A TRIDENT SCHOLAR PROJECT REPORT

NO. 461

Algebraic Structure of Dynamical Systems

by

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UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MARYLAND

USNA-1531-2

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
05-22-17

2. REPORT TYPE

3. DATES COVERED *(From - To)*

4. TITLE AND SUBTITLE
Algebraic Structure of Dynamical Systems

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
James P. Talisse

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Naval Academy
Annapolis, MD 21402

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
Trident Scholar Report no. 461 (2017)

12. DISTRIBUTION / AVAILABILITY STATEMENT

This document has been approved for public release; its distribution is UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

A dynamical system is a mathematical object which describes the motion of a set of points over time. Viewed as a purely mathematical object, one can ask questions about the behavior of the dynamical system based on the structure of algebraic objects associated with it. In this project we study two algebraic objects, centralizers and topological full groups, associated to symbolic dynamical systems. The centralizer group tells us about the symmetries a system possesses. Results relating to the centralizer historically have indicated that the more complex the dynamical system is, captured by the Topological Entropy, the more structure its centralizer has. Similarly, low complexity systems have been shown to have very simple centralizers. This seems to suggest that one can recover information about the dynamical system based upon its centralizer group. In particular, if a system is known to have a certain centralizer group, we might want to draw conclusions about the complexity of the system. In this project we present a class of high complexity systems which have a very rigid centralizer, which shows the relationship is more subtle than may have been originally thought.

We also study the topological full group of a dynamical system. This group completely defines the system up to time reversal. We apply numerical estimates to draw conclusions about the algebraic properties of this group. In particular, we seek to know when the topological full group of a dynamical system is amenable. Amenability is an algebraic property that can be thought of as having a probability measure on G. This measure would answer the question: given a subset A of G, what is the probability that a random element of G is in A? We apply Grigorchuk's amenability criterion to answer this question.

15. SUBJECT TERMS
Dynamical Systems, Toeplitz System, Centralizer, Odometer, Topological Full Group, Amenability

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**		54	**19b. TELEPHONE NUMBER** *(include area code)*

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

U.S.N.A — Trident Scholar project report; no. 461 (2017)

ALGEBRAIC STRUCTURE OF DYNAMICAL SYSTEMS

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Abstract

A dynamical system is a mathematical object which describes the motion of a set of points over time. Dynamical systems can be used to study differential equations, cryptography, computer science, and even biology. Viewed as a purely mathematical object, one can ask questions about the behavior of the dynamical system based on the structure of algebraic objects associated with it. In this project we study two algebraic objects, centralizers and topological full groups, associated to symbolic dynamical systems. The centralizer group tells us about the symmetries a system possesses. Results relating to the centralizer historically have indicated that the more complex the dynamical system is, captured by the Topological Entropy, the more structure its centralizer has. Similarly, low complexity systems have been shown to have very simple centralizers. This seems to suggest that one can recover information about the dynamical system based upon its centralizer group. In particular, if a system is known to have a certain centralizer group, we might want to draw conclusions about the complexity of the system. In this project we present a class of high complexity systems which have a very rigid centralizer, which shows the relationship is more subtle than may have been originally thought.

We also study the topological full group of a dynamical system. This group completely defines the system up to time reversal. We apply numerical estimates to draw conclusions about the algebraic properties of this group. In particular, we seek to know when the topological full group of a dynamical system is *amenable*. Amenability is an algebraic property that can be thought of as having a probability measure on G. This measure would answer the question: given a subset A of G, what is the probability that a random element of G is in A? We apply Grigorchuk's amenability criterion to answer this question.

Both these results provide us with information about the algebraic structure of dynamical systems. If we know certain information about the different groups associated with a dynamical system, we can make conclusions about the system itself. As such, questions about dynamical systems can now become questions about algebra, and vice versa. These results mostly reveal the structure of symbolic dynamical systems and address the fundamental question of mathematics about what is possible. However, our construction of a positive entropy system with trivial centralizer can be interpreted as the existence of an information channel with positive capacity that cannot be encrypted with substitution ciphers.

Key Terms. Dynamical Systems, Toeplitz System, Centralizer, Odometer, Topological Full Group, Amenability

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Acknowledgments

The author would like to acknowledge his adviser, Assistant Professor K. Medynets. He has devoted countless time and effort into helping make sure this is a successful project.

Additionally, the author would like to thank the Trident Scholar Committee for affording him the opportunity to perform this research.

1. Paper Layout

This paper is intended to be a self contained study on the Algebraic Structure of Dynamical Systems. The paper begins with some background and motivation in Section 2. In this section, we introduce where dynamical systems come from, and why they are important. Here we discuss the wide range of applications of dynamical systems from chemistry to physics, and computer science. It is important to note that the research performed in this project was not done with the intention of directly applying it to any of these fields. The main objective of this research project is to advance our knowledge and understanding of Symbolic Dynamical Systems and various algebraic object associated with them. However, we will discuss how one of our main results can be applied to cryptography.

In Section 3, we formulate three of our main results. A short discussion on a potential application of these results is contained in that section.

In Section 4, we provide some basic background on the main objects of our study. This section is intended to be accessible to a general audience.

In section 5, we provide a list of definitions which are important for the rest of the paper. This sections contains all definitions which are later referenced.

Section 6, contains rigorous mathematical arguments. In this section we prove one of our main results. In particular, we show that there is a class of multi-dimensional Toeplitz systems which has a trivial centralizer. This is labeled as Theorem 6.20.

Section 7, contains an explicit construction of a multidimensional Toeplitz system which is contained in the class of systems with a trivial centralizer, but which has positive entropy. In this section, we take a desired a property of a dynamical system, and construct a system that has that property. This differs from other fields of science in that we start with the desired property and construct a system to exhibit that, as opposed to observing properties in pre-defined systems or models. To the best of our knowledge, this construction is the first example of such a system ever constructed.

In Section 8, we introduce the notion of an amenable group and develop the theory of amenability.

In Section 9, we introduce the concept of the topological full group. We study the full group of the Fibonacci Substitution and provide evidence as to why this group is amenable. This evidence is provided in Appendix A as Mathematica code.

2. Introduction/Motivation

The study of general dynamical systems began in 1890 by Henri Poincaré [33]. This text sought to answer the three body problem in astronomy. Loosely, this problem seeks to describe the motion of three astronomical bodies acting under Newton's laws of motion, given some initial conditions about their mass and velocity. Poincaré looked at differential equations describing these bodies' motion, and developed the theory of dynamical systems to solve the three body problem without actually solving the differential equations. In this light, dynamical systems can be thought of the study of trajectories in the phase space of differential equations. Thus the field of dynamical systems was developed.

Gas Dynamics and Symbolic Dynamics. While dynamical systems were first developed to understand and solve differential equations, they have since been used for understanding a far wider range of physical and mathematical phenomena. For example, a dynamical system can arise from studying how gas particles move in an enclosed container. Suppose we have a closed container with a single gas molecule within. Furthermore, suppose we are tasked with tracking the location of the molecule. It would be almost impossible to determine three coordinates of the particle's location at all times. So we can split the container into two discrete halves. Next we label one half of the container A and the other B. After every second, we write down which section of the container the particle is in. What we are left with is a sequence of A's and B's. This process is shown in Figure 1. The sequence can be thought of as extending forever into the future, as well as forever in the past. What we

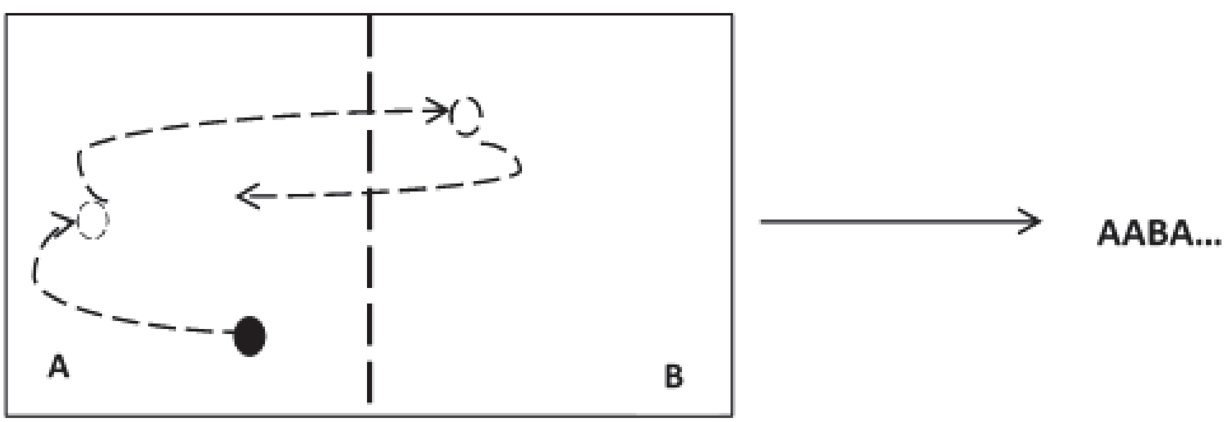

Figure 1. Gas Dynamics

are left with is a bi-infinite sequence of A's and B's. Changing the initial position of the gas particle will give us a different sequence. Collecting all such possible bi-infinite sequences, we have what is called the phase space. This will be denoted by X. We say X is acted on by the shift map T in the following way: for $x = (x_i)_{i=-\infty}^{\infty} \in X$, $(Tx)_i = x_{i+i}$. That is, the shift map T shifts every symbol in each sequence one place to the left. The resulting pair (X, T) is a dynamical system. This is one particular example of a dynamical system arising from a physical process. But dynamical systems can be studied much more generally, as will be done in the rest of the paper.

Quasicrystals. Quasicrystals are crystalline materials which are not perfect crystals. They display the property that they are ordered but have no translational symmetry. That is, given a quasicrystal it is impossible to slide it in any direction so that it perfectly matches itself. This property is called being aperiodic. Crystals, on the other hand, do possess translational symmetry. This sort of aperiodic structure was not thought to exist in nature, but its natural discovery was the subject of the 2011 Nobel Prize in Chemistry. Clearly this kind of structure is very important. It was found that the quasicrystalline structure can in fact be realized as a tiling of the plane. In particular, a specific type of tiling known as the Penrose Tiling is exactly a quasicrystal. Figure 2 depicts a Penrose Tiling of the

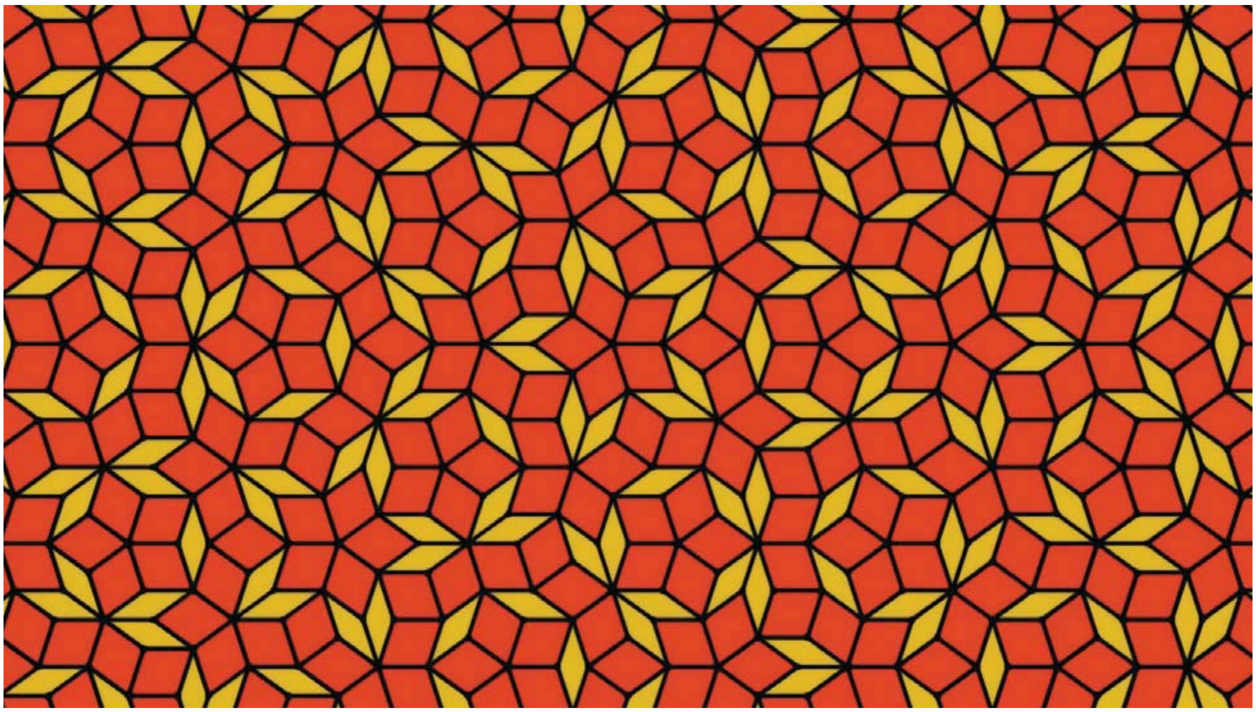

Figure 2. Penrose Tiling

two dimensional plane. We notice the striking structure in the tiling, but it is impossible to translate, or shift, the tiling and get back what we started with. So the study of aperiodic tilings such as the Penrose Tiling is equivalent to studying quasicrystals. Specifically, we will be considering sequences and tilings that have this aperiodicity, and the dynamical systems which arise from them.

