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by Terrence J Moore

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by Terrence J Moore
Computational and Information Sciences Directorate, ARL

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14. ABSTRACT

Solving certain sensor network coverage problems in a location-unaware environment has become feasible using algebraic topology techniques. The coverage of the network can be modeled using a simplicial complex, where simplices correspond to cliques in the communication graph. Homological methods can determine various properties of the coverage. One particular problem of interest is the sparse coverage problem (i.e., how many nodes are needed to maintain certain coverage quality). This can be translated to finding homology-preserving reduction algorithms. This report details 3 such distributive approaches based on greedy node removals. The first is a simple calculation of the potential change in homology locally. The second approach is based on strong collapsing, which has previously been applied in the hole localization problem. The last approach recognizes the connection between homology and the Euler characteristic, and calculates the potential change in the characteristic locally. Simulations are provided to validate each approach.

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1. Introduction

Practical applications incorporating topology appear in many fields, of which, obvious examples include the study of networks[1] (e.g., graph theory*), condensed matter physics (e.g., the recent Nobel prize-winning work on topological phase transitions by Thouless, Haldane, and Kosterlitz[3–7]), robotics,[8] and, more recently, topological data analysis.[9] Also recently, there has been a growing literature on the use of algebraic topological methods to solve coverage problems in sensor networks. This approach is particularly valuable when computational geometric approaches are insolvable or intractable due to the lack of location information at the sensor (e.g., in GPS-denied or GPS-spoofed environments). In general, the use of topology, which abstracts the geometry of the problem, enables the solvability of certain sensor coverage problems under a minimal set of assumptions. In this work, we are interested in this last application, in particular, as it pertains to the sparse coverage problem.

The practical application of homology in the sensor network coverage problem was introduced in the foundational work of Ghrist and Muhammad[10] and de Silva et al.[11–13] Therein, nontrivial members in the first homology groups of the Čech complex were shown to coincide with gaps in sensor coverage. Since the construction of this complex is impossible without sensor position information, homological coverage criteria are developed using Rips complexes that bound the Čech complex and that can be inferred directly from communication links (connectivity) among the sensors. These criteria on a simplicial complex representation of the sensor network coverage have contributed to unique approaches of solving several sensor network problems (e.g., coverage hole detection, coverage verification, hole localization, and so on; see Section 2 for more details).

The sparse coverage problem aims to find a sparse (much smaller) set of sensors that maintains the existing coverage or the desired coverage requirements. Contributions to the problem are relevant for power- and time-savings in sensor networks, extending mission lifetime. In the homological sensor network, the problem is equivalent to a homology-preserving reduction problem: How does one eliminate redundancy in the simplicial complex without changing homology? The problem is certainly feasible in a centralized fashion by "shrinking" found generators of nontrivial homology using boundary information.[12,13] Greedy single node removal approaches

*In fact, both graph theory and topology often are described to have originated from Euler's solution to the Königsberg bridge problem.[2]

based on homology calculations[14] or heuristic indicators[15] have also been developed. Gossip-type algorithms enable distributed solutions,[16,17] albeit at a significant communication cost. This report discusses in greater detail the simple distributed greedy approaches, presented previously,[18] tackling this problem in a fenced sensor network, including calculating homology changes locally, strong collapsing, and calculating Euler characteristic changes locally.

Due to the inherent computational complexity involved in computations on a combinatorial object like a simplicial complex, the sparse cover problem also generally contributes to answering Question 6 of de Silva and Ghrist[12]: “Is there a way to compress the Rips complex in a pre-processing step without changing the appropriate homology group?” The classical solution is elementary collapsing,[19] which removes pairs of simplices at a turn. But greedy node removal can be significantly more efficient since all the simplices created by the node can be removed at once. Amazingly, we discover that one of the approaches, strong collapsing, is both more efficient than elementary collapsing and has a minimal generator-preserving property. This last property is relevant to both the sparse cover problem and the hole localization problem in sensor networks.

The rest of the report is constructed as follows. The minimal assumptions needed for topological/homological methods in sensor networks and the basic sparse cover problem are presented in Section 2. A basic background of simplicial complexes, homology, and their application in the sensor network coverage problem are described in Section 3. The sparse coverage problem and 3 greedy distributed approaches are discussed in Section 4. Simulations results for a simple example are detailed in Section 5.

2. Sensor Networks with Minimal Assumptions

We start with a basic description of a planar sensor network with minimal assumptions. Let the sensors be denoted by $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$ with locations denoted by $\mathcal{X} = \{x_0, x_1, \ldots, x_n\} \subset \mathcal{D}$, where x_i corresponds to v_i for each i.

Assumption 1. Each sensor node has a unique ID.

Note, we do not assume that the sensors have knowledge of their locations. In fact, we assume the opposite. There exist numerous approaches for tackling various sensor coverage problems when the sensor locations are known. The goal under a minimal assumptions approach is to determine what is possible with less information.

Assumption 2. Each sensor node can communicate with other nodes within a radius r_c of its position.

Thus, if 2 nodes u and v are at positions x_u and x_v, they can communicate if $||x_u - x_v|| \leq r_c$. This enables information exchange between neighboring nodes. In particular, each node can share its ID with its neighbors (i.e., the other nodes in its communication range) and each node is able to learn the unique IDs of each of its neighbors as well as the unique IDs of all the neighbors of each of its neighbors, and so on.

In addition to a uniform communication range for the sensor network, we also assume a uniform disk graph model for the sensor network.

Assumption 3. Each sensor can make sensing observations within a radius r_s of its position.

Hence, the sensor cover is given by $\mathcal{U}(\mathcal{X}) = \bigcup_i B(x_i, r_s)$, where $B(x_i, r_s)$ is the closed disk of radius r_s centered at x_i. Without location information, we will eventually require some assumption comparing the communication radius r_c with the sensing radius r_s to determine what region $\mathcal{U}$ covers. But first, we require some assumptions about the coverage domain.

Assumption 4. Each sensor lies in a compact, connected domain $\mathcal{D} \subset \mathbb{R}^2$ and the boundary $\partial\mathcal{D}$ of this domain is connected.

Under these rather basic assumptions (and with additional enabling assumptions for the particular problem space or method), much has been accomplished in adapting, in particular, homology-based methods to various basic sensor coverage problems:

1. detection of coverage holes (i.e., detecting if a hole or gap in coverage exists),[10,17,20–25]
2. detection or verification of coverage (i.e., ensuring there is no coverage gap or hole),[11,12,26–29]

3. discovery of a sparse cover,[12,14–18]

4. hole coverage repair or patching,[12,21,30]

5. pursuit and evasion games,[12]

6. location of coverage holes (i.e., finding a minimal cycle of nodes that circumference a coverage hole or gap),[17,20,22,24,31,32]

7. event detection,[33] and

8. mapping and route planning in an unknown environment.[34–36]

This list merely includes a good sampling of the applications that use homology as a basis for tackling problems without location information and, in general, with minimal information. Some caveats require mention regarding the topological methods in solving these problems (for some early examples, refer to the questions in de Silva and Ghrist[12]). One example is that several of the methods for particular problems do not solve the complete problem. In fact, the initial coverage hole detection approach only satisfied a sufficiency criterion and so it was possible for imaginary coverage holes to be detected when the domain was, in fact, covered.[10] This made it a kind of over-coverage criterion. To address this and other related issues, there exist several works examining the accuracy[37–39] of various criteria or the vulnerability[40,41] of networks in the presence of sensor node failure.[42]

This report expands on the results reported previously[18] and focuses on the problem of finding a sparse cover using greedy approaches.

