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14. ABSTRACT

15. SUBJECT TERMS

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a. REPORT	b. ABSTRACT	c. THIS PAGE			Yen-Hsi Tsai
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RPPR Final Report
as of 14-Dec-2017

Agency Code:

Proposal Number: 62381MA **Agreement Number: W911NF-12-1-0519**

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Final Report for Period Beginning 07-Sep-2012 and Ending 06-Sep-2017
Title: Visibility-Based Goal Oriented Metrics and Application to Navigation and Path Planning Problems
Begin Performance Period: 07-Sep-2012 **End Performance Period:** 06-Sep-2017
Report Term: 0-Other
Submitted By: Yen-Hsi Tsai Email: ytsai@math.utexas.edu
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STEM Degrees: 1 **STEM Participants:** 3

Major Goals: The major goals of this project include:
(1) Study a class of visibility based metrics and their sensitivity to perturbations
(2) Application of the metric to un-supervised domain learning and navigation
(3) Develop and analyze a novel numerical algorithm for surface and line integrals defined on geometries which are only sampled by unstructured point clouds.

Accomplishments: 1. We have developed a new fully three dimensional visibility based metric for use in autonomous exploration and navigation suitable for small drones. The resulting algorithm is validated preliminarily on a previously developed platform that simulate LIDAR sensors and data on MOUT site. From our simulations, computations performed on a reasonable laptop can reach a spatial resolution of around 30cm.
2. We identified two important opportunities for further development: First we identify that the proposed algorithm requires integrations over interfaces formed by subgroups of lines-of-sights. To increase the robustness in real application, it is important to be able to compute such integrals efficiently. We thus developed a novel mathematical formulation for integration along non-parametrically defined surfaces (implicit surfaces or closest point representation). We prove that suitable products of the singular values of the Jacobian matrix of the closest point function yield correct weights for computing surface integrals with the measure that is induced by the Euclidean distance metric of the embedding space. This mathematical formulation opens doors to formulate and solve boundary integral equations without the need of parametrization. We have obtained and published some papers laying out the theoretical and computational foundations in this area. This formulation also seems to be robust in computing the integrals directly using point clouds sampled from the geometries, in a new multi-resolution fashion. In the last add-on funding period, we have concentrated in analyzing our algorithm for point clouds, in particular, those that contains points sampled from objects of different Hausdorff dimensions and have singularities (corners) or junctions (interacting surfaces and curves).
Motivated by our discussions with the previous project manager, John Lavery, we emphasized on the error estimates for the cases when the density of the point clouds varies greatly. In the attached report, we have included our latest working draft that documented the error bounds that we have obtained.

RPPR Final Report
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Training Opportunities: This grant has supported part of the PhD thesis work on Chieh Chen.
This grant has provided training for two other graduate students.

The PI has been collaborating extensively with a former post-doc, Catherine Kublik, who is now a junior faculty in University of Dayton, Ohio. The collaboration provided her with professional development, which is important for her up coming tenure in her university.
Through interaction with Kublik, the PI also indirectly provide training for an undergraduate student for analyzing point cloud data at Kublik's institute.

Results Dissemination: Results funded by this grant has been disseminated in the following manners:
1) Invited lectures in major conferences and workshop
2) Invited lectures in research seminars and colloquiums
3) Social media
4) Outreach activities

In (1), the PI has given lectures related to the funded results in various international conferences and workshops. The following is a selection of the PI's relevant activities.

• Plenary speaker in the 2016 Annual Meeting of the Taiwan Society for Industrial and Applied Mathematics, May 28 2016
• Invited speaker in International Conference on Applied Mathematics (ICAM), May 2016
• Invited speaker in “Shape Analysis and Learning by Geometry and Machine”, Institute of Pure and Applied Mathematics (IPAM), Feb 2016
• Invited speaker in “Frontiers of Applied and Computational Mathematics”, Peking University, July 2015
• Invited speaker, Sanya, China, March 2014
• Banff, Canada, March 31-April 5, 2013
• Algorithms for Threat Detection Workshop, Nov 28, 2012

In (2): the footprints of the PI presenting the funded results include:
• KTH Royal Institute of Technology, Sweden, 2017
• University of Coimbra, Portugal 2017
• National Ciao-Tung University and National Center for Theoretical Sciences, Taiwan, 2016
• Georgia Tech, Colloquium, March 13, 2015
• Umeå Universitet, Sweden, 06/09-06/10, 2014
• National Geospatial-Intelligence Agency, Dec 4, 2013

In (3): The PI actively maintains a representation on Google Scholar, ResearchGate, as well as a personal homepage hosted by The University of Texas. Particularly in the latter two media, the PI has created dedicated pages for the ARO project.

In (4), The PI has presented the findings of this project to undergraduate students at UT and in Taiwan, in outreach forums.

Honors and Awards: Simons Fellow in Mathematics, 2013-2014

Protocol Activity Status:

Technology Transfer: Nothing to Report

PARTICIPANTS:

Participant Type: PD/PI
Participant: Yen-Hsi Tsai
Person Months Worked: 7.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N

Other Collaborators:

Participant Type: Graduate Student (research assistant)
Participant: Chieh Chen
Person Months Worked: 10.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Graduate Student (research assistant)
Participant: Seong-Jun Kim
Person Months Worked: 5.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Graduate Student (research assistant)
Participant: Yijing Wu
Person Months Worked: 1.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

DISSERTATIONS:

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Institution: The University of Texas at Austin
Date Received: 29-Aug-2016 Completion Date: 12/15/15 7:19PM
Title: Implicit Boundary Integral Methods
Authors: Chieh, Chen
Acknowledged Federal Support: **Y**

WEBSITES:

URL: https://www.researchgate.net/project/Visibility-optimization-and-navigation-problems
Date Received: 04-Dec-2017
Title: Webpage on ResearchGate
Description: Project description on the social media: ResearchGate

RPPR Final Report
as of 14-Dec-2017

Project summary

- Objectives:
  - Formulate and analyze metrics for missions that require achieving "optimal" visibility (lines-of-sight) in a domain
  - Quantify the sensitivity of such metrics to perturbation
  - Design algorithms based on these metrics
- Approach:
  - Level set formulation (natural for the extensive Boolean operations on the occlusion sets)
  - Formulate suitable energies from integrations over the (boundary of) occlusion sets
  - Shape calculus: shape derivatives and topological derivatives
- Novelty:
  - Systematic, volumetric, and variational approach, allowing the use of calculus, to problems typically tackled (with difficulty) by approaches typically involving triangulations

Project summary

- Education:
  - Supported 2 PhD students.
  - Involved 2 PhD students from other university.
- Dissemination: Seminar at NGA and presentation in the Algorithm Workshop (NSF and DTRA), and other universities.
- Publications: one publication under revision, one under construction.
- Honors: The PI was awarded a Simons Foundation Fellowhsip for 2013-2014.

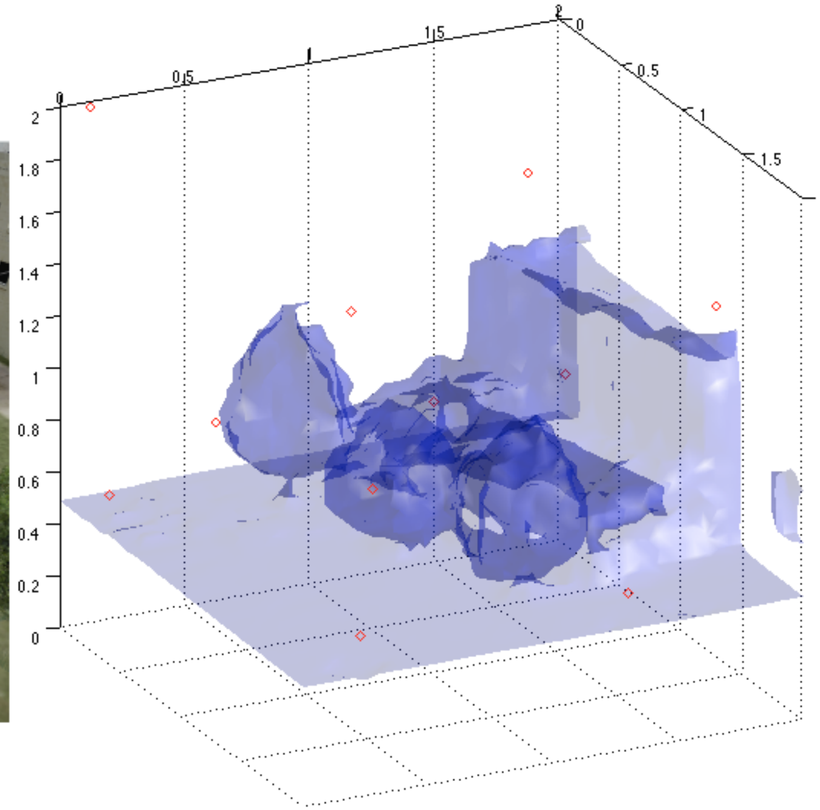
Project summary

- Mathematical results:
  - Formulated and implemented metrics based on "viewing angles" and the corresponding visibility fluxes across the occlusion boundaries.
  - Derived explicit formulae for the sensitivity of the metrics with respect to perturbation of the obstacles, and to distance to the vantage point
  - Formulate problems involving uncertainty in the scene. Some preliminary analysis is conducted.
- Computational results:
  - Fully automated, robust, and efficient simulations for complicated real 3D scenes.

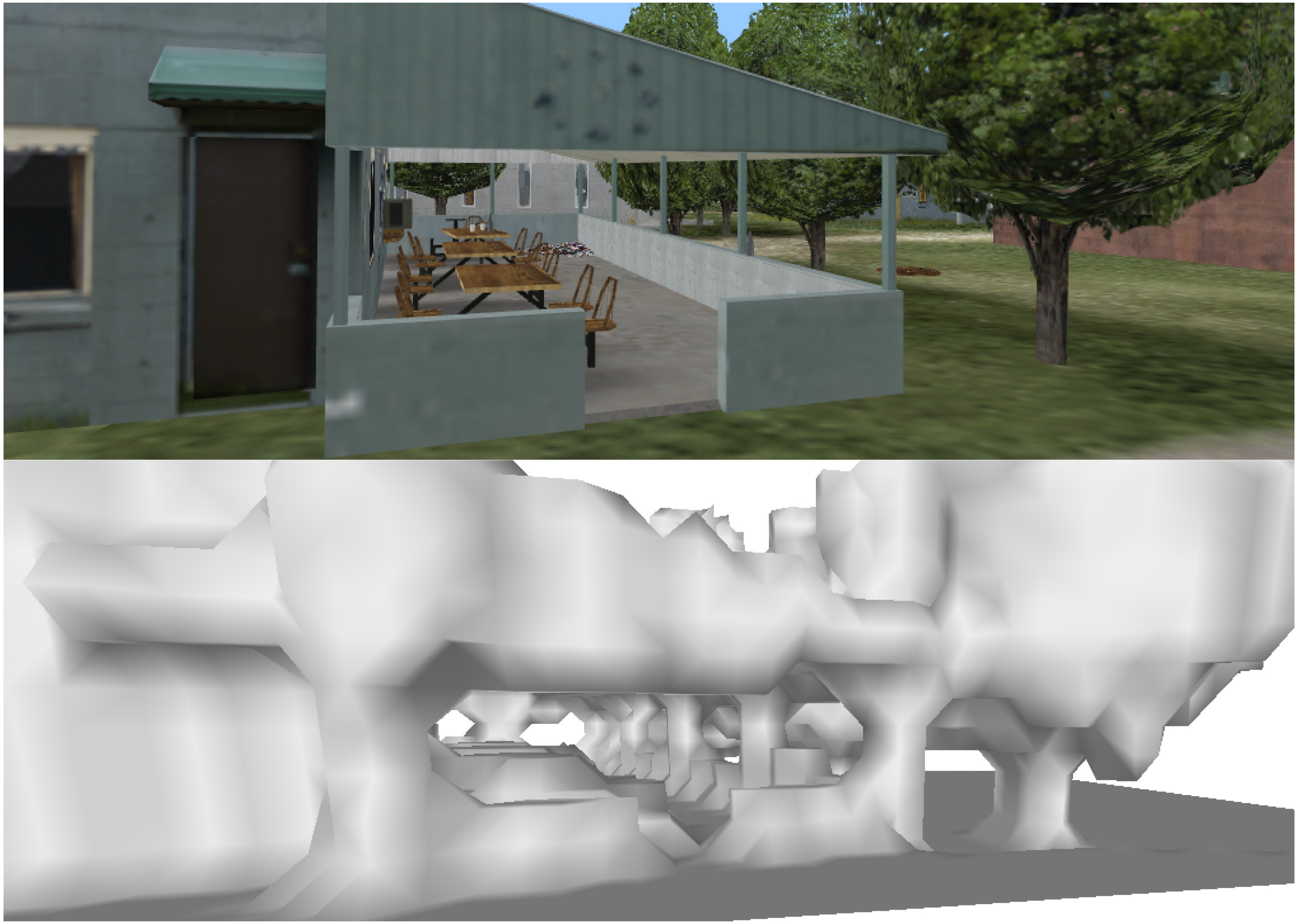
Exploration of an unknown domain

Iterative algorithm: determine an optimal location to explore the unknown part of the domain
Input: the views of a domain (in point clouds) from previous k vantage points
Output: coordinates of an optimal vantage point for viewing the unexplored region

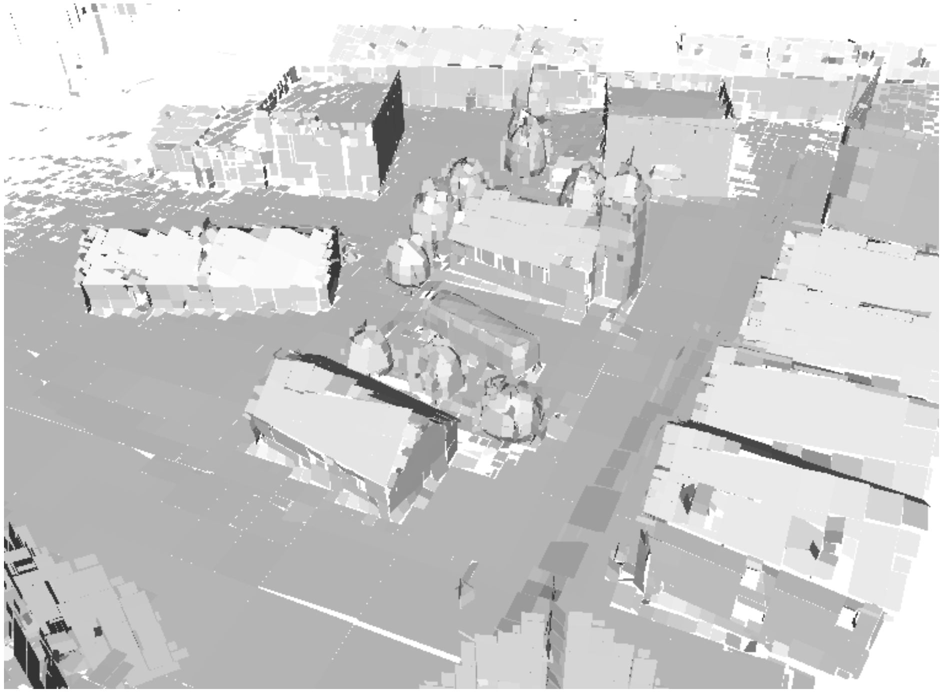
“Ground truth”: the MOUT site



Domain estimated from 11 vantage points



Algorithm was able to discover complicated scenes with minimal resolution and small number of vantage points.