Data Storage. Another motivating application comes from data storage in computers. These examples are due to Lind and Marcus ([27]). Data is stored as sequences of zeros and ones on a magnetic tape. As time moves forward, the computer reads the next bit in the data sequence. With each discrete time step, this can thought of as the whole magentic tape shifting over to the left. However the computer is not able to accurately read all sequences of data. For example, if the binary sequence has a very large number of zeros all in a row, it must keep track of how many have passed, which it does by keeping track of the time between instances one ones. However it is very likely that the computer can encounter clock drift. This means that the computer will lose accurate track of the time, hence distorting the data. Another problem that can be encountered is intersymbol interference. This is

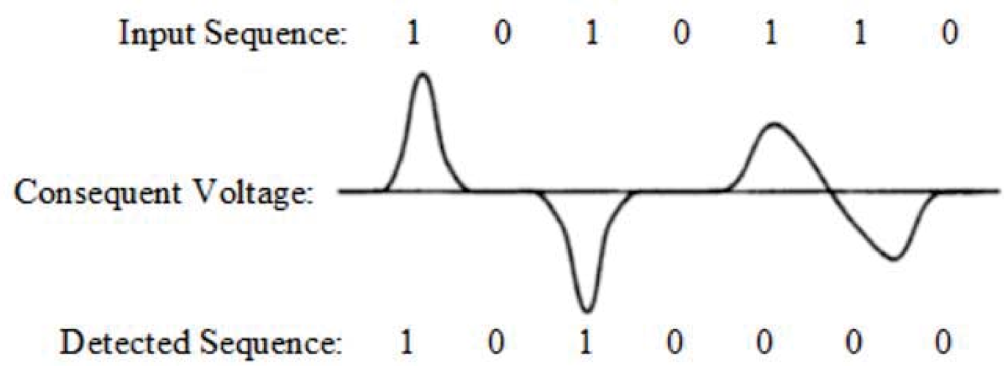

FIGURE 3. Intersymbol Interference

depicted in Figure 3. With each one detected by the computer in the binary sequence, there is an associated voltage drop. Each consecutive one has a voltage drop equal in magnitude but reversed in polarity to the one before it. So, if two ones are too close together, the opposing voltage drops can destructively interfere with each other causing the computer to detect two zeros instead.

These are two issues that computers can potentially face when storing data as binary sequences. In order to mitigate these risks, one can stipulate that the binary sequences used to store data can have no more than three zeros in a row, and no two ones consecutively. Placing these kinds of restrictions on the binary sequences can be studied mathematically. These constraints give rise to subshifts which can be studied with the rich field of dynamical systems.

Next, we will discuss substitution ciphers which can be used to encrypt messages. These types of ciphers have applications in symbolic dynamics. One of the main goals of this research is to determine what the symmetries of certain dynamical systems look like. When we consider data storage in computers as dynamical systems, as previously explained, we can conveniently think of the symmetries of these systems as substitution ciphers. In particular, the mechanism through which substitution ciphers are performed is exactly the same as symmetries of a symbolic dynamical system. One of our main results, which is the construction of a positive entropy subshift with no symmetries is analogous to the existence of an information channel with positive capacity, but which cannot be encrypted using substitution ciphers.

Cryptography and Substitution Ciphers. There are a number of methods which are used to encrypt data. One of the most basic encryption methods is known as the Caesar cipher. In this cipher, the whole alphabet is shifted a certain number of places, and each letter of the unencrypted message is replaced with the corresponding letter from the shifted alphabet. For example, we can shift the alphabet six places forward, as in Figure 4. The outer ring indicates the unencrypted alphabet, while the inner ring indicates what each letter maps to when encrypting. For example, the message "THIS IS A MESSAGE" would be encrypted to "ZNOY OY G SKYYGMK". Decryption is easy, as we then just go from the inside ring out.

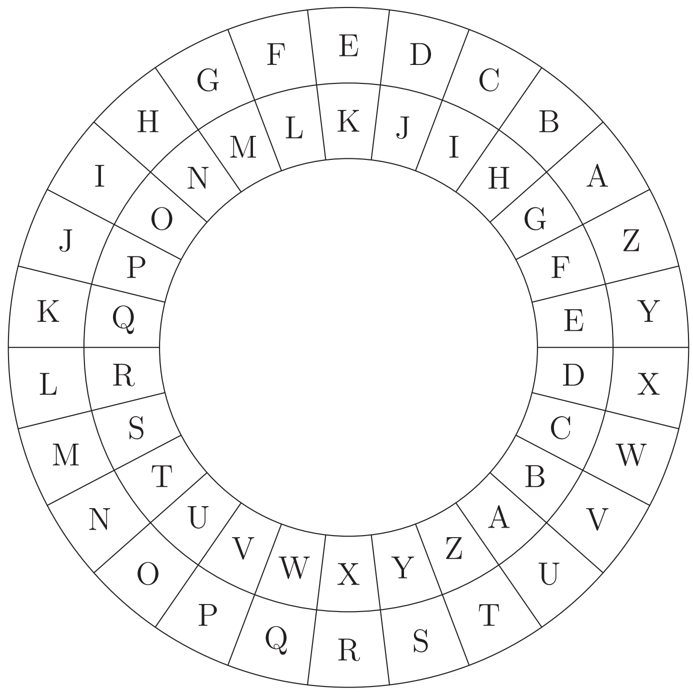

Figure 4. Caesar Cipher

This cipher is very easy to break. For example, we could just try rotating the disk up to 26 times until we had something sensible. Additionally, this is susceptible to frequency analysis. Since it is known which letters in the English language appear most frequently, a reasonable length encrypted message can be subject to frequency analysis.

Another similar type of cipher is known as the substitution cipher. Like the Caesar cipher each letter is replaced with another letter, but not in the same order as the alphabet. For example, in Figure 5 we have the key for a substitution cipher. The top row is the normal alphabet, and the bottom row is the alphabet jumbled up. This tells us what each letter must be substituted with when encrypting. Note that unlike the Caesar cipher, the encrypted alphabet is not in the same order. With this key, the message "THIS IS A MESSAGE" would be encrypted to "HADX DX B SKXXBNK".

Again, this cipher is very easy to crack for the same reasons as above. It is very susceptible to frequency analysis.

However, the substitution cipher can be enhanced where instead of substituting one letter with one letter, we substitute multiple letters with multiple letters. For example, we could

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
B	*V*	*G*	*Q*	*K*	*M*	*N*	*A*	*D*	*Z*	*C*	*W*	*S*	*E*	*O*	*Y*	*F*	*J*	*X*	*H*	*T*	*L*	*P*	*U*	*I*	*R*

Figure 5. Substitution Cipher

come up with a substitution rule that reassigns every pair of two letters with another pair of two letters. This is somewhat challenging in practice since there are $26^2 = 676$ combinations of two letters in the alphabet. Additionally, it is still susceptible to frequency analysis since combinations such as "TH" are more common in English than other two letter combinations. So shifting to even higher substitution blocks would require at least $26^3 = 17576$ substitution rules.

We get around this telescoping issue by limiting which alphabet we are using. In particular, we may only consider substitution ciphers on binary sequences. In this case, the alphabet is only two letters, namely 0 and 1. And so we can substitute blocks of three letters at a time, which would require only $2^3 = 8$ substitution rules.

Under this construct, consider having a binary sequence, which for illustration purposes we will show as the letters being the colors black and white. We can explicitly define a rule for each of the eight possible combinations of these colors over three spaces.

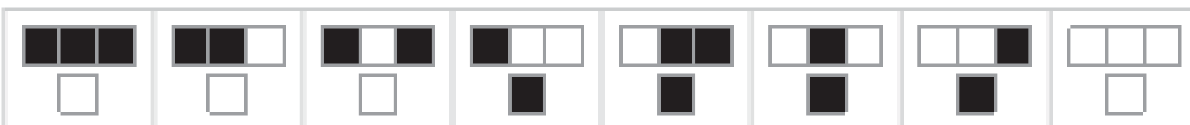

FIGURE 6. Substitution Rules

In Figure 6, we see an example of substitution rules that can be used to encode blocks of three letters at a time. Using these rules, the input sequence at the top of Figure 7 would be encrypted to the output sequence at the bottom. In order to encrypt, three letters at a time are considered, and the rules list is referenced to encrypt the sequence.

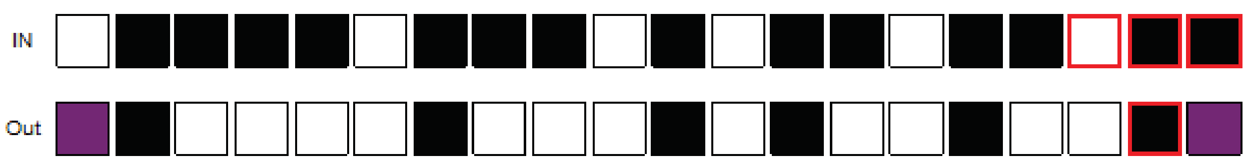

FIGURE 7. Encrypting a Sequence

3. RESULTS

Here we provide a list of our results, and how they can be applied. It is important to note that the mathematical results achieved in this research are intended purely for the development of knowledge. No specific applications were sought in pursuing these results. Often in mathematics results are developed with no specific application, and years later applications are found. For example, in ancient Greece, conic sections were studied as purely mathematical objects. That is, they studied mathematical objects such as circles, parabolas, hyperbolas, etc. purely for their mathematical merit. However, over 2000 years later these were found to have profound applications in the study of planetary orbits ([32]). This is not to suggest that the mathematics performed in this research will have such profound implications, but just to demonstrate that studying math for the sake of studying math can often reveal much about the world, even if not immediately obvious.

Throughout the paper, we prove the following three main theorems:

(1) **Theorem.** The centralizer of a Toeplitz system embeds into the centralizer of its maximal equicontinuous factor.

(2) **Theorem.** There is a class of multi-dimensional Toeplitz Systems which have a trivial centralizer.

(3) **Theorem.** This class of multi-dimensional Toeplitz Systems contains systems of positive entropy.

The second and third results can be interpreted in the context of information theory. The kinds of systems studied for these results can be thought of as information channels. Under this context, the entropy of these systems is exactly the Shannon Entropy, or channel capacity. So having positive entropy means that there is positive channel capacity in these systems, which in turn means that messages can be reliably sent and received. However, since these systems have a trivial centralizer, that means that these messages cannot be encoded using any substitution ciphers.

Additionally, in Section 9, we provide strong numerical evidence to suggest that the topological full group of the Fibonacci Substitution system is amenable. The kind of numerical methods used in this section can be used in the future to seek examples of groups which are simple, finitely presented, infinite and amenable. Finding an example of such a group is still an open problem in mathematics. The methods we used may be able to provide such an example by exploring the topological full group of multidimensional systems.

4. Background

In this section, we will provide some background and basic working definitions. This will allow the reader to understand and appreciate the results achieved.

Toeplitz Systems. A specific type of dynamical system that will be studied in this paper is a Toeplitz system. These systems are symbolic dynamical systems which arise from sequences satisfying certain properties. The sequences in these systems have the property that they look as if they repeat, or are periodic, but in fact never do. In particular, in one dimension, if you shift the sequence any number of times it will never come back to where it started, but for every position in the sequence there is an infinite arithmetic progression such that whichever symbol is in the first position is also in every position along that arithmetic progression. This is best illustrated through an example.

Example 4.1. (Due to Downarociz [16]). Consider the alphabet $A = \{0, 1\}$ and the following one-sided sequence on this alphabet

$$\omega = 0\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\ldots$$

We note that in the first position there is a zero. The arithmetic progression starting at the first position of $\{0, 2, 4, 6, \ldots\}$ contains a zero. In particular, the first position is a zero, and so is every other position after that. The second position contains a one, and every fourth position from there is also a one. We call ω a Toeplitz sequence.

