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Title	Combinatorial and Geometric Structures, Convex Optimization leading to Matrix Recovery, Efficient Combinatorial Algorithms and Min-Max Theorems
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Overview

The general context of this research proposal was that of mathematical optimization (continuous and discrete) and operations research. It had significant interaction with combinatorics and computational complexity as well as various other fields in mathematical sciences. Specifically, we were interested in designing efficient algorithms to find optimal or near optimal solutions for tractable classes of optimization problems. For problems that are provably hard to solve within any guaranteed bound, our focus was to develop practical heuristics and tools. We also addressed issues that arise from the nature of the way the data is collected in practice, such as robustness of algorithms under uncertain data, and hidden low dimensional information in high dimensional data. We outline each of the research areas and motivate their relevance to the research community and practitioners.

A. Matrix recovery via convex optimization.
In some applications, we deal with the phenomenon that a very small dimensional structure is responsible for the main behavior of a system, however, we can only observe the behavior of the system in terms of very high dimensional variables. Then the main task is, given a large number of observations of this very high dimensional data, deduce the low-dimensional

structure that gave rise to it. Applications are abound in signal and image processing, sparse signal recovery, compressed sensing, system identification and control.

In some other systems, we need to operate with extreme caution with respect to how close our solution is to the boundary conditions. In this latter case, we are interested in solutions that are far away from *every* boundary condition (or constraint) or a least from every element of a specific, critical subset of constraints. Typically, this can be achieved very naturally and efficiently via interior-point methods. If the variable space is embedded in a matrix space, this typically means, we want a matrix of maximum rank.

Interestingly, one of the best tools to approach both problems is convex optimization. Low rank matrix recovery was pursued via strong convex relaxations and the high rank matrix recovery via suitably adapted interior-point methods. We generalized the notion of Total Dual Integrality (TDI) from a linear programming setting to Semidefinite Programming problems (SDP) and convex optimization problems. Our work allows proving min-max theorems via semidefinite programming and convex optimization relaxations. We have designed new primal-dual interior-point algorithms for convex optimization in two separate but complementary bodies of work. One set of algorithms works with conic formulations while the other set of algorithms do not require a conic setting. We proved that both classes of algorithms attain the current best iteration complexity bounds and they lead to new algorithms even in the very special case of linear programming.

B. Geometry and boundary structure of convex cones.
Understanding the boundary structure of convex sets (and equivalently convex cones) is paramount in designing efficient algorithms for the solution of convex optimization problems and proving duality theorems. The goal of this module of the research program was to improve our understanding of the geometry and boundary structure of hyperbolic convex cones and their generalizations. In general convex cone setting, we looked for characterizations of facial exposedness and facial dual completeness as they relate to convex representations and duality theorems for convex optimization. We provided some of the strongest new characterizations of facial dual completeness in convex cones. We proved that strict complementarity (and facial exposedness in the dual) fails in MaxCut SDPs in a comparable way to some of the hardest, most ill-posed problems in convex optimization.

C. Mixed integer programming.
Branch and cut is the most widely used, and successful strategy for solving large scale integer programs. The performance of resulting algorithms depends in large part on the quality of the cutting planes generated at each iteration. The general goal is to find simple families of cutting planes that lead to tight linear programming relaxations. The goal of this research project was to use theory in guiding choices that practitioners have to make in selecting families of cutting planes to integrate in Branch and Cut algorithms.

D. Minimax theorems and combinatorial algorithms.
We gave a polynomial time algorithm for multi-commodity flows in graphs, that either finds a fractional flow, or shows that the cut condition is violated, or finds an obstruction for which

the cut-condition is not sufficient for the existence of a flow. This completes work that was announced in the previous proposal. Seymour's 1977 Flowing Conjecture remains one of the most important open problem in polyhedral combinatorics. We prove the conjecture for the case where the minimally non-ideal clutter has a triangle and leverage this result to prove that every counterexample to the aforementioned conjecture will contain a special obstruction. We prove Seymour's Cycling conjecture for lifts of graphic matroids. This completes work that was announced in the previous proposal. We give a polynomial time algorithm for the problem of recognizing if a binary matroid is a lift of a graphic (respectively co-graphic) matroid. We made substantial progress on the related problem of characterizing when two signed graphs have the same set of even parity cycles.

