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Circular Probable Error for Circular and Noncircular Gaussian Impacts

by David W. Webb

ARL-MR-830 **September 2012**

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REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704-0188

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1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED (From - To)
September 2012	Final	January 2012

4. TITLE AND SUBTITLE
Circular Probable Error for Circular and Noncircular Gaussian Impacts

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
David W. Webb

5d. PROJECT NUMBER
622618AH80

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
U.S. Army Research Laboratory
ATTN: RDRL-WML-A
Aberdeen Proving Ground, MD 21005-5066

8. PERFORMING ORGANIZATION REPORT NUMBER
ARL-MR-830

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Circular probable error (CEP), defined as the radius of a circle centered at the aimpoint, which has a 50% probability of hit, is one of many measures of precision used to characterize ballistic precision. When determining CEP for two-dimensional impact data in which both dimensions are assumed to be independent, the case where both dispersion components are equal is a straightforward application of the Rayleigh distribution. However, when the dispersion components differ, an approximate CEP formula is necessary. This document evaluates the accuracy of the approximate CEP formula using Monte-Carlo simulation and shows that its associated probability of hit is no more than 2.17% higher than the nominal value of 0.50.

15. SUBJECT TERMS
circular probable error, approximate formula, simulation, Rayleigh distribution, Wilson-Hilferty transformation

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			David W. Webb
Unclassified	Unclassified	Unclassified	UU	16	19b. TELEPHONE NUMBER (*Include area code*) 410-278-7014

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

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1. Introduction

In ballistic performance testing, many statistics have been proposed as measures of precision. Some of these statistics only consider the spread of the data in one direction, such as the horizontal standard deviation and the extreme vertical dispersion. Other measures attempt to capture a sense of the spread in both directions, e.g., extreme spread and circular probable error (CEP). Grubbs[1] describes each of these measures, along with several others in his seminal self-published reference, informally known as the "Red Book."

In this report, we examine the CEP, defined here as the radius of a circle centered at the aimpoint which, in the long run, is impacted by 50% of the projectiles. We assume that impacts land on a two-dimensional plane and that the horizontal (*x*) and vertical (*y*) locations are independent.[*] Furthermore, we assume that the impacts follow a Gaussian (normal) distribution centered at the aimpoint. We will first review the cases in which the only source of variation in each dimension is the round-to-round variation, σ_x and σ_y. Then, we augment Grubbs' approximate CEP formula[2] for those cases in which additional error sources contribute to the overall precision of the weapon system. Finally, we evaluate the accuracy of the approximate CEP formula using Monte-Carlo simulation.

2. CEP for Bivariate Independent Normal Impact Data

2.1 Case 1: Circular Data ($\sigma = \sigma_X = \sigma_Y$), Single Error Source

Because the impact locations are assumed to be centered at the aimpoint, we may set the aimpoint to be (0,0) without loss of generality. Then, assume that $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$ are independent normal random variables; therefore, $\frac{X}{\sigma} \sim N(0,1)$ and $\frac{Y}{\sigma} \sim N(0,1)$, and $\sqrt{\frac{X^2}{\sigma^2} + \frac{Y^2}{\sigma^2}}$ follows a Rayleigh distribution with parameter value 1. By definition, the CEP satisfies the following:

$$0.5 = P\left(\sqrt{X^2 + Y^2} < CEP\right) = P\left(\sqrt{\frac{X^2}{\sigma^2} + \frac{Y^2}{\sigma^2}} < \frac{CEP}{\sigma}\right) = F_R\left(\frac{CEP}{\sigma}\right), \quad (1)$$

[1] Grubbs, F. E. *Statistical Measures of Accuracy for Rifleman and Missile Engineers*; Havre de Grace, MD, 1964.

[*] In referring to the two directions as "horizontal" and "vertical," we imply that the target plane is upright, as is the case for many weapon systems, including calibers as large as tank ammunition. However, for artillery systems, the ground is the most likely target plane, and we instead refer to the *x* and *y* dimensions as "deflection" and "range," respectively.