We note that this example has in one dimension the kind of aperiodicity that the Penrose Tiling had in two dimensions. In order to get a Toeplitz System, we take a Toeplitz

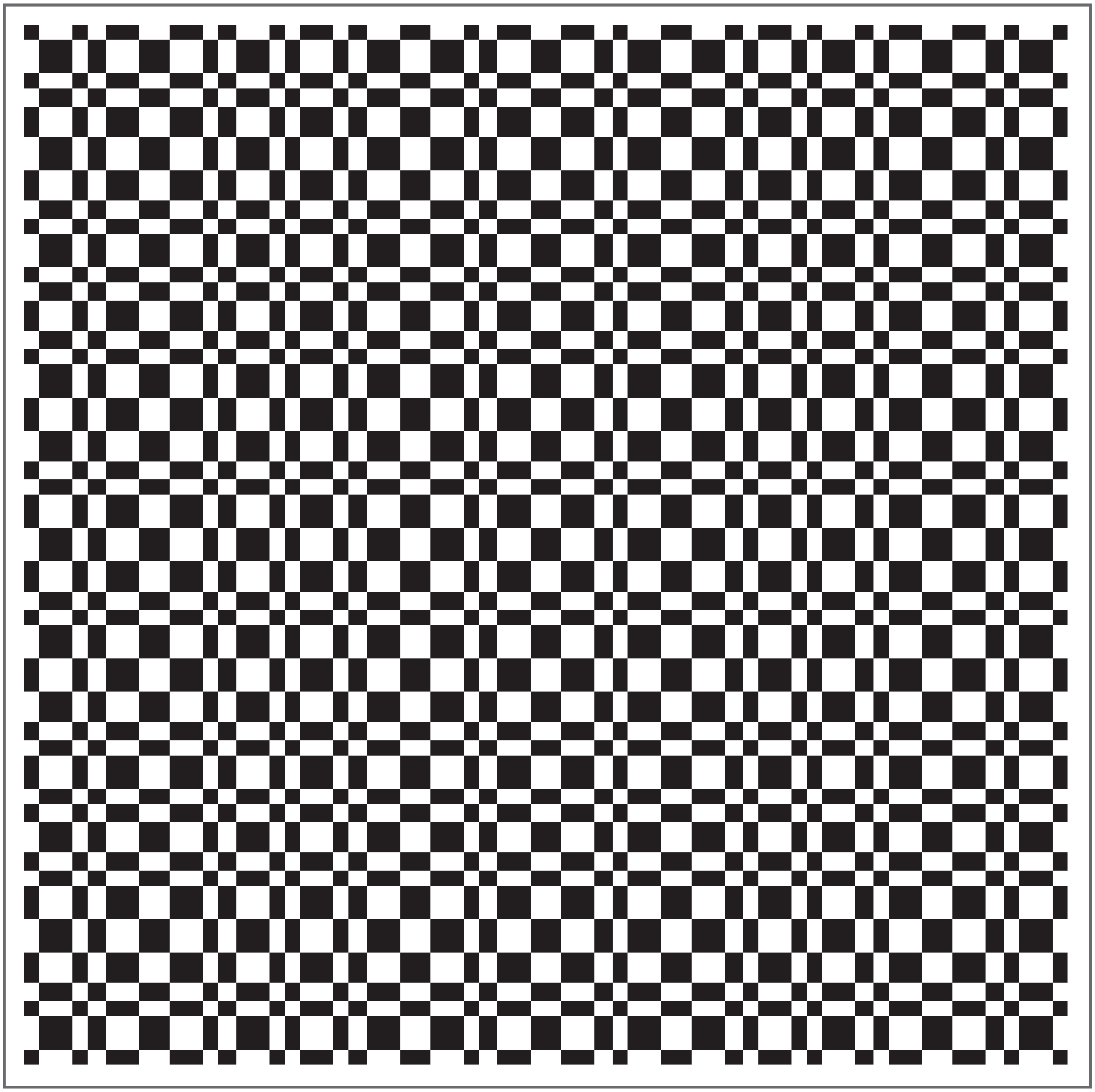

Figure 8. 2-dimensional Toeplitz Array

sequence as above. Next we shift it both forward and backwards infinitely, and collect the new sequence we have after each shift. We will have an infinite set of sequences. This set is called the orbit of the point ω. Specifically, we denote the orbit of ω as $O(\omega)$ and is defined as $O(\omega) = \{T^n(\omega)|n \in \mathbb{Z}\}$. Next we take the closure of this set. Loosely speaking, the closure of a set is all points in the set as well as all points which can be approached arbitrarily closely by points in the set. This is denoted as $\overline{O(\omega)}$. The pair $(\overline{O(\omega)}, T)$ is a Toeplitz system.

Additionally, Toeplitz systems can be constructed in any number of dimensions. In dimensions higher than one, we call a Toeplitz sequence a Toeplitz array to indicate the higher dimensional nature of it. In Figure 8, we have a two-dimensional Toeplitz array. This array is built using two colors, black and white, which function as our alphabet. This is an example of a substitution tiling of the plane.

Odometers and Their Almost One-to-One Extensions. An odometer is a specific kind of dynamical system. In order to understand what they are it is easiest to look at a specific example.

Example 4.2. We will start with the sequence $s = \{s_1, s_2, \ldots\} = \{2, 4, 8, 16, 32, \ldots\}$. We construct points $j = (j_1, j_2, \ldots)$ which themselves are sequences so that the following is true: when j_{i+1} is divided by s_i, the remainder is j_i. For example, $j = (1, 3, 3, 11, 27, \ldots)$. We note that $0 \leq j_i < s_i$. Another point in this odometer is $j' = (0, 0, 4, 12, 12, \ldots)$. Finally we can add these two points together in the expected way: $j + j' = (1, 2, 7, 7, 7, \ldots)$. The addition in the i^{th} component is the remainder when $j_i + j'_i$ is divided by s_i. For example, in the fourth component of $j + j'$ we have $11 + 12 = 23$. When 23 is divided by 16, the remainder is 7, and hence the fourth component of $j + j'$ is 7.

An odometer in a car works in a similar way. We consider the sequence $s = \{10, 10, 10, 10, \ldots\}$ and we will write points j from right to left, i.e. $j = (\ldots j_4, j_3, j_2, j_1)$. The car odometer starts at $j = (\ldots 0, 0, 0, 0)$ and each time one mile is driven, the number $j' = (\ldots, 0, 0, 0, 1)$ is added to the current reading of the odometer. The first nine miles will clearly bring the odometer reading up to $(\ldots, 0, 0, 0, 9)$. Driving one more mile will show a reading of $(\ldots, 0, 0, 1, 0)$, which we understand to mean 10 miles have been driven. If nine more miles are driven, the odometer will read $(\ldots, 0, 0, 1, 9)$ and then the next mile will turn it to $(\ldots, 0, 0, 2, 0)$. So the odometer shown in Example 4.2 works exactly the same as a car odometer, hence the name.

Example 4.2 is a simple example of an odometer, but they can be generalized much further. Odometers are related to Toeplitz systems in a very unique way. It turns out that a Toeplitz system is an almost one-to-one extension of an odometer. Before discussing what exactly this means, it is important to note that this gives us another way to understand Toeplitz systems. Before we discussed Toeplitz systems by starting with specific Toeplitz sequences or arrays, but here we can study Toeplitz system in more abstract ways.

Let us denote an odometer by (Y, S). To say that (X, T) is an almost one-to-one extension of (Y, S) means that there is a function $\pi : (X, T) \to (Y, S)$ such that for every point y in Y, there exists a point x in X such that $\pi(x) = y$. Furthermore, there must be at least one point y' in Y that has a unique point x' in X such that $\pi(x') = y'$. If these conditions are satisfied then (X, T) is an almost one-to-one extension of (Y, S). On the other hand, a one-to-one extension would mean that every point y in Y has a unique point x such that $\pi(x) = y$.

It can be shown that the almost one-to-one extensions of odometers are exactly Toeplitz systems, and Toeplitz systems are exactly almost one-to-one extensions of odometers.

Symmetries and Centralizers. The key question we explore in this paper is *What do the symmetries of a Toeplitz system look like?* Symmetries are very important in mathematics. A key object in mathematics is a group, and a fundamental example of a group is the group of symmetries of a shape. For example, in Figure 9, we see the six symmetries of an

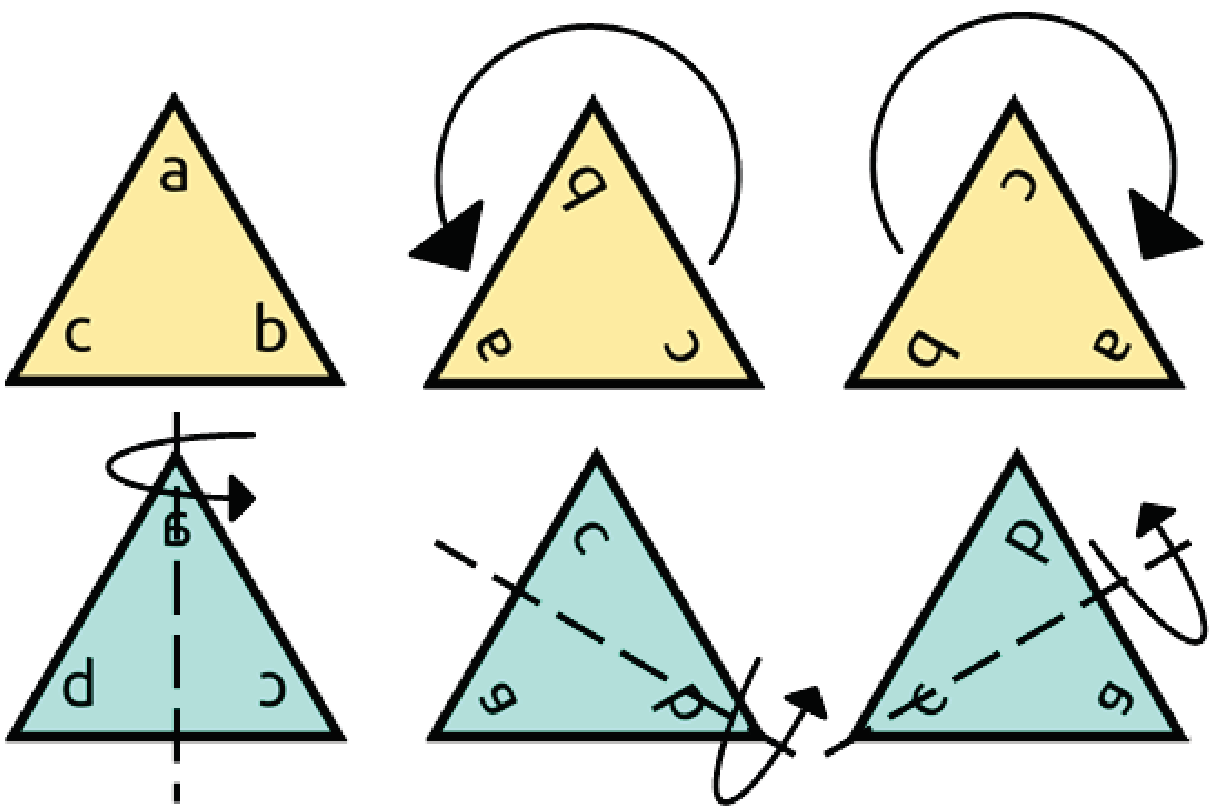

FIGURE 9. Symmetries of a Triangle

equilateral triangle. We can perform any one of these symmetries followed by another, and the end result will always be one of these six symmetries. Much like the symmetries of a triangle, the symmetries of a dynamical system form a group. We call this group the centralizer. However it is a little more difficult to visualize the symmetries of a general dynamical system. So we define symmetries of a dynamical system in more abstract terms, and try to understand how they behave. Specifically, a symmetry will be an invertible function on the dynamical system which commutes with the shift. These symmetries must also be continuous, and we must be able to reverse them continuously. This is called a *homeomorphism*. In particular, say (X, T) is a dynamical system. Then if $\varphi : X \to X$ is a homeomorphism, we have $\varphi \in C(T) \iff \forall x \in X, \varphi(Tx) = T(\varphi(x))$, where $C(T)$ is the centralizer of (X, T). In other words, we want to be able to apply a symmetry and then the shift and get the same result if we applied the shift first and then the symmetry. Visually,

$\varphi \in C(T)$ if the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ X \arrow[r, "T"'] & X \end{tikzcd}$$

An important result in symbolic dynamics, Hedlund [19], is that, every symmetry of a symbolic dynamical system is a *block code*. What this means is that, every symmetry of a symbolic dynamical system is a function based upon a finite window size of a sequence. Suppose we have a function φ which we know is a symmetry of a system (X, T). In order to understand what φ does to a point $x \in X$, without knowing anything else about φ or (X, T), we automatically know that in order to determine the symbol that appears at a location in $\varphi(x)$, we need to look at a finite window around the corresponding location in x. In particular, if we denote by x_i the symbol in the i^{th} location of x then

$$\varphi(x)_i = f(x_{i-m}, x_{i+k})$$

for some function f.

This means in order to determine $\varphi(x)_i$, the i^{th} coordinate of the image of x under the symmetry φ, we only need to look at the i^{th} coordinate in x, look back m spaces and forward k spaces. Based on this finite window, we can determine what $\varphi(x)_i$ is.

Example 4.3. Let (X, T) be a symbolic dynamical system. Let $x = \ldots 011011110001\ldots$. Suppose φ is a symmetry of (X, T) and we know that it acts according to the following rules:

$$\begin{gathered} 0110 \mapsto 0 \\ 1101 \mapsto 0 \\ 1011 \mapsto 1 \\ 0111 \mapsto 1 \\ 1111 \mapsto 0 \\ 1110 \mapsto 1 \\ 1100 \mapsto 1 \\ 1000 \mapsto 0 \\ 0001 \mapsto 1 \end{gathered}$$

Then We would have $\varphi(x) = \ldots 001101101\ldots$. The important fact to note is that for every single symmetry of any symbolic dynamical system, a list of rules can be written, as above, which tells how any finite block is changed.

We note that this is exactly what is happening in Figure 6 and Figure 7 above. Indeed symmetries of symbolic dynamical systems are analogous to substitution ciphers.

5. Definitions and Notation

In this section, we provide precise mathematical definitions of terms used throughout the paper. First, some basic notation will be explained.

