beginning with $\Theta_3^{\lambda_y} \tilde{\overleftrightarrow{G}}_{hh}\left(\substack{z=0\\ z'=0}\right)$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left[\frac{4\sin^2\left(\lambda_y \frac{h}{2}\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)} \right] \frac{\cancel{2}C\lambda_y^u \lambda_{z\alpha}^v \cos\left(\lambda_{z\alpha} d\right)}{\cancel{2}\lambda_\rho^2 \sin\left(\lambda_{z\alpha} d\right)} d\lambda_y \\
&= \int_{-\infty}^{\infty} \frac{4C\lambda_y^u \lambda_{z\alpha}^v \sin^2\left(\lambda_y \frac{h}{2}\right)\cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha} d\right)} d\lambda_y
\end{aligned}
$$

(F.88)

In order to evaluate the above integral using complex plane analysis, the integrand must decay to zero as $\lambda_y \to \infty$. First, examine

$$\begin{aligned}
\lim_{\lambda_y \to \infty} \lambda_{z\alpha} &= \lim_{\lambda_y \to \infty} \sqrt{k_t^2 - \frac{\mu_t}{\mu_z}\left(\lambda_x^2 + \lambda_y^2\right)} \\
&= \lim_{\lambda_y \to \infty} \lambda_y \sqrt{\cancelto{0}{\frac{k_t^2}{\lambda_y^2}} - \frac{\mu_t}{\mu_z}\left(\cancelto{0}{\frac{\lambda_x^2}{\lambda_y^2}} + 1\right)} \\
&= j\sqrt{\frac{\mu_t}{\mu_z}} \lim_{\lambda_y \to \infty} \lambda_y \to \infty
\end{aligned} \tag{F.89}$$

Therefore,

$$\begin{aligned}
\lim_{\lambda_y \to \infty} \frac{\cos\left(\lambda_{z\alpha} d\right)}{\sin\left(\lambda_z \alpha\right) d} &= \lim_{\lambda_y \to \infty} \tan\left(\lambda_{z\alpha} d\right) \\
&= \lim_{\lambda_y \to \infty} \tan\left(j\sqrt{\frac{\mu_t}{\mu_z}}\lambda_y d\right) \\
&= \lim_{\lambda_y \to \infty} j\cancelto{1}{\tanh\left(\sqrt{\frac{\mu_t}{\mu_z}}\lambda_y d\right)} \\
&= j
\end{aligned} \tag{F.90}$$

Next analyze

$$\lim_{\lambda_y \to \infty} \frac{4C\lambda_y^u \lambda_{z\alpha}^v}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)} = \frac{j}{\infty^{(6-u-v)}} \tag{F.91}$$

In this work, $u \leq 4$ and $v \in \{-1, 1\}$, therefore the above term always decays. However, the $\sin^2\left(\lambda_y \frac{h}{2}\right)$ component is neither guaranteed to decay nor bounded as $\Im\{k_y\} \to \infty$. Thus, the $\sin^2\left(\lambda_y \frac{h}{2}\right)$ term must be broken into its exponential form and the integral split to guarantee the proper decay of the integrands as $\Im\{k_y\} \to \infty$.

Therefore,

$$
\begin{aligned}
\sin^2\left(\lambda_y \frac{h}{2}\right) &= \left(\frac{e^{j\lambda_y \frac{h}{2}} - e^{-j\lambda_y \frac{h}{2}}}{j2}\right)\left(\frac{e^{j\lambda_y \frac{h}{2}} - e^{-j\lambda_y \frac{h}{2}}}{j2}\right) \\
&= \frac{e^{j\lambda_y h} - 1 - 1 + e^{-j\lambda_y h}}{-4} \\
&= \frac{1 - e^{j\lambda_y h}}{4} + \frac{1 - e^{-j\lambda_y h}}{4}
\end{aligned}
\tag{F.92}
$$

Thus the above integral becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{4C\lambda_y^u \lambda_{z\alpha}^v \sin^2\left(\lambda_y \frac{h}{2}\right)\cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha} d\right)} d\lambda_y \\
&= \int_{-\infty}^{\infty} \frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha} d\right)} d\lambda_y \\
&\quad + \int_{-\infty}^{\infty} \frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha} d\right)} d\lambda_y
\end{aligned}
\tag{F.93}
$$

There are five cases that must be explored when evaluating the above integral.

- Case 1: $p = q = 1$
- Case 2: $p = 1, q \neq 1$
- Case 3: $p \neq 1, q = 1$
- Case 4: $p = q \neq 1$
- Case 5: $p \neq q, p \neq 1, q \neq 1$

Case 1: $p = q = 1$**.**

In this case, $k^B_{yw_p} = k^B_{yw_q} = 0$, reducing (F.93) to

$$\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^{\alpha}}{2\lambda_\rho^2}d\lambda_y = \int_{-\infty}^{\infty} \frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}d\lambda_y + \int_{-\infty}^{\infty} \frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}d\lambda_y \tag{F.94}$$

On close inspection it can be seen that for each integral above there are possibly an order $4-u$ pole at $\lambda_y = 0$, and simple poles at $\lambda_y = j\lambda_x$ and $\lambda_{z\alpha}^v \sin\left(\lambda_{z\alpha}d\right) = 0$. The first integral requires UHPC and the second integral requires LHPC when employing CIT and CIF to solve the integrals. Fig. 22 depicts the contours required to account for each of these poles. There is a contour $C_0^{\pm}$ around the $4-u$-order pole at $\lambda_y = 0$, $C_{j\lambda_x}^{\pm}$ around simple poles at $\lambda_y = \pm j\lambda_x$, and $C_i^{\pm}$ around an infinite number of poles satisfying $\sin\left(\lambda_{z\alpha}d\right) = 0$.

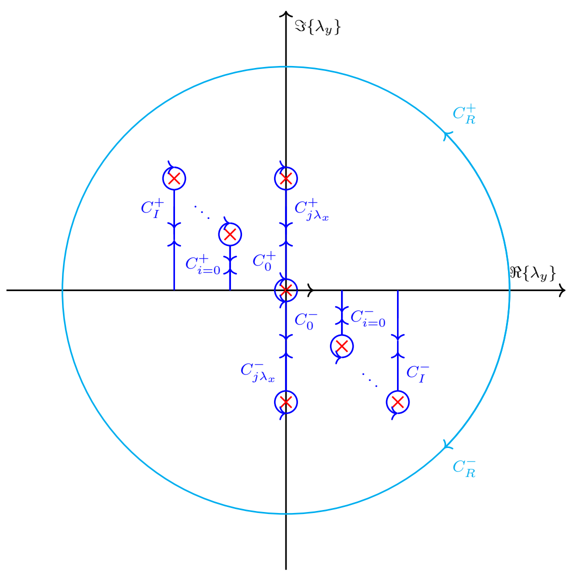

Figure 22. Complex poles (red) of the transverse spatial frequency domain principal scalar potential Green functions, deformation contours around those poles (blue) and closure contours as $R \to \infty$ (cyan) in the complex λ_y-plane.

Recall from Chapter II that for UHPC

$$\begin{aligned}
&\lim_{R\to\infty}\int_{-R}^{R} + \oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \oint_{\sum C_i^+} + \cancelto{0}{\oint_{C_R^+}} = 0 \\
&\Rightarrow \int_{-\infty}^{\infty} = \oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \sum_{i=-\infty}^{\infty}\oint_{C_i^+}
\end{aligned} \tag{F.95}$$

Also, recall from Chapter II that for LHPC

$$\begin{aligned}
&\lim_{R\to\infty}\int_{-R}^{R} + \oint_{C_0^-} + \oint_{C_{-j\lambda_x}^-} + \oint_{\sum C_i^-} + \cancelto{0}{\oint_{C_R^-}} = 0 \\
&\Rightarrow \int_{-\infty}^{\infty} = -\oint_{C_0^-} - \oint_{C_{-j\lambda_x}^-} - \sum_{i=-\infty}^{\infty}\oint_{C_i^-}
\end{aligned} \tag{F.96}$$

To evaluate the full integral with respect to λ_y, add the UHPC and LHPC components together such that

$$\int_{-\infty}^{\infty} = \oint_{C_0^+} + \oint_{C_{j\lambda_x}^+} + \sum_{i=-\infty}^{\infty}\oint_{C_i^+} - \oint_{C_0^-} - \oint_{C_{-j\lambda_x}^-} - \sum_{i=-\infty}^{\infty}\oint_{C_i^-} \tag{F.97}$$

It is important to note that the deformation contour C_0^+ around the pole at $\lambda_y = 0$ (or any purely-real pole, for that matter) only requires a semi-circular path in UHPC, thus the usual value obtained from CIF is halved. The other half of the deformation contour will be accounted for in LHPC since all poles residing on the real axis will be included in both UHPC and LHPC. Let us study each of these components' contributions, beginning with C_0^+.

Noting that $\left(1 - e^{j\lambda_y h}\right)$ is a zero of order 1 at $\lambda_y = 0$, this means that up to $u+1$ poles are ultimately removable. The value of the $(4-u)$-order pole contribution at

$\lambda_y = 0$ must be divided in half since the contour around the pole is semi-circular. Thus, from Appendix A,

$$\oint_{C_0^+} \frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^4 \left(\lambda_x^2 + \lambda_y^2\right) \sin\left(\lambda_{z\alpha} d\right)} d\lambda_y$$
$$= \frac{j\pi}{(3-u)!} \frac{\partial^{(3-u)}}{\partial \lambda_y{}^{(3-u)}} \left[\frac{C\lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_x^2 + \lambda_y^2\right) \sin\left(\lambda_{z\alpha} d\right)}\right]\Bigg|_{\lambda_y = 0} \tag{F.98}$$

To evaluate these derivatives, it is useful to first find the value of $\frac{\partial}{\partial \lambda_y} \lambda_{z\alpha}$

$$\frac{\partial}{\partial \lambda_y} \lambda_{z\alpha} = \frac{1}{2\lambda_{z\alpha}} \begin{cases} -\frac{2\mu_t}{\mu_z} \lambda_y \ldots \alpha = \theta \\ -\frac{2\epsilon_t}{\epsilon_z} \lambda_y \ldots \alpha = \psi \end{cases}$$
$$= -\frac{\lambda_y \tau_\alpha}{\lambda_{z\alpha}} \tag{F.99}$$

$$\tau_\alpha = \begin{cases} \frac{\mu_t}{\mu_z} \ldots \alpha = \theta \\ \frac{\epsilon_t}{\epsilon_z} \ldots \alpha = \psi \end{cases} \tag{F.100}$$

Now let us explore the cases where $u = \{0, 1, 2, 3\}$. First, note that when $u = 3$ the $\left(1 - e^{j\lambda_y h}\right)$ term goes to zero. Thus there is no contribution from C_0^+ when $u \geq 3$. When $u = 2$, it can be shown that

$$\oint_{C_0^+} = j\pi \frac{\partial}{\partial \lambda_y} \left[\frac{C\lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_x^2 + \lambda_y^2\right) \sin\left(\lambda_{z\alpha} d\right)}\right]\Bigg|_{\lambda_y = 0}$$
$$= j\pi C \left[\frac{-jh\lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)}\right]$$
$$= \pi C \left[\frac{h\lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)}\right] \tag{F.101}$$

where $\lambda_{z\theta}^* = \pm\sqrt{k_t^2 - \frac{\mu_t}{\mu_z} k_x^2}$ and $\lambda_{z\psi}^* = \pm\sqrt{k_t^2 - \frac{\epsilon_t}{\epsilon_z} k_x^2}$.