We now describe accomplishments in each of the aforementioned research areas in more details.

A. Matrix recovery via convex optimization

The first research problem in this part involved the design and analysis of algorithms for convex optimization problems which also contain matrix variables.

Problem A1. Investigate the new primal-dual scalings and the mathematical and computational properties of the resulting new interior-point algorithms.

In algorithm design and analysis, we have obtained very comprehensive results. With then PhD student Tor Myklebust, we provided iteration complexity analysis for primal-dual symmetric, v-space based, interior-point algorithms for convex optimization problems in conic form [37]. Our complexity bounds match the current best complexity bounds for the very special case of Semidefinite Programming (SDP). Our results also match and generalize a current best complexity result of Nesterov and Todd from 1997 [33]. Note that Nesterov and Todd's primal-dual symmetric interior-point algorithm only applies to the special (though very useful and popular) case of symmetric cone programming problems. In contrast, our iteration complexity results are new for all convex optimization problems in conic form where the underlying convex cone is not necessarily a symmetric cone, for primal-dual symmetric, v-space based algorithms. On the other hand, the class of algorithms that we analyze is so rich that their specialization to even Linear Programming problems or SDPs lead to new algorithms matching the current best iteration complexity for SDP.

Moreover, our approach has certain elements of Quasi-Newton methods naturally embedded in it. Due to this feature, our algorithms can run in a "first-order algorithm mode" (or even in a "hybrid mode," easily mixing some first-order information with some second-order information as appropriate) to solve very large scale convex optimization problems (of course the inclusion of less second order information to enable us to solve huge-scale big data instances comes at a cost in some accuracy). Tor Myklebust successfully defended his PhD thesis in August 2015 and has been working for a company in San Francisco area in California.

In addition to our progress on the design of new primal-dual interior-point algorithms for convex optimization problems in conic form, we have completed the design and analysis of other, new

primal-dual interior-point algorithms for convex optimization problems with
Domain-Driven formulations: [28]. Our second approach (i.e., Domain-Driven) also has many desired features

- Our algorithm attains the best, worst-case iteration complexity bound achieved to date and utilizes the theory of self-concordant convex functions.
- Our algorithm allows infeasible-start (we do not assume that the algorithm is provided with a starting feasible solution).
- Our algorithm can behave like a first-order algorithm if desired. Moreover, on the same problem, from one iteration to the next, we can change the way we compute the search directions (first-order vs. second-order), hence in effect turning our algorithms into hybrid first-or-second-order algorithms.
- Our formulations are *Domain-Driven* in the sense that we formulate convex optimization problems in their natural application domain without reformulating the problem in conic form. While the conic convex optimization enjoyed tremendous success, due to the nice primal-dual symmetry and the existence (at least in principle) of logarithmically homogeneous self-concordant barriers, enforcement of this conic structure is not free. It typically requires introduction of additional homogenizing variables and some absolute constants. This point was raised in some detail by Nemirovskii and PI Tunçel in a 2005 paper [32].

This latter class of algorithms and their complexity analysis are also a part of PhD Thesis work of Mehdi Karimi who defended his PhD Thesis successfully in August 2017 and took a postdoctoral appointment at the University of Waterloo. He continues to work with PI Tunçel on related research projects and completion of the resulting manuscripts.

With M.Math. student Miaolan Xie, PI Tunçel made progress on the following research problem A2.

Problem A2. Solve Todd's *dual ellipsoidal norms* problem for homogeneous cones; if successful then for the hyperbolic cones. Also, provide as geometric a proof as possible.

