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Itô-Taylor Expansion Moments for Continuous-Time State Propagation

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Itô-Taylor Expansion Moments for Continuous-Time State Propagation

David Frederic Crouse* *Member, IEEE*

***Abstract*—The state-prediction step of the continuous-discrete cubature Kalman filter by Arasaratnam, Haykin, and Hurd was derived using the mean and covariance matrix of an order-1.5 strong Itô-Taylor expansion for autonomous additive noise. In this note, we provide the mean and covariance matrix of a variety of strong and weak Itô-Taylor expansions, enabling the implementation of nine continuous-discrete cubature Kalman-filter variants.**

I. Introduction

Many target-tracking algorithms, such as the Kalman-filter variants discussed in [3], approximate the uncertainty in the target-state vector as a mean and a covariance matrix (a Gaussian approximation). With certain exceptions [5], [7], [12], [17], the time of a target's state associated with a particular measurement can be assumed to be known exactly and thus, one can separate the measurement update of a target state from the state-prediction step. Such a separation allows one to mix and match measurement-update and state-propagation routines, as one can do, for example, with the routines in the Tracker Component Library (TCL) [6], [18]. This note presents variants of the continuous-time state-propagation step of the continuous-discrete cubature Kalman filter (CKF), derived in [1]. The measurement-update step of the filter is not explicitly addressed. That is because any measurement-update routine that takes a mean and covariance matrix as inputs and outputs a mean and covariance matrix can be used.

As discussed in the tutorial [4], continuous-time stochastic dynamic models under Itô calculus are typically defined having the form:

$$d\mathbf{x}_t = \overbrace{\mathbf{a}(\mathbf{x}_t,t)dt}^{\substack{\text{What Physics}\\ \text{Tells You}}} + \overbrace{\mathbf{B}(\mathbf{x}_t,t)d\boldsymbol{\beta}_t}^{\substack{\text{Unknown}\\ \text{Perturbations}}}, \tag{1}$$

where $\mathbf{x}_t$ is the d_x-dimensional target state at time t, $d\boldsymbol{\beta}_t$ is the differential of a d_w-dimensional Wiener process, $\mathbf{a}$ is the $d_x \times 1$ drift function, and $\mathbf{B}$ is the $d_x \times d_w$ diffusion function.

When given a target-state probability density function (PDF) at time t, approximated as Gaussian with mean $\hat{\mathbf{x}}_t$ and covariance-matrix $\mathbf{P}_t$, a state-prediction step involves finding the mean and covariance matrix at time $t+T$ that is consistent with the continuous-time stochastic dynamic model described in (1). The most common approach to this problem involves formulating deterministic differential equations for $\hat{\mathbf{x}}_t$ and $\mathbf{P}_t$ (or in a square-root filter for $\mathbf{S}_t$ such that $\mathbf{P}_t = \mathbf{S}_t\mathbf{S}_t$) and integrating them using some type of Runge-Kutta method or other standard approach for solving the differential-equation initial-value problem. This is done for the continuous-discrete extended Kalman filter (EKF) in [8], [10] and for cubature filters in [1], [4], [15].

In contrast, the method presented in [1] replaces the solution with an approximate discretization of the form:

$$\mathbf{x}_{t+T} \approx \mathbf{f}(\mathbf{x}_t,t) + \mathbf{F}(\mathbf{x}_t,t)\tilde{\mathbf{w}} \tag{2}$$

where $\tilde{\mathbf{w}}$ is a zero-mean random vector whose dimensionality and distribution depends on the approach used to obtain the discretization, and $\mathbf{f}$ and $\mathbf{F}$ are functions of the state and time. Define the conditional expectations:

$$\boldsymbol{\mu}_{\mathbf{x}_t} \triangleq \mathrm{E}\left\{\mathbf{x}_{t+T} \middle| \mathbf{x}_t\right\} \tag{3}$$

$$\boldsymbol{\Sigma}_{\mathbf{x}_t} \triangleq \mathrm{Cov}\left\{\mathbf{x}_{t+T} \middle| \mathbf{x}_t\right\}. \tag{4}$$

The law of total covariance state, which comes from the law of total expectation, states that:

$$\mathrm{Cov}\left\{\mathbf{x}_{t+T}\right\} = \mathrm{E}\left\{\mathrm{Cov}\left\{\mathbf{x}_{t+T} \middle| \mathbf{x}_t\right\}\right\} + \mathrm{Cov}\left\{\mathrm{E}\left\{\mathbf{x}_{t+T} \middle| \mathbf{x}_t\right\}\right\}. \tag{5}$$

Consequently, the mean and covariance matrix associated with (2), given a Gaussian prior, are:

$$\hat{\mathbf{x}}_{t+T} = \mathrm{E}\left\{\mathbf{x}_{t+T}\right\} = \mathrm{E}\left\{\mathrm{E}\left\{\mathbf{x}_{t+T} \middle| \mathbf{x}_t\right\}\right\} \tag{6}$$

$$= \mathrm{E}\left\{\boldsymbol{\mu}_{\mathbf{x}_t}\right\} \tag{7}$$

$$\mathbf{P}_{t+T} = \mathrm{E}\left\{\boldsymbol{\Sigma}_{\mathbf{x}_t}\right\} + \mathrm{Cov}\left\{\boldsymbol{\mu}_{\mathbf{x}_t}\right\}. \tag{8}$$

The evaluation of $\hat{\mathbf{x}}_{t+T}$ and $\mathbf{P}_{t+T}$ involve integrals over $\mathcal{N}\left\{\mathbf{x}_t, \hat{\mathbf{x}}_t, \mathbf{P}_t\right\}$, the multivariate Gaussian distribution with mean $\hat{\mathbf{x}}_t$, and covariance matrix $\mathbf{P}_t$. As in [1], these can be approximated using cubature integration as follows:

$$\hat{\mathbf{x}}_{t+T} = \int_{\mathbf{x}\in\mathbb{R}^{d_x}} \boldsymbol{\mu}_{\mathbf{x}_t}\mathcal{N}\left\{\mathbf{x}_t, \hat{\mathbf{x}}_t, \mathbf{P}_t\right\} d\mathbf{x}_t \tag{9}$$

$$\approx \sum_{k=1}^{N} \omega_k \boldsymbol{\mu}_{\boldsymbol{\xi}_k} \tag{10}$$

$$\mathbf{P}_{t+T} = \int_{\mathbf{x}\in\mathbb{R}^{d_x}} \boldsymbol{\Sigma}_{\mathbf{x}_t}\mathcal{N}\left\{\mathbf{x}_t, \hat{\mathbf{x}}_t, \mathbf{P}_t\right\} d\mathbf{x}_t + \int_{\mathbf{x}\in\mathbb{R}^{d_x}} \left(\boldsymbol{\mu}_{\mathbf{x}_t} - \hat{\mathbf{x}}_{t+T}\right)\left(\boldsymbol{\mu}_{\mathbf{x}_t} - \hat{\mathbf{x}}_{t+T}\right)' d\mathbf{x}_t \tag{11}$$

$$\approx \sum_{k=1}^{N} \omega_k \left(\boldsymbol{\Sigma}_{\boldsymbol{\xi}_k} + \left(\boldsymbol{\mu}_{\boldsymbol{\xi}_k} - \hat{\mathbf{x}}_{t+T}\right)\left(\boldsymbol{\mu}_{\boldsymbol{\xi}_k} - \hat{\mathbf{x}}_{t+T}\right)'\right) \tag{12}$$

where $'$ denotes the transpose operator (See Section II for when this will have a different definition), and ω_k and $\boldsymbol{\xi}_k$ are the cubature weights and points for a Gaussian distribution with mean $\hat{\mathbf{x}}_t$ and covariance matrix $\mathbf{P}_t$.

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One is not limited to using the specific third-order cubature points that are given in the description of the continuous-discrete CKF in [1]; any cubature points of the required dimensionality for a Gaussian weighting function can be used. However, one should typically restrict the choice to cubature points with all-positive weights to guarantee that none of the eigenvalues of $\mathbf{P}_{t+T}$ is negative.

Cubature points and weights for the Gaussian distribution with zero mean and a covariance matrix of $\mathbf{I}$, the identity matrix, are given in the TCL [18].[1] To use such points for a Gaussian with mean $\hat{\mathbf{x}}_t$ and covariance matrix $\mathbf{P}_t$, just transform the normal $\mathbf{0}$,$\mathbf{I}$ cubature points $\tilde{\boldsymbol{\xi}}_k$ as

$$\boldsymbol{\xi}_k = \hat{\mathbf{x}}_t + \mathbf{S}_t\tilde{\boldsymbol{\xi}}_k \tag{13}$$

where $\mathbf{P}_t = \mathbf{S}_t\mathbf{S}_t'$. That is, $\mathbf{S}_t$ is a lower-triangular Cholesky decomposition of $\mathbf{P}_t$. Expressions and derivations for many cubature formulae are collected in [2], [14].