The MOUT site and the geometry learned by views from 20 vantage points

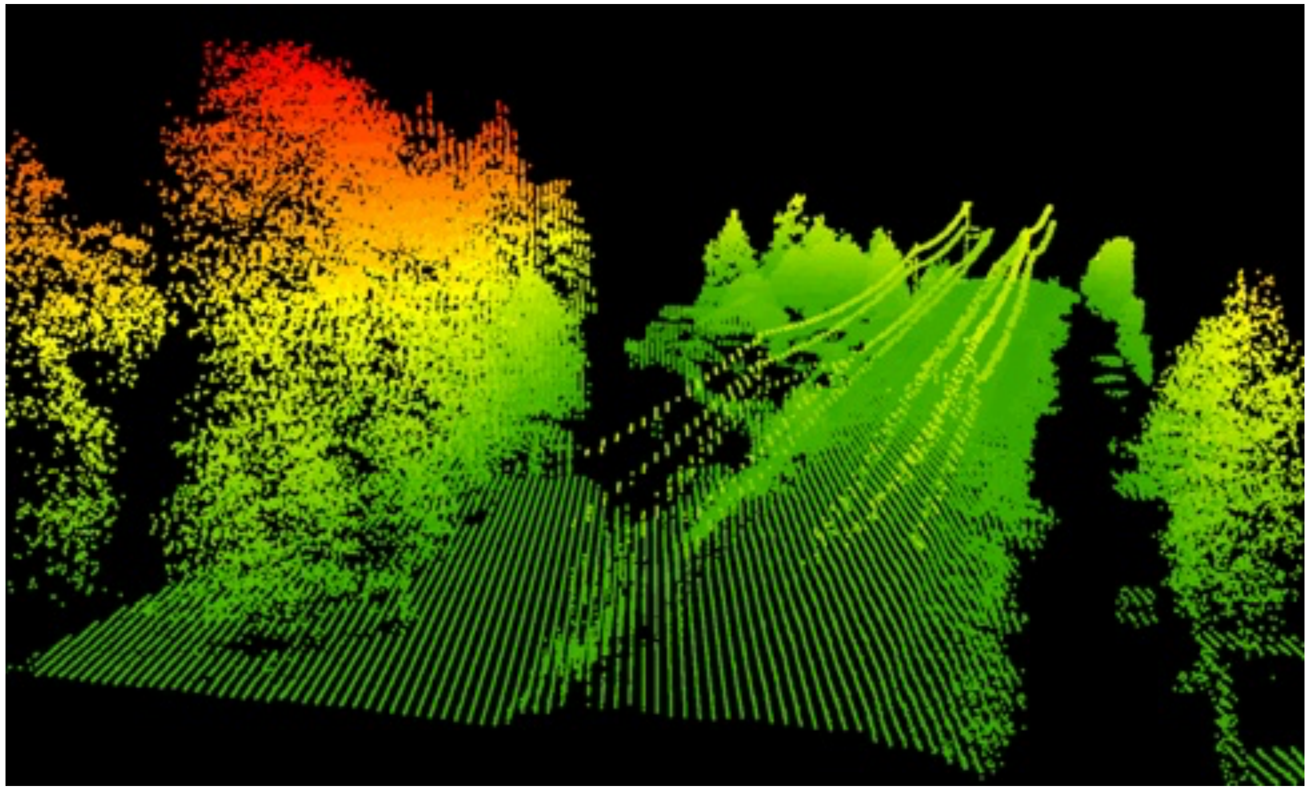
Summation of point clouds

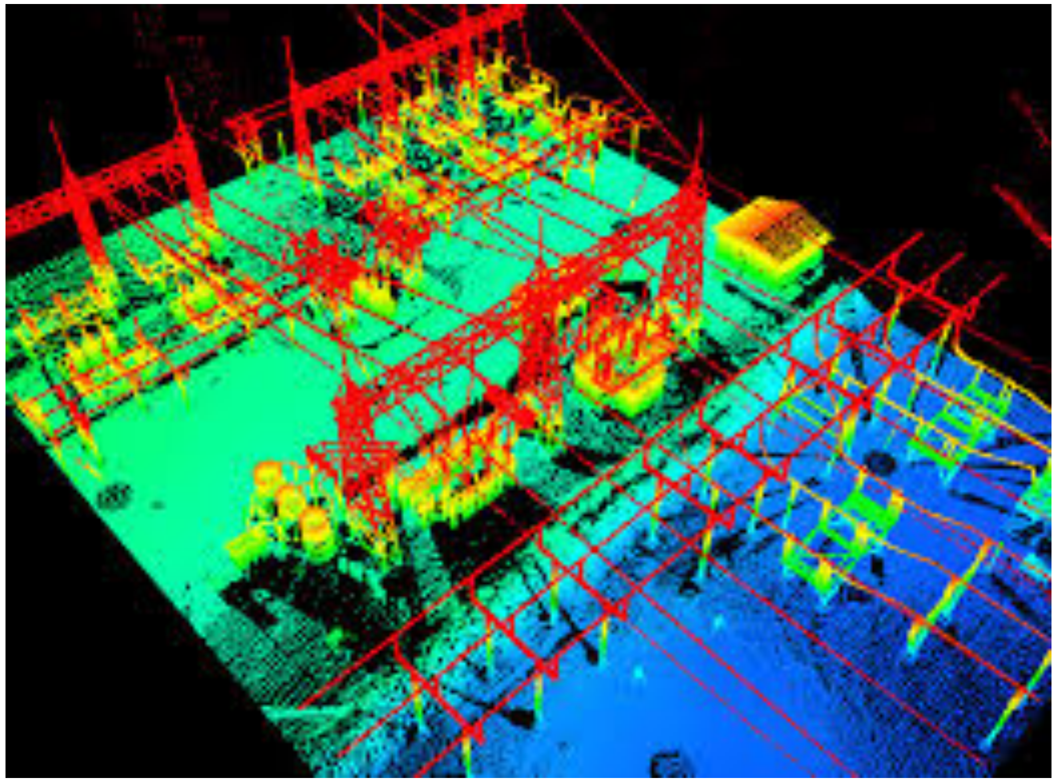
Richard Tsai

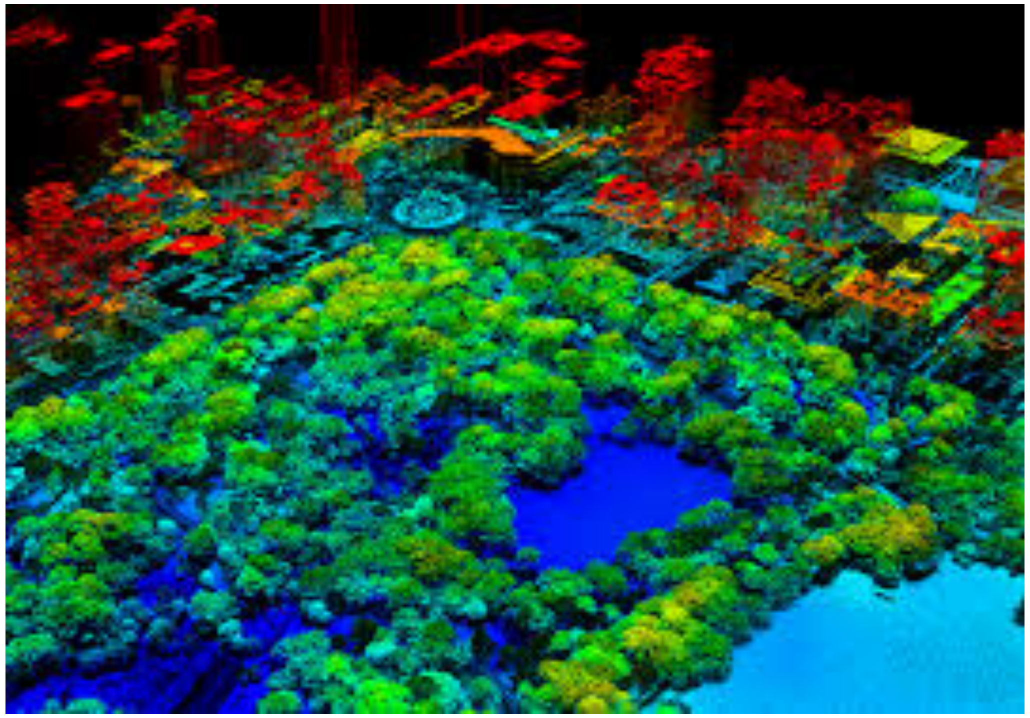
The University of Texas at Austin, USA

Research supported by NSF, ARO

Points sampled from imaging devices







What are the total length of the cables?

The total area of flat surfaces?

The electrical field generated from the cable?

Underlying objects have different dimensions

Setup

- Consider surfaces/curves as point sets $\Gamma_N \subset \Gamma$ (no parameterization)
- Closest point map $\mathcal{P}_{\Gamma_N}(x) := \arg\min_{y \in \Gamma_N} |x - y|$
  - contains a lot of information
  - can be computed easily

- How do we extract or infer information about Γ ? Information such as surface areas, curvatures, etc.

In this talk

The summation to be discussed:

$$S_N = S(\Gamma_N, \epsilon, h, d') = \sum_{\{x_j \in h\mathbb{Z}^d : \phi_N(x_j) < \epsilon\}} \omega(x_j, h, d', \mathcal{P}_{\Gamma_N})$$

$\phi_N(x) = |x - \mathcal{P}_{\Gamma_N} x|$ is the distance to the data set

---

Algorithm's complexity for N data points: $\sim (\frac{\epsilon}{\Delta x})^d N$

Base formulation

Theorem: [Kublik, T: 2015] Γ smooth surface (with boundary)

$$\int_\Gamma g(x)dS = \int_{\mathbb{R}^3} g(\mathcal{P}_\Gamma(x)) \prod_{j=1}^{d'} \sigma_j(x) K_{\epsilon,d'}(\phi_\Gamma(x))dx$$

d' is the Hausdorff dimension of Γ

$\sigma_j(x)$ is the jth singular value of the matrix $\mathcal{P}'_\Gamma(x)$

Simple quadratures

Mapping to the closest point in the given point set:

$$P_{\Gamma_N} : \mathbf{r} \in \mathbb{R}^d \mapsto \Gamma_N$$

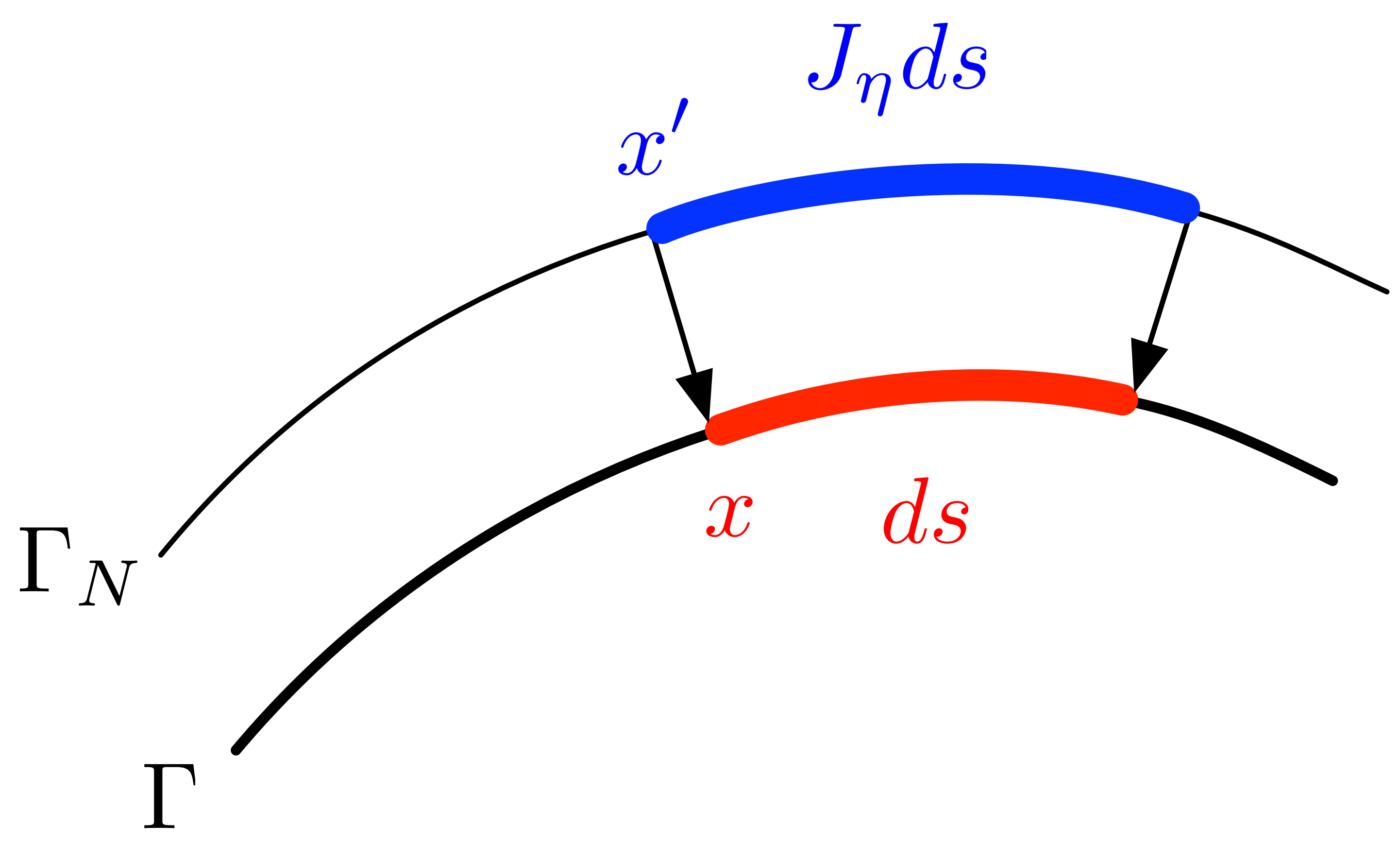
Central differencing matrix:

$$D_0^h \mathcal{P}_{\Gamma_N}(\mathbf{r}) := \frac{1}{2h}\left(\mathcal{P}_{\Gamma_N}(\mathbf{r}+h\mathbf{e_1}) - \mathcal{P}_{\Gamma_N}(\mathbf{r}-h\mathbf{e_1}), \mathcal{P}_{\Gamma_N}(\mathbf{r}+h\mathbf{e_2}) - \mathcal{P}_{\Gamma_N}(\mathbf{r}-h\mathbf{e_2})\right)$$