Miaolan Xie finished her M.Math. Thesis in April 2016 [48]. We computed the maximum volume ellipsoid contained in the cone of symmetric positive semidefinite matrices where the center of the ellipsoid is given by the user as any interior point of the positive semidefinite cone. We found that a previous claim in a published paper about such ellipsoids was false. Our new techniques seem generalizable to homogeneous cones. Therefore, we have made some demonstrable progress towards the solution of Problem A2, although the problem is still unsolved.

Problem A3. Design convex relaxations for the Low-rank Matrix Recovery problem and the related combinatorial optimization problems. Improve the existing sufficient conditions for LMR and derive verifiable sufficient conditions for LMR for these new convex relaxations.

With collaborator Fatma Kilinç-Karzan, we have designed convex relaxations for hard low-rank matrix recovery problems. We unified and generalized many of the existing techniques for various matrix norm optimization problems. A manuscript addressing these aspects of Problem A3. is in preparation.

With de Carli Silva, PI Tunçel utilized a modern convex optimization viewpoint and provided a set of minimal conditions (axioms) under which certain key, desired properties are generalized, including the main equivalent characterizations of the theta function, the theta body of graphs, and the corresponding antiblocking duality relations [11]. Our framework describes several semidefinite and polyhedral relaxations of the stable set polytope of a graph as generalized theta bodies. As a by-product of our approach, we introduce the notion of *Schur Lifting* of cones which is dual to PSD (positive semidefinite) Lifting (more commonly used in SDP relaxations of combinatorial optimization problems) in our axiomatic generalization. We also generalize the notion of complements of graphs to diagonally scaling-invariant polyhedral cones. For the final research problem in this section:

Problem A4. Develop a unifying theory of min-max theorems for combinatorial optimization based on SDP duality theory and convex optimization. possibly leading to combinatorial primal-dual algorithms for many problems.

With Dr. de Carli Silva, PI Tunçel has obtained a generalization of a key notion (in polyhedral combinatorics and combinatorial optimization) of Total Dual Integrality (TDI) to the setting of convex optimization in general and Semidefinite Programming in particular. This amounts to a major breakthrough in this part (but also in the next part, that is part B.) of the research program. We have completed writing our manuscript and submitted it in January 2018 [13]. Our results unify many min-max theorems including those arising via polyhedral theory and the conventional notion of TDIness. Moreover, our theory leads to a notion of TDIness that is applicable to extended formulations (lifted-representations) of polytopes. We also discovered a beautiful fact that "every pointed closed convex set is the intersection of all its rational supporting closed halfspaces" together with a short proof using elementary convex analysis. This result which is contained in a 1.5 page note we released in February 2018, is tight [14].

During the time period covered by this report, many other papers reporting results from this part of the research program (part A.) got published:

- With Muramatsu and Waki [36], we have proved a perturbed sum-of-squares theorem for polynomial optimization problems and then derived tractable convex relaxations (as SDPs). We tested our approach by computational experiments. An advantage of our SDP relaxations is that for many sparse instances of these semi-algebraic optimization problems, our SDP relaxations are of considerably smaller dimension than those originally proposed by Lasserre, and as a result lead to stronger bounds, faster.

- The recent joint paper Au and PI Tunçel appeared in SIDMA [**?**]. In this paper, We consider lift-and-project methods for combinatorial optimization problems and focus mostly on those lift-and-project methods which generate polyhedral relaxations of the convex hull of integer solutions. We introduce many new variants of Sherali–Adams and Bienstock–Zuckerberg operators. These new operators fill the spectrum of polyhedral lift-and-project operators in a way which makes all of them more transparent, easier to relate to each other, and easier to analyze. We provide new techniques to analyze the worst-case performances as well as relative strengths of these operators in a unified way. A referee has stated that our paper has the potential to become a landmark paper in the area.

- With research collaborators Bianchi, Escalante and Nasini from Argentina, PI Tunçel proposed the notion of LS_+-*perfect graphs* (a superclass of perfect graphs) based on the SDP relaxations of the optimal stable set problem originally proposed by Lovász and Schrijver in 1991. We were able make advances on the combinatorial characterizations of these graphs [9]. We established a new, close relationship among LS_+-perfect graphs, near-bipartite graphs and newly introduced concept of *full-support-perfect* graphs. This paper is already leading to some new research.

