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Experimental demonstration of quantum gain in a zero-sum game

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Abstract. We propose and experimentally demonstrate a zero-sum game that is in a fair Nash equilibrium for classical players, but has the property that a quantum player can always win using an appropriate strategy. The gain of the quantum player is measured experimentally for different quantum strategies and input states. It is found that the quantum gain is maximized by a maximally entangled state, but does not decrease to zero when entanglement disappears. Instead, it links with another kind of quantum correlation described by discord for the qubit case and the connection is demonstrated both theoretically and experimentally.

Game theory describes competition and collaboration between a number of agents and has found important applications in several branches of science [1]. With the development of quantum information theory, it has been noted that quantum mechanics can help game players under certain circumstances [2–5]. A central concept in classical game theory is the Nash equilibrium

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(or its various extensions, such as the correlated equilibrium) [1]. An equilibrium is a solution to a game where no player can gain by changing his strategy unilaterally when the other players maintain their strategies [1]. However, in a quantum version of the game, even in a Nash equilibrium where no classical players can benefit by changing their strategies unilaterally, a player who is allowed to play quantum mechanically can gain by unilaterally varying his strategy. In this paper, we propose and experimentally demonstrate a zero-sum quantum game that shows this kind of advantage for a quantum player who can win from a classical Nash equilibrium.

Quantum games have attracted significant interest in recent years. Under a model first proposed by Eisert *et al* [3] where a referee provides input states to the quantum players and collects output states for measurement, it is found that an undesirable equilibrium in a classical game (the prisoner's dilemma) can be avoided if each game player is allowed to choose from only a restricted set of quantum operations. This model has inspired many works and has been generalized to the multi-player case [4]. The ideas of these games have been demonstrated in nuclear magnetic resonance [6] and optical experiments [7, 8].

The game considered in this paper belongs to a different class. Firstly, we take a different model for the quantization of the game where the referee provides quantum correlated input states to the game players, but does not collect output states for measurement [5]. Instead, the game players directly choose a strategy to measure the state and get their payoff according to the measurement outcomes based on the classical payoff matrix. Following the convention in [5], a Nash or correlated equilibrium corresponds to an input state in which no player can apply any operation to achieve a positive gain, provided that all the other players directly measure in the computational basis. This definition is an analogue of the notion in the classical games where a referee provides classically correlated signals to the game players and an equilibrium corresponds to a certain correlated input signal [1]. In the language of game theory, this model corresponds to the strategic-form game, whereas the previous quantization model actually corresponds to the extensive-form games in which the game players choose a unitary operation instead of a strategy [3–5]. Secondly, we consider a zero-sum game which emphasizes competition in game theory. The gain of one player is necessarily at the expense of another. This helps us to show the advantage of a quantum player compared with the other classical players. In this regard, the game demonstrated here is similar in spirit to the pioneering penny-matching game proposed by Meyer [2]. The difference is that Meyer's game has an extensive form where the quantum players play two rounds while the classical player applies only one step of operation. Here, to ensure fairness in the game protocol, we assume that the referee sends only symmetric states to the game players, and both of the classical and the quantum players only have one round to choose their strategies with symmetric payoff matrix. So the advantage achieved by the quantum player is a hallmark of the quantum nature of the game.

We experimentally demonstrate this zero-sum quantum game in which the referee sends out quantum correlated photons to the game players. We choose the states sent out by the referee in such a way that they always correspond to a classical Nash equilibrium with each player having a 50% chance of winning when both the players play classically. However, if one player is allowed to play quantum mechanically, he can achieve the full quantum advantage by winning almost certainly when the state sent out by the referee has maximum entanglement. We demonstrate experimentally a winning chance of $(94.3 \pm 1.3)\%$ for the quantum player under an optimized strategy. To further understand the quantum nature of this game, we then investigate the origin of the gain of the quantum player. Is that entanglement in the input state or something