When $u = 1$, it can be shown that

$$\oint_{C_0^+} = \frac{j\pi}{2} \frac{\partial^2}{\partial {\lambda_y}^2} \left[\frac{C\lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_x^2 + \lambda_y^2\right) \sin\left(\lambda_{z\alpha} d\right)} \right] \Bigg|_{\lambda_y = 0}$$

$$= \frac{j\pi C}{2} \left[\frac{h^2 \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} \right] \tag{F.102}$$

Finally, when $u = 0$, it can be shown that

$$\oint_{C_0^+} = \frac{j\pi}{6} \frac{\partial^3}{\partial {\lambda_y}^3} \left[\frac{C\lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\left(\lambda_x^2 + \lambda_y^2\right) \sin\left(\lambda_{z\alpha} d\right)} \right] \Bigg|_{\lambda_y = 0}$$

$$= \frac{j\pi}{6} C \left\{ \frac{j h^3 \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} + \frac{j3h \left[v \lambda_{z\alpha}^{*(v-2)} \cos\left(\lambda_{z\alpha}^* d\right) - d \lambda_{z\alpha}^{*(v-1)} \csc\left(\lambda_{z\alpha}^*\right) \right] \tau_\alpha}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} \right.$$

$$\left. + \frac{j6h \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^4 \sin\left(\lambda_{z\alpha}^* d\right)} \right\}$$

$$= \frac{\pi C}{6\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} \left\{ -h^3 \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right) - 3h \left[v \lambda_{z\alpha}^{*(v-2)} \cos\left(\lambda_{z\alpha}^* d\right) \right. \right.$$

$$\left. \left. - d \lambda_{z\alpha}^{*(v-1)} \csc\left(\lambda_{z\alpha}^*\right) \right] \tau_\alpha - \frac{6h \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2} \right\} \tag{F.103}$$

Upon close inspection of the $u \in \{0, 1, 2\}$ cases, it can be shown that

$$\oint_{C_0^+} = \frac{-j^{(-u)} \pi h^{(3-u)} C \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right) (3-u)!} - \frac{\pi h C \delta_{u,0}}{2\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} \left(\frac{2\lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2} \right.$$

$$\left. + \left[v \lambda_{z\alpha}^{*(v-2)} \cos\left(\lambda_{z\alpha}^* d\right) - d \lambda_{z\alpha}^{*(v-1)} \csc\left(\lambda_{z\alpha}^*\right) \right] \tau_\alpha \right) \tag{F.104}$$

Similarly, it can be shown that under LHPC

$$\oint_{C_0^-} = \frac{j^u \pi h^{(3-u)} C \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right) (3-u)!} + \frac{\pi h C \delta_{u,0}}{2\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} \left(\frac{2\lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2} \right.$$

$$\left. + \left[v \lambda_{z\alpha}^{*(v-2)} \cos\left(\lambda_{z\alpha}^* d\right) - d \lambda_{z\alpha}^{*(v-1)} \csc\left(\lambda_{z\alpha}^*\right) \right] \tau_\alpha \right) \tag{F.105}$$

Next, analyze the $C^{+}_{j\lambda_x}$ contribution.

$$\oint\limits_{C^{+}_{j\lambda_x}} \frac{1}{(\lambda_y - j\lambda_x)} \left[\frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^4 \left(\lambda_y + j\lambda_x\right)\sin\left(\lambda_{z\alpha} d\right)}\right] d\lambda_y$$

$$= j2\pi \left[\frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^4 \left(\lambda_y + j\lambda_x\right)\sin\left(\lambda_{z\alpha} d\right)}\right]\Bigg|_{\lambda_y = j\lambda_x}$$

$$= j2\pi \left[\frac{Ck_t^v \left(1 - e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\left(j\lambda_x\right)^{4-u}\left(j2\lambda_x\right)\sin\left(k_t d\right)}\right]$$

$$= j^u \pi \left[\frac{Ck_t^v \left(1 - e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(5-u)}\sin\left(k_t d\right)}\right] \tag{F.106}$$

Similarly for the $C^{-}_{-j\lambda_x}$

$$\oint\limits_{C^{-}_{-j\lambda_x}} = j2\pi \left[\frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^4 \left(\lambda_y - j\lambda_x\right)\sin\left(\lambda_{z\alpha} d\right)}\right]\Bigg|_{\lambda_y = -j\lambda_x}$$

$$= j2\pi \left[\frac{C\left(-j\lambda_x\right)^u k_t^v \left(1 - e^{-j(-j\lambda_x)h}\right)\cos\left(k_t d\right)}{\left(-j\lambda_x\right)^4 \left(-j\lambda_x - j\lambda_x\right)\sin\left(k_t d\right)}\right]$$

$$= -\pi C \left[\frac{j^{3u} k_t^v \left(1 - e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(5-u)}\sin\left(k_t d\right)}\right] \tag{F.107}$$

Finally, analyze the $\sum C_i^+$ contribution at $\lambda_{z\alpha} = \frac{i\pi}{d}, i \in \mathbb{Z}$. First, find the value of $\lambda_{y\alpha i}$

$$\lambda_{z\alpha} = \begin{cases} \sqrt{k_t^2 - \frac{\mu_t}{\mu_z}\left(\lambda_x^2 + \lambda_y^2\right)} & ...\alpha = \theta \\ \sqrt{k_t^2 - \frac{\epsilon_t}{\epsilon_z}\left(\lambda_x^2 + \lambda_y^2\right)} & ...\alpha = \psi \end{cases}$$

$$\Rightarrow \left(\frac{i\pi}{d}\right)^2 = \begin{cases} k_t^2 - \frac{\mu_t}{\mu_z}\lambda_x^2 - \frac{\mu_t}{\mu_z}\lambda_y^2 & ...\alpha = \theta \\ k_t^2 - \frac{\epsilon_t}{\epsilon_z}\lambda_x^2 - \frac{\epsilon_t}{\epsilon_z}\lambda_y^2 & ...\alpha = \psi \end{cases}$$

$$\Rightarrow \lambda_{y\alpha i} = \begin{cases} \pm\sqrt{\frac{\mu_z}{\mu_t}\left[k_t^2 - \left(\frac{i\pi}{d}\right)^2\right] - \lambda_x^2} & ...\alpha = \theta \\ \pm\sqrt{\frac{\epsilon_z}{\epsilon_t}\left[k_t^2 - \left(\frac{i\pi}{d}\right)^2\right] - \lambda_x^2} & ...\alpha = \psi \end{cases} \tag{F.108}$$

Note that while technically i ranges in integer values from $-\infty$ to ∞, $\lambda_{-i} = \lambda_i, i \neq 0$. Therefore, if only the values of $i = 0, 1, ..., \infty$ are selected, all the unique values of $\lambda_{y\alpha i}$ are covered. There are two cases that need to be studied for this set of poles: $v \in \{-1, 1\}$.

When $v = -1$, we find from Appendix A that for UHPC

$$\begin{aligned} &\oint_{\sum C_i^+} \left[\frac{1}{\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)}\right]\left[\frac{C\lambda_y^u\left(1 - e^{j\lambda_y h}\right)\cos(\lambda_{z\alpha}d)}{\lambda_y^4\left(\lambda_x^2 + \lambda_y^2\right)}\right] d\lambda_y \\ &= \sum_{i=0}^{\infty} \left. \frac{j2\pi \frac{\partial^{(\sigma-1)}}{\partial {\lambda_y}^{(\sigma-1)}}\left[\frac{C\lambda_y^u\left(1-e^{j\lambda_y h}\right)\cos(\lambda_{z\alpha}d)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)}\right]}{\frac{\partial^\sigma}{\partial {\lambda_y}^\sigma}\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)} \right|_{\lambda_y = -\lambda_{y\alpha i}} \end{aligned} \tag{F.109}$$

where σ is the order of the pole $\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)$. When $i > 0$, $\sigma = 1$. When $i = 0$

$$\begin{aligned} \lambda_{z\alpha}\sin(\lambda_{z\alpha}d) &= 0 \\ \frac{\partial}{\partial\lambda_y}\lambda_{z\alpha}\sin(\lambda_{z\alpha}d) &= \sin(\lambda_{z\alpha}d)\frac{\partial}{\partial\lambda_y}\lambda_{z\alpha} + \lambda_{z\alpha}\cos(\lambda_{z\alpha}d)\frac{\partial}{\partial\lambda_y}\lambda_{z\alpha}d \\ &= -\lambda_y\left[\frac{d\sin(i\pi)}{i\pi} + d\cos(i\pi)\right]\tau_\alpha \\ &= -\lambda_y d\left[(-1)^i + \delta_{i,0}\right]\tau_\alpha \\ &\neq 0 \Rightarrow \sigma = 1 \end{aligned} \tag{F.110}$$

Therefore, no matter the value of i, $\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)$ is always a simple pole. Hence,

$$\oint_{\sum C_i^+} \left[\frac{1}{\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)}\right]\left[\frac{C\lambda_y^u\cos(\lambda_{z\alpha}d)\left(1-e^{j\lambda_y h}\right)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)}\right]d\lambda_y$$

$$= \sum_{i=0}^{\infty} \left.\frac{j2\pi\left[\frac{C\lambda_y^u\cos(\lambda_{z\alpha}d)\left(1-e^{j\lambda_y h}\right)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)}\right]}{\frac{\partial}{\partial\lambda_y}\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)}\right|_{\lambda_y=-\lambda_{y\alpha i}}$$

$$= -j2\pi\sum_{i=0}^{\infty}\left.\left\{\frac{C\lambda_y^u\cos(\lambda_{z\alpha}d)\left(1-e^{j\lambda_y h}\right)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)\lambda_y d\left[(-1)^i+\delta_{i,0}\right]\tau_\alpha}\right\}\right|_{\lambda_y=-\lambda_{y\alpha i}}$$

$$= -j2\pi\sum_{i=0}^{\infty}\left\{\frac{C(-1)^i\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{(-\lambda_{y\alpha i})^{(5-u)}\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)d\left[(-1)^i+\delta_{i,0}\right]\tau_\alpha}\right\}$$

$$= -j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\left\{\frac{\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{(-\lambda_{y\alpha i})^{(5-u)}\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left[1+\delta_{i,0}\right]\tau_\alpha}\right\} \tag{F.111}$$

In a similar way, it can be shown that the $\sum C_i^-$ contribution when $v=-1$ is

$$\oint_{\sum C_i^-} \left[\frac{1}{\lambda_{z\alpha}\sin(\lambda_{z\alpha}d)}\right]\left[\frac{C\lambda_y^u\cos(\lambda_{z\alpha}d)\left(1-e^{-j\lambda_y h}\right)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)}\right]d\lambda_y$$

$$= -j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\left[\frac{\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(5-u)}\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+3\delta_{i,0}\right)\tau_\alpha}\right] \tag{F.112}$$

Next, if $v=1$

$$\oint_{\sum C_i^+} \left[\frac{1}{\sin(\lambda_{z\alpha}d)}\right]\left[\frac{C\lambda_y^u\lambda_{z\alpha}\left(1-e^{j\lambda_y h}\right)\cos(\lambda_{z\alpha}d)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)}\right]d\lambda_y$$

$$= \sum_{i=0}^{\infty}\left.\frac{j2\pi\left[\frac{C\lambda_y^u\lambda_{z\alpha}\left(1-e^{j\lambda_y h}\right)\cos(\lambda_{z\alpha}d)}{\lambda_y^4\left(\lambda_x^2+\lambda_y^2\right)}\right]}{\frac{\partial}{\partial\lambda_y}\sin(\lambda_{z\alpha}d)}\right|_{\lambda_y=-\lambda_{y\alpha i}}$$

$$= \sum_{i=0}^{\infty} j2\pi \left[\frac{C\lambda_y^u \lambda_{z\alpha} \left(1 - e^{j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^4 \left(\lambda_x^2 + \lambda_y^2\right) d \cos\left(\lambda_{z\alpha} d\right) \frac{\partial}{\partial \lambda_y} \lambda_{z\alpha}} \right] \Bigg|_{\lambda_y = -\lambda_{y\alpha i}}$$

$$= -j \frac{2\pi C}{d} \sum_{i=0}^{\infty} \left[\frac{\lambda_{z\alpha}^2 \left(1 - e^{-j\lambda_{y\alpha i} h}\right)}{\left(-\lambda_{y\alpha i}\right)^{(5-u)} \left(\lambda_x^2 + \lambda_{y\alpha i}^2\right) \tau_\alpha} \right] \tag{F.113}$$