2.1 Sparse Coverage

The sparse coverage problem seeks to find as small a set as possible of existing sensor nodes that can still maintain the same level of coverage of the domain if the other nodes are removed. Generalizations or extensions of the problem include finding a set that almost maintains the same level of coverage, finding a set or sequence of (independent) sets that each maintains coverage, and finding a sequence of sets that maintains coverage over time but does not maintain coverage at any given time (i.e., adapting the pursuit and evasion problem to solve the sparse coverage problem). The motivation can be for power-savings and, hence, for extending the coverage lifetime, or for time-savings, enabling the "extra" nodes to be available for other non-sensing tasks.

This work only addresses the basic sparse coverage problem, as first presented by de Silva and Ghrist.[12] For this, we require several more simple assumptions about the coverage radii, the existence of fence nodes, and the cover.

Assumption 5. The sensing and communication radii satisfy the inequality

$$r_s \geq \frac{r_c}{\sqrt{3}}. \tag{1}$$

This assumption guarantees that any part of $\mathcal{D}$ that intersects with the convex hull of any collection of nodes that are pairwise no more than r_c distance apart (and, hence, can pairwise communicate) is covered by $\mathcal{U}(\mathcal{X})$. In particular, for any triple of sensor nodes that can pairwise communicate, the interior of the triangle of the induced communication subgraph is covered (Fig. 1). An alternative approach is to simply assume a "capture modality" that ensures the same guarantee.[17]

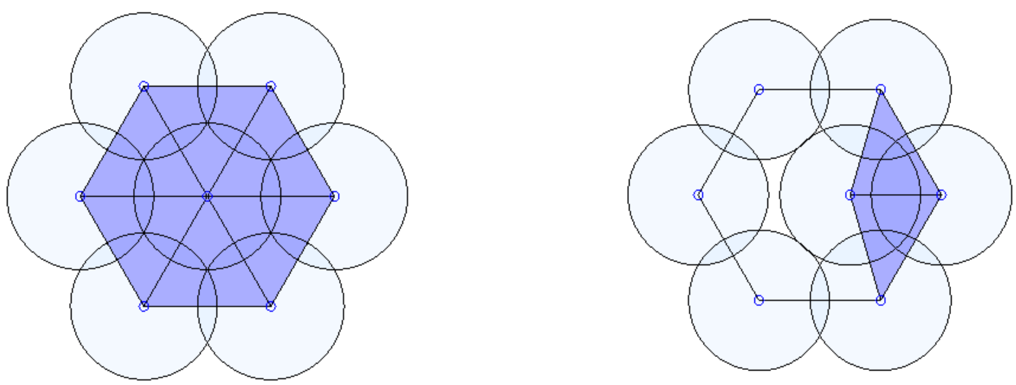

Fig. 1 Examples of communication links satisfying and failing the coverage criterion. Under Assumption 5, the convex hull of every set of nodes that are pairwise-connected in the communication graph is covered. Coverage in the Rips complex ensures coverage of the domain (left), where the simplices are the vertices, edges, and triangles (triples). The contrapositive of this statement is, if there is a gap in sensor coverage, then there will be a nontrivial homology class in the Rips complex. Above (right), the center node is slightly shifted, destroying the local triangle communication lattice structure.

Assumption 6. A subset of sensor nodes are recognized as fence nodes $\mathcal{F} = \{x_{f_0}, x_{f_1}, \ldots, x_{f_m}\} \subset \mathcal{X}$ that lie on (or close to) the boundary $\partial\mathcal{D}$ and $\mathcal{U}(\mathcal{F})$ covers $\partial\mathcal{D}$.

The fence nodes can recognize their membership in the fence using only the connectivity information in the communication graph.[43] This assumption prevents potential "edge effects" in the reduction/collapse processes presented later. Each node in $\mathcal{F}$ has knowledge of its membership in $\mathcal{F}$ and can share this information. Consequently, each node in $\mathcal{X}$ is aware which of its neighbors are fence nodes.

Assumption 7. $\mathcal{U}(\mathcal{X})$ covers $\mathcal{D}$.

In other words, coverage has already been verified. This is an assumption of convenience. An alternative approach if the domain was not covered is to detect the coverage holes and either repair or patch the coverage gaps or locate the coverage holes. If the former option, then Assumption 7 holds. If the latter option, then the sensor nodes "co-locating" the hole location can be treated as fence nodes in each of processes discussed later. However, this would also require some modification of Assumption 4 since $\partial\mathcal{D}$ would be piecewise connected.

3. Simplicial Complexes

This section provides a brief background on simplicial complexes (sufficient to understand this work) and the representations of sensor network coverage as simplicial complexes. For a more-detailed introductory treatment of simplicial complexes, the reader is referred to Munkres, Hatcher, and Ghrist, respectively.[44–46]

3.1 Basics of Simplicial Complexes

An *abstract simplicial complex* is a collection K of finite sets closed under the subset operation (i.e., if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$). An element σ of K is called a simplex. A subset τ of a simplex σ is called a face of that simplex. If a simplex is maximal (not a subset of any other simplex) in K, it is called a *facet*. A *subcomplex* Λ of K is an abstract simplicial complex such that every simplex in Λ also exists in K. The union of all the sets in K forms a *vertex set* $\mathcal{V}$ and its elements are the vertices of the complex. Hence, every simplex is a finite set of vertices and can be denoted by the vertices it contains (e.g., $\sigma = v_0 v_1 \ldots v_k$). The dimension of a simplex is one less than the number of its vertices (i.e., dim$(\sigma) = |\sigma| - 1$). In particular, we call a simplex a *k-simplex* if it has dimension k. The dimension of the complex is the supremum of the dimensions of its simplices.

An *oriented simplex* is a simplex with an orientation, denoted $[v_0, v_1, \ldots, v_k]$, such that 2 oriented simplices consisting of the same vertices are equal if they differ by an even number of permutations and they are the negative of the other if they differ by an odd number of permutations. This induces a sequence of vector spaces over $\mathbb{R}$, denoted $\mathcal{C}_k(K)$, with basis being the set of oriented k-simplices of K for each k.* The elements of $\mathcal{C}_k(K)$, written as $\sum_i c_i \sigma_i^{(k)}$ with each $c_i \in \mathbb{R}$, are called k-*chains*. The sequence of vector spaces on K is connected as a *chain complex* by the *boundary operators* (homomorphisms) $\partial_k : \mathcal{C}_k(K) \to \mathcal{C}_{k-1}(K)$ defined by

$$\partial_k[v_0, v_1, \ldots, v_k] = \sum_{j=0}^{k} (-1)^j [v_0, v_1, \ldots, \hat{v}_j, \ldots, v_k], \tag{2}$$

where $\hat{v}_j$ denotes that this vertex is missing from the oriented simplex. In $\mathcal{C}_k(K)$, we refer to the image of ∂_{k+1} as the subgroup of k-*boundaries*, denoted $\mathcal{B}_k(K)$, and we refer to the kernel of ∂_k as the subgroup of k-*cycles*, denoted $\mathcal{Z}_k(K)$. Since $\mathcal{B}_k(K) \subset \mathcal{Z}_k(K)$, the boundary operators induce a sequence of homology groups

$$\mathcal{H}_k(K) = \mathcal{Z}_k(K)/\mathcal{B}_k(K). \tag{3}$$

The homology of a simplicial complex determines the number of equivalence classes that do not bound (i.e., that are not contractible to a point in the topology). Formally, the number of nontrivial equivalence classes is the rank of the groups, called *Betti numbers*, denoted

$$b_k(K) = \text{rank}(\mathcal{H}_k(K)). \tag{4}$$

Note, in a geometric or network sense, $b_0(K)$ is the number of connected components of K, $b_1(K)$ is the number of 1-D "holes" in K, $b_2(K)$ is the number of 2-D voids in K, and so on.