- With past graduate students Karimi and Moazeni, PI Tunçel proposed a new approach to robust optimization which uses a utility theory viewpoint and employs the powerful theory of weighted centers from interior-point methods. Our paper [27] appeared in 2018. This paper has connections to research Problem A1. as well as research Problem A3. In this paper, we present a framework which allows a fine-tuning of the classical tradeoff between robustness and conservativeness by the Decision Maker (DM) and engages DM continuously and in a more effective way throughout the optimization process. We prove that under a suitable unifying framework, the classical robust optimization approach cannot obtain better solutions (with respect to DM's preferences) than our framework. We demonstrate that it is possible to efficiently perform optimization under this framework and finally, we illustrate some of our methods in our computational experiments. One of the main contributions of this paper is the development of cutting-plane algorithms for robust optimization using the notion of weighted analytic centers in a small dimensional weight space. Ultimately, we are proposing that our approach be used in practice with a small number (say somewhere in the order of 1 to 20) of driving factors that really matter to the DM. These driving factors are independent of the number of variables and constraints, and determine the dimension of the weight space (for interaction with the DM). Working in a low dimensional weight space not only simplifies the interaction for the DM, but also makes our cutting-plane algorithms more efficient. The algorithms we design in this paper make it possible to implement the ideas we mentioned above to help overcome some of the difficulties for robust optimization to reach a broader, practicing user base.

- With past graduate students Karimi and Luo, PI Tunçel completed another paper related to Problem A1 [26]. In this paper, we propose a family of search directions based on primal-dual entropy in the context of interior-point methods for linear optimization. Primal-dual entropy was proposed by PI Tunçel in 1993 as a potential function/barrier function (to obtain search directions, to obtain local metrics, and to perform line-search as well as complexity analysis) for interior-point algorithms (see [43, 45]). We show in this 2017 paper that by using primal-dual entropy-based search directions in the predictor step of a predictor-corrector algorithm together with a homogeneous self-dual embedding, we can achieve the current best iteration complexity bound for linear optimization. Then, we focus on some wide neighborhood algorithms and show that in our family of entropy-based search directions, we can find the best search direction and step size combination by performing a plane search at each iteration. For this purpose, we propose a heuristic plane search algorithm as well as an exact one. Finally, we perform computational experiments to study the performance of entropy-based search directions in wide neighborhoods of the central path, with and without utilizing the plane search algorithms.

- In relation to Problem A4., our paper (by Au and PI Tunçel) on certain elementary polytopes (those that are obtained from the unit hypercube in $\mathbb{R}^n$ by either removing a vertex by a cutting plane or by removing all vertices by 2^n cutting planes) with high lift-and-project ranks for strong positive semidefinite operators [6] appeared. Our results apply to the strongest lift-and-project operators such as SA_+ operator (this is a positive semidefinite strengthening of Sherali-Adams operator), BZ_+ (positive semidefinite version of Bienstock-Zuckerberg operator) and Lasserre operator.

B. Geometry and boundary structure of convex cones

Problem B1. Characterize the boundary structure of convex cones as they relate to facial exposedness and facial dual completeness properties.

With collaborator Dr. Vera Roshchina, we completed our manuscript [34]. In this manuscript, we were able to obtain some of the strongest results to date in characterizing facial dual completeness (FDC). FDC property is very important in duality theory. Presence of facial dual completeness makes various facial reduction algorithms behave well. FDC property is also relevant in the fundamental subject of closedness of the image of a convex set under a linear map. FDC property comes up in the area of lifted convex representations and in representations of a family of convex cones as a slice of another family of convex cones. FDC property seems to have a rather mysterious connection to facial exposedness of the underlying cone which is an intriguing and rather beautiful geometric property Moreover, better understanding of FDC property contributes to our understanding of the boundary structure of convex sets.