In a similar way, it can be shown that when $v = 1$, the $\sum C_i^-$ contribution is

$$\oint_{\sum C_i^-} \left[\frac{1}{\sin\left(\lambda_{z\alpha} d\right)} \right] \left[\frac{C\lambda_y^u \lambda_{z\alpha} \left(1 - e^{-j\lambda_y h}\right) \cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^4 \left(\lambda_x^2 + \lambda_y^2\right)} \right] d\lambda_y$$

$$= -j \frac{2\pi C}{d} \sum_{i=0}^{\infty} \left(\frac{i\pi}{d} \right)^2 \left[\frac{\left(1 - e^{-j\lambda_{y\alpha i} h}\right)}{\lambda_{y\alpha i}^{(5-u)} \left(\lambda_x^2 + \lambda_{y\alpha i}^2\right) \tau_\alpha} \right] \tag{F.114}$$

By inspecting both cases of $v \in \{-1, 1\}$, it can be more generally shown that

$$\oint_{\sum C_i^+} = -j \frac{2\pi C}{d} \sum_{i=0}^{\infty} \left(\frac{i\pi}{d} \right)^{(v+1)} \left[\frac{\left(1 - e^{-j\lambda_{y\alpha i} h}\right)}{\left(-\lambda_{y\alpha i}\right)^{(5-u)} \left(\lambda_x^2 + \lambda_{y\alpha i}^2\right) \left(1 + \delta_{i,0}\delta_{v,-1}\right) \tau_\alpha} \right] \tag{F.115}$$

$$\oint_{\sum C_i^-} = -j \frac{2\pi C}{d} \sum_{i=0}^{\infty} \left(\frac{i\pi}{d} \right)^{(v+1)} \left[\frac{\left(1 - e^{-j\lambda_{y\alpha i} h}\right)}{\lambda_{y\alpha i}^{(5-u)} \left(\lambda_x^2 + \lambda_{y\alpha i}^2\right) \left(1 + \delta_{i,0}\delta_{v,-1}\right) \tau_\alpha} \right] \tag{F.116}$$

Finally, it can be shown that when $p = q = 1$, the integral with respect to λ_y evaluates to

$$\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y} \lambda_y^u \lambda_{z\alpha}^v \Upsilon_{16}^\alpha}{2\lambda_\rho^2} d\lambda_y = \frac{\left[-j^{(-u)} - j^u\right] \pi h^{(3-u)} C \lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right) (3-u)!}$$
$$- \frac{\pi h C \delta_{u,0}}{\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)} \left(\frac{2\lambda_{z\alpha}^{*v} \cos\left(\lambda_{z\alpha}^* d\right)}{\lambda_x^2} + \left[v\lambda_{z\alpha}^{*(v-2)} \cos\left(\lambda_{z\alpha}^* d\right) - d\lambda_{z\alpha}^{*(v-1)} \csc\left(\lambda_{z\alpha}^*\right) \right] \tau_\alpha \right)$$
$$+ \pi C \left[\frac{j^u k_t^v \left(1 - e^{-\lambda_x h}\right) \cos\left(k_t d\right)}{\lambda_x^{(5-u)} \sin\left(k_t d\right)} \right] - \left\{ -\pi C \left[\frac{j^{3u} k_t^v \left(1 - e^{-\lambda_x h}\right) \cos\left(k_t d\right)}{\lambda_x^{(5-u)} \sin\left(k_t d\right)} \right] \right\}$$
$$+ (-j) \frac{2\pi C}{d} \sum_{i=0}^{\infty} \left(\frac{i\pi}{d} \right)^{(v+1)} \left[\frac{\left(1 - e^{-j\lambda_{y\alpha i} h}\right)}{\left(-\lambda_{y\alpha i}\right)^{(5-u)} \left(\lambda_x^2 + \lambda_{y\alpha i}^2\right) \left(1 + \delta_{i,0}\delta_{v,-1}\right) \tau_\alpha} \right]$$
$$- (-j) \frac{2\pi C}{d} \sum_{i=0}^{\infty} \left(\frac{i\pi}{d} \right)^{(v+1)} \left[\frac{\left(1 - e^{-j\lambda_{y\alpha i} h}\right)}{\lambda_{y\alpha i}^{(5-u)} \left(\lambda_x^2 + \lambda_{y\alpha i}^2\right) \left(1 + \delta_{i,0}\delta_{v,-1}\right) \tau_\alpha} \right]$$

$$\boxed{\begin{aligned}
&\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^\alpha}{2\lambda_\rho^2}d\lambda_y \\
&= -\pi C\left\{\frac{(j^u+j^{3u})(\delta_{u,0}+\delta_{u,1}+\delta_{u,2})\,h^{(3-u)}\lambda_{z\alpha}^{*v}\cos(\lambda_{z\alpha}^* d)}{\lambda_x^2\sin(\lambda_{z\alpha}^* d)(3-u)!}\right. \\
&\qquad + \frac{h\delta_{u,0}}{\lambda_x^2\sin(\lambda_{z\alpha}^* d)}\left(\frac{2\lambda_{z\alpha}^{*v}\cos(\lambda_{z\alpha}^* d)}{\lambda_x^2}\right. \\
&\qquad \left.\left.+\tau_\alpha\left[v\lambda_{z\alpha}^{*(v-2)}\cos(\lambda_{z\alpha}^* d) - d\lambda_{z\alpha}^{*(v-1)}\csc(\lambda_{z\alpha}^*)\right]\right)\right\} \\
&\qquad + \pi C\left[\frac{(j^u+j^{3u})\,k_t^v\left(1-e^{-\lambda_x h}\right)\cos(k_t d)}{\lambda_x^{(5-u)}\sin(k_t d)}\right] \\
&\qquad + j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\frac{\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)\left[1+(-1)^u\right]}{\lambda_{y\alpha i}^{(5-u)}\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)(1+\delta_{i,0}\delta_{v,-1})\,\tau_\alpha} \\
&...p=q=1, \tau_\alpha = \begin{cases} \frac{\mu_t}{\mu_z} & ...\alpha=\theta \\ \frac{\epsilon_t}{\epsilon_z} & ...\alpha=\psi \end{cases}
\end{aligned}} \tag{F.117}$$

Case 2: $p = 1, q \neq 1$**.**

In this case, $k_{yw_p}^B = 0$, reducing (F.93) to

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^\alpha}{2\lambda_\rho^2}d\lambda_y &= \int_{-\infty}^{\infty}\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{j\lambda_y h}\right)\cos(\lambda_{z\alpha}d)}{\lambda_y^2\left(\lambda_y+k_{yw_q}^B\right)\left(\lambda_y-k_{yw_q}^B\right)\left(\lambda_x^2+\lambda_y^2\right)\sin(\lambda_{z\alpha}d)}d\lambda_y \\
&\quad + \int_{-\infty}^{\infty}\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{-j\lambda_y h}\right)\cos(\lambda_{z\alpha}d)}{\lambda_y^2\left(\lambda_y+k_{yw_q}^B\right)\left(\lambda_y-k_{yw_q}^B\right)\left(\lambda_x^2+\lambda_y^2\right)\sin(\lambda_{z\alpha}d)}d\lambda_y
\end{aligned} \tag{F.118}$$

This analysis is very similar to Case 1, except there are now an order $2-u$ pole at $\lambda_y = 0$, and simple poles at $\lambda_y = \pm k_{yw_q}^B$, $\lambda_y = \pm j\lambda_x$, and $\lambda_{z\alpha} = \frac{i\pi}{d}$. Since one of the poles will ultimately be removable, it can be shown that when $u = 0$

$$\oint_{C_0^+} \frac{C\lambda_y^u \lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}{\lambda_y^2 \left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha} d\right)} d\lambda_y$$
$$= j\pi C \frac{\partial}{\partial \lambda_y}\left[\frac{\overbrace{\lambda_{z\alpha}^v \left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha} d\right)}^{N}}{\underbrace{\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha} d\right)}_{D}}\right]\Bigg|_{\lambda_y = 0} \tag{F.119}$$

It is important to note that as $\lambda_y \to 0$, the only surviving term in $\frac{\partial}{\partial \lambda_y} N$ is the term involving $\frac{\partial}{\partial \lambda_y}\left(1 - e^{j\lambda_y h}\right)$. Furthermore, it can be shown that as $\lambda_y \to 0$, $\frac{\partial}{\partial \lambda_y} D = 0$. Thus, from the quotient rule

$$\oint_{C_0^+} = j\pi C \left\{\frac{-jh\lambda_{z\alpha}^{*v}\cos\left(\lambda_{z\alpha}^* d\right) \cancel{k_{yw_q}^B \left(-k_{yw_q}^B\right)\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)}}{\left[\left(k_{yw_q}^B\right)\left(-k_{yw_q}^B\right)\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)\right]^{\cancel{2}}}\right\}$$
$$= -\pi C \left[\frac{h\lambda_{z\alpha}^{*v}\cos\left(\lambda_{z\alpha}^* d\right)}{k_{yw_q}^{B2}\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)}\right]\delta_{u,0} \tag{F.120}$$

Similarly, it can be shown that under LHPC

$$\oint_{C_0^-} = j\pi C \left\{\frac{jh\lambda_{z\alpha}^{*v}\cos\left(\lambda_{z\alpha}^* d\right) \cancel{k_{yw_q}^B \left(-k_{yw_q}^B\right)\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)}}{\left[\left(k_{yw_q}^B\right)\left(-k_{yw_q}^B\right)\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)\right]^{\cancel{2}}}\right\}$$
$$= \pi C \left[\frac{h\lambda_{z\alpha}^{*v}\cos\left(\lambda_{z\alpha}^* d\right)}{k_{yw_q}^{B2}\lambda_x^2 \sin\left(\lambda_{z\alpha}^* d\right)}\right]\delta_{u,0} \tag{F.121}$$

Next, analyze the $C_{\pm k_{yw_q}^B}^+$ contribution. It is important to note that $\pm k_{yw_q}^B$ are purely real. Therefore these poles lie on the real axis and should be treated in a similar fashion to the $\lambda_y = 0$ poles. Beginning with the pole at $\lambda_y = k_{yw_q}^B$, it can be shown that

$$\oint_{C^+_{k^B_{yw_q}}} = j\pi C \left[\frac{\lambda^v_{z\alpha}\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)} \right]\Bigg|_{\lambda_y = k^B_{yw_q}} \tag{F.122}$$

Again, the $\left(1-e^{j\lambda_y h}\right)$ term is zero when $\lambda_y = k^B_{yw_q}$. However, the denominator is non-zero when $\lambda_y = k^B_{yw_q}$. Therefore, there is no $C^+_{k^B_{yw_q}}$ contribution. Similarly, it can be shown that there is also no $C^-_{k^B_{yw_q}}$ contribution. Furthermore, it can be shown there are no $C^{\pm}_{-k^B_{yw_q}}$ contributions.