The calculation of these homologies[47] or Betti numbers is a simple (but not necessarily low complexity) matrix algebra task. Given an ordering of the k- and $(k-1)$-simplices, the boundary operator ∂_k can be represented in matrix form as $B^{(k)} : \mathbb{R}^{n_k} \to \mathbb{R}^{n_{k-1}}$ using Eq. 2, where n_k is the number of k-simplices. Algorithms for the computation of the homology groups can be found in Kaczynski et al.[47] The kth homology and Betti number can also be found via the matrix reduction

*The homology groups are more commonly developed over rings using the integers $\mathbb{Z}$ instead of fields using the reals $\mathbb{R}$. For rings, a sequence of modules is induced instead of a sequence of vector spaces.

method.[48] Specifically, note from Eqs. 4 and 3, that

$$\begin{aligned} b_k &= \text{rank}(\mathcal{Z}_k(K)) - \text{rank}(\mathcal{B}_k(K)) \\ &= n_k - \text{column rank}(B^{(k+1)}) - \text{column rank}(B^{(k)}) \\ &= n_k - (\text{\# of nonzero cols of } \tilde{B}^{(k+1)}) - (\text{\# of nonzero cols of } \tilde{B}^{(k)}), \quad (5) \end{aligned}$$

where $\tilde{B}(k)$ is the matrix reduction of $B^{(k)}$. This approach can be made simpler by using a much simpler field than $\mathbb{R}$ (e.g., $\mathbb{Z}_2$).

In this work, we are also interested in the relative homology $\mathcal{H}_k(K, \Lambda)$. This is obtained by considering the chains, cycles, and boundaries in Λ as part of their respective trivial groups. It can easily be calculated by placing a cone over the simplices in Λ (i.e., adding a new vertex ι to K and to each simplex in Λ) and calculating the usual homology one dimension higher.[16] In particular, when Λ has the topology of a cycle or loop and if $\mathcal{H}_1(K)$ is trivial (i.e., there is no nontrivial homology group), then the complex with a cone attached creates the topology of the surface of a sphere in $K \bigcup \{\iota\}$.

3.2 Simplicial Complexes in Sensor Networks

Since we lack the location information of the sensor nodes in our assumptions, we require some method to translate the geometric problem of the coverage problem to a strictly topological problem. Simplicial complexes, as a higher-dimensional topological space with a geometric realization, seem ideal for this translation provided a relation between the coverage in the geometric domain of the sensor nodes and a notion of a cover in the topological domain can be established. We rely on the criteria found in de Silva and Ghrist[12] and detailed as follows.

Recall, the (unknown) sensor node positions are given by $\mathcal{X} = \{x_0, x_1, \ldots, x_n\} \subset \mathcal{D}$, where x_i corresponds to v_i for each i and the sensor cover is given by $\mathcal{U}(\mathcal{X}) = \bigcup_i B(x_i, r_s)$, where $B(x_i, r_s)$ is the closed disk of radius r_s centered at x_i.

The proper complex for representing the cover is the *Čech complex* $\check{\mathcal{C}}_{r_s}(\mathcal{X})$, which is formed by considering the intersections of the sensing disks (i.e., $\check{\mathcal{C}}_{r_s}(\mathcal{X}) = \bigcap_i B(x_i, r_s)$). It has been shown that $\check{\mathcal{C}}_{r_s}(\mathcal{X})$ is homotopy equivalent to the cover $\mathcal{U}(\mathcal{X})$. This result is often called the Čech Theorem or Nerve Theorem.[49,50] In this construction, the simplex with vertices $v_0, v_1, \ldots, v_k$ exists if $\bigcap_{i=0}^k B(x_i, r_s)$ is nonempty. Unfortunately, this construction is impossible without location information, relying only on the connectivity information between the nodes in the network.

Instead, we construct the ***Rips complex*** $\mathcal{R}_{r_c}(\mathcal{X})$, which is formed by considering the cliques in the communication graph. In this construction, the simplex with vertices $v_0, v_1, \ldots, v_k$ exists if $||x_i - x_j|| \leq r_c$ for every $i, j = 0, 1, \ldots, k$. This construction is locally feasible from Assumptions 1 and 2 since each node can obtain awareness of its *star*[44] in the simplicial complex topology of $\mathcal{R}_{r_c}(\mathcal{X})$ (i.e., each node v_i has knowledge of which neighboring nodes are in each of the facets in which it is a member).

Via the Rips complex, we have a homology-based criterion for coverage. Given Assumptions 3–6, it can be shown[10–13] that

$$\mathcal{R}_{r_c}(\mathcal{X}) \subset \check{\mathcal{C}}_{r_s}(\mathcal{X}). \tag{6}$$

Therefore, if the first homology is trivial in $\mathcal{R}_{r_c}(\mathcal{X})$ (i.e., there exist no 1-D holes) then the first homology in $\check{\mathcal{C}}_{r_s}(\mathcal{X})$ is also trivial and the cover $\mathcal{U}(\mathcal{X})$ is hole-free in $\mathcal{D}$. $\check{\mathcal{C}}_{r_s}(\mathcal{X})$ may have nontrivial homology outside of $\mathcal{D}$ if the fence is pinched in some manner (e.g., this is possible if there exist non-neighboring fence nodes u and v, where $B(x_u, r_s) \bigcap B(x_v, r_s) \bigcap \mathcal{D}^c \neq \emptyset$ yet $B(x_u, r_c) \bigcap B(x_v, r_c) = \emptyset$).

An alternative homology-based criterion for verifying coverage is the existence of a generator in the second relative homology group $\mathcal{H}_2(\mathcal{R}_{r_c}(\mathcal{X}), \mathcal{R}_{r_c}(\mathcal{F}))$ that is nonzero.[12,13] Under Assumptions 1–7, since there is no hole in the domain, then the homology of the Rips complex with a cone attached to the fence creates the topology of the surface of a sphere. That is, since this complex with the cone has a nontrivial second homology group, then the complex has a trivial first homology group.

4. The Sparse Cover Problem

The general literature dealing with deciding which nodes are required to remain active and which nodes may sleep is rich.[51–55] The particular problem addressed here is the deterministic art-gallery-type problem as described in de Silva and Ghrist.[12] Given a cover of the domain, find a sparse (ideally, minimal) generator of the non-trivial homology classes in $\mathcal{H}_2(\mathcal{R}_{r_c}(\mathcal{X}), \mathcal{R}_{r_c}(\mathcal{F}))$.

Centralized approaches to reduce a given generator are, of course, available,[12] as are gossip-inspired distributive approaches.[16,17] The former may require a certain level of computational and memory capability (of at least one node) and the latter has a significant communication cost. The classical homology-preserving approach is elementary collapsing,[19] which has been used in the sensor network coverage verification problem[28] but not in the sparse cover problem yet. The elementary collapsing process removes 2 simplices (locally) at a time, which is similar to the "S-reduction pair" removal approach on "S-complexes" found in Dłotko et al.[56]

The 3 approaches discussed here are greedy node removal processes. These processes have the potential for finding a sparse cover faster at the potential cost of losing the optimal (minimal) cover. The first approach is just locally calculating the change in homology when a node is removed. The second approach utilizes the notion of strong collapsing.[57,58] The third approach uses the Euler characteristic.

4.1 Local Homology Changes

Calculating changes to the homology when a node is removed is the simplest greedy scheme, but also potentially the most computationally expensive. The premise is that the computation can be performed locally, under the specified assumptions. This approach has been used before for the sparse cover problem. Varposhti et al.[14] and Vergne et al.[15] relied on calculating the change in homology when a node is removed.* Both works presented a centralized algorithm, but the conversion to a distributed variant is trivial and is described in Algorithm 1.

