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A Note on Grinfelds' "Kelvin's Paradox"

by James Cazamias

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1. The Issue

Grinfeld and Grinfeld¹ state that the ubiquitous formula for ponderomotive forces due to a distribution of a polarized substance (Eq. 1) implies a nonvanishing self-force with a few exceptions, including spherically symmetric dipole distributions and constant distributions in elliptical domains where \mathbf{F} is the resulting net force, \mathbf{P} is the electric dipole distribution, \mathbf{E} is the electric field, and τ is the domain.

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla) \mathbf{E} d\tau . \quad (1)$$

The fundamental issue is that if $\mathbf{p} \cdot \hat{\mathbf{n}} \neq \mathbf{0}$, where $\hat{\mathbf{n}}$ is the surface normal, there is a bound surface charge resulting in a discontinuous \mathbf{E} field. Equation 1 requires the differentiation of this field, resulting in a delta function. The presence of this singularity on the boundary turns this into an improper integral. This does not mean that the integral diverges, but that care needs to be taken in performing the integration.

This behavior has been observed previously.

Griffiths, in the second edition of his textbook², has a homework problem that mentions the issue:

Problem 4.30 According to equation (4.5), the force on a single dipole is $(\mathbf{p} \cdot \nabla) \mathbf{E}$, so the total force on a piece of dielectric material must be

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla) \mathbf{E}_{\text{ext}} d\tau \quad (4.53)$$

[Here \mathbf{E}_{ext} is the field of everything *except* the dielectric. You might assume — as others have been known to⁹ — that it wouldn't matter if you used the *total* field; after all, the dielectric can't exert a force on *itself*. However, because the field of the dielectric is discontinuous at the location of any bound surface charge, the derivative introduces a spurious delta function, and you must either add a compensating surface term, or (better) stick with \mathbf{E}_{ext} , which suffers no such discontinuity.] Use equation (4.53) to determine the force on a tiny sphere of radius a , composed of linear dielectric material of susceptibility χ_e , which is situated a distance \mathbf{r} from a fine wire carrying a uniform line charge λ .

⁹ I thank Prof. M. Tiersten for pointing out this error in the first edition.

Tiersten³ states that the correct formulation is

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla) \mathbf{E}_{\text{ext}} d\tau = \int (\mathbf{P} \cdot \nabla) \mathbf{E} d\tau + \frac{1}{2\epsilon_0} \int (\mathbf{P} \cdot \hat{\mathbf{n}})^2 \hat{\mathbf{n}} da \quad (2)$$

and mentions a couple of example problems.

In the appendix of Zakharian et al.⁴ (see also Barnett and Loudon⁵) there is a discussion of issues associated with the discontinuity of \mathbf{E}_\perp at the boundary. Mansuripur⁶ states:

The basic problem is as follows: At the surface of a dielectric material, there is a discontinuity in the perpendicular component of the E-field (due to the presence of bound electric charges at the surface). Now, a dipole sitting just beneath the surface (and perpendicular to it) experiences a force on its positive end (which, we assume is located exactly at the dielectric surface), and also on its negative end (which is a little bit below the surface). The force on the negative end of the dipole is thus due to the local E-field, whatever that field happens to be. This E-field, of course, is the total field, which is the sum of the external and internal fields at the location of the negative end of the dipole under consideration. In contrast, the positive end of the dipole sees two different E-fields, one just above the surface, the other just below the surface – because, you recall, the E-field at this location is discontinuous. If you now use the average value of the E-field at the surface (i.e., $\frac{1}{2}$ of the sum of the E-fields immediately above and immediately below the surface), you'll find the correct E-field acting on the positive end of the dipole. The need for this averaging arises because the (bound) surface charges produce a perpendicular E-field that changes direction (but not magnitude) upon crossing the surface. Averaging cancels out this “self E-field,” and leaves behind only the external E-field at the location of the surface.