Next, analyze the $C^+_{j\lambda_x}$ contribution. It can be shown that

$$\begin{aligned}
\oint_{C^+_{j\lambda_x}} &= j2\pi \left.\frac{C\lambda_y^u\lambda^v_{z\alpha}\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\lambda_y^2\left(\lambda_y+k^B_{yw_q}\right)\left(\lambda_y-k^B_{yw_q}\right)\left(\lambda_y+j\lambda_x\right)\sin\left(\lambda_{z\alpha}d\right)}\right|_{\lambda_y=j\lambda_x} \\
&= j2\pi \frac{Ck_t^v\left(1-e^{jj\lambda_x h}\right)\cos\left(k_t d\right)}{\left(j\lambda_x\right)^{(2-u)}\left(j\lambda_x+k^B_{yw_q}\right)\left(j\lambda_x-k^B_{yw_q}\right)\left(j\lambda_x+j\lambda_x\right)\sin\left(k_t d\right)} \\
&= -\pi\frac{j^u Ck_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(3-u)}\left(\lambda_x^2+k^{B2}_{yw_q}\right)\sin\left(k_t d\right)}
\end{aligned} \tag{F.123}$$

Similarly,

$$\begin{aligned}
\oint_{C^-_{-j\lambda_x}} &= j2\pi \left.\frac{C\lambda_y^u\lambda^v_{z\alpha}\left(1-e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\lambda_y^2\left(\lambda_y+k^B_{yw_q}\right)\left(\lambda_y-k^B_{yw_q}\right)\left(\lambda_y-j\lambda_x\right)\sin\left(\lambda_{z\alpha}d\right)}\right|_{\lambda_y=-j\lambda_x} \\
&= j2\pi \frac{Ck_t^v\left(1-e^{-j(-j\lambda_x)h}\right)\cos\left(k_t d\right)}{\left(-j\lambda_x\right)^{(2-u)}\left(\left(-j\lambda_x\right)+k^B_{yw_q}\right)\left(\left(-j\lambda_x\right)-k^B_{yw_q}\right)\left(\left(-j\lambda_x\right)-j\lambda_x\right)\sin\left(k_t d\right)} \\
&= \pi C\frac{j^{3u}k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(3-u)}\left(\lambda_x^2+k^{B2}_{yw_q}\right)\sin\left(k_t d\right)}
\end{aligned} \tag{F.124}$$

Finally, analyze the $\sum C_i^+$ contribution. Similar to Case 1, it can be shown that

$$\oint\limits_{\sum C_i^+} = -j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\frac{\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\left(-\lambda_{y\alpha i}\right)^{(3-u)}\left(\lambda_{y\alpha i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \tag{F.125}$$

$$\oint\limits_{\sum C_i^-} = -j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\frac{\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(3-u)}\left(\lambda_{y\alpha i}+k_{yw_q}^{B}\right)\left(\lambda_{y\alpha i}-k_{yw_q}^{B}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \tag{F.126}$$

Combining all these residue contributions

$$\boxed{\begin{aligned}\int_{-\infty}^{\infty}\frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^\alpha}{2\lambda_\rho^2}d\lambda_y &= -2\pi C\left[\frac{h\lambda_{z\alpha}^{*v}\cos\left(\lambda_{z\alpha}^*d\right)}{k_{yw_q}^{B2}\lambda_x^2\sin\left(\lambda_{z\alpha}^*d\right)}\right]\delta_{u,0}\\ &-\pi C\left[\frac{\left(j^u+j^{3u}\right)k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(3-u)}\left(\lambda_x^2+k_{yw_q}^{B2}\right)\sin\left(k_t d\right)}\right]\\ &+j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\frac{\left[1+(-1)^u\right]\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(3-u)}\left(\lambda_{y\alpha i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha}\\ &\ldots p=1, q\neq 1\end{aligned}} \tag{F.127}$$

Case 3: $p\neq 1, q=1$**.**

By inspection from the previous case, it can be seen that

$$\boxed{\begin{aligned}\int_{-\infty}^{\infty}\frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^\alpha}{2\lambda_\rho^2}d\lambda_y &= -2\pi C\left[\frac{h\lambda_{z\alpha}^{*v}\cos\left(\lambda_{z\alpha}^*d\right)}{k_{yw_p}^{B2}\lambda_x^2\sin\left(\lambda_{z\alpha}^*d\right)}\right]\delta_{u,0}\\ &-\pi C\left[\frac{\left(j^u+j^{3u}\right)k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(3-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)\sin\left(k_t d\right)}\right]\\ &+j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\frac{\left[1+(-1)^u\right]\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(3-u)}\left(\lambda_{y\alpha i}^2-k_{yw_p}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha}\\ &\ldots p\neq 1, q=1\end{aligned}} \tag{F.128}$$

Case 4: $p = q \neq 1$.

In this case, $k_{yw_p}^B = k_{yw_q}^B \neq 0$, reducing (F.93) to

$$\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^{\alpha}}{2\lambda_\rho^2} d\lambda_y = \int_{-\infty}^{\infty} \frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y + k_{yw_p}^B\right)^2\left(\lambda_y - k_{yw_p}^B\right)^2\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)} d\lambda_y + \int_{-\infty}^{\infty} \frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1 - e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y + k_{yw_p}^B\right)^2\left(\lambda_y - k_{yw_p}^B\right)^2\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)} d\lambda_y \tag{F.129}$$

In this case, it is apparent that no poles exist at $\lambda_y = 0$, an order-2 pole exists at $\lambda_y = \pm k_{yw_p}^B$, and the familiar simple poles exist at $\lambda_y = \pm j\lambda_x$ and $\lambda_{z\alpha} = \frac{i\pi}{d}$. Begin by exploring the $C_{k_{yw_p}^B}^+$ contribution. Since this represents an order-2 pole on the real axis,

$$\oint_{C_{k_{yw_p}^B}^+} = j\pi \frac{\partial}{\partial\lambda_y}\left[\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1 - e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y + k_{yw_p}^B\right)^2\left(\lambda_y - k_{yw_p}^B\right)^2\left(\lambda_x^2 + \lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\Bigg|_{\lambda_y = k_{yw_p}^B} \tag{F.130}$$

Again, noting that the numerator term $\left(1 - e^{j\lambda_y h}\right)$ is a zero at $\lambda_y = k_{yw_p}^B$ implies that the only term that survives differentiation of the numerator is the chain rule term where the derivative of $\left(1 - e^{j\lambda_y h}\right)$ is taken. Thus,

$$\frac{\partial}{\partial\lambda_y} N = -jhC\lambda_y^u\lambda_{z\alpha}^v\cos\left(\lambda_{z\alpha}d\right) + \cancelto{0}{\cdots}\Big|_{\lambda_y = k_{yw_p}^B} \tag{F.131}$$

While there are no terms in the derivative of the denominator that go to zero after the substitution, multiplication with the zero in the numerator ultimately destroys the $N\frac{\partial}{\partial\lambda_y}D$ term from the quotient rule. Therefore, from the quotient rule

$$\begin{aligned}
\oint_{C^+_{k^B_{yw_p}}} &= j\pi C\left[\left.\frac{-jh\lambda_y^u\lambda_{z\alpha}^v e^{j\lambda_y h}\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y+k^B_{yw_p}\right)^2\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\right|_{\lambda_y=k^B_{yw_p}} \\
&= j\pi C\left[\frac{-jhk^{Bu}_{yw_p}\lambda_{z\alpha p}^v\cos\left(\lambda_{z\alpha p}d\right)}{\left(k^B_{yw_p}+k^B_{yw_p}\right)^2\left(\lambda_x^2+k^{B2}_{yw_p}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right] \\
&= \pi C\left[\frac{hk^{Bu}_{yw_p}\lambda_{z\alpha p}^v\cos\left(\lambda_{z\alpha p}d\right)}{4k^{B2}_{yw_p}\left(\lambda_x^2+k^{B2}_{yw_p}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right]
\end{aligned} \tag{F.132}$$

where $\lambda_{z\alpha p}=\sqrt{k_t^2-\tau_\alpha\left(\lambda_x^2+k^{B2}_{yw_p}\right)}$. Similarly, it can be shown that under LHPC

$$\oint_{C^-_{k^B_{yw_p}}} = -\pi C\left[\frac{hk^{Bu}_{yw_p}\lambda_{z\alpha p}^v\cos\left(\lambda_{z\alpha p}d\right)}{4k^{B2}_{yw_p}\left(\lambda_x^2+k^{B2}_{yw_p}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right] \tag{F.133}$$

Next, analyze the $C^+_{-k^B_{yw_p}}$ contribution.

$$\begin{aligned}
\oint_{C^+_{-k^B_{yw_p}}} &= j\pi\,\frac{\partial}{\partial\lambda_y}\left[\left.\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y-k^B_{yw_p}\right)^2\left(\lambda_y-k^B_{yw_p}\right)^2\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\right|_{\lambda_y=-k^B_{yw_p}} \\
&= j\pi C\left[\left.\frac{-jh\lambda_y^u\lambda_{z\alpha}^v e^{j\lambda_y h}\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y-k^B_{yw_p}\right)^2\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\right|_{\lambda_y=-k^B_{yw_p}} \\
&= j\pi C\left[\frac{-jhk^{Bu}_{yw_p}\lambda_{z\alpha p}^v\cos\left(\lambda_{z\alpha p}d\right)}{\left(-k^B_{yw_p}-k^B_{yw_p}\right)^2\left(\lambda_x^2+k^{B2}_{yw_p}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right] \\
&= \pi C\left[\frac{hk^{Bu}_{yw_p}\lambda_{z\alpha p}^v\cos\left(\lambda_{z\alpha p}d\right)}{4k^{B2}_{yw_p}\left(\lambda_x^2+k^{B2}_{yw_p}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right]
\end{aligned} \tag{F.134}$$

Similarly, it can be shown that under LHPC

$$\oint_{C^-_{-k^B_{yw_p}}} = -\pi C\left[\frac{hk^{Bu}_{yw_p}\lambda^v_{z\alpha p}\cos\left(\lambda_{z\alpha p}d\right)}{4k^{B2}_{yw_p}\left(\lambda_x^2+k^{B2}_{yw_p}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right] \tag{F.135}$$

Next, analyze the $C^+_{j\lambda_x}$ contribution. It can be shown that

$$\begin{aligned}\oint_{C^+_{j\lambda_x}} &= j2\pi\left[\frac{C\lambda_y^u\lambda^v_{z\alpha}\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y+k^B_{yw_p}\right)^2\left(\lambda_y-k^B_{yw_p}\right)^2\left(\lambda_y+j\lambda_x\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\Bigg|_{\lambda_y=j\lambda_x}\\ &= j2\pi\left[\frac{C\left(j\lambda_x\right)^u k_t^v\left(1-e^{jj\lambda_x h}\right)\cos\left(k_t d\right)}{\left(j\lambda_x+k^B_{yw_p}\right)^2\left(j\lambda_x-k^B_{yw_p}\right)^2\left(j\lambda_x+j\lambda_x\right)\sin\left(k_t d\right)}\right]\\ &= \pi C\left[\frac{j^u k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k^{B2}_{yw_p}\right)^2\sin\left(k_t d\right)}\right]\end{aligned} \tag{F.136}$$

Analyzing the $C^-_{-j\lambda_x}$ contribution, it can be shown that

$$\begin{aligned}\oint_{C^-_{-j\lambda_x}} &= j2\pi\left[\frac{C\lambda_y^u\lambda^v_{z\alpha}\left(1-e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y+k^B_{yw_p}\right)^2\left(\lambda_y-k^B_{yw_p}\right)^2\left(\lambda_y-j\lambda_x\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\Bigg|_{\lambda_y=-j\lambda_x}\\ &= j2\pi\left[\frac{C\left(-j\lambda_x\right)^u k_t^v\left(1-e^{-j(-j\lambda_x)h}\right)\cos\left(k_t d\right)}{\left(-j\lambda_x+k^B_{yw_p}\right)^2\left(-j\lambda_x-k^B_{yw_p}\right)^2\left(-j\lambda_x-j\lambda_x\right)\sin\left(k_t d\right)}\right]\\ &= -\pi C\left[\frac{j^{3u} k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k^{B2}_{yw_p}\right)^2\sin\left(k_t d\right)}\right]\end{aligned} \tag{F.137}$$

Finally, analyze the $\sum C_i^{\pm}$ contributions. By inspection from previous cases, it can be shown that

$$\oint\limits_{\sum C_i^+} = \sum_{i=0}^{\infty} \frac{\left(-j\frac{2\pi C}{d}\right)\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\left(-\lambda_{y\alpha i}\right)^{(1-u)}\left(\lambda_{y\alpha i}+k_{yw_p}^B\right)^2\left(\lambda_{y\alpha i}-k_{yw_p}^B\right)^2\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \tag{F.138}$$

$$\oint\limits_{\sum C_i^-} = -j\frac{2\pi C}{d}\sum_{i=0}^{\infty} \frac{\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(1-u)}\left(\lambda_{y\alpha i}+k_{yw_p}^B\right)^2\left(\lambda_{y\alpha i}-k_{yw_p}^B\right)^2\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \tag{F.139}$$