Consider if a non-fence node v is removed from the sensor network. Effectively, this removes the sensing range set $B(x_v, r_s)$ from the union of sets composing the cover $\mathcal{U}(\mathcal{X})$. In the simplicial complex representations, this removes any of the sim-

*Vergne et al.[15] selected the node for removal using a notion of "index", but still calculated the change in homology before removal.

plices containing v in the Čech and Rips complexes. Let $\mathcal{S}(v)$ denote the simplicial complex of the *closed star of* v (i.e., the minimal subcomplex of $\mathcal{R}_{r_c}(\mathcal{X})$ containing every simplex v is contained in). Let $\mathcal{L}(v)$ denote the simplicial complex of the *link of* v (i.e., the minimal subcomplex of $\mathcal{R}_{r_c}(\mathcal{X})$ containing the neighboring nodes of v).[44] Thus, the removal of node v from $\mathcal{R}_{r_c}(\mathcal{X})$ removes all of the simplices in the star $\mathcal{S}(v)$ except for those simplices in the link $\mathcal{L}(v)$. This gives the following result:

Theorem 1. Under Assumptions 1–7, the domain $\mathcal{D}$ remains covered with the removal of node v if $\text{rank}(\mathcal{H}_1(\mathcal{L}(v))) = 0$.

Proof. Suppose the removal of v causes a gap in sensor coverage. Then, by the Nerve Theorem,[49,50] there exists a nontrivial first homology group in the Čech complex without v. Since v is in the interior of $\mathcal{D}$ (it is not a fence node), then Eq. 1 implies the same for the Rips complex without v. Note that this implies there exists a cycle that previously bounded but now does not bound. Since the only change is the removal of simplices containing v, the cycle must previously have been a boundary of some subset of the removed simplices. Note, $\partial\mathcal{S}(v) = \mathcal{L}(v)$, so the now-nonbounding cycle must reside in the link of v. Hence, $\text{rank}(\mathcal{H}_1(\mathcal{L}(v))) \neq 0$. □

Since each node is aware of its local neighborhood by assumption, each node can detect if its removal causes a gap in the sensor coverage by detecting a change in the homology from $\mathcal{S}(v)$ to $\mathcal{L}(v)$. For the coverage problem here, since it is assumed the $\text{rank}(\mathcal{H}_1(\mathcal{S}(v))) = 0$, then only $\text{rank}(\mathcal{H}_1(\mathcal{L}(v)))$ needs to be checked. If $\text{rank}(\mathcal{H}_1(\mathcal{L}(v))) = 0$, then v is redundant to the cover; otherwise, v is necessary to the homological coverage criterion and likely necessary to the cover. Theorem 1 applies to the removal of a single vertex at a time in its local neighborhood. Neighboring nodes turned off simultaneously may affect the homological coverage, so nodes must cooperate in a decentralized manner to ensure adjacent nodes are not turned off simultaneously during each iteration. Algorithm 1 presents a scheme that satisfies this requirement.

The algorithm is only iterated on non-fence nodes to ensure coverage of the domain. At some iteration time, $\text{rank}(\mathcal{H}_1(\mathcal{L}(v))) \neq 0$ for every remaining non-fence node in the network since there are always a decreasing number of remaining nodes after each iteration. For the nodes to obtain a stop point, either a time-out or flooding protocol can be used when each node detects no changes in its local neighborhood.

Algorithm 1 Calculating homology changes locally

```
Calculate rank(H_1(L(v))).
if rank(H_1(L(v))) = 0 then
    Broadcast self as candidate for collapse to neighbors
    if All neighboring nodes broadcast themselves as non-candidates then
        v not needed for cover, broadcast sleep message (collapse)
    else if Some neighboring nodes also broadcast themselves as candidates for
    collapse then
        if Node v has a larger ID than all candidate neighbors then
            Broadcast sleep message (collapse)
        else if Candidate neighbor node u has a larger ID than v then
            Broadcast self as non-candidate (still needed for cover)
            if u broadcasts sleep/collapse message then
                Eliminate simplices containing u from the local information in v
                Recalculate rank(H_1(L(v)))
            end if
        end if
    end if
else
    Broadcast non-candidate status to neighbors
    if Neighbor u broadcast sleep/collapse message then
        Eliminate simplices containing u from the local information in v
        Recalculate rank(H_1(L(v)))
    end if
end if
```

The computational complexity of Algorithm 1 depends mostly on the complexity of calculating the homology. Essentially, this requires finding the rank of 2 matrices in Eq. 5 of size on the order of the number of 1- and 2-simplices. In the worst case, there are $O(d^3)$ 2-simplices, where d is the degree of the node, and by standard methods, such as Gaussian elimination, finding the rank is $O(d^9)$ per iteration at the node. The communication cost per node is at least one message sent per round, with additional messages required if the node is a candidate for removal.

4.2 Strong Collapsing in Sensor Networks

The second approach uses the notion of a *strong collapse*, which requires some further background. If every facet that contains a vertex $v \in K$ also contains the vertex w, then w is said to *dominate* v. If this is the case, then a *strong collapse* of the vertex v in K is the removal of every simplex that contains v from K with nothing else changed (Fig. 2). This is already proven by Barmak and Minian[57] and described by Wilkerson et al.,[58] but here we shall show a slightly stronger property first stated by Wilkerson et al.[58] To prove this stronger homology-preserving property of strong collapsing, we need to introduce the notion of relations and the conjugate complex.

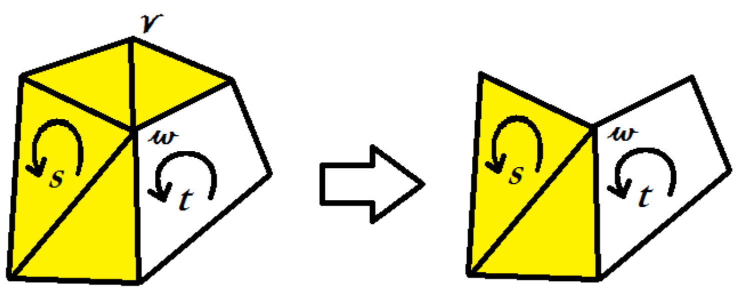

Fig. 2 Strong collapsing: A vertex is dominated by another vertex. Vertex v is dominated by vertex w. Cycle (*s*) bounds in the topology, whereas cycle (*t*) does not bound.

A *(binary) relation* R between the elements of 2 sets $\mathcal{A}$ and $\mathcal{B}$ induces 2 *labeled simplicial complexes*. For example, in one instance, we associate a simplex of elements of $\mathcal{B}$ if each element is related by R to an element $a \in \mathcal{A}$ (i.e., $\sigma_a = \{b \in \mathcal{B} : aRb\}$). This complex induced by this collection of sets is denoted as $K_{\mathcal{A}}(\mathcal{B}, R)$. For clarity, in this instance, we call $\mathcal{B}$ the vertex set and $\mathcal{A}$ the label set. This is equivalent to treating this relation as an incidence matrix where the labels (simplices) correspond to the rows and the vertices correspond to the columns of the matrix. Naturally, by this construction, every facet must have a label. However, not all labeled simplices are facets. Its *conjugate complex*, wherein the roles of the sets are reversed, is denoted $K_{\mathcal{B}}(\mathcal{A}, R^{-1})$. This is equivalent to the transpose of the incidence matrix. A classical result by Dowker[59] states that $K_{\mathcal{A}}(\mathcal{B}, R)$ and $K_{\mathcal{B}}(\mathcal{A}, R^{-1})$ have the same homology.

