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Basic Linear Cartesian Dynamic Models in Local Coordinates

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Basic Linear Cartesian Dynamic Models in Local Coordinates

David Frederic Crouse* *Member, IEEE*

***Abstract*—Dynamic models that are linear in Cartesian coordinates are nonlinear when transformed into other coordinate systems. This note goes through derivations of such models for constant-velocity problems in a variety of 2D polar and r-u coordinates systems and in 3D spherical and r-u-v coordinate systems, sparing tedious derivations for simple tracking problems. The conversions for r-u and r-u-v coordinate systems do not appear to have been previously published.**

I. Introduction

The target-tracking literature typically expresses dynamic models in Cartesian coordinates. Notable exceptions include the use of modified polar coordinates [5], log-polar coordinates [1], modified spherical coordinates [10], and log-spherical coordinates [7] in bearings-only passive tracking applications. A ballistic dynamic model including atmospheric drag, expressed in spherical coordinates, is also considered in [2]. However, when considering missile flight-control systems, where the tracking information is fed back to a controller, much of the literature focusses on the expression of the target state in a local spherical coordinate system with respect to missile guidance [6], [8], [9]. With the proliferation of small delivery drones, such guidance and tracking problems have increasingly non-military applications.

The expression of equations of motion for a deterministic constant-velocity Cartesian dynamic model in polar coordinates can be found in numerous classes on dynamics and mechanics. Consider for example, Lecture 5 of a Fall 2009 MIT OpenCourseWare course on dynamics [11]. However, expressions in spherical coordinates are harder to find (and derive) and it is doubtful that expressions in range-and-direction cosine coordinates (r-u-v coordinates), which are commonly used in radar target-tracking applications, have been published.

This paper provides linear dynamic models in polar (Sec. II), r-u (Sec. III), spherical (Sec. IV), and r-u-v (Sec. V) coordinate systems. Additionally, in each instance, expressions for converting the velocity into the local coordinate systems are provided. The results are presented in Sec. VI. The dynamic models are implemented in the free, open-source, copyleft-free Tracker Component Library (TCL) [4], [12]. Hopefully, this work can save others the tedium of deriving such transformations.

II. 2D Polar

The simplest case is polar coordinates, which consist of a range r and an angle θ. Here, we consider two types of polar coordinate systems. The Type-0 system measures the angle θ counterclockwise from the x-axis and the conversion of a position to Cartesian coordinates is:

$$x = r\cos(\theta) \qquad y = r\sin(\theta). \tag{1}$$

A Type-1 system measures the angle θ clockwise from the y-axis and is given as:

$$x = r\sin(\theta) \qquad y = r\cos(\theta). \tag{2}$$

A. Type-0 Polar Coordinates

Consider a 2D position-vector $\mathbf{r}$ in a polar coordinate system:

$$\mathbf{r} = r\mathbf{u}_r \tag{3}$$

where the unit-vector $\mathbf{u}_r$ and an orthonormal-vector $\mathbf{u}_\theta$ are

$$\mathbf{u}_r = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} \qquad \mathbf{u}_\theta = \frac{\partial \mathbf{u}_r}{\partial\theta} = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}. \tag{4}$$

The derivatives of the basis vectors with respect to time are:

$$\dot{\mathbf{u}}_r = \frac{\partial \mathbf{u}_r}{\partial\theta}\dot{\theta} = \dot{\theta}\mathbf{u}_\theta \qquad \dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u}_r, \tag{5}$$

so the velocity is:

$$\dot{\mathbf{r}} = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta. \tag{6}$$

Consequently, if one wished to convert a Cartesian velocity vector into polar coordinates, since $\mathbf{u}_r$ is orthonormal to $\mathbf{u}_\theta$, this can be done by taking the dot product of $\dot{\mathbf{r}}$ with its basis vectors:

$$\dot{r} = \dot{\mathbf{r}}'\mathbf{u}_r \qquad \dot{\theta} = \frac{1}{r}\dot{\mathbf{r}}'\mathbf{u}_\theta. \tag{7}$$