Combining all the residue contributions

$$\boxed{\begin{aligned}\int\limits_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^{\alpha}}{2\lambda_\rho^2}d\lambda_y &= \pi C\left[\frac{hk_{yw_p}^{Bu}\lambda_{z\alpha p}^v\cos\left(\lambda_{z\alpha p}d\right)}{k_{yw_p}^{B2}\left(\lambda_x^2+k_{yw_p}^{B2}\right)\sin\left(\lambda_{z\alpha p}d\right)}\right] \\ &+\pi C\left[\frac{\left(j^u+j^{3u}\right)k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)^2\sin\left(k_t d\right)}\right] \\ &+j\frac{2\pi C}{d}\sum_{i=0}^{\infty}\frac{\left[1+(-1)^u\right]\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(1-u)}\left(\lambda_{y\alpha i}^2-k_{yw_p}^{B2}\right)^2\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \\ &\ldots p=q\neq 1\end{aligned}} \tag{F.140}$$

Case 5: $p \neq q, p \neq 1, q \neq 1$**.**

In this case, (F.93) does not simplify at all. Therefore, there are no poles at $\lambda_y = 0$, as with Case 4. There are only simple poles at $\lambda_y = \pm k_{yw_p}^B$, $\lambda_y = \pm k_{yw_q}^B$, $\lambda_y = \pm j\lambda_x$, and $\lambda_{z\alpha} = \frac{i\pi}{d}$. Begin by analyzing the $C_{k_{yw_p}^B}^+$ contribution.

$$\oint\limits_{C_{k_{yw_p}^B}^+} = j\pi\left.\left[\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y+k_{yw_p}^B\right)\left(\lambda_y+k_{yw_q}^B\right)\left(\lambda_y-k_{yw_q}^B\right)\left(\lambda_x^2+\lambda_y^2\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\right|_{\lambda_y=k_{yw_p}^B} \tag{F.141}$$

Note that $\left(1-e^{j\pm k^B_{yw_p}h}\right)=0$. Therefore, it can be shown that there are no contributions from $C^{\pm}_{\pm k^B_{yw_p}}$. Furthermore, it is a trivial matter to show that the same is true for $C^{\pm}_{\pm k^B_{yw_q}}$ contributions.

Next, analyze the $C^+_{j\lambda_x}$ contribution.

$$\begin{aligned}
\oint_{C^+_{j\lambda_x}} &= j2\pi\left.\left[\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y^2-k^{B2}_{yw_p}\right)\left(\lambda_y^2-k^{B2}_{yw_q}\right)\left(\lambda_y+j\lambda_x\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\right|_{\lambda_y=j\lambda_x}\\
&= j2\pi\left[\frac{C\left(j\lambda_x\right)^u k_t^v\left(1-e^{jj\lambda_x h}\right)\cos\left(k_t d\right)}{\left(-\lambda_x^2-k^{B2}_{yw_p}\right)\left(-\lambda_x^2-k^{B2}_{yw_q}\right)\left(j\lambda_x+j\lambda_x\right)\sin\left(k_t d\right)}\right]\\
&= \pi C\left[\frac{j^u k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k^{B2}_{yw_p}\right)\left(\lambda_x^2+k^{B2}_{yw_q}\right)\sin\left(k_t d\right)}\right]
\end{aligned}\tag{F.142}$$

Similarly, for the $C^-_{-j\lambda_x}$ contribution it can be shown that

$$\begin{aligned}
\oint_{C^-_{-j\lambda_x}} &= j2\pi\left.\left[\frac{C\lambda_y^u\lambda_{z\alpha}^v\left(1-e^{-j\lambda_y h}\right)\cos\left(\lambda_{z\alpha}d\right)}{\left(\lambda_y^2-k^{B2}_{yw_p}\right)\left(\lambda_y^2-k^{B2}_{yw_q}\right)\left(\lambda_y-j\lambda_x\right)\sin\left(\lambda_{z\alpha}d\right)}\right]\right|_{\lambda_y=-j\lambda_x}\\
&= j2\pi\left[\frac{C\left(-j\lambda_x\right)^u k_t^v\left(1-e^{-j(-j\lambda_x)h}\right)\cos\left(k_t d\right)}{\left(-\lambda_x^2-k^{B2}_{yw_p}\right)\left(-\lambda_x^2-k^{B2}_{yw_q}\right)\left(-j\lambda_x-j\lambda_x\right)\sin\left(k_t d\right)}\right]\\
&= -\pi C\left[\frac{j^{3u} k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k^{B2}_{yw_p}\right)\left(\lambda_x^2+k^{B2}_{yw_q}\right)\sin\left(k_t d\right)}\right]
\end{aligned}\tag{F.143}$$

Finally, analyze the $\sum C_i^+$ contribution. By inspection from previous cases, it can be shown that

$$\oint\limits_{\sum C_i^+} = \sum_{i=0}^{\infty} \frac{\left(-j\frac{2\pi C}{d}\right)\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\left(-\lambda_{y\alpha i}\right)^{(1-u)}\left(\lambda_{y\alpha i}^2-k_{yw_p}^{B2}\right)\left(\lambda_{y\alpha i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \tag{F.144}$$

$$\oint\limits_{\sum C_i^-} = -j\frac{2\pi C}{d}\sum_{i=0}^{\infty} \frac{\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)}{\lambda_{y\alpha i}^{(1-u)}\left(\lambda_{y\alpha i}^2-k_{yw_p}^{B2}\right)\left(\lambda_{y\alpha i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)\tau_\alpha} \tag{F.145}$$

Combining all the residue contributions together

$$\boxed{\begin{aligned}&\int\limits_{-\infty}^{\infty}\frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^\alpha}{2\lambda_\rho^2}d\lambda_y\\ &\qquad\qquad = \pi C\left[\frac{\left(j^u+j^{3u}\right)k_t^v\left(1-e^{-\lambda_x h}\right)\cos\left(k_t d\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)\left(\lambda_x^2+k_{yw_q}^{B2}\right)\sin\left(k_t d\right)}\right]\\ &\quad+\sum_{i=0}^{\infty}\frac{\left(j\frac{2\pi C}{d}\right)\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)\tau_\alpha^{-1}}{\left(\lambda_{y\alpha i}\right)^{(1-u)}\left(\lambda_{y\alpha i}^2-k_{yw_p}^{B2}\right)\left(\lambda_{y\alpha i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)\left(1+\delta_{i,0}\delta_{v,-1}\right)}\\ &...p\neq q, p\neq 1, q\neq 1\end{aligned}} \tag{F.146}$$

Analyzing all five cases, one can see that the $\sum C_i^{\pm}$ contributions developed in Case 5 generalize down to all the other cases. This is intuitive since the configuration of these poles never changes. Therefore, a generalized solution to the λ_y integral can be formulated by switching different components on and off using delta functions for the different cases. Therefore

$$\boxed{\begin{aligned}
&\int_{-\infty}^{\infty}\frac{C\Theta_3^{\lambda_y}\lambda_y^u\lambda_{z\alpha}^v\Upsilon_{16}^{\alpha}}{2\lambda_\rho^2}d\lambda_y\\
&\quad=-\pi C\delta_{p,q}\delta_{p,1}\left\{\frac{(\delta_{u,0}+\delta_{u,1}+\delta_{u,2})\,(j^u+j^{3u})\,h^{(3-u)}\lambda_{z\alpha}^{*v}\cos(\lambda_{z\alpha}^*d)}{\lambda_x^2\sin(\lambda_{z\alpha}^*d)\,(3-u)!}\right.\\
&\qquad+\frac{h\delta_{u,0}}{\lambda_x^2\sin(\lambda_{z\alpha}^*d)}\left(\frac{2\lambda_{z\alpha}^{*v}\cos(\lambda_{z\alpha}^*d)}{\lambda_x^2}+\tau_\alpha\left[v\lambda_{z\alpha}^{*(v-2)}\cos(\lambda_{z\alpha}^*d)\right.\right.\\
&\quad\left.\left.\left.-d\lambda_{z\alpha}^{*(v-1)}\csc(\lambda_{z\alpha}^*)\right]\right)\right\}-2\pi C\delta_{u,0}\left[\frac{h\lambda_{z\alpha}^{*v}\cos(\lambda_{z\alpha}^*d)}{\lambda_x^2\sin(\lambda_{z\alpha}^*d)}\right]\left[\frac{\delta_{p,1}(1-\delta_{q,1})}{k_{yw_q}^{B2}}\right.\\
&\left.+\frac{\delta_{q,1}(1-\delta_{p,1})}{k_{yw_p}^{B2}}\right]+\delta_{p,q}(1-\delta_{p,1})(1-\delta_{q,1})\left[\frac{\pi Chk_{yw_p}^{Bu}\lambda_{z\alpha p}^v\cos(\lambda_{z\alpha p}d)}{k_{yw_p}^{B2}\left(\lambda_x^2+k_{yw_p}^{B2}\right)\sin(\lambda_{z\alpha p}d)}\right]\\
&\quad+\pi C\left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1}+\delta_{q,1})}(j^u+j^{3u})\,k_t^v\left(1-e^{-\lambda_xh}\right)\cos(k_td)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)\left(\lambda_x^2+k_{yw_q}^{B2}\right)\sin(k_td)}\right]\\
&\quad+\sum_{i=0}^{\infty}\frac{\left(j\frac{2\pi C}{d}\right)\left[1+(-1)^u\right]\left(\frac{i\pi}{d}\right)^{(v+1)}\left(1-e^{-j\lambda_{y\alpha i}h}\right)\tau_\alpha^{-1}}{(\lambda_{y\alpha i})^{(1-u)}\left(\lambda_{y\alpha i}^2-k_{yw_p}^{B2}\right)\left(\lambda_{y\alpha i}^2-k_{yw_q}^{B2}\right)\left(\lambda_x^2+\lambda_{y\alpha i}^2\right)(1+\delta_{i,0}\delta_{v,-1})}\\
&\{u\in\mathbb{N}|u\le 4\},\quad v\in\{-1,1\},\qquad\qquad \lambda_{z\alpha}^*=\sqrt{k_t^2-\tau_\alpha\lambda_x^2}\\
&\alpha\in\{\theta,\psi\},\qquad \lambda_{z\alpha p}=\sqrt{k_t^2-\tau_\alpha\left(\lambda_x^2+k_{yw_p}^{B2}\right)},\quad \tau_\alpha=\begin{cases}\frac{\mu_t}{\mu_z} & ...\alpha=\theta\\ \frac{\epsilon_t}{\epsilon_z} & ...\alpha=\psi\end{cases}
\end{aligned}}\tag{F.147}$$

Next, analyze the λ_y integral stemming from $\Theta_3^{\lambda_y}\overleftrightarrow{\tilde{G}}_{eh}\left(\begin{smallmatrix}z=0\\z'=0\end{smallmatrix}\right)$

$$\int_{-\infty}^{\infty}\left[\frac{4\sin^2\left(\lambda_y\frac{h}{2}\right)}{\left(\lambda_y+k_{yw_p}^B\right)\left(\lambda_y-k_{yw_p}^B\right)\left(\lambda_y+k_{yw_q}^B\right)\left(\lambda_y-k_{yw_q}^B\right)}\right]\frac{C\lambda_y^u\cancel{\sin(\lambda_{z\alpha}d)}}{2\lambda_\rho^2\cancel{\sin(\lambda_{z\alpha}d)}}d\lambda_y$$
$$=\int_{-\infty}^{\infty}\frac{4C\lambda_y^u\sin^2\left(\lambda_y\frac{h}{2}\right)}{\left(\lambda_y+k_{yw_p}^B\right)\left(\lambda_y-k_{yw_p}^B\right)\left(\lambda_y+k_{yw_q}^B\right)\left(\lambda_y-k_{yw_q}^B\right)\lambda_\rho^2}d\lambda_y$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{C\lambda_y^u \left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)} d\lambda_y \\
&\quad + \int_{-\infty}^{\infty} \frac{C\lambda_y^u \left(1 - e^{-j\lambda_y h}\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)} d\lambda_y
\end{aligned} \tag{F.148}
$$

Again, there are five cases that must be explored to find solutions to (F.148).