In the manuscript [34], we also introduced and studied a new geometric operation on convex sets and convex cones that we named *lexicographic tangents, and lexicographic tangent cones*. We further connected this notion to an analytic notion for functions (proposed earlier by Y. Nesterov) called *lexicographic derivatives*. It turns out our notion of tangential exposedness is also connected to the notion of *subtransversality* in variational analysis.

Problem B2. Determine whether tractable polytopes admit tractable self-concordant barriers. In particular, does the matching polytope admit an efficiently computable self-concordant barrier?

With Dr. de Carli Silva, we were able to solve an open problem from our earlier paper [10] in showing that strict complementarity (and facial exposedness in the dual) may fail in SDP relaxations of MaxCut problem. Then we were able to take this work a step further, and we were able to compute the probability of failure of these properties when the objective function is picked uniformly randomly from a class of natural linear objective functions. A manuscript containing these results has been finished in June 2018 [12] and submitted for publication. Among other results in the manuscript, we study how often strict complementarity holds or fails for the MaxCut SDP when a vertex of the feasible region is optimal, i.e., when the SDP relaxation is tight. While strict complementarity is known to hold when the objective function is in the interior of the normal cone at any vertex, we prove that it fails generically at the boundary of such normal cone. In this regard, the MaxCut SDP displays the nastiest behavior possible for a convex optimization problem.

Problem B3. Unify the boundary structure results on convex cones and sets.

As reported in part A (in relation to Problem A3.), with Dr. de Carli Silva, PI Tunçel has obtained a generalization of the key notion of Total Dual Integrality (TDI) to the setting of convex optimization in general and Semidefinite Programming in particular [13]. This amounts to a major breakthrough also in this part (that is part B.) of the research program. Our results unify many min-max theorems including those arising via polyhedral theory and the conventional notion of TDIness.

C. Mixed integer programming

We consider mixed integer linear programs with n unrestricted-in-sign integer variables x and k non-negative continuous variables s. We assume that the n variables x are expressed in terms of the variables s as follows,

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^n \\ s & \in & \mathbb{R}^k_+. \end{array} \tag{1}$$

We assume $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$, $k \geq 1$, and $r^j \in \mathbb{Q}^n \setminus \{0\}$ for all $j \in \{1, \ldots, k\}$. In particular, $s = 0$ is not a solution of (1). Denote by $\mathrm{R}(f; r^1, \ldots, r^k)$ the convex hull of all the points s for which there exists x where (x, s) satisfy (1). It follows from [31] that $\mathrm{R}(f; r^1, \ldots, r^k)$ is a polyhedron.

A *lattice-free convex set* is a convex set with no integral point in its relative interior. It is *maximal* if no proper superset is a lattice-free convex set. Lovász [30] proved that maximal lattice-free convex sets are polyhedra that have the property that every facet contains a lattice point in its relative interior.

The paper:

- Y. Awate, G. Cornuéjols, B. Guenin and L. Tunçel, On the relative strength of families of intersection cuts arising from pairs of tableau constraints in mixed integer programs, *Mathematical Programming A* 150 (2015) 459–489 appeared. In this paper, we proved with Yogesh Awate and Gérard Cornuéjols [8],

Theorem 1 *For $n = 2$, triangles approximate quadrilaterals within a factor $\frac{3}{2}$.*[1]

Problem C1. For values of $n \geq 3$ investigate the relation between the quality of the relaxation of $\mathrm{R}(f; r^1, \ldots, r^k)$ obtained from the set of all intersection cuts arising from a class of lattice free convex sets and the complexity of the family of lattice free convex sets. This is not only of interest from a theoretical point of view, but has practical consequences as well. It is likely that cuts arising by considering more than two constraints at the time are important in cutting plane algorithms. There are a lot of choices for selecting intersection cuts in those cases, and we wish to have some theoretical guidance on what intersection cuts will be most promising.