Taking one more derivative, the acceleration is:

$$\ddot{\mathbf{r}} = \ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta \tag{8}$$

$$= \underbrace{\left(\ddot{r} - r\dot{\theta}^2\right)}_{a_r}\mathbf{u}_r + \underbrace{\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)}_{a_\theta}\mathbf{u}_\theta. \tag{9}$$

For constant radial and cross-range acceleration components, a_r and a_θ, one gets the *nonlinear* dynamic model:

$$\ddot{r} = a_r + r\dot{\theta}^2 \qquad \ddot{\theta} = \frac{1}{r}\left(a_\theta - 2\dot{r}\dot{\theta}\right). \tag{10}$$

If one were to build a complete drift function for a continuous-time dynamic model, as in [3], then the drift function would

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be

$$\mathbf{a}(\mathbf{x},t)=\begin{bmatrix}\dot{r}\\ \dot{\theta}\\ a_r+r\dot{\theta}^2\\ \frac{1}{r}\left(a_\theta-2\dot{r}\dot{\theta}\right)\end{bmatrix}. \tag{11}$$

Given a target state, in this instance consisting of $\mathbf{x}=[r,\theta,\dot{r},\dot{\theta}]'$, the drift function is just an expression for $\dot{\mathbf{x}}$. Similar drift functions can be written for the other dynamic models in this paper, though they shall not be explicitely written out.

Using a similar approach, it is possible to obtain higher-order models. This is demonstrated in Appendix A for the constant-jerk model.

B. Type-1 Polar Coordinates

The equations for $\ddot{r}$ and $\ddot{\theta}$ for a Cartesian constant-velocity model in Type-1 polar coordinates are the same as those for type-0 coordinates (given in (10)). However, the elements of the basis vectors have changed order:

$$\mathbf{u}_r=\begin{bmatrix}\sin(\theta)\\ \cos(\theta)\end{bmatrix} \qquad \mathbf{u}_\theta=\frac{\partial\mathbf{u}_r}{\partial\theta}=\begin{bmatrix}\cos(\theta)\\ -\sin(\theta)\end{bmatrix}. \tag{12}$$

III. 2D Range-and-Direction Cosine

2D range-and-direction cosine measurements assume that the target is in front of the sensor with the pointing direction of the sensor specified by the y-axis. Rather than using an angle, a direction-cosine value $u\in[-1,1]$ is used. The direction-cosine u can be interpreted as the projection of a unit vector onto the x-axis. The conversion to Cartesian coordinates is thus:

$$x=ru \qquad y=r\sqrt{1-u^2}. \tag{13}$$

A position can be represented using the orthonormal basis vectors:

$$\mathbf{u}_r=\begin{bmatrix}u\\ \sqrt{1-u^2}\end{bmatrix} \tag{14}$$

$$\mathbf{u}_u=\sqrt{1-u^2}\frac{\partial\mathbf{u}_r}{\partial u}=\begin{bmatrix}\sqrt{1-u^2}\\ -u\end{bmatrix} \tag{15}$$

as

$$\mathbf{r}=r\mathbf{u}_r. \tag{16}$$

The derivatives of the basis vectors are:

$$\dot{\mathbf{u}}_r=\frac{\dot{u}}{\sqrt{1-u^2}}\mathbf{u}_u \tag{17}$$

$$\dot{\mathbf{u}}_u=\dot{u}\begin{bmatrix}-\frac{u}{\sqrt{1-u^2}}\\ -1\end{bmatrix}=-\frac{\dot{u}}{\sqrt{1-u^2}}\mathbf{u}_r \tag{18}$$

leading to the following expression for the first derivative of position (velocity):

$$\dot{\mathbf{r}}=\dot{r}\mathbf{u}_r+r\dot{\mathbf{u}}_r=\dot{r}\mathbf{u}_r+r\frac{\dot{u}}{\sqrt{1-u^2}}\mathbf{u}_u. \tag{19}$$