Case 1: $p = q = 1$**.**

When $p = q = 1$, $k_{yw_p}^B = k_{yw_q}^B = 0$. Thus, (F.148) reduces to

$$
\int_{-\infty}^{\infty} \frac{C\lambda_y^u \left(1 - e^{j\lambda_y h}\right)}{\lambda_y^4 \left(\lambda_x^2 + \lambda_y^2\right)} d\lambda_y + \int_{-\infty}^{\infty} \frac{C\lambda_y^u \left(1 - e^{-j\lambda_y h}\right)}{\lambda_y^4 \left(\lambda_x^2 + \lambda_y^2\right)} d\lambda_y \tag{F.149}
$$

From this, it can be seen there are up to 4 poles at $\lambda_y = 0$ and poles at $\lambda_y = \pm j\lambda_x$. First, analyze the C_0^+ contribution.

$$
\oint_{C_0^+} = \frac{j\pi}{(3-u)!} \frac{\partial^{(3-u)}}{\partial {\lambda_y}^{(3-u)}} \left[\frac{C\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} \tag{F.150}
$$

When $u = 3$

$$
\oint_{C_0^+} = j\pi \left[\frac{C\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} = 0 \tag{F.151}
$$

Thus, there is no C_0^+ contribution when $u \geq 3$. When $u = 2$

$$
\begin{aligned}
\oint_{C_0^+} &= j\pi \frac{\partial}{\partial \lambda_y} \left[\frac{C\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} \\
&= j\pi C \left[\frac{-jhe^{j\lambda_y h}}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0}
\end{aligned}
$$

$$= \pi C \frac{h}{\lambda_x^2} \tag{F.152}$$

When $u = 1$

$$\begin{aligned}
\oint_{C_0^+} &= \frac{j\pi}{2} \frac{\partial^2}{\partial {\lambda_y}^2} \left[\frac{C\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} \\
&= \frac{j\pi C}{2} \left[\frac{h^2 e^{jh\lambda_y}}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} \\
&= j\pi C \frac{h^2}{2\lambda_x^2}
\end{aligned} \tag{F.153}$$

Finally, when $u = 0$

$$\begin{aligned}
\oint_{C_0^+} &= \frac{j\pi}{6} \frac{\partial^3}{\partial {\lambda_y}^3} \left[\frac{C\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} \\
&= \frac{j\pi C}{6} \left[\frac{jh^3}{\lambda_x^2} + \frac{j6h}{\lambda_x^4} \right] \\
&= \frac{\pi C}{6} \left[\frac{-h^3}{\lambda_x^2} - \frac{6h}{\lambda_x^4} \right]
\end{aligned} \tag{F.154}$$

By inspection, it can be shown that

$$\oint_{C_0^+} = -\pi C \left(\delta_{u,0} + \delta_{u,1} + \delta_{u,2}\right) \left[\frac{j^{3u} h^{(3-u)}}{\lambda_x^2 (3-u)!} + \frac{h\delta_{u,0}}{\lambda_x^4} \right] \tag{F.155}$$

Now analyze the C_0^- contribution. When $u = 2$

$$\begin{aligned}
\oint_{C_0^-} &= j\pi \frac{\partial}{\partial \lambda_y} \left[\frac{C\left(1 - e^{-j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0} \\
&= j\pi C \left[\frac{jh}{\lambda_x^2} \right] \\
&= -\pi C \frac{h}{\lambda_x^2}
\end{aligned} \tag{F.156}$$

When $u = 1$

$$\oint_{C_0^-} = \frac{j\pi}{2} \frac{\partial^2}{\partial {\lambda_y}^2} \left[\frac{C\left(1 - e^{-j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right]\Bigg|_{\lambda_y = 0}$$

$$= j\pi C \frac{h^2}{2\lambda_x^2} \tag{F.157}$$

When $u = 0$

$$\oint_{C_0^-} = \frac{j\pi}{6} \frac{\partial^3}{\partial {\lambda_y}^3} \left[\frac{C\left(1 - e^{-j\lambda_y h}\right)}{\left(\lambda_x^2 + \lambda_y^2\right)} \right]\Bigg|_{\lambda_y = 0}$$

$$= \frac{j\pi C}{6} \left[-\frac{jh^3}{\lambda_x^2} - \frac{j6h}{\lambda_x^4} \right]$$

$$= \frac{\pi C}{6} \left[\frac{h^3}{\lambda_x^2} + \frac{6h}{\lambda_x^4} \right] \tag{F.158}$$

Therefore, by inspection

$$\oint_{C_0^-} = \pi C \left(\delta_{u,0} + \delta_{u,1} + \delta_{u,2}\right) \left[\frac{j^u h^{(3-u)}}{\lambda_x^2 \left(3 - u\right)!} + \frac{h\delta_{u,0}}{\lambda_x^4} \right] \tag{F.159}$$

Next, analyze the $C_{j\lambda_x}^+$ contribution.

$$\oint_{C_{j\lambda_x}^+} = j2\pi \left[\frac{C\lambda_y^u \left(1 - e^{j\lambda_y h}\right)}{\lambda_y^4 \left(\lambda_y + j\lambda_x\right)} \right]\Bigg|_{\lambda_y = j\lambda_x}$$

$$= j2\pi C \left[\frac{\left(j\lambda_x\right)^u \left(1 - e^{jj\lambda_x h}\right)}{\left(j\lambda_x\right)^4 \left(j\lambda_x + j\lambda_x\right)} \right]$$

$$= \pi C \left[\frac{j^u \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(5-u)}} \right] \tag{F.160}$$

Finally, analyze the $C^-_{-j\lambda_x}$ contribution.

$$\begin{aligned}
\oint\limits_{C^-_{-j\lambda_x}} &= j2\pi \left[\frac{C\lambda_y^u\left(1-e^{-j\lambda_y h}\right)}{\lambda_y^4\left(\lambda_y - j\lambda_x\right)}\right]\Bigg|_{\lambda_y=-j\lambda_x} \\
&= j2\pi \left[\frac{C\left(-j\lambda_x\right)^u\left(1-e^{-j(-j\lambda_x)h}\right)}{\left(-j\lambda_x\right)^4\left(-j\lambda_x - j\lambda_x\right)}\right] \\
&= -\pi C\left[\frac{j^{3u}\left(1-e^{-\lambda_x h}\right)}{\lambda_x^{(5-u)}}\right]
\end{aligned} \tag{F.161}$$

Combining all the residue contributions together implies that

$$\boxed{\begin{aligned}
\int\limits_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u\Upsilon_8^\alpha}{\lambda_\rho^2}d\lambda_y &= -\pi C\left(\delta_{u,0}+\delta_{u,1}+\delta_{u,2}\right)\left[\frac{\left(j^u+j^{3u}\right)h^{(3-u)}}{\lambda_x^2\left(3-u\right)!}\right. \\
&\quad \left.+\frac{2h\delta_{u,0}}{\lambda_x^4}\right] + \pi C\left[\frac{\left(j^u+j^{3u}\right)\left(1-e^{-\lambda_x h}\right)}{\lambda_x^{(5-u)}}\right] \\
&...p = q = 1
\end{aligned}} \tag{F.162}$$

Case 2: $p = 1, q \neq 1$**.**

In this case, (F.148) reduces to

$$\begin{aligned}
&\int\limits_{-\infty}^{\infty} \frac{C\lambda_y^u\left(1-e^{j\lambda_y h}\right)}{\lambda_y^2\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)\left(\lambda_x^2+\lambda_y^2\right)}d\lambda_y \\
&\quad + \int\limits_{-\infty}^{\infty} \frac{C\lambda_y^u\left(1-e^{-j\lambda_y h}\right)}{\lambda_y^2\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)\left(\lambda_x^2+\lambda_y^2\right)}d\lambda_y
\end{aligned} \tag{F.163}$$

Now it is evident that there are up to 2 poles at $\lambda_y = 0$, and simple poles at $\lambda_y = \pm k^B_{yw_q}$ and $\lambda_y = \pm j\lambda_x$. First, analyze the C_0^+ contribution. Note that if $u \geq 1$ there is no contribution due to the zeros in the numerator canceling them out. Thus, when $u = 0$

$$\oint\limits_{C_0^+} = j\pi \frac{\partial}{\partial \lambda_y} \left[\frac{C\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0}$$

$$= j\pi C \left[\frac{-jh}{\left(k^B_{yw_q}\right)\left(-k^B_{yw_q}\right)\lambda_x^2} \right]$$

$$= -\pi C \left[\frac{h}{k^{B2}_{yw_q}\lambda_x^2} \right] \tag{F.164}$$

Similarly, it can be shown that

$$\oint\limits_{C_0^-} = j\pi \frac{\partial}{\partial \lambda_y} \left[\frac{C\left(1 - e^{-j\lambda_y h}\right)}{\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = 0}$$

$$= j\pi C \left[\frac{jh}{\left(k^B_{yw_q}\right)\left(-k^B_{yw_q}\right)\lambda_x^2} \right]$$

$$= \pi C \left[\frac{h}{k^{B2}_{yw_q}\lambda_x^2} \right] \tag{F.165}$$

Next, analyze the $C^+_{k^B_{yw_q}}$ contribution.

$$\oint\limits_{C^+_{k^B_{yw_q}}} = j\pi \left[\frac{C\lambda_y^u\left(1 - e^{j\lambda_y h}\right)}{\lambda_y^2\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_x^2 + \lambda_y^2\right)} \right] \Bigg|_{\lambda_y = k^B_{yw_q}} = 0 \tag{F.166}$$

This implies that there is no contribution due to $C^+_{k^B_{yw_q}}$. In similar fashion, it can be shown that there are no $C^{\pm}_{\pm k^B_{yw_q}}$ contributions. Next, analyze the $C^+_{j\lambda_x}$ contribution.

$$\oint\limits_{C^+_{j\lambda_x}} = j2\pi \left[\frac{C\lambda_y^u\left(1 - e^{j\lambda_y h}\right)}{\lambda_y^2\left(\lambda_y + k^B_{yw_q}\right)\left(\lambda_y - k^B_{yw_q}\right)\left(\lambda_y + j\lambda_x\right)} \right] \Bigg|_{\lambda_y = j\lambda_x}$$

$$= j2\pi \left[\frac{C\,(j\lambda_x)^u \left(1 - e^{jj\lambda_x h}\right)}{(j\lambda_x)^2 \left(j\lambda_x + k^B_{yw_q}\right) \left(j\lambda_x - k^B_{yw_q}\right) (j\lambda_x + j\lambda_x)} \right]$$

$$= -\pi C \left[\frac{j^u \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(3-u)} \left(\lambda_x^2 + k^{B2}_{yw_q}\right)} \right] \tag{F.167}$$

Finally, analyze the $C^-_{-j\lambda_x}$ contribution.

$$\oint_{C^-_{-j\lambda_x}} = j2\pi \left[\frac{C\lambda_y^u \left(1 - e^{-j\lambda_y h}\right)}{\lambda_y^2 \left(\lambda_y + k^B_{yw_q}\right) \left(\lambda_y - k^B_{yw_q}\right) (\lambda_y - j\lambda_x)} \right] \Bigg|_{\lambda_y = -j\lambda_x}$$

$$= j2\pi \left[\frac{C\,(-j\lambda_x)^u \left(1 - e^{-j(-j\lambda_x)h}\right)}{(-j\lambda_x)^2 \left(-j\lambda_x + k^B_{yw_q}\right) \left(-j\lambda_x - k^B_{yw_q}\right) (-j\lambda_x - j\lambda_x)} \right]$$

$$= \pi C \left[\frac{j^{3u} \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(3-u)} \left(\lambda_x^2 + k^{B2}_{yw_q}\right)} \right] \tag{F.168}$$

Combining all the residue contributions implies that

$$\boxed{\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y} \lambda_y^u \Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y = -2\pi C \delta_{u,0} \left[\frac{h}{k^{B2}_{yw_q} \lambda_x^2} \right] - \pi C \left[\frac{\left(j^u + j^{3u}\right) \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(3-u)} \left(\lambda_x^2 + k^{B2}_{yw_q}\right)} \right] \quad ...p = 1, q \neq 1} \tag{F.169}$$

Case 3: $p \neq 1, q = 1$.