Theorem 2. Strong collapsing preserves homology.

Proof. Let v be dominated by w in K. Suppose v is contained in each of the facets $f_1, f_2, \ldots, f_m$ and w is contained in each of the facets $f_1, f_2, \ldots, f_m, \ldots, f_n$ with $n \geq m$ (i.e., w is contained in every facet containing v or w dominates v). Let $\mathcal{A}$ be the facet set and $\mathcal{B}$ the vertex set of K, so that $K = K_{\mathcal{A}}(\mathcal{B}, \ni)$. Then v and w, respectively, represent the labeled simplices $f_1 f_2 \ldots f_m$ and $f_1 f_2 \ldots f_n$ in the conjugate $K_{\mathcal{B}}(\mathcal{A}, \in)$. Since $\{f_1, f_2, \ldots, f_m\} \subset \{f_1, f_2, \ldots, f_n\}$, then the labeled set $v = f_1 f_2 \ldots f_m$ is a face of $w = f_1 f_2 \ldots f_n$ and is redundant to the collection of simplices in $\mathcal{B}$ that induces the simplicial complex $K_{\mathcal{B}}(\mathcal{A}, \in)$. Let $\mathcal{B}' = \mathcal{B} - \{v\}$ and let $\mathcal{A}' = \{\sigma - \{v\} : \sigma \in \mathcal{A}\}$. Then $K_{\mathcal{B}}(\mathcal{A}, \in) \cong K_{\mathcal{B}'}(\mathcal{A}', \in)$, which by Dowker[59] has the same homology as $K_{\mathcal{A}'}(\mathcal{B}', \ni)$. □

Corollary 3. Under Assumptions 1–7, the domain $\mathcal{D}$ remains covered with the strong collapse (removal) of a dominated node v.

In the sensor network topology, defining $\mathcal{A}$ to be the facets of the Rips complex $\mathcal{R}_{r_c}(\mathcal{X})$ and $\mathcal{B}$ to be the its vertices, we have $K_{\mathcal{A}}(\mathcal{B}, \ni) = \mathcal{R}_{r_c}(\mathcal{X})$. Strong collapsing can be implemented in a decentralized manner to eliminate redundancies in the homology. If $f_1, f_2, \ldots, f_m$ are the facets containing node v, then an algorithm for node v is described in Algorithm 2.

Before discussing the algorithm, we first discuss the stronger property of strong collapsing that is particularly applicable in sensor networks. We can show that strong collapsing preserves at least one *minimal cycle* generator for each nontrivial homology class, where a generator is a minimal k-cycle if there exists no other k-cycle consisting of fewer k-simplices.[58]

Theorem 4. Strong collapsing preserves a minimal cycle in each nontrivial homology class.

Proof. Let v be a vertex existing in a minimal k-cycle $\sum_i \sigma_i^{(k)}$ corresponding to a nontrivial class in $H_k(K)$ and suppose that v is dominated by w in K. Consider

Algorithm 2 Calculating node strong collapsibility

```
Set f = ⋂_i^m f_i
if f \ {v} ≠ ∅ then
    Choose largest alphanumeric ID w of the nodes in f \ {v}
    Broadcast request to collapse (not needed for cover) to w
    if w requests to collapse to v then
        Node with smallest ID among w and v collapses and broadcasts collapse message to neighbors
        Node with largest ID broadcasts non-candidate status for collapse (needed for cover)
        Node with largest ID ack request from collapsing node and updates facet list
    else
        Receive ack from w
        Broadcast collapse message to neighbors
    end if
else
    Broadcast non-candidate status for collapse (needed for cover)
    if Request received from neighboring node u to collapse to v or collapsing message received from u then
        Ack request from neighboring node u
        Update facet list, removing node u
    end if
end if
```

the subcomplex Λ consisting of the simplices $\sigma_i^{(k)}$ when $v \notin \sigma_i^{(k)}$ and the simplices $\{w\} \bigcup \sigma_i^{(k)}$ when $v \in \sigma_i^{(k)}$. The inclusion of w in Λ does not harm the k-cycle that doesn't bound, since it does not in K. Note that v is still dominated by w in Λ by construction. Define

$$\tau_i^{(k)} = \begin{cases} \sigma_i^{(k)} & \text{if } v \notin \sigma_i^{(k)} \\ (\sigma_i^{(k)} \bigcup \{w\}) - \{v\} & v \in \sigma_i^{(k)} \end{cases} \tag{7}$$

A strong collapse of v leaves Λ with only the facets $\tau_i^{(k)}$. Since strong collapsing preserved homology by Thm. 2, then $\sum_i \tau_i^{(k)}$ must be a minimal k-cycle corresponding to the nontrivial class in $H_k(\Lambda)$ otherwise, since each $\tau_i^{(k)} \subset K$, a cycle with fewer simplices exists in $H_k(K)$, contradicting the assumption that $\sum_i \sigma_i^{(k)}$ was minimal. □

Corollary 5. Under Assumptions 1–7, the minimal cover of the domain $\mathcal{D}$ remains with the strong collapse (removal) of a dominated node v.

Essentially, the vertex w replaces the dominated node v in the minimal cycle (i.e., if v is a dominated node in a minimal cycle, then there exists a homologous minimal cycle). This is a particularly relevant feature for sensor networks. For example, preserving the minimal cycle bounding a homological hole is critical for the hole localization problem[12] and the coverage repair problem.[30]

This property also has consequences for the sparse cover problem. Recall that if a cover exists in the usual first homology of $\mathcal{R}_{r_c}(\mathcal{X})$, then there exists a generator in the second relative homology group $\mathcal{H}_2(\mathcal{R}_{r_c}(\mathcal{X}), \mathcal{R}_{r_c}(\mathcal{F}))$.[12,13] This generator can be found by placing a "cone" over the fence subcomplex (i.e., adding an imaginary node ι to each simplex in the fence $\mathcal{R}_{r_c}(\mathcal{F})$). Since fence nodes are self-aware, they presume communication with this imaginary node to construct this extended Rips complex. This creates a sphere with the cover as part of the surface. The preservation of the cycle in the second homology of this extended Rips complex preserves a minimal cover of the Rips complex.

Theorem 6. Let $\mathcal{R}^+_{r_c}(\mathcal{X})$ denote the extended Rips complex with the imaginary node ι connected to each fence node in the communication graph. Under Assumptions 1–7, if a fence node v strongly collapses in $\mathcal{R}^+_{r_c}(\mathcal{X})$, it strongly collapses to another fence node.

Proof. Let v be a fence vertex dominated by w. Since v is a fence vertex, it is adjacent to the imaginary vertex ι in the augmented communication graph. So there exists a facet $\sigma \ni v$ that also contains ι. Since w dominates v, w must also reside in σ and, thus, be adjacent to ι in the augmented communication graph. Hence, w is a fence vertex. □

This implies that a cycle of fence nodes circumnavigating the perimeter of the domain remains after strong collapsing. Unlike other protocols, the fence nodes can participate, if desired, in the strong collapsing process with the guarantee that a cycle covering the boundary $\partial\mathcal{D}$ will remain in the collapsed complex.

Now, returning to Algorithm 2, first note that unlike the simple homology calculation approach in Algorithm 1, this strong collapsing algorithm allows multiple nodes to collapse within a single iteration. However, now care is taken in the case when 2 nodes dominate each other. In this case, only one can collapse since strong collapsing only preserves homology when the collapsing node's dominating vertex remains in the complex. In fact, the removal of 2 co-dominate vertices usually alters the homology.