Since $\mathbf{u}_r$ and $\mathbf{u}_u$ are orthonormal, if one wished to obtain $\dot{r}$ and $\dot{u}$ from $\dot{\mathbf{r}}$, it could be done using dot products of $\dot{\mathbf{r}}$ with its basis vectors:

$$\dot{r}=\dot{\mathbf{r}}'\mathbf{u}_r \qquad \dot{u}=\frac{\sqrt{1-u^2}}{r}\dot{\mathbf{r}}'\mathbf{u}_u. \tag{20}$$

The second derivative with respect to position is:

$$\begin{aligned}\ddot{\mathbf{r}}=&\dot{r}\dot{\mathbf{u}}_r+r\frac{\dot{u}}{\sqrt{1-u^2}}\dot{\mathbf{u}}_u+\ddot{r}\mathbf{u}_r\\ &+\left(\frac{\dot{r}\dot{u}+r\ddot{u}}{\sqrt{1-u^2}}+r\frac{u\dot{u}^2}{(1-u^2)^{\frac{3}{2}}}\right)\mathbf{u}_u \end{aligned} \tag{21}$$

$$=\underbrace{\left(\ddot{r}-r\frac{\dot{u}^2}{1-u^2}\right)}_{a_r}\mathbf{u}_r+\underbrace{\left(\frac{2\dot{r}\dot{u}+r\ddot{u}}{\sqrt{1-u^2}}+r\frac{u\dot{u}^2}{(1-u^2)^{\frac{3}{2}}}\right)}_{a_u}\mathbf{u}_u. \tag{22}$$

Assuming that the accelerations a_r and a_u are constant, then the differential equations for the linear Cartesian dynamic model in 2D range-and-direction cosine coordinates are:

$$\ddot{r}=a_r+\frac{r\dot{u}^2}{1-u^2} \tag{23}$$

$$\ddot{u}=\frac{1}{r}\left(a_u\sqrt{1-u^2}-2\dot{r}\dot{u}\right)-\frac{u\dot{u}^2}{1-u^2}. \tag{24}$$

IV. 3D Spherical

We consider three types of spherical coordinate systems that commonly arise. All systems have a range component r. The Type-0 measures the azimuth θ from the x-axis counterclockwise in the $x-y$ plane. The elevation ϕ is measured up from the $x-y$ plane towards the z-axis. The conversion is thus:

$$x=r\cos(\theta)\cos(\phi) \quad y=r\sin(\theta)\cos(\phi) \quad z=r\sin(\phi). \tag{25}$$

In the Type 1 system, azimuth is measured counterclockwise from the z-axis in the $z-x$ plane. Elevation is measured up from the $z-x$ plane towards the y-axis. The conversion is thus:

$$x=r\sin(\theta)\cos(\phi) \quad y=r\sin(\phi) \quad z=r\cos(\theta)\cos(\phi). \tag{26}$$

The Type-2 system is the same as the Type-0 system, except the definition of ϕ differs. In the Type-2 system, ϕ is measured down from the z-axis towards the $x-y$ plane rather than up from the $x-y$ plane. Thus

$$x=r\cos(\theta)\sin(\phi) \quad y=r\sin(\theta)\sin(\phi) \quad z=r\cos(\phi). \tag{27}$$

A. Type-0

A position can be represented using the orthonormal basis vectors:

$$\mathbf{u}_r=\begin{bmatrix}\cos(\theta)\cos(\phi)\\ \sin(\theta)\cos(\phi)\\ \sin(\phi)\end{bmatrix} \tag{28}$$

$$\mathbf{u}_\theta=\frac{1}{\cos(\phi)}\frac{\partial\mathbf{u}_r}{\partial\theta}=\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix} \tag{29}$$

$$\mathbf{u}_\phi = \frac{\partial \mathbf{u}_r}{\partial \phi} = \begin{bmatrix} -\cos(\theta)\sin(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\phi) \end{bmatrix} \tag{30}$$