else? We show that the maximum gain can only be achieved under a maximally entangled state; however, the gain does not decrease to zero when entanglement disappears. Instead, it links with another kind of quantum correlation characterized by discord [9]. For a game with qubits, the gain disappears when the quantum discord goes to zero. The discord is an important quantity introduced to characterize quantum correlation beyond entanglement [9] and has found interesting applications in quantum information protocols [10, 11]. For instance, the role of discord has been discussed in the context of one-bit mixed state quantum computation [10] and recently also in the prisoner's dilemma quantum game [11]. In our experiment, we measure the variation of the quantum gain under input states with different amounts of entanglement and discord. The experimental data clearly demonstrate the connection between quantum gain and discord.

We consider a zero-sum quantum game in the Hilbert space spanned by two qubits. The referee provides input states ρ_{ab} to the game players A and B, and ρ_{ab} is required to be symmetric to the game players under exchange of the subscripts a and b. For the classical game players, A and B can apply any classical operations (such as the bit-flip $X : |0\rangle \leftrightarrow |1\rangle$) and then measure in the computational basis $\{|0\rangle, |1\rangle\}$. We assume that A wins if the measurement outcomes are identical from A and B, and B wins otherwise. Clearly, only the diagonal elements of ρ_{ab} matter for the classical players. We assume that the game is in a classical Nash equilibrium with each player having a 50% chance to win. This requires the diagonal elements of ρ_{ab} to be $\{1/4, 1/4, 1/4, 1/4\}$; otherwise one of the two classical players could increase the winning probability by a bit-flip operation.

Now assume that player A has the secret power to play quantum mechanically. He can apply arbitrary single-bit operations before the measurement in the computation basis. How much can he gain from this classical balanced Nash equilibrium? The answer depends on the form of the state ρ_{ab}. Assume that ρ_{ab} takes the form of a symmetric maximally entangled state with $\rho_{ab} = \rho_{ab}^{\mathrm{e}} = |\Psi_1\rangle_{ab}\langle\Psi_1|$, where

$$|\Psi_1\rangle_{ab} = (|00\rangle_{ab} + |01\rangle_{ab} + |10\rangle_{ab} - |11\rangle_{ab})/2. \tag{1}$$

If A applies a Hadamard operation H before the measurement, the state $|\Psi_1\rangle_{ab}$ is transformed to $|\Psi_1'\rangle_{ab} = (|00\rangle_{ab} + |11\rangle_{ab})/\sqrt{2}$, for which A is certain to win. This shows that a quantum player can win in principle with 100% certainty if the symmetric state ρ_{ab} sent out by the referee has maximum entanglement.

To demonstrate experimentally the gain of the quantum player, we prepare entangled photons into the state described by equation (1) and send the photons to two separate game players. The photonic entanglement is generated through spontaneous parametric down conversion under the type-I configuration for phase matching [12]. The experimental setup is shown in figure 1. Ultrafast laser pulses (with pulse duration less than 150 fs and a repetition rate of 76 MHz) at the wavelength of 400 nm from a frequency doubled Ti:sapphire laser pump two joint beta-barium-borate (BBO) crystals, each of 0.6 mm depth with a perpendicular optical axis, to generate entangled photon pairs at the wavelength of 800 nm. The polarization of the pumping laser pulse is controlled by two half-wave plates (HWPs) and two quarter-wave plates (QWPs). The entangled photons emit into the modes a and b (fixed by two irises in figure 1) with an angle of about 3° from the pumping beam that is determined by the phase matching condition, and their effective state is described by $|\Phi\rangle_{ab} = \cos\alpha|HH\rangle_{ab} + \sin\alpha\, \mathrm{e}^{\mathrm{i}\varphi}|VV\rangle_{ab}$, where $|H\rangle$ ($|V\rangle$)