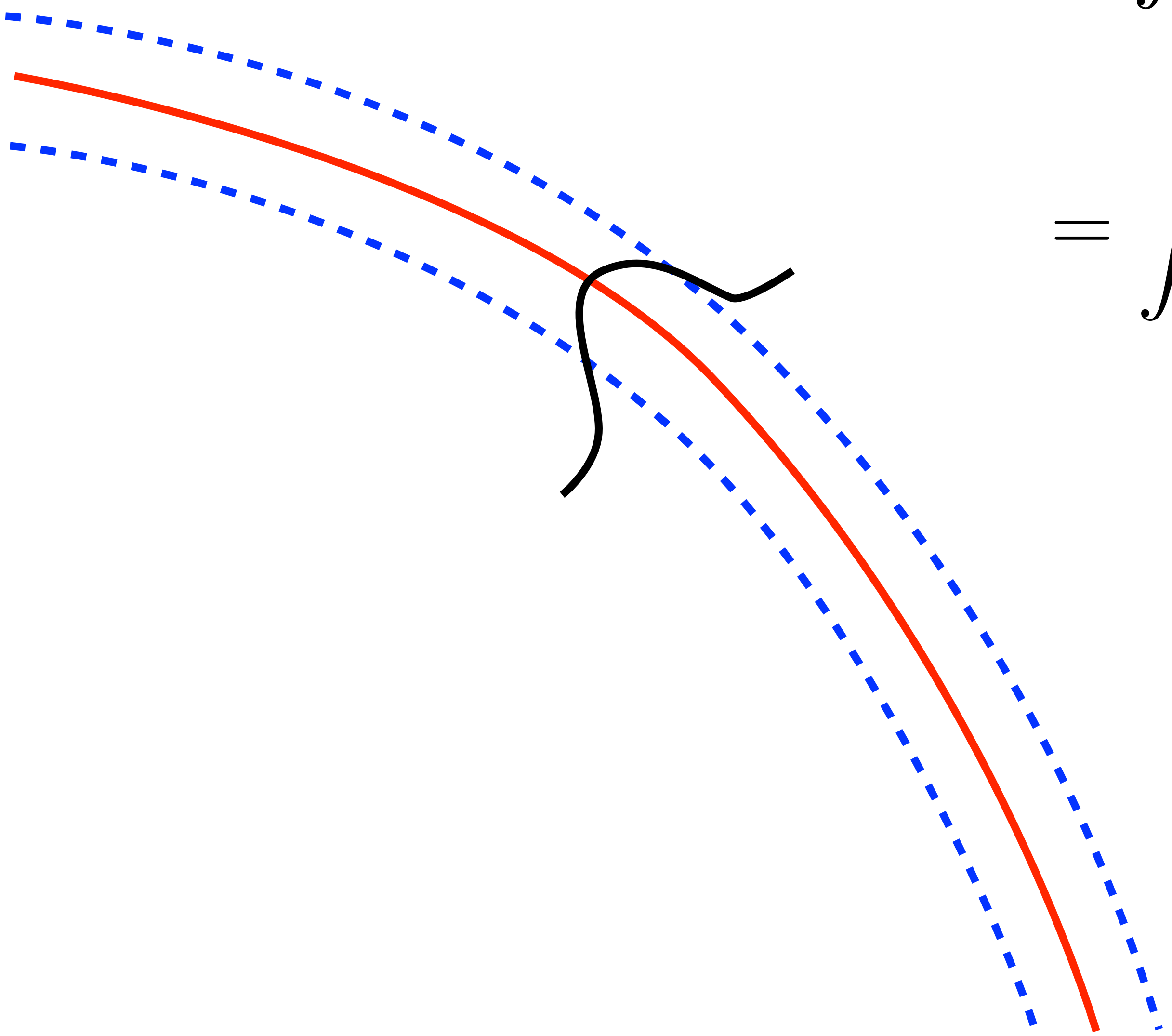
$$S_N = S(\Gamma_N, \epsilon, h, d') := \sum_{\mathbf{r}\in h\mathbb{Z}^d, \mathrm{dist}(\Gamma_N, \mathbf{r})\le\epsilon} \omega(\mathbf{r})\, \Pi_{j=1}^{d'} \sigma_j(D_0^h \mathcal{P}_{\Gamma_N}\mathbf{r})$$

Parametrizion by parallel level sets

$$I(0) = \int_{\Gamma} f(x) dS = \int_{\Gamma_\eta} f(\mathcal{P}_\Gamma x') J(x') dS' = I(\eta)$$



Average the identical integrals



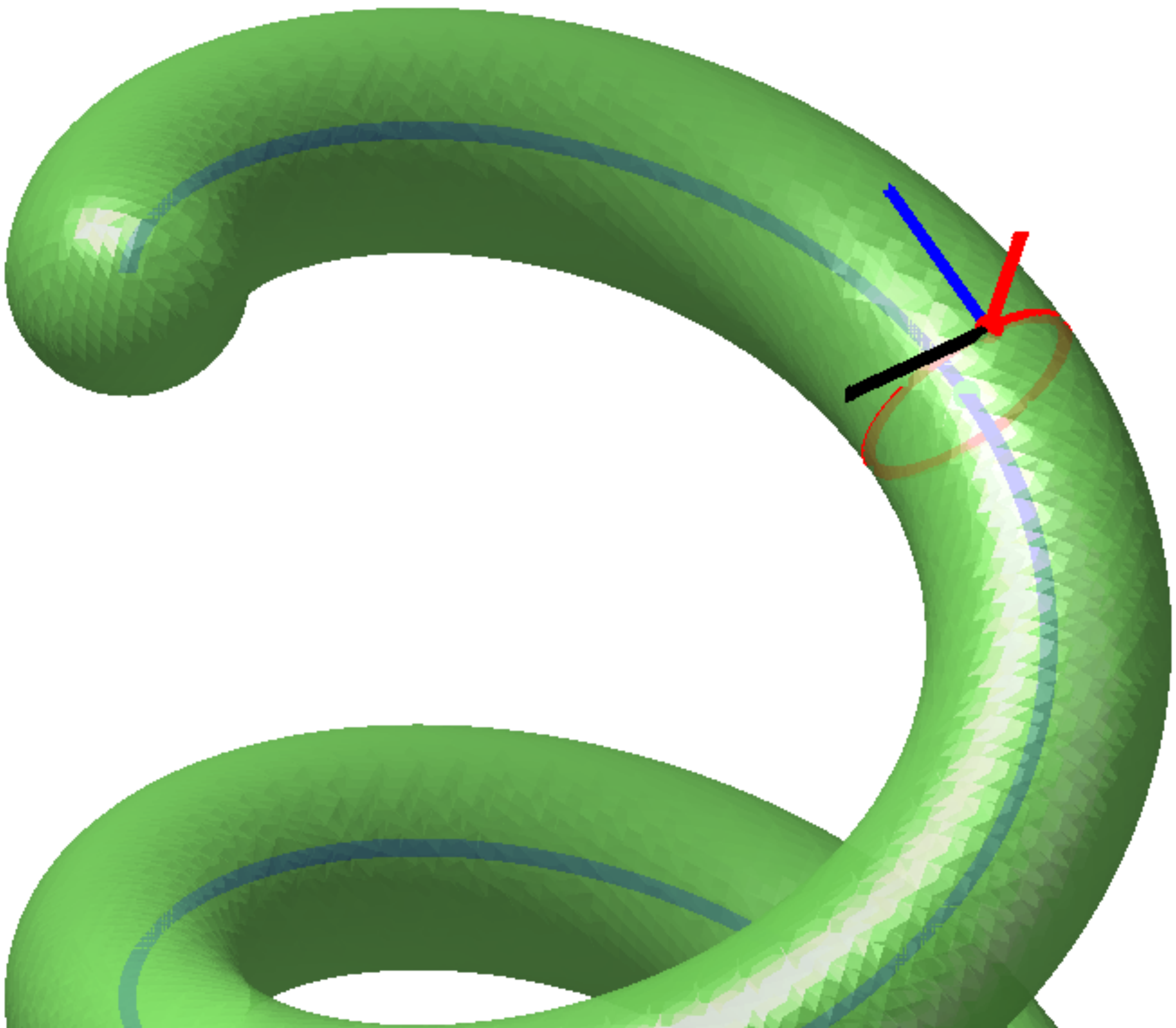
$$I(0) = \int_0^{\epsilon} I(0) K_\epsilon(\eta) d\eta = \int_0^{\epsilon} I(\eta) K_\epsilon(\eta) d\eta$$

$$= \int_{\mathbb{R}^d} f(\mathcal{P}_\Gamma x) J_\Gamma(x) K_\epsilon \circ \phi_\Gamma(x) dx$$

[Kublik-Tanushev-T:2013]

The role of singular values

Geometrical meaning easily seen by a convenient local coordinate system



Surfaces:

$$J_\Gamma(x) = \sigma_1\sigma_2(\mathcal{P}'_\Gamma(x)) \\ = 1 + \eta H(x) + \eta^2 G(x)$$

Curves:

$$\sigma_1 = (1 - \kappa\eta\cos\theta)^{-1}$$

$$\eta = |x - \mathcal{P}_\Gamma x|$$

distance to the manifold

$$\sigma_1\sigma_2(x) = 0, \qquad x \in \partial\Gamma$$

[T2013, Kublik-T2015]

Other applications

- Implicit Boundary Integral Methods (IBIMs):

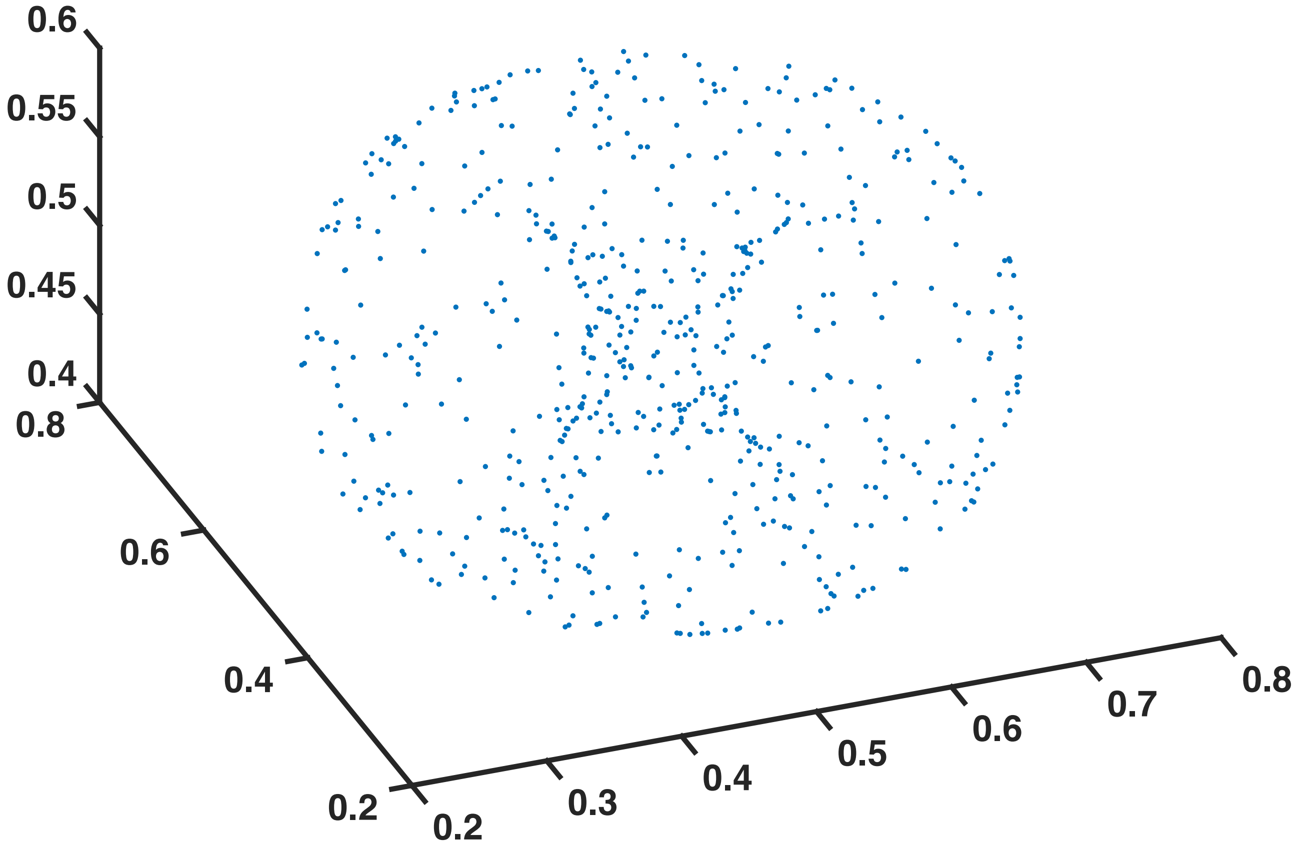
  Solve boundary integral equations volumetrically, *without parametrization*

  - High order nonlinear interface dynamics driven by bulk diffusion [Chen-Kublik-T]

  - Wave scattering with sound-hard boundaries:
    new regularization for hyper-sinuglar kernels [Chen-T]

- Possible generalization to higher dimensions via Weyl's tube formula

A fluke?



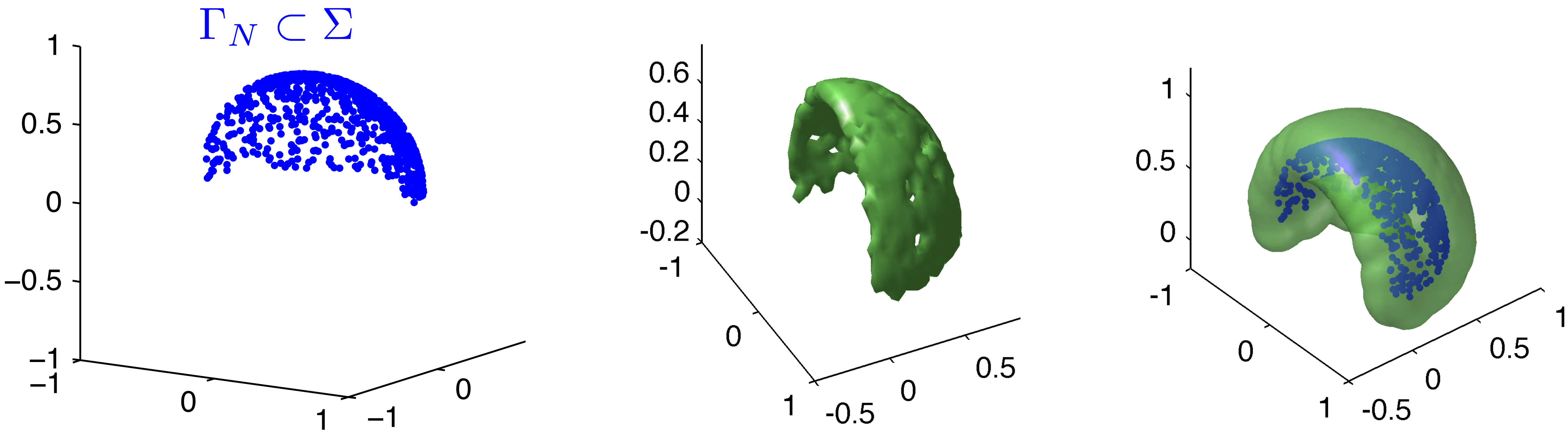
Total of 625 points

epsilon=0.1, dx=epsilon/5

Relative error: 0.07

$$S_N = S(\Gamma_N, \epsilon, h, d') := \sum_{\mathbf{r} \in h\mathbb{Z}^d, \mathrm{dist}(\Gamma_N, \mathbf{r}) \leq \epsilon} \omega(\mathbf{r}) \, \Pi_{j=1}^{d'} \sigma_j (D_0^h \mathcal{P}_{\Gamma_N} \mathbf{r})$$

Summation over point clouds



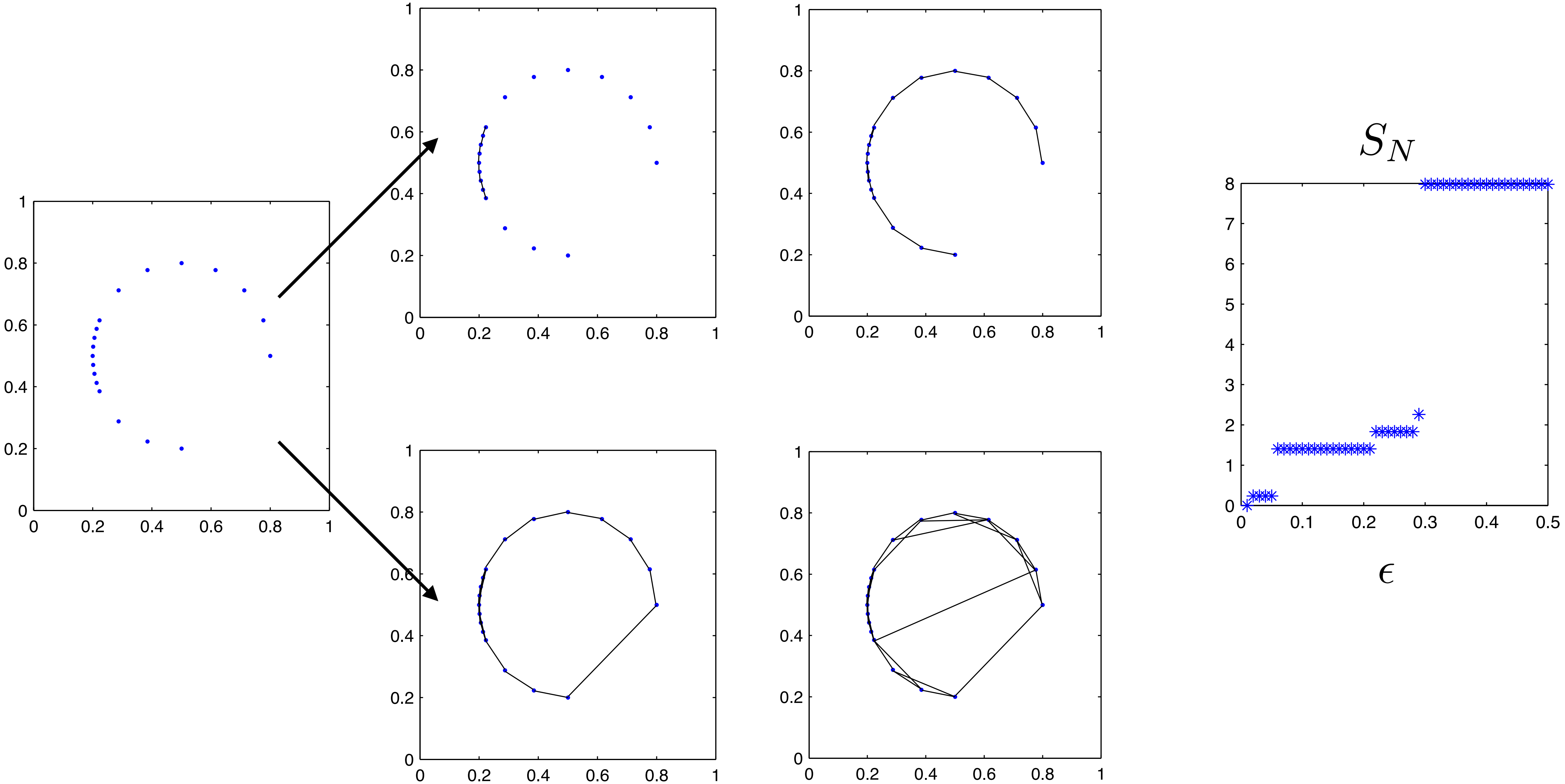
- 30×30 uniformly distributed point clouds sampling in spherical coordinate the quarter sphere patch.
- $50 \times 50 \times 50$ uniform Cartesian grid discretizing $[-1, 1]^3$.
- Relative error using $\epsilon = 0.05 = dx$: -0.56.
- Relative error using $\epsilon = 0.2 = 4dx$: -0.061.