as

$$\mathbf{r} = r\mathbf{u}_r. \tag{31}$$

The first derivatives of the basis vectors are:

$$\dot{\mathbf{u}}_r = \frac{\partial \mathbf{u}_r}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{u}_r}{\partial \phi}\dot{\phi} = \dot{\theta}\cos(\phi)\mathbf{u}_\theta + \dot{\phi}\mathbf{u}_\phi \tag{32}$$

$$\dot{\mathbf{u}}_\theta = \frac{\partial \dot{\mathbf{u}}_\theta}{\partial \theta}\dot{\theta} + \frac{\partial \dot{\mathbf{u}}_\theta}{\partial \phi}\dot{\phi} = -\dot{\theta}\cos(\phi)\mathbf{u}_r + \dot{\theta}\sin(\phi)\mathbf{u}_\phi \tag{33}$$

$$\dot{\mathbf{u}}_\phi = \frac{\partial \dot{\mathbf{u}}_\phi}{\partial \theta}\dot{\theta} + \frac{\partial \dot{\mathbf{u}}_\phi}{\partial \phi}\dot{\phi} = -\dot{\phi}\mathbf{u}_r - \dot{\theta}\sin(\phi)\mathbf{u}_\theta, \tag{34}$$

which leads to the expression for the velocity:

$$\dot{\mathbf{r}} = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r \tag{35}$$

$$= \dot{r}\mathbf{u}_r + r\dot{\theta}\cos(\phi)\mathbf{u}_\theta + r\dot{\phi}\mathbf{u}_\phi. \tag{36}$$

Due to the orthonormality of the basis vectors, the values $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ can be obtained from $\dot{\mathbf{r}}$ using dot products with the basis vectors:

$$\dot{r} = \dot{\mathbf{r}}'\mathbf{u}_r \qquad \dot{\theta} = \frac{1}{r\cos(\phi)}\dot{\mathbf{r}}'\mathbf{u}_\theta \qquad \dot{\phi} = \frac{1}{r}\dot{\mathbf{r}}'\mathbf{u}_\phi. \tag{37}$$

Taking a second derivative leads to the expression for the acceleration:

$$\begin{aligned}\ddot{\mathbf{r}} =& \dot{r}\dot{\mathbf{u}}_r + r\dot{\theta}\cos(\phi)\dot{\mathbf{u}}_\theta + r\dot{\phi}\dot{\mathbf{u}}_\phi \\ &+ \ddot{r}\mathbf{u}_r + \dot{r}\dot{\theta}\cos(\phi)\mathbf{u}_\theta + r\ddot{\theta}\cos(\phi)\mathbf{u}_\theta - r\dot{\theta}\dot{\phi}\sin(\phi)\mathbf{u}_\theta \\ &+ \dot{r}\dot{\phi}\mathbf{u}_\phi + r\ddot{\phi}\mathbf{u}_\phi \end{aligned} \tag{38}$$

$$\begin{aligned} =& \ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r + r\dot{\theta}\cos(\phi)\dot{\mathbf{u}}_\theta + r\dot{\phi}\dot{\mathbf{u}}_\phi \\ &+ \left(\dot{r}\dot{\theta}\cos(\phi) + r\ddot{\theta}\cos(\phi) - r\dot{\theta}\dot{\phi}\sin(\phi)\right)\mathbf{u}_\theta \\ &+ \left(\dot{r}\dot{\phi} + r\ddot{\phi}\right)\mathbf{u}_\phi \end{aligned} \tag{39}$$

$$\begin{aligned} =& \underbrace{\left(\ddot{r} - r\dot{\theta}^2\cos(\phi)^2 - r\dot{\phi}^2\right)}_{a_r}\mathbf{u}_r \\ &+ \underbrace{\left(\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos(\phi) - 2r\dot{\theta}\dot{\phi}\sin(\phi)\right)}_{a_\theta}\mathbf{u}_\theta \\ &+ \underbrace{\left(2\dot{r}\dot{\phi} + r\ddot{\phi} + r\dot{\theta}^2\cos(\phi)\sin(\phi)\right)}_{a_\phi}\mathbf{u}_\phi. \end{aligned} \tag{40}$$