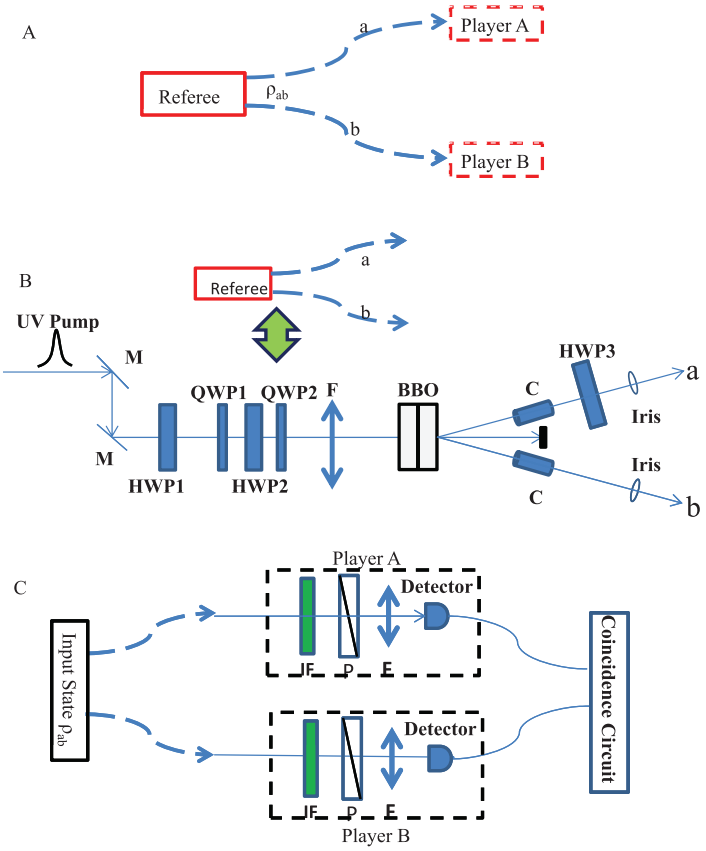

Figure 1. Experimental setup for demonstrating the quantum game. (A) The quantum game model taken in this paper, where a referee sends out correlated states, and the game players apply optimized measurement strategies and get their payoff according to the measurement outcomes. (B) The experimental setup for the referee to generate entangled states of photons. Ultraviolet laser pulses, after the mirrors M and the wave plates, are focused by a lens F into double BBO crystals placed in type-I configuration. The generated entangled photons, after the compensator C, are directed into the optical modes a and b through two irises. (C) Players A and B use a polarizer and a single-photon detector to implement their measurement strategies, and their measurement outcomes are compared through the coincidence circuit to determine their payoff. An interference filter (IF) of 3 nm bandwidth centered at 800 nm is used to filter out the background light.

stands for the horizontal (vertical) polarization and α, φ are independently controlled by HWP1 and HWP2 in figure 1. To observe entanglement, the temporal walk-off between the polarization components $|H\rangle$ and $|V\rangle$ resulting from birefringence in the BBO crystals is compensated for by two quartz rod compensators C in modes a and b, as shown in figure 1. Another HWP (HWP3) in mode a, setting at 22.5°, transforms the state $|\Phi\rangle_{ab}$ (with $\alpha = \pi/4$ and $\varphi = \pi$ set by HWP1 and HWP2) into the ideal form of $|\Psi_1\rangle_{ab}$ shown in equation (1), where $|0\rangle$ and $|1\rangle$ are identified with the polarization states $|H\rangle$ and $|V\rangle$, respectively. This serves as the state sent out by the referee to the game players A and B. The game players use a polarizer and