As before, the algorithm is guaranteed to obtain a steady-state where $f \setminus \{v\} = \emptyset$ for every remaining node in the domain, so only a (time-out or flood) protocol is required to end the strong collapsing process. The complexity of the algorithm is in searching the facet list for a node that dominates v and searching the facets for a dominated node u for removal. This depends on the data structure.* Intersecting 2 sets can be done in $O(p)$, where p is the size of the sets, in this case, the dimension of the complex. If m is the *facet degree* (number of facets connected to a node[60]), then finding f takes at worst $O(pm)$ for each node. For removal of vertices in a double incidence list data structure (i.e., for each neighbor of v a list is attached of labeled facets it is a member of and for each labeled facet a list is attached of the neighbors in each facet), the complexity is at worst $O(d + pm)$, where d is the degree of the node. For finding a dominating node of v, the complexity is $O(d)$ if each list of neighbors also records the number of its elements. Hence, the complexity is $O(d + pm)$ per iteration at the node. The communication cost of the algorithm requires at least one message sent per iteration if neither v collapses nor

*An alternative equivalent algorithm[32] can actually be executed simply from the neighbor information whereby each node compares its neighbor list with that of each of its neighbors. This can be done prior to the construction of any simplices.

any of its neighbors collapse. Otherwise, an additional message is needed for each vertex that collapses to v and at least 2 messages are needed if v collapses.

4.3 Euler Characteristic Changes

The last approach discussed here is heuristic, although simulations in the next section show it to be reliable. The Euler characteristic[45] is another topological invariant that categorizes classes of topological structure. This characteristic has 2 definitions for a simplicial complex K:

$$\chi_1(K) = \sum_k (-1)^k f_k, \tag{8}$$

$$\chi_2(K) = \sum_i (-1)^i b_i, \tag{9}$$

where f_k is the kth face number defined by $f_k = \left|\{\sigma^{(k)} \in K\}\right|$ and b_i is the ith Betti number defined in Eq. 4. It is trivial to show that $\chi_1(K) = \chi_2(K)$ (using an induction argument on the number of simplices).

The general greedy strategy for finding a sparse cover is for the node to perform the necessary calculations to determine if its removal changes the homology. If the homology does not change with a node removal, then it is clear from Eq. 9 that the Euler characteristic will also not change even though the number of simplices in the sum of Eq. 9 have changed. The converse statement is not generally true. The homology can change, but the Euler characteristic be preserved. A simple example is a node removal that simultaneously creates a cycle and disconnects a component (or creates a separate component).* However, in a planar geometry under Assumptions 1–7, this possibility may be implausible.

For the sensor coverage scenario, each node v can determine the number of simplices it resides and, hence, can count them. So if node v is removed, the change in

*A specific example is the complex generated by the facets $\{abc, acd, ade, abe, af\}$, which is a cone via the vertex a over a square cycle $\{bc, cd, de, eb\}$ and a vertex f. With a, the face numbers are $f_0 = 6, f_1 = 9, f_2 = 4$ and the Betti numbers are $b_0 = 1, b_1 = 0$. Without a, the face numbers are $f_0 = 5, f_1 = 4$ and the Betti numbers are $b_0 = 2, b_1 = 1$. In both cases, $\chi = 1$.

the Euler characteristic is given by

$$\begin{aligned}\Delta\chi(v) &= \chi(\mathcal{R}_{r_c}) - \chi(\mathcal{R}'_{r_c}) \\ &= \sum_k (-1)^k \left|\{\sigma^{(k)} \in \mathcal{R}_{r_c} : v \in \sigma^{(k)}\}\right|. \end{aligned} \quad (10)$$

Conjecture 1. Under Assumptions 1–7, the region remains covered with the removal of a non-fence node v if $\Delta\chi(v) = 0$.

Heuristically examining the scenario, the only way that b_0 can change is if the removal of node v separates a component into multiple components. This can only occur if the link of v is not connected (i.e., not a single component), but this is not possible by assumption. Either b_1 is unchanged (i.e., it remains $b_1 = 0$ as under the assumption and no coverage is lost by the removal of node v) or, if it is changed, then $b_1 > 0$ and there must exist some (even Betti numbers) b_{2i}s that also changed with the removal of v so that the Euler characteristic remains fixed. Any new higher-dimensional nonbounding cycle would cover v's position, which would prevent a nonbounding 1-cycle winding around v's position, and would/should necessarily include simplices covering the circumference of the sensor disk (acting as a cone over any potential cycle not winding around v's position). This argument is not a rigorous proof of the conjecture, but the conjecture holds with high probability for uniformly distributed vertices in a square domain, as discussed in the next section.

The implementation of this approach, described in Algorithm 3, is similar to that for calculating changes in homology in Algorithm 1 in that adjacent nodes cannot be simultaneously removed in the same iteration.

Similarly like the prior 2 greedy approaches, this algorithm will reach a steady-state where $\Delta\chi(v) \neq 0$ for every non-fence node v. The calculation of $\Delta\chi(v)$ in Eq. 10 is equivalent to calculating the Euler characteristic of the link of the vertex, which is $O(q2^{\min(d,m)})$ per iteration at each node, where d is the node degree, m is the facet degree, and q is a polynomial.[61] The communication cost is exactly the same as computing local homology changes.

Algorithm 3 Calculating Euler characteristic changes locally

```
Calculate Δχ(v).
if Δχ(v) = 0 then
    Broadcast self as candidate for collapse to neighbors
    if All neighboring nodes broadcast themselves as non-candidates then
        v not needed for cover, broadcast sleep message (collapse)
    else if Some neighboring nodes also broadcast themselves as candidates for
    collapse then
        if Node v has a larger ID than all candidate neighbors then
            Broadcast sleep message (collapse)
        else if Candidate neighbor node u has a larger ID than v then
            Broadcast self as non-candidate (still needed for cover)
            if u broadcasts sleep/collapse message then
                Eliminate simplices containing u from the local information in v
                Recalculate Δχ(v)
            end if
        end if
    end if
else
    Broadcast non-candidate status to neighbors
    if Neighbor u broadcast sleep/collapse message then
        Eliminate simplices containing u from the local information in v
        Recalculate Δχ(v)
    end if
end if
```

5. Simulations

We reconsider the scenario from Vergne et al.[15] and Moore,[18] wherein p nodes are randomly distributed in a square domain of length 2 with 8 fence nodes located at the corners and midpoints of the square edges. The communication network follows a unit disk graph model (i.e., the communication radius is $r_c = 1$), so that the fence nodes form a simply connected cycle that covers the boundary. On 1000 realizations for each case of $p = 18, 21, 24, 27, 30$ interior nodes (with the 8 fence nodes), the Rips complex is constructed from the communication graph. We check that the generated sensor network covers the domain in the homological sense under the conditions of Assumptions 1–7. On each sensor network realization, we implement the 3 approaches outlined in the previous sections.

For local homology calculations and strong collapsing, Thms. 1 and 2, respectively, guarantee that homology will be preserved. For Euler characteristic calculations, we do not have this guarantee. So for this last process, additional checks were performed to test if the homology $\mathcal{H}_1$ is changed at any point in the process (i.e., after every iteration). The $\mathcal{H}_1$ homology was preserved at each step throughout. This gives some indication that reduction via Euler characteristic calculations will at the very least preserve homology with high probability in planar networks.

5.1 An Illustrative Example

An illustrative example of a realization of the sensor placement, coverage, and communication connectivity is given in Fig. 3. The sensor positions are denoted by an $*$ (asterisk) symbol. When 2 sensor positions are separated by no more than $r_c = 1$ unit distance, a line is drawn to represent communication connectivity. Around each sensor, with radius $r_s = \sqrt{3}/3$, is a lightly shaded region denoted its sensing range. Darker regions indicate many overlaying sensing disks and, thus, the degree of redundancy even for this case of $p = 18$ interior sensors.