Interestingly, they also look at two different forms of the Lorentz force density: one based on treating the medium as formed from individual charges (Eq. 3) and one based on the medium as formed from individual dipoles (Eq. 4).

$$\mathbf{f}^c = -(\nabla \cdot \mathbf{P})\mathbf{E} + \dot{\mathbf{P}} \times \mathbf{B}. \quad (3)$$

$$\mathbf{f}^d = (\mathbf{P} \cdot \nabla)\mathbf{E} + \dot{\mathbf{P}} \times \mathbf{B}. \quad (4)$$

They find that while the total forces and torques are identical, the force densities associated with the volumes and surfaces may be different.

The observations of Tiersten and Mansuripur can be reconciled. Griffiths and Hnizdo⁷ integrate Eq. 1 by parts and use the standard method of using the average of the \mathbf{E} field above and below a charged surface² to get

$$\mathbf{F} = \int (-\nabla \cdot \mathbf{P})\mathbf{E}d\tau + \int (\mathbf{P} \cdot \hat{\mathbf{n}})\frac{1}{2}(\mathbf{E}_{above} + \mathbf{E}_{below})da. \quad (5)$$

Notice that the volumetric forces are not identical. They observe:

This raises a surprisingly delicate question: What do we mean by the “force density” inside the medium? Presumably we should (in the mind's eye) isolate an infinitesimal piece, of volume v , determine the force on it, and divided by v . But this little piece carries surface bound charge in addition to its volume bound charge, and it's easy to see ... that the total force is precisely [Eq. 1 in this note]. Of course, in the bulk material the surface charge on v is canceled by that on the adjacent inner surface of the surrounding medium – there is no net “surface” charge within the substance – but if we're interested in the force on v alone, its surface charge must not be ignored. The force density ... is incomplete, because it does not include this contribution.

Equation 2 may then be obtained by remembering that the \mathbf{E} field is discontinuous at the surface of the dielectric due to the bound surface charge with the boundary condition

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} . \quad (6)$$

Plugging in the boundary condition and noting that $\mathbf{P} \cdot \hat{\mathbf{n}} = \sigma$ gives

$$\begin{aligned} \mathbf{F} &= \int (-\nabla \cdot \mathbf{P}) \mathbf{E} d\tau + \int (\mathbf{P} \cdot \hat{\mathbf{n}}) \frac{1}{2} \left(\frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} + 2\mathbf{E}_{below} \right) da \\ &= \int (-\nabla \cdot \mathbf{P}) \mathbf{E} d\tau + \int (\mathbf{P} \cdot \hat{\mathbf{n}}) \frac{1}{2} \left(\frac{(\mathbf{P} \cdot \hat{\mathbf{n}})}{\epsilon_0} \hat{\mathbf{n}} + 2\mathbf{E}_{below} \right) da \\ &= \int (-\nabla \cdot \mathbf{P}) \mathbf{E} d\tau + \int (\mathbf{P} \cdot \hat{\mathbf{n}}) \mathbf{E}_{below} da + \int \frac{(\mathbf{P} \cdot \hat{\mathbf{n}})^2}{2\epsilon_0} \hat{\mathbf{n}} da . \end{aligned} \quad (7)$$

Unintegrating by parts the first two integrals gives

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla) \mathbf{E} d\tau + \frac{1}{2\epsilon_0} \int (\mathbf{P} \cdot \hat{\mathbf{n}})^2 \hat{\mathbf{n}} da , \quad (8)$$

where \mathbf{E} is now understood to be the continuous part of the electric field within the body.

Setting $4\pi\epsilon_0 = 1$ converts Eq. 8 in SI units to Eq. 9 in CGS units:

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla) \mathbf{E} d\tau + 2\pi \int (\mathbf{P} \cdot \hat{\mathbf{n}})^2 \hat{\mathbf{n}} da . \quad (\text{CGS}) \quad (9)$$

Examining the demonstration problem in Grinfeld and Grinfeld¹, the surface integral provides an additional force contribution of $2\pi(P_3^2(H) - P_3^2(0))LW$, which exactly cancels out the non-zero force.

The earliest derivation of Eq. 9 that the author is aware of is from 1949. Smith-White⁸ uses the fact that the total force on a body is composed of body forces and surface tractions and calculates the surface traction directly.

Equation 1 is an improper integral and thus requires careful consideration. Integration by parts and E field averaging are standard tools, and the use of them does not call into question the validity of Eq. 1.

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