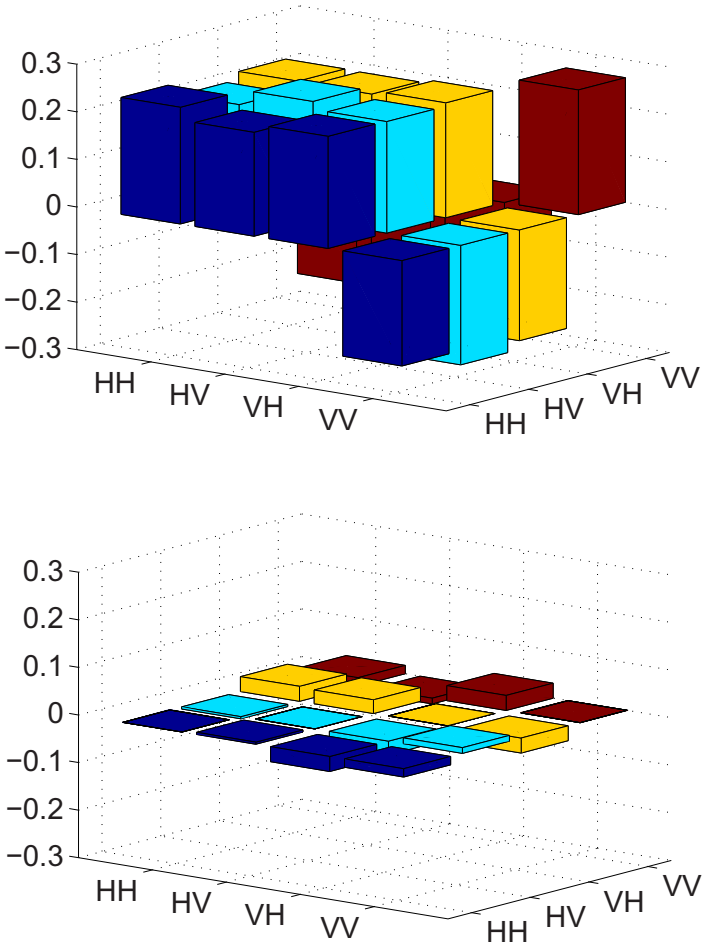

Figure 2. Real (A) and imaginary (B) parts of the density matrix elements of the experimental state reconstructed through the measurements of quantum state tomography.

a single-photon detector to implement their optimal measurement strategies and the detector outcomes are registered through the coincidence circuit.

To confirm experimentally that the state sent out by the referee indeed has the right form described by $|\Psi_1\rangle_{ab}$, we perform quantum state tomography to characterize this state. For two-qubit states, the quantum state tomography is done with 16 independent measurements in complementary bases and the density matrix is reconstructed using the maximum likelihood method [13]. The real and imaginary parts of all the elements of the two-qubit density matrix are shown in figures 2(a) and (b). The ideal density matrix corresponding to the state $|\Psi_1\rangle_{ab}$ has the form

$$\rho_{ab}^{\mathrm{e}} = |\Psi_1\rangle_{ab}\langle\Psi_1| = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. One can see from figure 2 that in the matrix ρ_{ab} reconstructed from experimental data, the imaginary parts are small and the real parts have

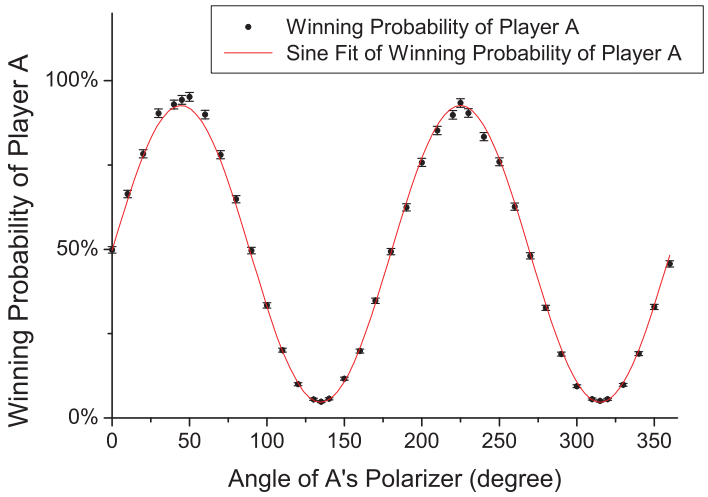

Figure 3. The winning chance of the quantum player A as he varies his measurement basis.