Given a set of cubature points, the continuous-time prediction T into the future is given by (10) and (12), which depend on $\boldsymbol{\mu}_{\mathbf{x}_t}$ and $\boldsymbol{\Sigma}_{\mathbf{x}_t}$, which are based on the discretization used. If T is large, one might wish to perform a number of small prediction steps rather than one large prediction step.

The following sections derive the conditional means $\boldsymbol{\mu}_{\mathbf{x}_t}$ and covariance matrices $\boldsymbol{\Sigma}_{\mathbf{x}_t}$ for many of the strong and weak Itô-Taylor discretizations that are given in [9].[2] Strong Itô-Taylor expansions are such that for a particular continuous-time realization of the Wiener process $\boldsymbol{\beta}_t$, the values taken by the expansion will equal the true values of the processes. In contrast, the values delivered by weak Itô-Taylor expansions do not necessarily converge to the true process values, but the moments of the values will converge to the true moments of the process. Weak Itô-Taylor expansions have been used before in target-tracking problems including [11].

The use of more general Itô-Taylor expansions goes beyond the order-1.5 strong Itô-Taylor discretization used in [1]. Indeed, in [15], it is criticized that the algorithm of [1] "cannot be easily extended to the case of non-additive noises." This paper addresses that shortcoming as a number of the expansions here can handle non-additive noise.[3] The results are summarized in Section II; Section III provides a simulation example of some of the results, and Section IV provides a conclusion. Expressions for random quantities and their expected values are given in Appendix A and these quantities are used in the derivations, which have been placed in Appendices B and C for the strong and weak expansions, respectively. The algorithms are implemented in the `stochTaylorCubPred` function in version 4.0 of the TCL [18].

II. Notation and Summary of Results

In the following sections, the drift function $\mathbf{a}(\mathbf{x}_t, t)$ and the diffusion function $\mathbf{B}(\mathbf{x}_t, t)$ from the stochastic differential equations (1) will only be considered evaluated at the state $\mathbf{x}_t$ and time t. Consequently, those function values shall be written as $\mathbf{a}$ and $\mathbf{B}$. The $'$ will denote the matrix transpose *except* when considering expansions where $d_x = d_w = 1$. In such instances, it denotes derivatives. Thus, a' and a'' are the first and second derivatives of the (scalar) drift function with respect to x_t. When considering individual elements of a vector or a matrix, subscripts on the vector or matrix, which will not be written in bold, will be used. For example, B_{j_1,j_2} refers to the item in row j_1 and column j_2 of $\mathbf{B}$.

The following derivative operators, defined in [9, Ch. 10.1] are used:

$$\mathbb{L}_0 \triangleq \frac{\partial}{\partial t} + \sum_{k=1}^{d_x} a_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{d_x}\sum_{i=1}^{d_x}\sum_{j=1}^{d_w} B_{k,j}B_{i,j}\frac{\partial^2}{\partial x_k \partial x_i} \tag{14}$$

$$\mathbb{L}^j \triangleq \sum_{k=1}^{d_x} B_{k,j}\frac{\partial}{\partial x_k}. \tag{15}$$

The remainder of this section presents the results for many of the expansions given in [9, Ch. 10.2]. Multiple variants of an expansion are given depending on assumptions on $\mathbf{a}$ and $\mathbf{B}$. The weak Itô-Taylor expansions given in [9, Ch. 14] can often be simulated with different random variables. However, these do not change the first and second moments.

A. Strong Expansions

1) **Euler-Maruyama Method** (order 0.5)
   The expansion is Eq. 2.1 in [9, Ch. 10.2].

$$\boldsymbol{\mu}_{\mathbf{x}_t} = \mathbf{x}_t + \mathbf{a}T \tag{16}$$
$$\boldsymbol{\Sigma}_{\mathbf{x}_t} = T\mathbf{B}\mathbf{B}'. \tag{17}$$

2) **Milstein Scheme** (order 1.0)
   All of the Milstein scheme variants considered have discretizations:

$$\boldsymbol{\mu}_{\mathbf{x}_t} = \mathbf{x}_t + \mathbf{a}T \tag{18}$$
$$\boldsymbol{\Sigma}_{\mathbf{x}_t} = T\mathbf{B}\mathbf{B}' + \frac{T^2}{2}\mathbf{D}\mathbf{D}'. \tag{19}$$

   where $\mathbf{D}$ depends on the scheme. For specific variants, the values of $\mathbf{D}$ are:

   a) Scalar noise.
      The expansion is Eq. 3.2 in [9, Ch. 10.3]. Assuming that $d_w = 1$,

$$\mathbf{D} = \sum_{i=1}^{d_x} B_{i,1}\frac{\partial \mathbf{B}}{\partial x_i}. \tag{20}$$

   b) General noise.
      The expansions are Eq. 3.3 in [9, Ch. 10.3] and Eq. 3.16 in [9, Ch. 10.3], respectively, though the equation for commutative noise must be modified for Itô calculus. General noise is any function for $\mathbf{B}$. Commutative noise is such that:

$$\mathbb{L}^{j_1}B_{k,j_2} = \mathbb{L}^{j_2}B_{k,j_1}. \tag{21}$$

[1] See the functions in Mathematical Functions/Numerical Integration/Cubature Points/Gaussian Weight/ in the TCL.

[2] It is worth noting that additional weak Itô-Taylor expansions exist in the literature [16].

[3] Additive noise is when $\mathbf{B}(\mathbf{x}_t, t)$ does not depend on $\mathbf{x}_t$.

In both instances, $\mathbf{D}$ has elements:

$$d_{k,j_1+(j_2-1)d_w} = \mathbb{L}^{j_1} B_{k,j_2}. \tag{22}$$

c) Diagonal noise.
The expansion is Eq. 3.12 in [9, Ch. 10.3]. $\mathbf{D}$ is a diagonal matrix where the kth element on the diagonal is:

$$D_{k,k} = B_{k,k} \frac{\partial B_{k,k}}{\partial x_k}. \tag{23}$$

3) **Order 1.5**

a) Additive noise.
The expansion is Eq. 4.10 in [9, Ch. 10.4].

$$\boldsymbol{\mu}_{\mathbf{x}_t} = \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}T^2 \left(\mathbb{L}_0 \mathbf{a}\right) \tag{24}$$

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{x}_t} =& T\mathbf{C}_1\mathbf{C}_1' \\ &+ \frac{1}{3}T^3\mathbf{C}_2\mathbf{C}_2' + \frac{1}{2}T^2\left(\mathbf{C}_1\mathbf{C}_2' + \mathbf{C}_2\mathbf{C}_1'\right)\end{aligned} \tag{25}$$

where

$$\mathbf{C}_1 = \mathbf{B} + T\frac{\partial \mathbf{B}}{\partial t} \tag{26}$$

$$\mathbf{C}_2 = (\mathbb{L}\mathbf{a}) - \frac{\partial \mathbf{B}}{\partial t}. \tag{27}$$

b) Autonomous scalar problems.
"Autonomous" means that $\mathbf{a}$ and $\mathbf{B}$ do not depend on t. The expansion is Eq. 4.1 in [9, Ch. 10.4].

$$\boldsymbol{\mu}_{\mathbf{x}_t} = x_t + aT + \frac{1}{2}\left(aa' + \frac{1}{2}B^2a''\right)T^2 \tag{28}$$

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{x}_t} =& \frac{1}{3}T\left(3c_1^2 + 3c_1\left(6c_3 + c_4\right)T\right) \\ &+ \frac{1}{3}T^2\left(6c_2^2 + \left(45c_3^2 + 9c_3c_4 + c_4^2\right)T\right)\end{aligned} \tag{29}$$

where

$$\begin{aligned}c_1 =& B + \left(aB' + \frac{1}{2}B^2B''\right)T \\ &- \frac{1}{2}B\left(BB'' + (B')^2\right)T\end{aligned} \tag{30}$$

$$c_2 = \frac{1}{2}BB' \tag{31}$$

$$c_3 = \frac{1}{6}B\left(BB'' + (B')^2\right) \tag{32}$$

$$c_4 = a'B - \left(aB' + \frac{1}{2}B^2B''\right). \tag{33}$$

B. Weak Expansions

1) **Order 2.0**

a) Scalar state and noise.
The non-simplified expansion is Eq. 2.1 in [9, Ch. 14.2]; the simplified expansion is Eq. 2.2 in [9, Ch. 14.2]. Both expansions have mean

$$\mu_{x_t} = x_t + aT + \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}\frac{\partial^2 a}{\partial x^2}B^2\right)T^2. \tag{34}$$