Even though $\{d_{\Gamma_N} = \eta\}$ has improved regularity, but that is not the reason.

Different regimes for the point density

- δ is the spacing of data points
- $h = \Delta x$ is the spacing of grid nodes
- $\delta = 0$: Γ_N the closest points of the grid nodes
- $\delta \ll \Delta x$: cloud is dense relative to the mesh
- $\delta > \Delta x$: fully discrete setting

Interpretations of an ill-posed problem



Analyze the central difference matrix

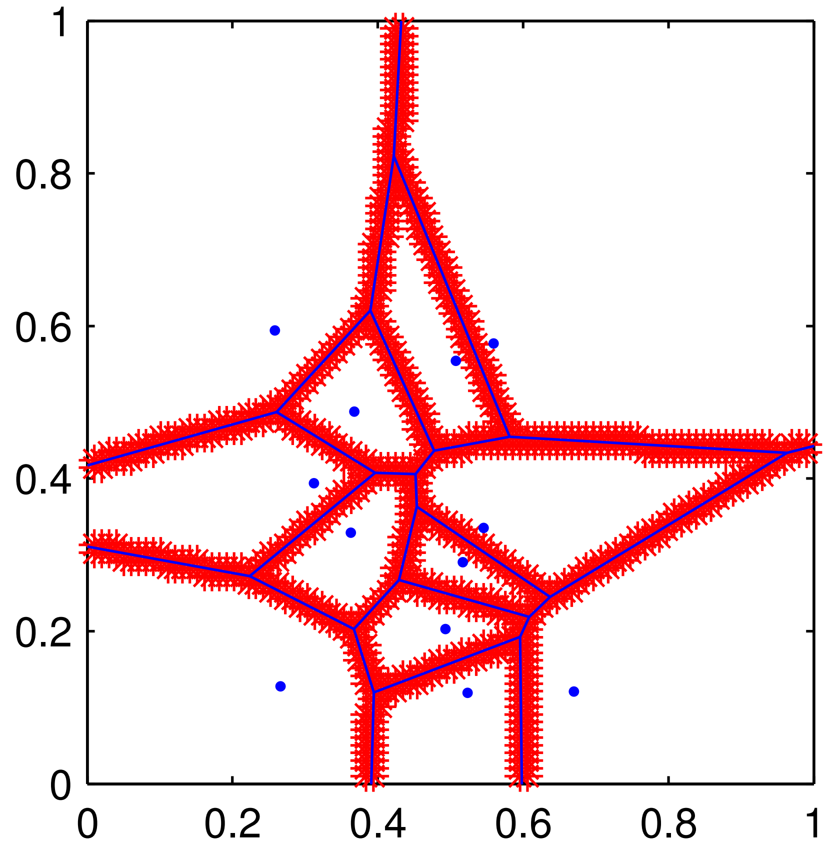
Mapping to the closest point in the given point set:

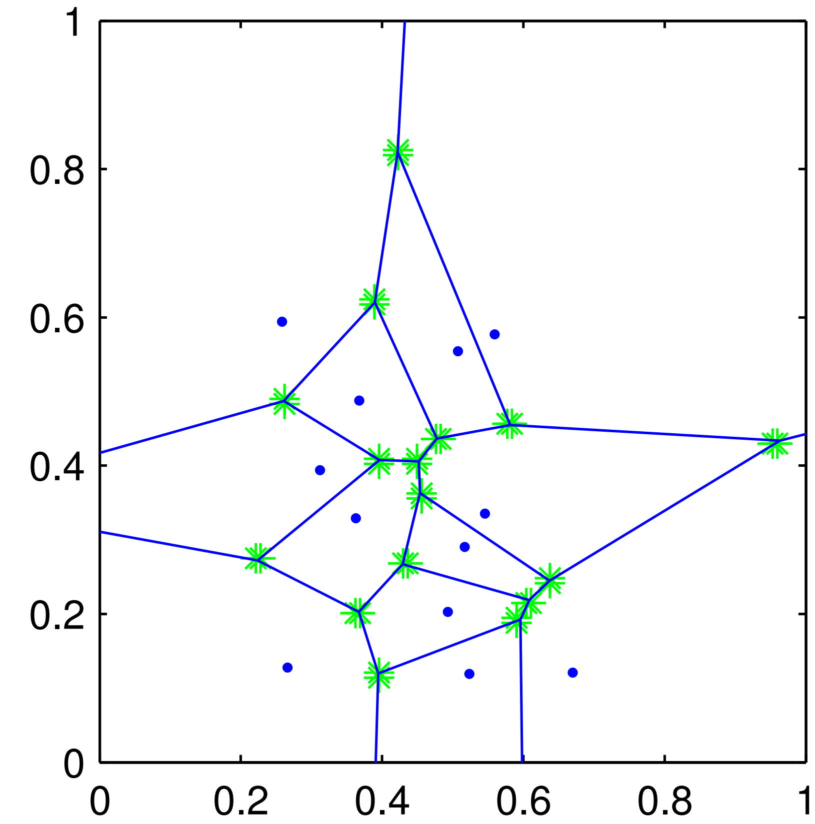
$$P_{\Gamma_N} : \mathbf{r} \in \mathbb{R}^d \mapsto \Gamma_N$$

Central differencing matrix:

$$D_0^h \mathcal{P}_{\Gamma_N}(\mathbf{r}) := \frac{1}{2h} \left(\mathcal{P}_{\Gamma_N}(\mathbf{r} + h\mathbf{e_1}) - \mathcal{P}_{\Gamma_N}(\mathbf{r} - h\mathbf{e_1}), \mathcal{P}_{\Gamma_N}(\mathbf{r} + h\mathbf{e_2}) - \mathcal{P}_{\Gamma_N}(\mathbf{r} - h\mathbf{e_2}) \right)$$

Singular values again





$$\sigma_1 = \sqrt{1+\alpha^2}\ |q_i - q_k|$$
$$\sigma_2 = 0$$

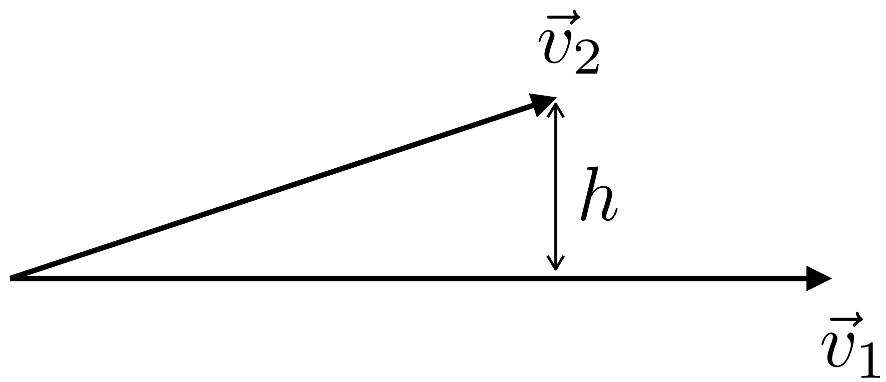
$$\alpha = 0, 1$$

$\sigma_1 \sigma_2 = \ 2\alpha$ area of the triangle/ quadrilateral

$$\alpha = 1, \sqrt{2}, \sqrt{3}$$

Connectivity of points related to their Voronoi diagram

Relating to length and area



$$D_0^h \mathcal{P}_{\Gamma_N} = (\vec{v}_1, \vec{v}_2)$$

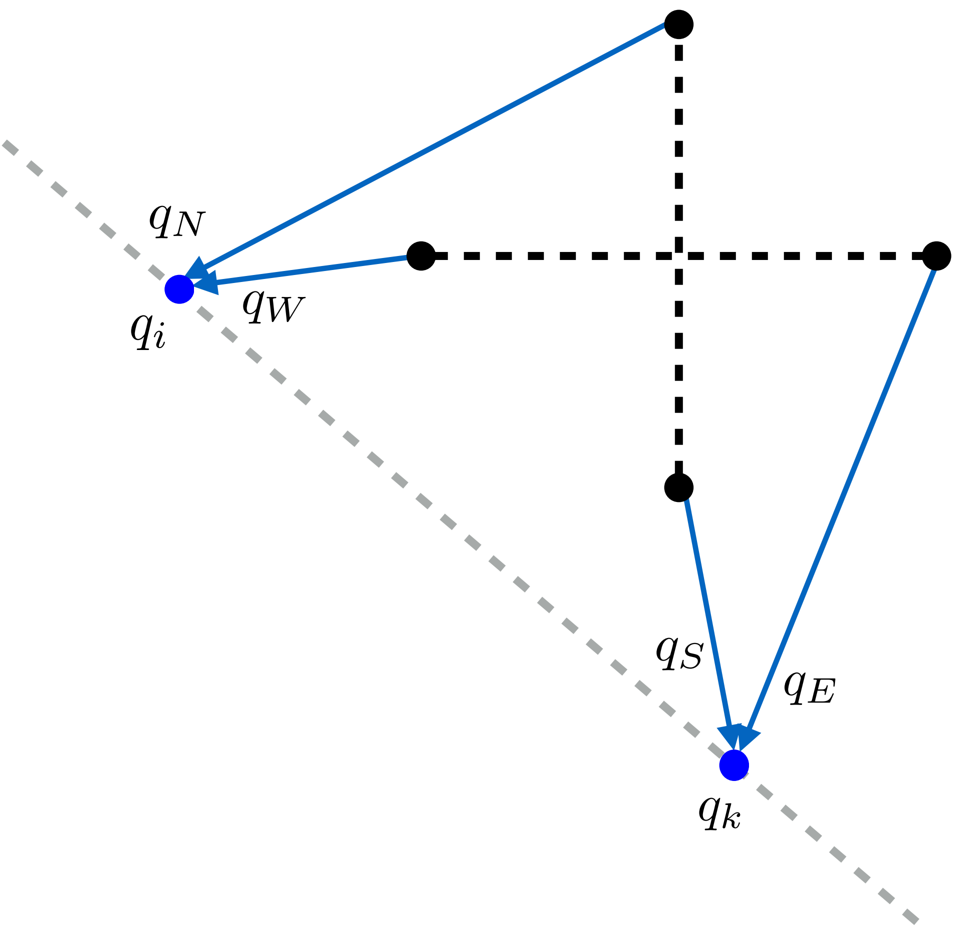
$$|\vec{v}_1| \geq |\vec{v}_2|$$

- rank=2: $\sigma_1 \sigma_2 = h|\vec{v}_1|$

$$\sigma_1 = \sqrt{|\vec{v}_1|^2 + |\vec{v}_2|^2} + \mathcal{O}(h^2)$$

- rank=1: $\sigma_1 = \alpha|\vec{v}_1|$

The first singular value



$$D_0^h \mathcal{P}_{\Gamma_N} = (\vec{v}_1, \vec{v}_2)$$

$$\vec{v}_1 = \pm \vec{v}_2 = \pm (q_i - q_k)$$

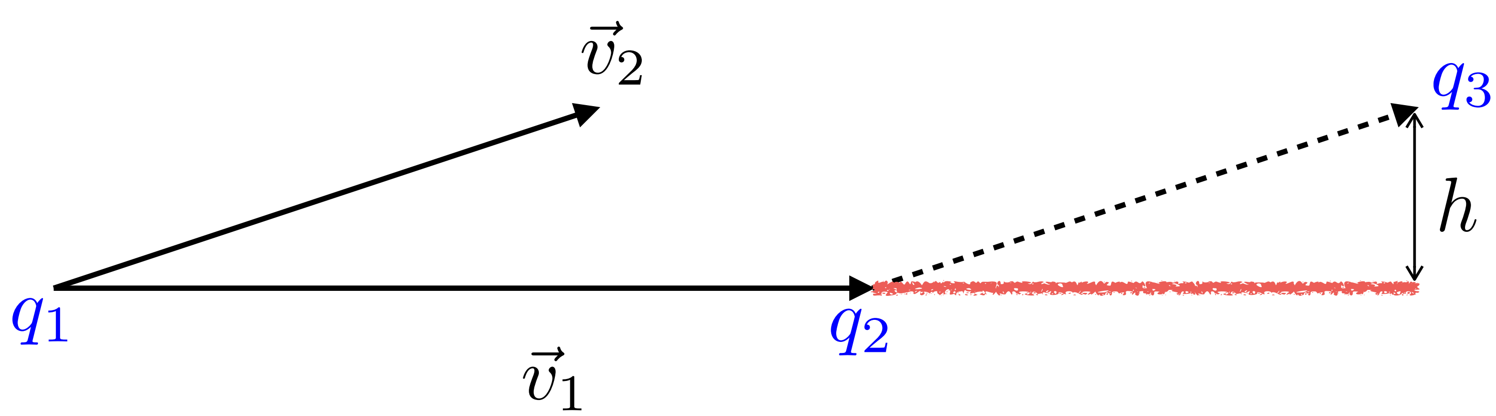
- rank=1: $\sigma_1 = \alpha |\vec{v}_1|, \quad \alpha = 1, \sqrt{2}$

The first singular value

$$D_0^h \mathcal{P}_{\Gamma_N} = (\vec{v}_1, \vec{v}_2)$$

- rank=2: $\sigma_1 = \sqrt{|q_1 - q_2|^2 + |q_2 - q_3|^2} + \mathcal{O}(h^2)$

h depends on the curvature of the underlying curve.