good correspondence with ρ_{ab}^{e}. We use the state fidelity $F = {}_{ab}\langle\Psi_1|\rho_{ab}|\Psi_1\rangle_{ab}$, defined as the overlap between the ideal state ρ_{ab}^{e} and the experimental state ρ_{ab}, to characterize the quality of the prepared state. The state fidelity calculated from the measurements of quantum state tomography is $F = (94.54 \pm 0.83)\%$, where the error bar comes from the statistical error associated with the photon counts (assuming the Poisson distribution). The degradation in fidelity is mainly due to the residual small mismatch of the pulse shapes for the frequency components $|H\rangle$ and $|V\rangle$ from birefringence in the nonlinear crystal.

The input state $|\Psi_1\rangle_{ab}$ corresponds to a Nash equilibrium for classical game players as all the diagonal elements of $\rho_{ab}^{e} = |\Psi_1\rangle_{ab}\langle\Psi_1|$ are $1/4$ and no classical players can gain by changing their measurement strategies (a bit-flip does not change the payoff). Now, for the quantum player A, he has more choices about his measurement strategies and can measure in any basis $\{\cos\theta|H\rangle_a + \sin\theta|V\rangle_a, -\sin\theta|H\rangle_a + \cos\theta|V\rangle_a\}$ by rotating a polarizer before the single-photon detector. Figure 3 shows the winning chance of player A when he rotates his polarizer. At an angle of $\theta = 45^{\circ}$, his winning chance is maximized, attaining $(94.3 \pm 1.3)\%$ in the experiment, representing a significant gain from the classical Nash equilibrium. So the experiment clearly demonstrates the advantage of the quantum player.

The power of the quantum player depends on the input state ρ_{ab} provided by the referee. If ρ_{ab} is an uncorrelated state with only diagonal elements (all of them are $1/4$ for the Nash equilibrium), clearly the quantum player cannot gain anything from whatever measurement strategy he chooses. An interesting question is, what property of ρ_{ab} characterizes the gain of the quantum player? In particular, is entanglement needed for any gain of the quantum player or just for the maximum gain? To characterize the gain of the quantum player under an arbitrary symmetric state ρ_{ab} corresponding to a classical Nash equilibrium, we consider its general form, which can be expressed as

$$\rho_{ab} = \begin{pmatrix} 1/4 & b & b & c \\ b^* & 1/4 & a & d \\ b^* & a^* & 1/4 & d \\ c^* & d^* & d^* & 1/4 \end{pmatrix}, \tag{2}$$

where a, b, c and d are arbitrary complex numbers (that make ρ_{ab} a legal quantum state). The quantum player needs to optimize his winning chance P_a by applying an optimized single-bit rotation U_a. Let us denote by $\rho' = U_a \otimes I_b \rho_{ab} U_a^\dagger \otimes I_b$ the density matrix after the operation U_a by the quantum player (I_b is the 2×2 identity matrix). The gain of the quantum player $G = P_a - P_b = 2P_a - 1$, defined as the difference of the winning probabilities for the quantum and the classical players, is expressed as $G = \rho'_{11} + \rho'_{44} - (\rho'_{22} + \rho'_{33})$ in terms of the matrix elements ρ'_{ij} $(i, j = 1, 2, 3, 4)$. We use $\Delta[M]$ to denote $M_{11} - M_{22}$ of a 2×2 matrix

$$M = \begin{pmatrix} \mathbf{M_{11}} & \mathbf{M_{12}} \\ \mathbf{M_{21}} & \mathbf{M_{22}} \end{pmatrix}.$$