[2] Grubbs, F. E. Approximate Circular and Noncircular Offset Probabilities of Hitting. *Operations Research* **1964**, *12*, 51–62.

where F_R is the Rayleigh cumulative distribution function. Since $F_R(x) = 1 - e^{-x^2/2}$, we have $0.5 = 1 - e^{-\left(\frac{CEP}{\sigma}\right)^2/2}$, which can be algebraically rearranged to obtain the following solution:

$$CEP = \sqrt{2ln(2)}\sigma \approx 1.1774\sigma. \tag{2}$$

2.2 Case 2: Noncircular Data ($\sigma_X \neq \sigma_Y$), Single Error Source

Assume that $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ are independent normal random variables; therefore, $\frac{X}{\sigma_X} \sim N(0,1)$ and $\frac{Y}{\sigma_Y} \sim N(0,1)$, and $U_X = \frac{X^2}{\sigma_X^2}$ and $U_Y = \frac{Y^2}{\sigma_Y^2}$ are independent chi-square random variables, each with one degree of freedom. Following a similar argument given by Grubbs,[2] we start by rewriting the probability statement as

$$P\left(\sqrt{X^2 + Y^2} < CEP\right) = P(X^2 + Y^2 < CEP^2) = P(\sigma_X^2 U_X + \sigma_Y^2 U_Y < CEP^2)\,. \tag{3}$$

The left side of this last inequality is a weighted sum of independent chi-square random variables, which, as Patnaik[3] argues, has a distribution that can be approximated by a chi-squared random variable when appropriately scaled by a positive constant *a*. This random variable is denoted as V_n, where *n* is the degrees of freedom. The approximation is found by matching the first two moments of $a(\sigma_X^2 U_X + \sigma_Y^2 U_Y)$ and V_n. That is, we seek *a* and *n* such that

$$E\left(a(\sigma_X^2 U_X + \sigma_Y^2 U_Y)\right) = E(V_n), \tag{4}$$

and

$$Var\left(a(\sigma_X^2 U_X + \sigma_Y^2 U_Y)\right) = Var(V_n). \tag{5}$$

The left side of equation 4 is

$$E\left(a(\sigma_X^2 U_X + \sigma_Y^2 U_Y)\right) = a\left(\sigma_X^2 E(U_X) + \sigma_Y^2 E(U_Y)\right) = a(\sigma_X^2 + \sigma_Y^2), \tag{6}$$

while the right side is

$$E(V_n) = n. \tag{7}$$

The left side of equation 5 is

$$Var\left(a(\sigma_X^2 U_X + \sigma_Y^2 U_Y)\right) = a^2\left(\sigma_X^4 Var(U_X) + \sigma_Y^4 Var(U_Y)\right) = 2a^2(\sigma_X^4 + \sigma_Y^4), \tag{8}$$

and the right side is

$$Var(V_n) = 2n. \tag{9}$$

[3]Patnaik, P. B. The Use of the Range as an Estimator of Variance in Statistical Tests. *Biometrika* **1950**, 37, 78–87.

Therefore, we have the following system of two equations in the two unknowns, *a* and *n*:

$$a(\sigma_X^2 + \sigma_Y^2) = n, \tag{10}$$

and

$$2a^2(\sigma_X^4 + \sigma_Y^4) = 2n. \tag{11}$$

The solution to this system is $a = \frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}$ and $n = \frac{\left(\sigma_X^2+\sigma_Y^2\right)^2}{\sigma_X^4+\sigma_Y^4}$.

Therefore,

$$a(\sigma_X^2 U_X + \sigma_Y^2 U_Y) = \frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}(X^2 + Y^2) \approxeq \chi_n^2. \tag{12}$$

By definition of the CEP, we seek a solution to

$$0.5 = P\left(\sqrt{X^2 + Y^2} < CEP\right), \tag{13}$$

which can be rewritten to obtain a probability statement based on the chi-square distribution in equation 12:

$$0.5 = P\left(\frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}(X^2 + Y^2) < \frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}CEP^2\right) = P\left(\chi_n^2 < \frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}CEP^2\right). \tag{14}$$

Next, we use the Wilson-Hilferty transformation, $\frac{\sqrt[3]{\chi_n^2/n}-\left(1-\frac{2}{9n}\right)}{\sqrt{\frac{2}{9n}}} \approxeq N(0,1)$, which relates a chi-square random variable to a standard normal.