Consider a maximal lattice-free set $\mathcal{B}$ that is a polytope. We say that $\mathcal{B}$ has *property (P)* if the following conditions hold:

- For every facet F of $\mathcal{B}$ pick a lattice point v_F in its relative interior and let $\mathcal{Q}$ denote the convex hull of all these points. We require the face lattice of $\mathcal{Q}$ to be the dual of the face lattice of $\mathcal{B}$.
- There exists a bijection between vertices of $\mathcal{B}$ and facets of $\mathcal{Q}$ obtained as follows: pick f a point in the interior of $\mathcal{Q}$, then for every vertex w of $\mathcal{B}$ there exists a unique facet F of $\mathcal{Q}$ intersecting the line segment between f and w.

For our purposes in this section, a *pyramid* in $\mathbb{R}^n$ is an n-dimensional, $(2n-1)$-faceted polytope that is obtained as the intersection of a pointed polyhedral cone with $(2n-2)$ facets and a closed half-space. The counterpart for $n \geq 3$ to triangles are pyramids, and the counterpart to maximal lattice-free quadrilateral are octahedra with property (P). With Gérard Cornuéjols and our PhD student Leanne Stuive [16] we extended the previous result [8] to arbitrary dimensions n, namely,

Theorem 2 *For $n \geq 2$, pyramids approximate octahedra with property (P) within a factor of $2 - \frac{1}{n}$.*

Problem C2. Provide a classification of all 3-dimensional maximal lattice-free convex bodies. This will involve among other things, writing a computer code to enumerate skeletons of all 3-dimensional maximal lattice-free polytopes with at most eight facets. A preliminary part of this work was already completed in our latest ONR grant cycle.

Our research in the directions given by research problems C1. and C2. led us to consider some other related open problems (described below) about maximal lattice-free convex sets rather than creating an "organized list" of all maximal lattice-free convex sets in arbitrary dimensions.

For dimension $n = 2$ maximal lattice-free polytopes are triangles and quadrilaterals. It is straightforward to verify that maximal lattice-free triangles and quadrilaterals have property (P).

[1] where triangles and quadrilaterals are maximal lattice-free sets

By slight abuse of language we call a cube any polytope with face lattice isomorphic that that of the cube. An octahedron is the polar of a cube.

Open problems:

a) Which maximal lattice-free polytopes have property (P)?

b) Does every maximal lattice-free cube have property (P)?

c) Does every maximal lattice-free octahedron have property (P)?

These problems are open even for dimension $n = 3$.

Another research problem considers hard MIPs which have a decoupling structure as exposed in detail in [35]. This structure involves 0,1 variables describing an *assignment* and continuous variables which define a tractable combinatorial optimization problem for every fixed value of the binary variables. In the cases considered in [35] the tractable combinatorial optimization problem (after the decoupling) becomes an *all-pairs shortest path problem*. Applications include the computation of Walrasian equilibria. PI Tunçel and Undergraduate Research Assistant Billy Zhengxu Jin worked on the following research problem during 2016.

Problem C3. Extend the decoupling property exposed in the MIPs in the manuscript [35] and the corresponding algorithms and results to those MIPs where the combinatorial optimization problem is a matching problem as well as to those where the decoupled problem is a minimum cost network flow problem.

We have not completed a newer publication beyond [35] yet. Billy Zhengxu Jin, after receiving many offers from PhD programs at top universities in North America, chose Cornell University to pursue his PhD studies.

D. Minimax theorems and combinatorial algorithms

Multi-commodity flows

In a multi-flow problem we are given a weighted graph where the edges are partitioned into *demand* and *capacity* edges. For a demand edge the weight is the amount of flow required between its endpoints and for a capacity edge the weight is the maximum amount of flow allowed on that edge. There exists a flow if the cut-condition holds and the flow instance does not contain a certain *bad instance* that arises from the complete graph on five vertices [19]. With PhD student Leanne Stuive we obtained an algorithmic counterpart to this result, namely there exists a polynomial time algorithm that takes as input a graph G with demand edges Σ and edge weights w and in polynomial time in the size of w and G either: finds a flow; or finds a cut that shows that the cut condition is violated; or finds where the bad instance is contained. The key step is an algorithmic counterpart to Lehman's theorem [29].