The gain G then has the form

$$\begin{aligned} G &= \Delta\left[U_a \begin{pmatrix} 1/4 & b \\ b^* & 1/4 \end{pmatrix} U_a^\dagger\right] - \Delta\left[U_a \begin{pmatrix} 1/4 & d \\ d^* & 1/4 \end{pmatrix} U_a^\dagger\right] \\ &= \Delta\left[U_a \begin{pmatrix} 0 & b-d \\ b^*-d^* & 0 \end{pmatrix} U_a^\dagger\right]. \end{aligned} \tag{3}$$

The optimal single-bit rotation U_a is just the one that diagonalizes the matrix

$$\begin{pmatrix} 0 & b-d \\ b^*-d^* & 0 \end{pmatrix}$$

and the optimal gain is $G = 2|b - d|$. Under the condition that ρ_{ab} has no entanglement (with positive partial transpose [14]), we numerically optimize the gain $G = 2|b - d|$ and find that its maximal value is 50%. This optimal value is achieved with the following unentangled state:

$$\rho^n_{ab} = (|H, +\rangle_{ab} \langle H, +| + |+, H\rangle_{ab} \langle +, H| + |V, -\rangle_{ab} \langle V, -| + |-, V\rangle_{ab} \langle -, V|)/4, \tag{4}$$

where $|\pm\rangle = (|H\rangle \pm |V\rangle)/\sqrt{2}$. This shows that entanglement is not necessary for the quantum player to achieve a positive gain; however, for any unentangled state, the gain G is limited to below 50% (in contrast to the maximum of 100% under the maximally entangled state $|\Psi_1\rangle_{ab}$).

The state in the form of ρ^{n}_{ab} has no entanglement; however, the discord for this state is nonzero, showing that the state still has some kind of quantum correlation. The concept of discord is introduced in [9]. It characterizes quantum correlation from the measurement point of view (in contrast to entanglement, which characterizes quantum correlation from the angle of state preparation). For the bipartite system AB, the discord is defined as $\delta(B/A) \equiv S(\rho_a) - S(\rho_{ab}) - \max_{\{\Pi^a_j\}}[S(\rho^d_a) - S(\rho^d_{ab})]$, where $S(\cdots)$ denotes the von Neumann entropy associated with the corresponding density operator, $\rho^{\mathrm{d}}_{ab} \equiv \sum_j \Pi^a_j \rho_{ab} \Pi^a_j$ ($\rho^{\mathrm{d}}_a = \mathrm{tr}_b \rho^{\mathrm{d}}_{ab}$), and Π^a_j denotes a set of von Neumann projectors corresponding to an orthogonal measurement on the subsystem A. For the state ρ^{n}_{ab}, the discord is found to be 0.31 [10]. For the bipartite qubit system, if the discord is zero, the symmetric state ρ_{ab} in equation (2) can be written into the form $\rho_{ab} = \cos^2\beta|+, +\rangle_{ab}\langle +, +| + \sin^2\beta|-, -\rangle_{ab}\langle -, -|$ under single-bit phase rotation, and in this case, one can see that the quantum player cannot gain anything from the classical Nash equilibrium.

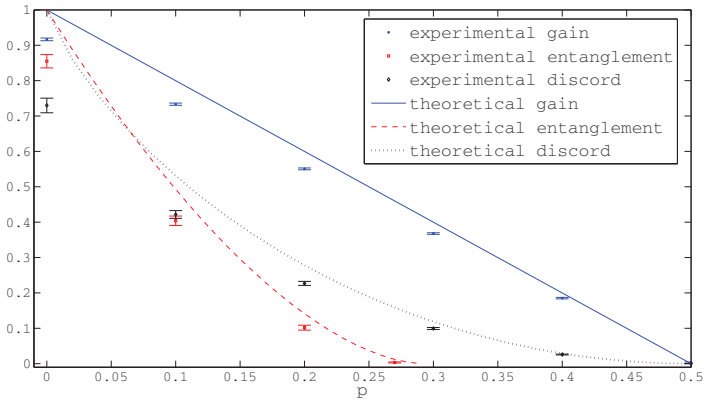

Figure 4. The gain of the quantum player versus the entanglement and the discord of the state sent out by the referee. The data points with error bars represent the experimentally measured values, whereas the solid lines correspond to the theoretically calculated values for the state in the form of equation (5) characterized by a single parameter p.