Assuming that the acceleration components a_r, a_θ, and a_ϕ are all constant, then the dynamic model is:

$$\ddot{r} = a_r + r\dot{\phi}^2 + r\dot{\theta}^2\cos(\phi)^2 \tag{41}$$

$$\ddot{\theta} = \frac{1}{r}\left(-2\dot{r}\dot{\theta} + \frac{a_\theta}{\cos(\phi)} + 2r\dot{\theta}\dot{\phi}\tan(\phi)\right) \tag{42}$$

$$\ddot{\phi} = \frac{1}{r}\left(a_\phi - 2\dot{r}\dot{\phi} - r\dot{\theta}^2\cos(\phi)\sin(\phi)\right). \tag{43}$$

B. Type-1

The dynamic equations in (41), (42), and (43) for the Type-0 spherical coordinate system still hold. However, the associated basis vectors have changed the ordering of their elements to:

$$\mathbf{u}_r = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix} \tag{44}$$

$$\mathbf{u}_\theta = \frac{1}{\cos(\phi)}\frac{\partial \mathbf{u}_r}{\partial \theta} = \begin{bmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{bmatrix} \tag{45}$$

$$\mathbf{u}_\phi = \frac{\partial \mathbf{u}_r}{\partial \phi} = \begin{bmatrix} -\sin(\theta)\sin(\phi) \\ \cos(\phi) \\ -\cos(\theta)\sin(\phi) \end{bmatrix}. \tag{46}$$

C. Type-2

The Type-2 spherical coordinate system differs from the Type-0 system only in that ϕ has been replaced by $\pi/2 - \phi$. Thus, the new set of orthonormal basis vectors is:

$$\mathbf{u}_r = \begin{bmatrix} \cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\phi) \end{bmatrix} \tag{47}$$

$$\mathbf{u}_\theta = \frac{1}{\sin(\phi)}\frac{\partial \mathbf{u}_r}{\partial \theta} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} \tag{48}$$

$$\mathbf{u}_\phi = \frac{\partial \mathbf{u}_r}{\partial \phi} = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ -\sin(\phi) \end{bmatrix}, \tag{49}$$

which modified the velocity expression to:

$$\dot{\mathbf{r}} = \dot{r}\mathbf{u}_r + r\dot{\theta}\sin(\phi)\mathbf{u}_\theta + r\dot{\phi}\mathbf{u}_\phi \tag{50}$$

so

$$\dot{r} = \dot{\mathbf{r}}'\mathbf{u}_r \qquad \dot{\theta} = \frac{1}{r\cos(\phi)}\dot{\mathbf{r}}'\mathbf{u}_\theta \qquad \dot{\phi} = \frac{1}{r}\dot{\mathbf{r}}'\mathbf{u}_\phi. \tag{51}$$

Assuming that the acceleration components a_r, a_θ, and a_ϕ are all constant, the modified dynamic model is:

$$\ddot{r} = a_r + r\dot{\phi}^2 + r\dot{\theta}^2\sin(\phi)^2 \tag{52}$$

$$\ddot{\theta} = \frac{1}{r}\left(-2\dot{r}\dot{\theta} + \frac{a_\theta}{\sin(\phi)} - 2r\dot{\theta}\dot{\phi}\frac{1}{\tan(\phi)}\right) \tag{53}$$

$$\ddot{\phi} = \frac{1}{r}\left(a_\phi - 2\dot{r}\dot{\phi} + r\dot{\theta}^2\cos(\phi)\sin(\phi)\right). \tag{54}$$