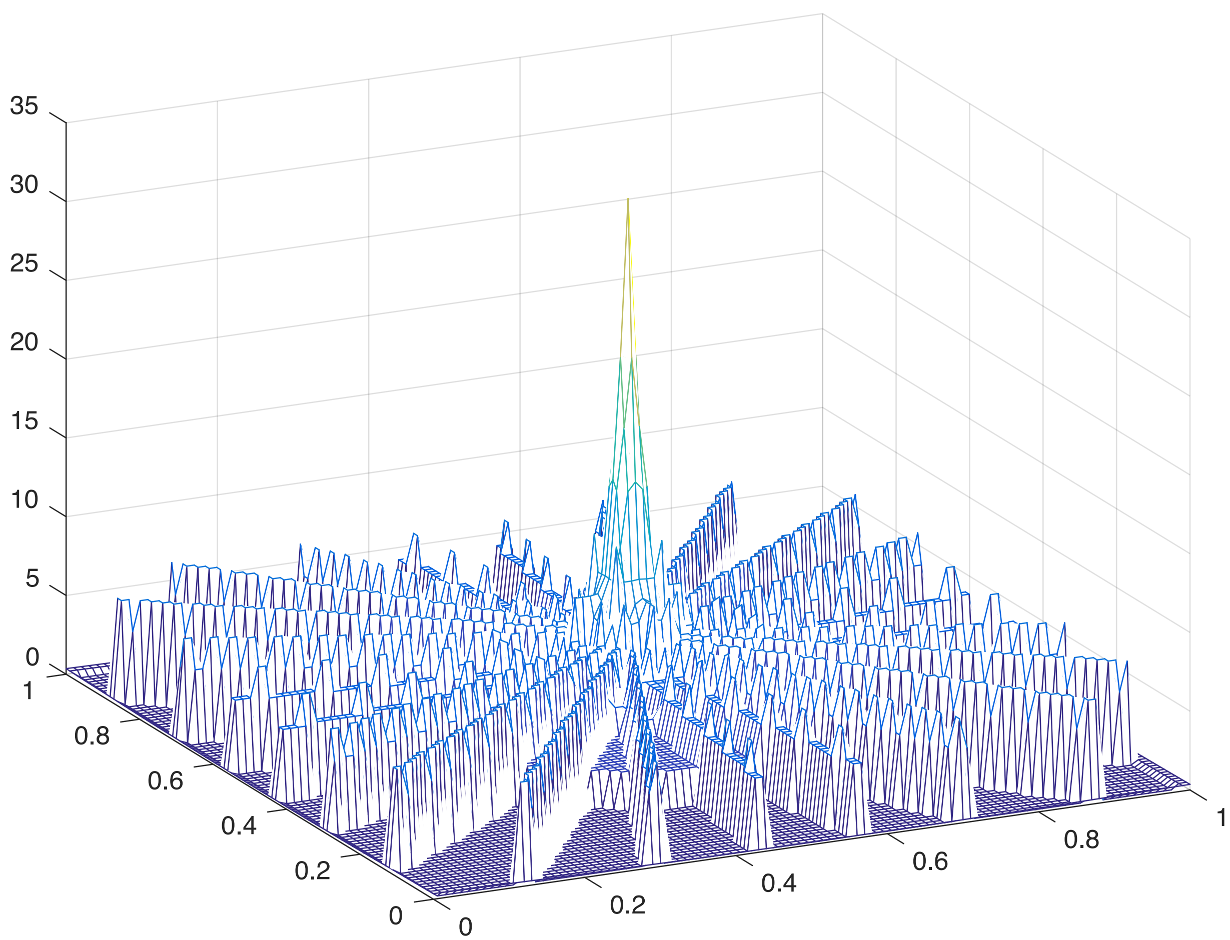


$\sigma_1 \approx |q_1 - q_2| + |q_2 - q_3| + \mathcal{O}(h^2)$

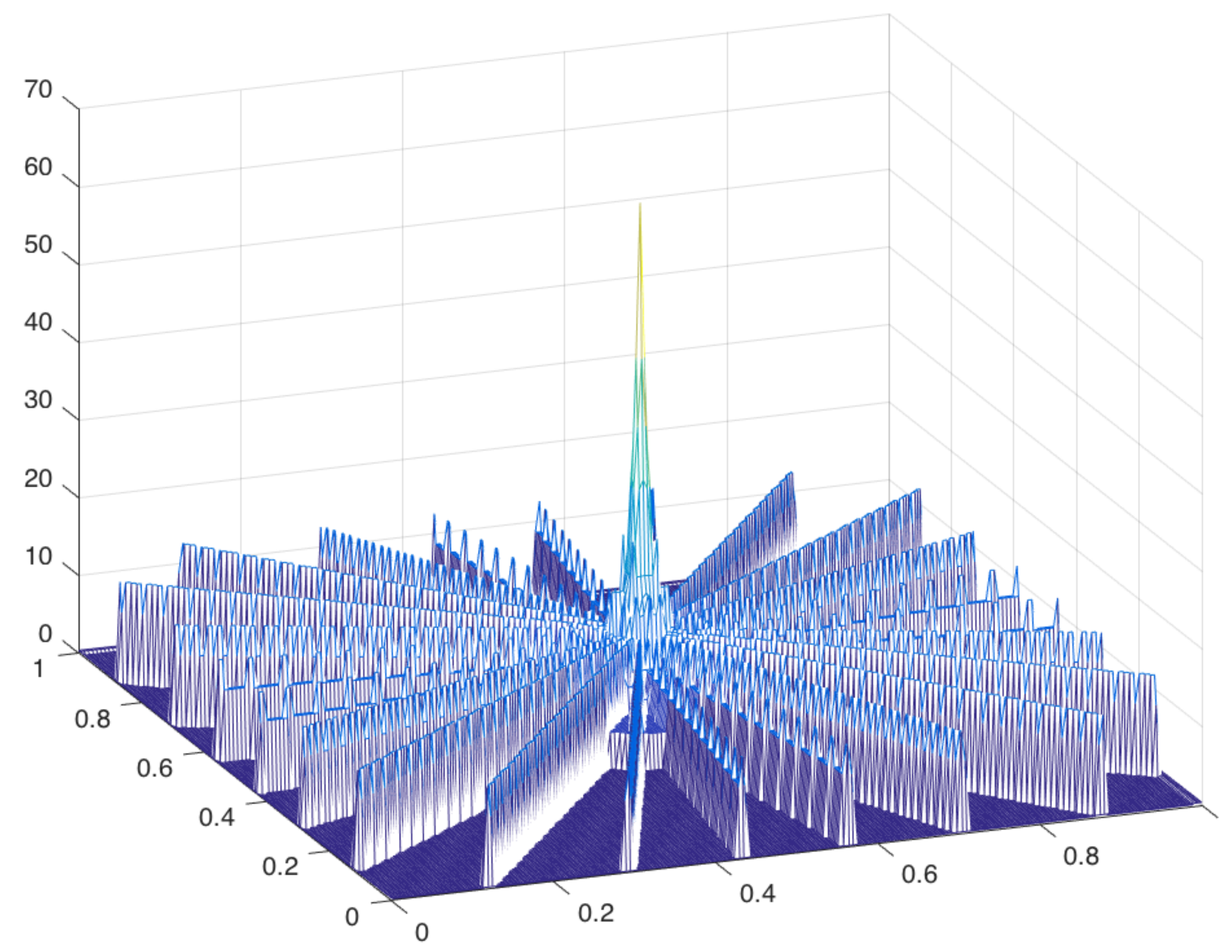
- Similar properties in 3D.

Concentration of the first singular value

25 points from a circle



M=100



M=200

An algorithm that we disliked

- For the regime: $\delta > \Delta x$
- Build a directed graph, connecting the data points based on the analysis by the singular values
- Sum the length of the edges or the area of triangles defined by the graph
- Sums are exact for "nice" data sets: $\delta_{max} < \epsilon < \kappa^{-1}$

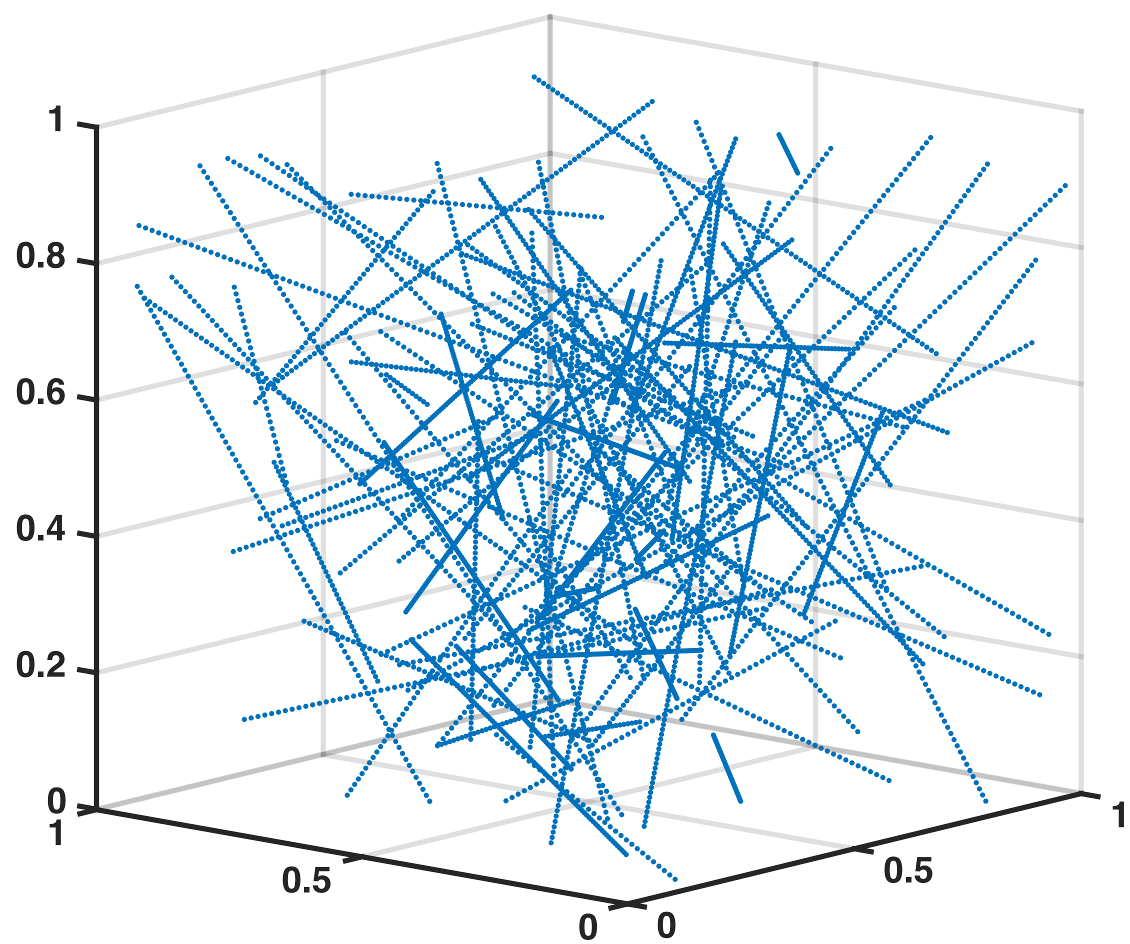
  $\Delta x \ll \delta_{min}$
- More general data require further (ad-hoc) surgeries to the graph

[T-Wu:2015]

The proposed “quadrature”

$$S_N = \sum_{\{x_j \in h\mathbb{Z}^d : \phi_N(x_j) < \epsilon\}} \alpha(\mathbf{r}, \sigma_1, \cdots, \sigma_d) K_\epsilon(|\mathbf{r} - \mathcal{P}_{\Gamma_N}\mathbf{r}|) \prod_{j=1}^{d'} \sigma_j (D_0^{\Delta x} \mathcal{P}_{\Gamma_N}\mathbf{r}) \Delta x^3$$

Total length of 100 filaments



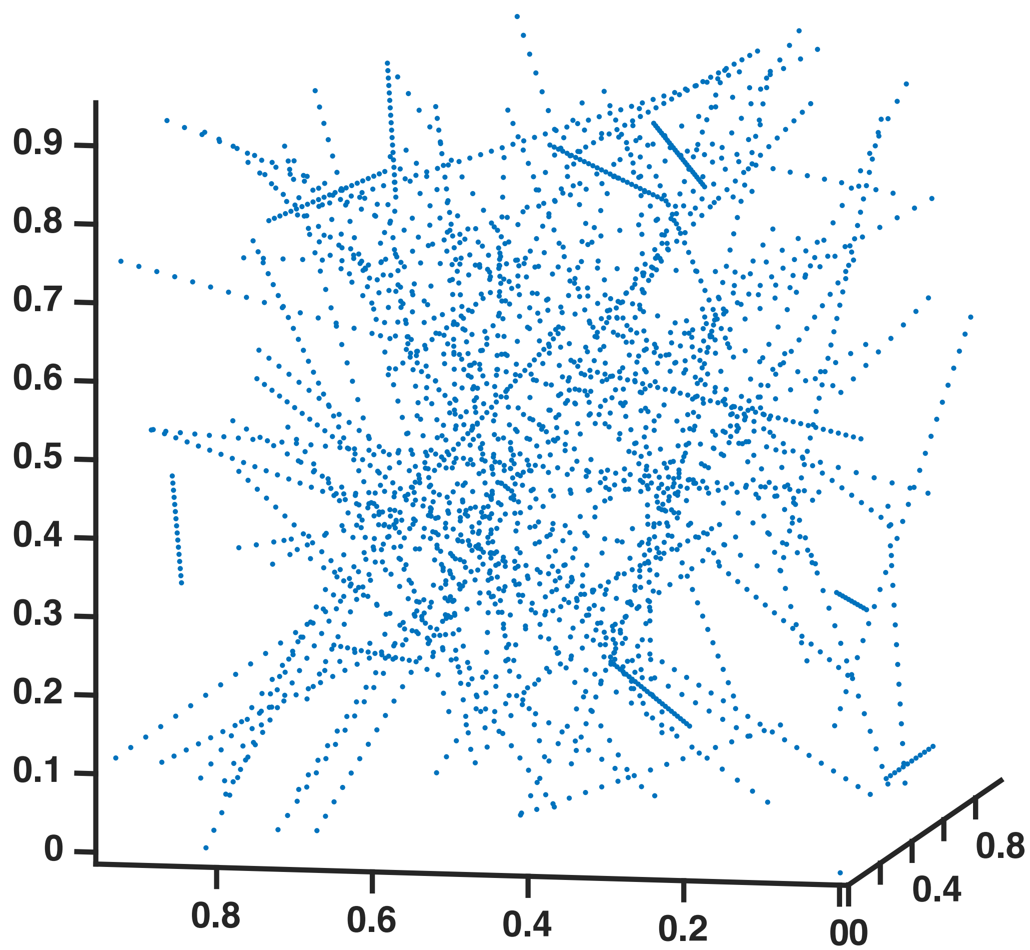
Geometries resolved.