A set S is a collection of elements. There can be infinitely many or finitely many elements in a set. We denote the number of elements in S by $|S|$. This is sometimes called the cardinality or the order of S. If s is an element of S, we write $s \in S$. If S' is another set, and every $s \in S'$ is also in S, we say that S' is a subset of S and is written $S' \subseteq S$. We say S' is a *proper subset* of S if $S' \neq S$. Two subsets are said to be disjoint if they share no common elements. The power set of a set S, labeled $\mathcal{P}(S)$ is the set of all subsets of S. There are $2^{|S|}$ elements in the power set of S.

In order to define a set S, we can write $S = \{s \mid s \text{ has some property } P\}$. This means the set S consists of all elements s which satisfy the property P. For example, we can define $S = \{s \mid s \text{ is blue}\}$ and S would be the set of everything that is blue.

Given a set S, the set $S \times S$ is the set of all ordered pairs of elements from S. Specifically, $S \times S = \{(s_1, s_2) \mid s_1, s_2 \in S\}$. Let S and T be two sets. The *union* of these sets, written $S \cup T$ is the set of all elements of S and T. In particular, $S \cup T = \{s \mid s \in S \text{ or } s \in T\}$. The *intersection* of S and T written $S \cap T$ is the set consisting of the elements which are both in S and T, i.e. $S \cap T = \{s \mid s \in S \text{ and } s \in T\}$.

Example 5.1. Let $S = \{0, 1, 2, 3\}$ and $T = \{3, 4, 5, 6\}$. Then $S \cup T = \{0, 1, 2, 3, 4, 5, 6\}$ and $S \cap T = \{3\}$.

The symbol $\forall$ means 'for all'. For example, $\forall s \in S$ means every element s in S. The symbol $\exists$ means 'there exists'. For example $\exists s \in S$ means that there is an element s in S.

Definition 5.2 (Partially Ordered Set)**.** Let S be a set, and $\leq$ be a relation on S which tells us how two elements of the set are related. We call $(S, \leq)$ a *partially ordered set* if for all $a, b, c, \in S$ the following hold:

(1) $a \leq a$. This is known as *reflexivity*

(2) If $a \leq b$ and $b \leq c$ then $a \leq c$. This is known as *transitivity*

(3) If $a \leq b$ and $b \leq a$ then $a = b$. This is known as *antisymmetry*

Definition 5.3 (Groups)**.** Let G be a set and $\cdot$ be a function from $G \times G$ to G such that the following hold:

(1) $\exists e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$. This element e is unique, and is called the identity.

(2) $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is called associativity.

(3) For every $a \in G$, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$. This is called the inverse of a

If, additionally, we have that $a \cdot b = b \cdot a$ for all $a, b \in G$, we say that the group G is abelian. In this case, we say the group commutes.

If $H \subseteq G$, and H itself is a group, we say that H is a subgroup of G. When there can be no confusion, this is also denoted as $H \subseteq G$.

Example 5.4. Here we provide two examples of groups, one finite and one infinite, as well as an example of something that is not a group.

- Consider the set of integers $\mathbb{Z}$ along with the operation of addition. We note that 0 is the identity element, and that the operation is associative. Also, the negative of a number is its inverse. Furthermore, the order of addition does not matter, so this is also an abelian group.
- The set of integers $\mathbb{Z}$ with the operation of multiplication is *not* a group. Note that the identity element would be 1, but there is no element of this set that when multiplied by 0 yields 1, i.e. 0 has no inverse, so this cannot be a group.
- Consider the set $\{0, 1, 2\}$ with addition modulo 3. That is, when adding two numbers, take the remainder of the results when divided by three. For example, $7 + 6 = 13$ which when divided by three has remainder 1. So in our set, 0 is the identity and the addition is associative. Additionally the inverse of 0 is 0 (the inverse of the identity is always itself), the inverse of 1 is 2, because $1+2 = 3$ which has no remainder when divided by 3, and the inverse of 2 is 1. So this is a group. This group is called the cyclic group of order 3 and is sometimes denoted $\mathbb{Z}_3$

Groups can be defined explicitly by their members and the function of multiplication. Additionally they can be defined as a combination of *generators* and *relations*.

Definition 5.5 (Generators and Relations)**.** Let $x_1, x_2, \ldots, x_t$ be symbols and $r_1, r_2, \ldots$ concatenation of these symbols. The group $G = \langle x_1, x_2, \ldots, x_t \mid r_1, r_2, \ldots \rangle$ has generators $x_1, x_2, \ldots, x_t$ and relations $r_1, r_2, \ldots$ and is defined as concatenations of the generators, known as *words*, but if any of the relations appear in a word, that is reduced to the identity.

This is best illustrated in an example.

Example 5.6. Let $G = \langle a, b \mid ab = ba \rangle$. So, this group is generated by a, b subject to the relation $ab = ba$. We can interpret this relation as $aba^{-1}b^{-1} = e$, where e is the identity element in the group. So a is a word in the group, and so $ababbba$. But the word $abbaba^{-1}b^{-1}ab$ can be *reduced* to $abbab$, since the $aba^{-1}b^{-1}$ in the middle can be reduced to the identity. This group is actually $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

Definition 5.7 (Quotient Group)**.** Let G be a group and $H \subseteq G$ a subgroup. For $g \in G$, we define the coset $gH = \{gh \mid h \in H\}$. The set Hg is defined in a similar way. We define the quotient group as $G/H = \{gH \mid g \in G\}$, i.e. it is the set of cosets of the subgroup H. This set itself is a group with the group operation being $(gH)(g'H) = (gg')H$.

If $gH = Hg$ for all $g \in G$, we call H a normal subgroup.

Definition 5.8 (Isomorphism)**.** Let G, G' be two groups. We say G is isomorphic to G' if there exists a function $\varphi : G \to G'$ such that the following hold:

(1) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for all $a, b \in G$.

(2) $\forall g' \in G'$ there exists a $g \in G$ such that $\varphi(g) = g'$

(3) $\forall g_1, g_2 \in G$, we have $\varphi(g_1) = \varphi(g_2) \Rightarrow g_1 = g_2$

The first condition ensures that φ is a group homomorphism, while the last two conditions ensure φ is a bijection.

When two groups are isomorphic they are considered mathematically indistinguishable.

Definition 5.9 (Metric Space)**.** A set X is called a metric space if there exists a distance function $d : X \times X \to \mathbb{R}$ such that the following hold:

(1) $d(x, y) \geq 0$ for all $x, y \in X$ with $d(x, y) = 0 \Leftrightarrow x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. This is called the triangle inequality.

A set on which a metric can be defined is called metrizable.

Example 5.10. The real numbers $\mathbb{R}$ equipped with the distance function $d(x, y) = |x - y|$ is a metric space.

Definition 5.11 (Open, Closed, Clopen Sets)**.** Let (X, d) be a metric space. An *open set* is a set U such that $\forall x \in U$, there exists $r > 0$ such that $\{y \in X \mid d(x, y) < r\} \subseteq U$.

A *closed set* is the complement of an open set.

A *clopen set* is a set which is both closed and open.

Example 5.12. Let (X, d) be $\mathbb{R}$ with the usual distance metric. Then $(0, 1)$ is an open set, i.e. the interval from 0 to 1 not including the points 0 and 1. The set $[0, 1]$ is a closed set, i.e. the interval from 0 to 1 containing both 0 and 1. The empty set $\emptyset$ is clopen, i.e. it is both closed and open.

Definition 5.13 (Continous Functions)**.** A function $f : A \to B$ is continuous if the inverse image of an open set in B is open in A. We say f is a homeomorphism is f is continuous, and f has a continuous inverse.

Example 5.14. The function $f : [0, 1] \to [0, 2]$ defined by $f(x) = 2x$ is a homeomorphism. This tells us that the sets $[0, 1]$ and $[0, 2]$ can be considered, in many senses, to be the same.

Definition 5.15 (Compact Space)**.** A set X is called compact if for every open cover, there is a finite subcover. That is for every collection of open sets that cover the entire space X, we can pick finitely many of these open sets that still cover X. On the real line, $\mathbb{R}$ with the absolute value metric as above, this is equivalent to a set being closed and bounded.

Definition 5.16 (Closure)**.** Let (X, d) be a metric space. The closure of X is denoted by $\overline{X}$. We say $y \in \overline{X}$ if and only if there exists a sequence $\{x_i\} \subseteq X$ such that $x_i \to y$. That is, y is in the closure of X if there is a sequence of points in X which approach y arbitrarily closely. Informally, a point is in the closure of a set if the set gets arbitrarily close to that point.

Example 5.17. The following is an example of a set and its closure. Consider the set $(0, 1)$. That is the set of all real numbers between 0 and 1, not including 0 and 1. The points in this set get arbitrarily close to both 0 and 1 however, so the closure of $(0, 1)$ is $[0, 1]$.

Definition 5.18 (Density)**.** Let $A \subseteq X$. We say A is dense in X if $\overline{A} = X$.

Example 5.19. Building on the previous example, the set $(0, 1)$ is dense in $[0, 1]$, since $\overline{(0, 1)} = [0, 1]$.

Definition 5.20 (Topological Space)**.** A *topological space* is a set X endowed with a *topology*. That is, a collection of subsets of X which are closed under arbitrary unions and finite intersections, as well as containing $\emptyset$ and X. These sets are understood to be open.

Example 5.21. Let $X = \{0, 1, 2\}$. The collection of all subsets of X forms a topology on X, and is called the *discrete topology*. In particular,

$$\tau = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

is a topology on X.

Definition 5.22 (Basis)**.** Given a set X endowed with a topology τ, we say that B is a *basis* for the topology τ if every member of τ can be realized as the union of members of B.

Example 5.23. Continuing with Example 5.21, the collection

$$B = \{\{0\}, \{1\}, \{2\}\}$$

forms a basis for the discrete topology τ.

Definition 5.24 (Cylinder Set)**.** Let $\{X_i\}_{\imath=0}^{\infty}$ be a collection of topological spaces. The basis for a topology on $\Pi_{i=0}^{\infty} X_i$, the Cartesian product of these topological spaces, contains *cylinder sets*. That is, sets whose where a finite number of coordinates are fixed.

Definition 5.25 (Symbolic Dynamics and Subshifts)**.** Let A be a finite set with $|A| = a$. The full a-shift is the set of all bi-infinte sequences $(x_i)_{i=-\infty}^{\infty}$ where $x_i \in A$ for all i. This is denoted as $A^{\mathbb{Z}}$. The shift operator T is defined on $A^{\mathbb{Z}}$ as $(Tx)_i = x_{i+1}$ for all $x \in A^{\mathbb{Z}}$ and all i. A subset $X \subseteq A^{\mathbb{Z}}$ is a subshift if $TX \subseteq X$, i.e. X is T-invariant. This is a symbolic dynamical system and is denoted as (X, T).

In general, we denote the full d dimensional n-shift by $A^{\mathbb{Z}^d}$. This set is endowed with the product topology from the discrete topology in each coordinate. Cylinder sets in which we fix a finite number of coordinates form a basis for the topology. For $x \in A^{\mathbb{Z}^d}$ we write $x = \{x(v)\}_{v \in \mathbb{Z}^d}$. We call x a $\mathbb{Z}^d$ array. The group $\mathbb{Z}^d$ acts on $A^{\mathbb{Z}^d}$, denoted by $T^z(x)$ for $z \in \mathbb{Z}^d$ and $x \in A^{\mathbb{Z}^d}$ as follows: $T^z(x) = \{x(z + v)\}_{v \in \mathbb{Z}^d}$. The orbit of an array is $\{T^v(x) : v \in \mathbb{Z}^d\}$. A subset $X \subseteq A^{\mathbb{Z}^d}$ is called a *subshift* if it is closed under the action of $\mathbb{Z}^d$.

For the sake of completeness, we note that symbolic dynamics can be studied over general, discrete groups. In this case, let G be a discrete group. Then Σ^G is acted on by the group G. While in this paper we restrict our study of symbolic dynamics to $\mathbb{Z}^d$ systems, we note that many of the results can be extended to G systems for more general groups G.

It is important to note that subshifts are both compact and metrizable.

Definition 5.26 (Entropy)**.** Let (X, T) be a subshift. Let $B_n(X)$ be the number of words of length n which occur in X. Then the entropy, $h(X)$ is

$$h(X) = \lim_{n \to \infty} \frac{\log(|B_n(X)|)}{n}$$

Example 5.27. Let (X, T) be the full 2-shift. We note that $|B_n(X)| = 2^n$ since for each position in a block of length n, we have two choices. So,

$$h(X) = \lim_{n\to\infty} \frac{\log(|B_n(X)|)}{n} = \lim_{n\to\infty} \frac{\log(2^n)}{n} = \log(2)$$

The base of the logarithm isn't really important, but sometimes we will define it to be base 2 so that the entropy of the full shift is 1.

Definition 5.28 (Toeplitz Sequence)**.** A sequence $\omega \in A^{\mathbb{Z}}$ is a Toeplitz sequence if $(\forall n \in \mathbb{Z})(\exists l \in \mathbb{N})(\forall k \in \mathbb{N})\ \ \omega(n) = \omega(n + kl)$

We refer the reader to Example 4.1.