The unsimplified expansion has variance

$$\begin{aligned}\Sigma_{x_t} =& c_0^2 + (c_1^2 + 2c_0c_2)T + \left(3c_2^2 + c_1c_3\right)T^2 \\ &+ \frac{1}{3}c_3^2T^3\end{aligned} \tag{35}$$

where

$$c_0 = -\frac{T}{2}B\frac{\partial B}{\partial x} \tag{36}$$

$$c_1 = B + T\left(a\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}B^2\right) \tag{37}$$

$$c_2 = \frac{1}{2}B\frac{\partial B}{\partial x} \tag{38}$$

$$c_3 = \frac{\partial a}{\partial x}B - \left(a\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}B^2\right). \tag{39}$$

The simplified expansion has variance

$$\Sigma_{x_t} = c_0^2 + (c1^2 + 2c_0c_2)T + 3c_2^2T^2 \tag{40}$$

where

$$c_0 = -\frac{1}{2}B\frac{\partial B}{\partial x} \tag{41}$$

$$c_1 = B + \frac{T}{2}\left(\frac{\partial a}{\partial x}B + a\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}B^2\right) \tag{42}$$

$$c_2 = \frac{1}{2}B\frac{\partial B}{\partial x}. \tag{43}$$

b) General noise.
The unsimplified expansion is Eq. 2.6 in [9, Ch. 14.2] and the simplified expansion is Eq. 2.7 in [9, Ch. 14.2]. In both instances,

$$\boldsymbol{\mu}_{\mathbf{x}_t} = \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}\left(\mathbb{L}^0\mathbf{a}\right)T^2. \tag{44}$$

When using the unsimplified scheme:

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{x}_t} =& T\mathbf{B}\mathbf{B}' + \frac{T^3}{3}\left(\mathbf{C}^{(1)}\left(\mathbf{C}^{(1)}\right)' + \mathbf{C}^{(2)}\left(\mathbf{C}^{(2)}\right)'\right) \\ &+ \frac{T^3}{6}\left(\mathbf{C}^{(1)}\left(\mathbf{C}^{(2)}\right)' + \mathbf{C}^{(2)}\left(\mathbf{C}^{(1)}\right)'\right) \\ &+ \frac{T^2}{2}\left(\mathbf{C}^{(3)}\left(\mathbf{C}^{(3)}\right)' + \mathbf{B}\left(\mathbf{C}^{(1)}\right)'\right) \\ &+ \frac{T^2}{2}\left(\mathbf{B}\left(\mathbf{C}^{(2)}\right)' + \mathbf{C}^{(1)}\mathbf{B}' + \mathbf{C}^{(2)}\mathbf{B}'\right)\end{aligned} \tag{45}$$

where

$$\mathbf{C}_{k,j}^{(1)} = \mathbb{L}^0 B_{k,j} \tag{46}$$

$$\mathbf{C}_{k,j}^{(2)} = \left(\mathbb{L}^j a_k\right) \tag{47}$$

$$\mathbf{C}_{k,j_1+(j_2-1)m}^{(3)} = \mathbb{L}^{j_1} B_{k,j_2}. \tag{48}$$

When using the simplified scheme:

$$\boldsymbol{\Sigma}_{\mathbf{x}_t} = T\mathbf{C}^{(1)}\left(\mathbf{C}^{(1)}\right)' + 2T^2\mathbf{C}^{(2)}\left(\mathbf{C}^{(2)}\right)' \tag{49}$$

where

$$\mathbf{C}^{(1)} = \mathbf{B} + \frac{T}{2}\left(\left(\mathbb{L}^0\mathbf{B}\right) + (\mathbb{L}\mathbf{a})\right) \tag{50}$$

and $\mathbf{C}^{(2)}$ has elements

$$C^{(2)}_{k,j_1+(j_2-1)d_w} = \frac{1}{2}\mathbb{L}^{j_1} B_{k,j_2}. \quad (51)$$

c) Scalar noise.
This requires that $d_w = 1$. The expansion is Eq. 2.5 in [9, Ch. 14.2]:

$$\boldsymbol{\mu}_{\mathbf{x}_t} = \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}\left(\mathbb{L}^0\mathbf{a}\right)T^2 \quad (52)$$

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{x}_t} =& \mathbf{c}_0\mathbf{c}_0' + \mathbf{c}_2\mathbf{c}_2' 3T^2 + \mathbf{c}_3\mathbf{c}_3'\frac{T^3}{3}\\ &+ \left(\mathbf{c}_1\mathbf{c}_1' + \mathbf{c}_0\mathbf{c}_2' + \mathbf{c}_2\mathbf{c}_0'\right)T\\ &+ \left(\mathbf{c}_1\mathbf{c}_3' + \mathbf{c}_3\mathbf{c}_1'\right)\frac{T^2}{2}\end{aligned} \quad (53)$$

where

$$\mathbf{c}_0 = -\frac{T}{2}\left(\mathbb{L}\mathbf{B}\right) \quad (54)$$

$$\mathbf{c}_1 = \mathbf{B} + T\left(\mathbb{L}^0\mathbf{B}\right) \quad (55)$$

$$\mathbf{c}_2 = \frac{1}{2}\left(\mathbb{L}\mathbf{B}\right) \quad (56)$$

$$\mathbf{c}_3 = \left(\mathbb{L}\mathbf{a}\right) - \left(\mathbb{L}^0\mathbf{B}\right). \quad (57)$$

III. Simulation Example

To demonstrate the utility of some of the new continuous-time prediction schemes, we consider a multivariate dynamic model with non-additive noise, as it cannot be handled by [1]. However, we specifically choose a model for which an exact solution is available. Thus, we choose the geometric Brownian model, also known as the Black-Scholes model. This dynamic model is a multivariate form of the model as expressed by the stochastic differential equation under the Itô calculus with drift and diffusion terms [13, Ch. 2.4],

$$\mathbf{a}(\mathbf{x}_t, t) = \operatorname{diag}\left(\mathbf{x}_t\right)\mathbf{m} \quad (58)$$

$$\mathbf{B}(\mathbf{x}_t, t) = \operatorname{diag}\left(\mathbf{x}_t\right)\mathbf{D} \quad (59)$$

where $\mathbf{m}$ is a $d_x \times 1$ vector and $\mathbf{D}$ is a $d_x \times d_w$ matrix. The expected value and the elements of the covariance matrix of the prediction of this system conditioned on $\mathbf{x}_t$ are given in [4][4] based on [13, Ch. 2.4] and are

$$\boldsymbol{\mu}_{\mathbf{x}_t} = \operatorname{diag}\left(\mathbf{x}_t\right) e^{\boldsymbol{\mu} + \frac{1}{2}\operatorname{diag}\left(\tilde{\boldsymbol{\Sigma}}\right)}, \quad (60)$$

$$\Sigma_{\mathbf{x}_t,i,j} = x_{t,i}x_{t,j}e^{\mu_i+\mu_j+\frac{1}{2}\left(\tilde{\Sigma}_{i,i}+\tilde{\Sigma}_{j,j}\right)}\left(e^{\tilde{\Sigma}_{i,j}} - 1\right) \quad (61)$$

where the exponential is element-by-element (not a matrix exponential), and

$$\boldsymbol{\mu} = \left(\mathbf{m} - \frac{1}{2}\operatorname{diag}\left(\mathbf{D}\mathbf{D}'\right)\right)T \quad (62)$$

$$\tilde{\boldsymbol{\Sigma}} = \mathbf{D}\mathbf{D}'T. \quad (63)$$

Additionaly, the diag operator is defined such that diag applied to a vector returns a matrix with the vector as its main diagonal and diag of a matrix returns a vector holding the main diagonal of the matrix.

[4]In [4], it is incorrectly claimed that the matrix $\mathbf{D}$ must be square. That, however, is not the case.

TABLE I
The peak relative errors of the different prediction algorithms predicting the geometric Brownian model. The higher order the formula, the better the results.

Expansion	pErr_x	pErr_P
Euler-Maruyama	0.1219	0.5056
Milstein	0.1219	0.3940
Weak Order 2.0	0.0072	0.0819

Given (60) and (61), unlike with general nonlinear models, it is possible to explicitly evaluate the expected values for $\hat{\mathbf{x}}_{T+t}$ and $\mathbf{P}_t$ in (9) and (11) cubature integration. These are:

$$\hat{\mathbf{x}}_{t+T} = \operatorname{diag}\left\{e^{\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}\left(\tilde{\boldsymbol{\Sigma}}\right)}\right\}\hat{\mathbf{x}}_t \quad (64)$$

$$\begin{aligned}\mathbf{P}_{t+T} =& \tilde{\mathbf{P}}_{t+T}\\ &+ \operatorname{diag}\left\{e^{\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}\left(\tilde{\boldsymbol{\Sigma}}\right)}\right\}\mathbf{P}_t\operatorname{diag}\left\{e^{\boldsymbol{\mu}+\frac{1}{2}\operatorname{diag}\left(\tilde{\boldsymbol{\Sigma}}\right)}\right\}\end{aligned} \quad (65)$$

$$\tilde{P}_{t+T,i,j} = \left(P_{t,i,j} + \hat{x}_{t,i}\hat{x}_{t,j}\right)e^{\mu_i+\mu_j+\frac{1}{2}\left(\tilde{\Sigma}_{i,i}+\tilde{\Sigma}_{j,j}\right)}\left(e^{\tilde{\Sigma}_{i,j}} - 1\right) \quad (66)$$

where the Law of Total Covariance was used to find (65).