By inspection from Case 2

$$\boxed{\int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y} \lambda_y^u \Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y = -2\pi C \delta_{u,0} \left[\frac{h}{k^{B2}_{yw_p} \lambda_x^2} \right] - \pi C \left[\frac{\left(j^u + j^{3u}\right) \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(3-u)} \left(\lambda_x^2 + k^{B2}_{yw_p}\right)} \right] \quad ...p \neq 1, q = 1} \tag{F.170}$$

Case 4: $p = q \neq 1$**.**

In this case, (F.148) reduces to

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{C\lambda_y^u\left(1-e^{j\lambda_y h}\right)}{\left(\lambda_y+k_{yw_p}^B\right)^2\left(\lambda_y-k_{yw_p}^B\right)^2\left(\lambda_x^2+\lambda_y^2\right)}d\lambda_y\\&+\int_{-\infty}^{\infty} \frac{C\lambda_y^u\left(1-e^{-j\lambda_y h}\right)}{\left(\lambda_y+k_{yw_p}^B\right)^2\left(\lambda_y-k_{yw_p}^B\right)^2\left(\lambda_x^2+\lambda_y^2\right)}d\lambda_y\end{aligned} \tag{F.171}$$

Now it is evident there are order-2 real poles at $\lambda_y = \pm k_{yw_p}^B$ and simple poles at $\lambda_y = \pm j\lambda_x$. First, analyze the $C_{k_{yw_p}^B}^+$ contribution.

$$\begin{aligned}\oint_{C_{k_{yw_p}^B}^+} &= j\pi\frac{\partial}{\partial\lambda_y}\left[\frac{C\lambda_y^u\left(1-e^{j\lambda_y h}\right)}{\left(\lambda_y+k_{yw_p}^B\right)^2\left(\lambda_x^2+\lambda_y^2\right)}\right]\Bigg|_{\lambda_y=k_{yw_p}^B}\\&= j\pi C\left[\frac{-jhk_{yw_p}^{Bu}}{\left(k_{yw_p}^B+k_{yw_p}^B\right)^2\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right]\\&= \pi C\left[\frac{h}{4k_{yw_p}^{B(2-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right]\end{aligned} \tag{F.172}$$

Similarly, it can be shown that under LHPC

$$\oint_{C_{k_{yw_p}^B}^-} = -\pi C\left[\frac{h}{4k_{yw_p}^{B(2-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right] \tag{F.173}$$

Next, analyze the $C_{-k_{yw_p}^B}^+$ contribution.

$$\oint_{C_{-k_{yw_p}^B}^+} = j\pi\frac{\partial}{\partial\lambda_y}\left[\frac{C\lambda_y^u\left(1-e^{j\lambda_y h}\right)}{\left(\lambda_y-k_{yw_p}^B\right)^2\left(\lambda_x^2+\lambda_y^2\right)}\right]\Bigg|_{\lambda_y=-k_{yw_p}^B}$$

$$= j\pi C\left[\frac{-jh\left(-k_{yw_p}^{B}\right)^{u}}{\left(-k_{yw_p}^{B}-k_{yw_p}^{B}\right)^{2}\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right]$$

$$= \pi C\left[\frac{h\left(-1\right)^{u}}{4k_{yw_p}^{B(2-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right] \tag{F.174}$$

Similarly, it can be shown that under LHPC

$$\oint\limits_{C^{-}_{-k^{B}_{yw_p}}} = -\pi C\left[\frac{h\left(-1\right)^{u}}{4k_{yw_p}^{B(2-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)}\right] \tag{F.175}$$

Next, analyze the $C^{+}_{j\lambda_x}$ contribution.

$$\oint\limits_{C^{+}_{j\lambda_x}} = j2\pi\left[\frac{C\lambda_y^u\left(1-e^{j\lambda_y h}\right)}{\left(\lambda_y+k_{yw_p}^{B}\right)^{2}\left(\lambda_y-k_{yw_p}^{B}\right)^{2}\left(\lambda_y+j\lambda_x\right)}\right]\Bigg|_{\lambda_y=j\lambda_x}$$

$$= j2\pi C\left[\frac{\left(j\lambda_x\right)^{u}\left(1-e^{jj\lambda_x h}\right)}{\left(j\lambda_x+k_{yw_p}^{B}\right)^{2}\left(j\lambda_x-k_{yw_p}^{B}\right)^{2}\left(j\lambda_x+j\lambda_x\right)}\right]$$

$$= \pi C\left[\frac{j^{u}\left(1-e^{-\lambda_x h}\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2+k_{yw_p}^{B2}\right)^{2}}\right] \tag{F.176}$$

Finally, analyze the $C^{-}_{-j\lambda_x}$ contribution.

$$\oint\limits_{C^{-}_{-j\lambda_x}} = j2\pi\left[\frac{C\lambda_y^u\left(1-e^{-j\lambda_y h}\right)}{\left(\lambda_y+k_{yw_p}^{B}\right)^{2}\left(\lambda_y-k_{yw_p}^{B}\right)^{2}\left(\lambda_y-j\lambda_x\right)}\right]\Bigg|_{\lambda_y=-j\lambda_x}$$

$$= j2\pi\left[\frac{C\left(-j\lambda_x\right)^{u}\left(1-e^{-j(-j\lambda_x)h}\right)}{\left(-j\lambda_x+k_{yw_p}^{B}\right)^{2}\left(-j\lambda_x-k_{yw_p}^{B}\right)^{2}\left(-j\lambda_x-j\lambda_x\right)}\right]$$

$$= -\pi C \left[\frac{j^{3u}\left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2 + k_{yw_p}^{B2}\right)^2} \right] \tag{F.177}$$

Combining all the residue contributions implies that

$$\boxed{\begin{aligned} \int_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y}\lambda_y^u \Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y &= \pi C \left[\frac{\left(1 + (-1)^u\right) h}{2k_{yw_p}^{B(2-u)}\left(\lambda_x^2 + k_{yw_p}^{B2}\right)} \right] \\ &+ \pi C \left[\frac{\left(j^u + j^{3u}\right)\left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(1-u)}\left(\lambda_x^2 + k_{yw_p}^{B2}\right)^2} \right] \\ ...p = q \neq 1 & \end{aligned}} \tag{F.178}$$

Case 5: $p \neq q, p \neq 1, q \neq 1$.

In this case, (F.148) does not simplify at all. Thus, it is evident that only simple poles exist at $\lambda_y = \pm k_{yw_p}^B$, $\lambda_y = \pm k_{yw_q}^B$, and $\lambda_y = \pm j\lambda_x$. First, analyze the $C_{k_{yw_p}^B}^+$ contribution.

$$\oint_{C_{k_{yw_p}^B}^+} = j\pi \left[\left[\frac{C\lambda_y^u\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_x^2 + \lambda_y^2\right)} \right] \right] \Bigg|_{\lambda_y = k_{yw_p}^B} = 0 \tag{F.179}$$

Therefore, there is no contribution from $C_{k_{yw_p}^B}^+$. In a similar fashion, it can be shown that there are no contributions from $C_{\pm k_{yw_p}^B}^\pm$ or $C_{\pm k_{yw_q}^B}^\pm$. Next, analyze the $C_{j\lambda_x}^+$ contribution.

$$\begin{aligned} \oint_{C_{j\lambda_x}^+} &= j2\pi \left[\left[\frac{C\lambda_y^u\left(1 - e^{j\lambda_y h}\right)}{\left(\lambda_y + k_{yw_p}^B\right)\left(\lambda_y - k_{yw_p}^B\right)\left(\lambda_y + k_{yw_q}^B\right)\left(\lambda_y - k_{yw_q}^B\right)\left(\lambda_y + j\lambda_x\right)} \right] \right] \Bigg|_{\lambda_y = j\lambda_x} \\ &= j2\pi \left[\frac{C\left(j\lambda_x\right)^u\left(1 - e^{jj\lambda_x h}\right)}{\left(j\lambda_x + k_{yw_p}^B\right)\left(j\lambda_x - k_{yw_p}^B\right)\left(j\lambda_x + k_{yw_q}^B\right)\left(j\lambda_x - k_{yw_q}^B\right)\left(j\lambda_x + j\lambda_x\right)} \right] \end{aligned}$$

$$= \pi C \left[\frac{j^u \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(1-u)} \left(\lambda_x^2 + k_{yw_p}^{B2}\right) \left(\lambda_x^2 + k_{yw_q}^{B2}\right)} \right] \tag{F.180}$$

Finally, analyze the $C^-_{-j\lambda_x}$ contribution.

$$\oint\limits_{C^-_{-j\lambda_x}} = j2\pi \left[\left. \left[\frac{C\lambda_y^u \left(1 - e^{-j\lambda_y h}\right)}{\left(\lambda_y + k_{yw_p}^B\right) \left(\lambda_y - k_{yw_p}^B\right) \left(\lambda_y + k_{yw_q}^B\right) \left(\lambda_y - k_{yw_q}^B\right) \left(\lambda_y - j\lambda_x\right)} \right] \right|_{\lambda_y = -j\lambda_x} \right.$$

$$= j2\pi \left[\frac{C \left(-j\lambda_x\right)^u \left(1 - e^{-j(-j\lambda_x)h}\right)}{\left(-\lambda_x^2 - k_{yw_p}^{B2}\right) \left(-\lambda_x^2 - k_{yw_q}^{B2}\right) \left(-j\lambda_x - j\lambda_x\right)} \right]$$

$$= -\pi C \left[\frac{j^{3u} \left(1 - e^{-\lambda_x} h\right)}{\lambda_x^{(1-u)} \left(\lambda_x^2 + k_{yw_p}^{B2}\right) \left(\lambda_x^2 + k_{yw_q}^{B2}\right)} \right] \tag{F.181}$$

Combining all the residue contributions implies that

$$\boxed{\begin{aligned} &\int\limits_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y} \lambda_y^u \Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y = \pi C \left[\frac{\left(j^u + j^{3u}\right) \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(1-u)} \left(\lambda_x^2 + k_{yw_p}^{B2}\right) \left(\lambda_x^2 + k_{yw_q}^{B2}\right)} \right] \\ &...p \neq q, p \neq 1, q \neq 1 \end{aligned}} \tag{F.182}$$

Inspecting all the cases, it can be seen that in general

$$\boxed{\begin{aligned} \int\limits_{-\infty}^{\infty} \frac{C\Theta_3^{\lambda_y} \lambda_y^u \Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y &= -\pi C \delta_{p,q} \delta_{p,1} \left(\delta_{u,0} + \delta_{u,1} + \delta_{u,2}\right) \left[\frac{\left(j^u + j^{3u}\right) h^{(3-u)}}{\lambda_x^2 \left(3-u\right)!} \right. \\ &\left. + \frac{2h\delta_{u,0}}{\lambda_x^4} \right] - 2\pi C \left[\frac{\delta_{p,1} \left(1 - \delta_{q,1}\right) \delta_{u,0} h}{k_{yw_q}^{B2} \lambda_x^2} \right] - 2\pi C \left[\frac{\delta_{q,1} \left(1 - \delta_{p,1}\right) \delta_{u,0} h}{k_{yw_p}^{B2} \lambda_x^2} \right] \\ &+ \pi C \delta_{p,q} \left(1 - \delta_{p,1}\right) \left(1 - \delta_{q,1}\right) \left[\frac{\left(1 + (-1)^u\right) h}{2k_{yw_p}^{B(2-u)} \left(\lambda_x^2 + k_{yw_p}^{B2}\right)} \right] \\ &+ \pi C \left[\frac{(-1)^{(1-\delta_{p,q})(\delta_{p,1} + \delta_{q,1})} \left(j^u + j^{3u}\right) \left(1 - e^{-\lambda_x h}\right)}{\lambda_x^{(1-u)} \left(\lambda_x^2 + k_{yw_p}^{B2}\right) \left(\lambda_x^2 + k_{yw_q}^{B2}\right)} \right] \end{aligned}} \tag{F.183}$$