Relative error: 1.8e-3

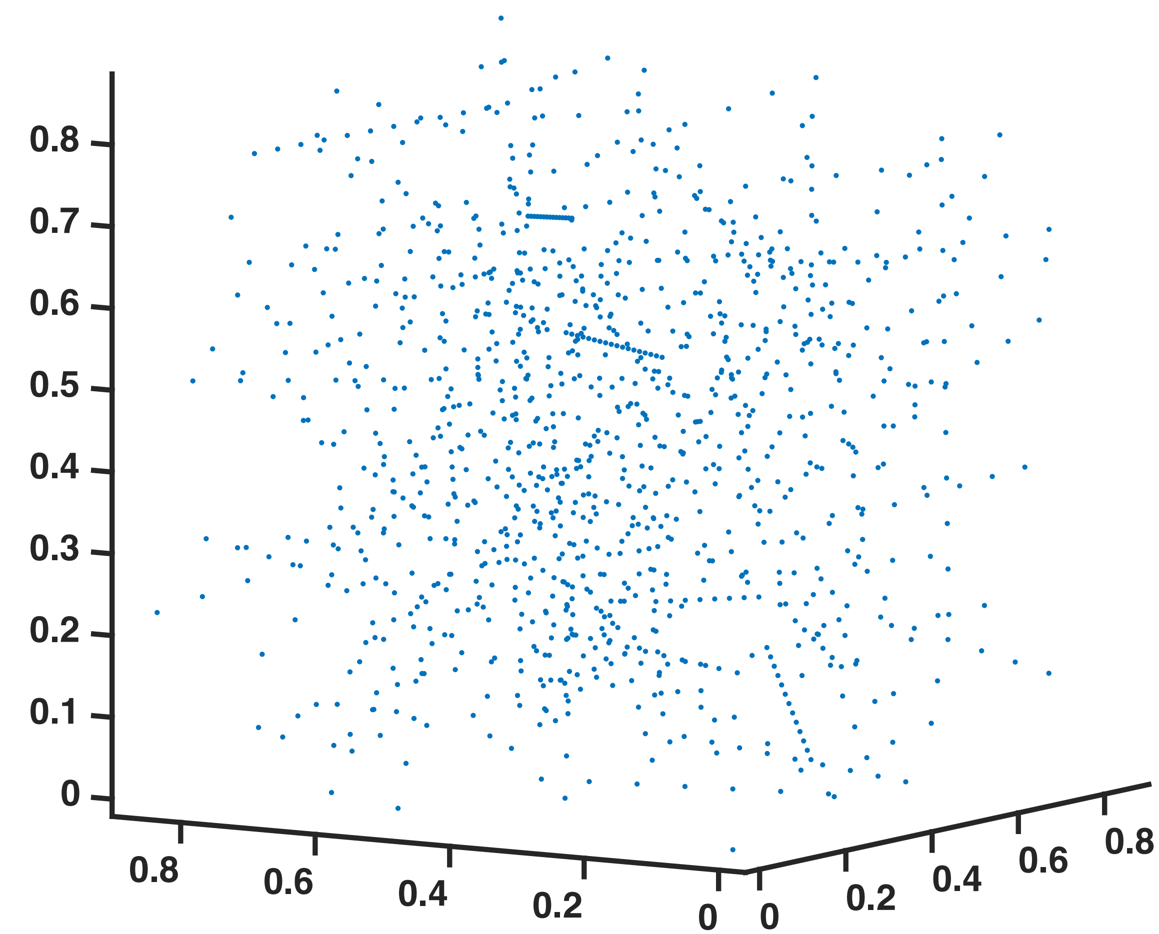
$$\frac{|S_N - |\Gamma||}{|\Gamma|}$$

Total length of 100 filaments

Geometries under-resolved



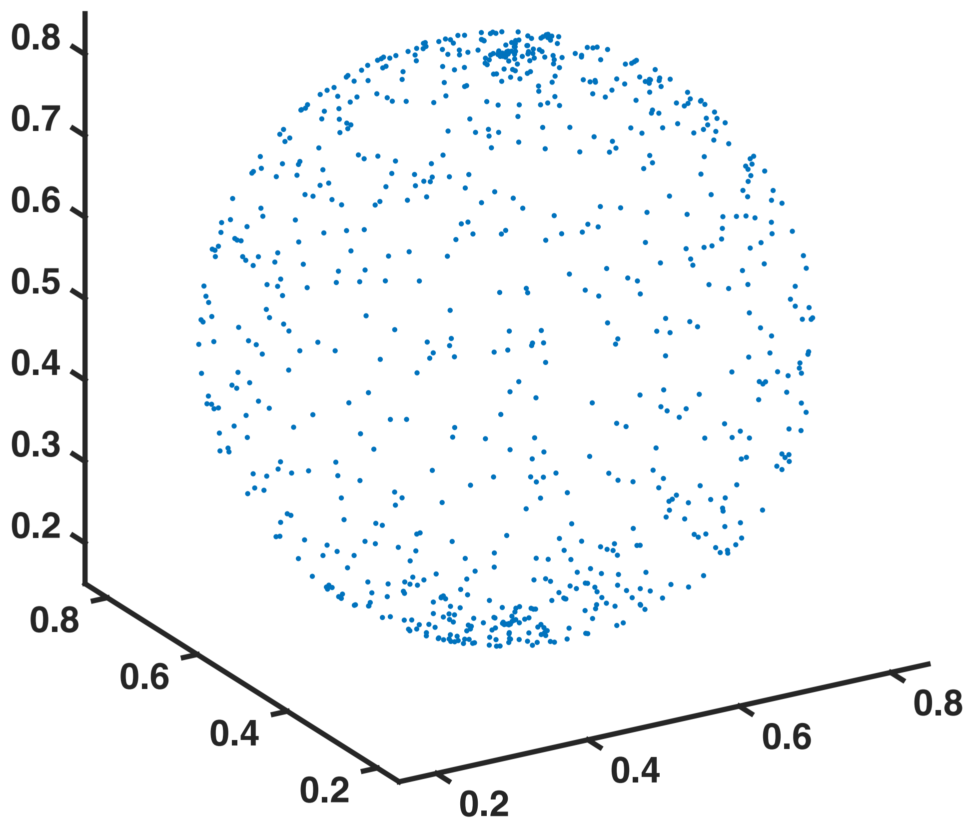
Relative error: 0.03

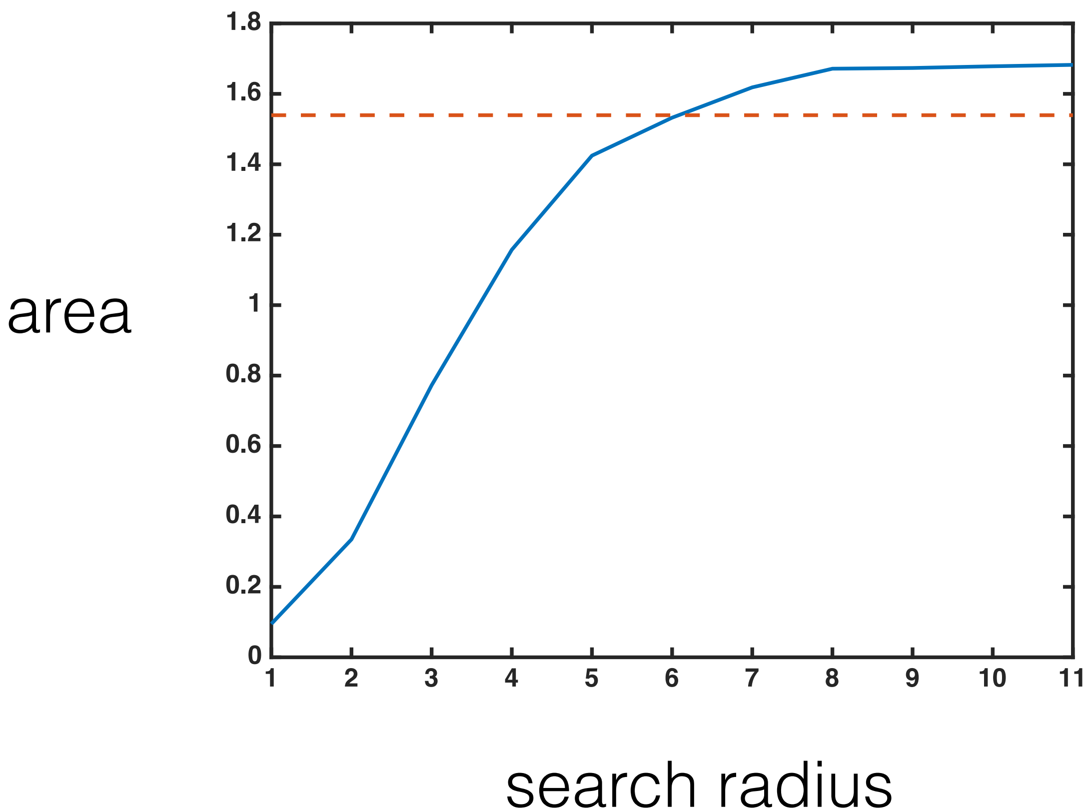


Relative “error”: 0.11

Surface area of a sphere

800 points randomly sampled in the spherical coordinates

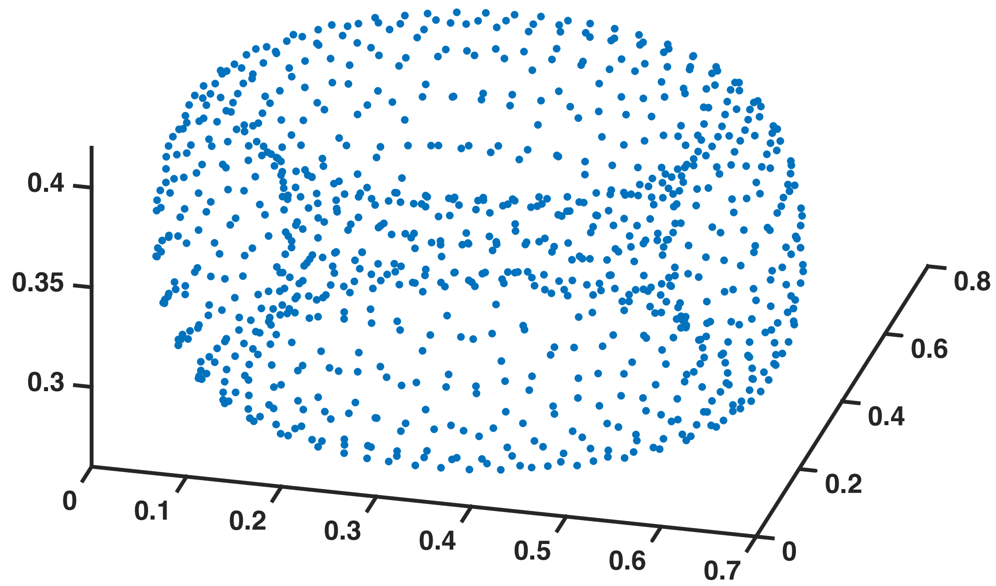




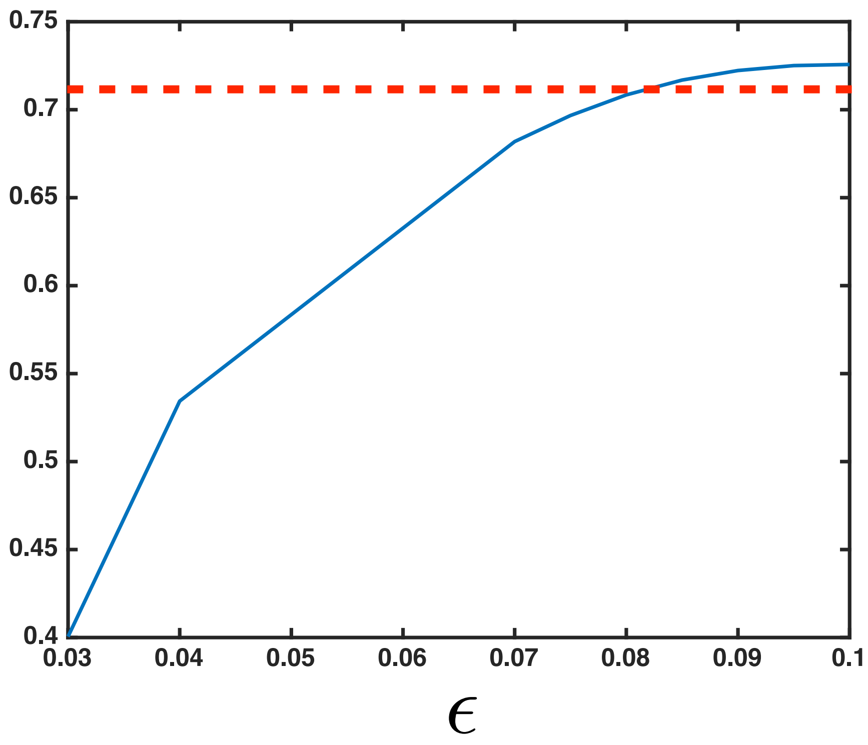
There is very large gap between δ_{min} and δ_{max}