Definition 5.29 (Dynamical Systems)**.** A dynamical system is a pair (X, G) where X is a compact metrizable space, and G acts on X by homeomorphism. We denote the action of $g \in G$ on a point $x \in X$ as gx. In the case that G is $\mathbb{Z}$, we often write (X, T) where T is understood to be the generating action of $\mathbb{Z}$. This draws the connection between dynamical systems and subshifts. Indeed, subshifts are dynamical systems. We denote the orbit of a point $x \in X$ as $O_G(x) = \{gx \mid g \in G\}$. If the group G is $\mathbb{Z}$, then the orbit is $O_T(x) = \{T^n(x) \mid n \in \mathbb{Z}\}$. When the group is understood, the orbit will be denoted simply $O(x)$. A dynamical system is minimal if every orbit is dense. In fact it is enough to show that there is one dense orbit.

We note that if we take the orbit closure, i.e. $\overline{O(x)}$ of a single point $x \in A^{\mathbb{Z}}$ in the full shift, we will always have a subshift.

Definition 5.30 (Toeplitz System)**.** A Toeplitz system is the orbit closure of a Toeplitz sequence.

Definition 5.31 (Centralizer)**.** Let (X, G) be a dynamical system and $Homeo(X)$ be the set of all homeomorphisms from X to itself. The centralizer $C(G)$ is defined as $C(G) = \{\varphi \in Homeo(X) \mid \varphi g = g\varphi\ \forall g \in G\}$. That is, the centralizer of a dynamical system is the set of all homemorphisms of the system which commute with the group action. When the group is $\mathbb{Z}$, this is equivalent to the set of homeomorphims which commute with the shift action T. It can be shown that the centralizer of any dynamical system is a group with the operation of composition.

Given a dynamical system (X, T) acted on by $\mathbb{Z}$ we say that the centralizer $C(T)$ is rigid if $\forall \varphi \in C(T)$, $\varphi(x) = T^n(x)$ for all $x \in X$ and some $n \in \mathbb{Z}$.

Definition 5.32 (Factor Map)**.** Let X, Y be sets and $\pi : X \to Y$ be a function. We say π is a factor map if for all $y \in Y$, there exists an $x \in X$ such that $\pi(x) = y$. This is also called a surjection or onto map.

Definition 5.33 (Equicontinuity)**.** A system (X, G) is *equicontinuous* if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in X$ if $d(x, y) < \epsilon$, then $d(g \cdot x, g \cdot y) < \delta$ for all $g \in G$.

Definition 5.34 (Extension)**.** Let (X, G), and (Y, G) be two minimal systems. If there exists a continuous surjection $\pi : X \to Y$ which preserves the action of G, we say that X is an *extension* of Y, and that Y is a *factor* of X. See Definition 5.32.

Given two factor maps π and π', we say that π is *larger* than π' if there exists a third factor map π'' such that $\pi' = \pi'' \circ \pi$. As such, we can discuss the *maximal* factor of a system. It is a known fact that every dynamical system has a maximal equicontinuous factor.

Definition 5.35 (Cantor Set)**.** A topological zero-dimensional compact metric space without isolated points is called a *Cantor Set*.

6. Toeplitz Systems

In this section, we provide rigorous mathematical arguments concerning results about Toeplitz Systems. We start with some basic definitions and then prove results about odometers. Here we classify Toeplitz systems as almost $1-1$ extensions of odometers. Next we present Toeplitz systems as symbolic dynamical systems. Here we prove that there is a class of multi-dimensional Toeplitz systems with a trivial centralizer.

Historical Account and Previous Results. Toeplitz dynamical systems were first introduced by Jacobs and Keane [22]. They provided a classical definition for a Toeplitz sequence over $\{0, 1\}$. Markley [28] studied these sequences and showed the equivalence of various definitions of them. The orbit closure of a Toeplitz sequence is regarded as a Toeplitz flow. Markley and Paul [29] showed that these flows were exactly almost one-to-one extensions of odometers, or the group of p-adic integers. See Hewitt [20] for a general discussion of the group theoretic properties of the group of p-adic integers. For a general survey of symbolic dynamics, we refer the reader to Kitchens [24]. For a good survey on $\mathbb{Z}$ odometers and Toeplitz flows, the reader is referred to Downarowicz [16]. Recently the definition of Toeplitz flows was extended to flows over $\mathbb{Z}^d$ by Cortez [10], and then to flows over general groups Cortez [11], and Krieger [25].

Sometimes called the *automorphism group* of the dynamical system in the literature, the centralizer of a dynamical system has an intricate relationship with its parent dynamical system. For example, Boyle, Lind and Rudolph [5] study the centralizer of shifts of finite type and show that they are countable, residually finite and contain the free group on two generators. Several results have been shown by Cyr and Kra ([12], [13], [14]) which relate varying levels of complexity of symbolic dynamical systems to algebraic properties of their centralizers. We notice that systems with positive entropy tend to have very large centralizers. For example, the centralizer of the full shift contains every finite group and the free group on two generators. On the other hand, Donoso, Durand and Petite [15] showed that some classes of low complexity symbolic dynamical systems have very small centralizers, in the sense that they consist only of powers of T. Bulatek and Kwiatkowski [7], [8] study the centralizer of a class of high complexity Toeplitz systems. The centralizer of multidimensional symbolic dynamical systems is studied by Hochman [21]. For example, he shows that the centralizer of a positive entropy multidimensional shift of finite type contains a copy of every finite group.

The main question this section seeks to answer is whether there are multidimensional systems with a trivial centralizer and positive entropy. Following the ideas of Bulatek and Kwiatkowski in [8], which developed this result in one dimension, we establish this result with a constructive proof.

In Section 6 we present main facts with proofs regarding general G-odometers, where G is a residually finite group. For the reader's convenience, we include the proofs, otherwise scattered across multiple sources. In particular we show that the centralizer group of $\mathbb{Z}^d$-Toeplitz systems embeds into the centralizer group of its maximal equicontinuous factor, which is a $\mathbb{Z}^d$-odometer, and so is abelian. In section 6, we construct a class of $\mathbb{Z}^d$-Toeplitz systems that have trivial centralizers. Then in Section 7, we show that this class contains systems of positive entropy. In this section we provide an explicit construction of a two dimensional Toeplitz of positive entropy.

The topological spaces discussed in this section will be Cantor sets. Notice that by a theorem of Brouwer [6], every Cantor set is homeomorphic to the middle-thirds Cantor set, and so all Cantor sets are homeomorphic.

Odometers. In this section, we will recall some basic facts about odometers and their almost $1-1$ extensions. In particular, we show that the centralizer of an odometer is abelian, and the centralizer of the almost $1-1$ extension of an odometer is also abelian. These results are mostly known, but are scattered. In particular, the proof of Lemma 6.11 appears in Veech [34] and the proof of Proposition 6.12 appears in Olli [30]. We present slightly modified proofs for clarity and the reader's convenience.

Definition 6.1. A group G is called *residually finite* if the intersection of all its finite index normal subgroups is normal.

Definition 6.2. Let G be a residually finite group, and $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ be nested normal subgroups such that $\bigcap G_n = \{0\}$. Let π_n be the natural homomorphism from G/G_n onto G/G_{n-1}, i.e. $\pi_n(hG_n) = hG_{n-1}$ for $h \in G$. The *G-odometer*, $\overline{G}$, is the inverse limit

$$\overline{G} = \varprojlim(G/G_i; \pi_i) = \{(g_k)_{k=0}^{\infty} \in \prod_{k=0}^{\infty} G/G_k \mid \pi_n(g_n) = g_{n-1} \text{ for all } n \geq 1\}$$

An element $g \in G$ acts on an element $y = (y_i)_{i=0}^{\infty} \in \overline{G}$ as $g \cdot y = (g \cdot y_i)_{i=0}^{\infty}$.

First we prove that G embeds into $\overline{G}$.

Lemma 6.3. *Let $\phi : G \to \overline{G}$ be defined as $g \mapsto (gG_1, gG_2, \ldots)$. Then ϕ is an embedding.*

Proof. Let $g_1, g_2 \in G$. Suppose

$$\phi(g_1) = (g_1G_1, g_1G_2, g_1G_3, \ldots) = (g_2G_1, g_2G_2, g_2G_3, \ldots) = \phi(g_2)$$

So $g_1G_i = g_2G_i$ for all i. Therefore $g_1^{-1}g_2 \in G_i$ for all i, and so $g_1^{-1}g_2 \in \bigcap G_i = \{0\}$. So $g_1 = g_2$. ☐

So we have shown that G embeds into $\overline{G}$ in a natural way. We now prove that $(\overline{G}, G)$ is minimal.

Lemma 6.4. *The system $(\overline{G}, G)$ is minimal.*

Proof. Consider the identity element, $e \in \overline{G}$. In particular, $e = (G_1, G_2, G_3, \ldots)$. Let $y = (y_i)_{i=0}^{\infty} \in \overline{G}$. So, for each n, we have $y_n = \overline{y_n} G_n$, where $\overline{y_n} \in G$ is a representative of the coset. Note

$$\begin{aligned}\overline{y_n} \cdot e &= \overline{y_n}(G_1, G_2, G_3, \ldots, G_n, \ldots) \\ &= (\overline{y_n}G_1, \overline{y_n}G_2, \overline{y_n}G_3, \ldots, \overline{y_n}G_n, \ldots) \\ &= (\overline{y_1}G_1, \overline{y_2}G_2, \ldots, \overline{y_n}G_n, \ldots) = (y_1, y_2, \ldots, y_n, \ldots)\end{aligned}$$

So $e \cdot \overline{y_n}$ agrees with y in the first n coordinates. And so we can get arbitrarily close to y as we increase n. Hence e has a dense orbit.

Now let $a, b \in \overline{G}$. Note we can find a sequence b_n such that $b_n \cdot e \to ab^{-1}$, since e has a dense orbit. Then $(b_n \cdot e) \cdot b \to a$ so $b_n \cdot b \to a$. Therefore b has dense orbit. □

Definition 6.5 (Centralizer)**.** Let (X, G) be a dynamical system. The *centralizer*, $C(G)$ is defined as

$$C(G) = \{\varphi \in \text{Homeo}(X) \mid g\varphi = \varphi g \text{ for all } g \in G\}$$

That is, the centralizer of a system consists of all homeomorphisms of the system which commute with the group action. It can be checked that this is a group under composition.

Next we show that elements of the centralizer of an odometer act as translations of the odometer.

Lemma 6.6. *Let $\varphi \in C(\overline{G}, G)$. There exists $g_0 \in \overline{G}$ such that $\varphi(x) = x \cdot g_0$ for all $x \in \overline{G}$.*

Proof. Let $x \in \overline{G}$. Since the orbit of e is dense, by Lemma 6.4, there exists a sequence $\{g_n\} \subseteq G$ such that $g_n \cdot e \to x$. Since φ is continuous, $\varphi(g_n \cdot e) \to \varphi(x)$. But $\varphi(g_n \cdot e) = g_n \cdot \varphi(e)$ for all n. Since $g_n \cdot e \to x$, we have $g_n \to x$. So $\varphi(g_n \cdot e) \to x \cdot \varphi(e)$. Therefore $\varphi(x) = x \cdot \varphi(e)$. □

We are now ready to prove the following Proposition. In the following, G is an abelian group.

Proposition 6.7. *The centralizer $C(\overline{G}, G) = \{\varphi : \overline{G} \to \overline{G} \mid \varphi g = g\varphi \; \forall g \in G\}$ of an odometer $\overline{G}$ is isomorphic to $\overline{G}$.*

Proof. Define $\psi : C(\overline{G}, G) \to \overline{G}$ as $\psi(\varphi) = \varphi(e)$ for all $\varphi \in C(\overline{G}, G)$. Let $\varphi_1, \varphi_2 \in C(\overline{G}, G)$. Then

$$\begin{aligned}\psi(\varphi_1 \circ \varphi_2) &= \varphi_1 \circ \varphi_2(e) \\ &= \varphi_1(\varphi_2(e)) \\ &= \varphi_2(e) \cdot \varphi_1(e) \\ &= \varphi_1(e) \cdot \varphi_2(e) \\ &= \psi(\varphi_1)\psi(\varphi_2)\end{aligned}$$

So ψ is a homomorphism. Let $y \in \overline{G}$. Let $\varphi_y(x) = x \cdot y$ for all $x \in \overline{G}$. Note, for $g \in \overline{G}$, we have $\varphi_y(gx) = g\varphi_y(x)$ so $\varphi_y \in C(\overline{G}, G)$. Also, $\psi(\varphi_y) = y$, so ψ is onto. Suppose $\psi(\varphi_1) = \psi(\varphi_2)$. Then $\varphi_1(e) = \varphi_2(e)$. So for any $x \in \overline{G}$, $\varphi_1(x) = x \cdot \varphi_1(e) = x \cdot \varphi_2(e) = \varphi_2(x)$. Therefore ψ is an isomorphism. □

We now turn our attention to almost $1-1$ extensions of odometers.

Definition 6.8. We say (X, G) is an *almost* $1-1$ *extension* of (Y, G) if there is a factor map $\pi : X \to Y$ such that there is at least one $y \in Y$ so that $\pi^{-1}y$ is singleton. Almost $1-1$ extensions of odometers are also called *Toeplitz Systems*.