Given explicit solutions for the predicted moments, we consider a Black-Scholes system with

$$\mathbf{m} = \begin{bmatrix}0.9\\1.7\\1.3\\0.1\end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix}1.6154 & 0.0284\\0.1034 & 0.4361\\0.9386 & 0.0641\\1.1955 & 0.4186\end{bmatrix} \quad (67)$$

being predicted $T = 1/5$ forward in time.

In comparison, we consider the solutions for the strong Euler-Maruyama expansion, using (16) and (17), the strong Milstein scheme for general noise using (18), (19), and (22), and the unsimplified weak order-2.0 expansion for general noise using (44) and (45). For cubature points, we use fifth-order cubature points corresponding to the algorithm named $E_n^{r^2}$ 5-3, on pg. 317 of [14]. These are reproduced in Appendix D, as was also done in [4]. These cubature points are implemented as one of the choices in `fifthOrderCubPoints` in the TCL [18].

For each of the expansions, the prediction was done using ten steps to go from $t = 0$ to $t = T = 1/5$. The peak relative error of the estimates is considered. If $\hat{\mathbf{x}}_{t+T}$ is the true mean obtained from (64), and $\hat{\hat{\mathbf{x}}}_{t+T}$ is the estimate from one of the aforementioned methods, and $\mathbf{P}_{t+T}$ and $\hat{\mathbf{P}}_{t+T}$ are the analogous quantities for the covariance matrix, then the peak relative errors are defined to be

$$\text{pErr}_x \triangleq \max\left\{\left(\hat{\hat{\mathbf{x}}}_{t+T} - \hat{\mathbf{x}}_{t+T}\right)/\hat{\mathbf{x}}_{t+T}\right\} \quad (68)$$

$$\text{pErr}_P \triangleq \max\left\{\left(\hat{\mathbf{P}}_{t+T} - \mathbf{P}_{t+T}\right)/\mathbf{P}_{t+T}\right\} \quad (69)$$

where the division is element-by-element and the maximum operator is taken over all elements of the vector or matrix.

Table I shows the maximum relative errors for each of the algorithms. It can be seen that the higher the approximation order, the more accurate the results are. The Euler-Maruyama and Milstein methods differ only in the covariance estimates with better performance from the higher-order Milstein method. The order-2.0 weak scheme outperforms the other methods in both respects.

IV. Conclusions

The logic of [1] was applied to obtain new continuous-time prediction steps for CKFs based on Itô-Taylor expansions beyond the one considered in [1]. An example was presented demonstrating improved performance with higher expansion orders when applied to a geometric Brownian model, for which an explicit propagation solution is available. This goes beyond the capabilities of the continuous-time propagation algorithm of [1], which cannot handle dynamic models with non-additive noise. The algorithms are implemented in the `stochTaylorCubPred` function in version 4.0 of the TCL [18].

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Appendix A
Random Quantities and Moments

The integral over the Wiener process from t to T is a multivariate Gaussian random variable:

$$\mathbf{w} \triangleq \int_t^T d\boldsymbol{\beta} \tag{70}$$

such that

$$\mathrm{E}\{\mathbf{w}\} = \mathbf{0} \tag{71}$$
$$\mathrm{E}\{\mathbf{w}\mathbf{w}'\} = T\mathbf{I}. \tag{72}$$

Consequently, when scalar, one can use the following moments for Gaussian random variables:

$$\mathrm{E}\left\{w^2\right\} = T \tag{73}$$
$$\mathrm{E}\left\{w^3\right\} = 0 \tag{74}$$
$$\mathrm{E}\left\{w^4\right\} = 3T^2 \tag{75}$$
$$\mathrm{E}\left\{w^5\right\} = 0 \tag{76}$$
$$\mathrm{E}\left\{w^6\right\} = 15T^3. \tag{77}$$

A number of the expansions use a second Gaussian random-variable $\mathbf{z}$ that is correlated with $\mathbf{w}$ such that:

$$\mathrm{E}\{\mathbf{z}\} = \mathbf{0} \tag{78}$$
$$\mathrm{E}\{\mathbf{z}\mathbf{z}'\} = \frac{1}{3}T^3\mathbf{I} \tag{79}$$
$$\mathrm{E}\{\mathbf{w}\mathbf{z}'\} = \frac{1}{2}T^2\mathbf{I}. \tag{80}$$

In the scalar case, the following expected values are also needed:

$$\mathrm{E}\left\{w^2 z\right\} = 0 \tag{81}$$
$$\mathrm{E}\left\{w^3 z\right\} = \frac{3}{2}T^3. \tag{82}$$

Additionally, multiple Itô integrals of the noise process appear in many of the expansions. In the expansions chosen for this paper, only Itô integrals with one or two subscripts appear. However, when computing $\boldsymbol{\Sigma}_{\mathbf{x}_t}$ in subsequent sections, values with up to four subscripts appear. The multiple Itô integrals arising in this paper are:

$$I_j = \int_t^T d\beta_{s,j} = w_j \tag{83}$$
$$I_{0,0} = \int_t^T \int_t^{s_2} ds_1\, ds_2 = \frac{1}{2}T^2 \tag{84}$$
$$I_{0,j} = \int_t^T \int_t^{s_2} ds_1\, d\beta_{s_2,j} \tag{85}$$
$$I_{j,0} = \int_t^T \int_t^{s_2} d\beta_{s_1,j}\, ds_2 \tag{86}$$
$$I_{j_1,j_2} = \int_t^T \int_t^{s_2} d\beta_{s_1,j_1}\, d\beta_{s_2,j_2} \tag{87}$$
$$I_{j_1,j_2,j_3} = \int_t^T \int_t^{s_3} \int_t^{s_2} d\beta_{s_1,j_1}\, d\beta_{s_2,j_2}\, d\beta_{s_3,j_3} \tag{88}$$
$$I_{j_1,j_2,j_3,j_4} = \int_t^T \int_t^{s_4} \int_t^{s_3} \int_t^{s_2} d\beta_{s_1,j_1}\, d\beta_{s_2,j_2}\, d\beta_{s_3,j_3}\, d\beta_{s_4,j_4} \tag{89}$$

and are defined in [9, Ch. 5.2]. For the purpose of this paper, there is no need to evaluate these stochastic integrals. Rather, only various expected values involving them are needed. First, due to Lemma 5.7.1 of [9, Ch. 5.7], the expected values of all of those multiple Itô integrals are zero. Also, using Lemma 5.7.2 of [9, Ch. 5.7], we know that

$$\mathrm{E}\left\{I_{j_1,j_2} I_{j_3,j_4}\right\} = \begin{cases} 0 & \text{if } j_1 \neq j_3 \text{ or } j_2 \neq j_4 \\ \frac{1}{2}T^2 & \text{otherwise.} \end{cases} \tag{90}$$

Using Eq. 2.16 in [9, Ch. 5.2]

$$w_k I_{j_1,j_2} = I_{k,j_1,j_2} + I_{j_1,k,j_2} + I_{j_1,j_2,k} + \delta\{j_1 - k\} I_{0,j_2} + \delta\{j_2 - k\} I_{j_1,0}. \tag{91}$$

Consequently,

$$\mathrm{E}\left\{w_k I_{j_1,j_2}\right\} = 0. \tag{92}$$

Similarly, Eq. 2.16 in [9, Ch. 5.2]

$$w_k I_{0,j} = I_{k,0,j} + I_{0,k,j} + I_{0,j,k} + \delta\{k - j\} I_{0,0} \tag{93}$$
$$w_k I_{j,0} = I_{k,j,0} + I_{j,k,0} + I_{j,0,k} + \delta\{k - j\} I_{0,0} \tag{94}$$

where δ indicates the Dirac delta function. Consequently,

$$\mathrm{E}\left\{w_k I_{j,0}\right\} = \mathrm{E}\left\{w_k I_{0,j}\right\} = \delta\{k - j\}\frac{1}{2}T^2. \tag{95}$$