$$\begin{aligned}
0.5 &= P\left(\chi_n^2 < \frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}CEP^2\right) \\
&\cong P\left(\frac{\sqrt[3]{\chi_n^2/n} - \left(1 - \frac{2}{9n}\right)}{\sqrt{\frac{2}{9n}}} < \frac{\sqrt[3]{\frac{\sigma_X^2 + \sigma_Y^2}{\sigma_X^4 + \sigma_Y^4}CEP^2/n} - \left(1 - \frac{2}{9n}\right)}{\sqrt{\frac{2}{9n}}}\right) \\
&\cong P\left(Z < \frac{\sqrt[3]{\frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4}CEP^2/n}-\left(1-\frac{2}{9n}\right)}{\sqrt{\frac{2}{9n}}}\right).
\end{aligned} \tag{15}$$

Since the median of the standard normal distribution is 0, the numerator of the right side of the inequality needs to be set to 0 to find a solution to the CEP. That is,

$$0 \cong \sqrt[3]{\frac{\sigma_X^2+\sigma_Y^2}{\sigma_X^4+\sigma_Y^4} CEP^2/n} - \left(1 - \frac{2}{9n}\right), \tag{16}$$

or

$$CEP \cong \sqrt{n \frac{\sigma_X^4+\sigma_Y^4}{\sigma_X^2+\sigma_Y^2}\left(1 - \frac{2}{9n}\right)^3}. \tag{17}$$

Substituting for *n*, we arrive at the final solution of

$$CEP \cong \sqrt{(\sigma_X^2 + \sigma_Y^2)\left(1 - \frac{2(\sigma_X^4+\sigma_Y^4)}{9(\sigma_X^2+\sigma_Y^2)^2}\right)^3}. \tag{18}$$

2.3 Case 3: Circular Data, Multiple Error Sources

When multiple sources contribute to overall weapon system accuracy (e.g., occasion-to-occasion, location, crosswind), equation 18 still applies as long as these sources are uncorrelated and the vertical and horizontal impacts are independent. However, in lieu of the single error source, σ, we use the total system error calculated from the usual root-sum-square formula, $\sigma_{TOTAL} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2}$.

2.4 Case 4: Noncircular Data, Multiple Error Sources

When the weapon system accuracy differs in the vertical and horizontal directions, the presence of multiple error sources does not change equation 18 other than σ_X and σ_Y are replaced by $\sigma_{X-TOTAL} = \sqrt{\sigma_{X1}^2 + \sigma_{X2}^2 + \cdots + \sigma_{Xm}^2}$ and $\sigma_{Y-TOTAL} = \sqrt{\sigma_{Y1}^2 + \sigma_{Y2}^2 + \cdots + \sigma_{Ym}^2}$, respectively.

3. Accuracy of the CEP Approximation for Noncircular Data

The accuracy of the approximate formula for CEP (equation 18) can be determined using Monte-Carlo simulation by randomly generating impact data under various assumed values of σ_X and σ_Y.* The frequency with which the impacts fall within a circle of radius CEP should be close to 0.5. The MATLAB† code used to conduct this Monte-Carlo study is given in the appendix. Figure 1 is a plot of values of $log_{10}(\sigma_{max}/\sigma_{min})$ vs. the frequency of hitting a circle of radius CEP, based on 100,000,000 simulated impacts.

*If both σ_X and σ_Y are increased by a common factor "*k*," then it can be shown that the CEP will also increase by a factor of *k*. Therefore, it is sufficient to just consider the ratio σ_X/σ_Y. In the simulation study, we fixed $\sigma_Y = 1$ and varied σ_X.

†MATLAB is a trademark of The MathWorks, Inc.

Since the estimated hit probabilities are all slightly higher than 0.5, we see that Grubbs' CEP approximation is a slight overestimate. The approximation has an error between 0.47% and 2.17% when the horizontal and vertical errors are within two orders of magnitude of each other. The relationship appears to have an asymptotic value of about 0.5073. A similar Monte-Carlo study in which the order of magnitude of the error ratio was increased to as large as 6 upheld this asymptotic behavior.