We make use of the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{G} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[G]{} & Y \end{array}$$

Sometimes the context will deem the action of G on X or Y ambiguous, so we will use $T^g x$ to denote the action of the group element $g \in G$ on $x \in X$ and $S^g y$ to denote the action of g on $y \in Y$. In particular, $\pi \circ T^g = S^g \circ \pi$. If the context is clear, the action of g on a point x will be denoted $g \cdot x$.

If (X, G) is a minimal almost $1-1$ extension of a minimal equicontinuous system, (Y, G), then it is known that (Y, G) is the maximal equicontinuous factor of (X, G) ([1]). As such, the odometer of which a Toeplitz system (X, G) is an almost $1-1$ extension is its maximal equicontinuous factor.

We will be considering almost $1-1$ extensions of $\mathbb{Z}^d$-odometers. In this context, we will the following proposition.

Proposition 6.9. *The centralizer $C(G)$ of the almost $1-1$ extension of a $\mathbb{Z}^d$-odometer is abelian.*

To prove Proposition 6.9, we show that the centralizer of the almost $1-1$ extension of an odometer embeds into the centralizer of its maximal equicontinuous factor, which we have already shown to be isomorphic to the odometer, which is abelian in the case of $G = \mathbb{Z}^d$.

Definition 6.10 ([34])**.** Given a dynamical system (X, G) and a metric d compatible with the topology on X, two points x_1, $x_2 \in X$ are called *proximal* if

$$\inf_{g \in G} d(g \cdot x_1, g \cdot x_2) = 0$$

Lemma 6.11. *Let (X, G) be an almost $1-1$ extension of an odometer $(\overline{G}, G)$ via the factor map π. Then points of X are proximal if and only if they are in the same π fiber.*

Proof. Let $x_1, x_2 \in X$ be in the same π fiber, i.e. $\pi(x_1) = \pi(x_2)$. Let $y \in \overline{G}$ be such that $\pi^{-1}y$ is a singleton. Since $(\overline{G}, G)$ is minimal, there exists a sequence $\{g_n\}$ such that $\lim_{n\to\infty} S^{g_n}\pi x_1 = y$ and so $\lim_{n\to\infty} S^{g_n}\pi x_2 = y$. Since X is compact, there is a subsequence $\{T^{g_{n_k}}\}$ such that $T^{g_{n_k}}x_1$ converges. Suppose $\lim_{n\to\infty} T^{g_{n_k}}x_1 = z$. Applying π, we have

$$\lim_{n\to\infty} \pi T^{g_{n_k}} x_1 = \pi z$$
$$\lim_{n\to\infty} S^{g_{n_k}} \pi x_1 = \pi z = y$$

So we also have $\lim_{n\to\infty} S^{g_{n_k}}\pi x_2 = \pi z = y$. Since $\pi^{-1}y$ is a singleton, z is unique. Now, $d(g_n \cdot x_1, g_n \cdot x_2) \le d(g_n \cdot x_1, z) + d(z, g_n \cdot x_2)$. So,

$$\begin{aligned}\limsup_{g\in G} d(g_n \cdot x_1, g_n \cdot x_2) &\le \limsup_{g\in G}(d(g_n \cdot x_1, z) + d(z, g_n \cdot x_2)) \\ &\le \limsup_{g\in G} d(g_n \cdot x_1, z) + \limsup_{g\in G} d(z, g_n \cdot x_2) = 0\end{aligned}$$

So the points x_1 and x_2 are proximal.

Now suppose $x_1, x_2 \in X$ are proximal. Assume that $\pi x_1 \neq \pi x_2$, i.e. they are not in the same π fiber. Since x_1, x_2 are proximal, there is a sequence $\{g_n\} \subseteq G$ such that $\lim_{n\to\infty} T^{g_n}x_1 = \lim_{n\to\infty} T^{g_n}x_2 = z$. Applying π, we have $\lim_{n\to\infty} \pi T^{g_n}x_1 = \lim_{n\to\infty} \pi T^{g_n}x_2 = \pi z$. So $\lim_{n\to\infty} S^{g_n}\pi x_1 = \lim_{n\to\infty} S^{g_n}\pi x_2$ which implies $\pi x_1, \pi x_2 \in \overline{G}$ are proximal. But $\overline{G}$ has no proximal points, so $\pi x_1 = \pi x_2$. □

Finally, we prove that the centralizer of X embeds into the centralizer of the odometer Y.

Proposition 6.12. *Let (X, G) be an almost $1-1$ extension of a $G-$odometer (Y, G). Every element $\varphi \in C(X, G)$ determines $\psi_\varphi \in C(Y, G)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[\psi_\varphi]{} & Y \end{array}$$

Additionally, this relationship is an embedding, i.e. $\psi_{\varphi_1} = \psi_{\varphi_2} \Rightarrow \varphi_1 = \varphi_2$.

Proof. Let $\varphi \in C(X, G)$. Let $x_1, x_2 \in X$ be proximal. So $\pi x_1 = \pi x_2$. Since x_1 and x_2 are proximal, $\inf_{g\in G} d(g \cdot x_1, g \cdot x_2) = 0$. Thus $\inf_{g\in G} d(\varphi(g \cdot x_1), \varphi(g \cdot x_2)) = 0$ which, by Lemma 6.11, implies that $\varphi(x_1), \varphi(x_2)$ are proximal. So φ preserves the proximal relationship, and so it preserves π fibers. Define $\psi_\varphi : Y \to Y$ as $\psi_\varphi = \pi \circ \varphi \circ \pi^{-1}$. This map is well defined because φ preserves the π fibers. Suppose $\psi_\varphi(y_1) = \psi_\varphi(y_2)$ for $y_1, y_2 \in Y$. So $\pi \circ \varphi \circ \pi^{-1}(y_1) = \pi \circ \varphi \circ \pi^{-1}(y_2)$, and so $\varphi \circ \pi^{-1}(y_1)$ and $\varphi \circ \pi^{-1}(y_2)$ are in the same π fibers. Since φ preserves the π fibers, $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ are in the same π fibers, and so it is clear that $y_1 = y_2$. Therefore ψ_φ is $1-1$. Also, ψ_φ is continuous, so it is a homeomorphism, i.e. $\psi_\varphi \in C(Y, G)$.

Now suppose $\psi_{\varphi_1} = \psi_{\varphi_2}$. Let $y \in Y$ be such that $\pi^{-1}y = \{x\}$ is a singleton. Then $\varphi_1(x) = \pi^{-1}(\psi_{\varphi_1}(y))$ and $\varphi_2(x) = \pi^{-1}(\psi_{\varphi_2}(y))$. Since φ_i preserves π fibers, for $i \in \{1, 2\}$, these are singletons. In particular, $\varphi_1(x) = \varphi_2(x)$. So it is clear then that $g\cdot\varphi_1(x) = g\cdot\varphi_2(x)$ for all $g \in G$, and so $\varphi_1(g \cdot x) = \varphi_2(g \cdot x)$ for all $g \in G$. But every orbit is dense, so φ_1 and φ_2 agree on a dense subset of X, and hence agree everywhere. □

Finally we prove Proposition 6.9.

Proof. We have shown in Proposition 6.12 that $C(X, G)$ embeds into $C(Y, G)$ and by Proposition 6.7 $C(Y, G)$ is abelian, so $C(X, G)$ is abelian. □

$\mathbb{Z}^d$**-Toeplitz Systems.** In this section, we study Toeplitz systems over $\mathbb{Z}^d$ and generalize the construction of Bulatek and Kwiatkowski. In particular, we present a class of Toeplitz systems over $\mathbb{Z}^d$ with a trivial centralizer and positive entropy.

Let $x \in \Sigma^{\mathbb{Z}^d}$. Note that the topological closure of the orbit of x, $\overline{O(x)}$ is closed and T-invariant. So $(\overline{O(x)}, T)$ is a subshift. This is called the *orbit closure* of x.

Definition 6.13. The centralizer of a symbolic dynamical system is called *trivial* if every element S of the centralizer is T^g for some $g \in \mathbb{Z}^d$.

Let $Z \subseteq \mathbb{Z}^d$ be a finite index subgroup of $\mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$. For $x \in \Sigma^{\mathbb{Z}^d}$ and $\sigma \in \Sigma$, define

$$Per(x, Z, \sigma) = \{w \in \mathbb{Z}^d | x(w + z) = \sigma \ \forall z \in Z\}$$

And,

$$Per(x, Z) = \bigcup_{\sigma \in \Sigma} Per(x, Z, \sigma)$$

Definition 6.14. We say that $x \in \Sigma^{\mathbb{Z}^d}$ is a *Toeplitz array* if for all $v \in \mathbb{Z}^d$, there exists a finite index subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that $v \in Per(x, Z)$.

It can be shown that the orbit closure of a Toeplitz Array is an almost one-to-one extension of a $\mathbb{Z}^d$ odometer. For details, the reader is referred to Theorem 7 and Proposition 21 in [10]. In fact, almost one-to-one extensions of odometers are exactly those systems which are orbit closures of Toeplitz Arrays. In particular, defining a Toeplitz System as the orbit closure of a Toeplitz Array is equivalent to Definition 6.8.

We now show how Toeplitz Arrays can be constructed over an alphabet Σ borrowing ideas from Downarowicz [16]:

Let $\{p_{t,i}\}_{t=0}^{\infty}, 1 \leq i \leq d$ be d sequences of positive integers such that $p_{0,i} \geq 2$ and $p_{t,i}$ divides $p_{t+1,i}$ for all $0 \leq i \leq d$. Define $\lambda_{t,i} = p_{t+1,i}/p_{t,i}$ and $\lambda_{0,i} = p_{0,i}$ for all $1 \leq i \leq d$ and $t \geq 0$.

An array of $\mathbb{Z}^d$ is a point in our system. Any finite rectangular block consisting of letters from our alphabet is called a *finite block*. For a finite block D in d dimensions, we denote the size of D along the i^{th} dimension as $|D|_i$. We identify the element in the $(i_1, i_2, \ldots, i_d)$ position as $D(i_1, i_2, \ldots, i_d)$ with the standard Cartesian coordinate system, i.e. the left most and bottom-most entry of D is identified with $D(0, 0, \ldots, 0)$.

Specify blocks A_t as follows:

(1) $|A_t|_i = p_{t,i}$

(2) Some spaces in A_t are filled with elements from Σ and others are left unfilled. The unfilled spaces are called *holes*.

(3) The block A_{t+1} is the concatenation of $\lambda_{t+1,i}$ copies of A_t along the i^{th} dimension for all $1 \leq i \leq d$, where some holes are filled by symbols from Σ.

(4) For every $(i_1, i_2, \ldots, i_d) \in \mathbb{N}^d$ there exists a $t \geq 0$ such that $A_t(i_1, i_2, \ldots, i_d) \in \Sigma$ and $A_t(p_{t,1} - i_1, p_{t,2} - i_2, \ldots, p_{t,d} - i_d) \in \Sigma$.

We obtain a Toeplitz array $\omega \in \Sigma^{\mathbb{Z}^d}$ by continually repeating the above process. Note that the process described will only tile the first orthant. So to tile the entire space, at each step we shift the origin to be located in the center of our block A_t. Continuing this process, we will tile the whole space.

The fourth condition assures that all holes are eventually filled. Note that if after any finite step all holes are filled we will have a periodic array.

Essentially, in this construction we build finite blocks, each of which contains multiple copies of the block built in the previous step. As we copy theses blocks, we fill in some the holes, and leave some them as holes. As we continue this process forever, we will have a Toeplitz Array covering the whole plane.

Example 6.15 (One dimensional Toeplitz array)*.* (Due to Downarowicz [16])
We will construct a Toeplitz array over $\mathbb{Z}$ from the alphabet $\Sigma = \{0, 1\}$. Let $\{p_t\} = \{2, 4, 8, 16, \dots\}$ and so $\lambda_t = 2$ for all $t \geq 0$. Let $A_0 = 0\_$, where the $\_$ symbol indicates a hole. To get A_1, we copy A_0 twice and fill in some of the holes. Say $A_1 = 0\underline{1}0\_$. The underline indicates a hole that was filled at that step. In each step, we will have two holes. For this construction, at each step we will alternately fill in the first hole with 0 and 1. Let the limiting sequence of this process be ω. Continuing, we have

$$\begin{aligned} A_2 &= 010\underline{0}010\_ \\ A_3 &= 0100010\underline{1}0100010\_ \\ A_4 &= 010001010100010\underline{0}010001010100010\_ \\ &\vdots \\ \omega &= 0100010101000100010001010100010101000101010001000100010101000100\dots \end{aligned}$$

And so we have a Toeplitz array ω. The orbit closure of this point is a Topelitz system.

Example 6.16 (Two dimensional Toeplitz array)*.* Again we will use the alphabet $\Sigma = \{0, 1\}$ and we will construct a Toeplitz array over $\mathbb{Z}^2$. Let $\{p_{t,1}\} = \{p_{t,2}\} = \{2, 4, 8, 16, \dots\}$. Then $\lambda_{t,1} = \lambda_{t,2} = 2$ for all $t \geq 0$.

Let

$$A_0 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & \blacksquare \\ \hline \end{array}$$

$$A_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 0 & \blacksquare & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & \blacksquare \\ \hline \end{array}$$

$$A_2 = \begin{array}{|cccc|cccc|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \blacksquare & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & \blacksquare & 0 & 1 & 0 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \blacksquare & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & \blacksquare \\
\hline
\end{array}$$

The black squares indicate where the holes are. Continuing this process, we will have a coloring of the whole plane, which will be a Toeplitz array, say ω.