Using Lemma 5.7.2 of [9, Ch. 5.7] one can also determine the cross terms:

$$\mathrm{E}\left\{I_{0,j_1} I_{0,j_2}\right\} = \mathrm{E}\left\{I_{j_1,0} I_{j_2,0}\right\} = \begin{cases} 0 & \text{if } j_1 \neq j_2 \\ \frac{1}{3}T^3 & \text{otherwise} \end{cases} \tag{96}$$

$$\mathrm{E}\left\{I_{j_1,0} I_{0,j_2}\right\} = \mathrm{E}\left\{I_{0,j_1} I_{j_2,0}\right\} = \begin{cases} 0 & \text{if } j_1 \neq j_2 \\ \frac{1}{6}T^3 & \text{otherwise.} \end{cases} \tag{97}$$

Another expected value that will arise more than once is:

$$\begin{aligned}&\frac{1}{4}\operatorname{E}\left\{\left(w_{j_1}w_{j_2}-\delta\{j_1-j_2\}T\right)\left(w_{j_3}w_{j_4}-\delta\{j_3-j_4\}T\right)\right\}\\ =&\frac{1}{4}\operatorname{E}\left\{w_{j_1}w_{j_2}w_{j_3}w_{j_4}+\delta\left\{j_1-j_2\right\}\delta\left\{j_3-j_4\right\}T^2\right\}\\ &+\frac{1}{4}\operatorname{E}\left\{-\delta\left\{j_3-j_4\right\}Tw_{j_1}w_{j_2}-\delta\left\{j_1-j_2\right\}Tw_{j_3}w_{j_4}\right\}\end{aligned} \tag{98}$$

$$=\begin{cases}0 & \text{if } j_1\neq j_3 \text{ or } j_2\neq j_4\\ \frac{1}{2}T^2 & \text{otherwise.}\end{cases} \tag{99}$$

The second condition in (99) is a simplification of $\operatorname{E}\left\{w_j^4\right\}+T^2-2T\operatorname{E}\left\{w_j^2\right\}$.

Finally, a random-matrix $\mathbf{V}$ that is independent of all of the aforementioned random variables arises in a simplified weak scheme. This symmetric matrix has a deterministic diagonal and is defined with elements such that:

$$V_{j_2,j_1}=\begin{cases}-T & \text{if } j_1=j_2\\ V_{j_2,j_1} & \text{if } j_1<j_2\\ \pm T & \text{if } j_1>j_2\text{; the signs are equiprobable.}\end{cases} \tag{100}$$

The following expectations involving the elements of this random matrix will arise:

$$\operatorname{E}\left\{w_{j_1}\left(w_{j_2}w_{j_3}+V_{j_2,j_3}\right)\right\}=0 \tag{101}$$

and

$$\operatorname{E}\left\{\left(w_{j_1}w_{j_2}+V_{j_1,j_2}\right)\left(w_{j_3}w_{j_4}+V_{j_3,j_4}\right)\right\}=\begin{cases}0 & \text{if } j_1\neq j_3 \text{ or } j_2\neq j_4\\ 2T^2 & \text{otherwise.}\end{cases} \tag{102}$$

APPENDIX B
STRONG ITÔ-TAYLOR SCHEMES

A. Order 0.5

1) Euler-Maruyama Method: The expansion from Eq. 2.1 in [9, Ch. 10.2] is:

$$\mathbf{x}_{t+T}=\mathbf{x}_t+\mathbf{a}T+\mathbf{B}\mathbf{w}. \tag{103}$$

The mean and covariance matrix conditioned on $\mathbf{x}_t$ in (16) and (17) come directly from the definitions of the expected value and covariance, and from the moments (71) and (72).

B. Strong Order 1.0

1) The Milstein Scheme for Scalar Noise: The expansion from Eq. 3.2 in [9, Ch. 10.3] is:

$$\mathbf{x}_{t+T}=\mathbf{x}_t+\mathbf{a}T+\mathbf{B}w+\frac{1}{2}\left(\sum_{i=1}^{d_x}b_i\frac{\partial\mathbf{B}}{\partial x_i}\right)\left(w^2-T\right). \tag{104}$$

Using (71) and (72), one obtains the expected value in (18).

For the covariance matrix, define

$$\mathbf{C}_1\triangleq\mathbf{B} \tag{105}$$

$$\mathbf{C}_2\triangleq\frac{1}{2}\left(\sum_{i=1}^{d_x}b_i\frac{\partial\mathbf{B}}{\partial x_i}\right) \tag{106}$$

$$\mathbf{C}_3\triangleq-\mathbf{C}_2T. \tag{107}$$

The covariance matrix is

$$\mathbf{\Sigma}_{\mathbf{x}_t}=\operatorname{E}\left\{\left(\mathbf{C}_1w+\mathbf{C}_2w^2+\mathbf{C}_3\right)\left(\mathbf{C}_1w+\mathbf{C}_2w^2+\mathbf{C}_3\right)'\right\} \tag{108}$$

$$\begin{aligned}=&\mathbf{C}_1\mathbf{C}_1'\operatorname{E}\left\{w^2\right\}+\mathbf{C}_2\mathbf{C}_2'\operatorname{E}\left\{w^4\right\}+\mathbf{C}_3\mathbf{C}_3'\\ &+\left(\mathbf{C}_1\mathbf{C}_2'+\mathbf{C}_2\mathbf{C}_1'\right)\operatorname{E}\left\{w^3\right\}+\mathbf{C}_1\mathbf{C}_3'\operatorname{E}\left\{w\right\}\\ &+\mathbf{C}_3\mathbf{C}_1'\operatorname{E}\left\{w\right\}+\mathbf{C}_2\mathbf{C}_3'\operatorname{E}\left\{w^2\right\}+\mathbf{C}_3\mathbf{C}_2'\operatorname{E}\left\{w^2\right\}.\end{aligned} \tag{109}$$

Using (71), (73), (74), and (75), one obtains the expression for the covariance matrix in (19) with $\mathbf{D}$ given in (20).

2) The Milstein Scheme for General Noise: The expansion from Eq. 3.3 in [9, Ch. 10.3] has its kth elements as

$$x_{t+T,k}=x_t+a_kT+\sum_{j=1}^{d_w}B_{k,j}w_j+\sum_{j_1=1}^{d_w}\sum_{j_2=1}^{d_w}\left(\mathbb{L}^{j_1}B_{k,j_2}\right)I_{j_1,j_2}. \tag{110}$$

Define a matrix $\mathbf{D}$ with elements as in (22), and define a vector $\mathbf{i}$ with elements such that

$$i_{j_1+(j_2-1)d_w}\triangleq I_{j_1,j_2}. \tag{111}$$

Equation (110) can be written in vector form as:

$$\mathbf{x}_{t+T}=\mathbf{x}_t+\mathbf{a}T+\mathbf{B}\mathbf{w}+\mathbf{D}\mathbf{i}. \tag{112}$$

As noted in Appendix A, $\operatorname{E}\left\{I_{j_1,j_2}\right\}=0\,\forall j_1,j_2\geq 1$. Consequently, one obtains the expected value in (18).

The covariance matrix is

$$\mathbf{\Sigma}_{\mathbf{x}_t}=\operatorname{E}\left\{\left(\mathbf{B}\mathbf{w}+\mathbf{D}\mathbf{i}\right)\left(\mathbf{B}\mathbf{w}+\mathbf{D}\mathbf{i}\right)'\right\} \tag{113}$$

$$\begin{aligned}=&\mathbf{B}\operatorname{E}\left\{\mathbf{w}\mathbf{w}'\right\}\mathbf{B}'+\mathbf{D}\operatorname{E}\left\{\mathbf{i}\mathbf{i}'\right\}\mathbf{D}'+\mathbf{B}\operatorname{E}\left\{\mathbf{w}\mathbf{i}'\right\}\mathbf{D}'\\ &+\mathbf{D}\operatorname{E}\left\{\mathbf{i}\mathbf{w}'\right\}\mathbf{B}'.\end{aligned} \tag{114}$$

Using the expected value in (90), we know that

$$\operatorname{E}\left\{\mathbf{i}\mathbf{i}'\right\}=\frac{1}{2}T^2\mathbf{I}. \tag{115}$$

Using (92), we know that

$$\operatorname{E}\left\{\mathbf{w}\mathbf{i}'\right\}=\mathbf{0}. \tag{116}$$

Finally with (72), the covariance matrix simplifies to the formulation in (19) with $\mathbf{D}$ given by (22).