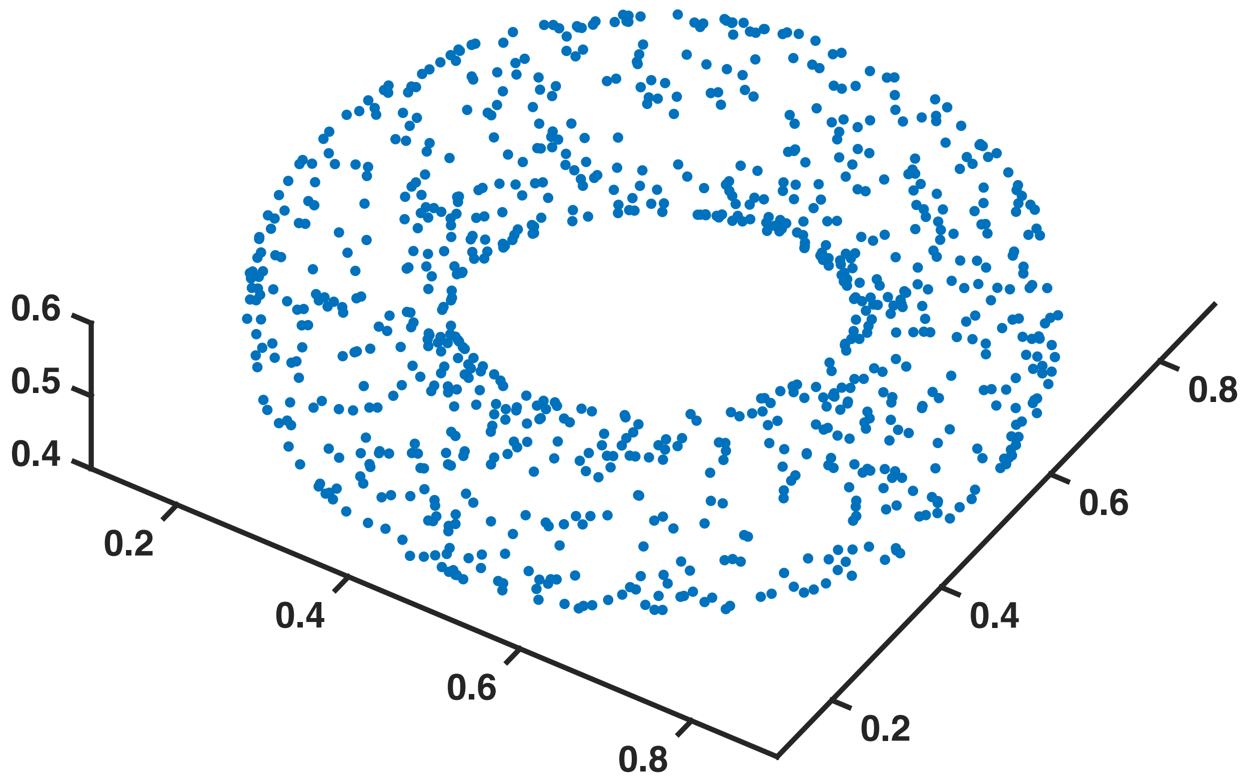
Uniformly sampled ellipse

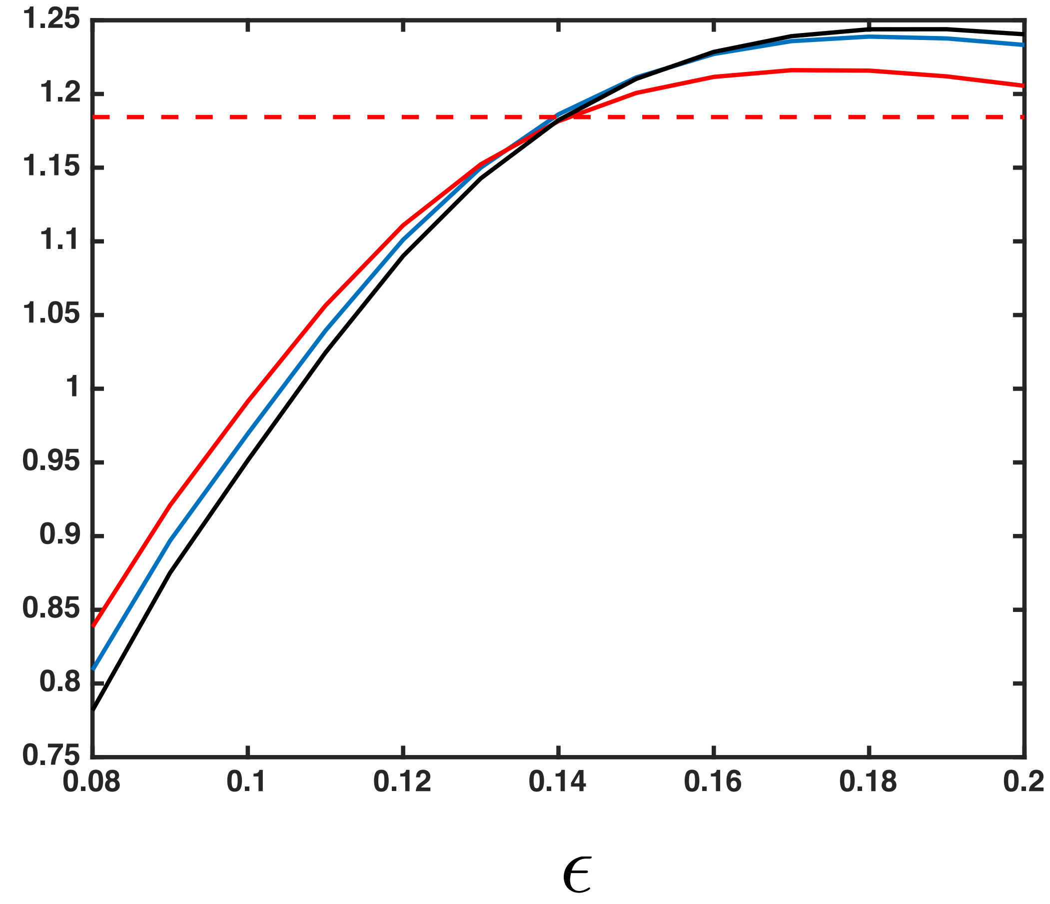
896 points

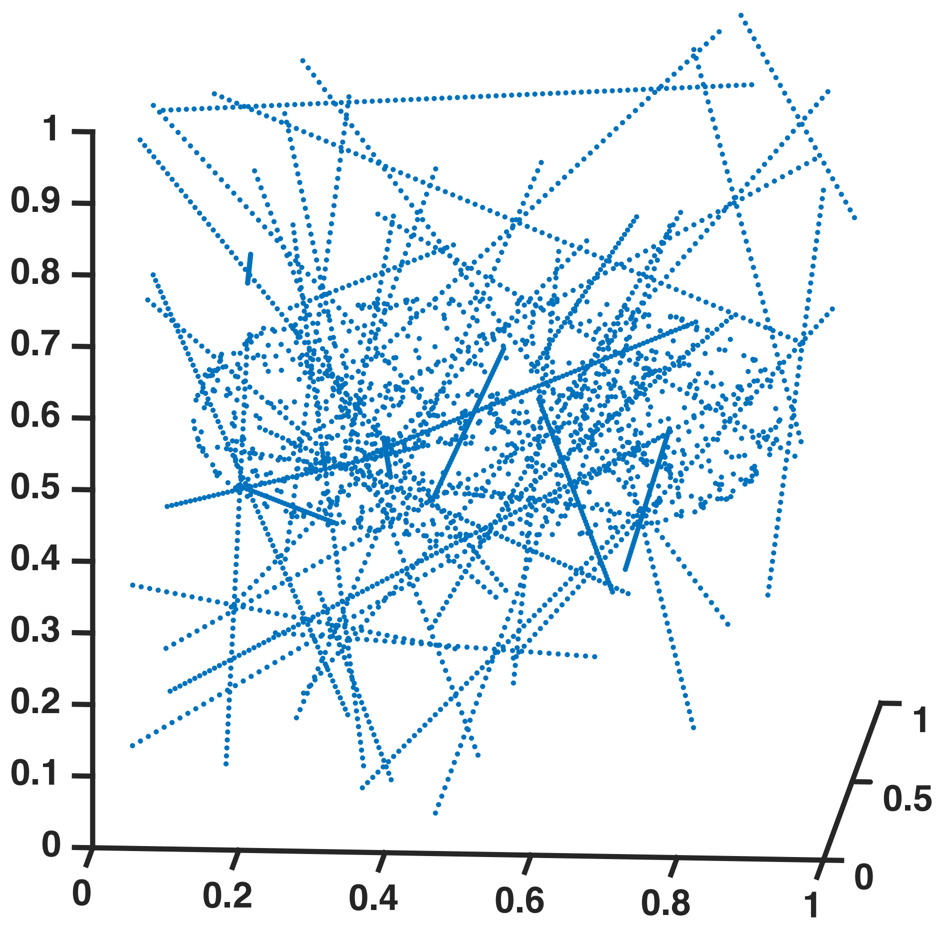


Computed sum




0.6
0.5
0.4
0.2
0.4
0.6
0.8
0.2
0.4
0.6
0.8


1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
0.75
0.08
0.1
0.12
0.14
0.16
0.18
0.2
ϵ



Total number of points: 3654

2754 points from the line segments

900 points on the torus

Relative errors

- in the computed length: 0.096
- in the computed area: 0.013

Summary and discussion

- This is an on-going investigation. Much to be improved.
- An algorithm derived from volume integral formulation of line/surface integrals
- Use closest point mapping, and the singular values of its derivative
- A finite difference stencil on a grid is used to explore the data
- $\sigma_1(D_0^h \mathcal{P}_{\Gamma_N}) \rightharpoonup \alpha\delta_V$ measures with weights discussed above as $\Delta x \to 0$
- Use such insights to improve the summation algorithm
- Solve integral equations/PDEs on point clouds directly?

Point Clouds

September 29, 2017

1 Analysis of the continuous KTT scheme

1.1 Some results on the continuous KTT for dense point clouds

Let Γ be a curve or a surface and let Γ_N be its sampled version. Also we let D_h^0 denote the approximation of the Jacobian matrix of the closest point mapping P_Γ using central differencing.

Lemma 1. *Let Γ be a straight line with angle $0 \le \theta < \frac{\pi}{2}$ from the horizontal. Then the non zero singular value of $D_h^0(P_\Gamma)$ is $\sigma_1 = 1$.*

Proof. We have

$$\begin{aligned} D_h^0(P_\Gamma(\cdot)) &= \left(\frac{P_\Gamma(x_{i+1,j}) - P_\Gamma(x_{i-1,j})}{2h}, \frac{P_\Gamma(x_{i,j+1}) - P_\Gamma(x_{i,j-1})}{2h} \right) \\ &= \frac{1}{2h}\left(\vec{AB}, \vec{CD}\right), \end{aligned}$$

where the point A is the projection of $x_{i-1,j}$, B is the projection of $x_{i+1,j}$, C is the projection of $x_{i,j-1}$ and D is the projection of $x_{i,j+1}$. We denote the distance between the point where the line crosses a vertical grid line and its adjacent grid node above is αh with $0 < \alpha < 1$. We need a picture here. Now

$$\begin{aligned} |\vec{G_3 A}| &= h(1 + \alpha - \tan\theta)\sin\theta \\ |\vec{G_2 D}| &= (2h + \alpha h)\sin\theta \\ |G_2 C| &= \alpha h \sin\theta \\ |\vec{G_1 B}| &= (2h + h\tan\theta - (1-\alpha)h)\sin\theta \\ |\vec{BG_2}| &= \frac{h}{\cos\theta} - |\vec{G_1 B}| \\ |\vec{CG_3}| &= \frac{h}{\cos\theta} - |\vec{G_2 C}| \\ |\vec{G_3 D}| &= |\vec{G_2 D}| - \frac{h}{\cos\theta}. \end{aligned}$$

Now since $|\vec{AB}| = |\vec{AG_3}| + |\vec{G_2G_3}| + |\vec{G_2B}|$ and $|\vec{CD}| = |\vec{CG_3}| + |\vec{G_3D}|$, we obtain

$$\begin{aligned}|\vec{AB}| &= h(1+\alpha-\tan\theta)\sin\theta + \frac{2h}{\cos\theta} - (2h - h\tan\theta - (1-\alpha)h)\sin\theta \\ &= -2h\tan\theta\sin\theta + \frac{2h}{\cos\theta} \\ &= 2h\cos\theta,\end{aligned}$$

and

$$|\vec{CD}| = \frac{h}{\cos\theta} - \alpha h\sin\theta + (2h+\alpha h)\sin\theta - \frac{h}{\cos\theta} = 2h\sin\theta.$$

Thus, if we let $\vec{u} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ be the direction of the line, we have

$$D_h^0(P_\Gamma(\cdot)) = \frac{1}{2h}\left(\vec{AB}, \tan\theta\vec{AB}\right).$$

It follows that the non zero singular value of $D_h^0(P_\Gamma(\cdot))$ is

$$\frac{1}{2h}\sqrt{1+\tan^2\theta}|\vec{AB}| = \frac{1}{2h}\frac{1}{|\cos\theta|}2h\cos\theta = 1,$$

since $0 \le \theta < \frac{\pi}{2}$. For a line, the non zero singular value of P'_Γ is also 1. Thus, if we have a curve with no curvature, the non singular value of the approximate $D_h^0(P_\Gamma)$ is the same as the non zero singular value of the exact P'_Γ . □

Lemma 2. *Let Γ be a straight line with angle $0 \le \theta < \frac{\pi}{2}$ from the horizontal and suppose that we have N points sampled from Γ . We denote that point set by Γ_N. In addition, we assume that the average distance between points in the point set is $\delta > 0$. Then the non zero singular value of $D_h^0(P_{\Gamma_N})$ is*

$$1 + O\left(\frac{\delta}{h}\right).$$

Proof. Let $\vec{u} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ be the direction of the line. Then we have

$$\begin{aligned}P_{\Gamma_N}(x_{i+1,j}) &= P_\Gamma(x_{i+1,j}) + \mu_1\vec{u} \\ P_{\Gamma_N}(x_{i-1,j}) &= P_\Gamma(x_{i-1,j}) + \mu_2\vec{u} \\ P_{\Gamma_N}(x_{i,j+1}) &= P_\Gamma(x_{i,j+1}) + \mu_3\vec{u} \\ P_{\Gamma_N}(x_{i,j-1}) &= P_\Gamma(x_{i,j-1}) + \mu_4\vec{u}.\end{aligned}$$

Then

$$\frac{P_{\Gamma_N}(x_{i+1,j}) - P_{\Gamma_N}(x_{i-1,j})}{2h} = \frac{P_\Gamma(x_{i+1,j}) - P_\Gamma(x_{i-1,j})}{2h} + \frac{\mu_1-\mu_2}{2h}\vec{u},$$

and

$$\frac{P_{\Gamma_N}(x_{i,j+1}) - P_{\Gamma_N}(x_{i,j-1})}{2h} = \frac{P_\Gamma(x_{i,j+1}) - P_\Gamma(x_{i,j-1})}{2h} + \frac{\mu_3 - \mu_4}{2h}\vec{u}.$$

Thu

$$\begin{aligned}
D_h^0(P_{\Gamma_N}(x_{i,j})) &= \left(\frac{\vec{AB}}{2h} + \frac{\mu_1 - \mu_2}{2h}\vec{u}, \frac{\vec{CD}}{2h} + \frac{\mu_3 - \mu_4}{2h}\vec{u}\right) \\
&= \frac{1}{2h}\left(\vec{AB} + (\mu_1 - \mu_2)\vec{u}, \tan\theta\vec{AB} + (\mu_3 - \mu_4)\vec{u}\right) \\
&= \frac{1}{2h}\left((2h\cos\theta + (\mu_1 - \mu_2))\vec{u}, (2h\sin\theta + (\mu_3 - \mu_4))\vec{u}\right) \\
&= \left(\left(\cos\theta + \frac{\mu_1 - \mu_2}{2h}\right)\vec{u}, \left(\sin\theta + \frac{\mu_3 - \mu_4}{2h}\right)\vec{u}\right).
\end{aligned}$$

Thus, the non zero value of $D_h^0(P_{\Gamma_N}(x_{i,j}))$ is

$$\begin{aligned}
\sigma_1 &= \sqrt{(\cos^2\theta + \sin^2\theta)(\left(\cos\theta + \frac{\mu_1 - \mu_2}{2h}\right)^2 + \left(\sin\theta + \frac{\mu_3 - \mu_4}{2h}\right)^2)} \\
&= \sqrt{1 + \frac{\mu_1 - \mu_2}{h}\cos\theta + \frac{\mu_3 - \mu_4}{h}\sin\theta + \left(\frac{\mu_1 - \mu_2}{2h}\right)^2 + \left(\frac{\mu_3 - \mu_4}{2h}\right)^2}.
\end{aligned}$$

We have $\mu_i < \delta$ for $1 \leq i \leq 4$. Now if we assume that $\delta \ll h$, i.e. if the point cloud is dense with respect to the grid, then $\frac{\mu_1 - \mu_2}{2h} < \frac{\delta}{h} \ll 1$. Similarly for $\frac{\mu_3 - \mu_4}{h}$. Thus we can do a Taylor series expansion of σ_1 above to obtain

$$\sigma_1 = 1 + \frac{1}{2}\cos\theta\frac{\mu_1 - \mu_2}{h} + \frac{1}{2}\sin\theta\frac{\mu_3 - \mu_4}{h} + O\left(\frac{\delta}{h}\right)^2 \approx 1 + O\left(\frac{\delta}{h}\right).$$

Now if $h < \delta$, then the stencil will tend to see discrete points instead of the underlying curve. Mathematically, this means that the Taylor expansion is not valid and there is a larger error between the exact singular value (which is 1) and σ_1. □

Lemma 3. *Let Γ be the semi circle centered at the origin with radius R. Then the non zero singular value of $D_h^0(P_\Gamma)$ is*

$$\sigma_1 = \frac{R}{R + \eta} + O(h^2),$$

where the non zero singula value of P'_Γ is $\frac{R}{R+\eta}$.

Proof. For simplicity in the calculations and WLOG we take the grid node $x_{i,j}$ to be the point $(0, R+\eta)$. Then we have $P_\Gamma(x_{i,j-1}) = P_\Gamma(x_{i,j+1}) = (0, R)$. Also we have

$$P_\Gamma(x_{i-1,j}) = \left(R\sin\left(\arctan\left(\frac{h}{R+h}\right)\right), R\cos\left(\arctan\left(\frac{h}{R+h}\right)\right)\right)$$
$$P_\Gamma(x_{i+1.j}) = \left(-R\sin\left(\arctan\left(\frac{h}{R+h}\right)\right), R\cos\left(\arctan\left(\frac{h}{R+h}\right)\right)\right),$$

which can be rewritten as

$$P_\Gamma(x_{i-1,j}) = \left(\frac{Rh}{\sqrt{(R+\eta)^2+h^2}}, \frac{R^2+R\eta}{\sqrt{(R+\eta)^2+h^2}}\right)$$
$$P_\Gamma(x_{i+1.j}) = \left(\frac{-Rh}{\sqrt{(R+\eta)^2+h^2}}, \frac{R^2+R\eta}{\sqrt{(R+\eta)^2+h^2}}\right),$$

leading to

$$P_\Gamma(x_{i+1,j}) - P_\Gamma(x_{i-1,j}) = \begin{pmatrix} \frac{2Rh}{\sqrt{(R+\eta)^2+h^2}} \\ 0 \end{pmatrix},$$

and therefore to

$$D_h^0(P_\Gamma(x_{i,j})) = \begin{pmatrix} \frac{R}{\sqrt{(R+\eta)^2+h^2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the non zero singular value of $D_h^0(P_\Gamma(x_{i,j}))$ is

$$\sigma_1 = \frac{R}{\sqrt{(R+\eta)^2+h^2}}.$$

Since $h \to 0$, we can use a Taylor expansion on σ_1 to obtain

$$\sigma_1 = \frac{R}{R+\eta}\left(1 - \frac{1}{2}\frac{h^2}{(R+\eta)^2} + O(h^4)\right) = \frac{R}{R+\eta} + O(h^2).$$

□

Lemma 4. *Let Γ be the semi circle centered at the origin with radius R and suppose that we have N points sampled from Γ. We denote that point set by Γ_N. In addition, we assume that the average distance between points in the point set is $\delta > 0$. Then the non zero singular value of $D_h^0(P_{\Gamma_N})$ is*

$$\sigma_1 \approx \frac{R}{R+\eta} + O(h^2) + O\left(\frac{\delta}{h}\right).$$

Proof. For simplicity in the calculations and WLOG we take the grid node $x_{i,j}$ to be the point $(0, R+\eta)$. η also corresponds to the distance between the grid point $x_{i,j}$ and Γ. Then we have $P_{\Gamma_N}(x_{i,j-1}) = P_{\Gamma_N}(x_{i,j+1}) = P_\Gamma(x_{i,j-1}) + \mu_1\vec{v}$, where $\vec{v}$ is a unit vector the direction of which is the same as the vector between the point $P_\Gamma(x_{i,j-1})$ which is the exact closest point to either $x_{i,j-1}$ or $x_{i,j+1}$ and the closest point on the point set $P_{\Gamma_N}(x_{i,j-1})$ (or $P_{\Gamma_N}(x_{i,j+1})$).