V. 3D Range-and-Direction Cosines

Like 2D range-and-direction cosine measurements, 3D range-and-direction cosine measurements assume that the target is in front of the sensor. Here, the pointing direction of the sensor is given by the z-axis. Rather than using an angle, a pair of direction-cosine values $u, v \in [-1, 1]$ are used. The pair (u, v) can be interpreted as the the first two elements of a unit vector pointing from the receiver to the target. The conversion into Cartesian coordinates is:

$$x = ru \qquad y = rv \qquad z = r\sqrt{1 - u^2 - v^2}. \tag{55}$$

As in all of the other coordinate systems, one might wish to choose a set of orthonormal bases. The first basis vector is :

$$\mathbf{u}_1 = \begin{bmatrix} u \\ v \\ \sqrt{1-u^2-v^2} \end{bmatrix} \quad (56)$$

so that, as in the other systems,

$$\mathbf{r} = r\mathbf{u}_1. \quad (57)$$

A second basis vector can be a scaled version of $\frac{\partial \mathbf{u}_1}{\partial u}$, so

$$\mathbf{u}_2 = \sqrt{\frac{1-u^2-v^2}{1-v^2}}\frac{\partial \mathbf{u}_1}{\partial u} = \begin{bmatrix} \sqrt{\frac{1-u^2-v^2}{1-v^2}} \\ 0 \\ -\frac{u}{\sqrt{1-v^2}} \end{bmatrix} \quad (58)$$

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{bmatrix} -\frac{uv}{\sqrt{1-v^2}} \\ \sqrt{1-v^2} \\ -v\sqrt{\frac{1-u^2-v^2}{1-v^2}} \end{bmatrix}. \quad (59)$$

As in the previous sections, one might assume that a scaled version of $\frac{\partial \mathbf{u}_1}{\partial v}$ would finish an orthonormal set of basis vectors. However, $\frac{\partial \mathbf{u}_1}{\partial v}$ is not orthogonal to $\frac{\partial \mathbf{u}_1}{\partial u}$. An orthonormal third basis vector is the cross product:

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2. \quad (60)$$

The variables u and v are symmetric in $\mathbf{u}_1$. Thus, when performing simplifications, we know that if a simple solution is obtained for derivatives of u (or v) then there will exist an equally simple solution for derivatives of v (or u) that can be obtained by switching u and v and x and y. In such an instance, the following switched versions of $\mathbf{u}_2$ and $\mathbf{u}_3$ can be useful:

$$\mathbf{u}_2^s = \begin{bmatrix} \sqrt{\frac{1-u^2-v^2}{1-u^2}} \\ 0 \\ -\frac{v}{\sqrt{1-u^2}} \end{bmatrix} \quad \mathbf{u}_3^s = \begin{bmatrix} \sqrt{1-u^2} \\ -\frac{uv}{\sqrt{1-u^2}} \\ -u\sqrt{\frac{1-u^2-v^2}{1-u^2}} \end{bmatrix}. \quad (61)$$

The time derivatives of the basis vectors are:

$$\dot{\mathbf{u}}_1 = \frac{\partial \mathbf{u}_1}{\partial u}\dot{u} + \frac{\partial \mathbf{u}_1}{\partial v}\dot{v} = c_1\mathbf{u}_2 + c_2\mathbf{u}_3 \quad (62)$$

$$\dot{\mathbf{u}}_2 = \frac{\partial \dot{\mathbf{u}}_2}{\partial u}\dot{u} + \frac{\partial \dot{\mathbf{u}}_2}{\partial v}\dot{v} = c_3\mathbf{u}_1 + c_4\mathbf{u}_3 \quad (63)$$

$$\dot{\mathbf{u}}_3 = \frac{\partial \dot{\mathbf{u}}_3}{\partial u}\dot{u} + \frac{\partial \dot{\mathbf{u}}_3}{\partial v}\dot{v} = -c_2\mathbf{u}_1 - c_4\mathbf{u}_2 \quad (64)$$

where

$$c_1 = \frac{\dot{u}(1-v^2) + uv\dot{v}}{\sqrt{(1-v^2)(1-u^2-v^2)}} \quad (65)$$

$$c_2 = \frac{\dot{v}}{\sqrt{1-v^2}} \quad (66)$$

$$c_3 = -\frac{\left(\sqrt{1-u^2-v^2} + u^2\sqrt{\frac{1}{1-u^2-v^2}}\right)\left(\dot{u}\left(1-v^2\right) + uv\dot{v}\right)}{\left(1-v^2\right)^{\frac{3}{2}}} \quad (67)$$

$$c_4 = \frac{v\left(\dot{u}\left(1-v^2\right) + uv\dot{v}\right)}{\left(1-v^2\right)\sqrt{1-u^2-v^2}}. \quad (68)$$