F.4 Evaluation of λ_x Integrals

The λ_y integrals involving Υ_8^α have no multivalued terms. Thus, no branch cuts arise, making it possible to evaluate the λ_x integrals using complex analysis. Thus, begin by analyzing the most general form of the λ_x integrals. It is noted in the main body of the text that the terms from the λ_y integral resulting from poles at $\lambda_y = \pm j\lambda_x$ ultimately cancel out when TM^z and TE^z terms are added together. Thus,

$$C\int_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^v\int_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^u\Upsilon_8^\alpha}{\lambda_\rho^2}d\lambda_y d\lambda_x = C\int_{-\infty}^{\infty}\frac{4\lambda_x^v\cos^2\left(\lambda_x\frac{a}{2}\right)}{\left(\lambda_x^2-k_{xv_m}^2\right)\left(\lambda_x^2-k_{xv_n}^2\right)}$$
$$\cdot\left\{-\pi\delta_{p,q}\delta_{p,1}\left(\delta_{u,0}+\delta_{u,1}+\delta_{u,2}\right)\left(\frac{\left(j^u+j^{3u}\right)\left[(b+h)^{(3-u)}-(b-h)^{(3-u)}\right]}{2^{(3-u)}\,(3-u)!\lambda_x^2}\right.\right.$$
$$\left.\left.+\frac{2h\delta_{u,0}}{\lambda_x^4}\right)-2\pi\delta_{p,1}\left(1-\delta_{q,1}\right)\delta_{u,0}\left[\frac{h}{k_{yw_q}^{B2}\lambda_x^2}\right]-2\pi\delta_{q,1}\left(1-\delta_{p,1}\right)\delta_{u,0}\left[\frac{h}{k_{yw_p}^{A2}\lambda_x^2}\right]\right\}d\lambda_x \tag{F.184}$$

It is important to note that in this work $u+v=4$ and $u,v\in 2\mathbb{N}$. Knowing this, it can be shown that there will be no simple poles at $\lambda_x = 0$ and the above simplifies to

$$C\int_{-\infty}^{\infty}\Theta_3^{\lambda_x}\lambda_x^v\int_{-\infty}^{\infty}\frac{\Theta_3^{\lambda_y}\lambda_y^u\Upsilon_8^\alpha}{\lambda_\rho^2}d\lambda_y d\lambda_x = -2\pi C\int_{-\infty}^{\infty}\frac{4\lambda_x^v\cos^2\left(\lambda_x\frac{a}{2}\right)}{\left(\lambda_x^2-k_{xv_m}^2\right)\left(\lambda_x^2-k_{xv_n}^2\right)}$$
$$\cdot\left\{\delta_{p,q}\delta_{p,1}\left(\delta_{u,0}+\delta_{u,2}\right)\left(\frac{j^u\left[(b+h)^{(3-u)}-(b-h)^{(3-u)}\right]}{2^{(3-u)}\,(3-u)!\lambda_x^2}+\frac{h\delta_{u,0}}{\lambda_x^4}\right)\right.$$
$$\left.+\delta_{p,1}\left(1-\delta_{q,1}\right)\delta_{u,0}\left[\frac{h}{k_{yw_q}^{B2}\lambda_x^2}\right]+\delta_{q,1}\left(1-\delta_{p,1}\right)\delta_{u,0}\left[\frac{h}{k_{yw_p}^{A2}\lambda_x^2}\right]\right\}d\lambda_x$$

$$= -2\pi C \int_{-\infty}^{\infty} \frac{\lambda_x^v \left[\left(1 + e^{j\lambda_x a}\right) + \left(1 + e^{-j\lambda_x a}\right)\right]}{\left(\lambda_x^2 - k_{xv_m}^2\right)\left(\lambda_x^2 - k_{xv_n}^2\right)}$$

$$\cdot \left\{ \delta_{p,q}\delta_{p,1}\left(\delta_{u,0} + \delta_{u,2}\right)\left(\frac{j^u\left[(b+h)^{(3-u)} - (b-h)^{(3-u)}\right]}{2^{(3-u)}(3-u)!\lambda_x^2} + \frac{h\delta_{u,0}}{\lambda_x^4}\right)\right. \tag{F.185}$$

$$\left. + \delta_{p,1}\left(1 - \delta_{q,1}\right)\delta_{u,0}\left[\frac{h}{k_{yw_q}^{B2}\lambda_x^2}\right] + \delta_{q,1}\left(1 - \delta_{p,1}\right)\delta_{u,0}\left[\frac{h}{k_{yw_p}^{A2}\lambda_x^2}\right]\right\} d\lambda_x$$

There are three cases that must be studied where $v \in \{0, 2, 4\}$. If $v = 0$, that implies $u = 4$, which results in zero contribution. If $v = 2$, that implies $u = 2$, therefore

$$C \int_{-\infty}^{\infty} \Theta_3^{\lambda_x}\lambda_x^2 \int_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y}\lambda_y^2\Upsilon_8^{\alpha}}{\lambda_\rho^2} d\lambda_y d\lambda_x = -2\pi C \int_{-\infty}^{\infty} \frac{\lambda_x^2 \left[\left(1 + e^{j\lambda_x a}\right) + \left(1 + e^{-j\lambda_x a}\right)\right]}{\left(\lambda_x^2 - k_{xv_m}^2\right)\left(\lambda_x^2 - k_{xv_n}^2\right)}$$

$$\cdot \left\{ \delta_{p,q}\delta_{p,1}\left(\frac{j^2\left[(b+h)^{(3-2)} - (b-h)^{(3-2)}\right]}{2^{(3-2)}(3-2)!\lambda_x^2}\right)\right\} d\lambda_x$$

$$= 2\pi C h \delta_{p,q}\delta_{p,1}\left[\int_{-\infty}^{\infty} \frac{\left(1 + e^{j\lambda_x a}\right)}{\left(\lambda_x^2 - k_{xv_m}^2\right)\left(\lambda_x^2 - k_{xv_n}^2\right)} d\lambda_x + \int_{-\infty}^{\infty} \frac{\left(1 + e^{-j\lambda_x a}\right)}{\left(\lambda_x^2 - k_{xv_m}^2\right)\left(\lambda_x^2 - k_{xv_n}^2\right)} d\lambda_x\right] \tag{F.186}$$

In this case, there are poles at $\lambda_x = \pm k_{xv_m}$ and $\lambda_x = \pm k_{xv_n}$. If $m \neq n$, it can be shown that there is only a simple pole at each of those locations. Thus, under UHPC, the $\left(1 + e^{j\lambda_x a}\right)$ evaluates to zero. However, when $m = n$, there is a secnd-order pole at $\lambda_x = \pm k_{xv_m}$. Studying the contribution from the pole at $\lambda_x = k_{xv_m}$,

$$\oint_{C_{k_{xv_m}}^+} = j\pi \frac{\partial}{\partial \lambda_x}\left[\frac{1 + e^{j\lambda_x a}}{\left(\lambda_x + k_{xv_m}\right)^2}\right]\Bigg|_{\lambda_x = k_{xv_m}}$$

$$= j\pi \left[\frac{jae^{jk_{xv_m}a}}{\left(k_{xv_m} + k_{xv_m}\right)^2}\right]$$

$$= \frac{\pi a}{4k_{xv_m}^2} \tag{F.187}$$

Under LHPC, it can be shown that

$$\begin{aligned}
\oint\limits_{C^-_{k_{xv_m}}} &= j\pi \frac{\partial}{\partial \lambda_x}\left[\frac{1+e^{-j\lambda_x a}}{(\lambda_x + k_{xv_m})^2}\right]\Bigg|_{\lambda_x = k_{xv_m}} \\
&= j\pi\left[\frac{-jae^{-jk_{xv_m}a}}{(k_{xv_m}+k_{xv_m})^2}\right] \\
&= -\frac{\pi a}{4k_{xv_m}^2} \qquad \text{(F.188)}
\end{aligned}$$

Further, it can be shown that the contributions for $\lambda_x = -k_{xv_m}$ are

$$\oint\limits_{C^+_{-k_{xv_m}}} = \frac{\pi a}{4k_{xv_m}^2} \tag{F.189}$$

$$\oint\limits_{C^-_{-k_{xv_m}}} = -\frac{\pi a}{4k_{xv_m}^2} \tag{F.190}$$

Combining all these contribution, we find that for $v = 2$

$$C\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x}\lambda_x^2 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y}\lambda_y^u\Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y d\lambda_x = \frac{\pi a \delta_{m,n}}{k_{xv_m}} \tag{F.191}$$

Finally, when $v = 4$, that implies $u = 0$. Therefore

$$\begin{aligned}
C\int\limits_{-\infty}^{\infty} \Theta_3^{\lambda_x}\lambda_x^4 \int\limits_{-\infty}^{\infty} \frac{\Theta_3^{\lambda_y}\lambda_y^0\Upsilon_8^\alpha}{\lambda_\rho^2} d\lambda_y d\lambda_x &= -2\pi C\int\limits_{-\infty}^{\infty} \frac{\lambda_x^4\left[\left(1+e^{j\lambda_x a}\right)+\left(1+e^{-j\lambda_x a}\right)\right]}{\left(\lambda_x^2 - k_{xv_m}^2\right)\left(\lambda_x^2 - k_{xv_n}^2\right)} \\
&\cdot\left\{\delta_{p,q}\delta_{p,1}\left(\frac{j^0\left[(b+h)^{(3-0)} - (b-h)^{(3-0)}\right]}{2^{(3-0)}(3-0)!\lambda_x^2} + \frac{h}{\lambda_x^4}\right)\right. \\
&\left. +\delta_{p,1}\left(1-\delta_{q,1}\right)\left[\frac{h}{k_{yw_q}^{B2}\lambda_x^2}\right] + \delta_{q,1}\left(1-\delta_{p,1}\right)\left[\frac{h}{k_{yw_p}^{A2}\lambda_x^2}\right]\right\} d\lambda_x
\end{aligned}$$

$$
\begin{aligned}
&= -2\pi C h \int_{-\infty}^{\infty} \frac{\left[\left(1+e^{j\lambda_x a}\right)+\left(1+e^{-j\lambda_x a}\right)\right]}{\left(\lambda_x^2-k_{xv_m}^2\right)\left(\lambda_x^2-k_{xv_n}^2\right)} \left\{\delta_{p,q}\delta_{p,1}\left(\frac{\lambda_x^2\left(3b^2+h^2\right)}{24}+1\right)\right. \\
&\qquad \left. +\delta_{p,1}\left(1-\delta_{q,1}\right)\left[\frac{\lambda_x^2}{k_{yw_q}^{B2}}\right]+\delta_{q,1}\left(1-\delta_{p,1}\right)\left[\frac{\lambda_x^2}{k_{yw_p}^{A2}}\right]\right\} d\lambda_x
\end{aligned} \tag{F.192}
$$

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14. ABSTRACT

First, this dissertation presents motivation and background on complex media and their characterization. Next, a scalar-potential formulation is presented to derive Green functions describing a parallel plate region containing two layers of uniaxial material. Two measurement techniques based on those Green functions are then developed and analyzed via uncertainty analysis. One technique is validated using laboratory measurements compared to those from a mature destructive technique. Next, the advantages and disadvantages of both proposed techniques are discussed as well as suggested areas of promising future research. Ultimately, this work demonstrates that nondestructive characterization of conductor-backed uniaxial materials is not only possible, but can be achieved in an efficient, practical manner with results on par with mature destructive techniques.

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