An extended abstract of this result appeared in the proceedings of the Integer Programming and Combinatorial Optimization conference [20]. The complete paper (which was extensively revised during the tenure of this grant) appeared in the SIAM J. of Discrete Math. [21]. PI Guenin completed a survey on flows in graphs and matroids that appeared in Discrete Applied Mathematics [18].

The Flowing conjecture

A clutter $\mathcal{C}$ is a family of sets over a ground set $E(\mathcal{C})$ with the property that no set properly contains another. The *blocker* $b(\mathcal{C})$ of clutter $\mathcal{C}$ is the set of all inclusion-wise minimal sets that intersect every set in $\mathcal{C}$. A clutter $\mathcal{C}$ is *binary* if for every set $S \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|S \cap B|$ is odd. Let

$$P(\mathcal{C}) := \left\{ x \in \mathbb{R}_+^{E(\mathcal{C})} : \sum (x_e : e \in S) \geq 1, \text{for all } S \in \mathcal{C} \right\}.$$

A clutter $\mathcal{C}$ is *ideal*, if $P(\mathcal{C})$ is an integral polyhedron. The polyhedron obtained from $P(\mathcal{C})$ by setting some variables to zero and projecting variables is another set covering polyhedron $P(\mathcal{D})$ for some clutter $\mathcal{D}$. We say that $\mathcal{D}$ is a *minor* of $\mathcal{C}$. A clutter is *minimally non-ideal* if it is non-ideal, but every minor is ideal. $\mathcal{O}_5$ is the clutter corresponding to triangles and pentagons of K_5 and $\mathcal{L}_7$ is the clutter corresponding to the lines of the Fano matroid. If a clutter is ideal (resp. binary) then so is any minor and so is its blocker. In 1977, Seymour [41] proposed the

Flowing Conjecture. *If $\mathcal{C}$ is a minimally non-ideal binary clutter, then $\mathcal{C}$ or $b(\mathcal{C})$ is $\mathcal{O}_5$ or $\mathcal{L}_7$.*

A *triangle* in clutter $\mathcal{F}$ is a set of cardinality three. With PhD student Ahmad Abdi we proved that the only minimally non-ideal binary clutters that have a triangle are $\mathcal{L}_7$ and $\mathcal{O}_5$. The result will appear in Combinatorica [1].

The *two-point Fano clutter*, denoted by $\mathcal{D}_7$, is the clutter whose sets are the lines and their complements of the Fano plane, that intersect two fixed elements exactly once. We showed that,

Theorem 3 *If $\mathcal{C}$ is a minimally non-ideal binary clutter, then $\mathcal{C}$ or $b(\mathcal{C})$ is $\mathcal{O}_5$ or $\mathcal{L}_7$ or $\mathcal{D}_7$.*

An extended abstract has appeared in the Integer Programming and Combinatorial Optimization conference [2]. A full version of the paper will appear in Combinatorica [3].

The Cycling conjecture

Let $\mathcal{C}$ be a clutter. We say that $\mathcal{C}$ *packs* if the maximum number of pairwise disjoint sets of $\mathcal{C}$ is equal to the cardinality of the smallest set in the blocker of $\mathcal{C}$. A binary clutter is *Eulerian* if the parity of the cardinality of every set in the blocker is the same. Denote by $\mathcal{P}_{10}$ the clutter whose sets are the postman sets of the Petersen graph. In 1977, Seymour [41] proposed the

Cycling Conjecture. *Eulerian binary clutters without $\mathcal{L}_7$, $\mathcal{O}_5$, $b(\mathcal{O}_5)$, or $\mathcal{P}_{10}$ minor, pack.*

With PhD student Ahmad Abdi we proved the Cycling Conjecture for clutters of odd cardinality T-joins where $|T| \leq 2$. Ahmad Abdi was awarded the University of Waterloo 2014 Alumni gold medal for this work. An extended abstract of this result appeared in the proceedings of the Integer Programming and Combinatorial Conference [4]. The complete paper (which was extensively revised during the tenure of this grant) appeared in the J. of Graph Theory [5].

