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by Michael Grinfeld and Pavel Grinfeld

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by Michael Grinfeld
Weapons and Materials Research Directorate, CCDC Army Research Laboratory

Pavel Grinfeld
Drexel University

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14. ABSTRACT Previously we introduced the so-called cardinal tensors for elastic polarizable substances. Our analysis was based on the variational principle, which provides its logical self-consistency and compatibility with basic principles of continuum physics. It used some technics, though, which are rigorously applicable only to isotropic substances. In this technical note, we present the required modifications making the earlier results applicable to anisotropic elastic polarizable solids.			
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1. Introduction

When dealing with anisotropic polarizable substances, it is convenient to use the mixed Eulerian–Lagrangian description of continuum media.

Consider the immobile spatial coordinate system referred to the coordinates z^i (the reference indexes from the middle of the Latin alphabet *i, j, k* run the values 1, 2, 3) and assume that our space is Euclidean. In this space, we consider a material body *B*, referred to the material coordinates x^a (the material indexes from the beginning of the Latin alphabet *a, b, c* run the values 1, 2, 3 as well). We accept the standard concepts of the covariant and contravariant indexes, and accept the standard agreement of summation over the repeat covariant and contravariant indexes of the same type (i.e., of the reference or material type).

As always in mechanics of deformable solids, we distinguish between the initial and the current configurations of the body. Let $z^i = z^i\left(x^a\right)$ be the Eulerian coordinates in the current configuration of the material point x^a (we use the notation $x^a = x^a\left(z^i\right)$ for the inverse of $z^i\left(x^a\right)$. Let us use the notation Z_{ij} for deformation-independent metrics and the notation X_{ab} for the deformation-sensitive metrics of the actual material configuration. These two metrics are connected by the relationships

$$X_{ab} = Z_{ij} z^{i.}_{.a} z^{j.}_{.b} \ , \ Z_{ij} = X_{ab} x^{a.}_{.i} x^{a.}_{.i} \ , \tag{1}$$

where the mixed shift-tensors $z^{i.}_{.a}$ and $x^{a.}_{.i}$ are defined as

$$z^{i.}_{.a} \equiv \frac{\partial z^i\left(x\right)}{\partial x^a}, \ x^{a.}_{.i} \equiv \frac{\partial x^a\left(z\right)}{\partial z^i} \ . \tag{2}$$

The reference and the coordinate systems are characterized by the current covariant bases $\mathbf{Z}_i$ and $\boldsymbol{X}_a$ and contravariant bases $\boldsymbol{Z}^i$ and $\boldsymbol{X}^a$, respectively

We use the standard notation ∇_i and ∇_a for the reference and material contravariant differentiation in the metrics of the actual configuration.

2. Polarizable Elastic Substance

As long as we deal with the statics in the absence of electric current, the formal technics of electrostatics and magnetostatics are almost indistinguishable. For the sake of brevity and definiteness, let us consider electrostatics. Polarization is a vector quantity. A distributed polarization field is characterized by the density per unit mass $\boldsymbol{\Pi}$ or per unit volume $\boldsymbol{P} = \rho\boldsymbol{\Pi}$, where ρ is the mass density. Vectors $\boldsymbol{\Pi}$ and $\boldsymbol{P}$ can be decomposed with respect to the material basis $\boldsymbol{X}^a$:

$$\boldsymbol{P} = P^a \boldsymbol{X}_a , \ \boldsymbol{\Pi} = \Pi^a \boldsymbol{X}_a . \tag{3}$$

The bulk energy density per unit mass Ψ is given as a function of the actual material metrics X_{ab}, the Lagrangian components Π_a of the polarization vector per unit mass, and permanent constant material tensors and constants, which we do not mention explicitly in the following:

$$\Psi = \Psi\left(X_{ab}, \Pi_a\right) . \tag{4}$$

There are several other reasonable substitutes for $\Psi\left(X_{ab}, \Pi_a\right)$; for instance, the bulk energy density per unit mass ψ as a function of the Lagrangian components P_a of the polarization vector per unit volume.

$$\psi = \psi\left(X_{ab}, P_a\right) . \tag{5}$$

It was demonstrated in Grinfeld and Grinfeld[1–3] how to derive the cardinal tensors for the polarizable elastic substance based on the minimum energy or the Gibbs principles. The analysis of Grinfeld and Grinfeld[1–3] is applicable to isotropic substances. For anisotropic substances we have to use the bulk energy densities of the form Eq. 4 or 5.