3) The Milstein Scheme for Commutative Noise: The expansion from Eq. 3.16 in [9, Ch. 10.3], modified to be for Itô calculus rather than Stratonovich calculus, is:

$$\begin{aligned}x_{t+T,k}=&x_t+a_kT+\sum_{j=1}^{d_w}B_{k,j}w_j\\ &+\frac{1}{2}\sum_{j_1=1}^{d_w}\sum_{j_2=1}^{d_w}\left(\mathbb{L}^{j_1}B_{k,j_2}\right)\left(w_{j_1}w_{j_2}-\delta\{j_1-j_2\}T\right).\end{aligned} \tag{117}$$

Define a matrix $\mathbf{D}$ with elements as in (22), and define a vector $\mathbf{i}$ with elements such that

$$\mathbf{i}_{j_1+(j_2-1)m}\triangleq\frac{1}{2}\left(w_{j_1}w_{j_2}-\delta\{j_1-j_2\}T\right). \tag{118}$$

Equation (117) can be written in vector form as:

$$\mathbf{x}_{t+T}=\mathbf{x}_t+\mathbf{a}T+\mathbf{B}\mathbf{w}+\mathbf{D}\mathbf{i}. \tag{119}$$

Using (72), we know that

$$\mathrm{E}\left\{\mathbf{i}\right\} = \mathbf{0}. \tag{120}$$

Thus, the expected value is given by (18).

Prior to simplification, the covariance matrix has the same form as (114). However, the expected values involving $\mathbf{i}$ have to be determined. These are given by (99). All of the elements in $\mathbf{wi}'$ have odd powers of the elements of $\mathbf{w}$. Consequently, the expected value is zero. Thus, the covariance matrix is again (19) with the matrix $\mathbf{D}$ given by (22).

4) The Milstein Scheme for Diagonal Noise: The expansion from Eq. 3.12 in [9, Ch. 10.3] has its kth element given by

$$x_{t+T,k} = x_t + a_k T + B_{k,k} w_k + \frac{1}{2} B_{k,k} \frac{\partial B_{k,k}}{\partial x_k}\left(w_k^2 - T\right). \tag{121}$$

Define the diagonal matrix $\mathbf{D}$ with the kth element given by (23), and define a vector $\mathbf{i}$ such that the kth element is

$$i_k = \frac{1}{2}\left(w_k^2 - T\right). \tag{122}$$

Equation (121) can now be written in vector form as

$$\mathbf{x}_{t+T} = \mathbf{x}_t + \mathbf{a}T + \mathbf{B}\mathbf{w} + \mathbf{D}\mathbf{i}. \tag{123}$$

The expected value is given by (18).

Before simplification, the covariance matrix has the same form as in (114). However, the expected values involving $\mathbf{i}$ have to be determined. The elements of $\mathrm{E}\left\{\mathbf{ii}'\right\}$ have the form

$$\begin{aligned}
\mathrm{E}\left\{i_{j_1} i'_{j_2}\right\} =& \frac{1}{4}\mathrm{E}\left\{\left(w_{j_1}^2 - T\right)\left(w_{j_2}^2 - T\right)\right\} && (124)\\
=& \frac{1}{4}\left(T^2 - T\,\mathrm{E}\left\{w_{j_1}^2\right\} - T\,\mathrm{E}\left\{w_{j_2}^2\right\}\right)\\
&+ \frac{1}{4}\mathrm{E}\left\{w_{j_1}^2 w_{j_2}^2\right\} && (125)\\
=& \frac{1}{4}\left(T^2 - 2T^2 + \mathrm{E}\left\{w_{j_1}^2 w_{j_2}^2\right\}\right) && (126)\\
=& \frac{1}{4}\left(\mathrm{E}\left\{w_{j_1}^2 w_{j_2}^2\right\} - T^2\right) && (127)\\
=& \begin{cases} 0 & \text{if } j_1 \neq j_2 \\ \frac{1}{2}T^2 & \text{otherwise} \end{cases} && (128)
\end{aligned}$$

where the second condition in (128) comes using (75).

The cross terms $\mathbf{iw}'$ have an odd number of products of elements of $\mathbf{w}$ and are thus zero. Consequently, the covariance matrix simplifies to (19) with $\mathbf{D}$ given by (23).

C. Strong Order 1.5

1) The Order-1.5 Strong Taylor Scheme for Additive Noise: The expansion in Eq. 4.10 in [9, Ch. 10.4] is

$$\begin{aligned}
\mathbf{x}_{t+T} =& \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}T^2\left(\mathbb{L}_0\mathbf{a}\right) + \mathbf{B}\mathbf{w} + \left(\mathbb{L}\mathbf{a}\right)\mathbf{z}\\
&+ \frac{\partial \mathbf{B}}{\partial t}\left(\mathbf{w}T - \mathbf{z}\right). && (129)
\end{aligned}$$

Define

$$\mathbf{c}_0 \triangleq \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}T^2\left(\mathbb{L}_0\mathbf{a}\right) \tag{130}$$

and $\mathbf{C}_1$ and $\mathbf{C}_2$ as given by (26) and (27). Equation (129) can thus be written

$$\mathbf{x}_{t+T} = \mathbf{c}_0 + \mathbf{C}_1\mathbf{w} + \mathbf{C}_2\mathbf{z}. \tag{131}$$

The expected value, conditioned on $\mathbf{x}_t$, is then given by (24).

The covariance matrix is:

$$\begin{aligned}
\mathbf{\Sigma}_{\mathbf{x}_t} =& \mathrm{E}\left\{\left(\mathbf{C}_1\mathbf{w} + \mathbf{C}_2\mathbf{z}\right)\left(\mathbf{C}_1\mathbf{w} + \mathbf{C}_2\mathbf{z}\right)'\right\} && (132)\\
=& \mathbf{C}_1\,\mathrm{E}\left\{\mathbf{ww}'\right\}\mathbf{C}_1' + \mathbf{C}_2\,\mathrm{E}\left\{\mathbf{zz}'\right\}\mathbf{C}_2'\\
&+ \mathbf{C}_1\,\mathrm{E}\left\{\mathbf{wz}'\right\}\mathbf{C}_2' + \mathbf{C}_2\,\mathrm{E}\left\{\mathbf{zw}'\right\}\mathbf{C}_1' && (133)
\end{aligned}$$

where a final simplification using (72), (80), and (79) leads to (25).

2) The Order-1.5 Strong Taylor Scheme for Autonomous Scalar Problems: The expansion in Eq. 4.1 in [9, Ch. 10.4] is of the form

$$x_{t+T} = c_0 + c_1 w + c_2(w^2 - T) + c_3 w^3 + c_4 z \tag{134}$$

where

$$c_0 = x_t + aT + \frac{1}{2}\left(aa' + \frac{1}{2}b^2 a''\right)T^2 \tag{135}$$

and c_1, c_2, c_3, and c_4 are given by (30), (31), (32), and (33). With this format, the mean is (28).

The variance is

$$\Sigma_{x_t} = \mathrm{E}\left\{\left(c_1 w + c_2(w^2 - T) + c_3 w^3 + c_4 z\right)^2\right\}. \tag{136}$$

To evaluate the variance, use (73), (74), (75), (76), (77), (79), (80), (81), and (82). After substitution, one gets the expression of (29).

Appendix C
Weak Itô-Taylor Schemes

A. Weak Order 2.0

1) The Order-2.0 Weak Itô-Taylor Scheme for Scalar Problems: There are two such order-2.0 expansions: a non-simplified and a simplified expansion. We consider the non-simplified expansion first. The non-simplified expansion in Eq. 2.1 in [9, Ch. 14.2] is

$$\begin{aligned}
x_{t+T} =& x_t + aT + Bw + \frac{1}{2}b\frac{\partial b}{\partial x}\left(w^2 - T\right)\\
&+ \frac{\partial a}{\partial x}bz + \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}\frac{\partial^2 a}{\partial x^2}b^2\right)T^2\\
&+ \left(a\frac{\partial b}{\partial x} + \frac{1}{2}\frac{\partial^2 b}{\partial x^2}b^2\right)\left(wT - z\right). && (137)
\end{aligned}$$

The expected value is thus given by (34).

The covariance matrix is:

$$\begin{aligned}
\Sigma =& \mathrm{E}\Bigg\{\Bigg(\frac{1}{2}b\frac{\partial b}{\partial x}\left(w^2 - T\right) + \frac{\partial a}{\partial x}bz\\
&+ \left(a\frac{\partial b}{\partial x} + \frac{1}{2}\frac{\partial^2 b}{\partial x^2}b^2\right)\left(wT - z\right)\Bigg)^2\Bigg\} && (138)\\
=& \mathrm{E}\left\{\left(c_0 + c_1 w + c_2 w^2 + c_3 z\right)^2\right\} && (139)\\
=& c_0^2 + 2c_0c_1\,\mathrm{E}\left\{w\right\} + \left(c_1^2 + 2c_0c_2\right)\mathrm{E}\left\{w^2\right\}\\
&+ 2c_2c_3\,\mathrm{E}\left\{w^2 z\right\} + 2c_1c_2\,\mathrm{E}\left\{w^3\right\} + c_2^2\,\mathrm{E}\left\{w^4\right\}
\end{aligned}$$

$$+ 2c_1c_3 \operatorname{E}\{wz\} + 2c_0c_3 \operatorname{E}\{z\} + c_3^2 \operatorname{E}\{z^2\} \tag{140}$$

with c_0, c_1, c_2, and c_3 given by (36), (37), (38), (39). The expected value terms are given by (71), (73), (74), (75), (78), (80), (81), and (79). After substitution, one gets (34).