The velocity is the first derivative of (57):

$$\dot{\mathbf{r}} = \dot{r}\mathbf{u}_1 + r\dot{\mathbf{u}}_1 \quad (69)$$

$$= \dot{r}\mathbf{u}_1 + rc_1\mathbf{u}_2 + rc_2\mathbf{u}_3. \quad (70)$$

Since $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are orthonormal, taking the dot product of $\dot{\mathbf{r}}$ with each of the basis vectors produces the value of the coefficient of that vector in (70). From the coefficient for $\mathbf{u}_1$ and $\mathbf{u}_3$, one can easily obtain expressions for $\dot{r}$ and $\dot{v}$. The expression for $\dot{u}$ is more complicated, but due to the symmetry of u and v, we can use an equivalent for as for $\dot{v}$:

$$\dot{r} = \dot{\mathbf{r}}'\mathbf{u}_1 \quad (71)$$

$$\dot{v} = \frac{\sqrt{1-v^2}}{r}\dot{\mathbf{r}}'\mathbf{u}_3 \quad (72)$$

$$\dot{u} = \frac{\sqrt{1-u^2}}{r}\dot{\mathbf{r}}'\mathbf{u}_3^s. \quad (73)$$

The acceleration is the second derivative:

$$\begin{aligned}\dot{\mathbf{r}} =& \ddot{r}\mathbf{u}_1 + \dot{r}\dot{\mathbf{u}}_1 + \left(\dot{r}c_1 + r\dot{c}_1\right)\mathbf{u}_2 + rc_1\dot{\mathbf{u}}_2 \\ &+ \left(\dot{r}c_2 + r\dot{c}_2\right)\mathbf{u}_3 + rc_2\dot{\mathbf{u}}_3 \quad (74)\\ =& \ddot{r}\mathbf{u}_1 + \dot{r}\left(c_1\mathbf{u}_2 + c_2\mathbf{u}_3\right) + \left(\dot{r}c_1 + r\dot{c}_1\right)\mathbf{u}_2 \\ &+ rc_1\left(c_3\mathbf{u}_1 + c_4\mathbf{u}_3\right) + \left(\dot{r}c_2 + r\dot{c}_2\right)\mathbf{u}_3 \\ &+ rc_2\left(-c_2\mathbf{u}_1 - c_4\mathbf{u}_2\right) \quad (75)\\ =& \underbrace{\left(\ddot{r} + r\left(c_1c_3 - c_2^2\right)\right)}_{a_1}\mathbf{u}_1 + \underbrace{\left(2\dot{r}c_1 + r\left(\dot{c}_1 - c_2c_4\right)\right)}_{a_2}\mathbf{u}_2 \\ &+ \underbrace{\left(2\dot{r}c_2 + r\left(\dot{c}_2 + c_1c_4\right)\right)}_{a_3}\mathbf{u}_3. \quad (76)\end{aligned}$$

The derivative of c_1 will turn out to be unnecessary for the solution and is omitted. The derivative of c_2 is:

$$\dot{c}_2 = \frac{v\dot{v}^2 + \left(1-v^2\right)\ddot{v}}{\left(1-v^2\right)^{\frac{3}{2}}}. \quad (77)$$