To experimentally investigate the gain of the quantum player under quantum states with different amounts of entanglement and discord, we prepare the following mixed state

$$\rho_{ab}^{\mathrm{m}} = (1-p)^2 \left|\Psi_1\right\rangle_{ab} \left\langle\Psi_1\right| + p^2 \left|\Psi_4\right\rangle_{ab} \left\langle\Psi_4\right| + 2p\,(1-p)\,(\left|\Psi_2\right\rangle_{ab} \left\langle\Psi_2\right| + \left|\Psi_3\right\rangle_{ab} \left\langle\Psi_3\right|), \tag{5}$$

where $0 \leqslant p \leqslant 1/2$, $\left|\Psi_1\right\rangle_{ab}$ is given by equation (1), and $\left|\Psi_2\right\rangle_{ab} = (\left|00\right\rangle_{ab} + \left|01\right\rangle_{ab} - \left|10\right\rangle_{ab} + \left|11\right\rangle_{ab})/2$, $\left|\Psi_3\right\rangle_{ab} = (\left|00\right\rangle_{ab} - \left|01\right\rangle_{ab} + \left|10\right\rangle_{ab} + \left|11\right\rangle_{ab})/2$, $\left|\Psi_4\right\rangle_{ab} = (-\left|00\right\rangle_{ab} + \left|01\right\rangle_{ab} + \left|10\right\rangle_{ab} + \left|11\right\rangle_{ab})/2$. The states $\left|\Psi_2\right\rangle_{ab}$, $\left|\Psi_3\right\rangle_{ab}$ and $\left|\Psi_4\right\rangle_{ab}$ can be obtained experimentally from $\left|\Psi_1\right\rangle_{ab}$ by rotating HWPs 1 and 2 shown in figure 1. If we mix up the measurement outcomes corresponding to the states $\left|\Psi_i\right\rangle_{ab}$ $(i = 1, 2, 3, 4)$ with the weight $(1-p)^2$, $p(1-p)$, $p(1-p)$, p^2, respectively, we get the experimental outcomes corresponding to the mixed state ρ_{ab}^{m}. The state ρ_{ab}^{m} has the form of equation (2) with $b = -d = (p-1/2)/2$ and $a = -c = (p-1/2)^2$. In figure 4, we show the measured gain G of the quantum player versus entanglement and discord of the state ρ_{ab}^{m} as one varies the parameter p. The gain G is measured directly as the difference of the winning chances between the quantum and the classical players, and the entanglement E (characterized by the concurrence [14]) and the discord δ are calculated from the experimental density matrix reconstructed with the maximum likelihood method from the measurements of quantum state tomography [13]. The solid curves represent the corresponding theoretical results under the state ρ_{ab}^{m}. From the figure, one can see that the experimental data in general agree with the theoretical prediction. In particular, the experiment confirms that the gain G is maximal under the maximally entangled state, remains positive when entanglement disappears at $p \geqslant 0.3$ and approaches zero along with the discord.

In summary, we have experimentally demonstrated a new type of quantum game, where the quantum player can gain from the classical Nash equilibrium. The advantage of the quantum player is demonstrated by using entangled photons, and his winning chance attains $(94.3 \pm 1.3)\%$ in experiment under a near-maximally entangled state. The gain of the quantum player is linked with quantum correlation in the input state provided by the referee, and this connection is confirmed experimentally by observing the variation of the quantum gain as one changes entanglement or discord in the input state.

Acknowledgments

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