Also illustrated in Fig. 3 is the application of strong collapsing for this realization. The sensor nodes that are turned off (or put in a sleep state) using Algorithm 2 are now denoted with a $\circ$ (circle) symbol, while the nodes that are still needed to maintain full homological coverage are still denoted with a $*$ symbol. For this particular realization, only 5 of the original 18 nodes are required for a sparse cover. Moreover, the result shown is also a minimal cover since we know that strong collapsing

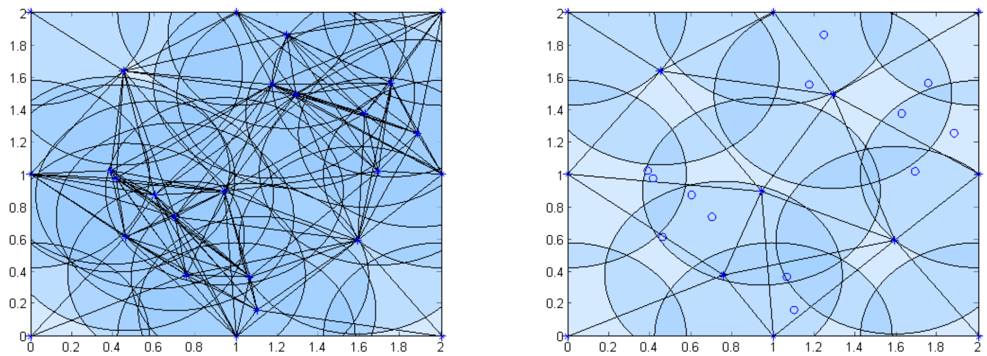

Fig. 3 Sensor network coverage/communication realization and the associated (strong collapsing) sparse cover. A realization of a sensor network with $p = 18$ interior nodes before (left) and only 5 interior nodes after strong collapsing (right).

preserves such a cover from Thm. 4 and by observation it is clear that if any node is removed, a coverage hole or gap would be formed.*

Results of sparse covers found by local calculations of homology changes and Euler characteristic changes are not shown since this is just an illustrative example of a sparse cover outcome.

5.2 Distributions of Sparse Covers Results

Figure 4 shows the mean and median size of the sparse covers found for the realizations via the 3 methods: local homology calculations (blue), strong collapsing (black), and Euler characteristic calculations (red). The results for local homology calculations and Euler characteristic calculations are identical for the median and nearly so for the mean. Strong collapsing, while preserving a minimal cover (Thm. 4), typically results in a larger sparse cover compared to the other 2 approaches.

The reasons for both of these observations are explained by the illustrations in Fig. 5. First, note that Algorithm 1 does not calculate the homology (or even the rank) in every dimension. Since the region of interest is planar, it is only necessary to check $\mathcal{H}_1$. Hence, it is possible that removing a node can change the homology in a higher dimension ($\mathcal{H}_0$ cannot change since the preservation of the fence nodes and coverage assumption ensures connectivity). In the first illustration, the removal of the central node g creates an octahedron, which has a void in the interior. Since it

*The measure of minimal for the cover is measured in the number of 2-simplices, or triangles, and not the number of nodes.

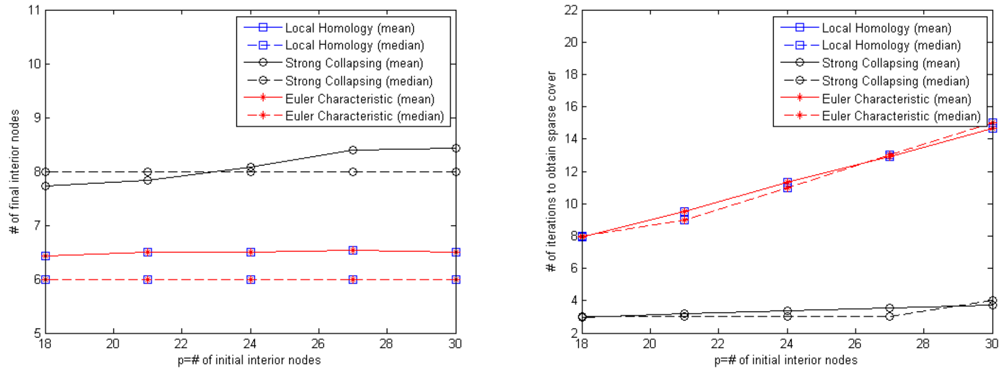

Fig. 4 Final number of nodes in the sparse covers and iterations needed to obtain covers. Mean and median values for the final number of interior nodes in the sparse cover (left) and the number of iterations required (right) to reach that state for each method: local homology calculations (blue), strong collapsing (black), and Euler characteristic calculations (red).

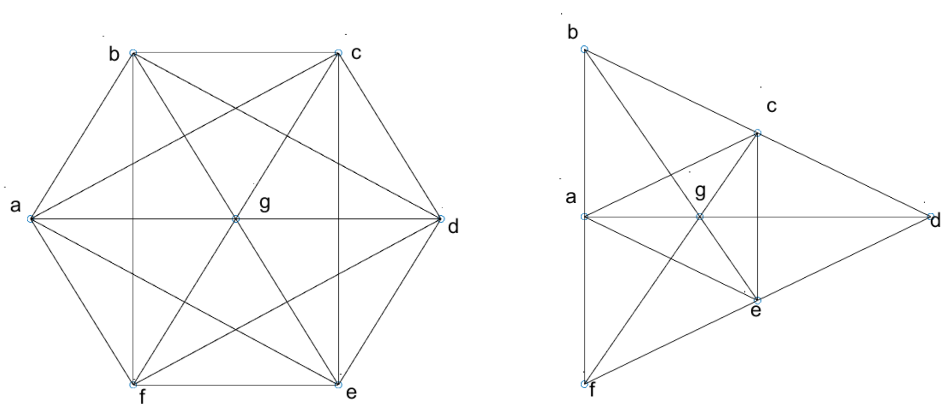

Fig. 5 Examples demonstrating feasibility relations among local homology calculations, strong collapsing, and Euler characteristic calculations. It is possible for the removal of a node (g) via local homology calculations when it cannot happen via Euler characteristic calculations or strong collapsing (left). Moreover, is it possible for the removal of a node (g) via both of these methods when it cannot happen via strong collapsing (right). Only the communication links are shown for clarity; the Rips complex consists of the cliques.

only changes $\mathcal{H}_2$, then the central node is a candidate for removal in Algorithm 1. However, the addition of a nontrivial group in $\mathcal{H}_2$ changes b_2 and the Euler characteristic via Eq. 9. So the central node is not a candidate for removal in Algorithm 3. However, this situation is rare as indicated in Fig. 4.

Second, note that since strong collapsing preserves homology (Thm. 2), then when a node can be removed in Algorithm 2 the homology and Euler characteristic will be unchanged. Hence, the node can also be removed by Algorithms 1 and 3. The converse is not true. This is illustrated in the second example, where the central node g is part of three 3-simplices that each have a unique node from the other pair. So the central node is not dominated, but its removal is clearly possible since $\text{rank}(\mathcal{H}_1(\mathcal{L}(g))) = 0$ and $\Delta\chi(g) = 0$. This occurs frequently as indicated in Fig. 4.

In each method, Fig. 4 shows that the median sparse cover size remains unchanged and the mean nearly so over the range of initial interior nodes p. This result is not surprising, since the size of the domain remains fixed and the typical number of nodes necessary for a cover should not change. There is a slight growth in the mean size in the sparse cover after strong collapsing, which can likely be attributed to increasing occurrences of instances similar to that described in Fig 5.