Assuming that $a_1 = a_2 = a_3 = 0$, the second derivatives of $\ddot{r}$ and $\ddot{v}$ can be found from the expressions for a_1 and a_3:

$$\ddot{r} = r\left(c_2^2 - c_1c_3\right) \quad (78)$$

$$= r\frac{\dot{u}^2(1-v^2) + 2uv\dot{u}\dot{v} + \dot{v}^2(1-u^2)}{1-u^2-v^2} \quad (79)$$

$$\ddot{v} = -\sqrt{1-v^2}\left(c_1c_4 + \frac{2c_2\dot{r}}{r} + \frac{v\dot{v}^2}{\left(1-v^2\right)^{\frac{3}{2}}}\right) \quad (80)$$

$$= -\frac{2\dot{r}\dot{v}}{r} - \frac{v\left(\dot{u}^2\left(1-v^2\right) + \dot{v}^2\left(1-u^2\right) + 2uv\dot{u}\dot{v}\right)}{1-u^2-v^2}. \quad (81)$$

Using the symmetry of u and v in $\mathbf{u}_1$, the expression for $\ddot{v}$ comes from switching u and v in (81) to get:

$$\ddot{u} = -\frac{2\dot{r}\dot{u}}{r} - \frac{u\left(\dot{v}^2\left(1-u^2\right) + \dot{u}^2\left(1-v^2\right) + 2uv\dot{u}\dot{v}\right)}{1-u^2-v^2}. \quad (82)$$

VI. Conclusion

Expressions for the acceleration terms necessary for drift functions, like that written out in (11), representing constant-velocity motion in non-Cartesian coordinate systems were presented. The expressions for dynamics in r-u and r-u-v coordinates do not appear to have been previously published. The functions are implemented as `aCVPolar`, `aCVRu2D`, `aCVSpherical`, and `aCVRuv` in the TCL [4], [12]. Additionally, this paper presents expressions for the velocity coordinates in non-Cartesian coordinate systems. These are used to implement full state conversions in the TCL via the functions in /Coordinate Systems/State Coordinate System Conversion/. Appendix A contains an example of a constant-acceleration linear dynamic model converted into polar coordinates to illustrate how the general approach can be applied to higher-order moments.

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Appendix A
A 2D Polar Constant-Jerk Model

Starting from the expression for the acceleration given in polar coordinates in (9), we take another derivative to obtain jerk:

$$\dddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\dot{\mathbf{u}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\dot{\mathbf{u}}_\theta + \left(\dddot{r} - \dot{r}\dot{\theta}^2 - 2r\dot{\theta}\ddot{\theta}\right)\mathbf{u}_r + \left(\dot{r}\ddot{\theta} + r\dddot{\theta} + 2\ddot{r}\dot{\theta} + 2\dot{r}\ddot{\theta}\right)\mathbf{u}_\theta \tag{83}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\dot{\theta}\mathbf{u}_\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\dot{\theta}\mathbf{u}_r + \left(\dddot{r} - \dot{r}\dot{\theta}^2 - 2r\dot{\theta}\ddot{\theta}\right)\mathbf{u}_r + \left(\dot{r}\ddot{\theta} + r\dddot{\theta} + 2\ddot{r}\dot{\theta} + 2\dot{r}\ddot{\theta}\right)\mathbf{u}_\theta \tag{84}$$

$$= \underbrace{\left(\dddot{r} - 3\dot{\theta}\left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\right)}_{j_r}\mathbf{u}_r + \underbrace{\left(3\dot{r}\ddot{\theta} + 3\ddot{r}\dot{\theta} + r\left(\dddot{\theta} - \dot{\theta}^3\right)\right)}_{j_\theta}\mathbf{u}_\theta. \tag{85}$$

If the j_r and j_θ components of jerk are constant, then one gets the dynamic model:

$$\dddot{r} = j_r + 3\dot{\theta}\left(\dot{r}\dot{\theta} + r\ddot{\theta}\right) \tag{86}$$

$$\dddot{\theta} = \dot{\theta}^3 + \frac{1}{r}\left(j_\theta - 3\dot{r}\ddot{\theta} - 3\ddot{r}\dot{\theta}\right). \tag{87}$$

Similar techniques could be used to obtain constant-jerk dynamic models for other coordinate systems.

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