Definition 6.17. We call subblocks of A_{t+1} which coincide with indices of the location of concatenated A_t blocks *t-blocks*.

We note that ω consists of the concatenation of A_t blocks in all directions for any t, where all t-blocks agree in all locations except for where the holes were. In Example 6.16, the thick lines in A_1 indicate the 0-blocks, and the thick lines in A_2 indicate the 1-blocks.

We introduce a condition on constructing Toeplitz arrays which will give rise to Toeplitz Systems with a trivial centralizer.

Definition 6.18. We say a Toeplitz Array satisfies the condition $(*)$ if:

- Every t-block in A_{t+1} is composed of A_t or A_t with all holes filled
- The perimeter of A_{t+1} is composed of t-blocks which are all filled in

Let $e_1, e_2, \ldots, e_d$ be the generators of $\mathbb{Z}^d$. For $1 \leq i \leq d$, let T_i denote a shift by the vector e_i. In this context, the shift action on the system can be considered d independent shift actions, i.e. $T^g = T^{(g_1, g_2, \ldots, g_d)} = T_1^{g_1} \times T_2^{g_2} \times \ldots \times T_d^{g_d}$.

Definition 6.19. Given a finite alphabet Σ, a *patch* is a pair $(P, \mathcal{L})$, where $P \subseteq \mathbb{Z}^d$ and $\mathcal{L} : P \to \Sigma$ is a labeling of P. For the purposes of this paper, we will only consider rectangular patches which can be defined by d vectors parallel to the coordinate axes.

Given a patch $(P, \mathcal{L})$, we denote the the coordinate closest to the origin in Cartesian space by $P[0]$. Any other location in the patch is denoted by $P[i]$ where $i \in \mathbb{Z}^d$ is a vector pointing to that location, as referenced from $P[0]$. A square block within P is denoted by $P[i-l, i+k]$ where $k, l \in \mathbb{Z}$ and is the (hyper)cube in P located between $P[i - l\bar{1}]$ and $P[i + k\bar{1}]$, where $\bar{1} = (1, 1, \ldots, 1)$.

Theorem 6.20. *Let ω be a Toeplitz array satisfying the condition $(*)$. Then the centralizer $C(T)$ of $(\overline{O(\omega)}, T)$ is trivial.*

Proof. Let $(\overline{G}, T_1 \times T_2 \times \ldots \times T_d)$ be the maximal equicontinuous factor of $(\overline{O(\omega)}, T)$. Denote by $\pi : (\overline{O(\omega)}, T) \to (\overline{G}, T_1 \times T_2 \times \ldots \times T_d)$ the almost one-to-one factor map. Let $S \in C(T)$. By Proposition 6.12, this determines an element $S' \in C(\overline{G}, T_1 \times T_2 \times \ldots \times T_d)$ which acts as a translation by some element $h \in G$ (see Lemma 6.6). By a result of Hedlund [19], we note S is determined by a block code f of window size $k \in \mathbb{N}$. In particular, if $u \in \overline{O(\omega)}$, and $z = S(u)$, then

$$z[i] = f(u[i-k, i+k]) \text{ for all } i \in \mathbb{Z}^d$$

In particular, the automorphism determines what to put in a specific location by looking at a block around that location in the preimage. By choosing appropriate $j \in \mathbb{Z}^d$, we can define $\tilde{S} = S \circ T^j$ which would require $\tilde{S}$ to only look forward. Specifically, for $u \in \overline{O(\omega)}$, and $z = \tilde{S}(u)$ we have

$$z[i] = f(u[i, i+k]) \text{ for all } i \in \mathbb{Z}^d \tag{1}$$

for some $k \in \mathbb{N}$. As such, we can assume S is defined as a block map as in (1).

Note that G is a product odometer, so $h = (h_1, h_2, \ldots, h_d)$ where $h_i = \sum_{t=0}^{\infty} h_{t,i} p_{t-1,i}$ for $1 \leq i \leq d$ with $0 \leq h_{t,i} \leq \lambda_{t,i} - 1$. Each h_i is an element of the one dimensional odometer occurring in the i^{th} coordinate of h. Let $m_{t,i} = \sum_{j=0}^{t} h_{j,i} p_{j-1,i}$ and $m_t = (m_{t,1}, m_{t,2}, \ldots, m_{t,d}) \in \mathbb{Z}^d$. Let $Q_t \subseteq \Sigma^{\mathbb{Z}^d}$ be the clopen cylinder set with 0's located in a t-dimensional hybercube about the origin. Then $h \in T_1^{m_{t,1}} T_2^{m_{t,2}} \ldots T_d^{m_{t,d}} Q_t$.

We claim that for all $1 \leq i \leq d$ either $m_{t,i} \leq k$ or $m_{t,i} \geq p_{t,i} - k - 1$.

Let $x \in \overline{O(\omega)}$ and $y = S(x)$. Suppose that x has a $(t+1)$-block appearing at a location $x[i]$. Then by the construction of Toeplitz subshifts and almost one-to-one extensions, y necessarily has a $(t+1)$-block at the location $y[i - m_t]$.

Let A denote any $(t+1)$-block of x. Note that $A = x[i, i+k]$ for some $i \in \mathbb{Z}^d$ and $k \in \mathbb{N}$. This block looks like A_{t+1}, which in turn is the concatenation of A_t blocks. In particular, all t-blocks are the same, except they may disagree where the holes are located. Specifically, suppose C is a t-block and $C[i]$ is the location of the hole in C that is closest to the bottom left corner. In general, we choose the hole whose location vector i has the minimum length. If there is more than one hole with the same minimum length location vector, then we just choose one at random. Note $C[0, i-1]$ is completely determined, and is the same in those locations as every other t-block in x. The only place where t-blocks may potentially disagree is at the holes.

Let B be the $(t+1)$-block in y starting at location $y[i - m_t]$. Suppose the first hole in B occurs at $B[j]$ for some $j \in \mathbb{Z}^d$. This hole occurs at $A[j + m_t]$ in A. In order to determine what is at this location in B, S looks at a hypercube of side length k around $A[j]$. In view of Equation (1), $B[j]$ is determined by $A[j, j+k]$. We note that if $m_{t,i} > k$ for any $1 \leq i \leq d$, then this window would not overlap the hole at $A[j + m_t]$. And since this hole was the hole closest to the bottom left corner, everything in the window $A[j, j+k]$ is not a hole. And so $B[j]$ is uniquely determined, and is not a hole. Since A was an arbitrary $(t+1)$-block, every $(t+1)$-block will have the symbol $B[j]$ located the relative position j. In particular, $A[j + m_t] = B[j]$.

We can continue to the next hole in A on the same horizontal level, and the same argument would show that this hole is completely determined. Continuing this argument for every hole, we see that the entire block is completely filled in, and so then y is periodic, which is a contradiction. In general, in d dimensions, we move along hyperplanes in $d-1$ dimensions which are parallel to the coordinate hyperplanes. We fill in all the holes on a constant hyperplane, and then increase levels by one, until we fill in all the holes.

On the other hand, suppose that B is a $(t+1)$-block in y starting at location $y[i+m_t]$. Note that S^{-1} is also determined by a block map. If S is looking forward, then taking a larger k if needed one can show that S^{-1} is a "past looking" map determined by $z[i] = g([u[i-k,i]])$. Changing the role of x and y and using S^{-1} for S and using the argument similar to the one above, we can show that $m_t < p_t - k$.

The first case is demonstrated for the two dimensional case in Figure 10. Here, $\dot{A}_t$ indicates A_t blocks with all holes filled and the solid black and red squares indicate a hole in A and B, respectively.

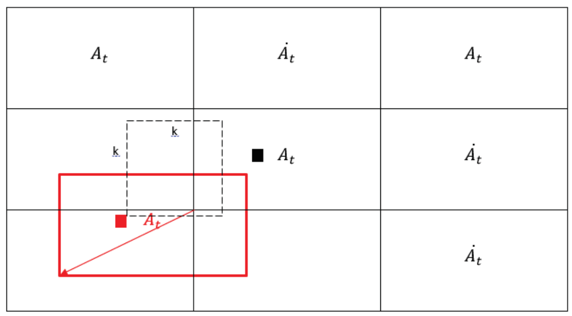

Figure 10

Now note that $h \in T_1^{m_{t,1}} T_2^{m_{t,2}} \dots T_d^{m_{t,d}} Q_t$ for all t. And since $|m_{t,i}| \leq k \mod p_{t,i}$, we have that h is in the orbit of 0. In particular,

$$h \in \bigcap_t T_1^{h_1} T_2^{h_2} \dots T_d^{h_d}(0) = T^m\{0\}$$

where $||m||_\infty = \max |m_{t,i}| \leq k$. So, $h = S'(0) = T^h(0)$, i.e. S' and T agree on one point. Furthermore, S' agrees with the action of $h \in \mathbb{Z}^d$ on the entire orbit of 0, which is dense. Therefore S' is a power of the shift, i.e. $S' = T^h$.

Let α be in the orbit of ω in $(\overline{O(\omega)}, T)$, i.e. $\alpha = T^g \omega$ for some $g \in \mathbb{Z}^d$. Note

$$\begin{aligned}
\pi S(\alpha) &= \pi S(T^g \omega) \\
&= S' \pi(T^g \omega) \\
&= S' T^g(0) \\
&= T^h T^g(0) \\
&= \pi T^h T^g \omega = \pi T^h(\alpha)
\end{aligned}$$

So $S(\alpha)$ and $T^h(\alpha)$ are in the same π fiber. Since α is in the orbit of ω, it has a unique preimage under π. Therefore $S(\alpha) = T^h(\alpha)$. And so S and T^h agree on the entire orbit of ω, which is dense. So $S = T^h$. $\square$

7. Positive Entropy Toeplitz Subshift

We now construct an explicit example of a two dimensional Toeplitz subshift which has positive entropy. This example is constructed so that it obeys the $(*)$ condition, thus ensuring that it has a trivial centralizer.

Let $h > 0$ and choose l_0 such that $\log(l_0 - 1) \leq h \leq \log(l_0)$. For $i \geq 0$, let $\varepsilon_i > 0$ and $\{\varepsilon_i\}$ be such that $\sum_{i=0}^{\infty} \varepsilon_i < \frac{h}{2}$.

We note that for any l and any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ sufficiently large such that

$$\frac{\log\left(l^{n^2}\right)}{(n+2)^2} \geq \log(l) - \varepsilon \tag{2}$$

since $\left(\frac{n}{n+2}\right)^2 \to 1$.

Let q_0 be chosen so that

$$\frac{\log({l_0}^{q_0^2})}{(q_0+2)^2} \geq \log(l_0) - \frac{\varepsilon_0}{2}$$

Also require $q_0^2 \geq l_0$. Define $l_1 = l_0^{q_0^2}$. We notice that there are $l_0^{q_0^2}$ square blocks of side length q_0 over the alphabet $\{0, 1, \ldots, l_0 - 1\}$. We enumerate these blocks as $B_i^{(0)}$ for $0 \leq i \leq l_1 - 1$. Furthermore, we require that $B_0^{(0)}$ and $B_1^{(0)}$ contain every letter from the alphabet. Let $C_i^{(0)}$ be the square block of side length $q_0 + 2$ with the block $B_i^{(0)}$ surrounded by a 0 in the top left corner, a 1 in the bottom right corner, and 0's below the main diagonal and 1's above it, as in the diagram below. We will denote this as $C_i^{(0)} = 0B_i^{(0)}1$ for $0 \leq i \leq l_1 - 1$.

$$C_i^{(0)} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & 1 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$B_i^{(0)}$}} & \vdots \\ \cline{1-1}\cline{4-4} \vdots & \multicolumn{2}{c|}{} & 1 \\ \hline 0 & \cdots & 0 & 1 \\ \hline \end{array}$$

For $k \geq 1$, define $l_k = l_{k-1}^{q_{k-1}^2}$ and let q_k be such that

$$\frac{\log({l_k}^{q_k^2})}{(q_k+2)^2} \geq \log(l_k) - \frac{\varepsilon_k}{2} \tag{3}$$

Additionally, require that $q_k^2 \geq l_k$. Let $B_i^{(k)}$ be all the square blocks of side length q_k over the alphabet $\{0, 1, \ldots, l_k - 1\}$ for $0 \leq i \leq l_{k+1} - 1$. Require that $B_0^{(k)}$ and $B_1^{(k)}$ contain every letter from the alphabet. Let $C_i^{(k)} = 0B_i^{(k)}1$ for $0 \leq i \leq l_{k+1} - 1$. Define $\lambda_k = q_k + 2$ and $p_k = \lambda_1 \lambda_2 \ldots \lambda_k$.

Consider the following operation on finite blocks: let $\{A_1, A_2, \ldots, A_n\}$ be square blocks of the same side length, A over some alphabet. Let B be a square block whose side length is at least $\sqrt{n}$ over an alphabet containing $\{1, 2, \ldots, n\}$. We define the block

$$C = \{A_1, A_2, \ldots, A_n\} * B$$

as $C[i,j] = A_{B[i.j]}$. In particular, C will be a square block of side length $|B| \cdot A$.