The distributions of the sparse cover sizes for the realizations in the cases of $p = 18$ and $p = 30$ are shown in Fig. 6. This illustrates the slight rightward shift in the sparse cover distributions as the number of initial nodes is increased. The shift is more significant for strong collapsing, where the number of sparse covers at least as large as 10 interior nodes doubles. For much denser networks, if one seeks a smaller cover, then local homology and Euler characteristic calculations may be preferred.

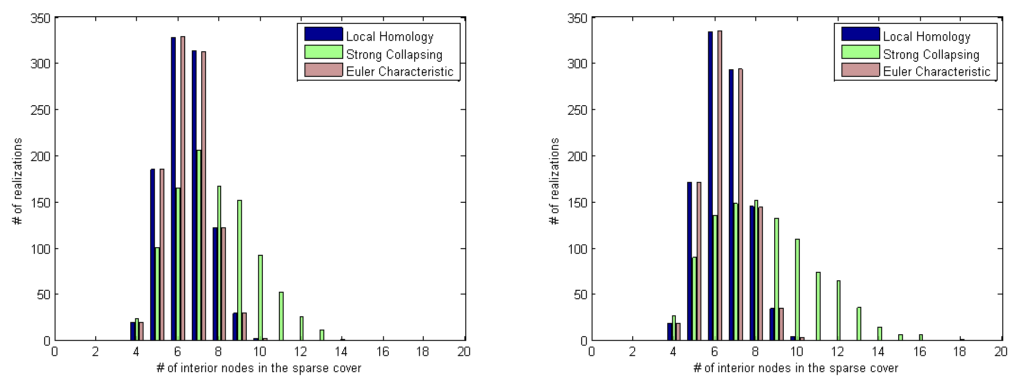

Fig. 6 Sparse cover size distributions. Distribution of the final number of interior nodes remaining in the sparse cover for local homology calculations (blue), strong collapsing (green), and Euler characteristic calculations (pink) in the case of $p = 18$ initial nodes (left) and $p = 30$ initial nodes (right).

We also calculated (not shown) the percentage when the various methods obtained a minimal cover (in terms of the number of nodes). For $p = 18$, strong collapsing finds a minimal cover for 23.5% of the realizations. This nearly doubles to 49.1% for local homology and Euler characteristic calculations. For $p = 30$, these rates drop to 8% and 17%, respectively. Also, we found that after strong collapsing, on average, more than 20% (when $p = 18$) to 35% (when $p = 30$) of the nodes would be candidates for removal via local homology or Euler characteristic calculations.

The mean and median number of iterations required for each approach are also shown in Fig. 4: local homology calculations (blue), strong collapsing (black), and Euler characteristic calculations (red). Again, we note that the difference between local homology and Euler characteristic calculations is minimal. Both methods indicate predictable near linear growth with respect to p in the number of iterations required to obtain the sparse cover (i.e., the methods require a little greater than one more iteration for each additional 2 nodes). On the other hand, the method of strong collapsing obtains its sparse cover nearly independently with respect to p. This is because, unlike the other 2 methods, strong collapsing allows adjacent nodes to collapse simultaneously to a dominating node.

The distributions of the number of iterations required to obtain the sparse cover for the realizations in the cases of $p = 18$ and $p = 30$ are shown in Fig. 7. Strong collapsing clearly performs best in terms of the number of iterations. Fewer than 3% of realizations of the local homology or Euler characteristic calculations took as few iterations as the worst-performing realizations for strong collapsing when $p = 18$. This difference becomes more stark as p grows. When $p = 30$, none of the realizations of strong collapsing took as many as 8 iterations, whereas none of the realizations of local homology and Euler characteristic calculations took as few as 8 iterations.

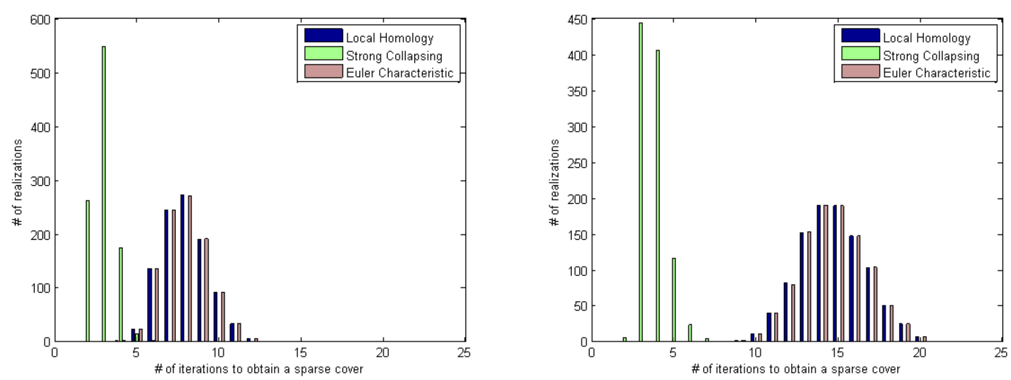

Fig. 7 Total iterations required to obtain cover distributions. Distribution of the number of iterations required to obtain the sparse cover for local homology calculations (blue), strong collapsing (green), and Euler characteristic calculations (pink) in the case of $p = 18$ initial nodes (left) and $p = 30$ initial nodes (right).

5.3 Comments on Computational Complexity

The comparisons of total iterations needed to obtain a sparse cover for the various methods in Fig. 7 is not the fairest comparison. As discussed for each method in Section 4, while the communication costs are similar, the computational complexity can be very different depending on the approach and on the basic network characteristics (e.g., average or maximum values of node degree, facet degree, and size of the facets).

Note the size of the facets is generally bounded by the maximum degree. So under the assumption that the facet degree scales with the node degree, then it is easy to see that the worst-case complexity of the strong collapsing computations will be less than that of the other 2 methods. It is always true that $d + pm < d^9$, and except for low values of d, p, m, it is also true that $d + pm < q2^{\min(d,m)}$. (This last inequality can be inferred from the fact that $\log(x)/x$ is a decreasing function for large x.) This is confirmed experimentally in the simulations. Strong collapsing took orders of magnitude less computation time than local homology calculations, and Euler characteristic calculations took less time than strong collapsing when $p = 18$ and more time with $p = 30$.

Comparing local homology calculations with Euler characteristic equations, note that for $d^9 < 2^{\min(d,m)}$, then the degree must satisfy $\log(d)/d < log(2)/9$. This implies the degree needs to be quite large (i.e., $d > 50$) before the counting problem of finding the Euler characteristic surpasses the rank problem of determining the number of homology classes b_1. (This, of course, assumes worst-case complexity and ignores constant coefficients.) For the sensor network realizations examined in this report, this never occurred and Euler characteristic computations took less time than local homology computations.

6. Conclusion

Several simple greedy methods of determining a sparse cover in a location-unaware fenced sensor network via homological methods are presented. None of these approaches is presumed or proposed as a standalone process for finding a cover, but as preprocessing steps, potentially including other approaches,[12,14–17] selected based on the sensor network density characteristics (i.e., node degree, facet degree, and facet size). Strong collapsing seems a particularly attractive preprocessing option for several desirable features. It preserves a minimal cover (Thm. 4). It can be executed prior to the construction of the simplicial complex.[32] Unlike the other methods presented here, it enables the greedy removal of adjacent nodes. Calculating changes in the Euler characteristic also seems a potentially promising option. The justification for the process (i.e., that a nonbounding 1-cycle will not be created [Conj. 1]) still needs to be proven rigorously. However, simulation results are encouraging in that the method at least works with high probability.

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