The simplified expansion in Eq. 2.2 in [9, Ch. 14.2] is:

$$\begin{aligned} x_{t+T} =& x_t + aT + Bw + \frac{1}{2} b \frac{\partial b}{\partial x}\left(w^2 - T\right) \\ &+ \frac{1}{2}\left(\frac{\partial a}{\partial x} b + a\frac{\partial b}{\partial x} + \frac{1}{2}\frac{\partial^2 b}{\partial x^2} b^2\right) wT \\ &+ \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}\frac{\partial^2 a}{\partial x^2} b^2\right) T^2. \end{aligned} \tag{141}$$

The mean is thus given by (34) again.

The variance is

$$\Sigma_{x_t} = \operatorname{E}\left\{\left(c_0 + c_1 w + c_2 w^2\right)^2\right\} \tag{142}$$

$$\begin{aligned} =& c_0^2 + 2c_0c_1 \operatorname{E}\{w\} + (c1^2 + 2c_0c_2) \operatorname{E}\{w^2\} \\ &+ 2c_1c_2 \operatorname{E}\{w^3\} + c_2^2 \operatorname{E}\{w^4\} \end{aligned} \tag{143}$$

where c_0, c_1, and c_2 are given by (41), (42), and (43). Consequently, using (71), (73), (74), and (75), one gets the solution in (40).

2) The Order-2.0 Weak Taylor Scheme for General Noise: There are two such order-2.0 expansions: a non-simplified and a simplified expansion. We consider the non-simplified expansion first. The unsimplified expansion in Eq. 2.6 in [9, Ch. 14.2] is:

$$\begin{aligned} \mathbf{x}_{t+T} =& \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}\left(\mathbb{L}^0 \mathbf{a}\right) T^2 \\ &+ \sum_{j=1}^{d_w}\left(\mathbf{b}_j w_j + \left(\mathbb{L}^0 \mathbf{b}_j\right) I_{0,j} + \left(\mathbb{L}^j \mathbf{a}\right) I_{j,0}\right) \\ &+ \sum_{j_1=1}^{d_w}\sum_{j_2=1}^{d_w}\left(\mathbb{L}_{j_1}\mathbf{b}_{j_2}\right) I_{j_1,j_2}. \end{aligned} \tag{144}$$

This can be rewritten as

$$\begin{aligned} \mathbf{x}_{t+T} =& \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}\left(\mathbb{L}^0 \mathbf{a}\right) T^2 + \mathbf{B}\mathbf{w} + \mathbf{C}^{(1)}\mathbf{i}^{(1)} \\ &+ \mathbf{C}^{(2)}\mathbf{i}^{(2)} + \mathbf{C}^{(3)}\mathbf{i}^{(3)} \end{aligned} \tag{145}$$

where $\mathbf{C}^{(1)}$, $\mathbf{C}^{(2)}$, and $\mathbf{C}^{(3)}$ are given by (46), (47), and (48) and

$$i_j^{(1)} \triangleq I_{0,j} \tag{146}$$

$$i_j^{(2)} \triangleq I_{j,0} \tag{147}$$

$$i_{j_1+(j2-1)m}^{(3)} \triangleq I_{j_1,j_2}. \tag{148}$$

The mean is thus (44).

The covariance matrix follows as

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{x}_t} =& \operatorname{E}\Big\{\left(\mathbf{B}\mathbf{w} + \mathbf{C}^{(1)}\mathbf{i}^{(1)} + \mathbf{C}^{(2)}\mathbf{i}^{(2)} + \mathbf{C}^{(3)}\mathbf{i}^{(3)}\right) \\ &\cdot\left(\mathbf{B}\mathbf{w} + \mathbf{C}^{(1)}\mathbf{i}^{(1)} + \mathbf{C}^{(2)}\mathbf{i}^{(2)} + \mathbf{C}^{(3)}\mathbf{i}^{(3)}\right)'\Big\}. \end{aligned} \tag{149}$$

Using (95), one can evaluate the following expectations:

$$\operatorname{E}\left\{\mathbf{w}\left(\mathbf{i}^{(1)}\right)'\right\} = \operatorname{E}\left\{\mathbf{w}\left(\mathbf{i}^{(2)}\right)'\right\} = \operatorname{E}\left\{\mathbf{i}^{(1)}\mathbf{w}'\right\}$$

$$= \operatorname{E}\left\{\mathbf{i}^{(2)}\mathbf{w}'\right\} = \frac{T^2}{2}\mathbf{I}. \tag{150}$$

Using (92), $\operatorname{E}\left\{\mathbf{w}\left(\mathbf{i}^{(3)}\right)'\right\} = \mathbf{0}$ and $\operatorname{E}\left\{\mathbf{i}^{(3)}\mathbf{w}'\right\} = \mathbf{0}$. Also, the expected values of $\mathbf{i}^{(1)}\mathbf{i}^{(3)}$ and $\mathbf{i}^{(2)}\mathbf{i}^{(3)}$ are zero, because the expected value of all of the multiple Itô integrals, except $I_{0,0}$ in Appendix A are zero. Consequently,

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{x}_t} =& \mathbf{B}\operatorname{E}\{\mathbf{w}\mathbf{w}'\}\mathbf{B}' + \mathbf{C}^{(1)}\operatorname{E}\left\{\mathbf{i}^{(1)}\left(\mathbf{i}^{(1)}\right)'\right\}\left(\mathbf{C}^{(1)}\right)' \\ &+ \mathbf{C}^{(2)}\operatorname{E}\left\{\mathbf{i}^{(2)}\left(\mathbf{i}^{(2)}\right)'\right\}\left(\mathbf{C}^{(2)}\right)' \\ &+ \mathbf{C}^{(3)}\operatorname{E}\left\{\mathbf{i}^{(3)}\left(\mathbf{i}^{(3)}\right)'\right\}\left(\mathbf{C}^{(3)}\right)' \\ &+ \mathbf{C}^{(1)}\operatorname{E}\left\{\mathbf{i}^{(1)}\left(\mathbf{i}^{(2)}\right)'\right\}\left(\mathbf{C}^{(2)}\right)' \\ &+ \mathbf{C}^{(2)}\operatorname{E}\left\{\mathbf{i}^{(2)}\left(\mathbf{i}^{(1)}\right)'\right\}\left(\mathbf{C}^{(1)}\right)' \\ &+ \mathbf{B}\operatorname{E}\left\{\mathbf{w}\left(\mathbf{i}^{(1)}\right)'\right\}\left(\mathbf{C}^{(1)}\right)' \\ &+ \mathbf{B}\operatorname{E}\left\{\mathbf{w}\left(\mathbf{i}^{(2)}\right)'\right\}\left(\mathbf{C}^{(2)}\right)' \\ &+ \mathbf{C}^{(1)}\operatorname{E}\left\{\mathbf{i}^{(1)}\mathbf{w}'\right\}\mathbf{B}' + \mathbf{C}^{(2)}\operatorname{E}\left\{\mathbf{i}^{(2)}\mathbf{w}'\right\}\mathbf{B}'. \end{aligned} \tag{151}$$

For the expected value of the outer product of $\mathbf{i}^{(3)}$, we use the result of (99) to get

$$\operatorname{E}\left\{\mathbf{i}^{(3)}\left(\mathbf{i}^{(3)}\right)'\right\} = \frac{1}{2}T^2\mathbf{I}. \tag{152}$$

For the outer products involving $\mathbf{i}^{(1)}$ and $\mathbf{i}^{(2)}$, we use (96) and (97) to get the expression for the covariance matrix in (45).

The simplified expansion in Eq. 2.7 in [9, Ch. 14.2] is:

$$\begin{aligned} \mathbf{x}_{t+T} =& \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}\left(\mathbb{L}^0\mathbf{a}\right)T^2 + \left(\mathbf{B} + \frac{T}{2}\left(\left(\mathbb{L}^0\mathbf{B}\right) + \left(\mathbb{L}\mathbf{a}\right)\right)\right)\mathbf{w} \\ &+ \frac{1}{2}\sum_{j_1=1}^{d_w}\sum_{j_2=1}^{d_w}\left(\mathbb{L}^{j_1}\mathbf{b}_{j_2}\right)\left(w_{j_1}w_{j_2} + V_{j_1,j_2}\right). \end{aligned} \tag{153}$$

The mean is thus (44).

To determine the covariance matrix, we rewrite the equation in the form:

$$\mathbf{x}_{t+T} = \mathbf{x}_t + \mathbf{a}T + \frac{1}{2}\left(\mathbb{L}^0\mathbf{a}\right)T^2 + \mathbf{C}^{(1)}\mathbf{w} + \mathbf{C}^{(2)}\mathbf{i} \tag{154}$$

where

$$i_{j1+(j_2-1)m} = w_{j_1}w_{j_2} + V_{j_1,j_2} \tag{155}$$

and $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ are given by (50), and (51).