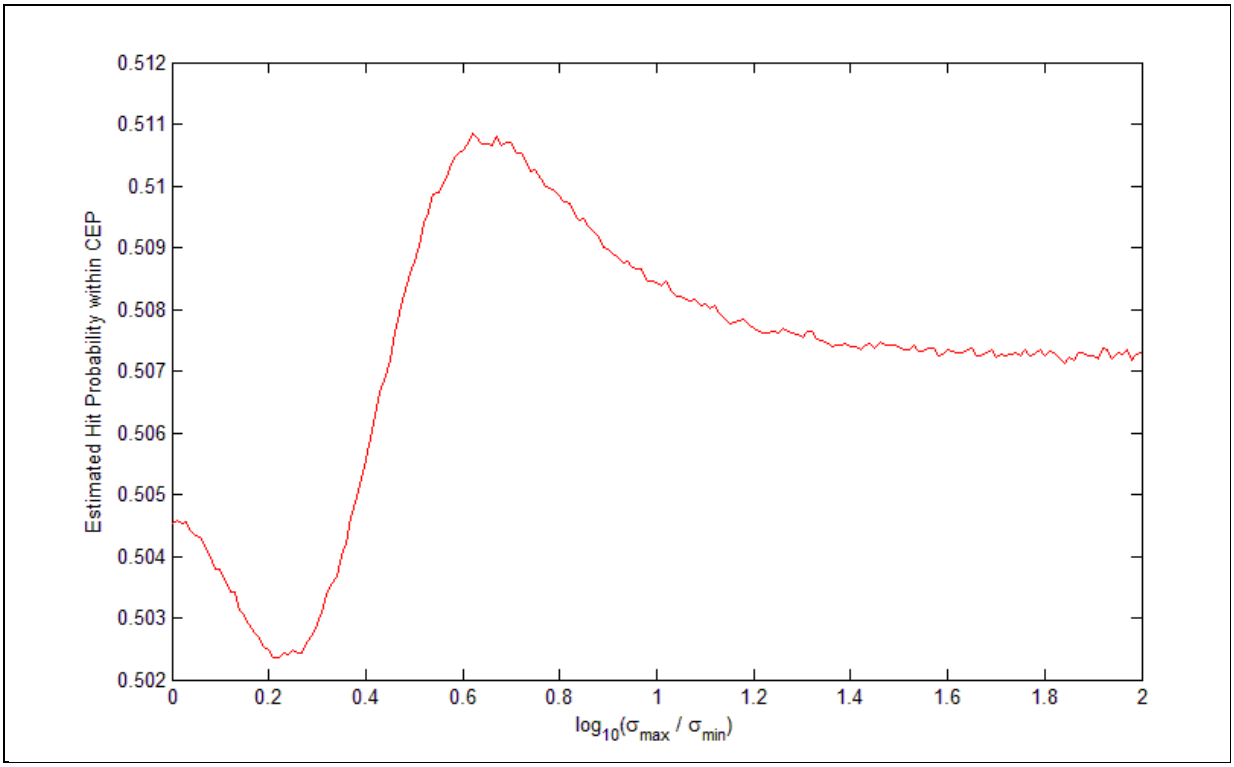

Figure 1. Relationship between relative size of horizontal and vertical errors and the hit probability of a circle centered at the aimpoint and having radius given by Grubbs' approximate formula.

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Appendix. MATLAB Code for Evaluating the Accuracy of Grubbs' Approximate Circular Probable Error (CEP) Formula

This appendix appears in its original form, without editorial change.

```
% OBTAIN P(HIT) ESTIMATES FOR HORIZONTAL ERRORS OF 1 TO 100
% KEEP VERTICAL ERROR EQUAL TO 1
i=[0:.01:2];                              % exponent values
for j=1:length(i)
   sigx(j)=10^i(j)                        % horizontal error
   sig=sqrt(sigx(j)*sigx(j)+1);           % root sum squares; NOTE sigy=1
   v=2*(((sigx(j)^4)+1)/(sig^4));         % df from chi-square in Eq (6)
   CEP=sig*(1-v/9)^1.5;
   %
   clear dat
   for k=1:100                            % generate 100 Ph estimates
      imp=mvnrnd([0 0],[sigx(1)^2 0;0 1],1000000);    % 1M simulated impacts
      ph(k)=mean(imp(:,1).^2+imp(:,2).^2<=CEP^2);     % hit frequency on CEP
   end
   phit(j)=mean(ph);                      % avg 100 hit frequencies to "incr n"
end

% GRAPHICS
plot(i, phit,'r-');                       % error exponent versus Ph estimate
ylabel('Estimated Hit Probability within CEP')
xlabel('log_{10}(\sigma_{max} / \sigma_{min})')
```

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