Using the relationship in Eq. 4 we arrive at the following formula of the Aleph tensor $\aleph^{ij}$:

$$\aleph^{ij} \equiv 2\rho \frac{\partial \Psi}{\partial X_{(cd)}} z^{i\cdot}_{.c} z^{j\cdot}_{.d} + \frac{1}{4\pi} E^i E^j - \frac{1}{8\pi} z^{ij} E_k E^k \ , \tag{6}$$

where E^i is the electric field.

Using the relationship in Eq. 5 we arrive at the following formula of the Aleph tensor $\aleph^{ij}$:

$$\aleph^{ij} = 2\rho \tfrac{\partial \psi}{\partial X_{(ab)}} z^{i\cdot}_{.a} z^{j\cdot}_{.b} + \tfrac{1}{4\pi}\left(D^i E^j + D^j E^j - E^i E^j\right) - \left(\tfrac{1}{4\pi} E_k D^k - \tfrac{1}{8\pi} E_k E^k\right) z^{ij} \ . \tag{7}$$

In vacuum, both formulae of the Aleph tensor reduce to the Maxwell tensor:

$$\aleph_{vac}^{ij} \equiv \frac{1}{4\pi} E^i E^j - \frac{1}{8\pi} z^{ij} E_k E^k . \tag{8}$$

In the absence of the electrostatic field, the Aleph tensor reduces to the ordinary stress tensor:

$$\aleph_{mech}^{ij} \equiv 2\rho \frac{\partial \Psi}{\partial X_{(cd)}} z_{.c}^{i.} z_{.d}^{j.} . \tag{9}$$

Within the bulk we postulate the following equilibrium equations:

$$\nabla_i \left(2\rho \frac{\partial \Psi}{\partial X_{(cd)}} z_{.c}^{i.} z_{.d}^{j.} + \frac{1}{4\pi} E^i E^j - \frac{1}{8\pi} z^{ij} E_k E^k \right) = 0 . \tag{10}$$

At the voids-free interface between polarizable solids, we postulate the equilibrium equations

$$\left[2\rho \frac{\partial \Psi}{\partial X_{(cd)}} z_{.c}^{i.} z_{.d}^{j.} + \frac{1}{4\pi} E^i E^j - \frac{1}{8\pi} z^{ij} E_k E^k \right]_{-}^{+} N_i = 0 . \tag{11}$$

Of course, Eqs. 10 and 11 should be amended with the standard equation of electrostatics,

$$\nabla_i \left(E^i + 4\pi P^i \right) = 0 , \tag{12}$$

and the standard boundary conditions and conditions at infinity.

3. Conclusion

The Aleph cardinal tensor $\aleph^{ij}$ appears in a natural way when applying the minimum energy variational approach to electrostatics or magnetostatics. The Aleph tensor combines the key features of the stress tensors of simple elastic solids and the Maxwell stress tensor of electromagnetic field. In this technical note we generalized our earlier results for the case of piezoelectric elastic media of arbitrary symmetry. The specific form of the Aleph tensor depends essentially on the choice of the internal energy of the substance and varies quite significantly when passing from the choice of Eq. 4 to the choice of Eq. 5. However, regardless of this choice, the Aleph tensor appears to be symmetric for the solids of arbitrary physical symmetry, and it allows us to formulate the closed system of piezoelectric or piezomagnetic equilibrium.

4. References

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