We also have

$$\begin{aligned} P_{\Gamma_N}(x_{i+1,j}) &= P_\Gamma(x_{i+1,j}) + \mu_2 \vec{w}, \\ P_{\Gamma_N}(x_{i-1,j}) &= P_\Gamma(x_{i-1,j}) + \mu_3 \vec{z}, \end{aligned}$$

where both $\vec{w}$ and $\vec{z}$ are unit vectors the direction of which is the same as the direction of the vector between the point $P_{\Gamma_N}(x_{i+1,j})$ and $P_\Gamma(x_{i+1,j})$ (respectively between $P_{\Gamma_N}(x_{i-1,j})$ and $P_\Gamma(x_{i-1,j})$). Thus

$$\begin{aligned} \frac{P_{\Gamma_N}(x_{i+1,j}) - P_{\Gamma_N}(x_{i-1,j})}{2h} &= \frac{P_\Gamma(x_{i+1,j}) - P_\Gamma(x_{i-1,j})}{2h} + \mu_2\vec{w} - \mu_3\vec{z}, \\ \frac{P_{\Gamma_N}(x_{i,j+1}) - P_{\Gamma_N}(x_{i,j-1})}{2h} &= \vec{0}. \end{aligned}$$

Note that μ_2 is the length along a straight line between $P_{\Gamma_N}(x_{i+1,j})$ and $P_\Gamma(x_{i+1,j})$, and similarly for μ_3. If we want to know the length between $P_{\Gamma_N}(x_{i+1,j})$ and $P_\Gamma(x_{i+1,j})$ along the curve (here the semi circle) we can use the Al Kashi Theorem to obtain

$$s_2 = R \arccos\left(1 - \frac{\mu_2^2}{2R^2}\right).$$

Since μ_2 is small, we can do a Taylor expansion of that expression to obtain

$$s_2 = \sqrt{2\frac{\mu_2^2}{2R^2}} + O\left(\mu_2^3\right) = \mu_2 + O(\mu_2^3),$$

which is good because we expect s_2 to be of order μ_2. Same result for μ_3 and s_3. Thus we don't lose any relevant information by looking at distances along straight lines instead of along the curve.

Now let $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then we can write

$$D_h^0(P_{\Gamma_N}(x_{i,j})) = \begin{pmatrix} \frac{R}{\sqrt{(R+\eta)^2+h^2}} + \frac{1}{2h}\left(\mu_2 w_1 - \mu_2 z_1\right) & 0 \\ \frac{\mu_2 w_2 - \mu_3 z_2}{2h} & 0 \end{pmatrix}.$$

It follows that the non zero singular value of $D_h^0(P_{\Gamma_N}(x_{i,j}))$ can be written as

$$\begin{aligned} \sigma_1 &= \sqrt{\left(\frac{R}{\sqrt{(R+\eta)^2+h^2}} + \frac{1}{2h}\left(\mu_2 w_1 - \mu_2 z_1\right)\right)^2 + \left(\frac{\mu_2 w_2 - \mu_3 z_2}{2h}\right)^2} \\ &= \sqrt{\left(\frac{R}{\sqrt{(R+\eta)^2+h^2}}\right)^2 + 2\frac{R}{\sqrt{(R+\eta)^2+h^2}}\frac{1}{2h}\left(\mu_2 w_1 - \mu_2 z_1\right) + \left(\frac{\mu_2 w_1 - \mu_3 z_1}{2h}\right)^2 + \left(\frac{\mu_2 w_2 - \mu_3 z_2}{2h}\right)^2} \\ &= \frac{R}{\sqrt{(R+\eta)^2+h^2}}\sqrt{1 + \frac{\mu_2 w_1 - \mu_3 z_1}{hR}\sqrt{(R+\eta)^2+h^2} + \frac{(R+\eta)^2+h^2}{R^2}\left(\left(\frac{\mu_2 w_1 - \mu_3 z_1}{2h}\right)^2 + \left(\frac{\mu_2 w_2 - \mu_3 z}{2h}\right.\right.} \end{aligned}$$

If we assume that the point cloud is dense relative to the grid, i.e. $\mu_i \ll h$, we can use a Taylor series expansion to simplify the above formula and see the dominating term. In that case, we obtain

$$\begin{aligned}\sigma_1 &\approx \frac{R}{\sqrt{(R+\eta)^2+h^2}}\left(1+\frac{1}{2}\left(\frac{\mu_2 w_1-\mu_3 z_1}{hR}\sqrt{(R+\eta)^2+h^2}+\frac{(R+\eta)^2+h^2}{R^2}\left(\left(\frac{\mu_2 w_1-\mu_3 z_1}{2h}\right)^2+\left(\frac{\mu_2 w_2 -}{2h}\right.\right.\right.\right. \\ &\approx \frac{R}{\sqrt{(R+\eta)^2+h^2}}+\frac{\mu_2 w_1-\mu_3 z_1}{2h} \\ &\approx \frac{R}{R+\eta}+O(h^2)+O\left(\frac{\delta}{h}\right),\end{aligned}$$

since μ_i are of the same order as δ, the average distance between points in the clouds. Of course, if $h > \mu_i$, then the above Taylor expansion does not hold. □

Lemma 5. *Let Γ be the semi circle centered at the origin with radius R and suppose that we have N points sampled from Γ. We denote that point set by Γ_N. In addition, we assume that the average distance between points in the point set is $\delta > 0$ and that the point set is dense with respect to the grid, i.e. $\delta \ll h$. Then the error in the KTT formulation is*

$$O(h^2)+O\left(\frac{\delta}{h}\right).$$

Proof. For Γ, the KTT formulation can be written as

$$h^2 \sum_{x:d(x)\leq\epsilon} f(d(x))\sigma(x)K_\epsilon(d(x)),$$

where $\epsilon > 0$ represents the half width of the tubular neighborhood around Γ, f is the function we are integrating along Γ , d is the distance function to Γ , $\sigma(x)$ is the non zero singular value of the Jacobian matrix of the closest point mapping at x and K_ϵ is an averaging kernel. For the point set Γ_N we will be summing all grid points that are located at a distance less or equal to the point set. In general, this will not be a tube around the point cloud but if the pont cloud is dense enough (which is our assumption) it will. We have the following:

$$K_\epsilon(d(x)) = \frac{1}{\epsilon}K\left(\frac{d(x)}{\epsilon}\right),$$

where $K : [0,1] \mapsto \mathbb{R}$ is bounded. Now at any grid point, from the previous theorem we know that

$$\sigma(x_{i,j}) = \sigma_1 = \frac{R}{R+\eta}+O(h^2)+O\left(\frac{\delta}{h}\right),$$

thus it follows that if we assume f is continuous, we have

$$\begin{aligned} KTT &\approx \frac{h^2}{\epsilon} \sum_{x_i : d(x_i) \le \epsilon} \sigma_1 \\ &\approx \frac{h^2}{\epsilon} \sum_{x_i : d(x_i) \le \epsilon} \left(\frac{R}{R+\eta} + O(h^2) + O\left(\frac{\delta}{h}\right)\right). \end{aligned}$$

Now how many points do we have in the "tubular neighborhood" of Γ_N? Let n_g be the number of grid points in the set . If we assume that the point cloud is dense with respect to the grid size h, then can consider the tube around Γ_N to be similar to the tube around Γ. Then in that case (since we have a semi circle), we have

$$2\epsilon 2\pi R + \pi\epsilon^2 \approx n_g h^2,$$

and thus $n_g \approx O\left(\frac{\epsilon}{h^2}\right) + O\left(\frac{\epsilon^2}{h^2}\right)$. If we assume that $\epsilon = O(h)$, then $\frac{\epsilon}{h^2}$ dominates and thus

$$n_g \approx O\left(\frac{\epsilon}{h^2}\right).$$

Thus we obtain

$$KTT \approx \frac{h^2}{\epsilon} \left(\sum_{x_i : d(x_i) \le \epsilon} \frac{R}{R+\eta} + \frac{\epsilon}{h^2} \left(O(h^2) + O\left(\frac{\delta}{h}\right)\right)\right) \approx \frac{h^2}{\epsilon} \sum_{x_i : d(x_i) \le \epsilon} \frac{R}{R+\eta} + O(h^2) + O(\frac{\delta}{h}).$$

□

1.2 Looking at a hole in 2D

We are in the regime $h < \delta < \epsilon$.

1.2.1 A line

Suppose now that Γ is a line and consider two points in the point set Γ_N. We want to estimate the order of accuracy of KTT for grid points that see those two points. WLOG, we assume $x_1 = (-\frac{\delta}{2}, 0)$ and $x_2 = (\frac{\delta}{2}, 0)$. Since the search radius is $\epsilon > \delta$, the grid points that see these two points will be inside the intersection of the circle C_1 centered at x_1 with radius ϵ and the circle C_2 centered at x_2 with radius ϵ. We want to estimate the number of grid points that only see the two points x_1 and x_2, i.e. the grid points that straddle the Voronoi boundary between the Voronoi cell that contains x_1 and the Voronoi cell that contains x_2. For the line, the Voronoi boundary will be the line perpendicular to the line segment $x_1 x_2$. To estimate the number of grid points that straddle this Voronoi boundary we first estimate the area of the region inside $C_1 \cap C_2$ centered at $(0, 0)$ and with width h in the horizontal direction. This area can be calculated as

$$
\begin{aligned}
A &= \int_0^h \sqrt{\epsilon^2 - (x + \frac{\delta}{2})^2} dx \\
&= \frac{\epsilon^2}{2}\left(\arcsin\left(\frac{h}{\epsilon} + \frac{\delta}{2\epsilon}\right) - \arcsin\left(\frac{\delta}{2\epsilon}\right)\right) + \frac{\epsilon^2}{4}\left(\sin\left(2\arcsin\left(\frac{h}{\epsilon} + \frac{\delta}{2\epsilon}\right)\right) - \sin\left(2\arcsin\left(\frac{\delta}{2\epsilon}\right)\right)\right)
\end{aligned}
$$

Note that this area A is only a quarter of the total area where the grid points straddling the Voronoi boundary are located. Thus, an estimate for the number of grid points straddling the Voronoi boundary is

$$
N = \left\lfloor \frac{4A}{h^2} \right\rfloor.
$$

Now, since $h < \delta < \epsilon$, we have $\frac{h}{\epsilon} < 1$ and $\frac{\delta}{\epsilon} < 1$. Suppose that $h < \delta < \epsilon$ are such that $\frac{h}{\epsilon} + \frac{\delta}{2\epsilon} < 1$, then we use the following series expansions

$$
\arcsin(x) = x + O(x^3) \sin(2\arcsin(x)) = 2x + O(x^3),
$$

we obtain the following estimate for A :

$$
A = \frac{\epsilon^2}{2}\left(\frac{h}{\epsilon} + \frac{\delta}{2\epsilon} - \frac{\delta}{2\epsilon} + O\left(\left(\frac{\delta}{\epsilon}\right)^3\right)\right) + \frac{\epsilon^2}{4}\left(2\left(\frac{h}{\epsilon} + \frac{\delta}{2\epsilon}\right) - 2\frac{\delta}{2\epsilon} + O\left(\frac{\delta^3}{\epsilon}\right)\right) = \epsilon h + O\left(\frac{\delta^3}{\epsilon}\right)
$$

Thus an estimate for the number of grid points is

$$
N = 4\left\lfloor \frac{\epsilon}{h} + O\left(\frac{\delta^3}{h^2\epsilon}\right) \right\rfloor.
$$

Now it is easy to see that the non zero singular value of $D_h^0(P_{\Gamma_N}(x_{i,j}))$ is

$$
\sigma_1 = \frac{|\vec{BA}|}{2h},
$$

where $x_{i,j}$ is a grid point straddling the Voronoi boundary. Thus when using KTT we obtain the following estimate

$$
KTT = O\left(\frac{h^2}{\epsilon}\sigma_1 N\right) = O\left(\frac{h^2}{\epsilon}\frac{\delta}{h}\frac{\epsilon}{h}\right) = O(\delta).
$$

So KTT calculates an expression of order δ which is the length of the line between those two points x_1 and x_2.

1.2.2 A circle of radius R

Now we want to look at the same situation of a hole but with a curve that has curvature. For simplicity, let's consider Γ to be a circle of radius R, and consider two consecutive points on the point set Γ_N. Assume that the length of the line segment between x_1 and x_2 is δ. We will try to estimate the number of points that only see those two points on Γ_N. The Voronoi boundary is part of the semi-line perpenticular to the line segment x_1x_2 starting at the center of the circle. To estimate the number of grid points that straddle the Voronoi boundary as again estimate the area of the region where these grid points are located. Note that the calculation is the same as previously for the area that is "outside" the circle. For the area "inside" the circle, it is a little bit different if $R < \epsilon$, i.e. the search radius is larger than the curvature of the curve.

If $R > \epsilon$, then the calculations are the same as above and the calculations at these grid points using KTT are $O(\delta)$.

If $R < \epsilon$, then we need to alter our calculations of the area "inside" the circle. In that case, instead of looking at the circles of radius ϵ centered at x_1 and x_2 , we look at the circles of radius R centered at x_1 and x_2.

The calculations are thus similar as above but ϵ is now replaced by R. Thus we get the following estimate for the total area (corresponding to $4A$ for the line):

$$A = 2\epsilon h + 2Rh + O\left(\frac{\delta^3}{\epsilon}\right).$$

Thus an estimate for the number of grid points that straddle the Voronoi boundary is

$$N = \left\lfloor \frac{A}{h^2} \right\rfloor = 2\frac{\epsilon}{h} + 2\frac{R}{h} + O\left(\left(\frac{\delta^3}{\epsilon h^2}\right)\right).$$

Now, similar to the case of a line (since the grid points do not see the curvature of the underlying curve), the non zero singular value of $D_h^0(P_{\Gamma_N}(x_{i,j}))$ is still

$$\sigma_1 = \frac{|\vec{BA}|}{2h},$$

where $x_{i,j}$ is a grid point straddling the Voronoi boundary. Thus when using KTT, we obtain the following estimate

$$KTT = O\left(\frac{h^2}{\epsilon}\sigma_1 N\right) = O\left(\frac{h^2}{\epsilon}\frac{\delta}{h}\left(\frac{\epsilon}{h} + \frac{R}{h}\right)\right) = O\left(\delta\right) + O\left(\frac{\delta R}{\epsilon}\right).$$

Since $R < \epsilon$, we have that $\frac{R}{\epsilon} < 1$ and therefore

$$KTT = O(\delta)$$

just like the line. So even with curvature, KTT calculates an expression of order δ which is the length of the line between the two points x_1 and x_2 on the point set Γ_N.