The covariance matrix has the form:

$$\boldsymbol{\Sigma}_{\mathbf{x}_t} = \operatorname{E}\left\{\left(\mathbf{C}^{(1)}\mathbf{w} + \mathbf{C}^{(2)}\mathbf{i}\right)\left(\mathbf{C}^{(1)}\mathbf{w} + \mathbf{C}^{(2)}\mathbf{i}\right)'\right\} \tag{156}$$

$$\begin{aligned} =& \mathbf{C}^{(1)}\operatorname{E}\{\mathbf{w}\mathbf{w}'\}\left(\mathbf{C}^{(1)}\right)' + \mathbf{C}^{(2)}\operatorname{E}\{\mathbf{i}\mathbf{i}'\}\left(\mathbf{C}^{(2)}\right)' \\ &+ \mathbf{C}^{(1)}\operatorname{E}\{\mathbf{w}\mathbf{i}'\}\left(\mathbf{C}^{(2)}\right)' + \mathbf{C}^{(2)}\operatorname{E}\{\mathbf{i}\mathbf{w}'\}\left(\mathbf{C}^{(1)}\right)'. \end{aligned} \tag{157}$$

To simplify this, we use (101) and (102) to get

$$\mathrm{E}\left\{\mathbf{i}\mathbf{i}'\right\} = 2T^2\mathbf{I}. \tag{158}$$

So, the covariance matrix simplifies to (49).

3) The Order-2.0 Weak Taylor Scheme for Scalar Noise: The expansion in Eq. 2.5 in [9, Ch. 14.2] is:

$$\mathbf{x}_{t+T} = \mathbf{x}_t + \mathbf{a}T + \mathbf{B}w + \frac{1}{2}\left(\mathbb{L}\mathbf{B}\right)\left(w^2 - T\right) + \frac{1}{2}\left(\mathbb{L}^0\mathbf{a}\right)T^2 + \left(\mathbb{L}^0\mathbf{B}\right)\left(wT - z\right) + \left(\mathbb{L}\mathbf{a}\right)z. \tag{159}$$

The mean is consequently (52).

The covariance matrix is

$$\Sigma_{\mathbf{x}_t} = \mathrm{E}\left\{\left(\mathbf{B}w + \frac{1}{2}\left(\mathbb{L}\mathbf{B}\right)\left(w^2 - T\right) + \left(\mathbb{L}^0\mathbf{B}\right)\left(wT - z\right) + \left(\mathbb{L}\mathbf{a}\right)z\right) \cdot \left(\mathbf{B}w + \frac{1}{2}\left(\mathbb{L}\mathbf{B}\right)\left(w^2 - T\right) + \left(\mathbb{L}^0\mathbf{B}\right)\left(wT - z\right) + \left(\mathbb{L}\mathbf{a}\right)z\right)'\right\} \tag{160}$$

$$= \mathrm{E}\left\{\left(\mathbf{c}_0 + \mathbf{c}_1 w + \mathbf{c}_2 w^2 + \mathbf{c}_3 z\right)\left(\mathbf{c}_0 + \mathbf{c}_1 w + \mathbf{c}_2 w^2 + \mathbf{c}_3 z\right)'\right\} \tag{161}$$

where $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are given by (54), (55), (56), and (57).

Omitting all terms containing a coefficient of just w, w^3 or z, whereby the expected value is zero, the covariance matrix expands to

$$\begin{aligned}\mathbf{\Sigma}_{\mathbf{x}_t} =& \mathbf{c}_0\mathbf{c}_0' + \mathbf{c}_1\mathbf{c}_1'\,\mathrm{E}\left\{w^2\right\} + \mathbf{c}_2\mathbf{c}_2'\,\mathrm{E}\left\{w^4\right\} + \mathbf{c}_3\mathbf{c}_3'\left\{z^2\right\}\\ &+ \left(\mathbf{c}_0\mathbf{c}_2' + \mathbf{c}_2\mathbf{c}_0'\right)\mathrm{E}\left\{w^2\right\} + \left(\mathbf{c}_1\mathbf{c}_3' + \mathbf{c}_3\mathbf{c}_1'\right)\mathrm{E}\left\{wz\right\}\\ &+ \left(\mathbf{c}_2\mathbf{c}_3' + \mathbf{c}_3\mathbf{c}_2'\right)\mathrm{E}\left\{w^2 z\right\}.\end{aligned} \tag{162}$$

Substituting (73), (75), (79), (80), and (81), this simplifies to (53).

Appendix D
Fifth-Order Cubature Points

Fifth-Order Cubature Points and Weights

Weight (ω_i)	Point ($\boldsymbol{\xi}_i$)
$\frac{4}{(d+2)^2}$	$[\pm a]$
$\frac{(d-2)^2}{2^d(d+2)^2}$	$(\pm b, \pm b, \ldots, \pm b)$

The cubature points and weight of [14, pg. 317, No. 5-3] are given as shown above, where

$$a = \sqrt{\frac{d+2}{2}} \qquad b = \sqrt{\frac{d+2}{d-2}}, \tag{163}$$

and d is the dimensionality of the points generated. The $\pm$ indicates that all possible combinations of negative and positive elements should be used. The bracket notation indicates that all possible vectors with that nonzero element should be generated. There are $2d$ points of the first type and 2^d of the second type. These points can be used for integrals involving an arbitrary Gaussian weighting with $d > 2$.

References

[1] I. Arasaratnam, S. Haykin, and T. R. Hurd, "Cubature Kalman filtering for continuous-discrete systems: Theory and simulations," *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 4977–4993, Oct. 2010.

[2] R. Cools and P. Rabinowitz, "Monomial cubature rules since "Stroud": A compilation," *Journal of Computational and Applied Mathematics*, vol. 48, no. 3, pp. 309–326, Nov. 1993.

[3] D. F. Crouse, "Basic tracking using nonlinear 3D monostatic and bistatic measurements," *IEEE Aerospace and Electronic Systems Magazine*, vol. 29, no. 8, pp. 4–53, Aug. 2014.

[4] ——, "Basic tracking using nonlinear continuous-time dynamic models," *IEEE Aerospace and Electronic Systems Magazine*, vol. 30, no. 2, Part II, pp. 4–41, Feb. 2015.

[5] ——, "On measurement-based light-time corrections for bistatic orbital debris tracking," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 51, no. 3, pp. 2502–2518, Jul. 2015.

[6] ——, "The tracker component library: Free routines for rapid prototyping," *IEEE Aerospace and Electronic Systems Magazine*, vol. 32, no. 5, pp. 18–27, May 2017.

[7] A. M. Fosbury, "Estimation with multitemporal measurements," *Journal of Guidance, Control, and Dynamics*, vol. 33, no. 5, pp. 1518–1526, Sep.–Oct. 2010.

[8] P. Frogerais, J. Bellanger, and L. Senhadji, "Various ways to compute the continuous-discrete extended Kalman filter," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 1000–1004, Apr. 2012.

[9] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*. Berlin: Springer, 1999.

[10] G. Y. Kulikov and M. V. Kulikova, "Accurate numerical implementation of the continuous-discrete extended Kalman filter," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 273–279, Jan. 2014.

[11] D. Laneuville, "New models for 3D maneuvering target tracking," in *New Models for 3D Maneuvering Target Tracking*, Big Sky, MT, 1–8 Mar. 2014.

[12] S. Mohiuddin and M. L. Psiaki, "Continuous-time Kalman filtering with implicit discrete measurement times," *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 1, pp. 148–163, Jan.–Feb. 2010.

[13] E. Platen and N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*. Berlin: Springer, 2010.

[14] A. H. Stroud, *Approximate Calculation of Multiple Integrals*. Edgewood Cliffs, NJ: Prentice-Hall, Inc., 1971.

[15] S. Särkkä and A. Solin, "On continuous-discrete cubature Kalman filtering," *IFAC Proceedings Volumes*, vol. 45, no. 16, pp. 1221–1226, Jul. 2012.

[16] A. Tocino and J. Vigo-Aguiar, "New Itô-Taylor expansions," *Journal of Computational and Applied mathematics*, vol. 158, no. 1, pp. 169–185, 1 Sep. 2003.

[17] R. Yang, Y. Bar-Shalom, A.-C. Pérez, and G. W. Ng, "Maneuvering target tracking using continuous wave bistatic sonar with propagation delay," *Journal of Advances in Information Fusion*, vol. 13, no. 1, pp. 36–49, Jun. 2018.

[18] The tracker component library. [Online]. Available: https://github.com/USNavalResearchLaboratory/TrackerComponentLibrary