Recognition of even-cycle and even-cut matroids

A signed graph is a pair (G, Σ) where $\Sigma \subseteq E(G)$. A cycle is *even* if it has an even number of edges in Σ. A binary matroid is an *even-cycle* matroid if its cycles correspond to the even cycles of some signed-graph. Let $\mathcal{M}$ be a class of binary matroid. By the *membership* problem for $\mathcal{M}$ we mean the following: we are given a $(0,1)$-matrix representation A of a binary matroid M, we return either YES if $M \in \mathcal{M}$ and NO if $M \notin \mathcal{M}$. An algorithm for testing membership problem is polynomial if its running time is polynomial in the size of A. In [46], Tutte gave a polynomial time algorithm to test membership for the class of graphic (and hence co-graphic) matroids. Together with PhD student Cheolwon Heo we resolved a long standing problem by finding a polynomial time algorithm for testing membership in the class of even-cycle matroids. We also prove the analogeous result for even-cut matroids. The results appear in Cheolwon Heo's thesis [25]. We have since found a rather different (and simpler) algorithm and are finalizing the paper.

Whitney type results

Whitney proved that if two 3-connected graphs G_1, G_2 have the same cycle space then $G_1 = G_2$. This explains why the membership algorithm for the class of graphic matroids is comparably easy. In his PhD thesis Shih [42] characterized the relationship between a graph and a signed graph when the cycle space of the graph equals the even-cycle space of the signed graph. Alas the proof is complicated and the result never appeared in a refereed journal. With a former undergraduate researcher Zouhaier Ferchiou we found a shorter proof of this result as well as several restatements that are of independent interest. The paper has been submitted for publication [17]. The problem of characterizing when two signed graphs have the same even-cycle space remains open and is considerably harder. With Irene Pivotto, we give a characterization for the case where the graphs are 4-connected [23]. The same result also characterizes when two grafts have the same even-cut space when the graphs are 4-connected. Jointly with PhD student Cheolwon Heo, we believe that we know how to remove the connectivity assumption. We therefore decided to postpone the submission of the aforementioned paper and instead write a new one with the stronger result.

Awards

- Ahmad Abdi was awarded the 2014 University of Waterloo Alumni Gold Medal for best Master's thesis (all disciplines combined).
- Mehdi Karimi won the 2016-17 OGS scholarship for international graduate students at University of Waterloo. This is a very competitive scholarship. There are only five such scholarships per year for the whole university (University of Waterloo has approximately 6000 graduate students). This is the third year in a row that Mehdi Karimi earned this very prestigious scholarship.
- In June 2016 convocation at the University of Waterloo, Tor Myklebust was honored with an Outstanding Achievement in Graduate Studies Award for PhD. In each convocation, there is only one such award for a graduating PhD student in each Faculty. Faculty of Mathematics has more than 1000 graduate students (although more than 500 are in Master's programs).
- Ahmed Abdi earned an NSERC Doctoral scholarship, and was nominated for the 2016 Murray Martin Prize for the Best Research Paper by a Mathematics Graduate student.
- Leanne Stuive earned (three, merit-based awards): an NSERC Doctoral scholarship, Barbara Hayes-Roth scholarship, and Cotton Family Women in Math scholarship.
- Ahmad Abdi was a recipient of the 2018 Huawei Prize. This award provides one or two graduate students in the Faculty of Mathematics $4,000 each for the demonstration of outstanding achievement in research through a peer-reviewed paper.

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14. ABSTRACT

The general context of this research project is that of mathematical optimization (continuous and discrete) and operations research. It has significant interaction with combinatorics and computational complexity as well as various other fields in mathematical sciences. Specifically, we are designing efficient algorithms to find optimal or near optimal solutions for tractable classes of optimization problems. For problems that are provably hard to solve within any guaranteed bound, our focus is to develop practical heuristics and tools. We also address issues that arise from the nature of the way the data is collected in practice, such as robustness of algorithms under uncertain data, and hidden low dimensional information in high dimensional data.

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