We are constructing a tiling of $\mathbb{Z}^2$ using k-blocks as building blocks. Additionally, we must construct these blocks so that they satisfy the $(*)$ condition. As such we define k-blocks in the following way: Let $A_i^{(0)} = C_i^{(0)}$ and

$$A_i^{(k)} = \{A_0^{(k-1)}, A_1^{(k-1)}, \ldots, A_{l_k-1}^{(k-1)}\} * C_i^{(k)}$$

We note that since $C_0^{(0)}$ and $C_1^{(0)}$ have every letter of the alphabet $\{0, 1, \ldots, l_0 - 1\}$, the blocks $A_0^{(1)}$ and $A_1^{(0)}$ will have every $0-$block as a subblock. Similarly, $C_0^{(1)}$ and $C_1^{(1)}$ contain every letter in $\{0, 1, \ldots, l_1 - 1\}$ and so the blocks $A_0^{(2)}$ and $A_1^{(2)}$ will contain every $1-$ block as a subblock. In general, we note that each block $A_i^{(k)}$ for $i = 0, 1$ has every $(k-1)-$block as a subblock.

We let

$$A_0 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & 1 \\ \hline 0 & & & \vdots \\ \cline{1-1} \cline{4-4} \vdots & \multicolumn{2}{c|}{-} & 1 \\ \hline 0 & \cdots & 0 & 1 \\ \hline \end{array}$$

where the side length of the square box A_0 is $q_0 + 2$, and the dash in the center square indicates a square of side length q_0 consisting of all holes.

Define

$$A_{k+1} = \begin{array}{|c|c|c|c|c|} \hline A_0^{(k)} & A_1^{(k)} & \cdots & A_1^{(k)} & A_1^{(k)} \\ \hline A_0^{(k)} & A_k & \cdots & A_k & A_1^{(k)} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline A_0^{(k)} & A_k & \cdots & A_k & A_1^{(k)} \\ \hline A_0^{(k)} & A_0^{(k)} & \cdots & A_0^{(k)} & A_1^{(k)} \\ \hline \end{array}$$

where there is a square block consisting of q_k^2 copies of A_k surrounded by $4q_k + 4$ copies of $A_i^{(k)}$ for $i = 0$ or 1 on each side. Notice that $A_0^{(k)}$ and $A_1^{(k)}$ have no holes, so all the holes are contained in the middle block of A_k blocks.

Let ω be the limiting array from the above process. We note here that ω satisfies the $(*)$ condition.

Proposition 7.1. *The Toeplitz system $(\overline{O(\omega)}, T)$ has positive entropy.*

Proof. Let h_ω be the entropy of $(\overline{O(\omega)}, T)$ and let $\Theta(n)$ be the number of square blocks of side length n appearing in ω. We note that $h_\omega = \lim_{n\to\infty} \frac{\log(\Theta(n))}{n^2} = \lim_{k\to\infty} \frac{\log(\Theta(p_k))}{p_k^2}$, by switching to a subsequence.

There are l_{k+1} many $k-$blocks. We note that every A_k block contains every $(k-1)-$ block as a subblock. This is because the blocks $C_i^{(k)}$ for $i = 0$ or $i = 1$ contain every letter of the alphabet in them. This means that as we do the shuffling process described above, the blocks $A_i^{(k)}$ for $i=0$ or $i=1$ contain every single block $A_i^{(k-1)}$ for $0 \leq i \leq l_k - 1$. The blocks $A_i^{(k)}$ for $i=0$ or $i=1$ are exactly those which occur in the $k-$blocks, and so they contain every $(k-1)-$block as a subblock. Furthermore, since $k-$blocks are squares of side length p_k, there are at least as many blocks of side length p_k occurring in ω as there are $k-$blocks. Specifically, square blocks of length p_k can occur at any position within ω, while $k-$blocks only occur at specific positions. Hence we have

$$\Theta(p_k) \geq l_{k+1} \tag{4}$$

So we have

$$h_\omega \geq \limsup_{k\to\infty} \frac{\log(l_{k+1})}{p_k^2} \tag{5}$$

By (3) we have that

$$\frac{\log(l_{k+1})}{\lambda_k^2} \geq \log(l_k) - \frac{\epsilon_k}{2}$$

It then follows, and by (2), that

$$\frac{\log(l_{k+1})}{p_k^2} \geq \frac{\lambda_k^2(\log(l_k) - \frac{\varepsilon_k}{2})}{p_k^2} = \frac{\log(l_k) - \frac{\varepsilon_k}{2}}{p_{k-1}^2} \geq \frac{\log(l_k)}{p_{k-1}^2} - \varepsilon_k$$

Continuing, we have

$$\frac{\log(l_{k+1})}{p_k^2} \geq h - \sum_{i=0}^{k} \varepsilon_i$$

Taking the limit as $k \to \infty$, from (5), we have $h_\omega \geq h/2 > 0$.

It is a basic fact that every Toeplitz system is minimal, so this system is minimal. It is either finite or uncountable, and since it has positive entropy, it cannot be finite. So this is an infinite minimal Toeplitz system.

□

8. Amenable Groups

Groups in mathematics can have a number of different properties. These are called the algebraic property of the group. For example, a group can be *finite*, meaning it only has a finite number of members. Some other examples include being *abelian*, which, from Definition 5.3, means that the order of the multiplication within the group does not matter. These are two properties that make groups very easy to work with. Specifically, if we know that a group is abelian, we do not need to keep track of the order in which we are multiplying group elements. This property makes working with them far more desirable.

Another desirable property of groups is known as *amenability*. Intuitively speaking, a group is amenable if averages can be taken within that group. The meaning of this is a little vague, so we will formalize it shortly. In order to motivate the definition of amenability, we will present a classic paradox, known as the Banach-Tarski paradox, first discovered in 1924 [2]. This paradox states that a ball with a finite volume, say V can be torn apart into pieces, and those pieces can be rearranged to give two identical copies of the original ball, as shown in Figure 11. While this sounds like an outlandish claim, it can be proven rigorously. Here we will sketch the key part of the proof:

Figure 11. The Banach-Tarski Paradox

We start with the *free group on two generators*. The free group on two generators, denoted by $\mathbb{F}_2$ is the set of all finite combinations of two symbols (i.e. generators), say a and b as well as a^{-1} and b^{-1} such that a^{-1} never occurs next to a, and b^{-1} never occurs next to b. In particular, $\mathbb{F}_2$ is the group on two generators with no relations.

Example 8.1. The words $a, ababab, aba^{-1}b^{-1}aab, b^{-1}$ are in $\mathbb{F}_2$. The word $abb^{-1}a$ can be *reduced* to aa, so to avoid redundancy we do not count it in $\mathbb{F}_2$.

Define $S(a)$ to be every word that starts with a, $S(b)$ to be every word that starts with b, and $S(a^{-1})$ and $S(b^{-1})$ are defined similarly. So, $S(a) = \{a, aa, ab, ab^{-1}, aab, aab^{-1}, aba, \dots\}$. As a technical point, we also need to include the *empty word*, denoted ϵ. This is just a word that has no letters in it. In particular, $\epsilon = aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b$. We notice now that we have completely accounted for every word in $\mathbb{F}_2$. That is,

$$\mathbb{F}_2 = \{\epsilon\} \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$$

Now, take $S(a^{-1})$ and add the letter a to the beginning of each word. Denote this by $aS(a^{-1})$. This will give us every word that does not begin with a. And so we also have

$$\mathbb{F}_2 = aS(a^{-1}) \cup S(a)$$

Similarly, we have

$$\mathbb{F}_2 = bS(b^{-1}) \cup S(b)$$

This step is the most important part of the proof. Essentially we have broken up the group $\mathbb{F}_2$ first into four pieces all having the same size (as well as $\{\epsilon\}$ which we will say has no size). But then we broke it in a different way into two pieces which each have the same size as the pieces from the first division. Intuitively, we can think of the ball as being $\mathbb{F}_2$. First we will break it into four sections, each of which has many small pieces contained in it, and notice that we need only two of these sections to reconstruct the original ball. And since we have four, we can make two balls, each the same size as the original. It takes some additional rigor to show why we can think of the ball as being analogous to $\mathbb{F}_2$, but for the purposes of this we have completed the crucial step.

Figure 12 shows the *Cayley graph* of $\mathbb{F}_2$. This is a graphical representation of the group $\mathbb{F}_2$. Starting in the middle, at the identity element, if we want an a we go to right, we go up for a

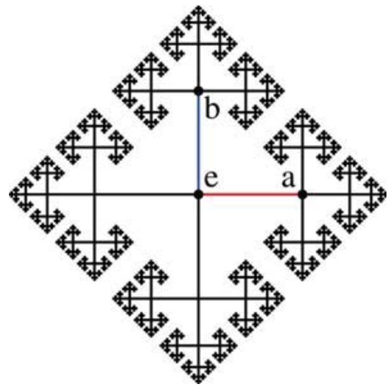

FIGURE 12. The Cayley Graph of $\mathbb{F}_2$

b, left for a^{-1} and down for b^{-1}. After this first step, there are now three choices depending on the first choice. For example, if we started by going right, then we can't immediately follow that with left, as this would correspond to aa^{-1} which just takes us back to the start. So if we started by going to the right, then went up, and then left, this would correspond to aba^{-1}. Now, we note that the set $S(a)$ is the main branch of the graph off to the right, as this represents all the words that start with a. Similarly, $S(b)$ is the branch going up, $S(a^{-1})$ is the branch going left and $S(b^{-1})$ is the branch going down. So the group $\mathbb{F}_2$ is split into these four branches in the Cayley graph, and we notice that these branches don't overlap each other. We note the self symmetry of the graph. For example, the left branch, which is $S(a^{-1})$, is a rescaled version of the upward, downward, and left facing branches. In particular, if we shift the left branch over to the right and scale it up, we will exactly overlap the branches corresponding to $S(b)$, $S(b^{-1})$, and $S(a^{-1})$. This shifting is exactly the same as adding an a to the beginning of every word in the original branch. So we have just shown graphically why $aS(a^{-1})$ is the same as $S(b) \cup S(a^{-1}) \cup S(b^{-1})$.

This is called a *paradoxical decomposition* and it really challenges our intuitive understanding of the physical world. Certainly this kind of decomposition is not actually physically possible with a real ball, but there is nothing in mathematics that prohibits it. Some would argue that occurrences like this lie at the heart of the beauty of mathematics. However, this sometimes makes even mathematicians uncomfortable, so the concept of amenability was developed. The main reason this paradoxical decomposition was able to occur is because $\mathbb{F}_2$ is not amenable.

In what follows, we will provide some results about amenability as well as some different but equivalent definitions. The study of amenability is a rich field of mathematics, and what we have here only scratches the surface. A good survey of amenability can be found in Paterson [31].

Definitions of Amenability. There are several equivalent definitions for amenability. Here we will provide two, and show that they are in fact equivalent. The concept of amenability is complicated, so first, we will provide some basic definitions and examples. In what follows we will always assume G is a discrete group. We will denote by $\mathcal{P}(G)$ the *power set* of G; that is, the set of all subsets of G.

Definition 8.2 (Measure)**.** A *measure* on a group G is a function $\mu : \mathcal{P}(G) \to \mathbb{R}$ such that $\forall X \in \mathcal{P}(G)$ the following hold:

(1) $\mu(X) \geq 0$

(2) $\mu(\emptyset) = 0$

(3) For a countable collection of pairwise disjoint subsets $\{X_i\}_{i=0}^{\infty} \subseteq \mathcal{P}(G)$, we have

$$\mu\left(\bigcup_{i=0}^{\infty} X_i\right) = \sum_{i=0}^{\infty} \mu(X_i)$$

Example 8.3. $G = \mathbb{Z}$. Define $\mu(X)$ to be the number of elements of X, for any $X \subseteq \mathcal{P}(\mathbb{Z})$. So $\mu(\{0,1,2,3\}) = 4$, and $\mu(\{9,14,2017\}) = 3$. Then μ is a measure on $\mathbb{Z}$.

We now introduce some additional properties that a measure can attain.

Definition 8.4. Let G be a group, and μ a measure on G.

- μ is called a *probability measure* if $\mu(G) = 1$ and for any $X \subseteq G$, $0 \leq \mu(X) \leq 1$.
- μ is called *finitely additive* if for any finite collection of disjoint subsets of G, say $\{X_i\}_{i=0}^{n}$ we have

$$\mu\left(\bigcup_{i=0}^{n} X_i\right) = \sum_{i=0}^{n} \mu(X_i)$$

Example 8.5. Let $G = \{0,1,2\}$. Define μ as follows:

$$\mu(\emptyset) = 0$$
$$\mu(\{0\}) = \mu(\{1\}) = \mu(\{2\}) = 1/3$$
$$\mu(\{0,1\}) = \mu(\{0,2\}) = \mu(\{1,2\}) = 2/3$$
$$\mu(\{0,1,2\}) = 1$$

Here we note that μ is both a probability measure and finitely additive.

We now introduce the concept of a coset.

Definition 8.6. Let G be a group, H a subgroup and g be a group element not in H. The the *left coset* of H, denoted gH is

$$gH = \{gh \mid h \in H\}$$

Example 8.7. Let $G = \mathbb{Z}$ and $H = 2\mathbb{Z} = \{\ldots,-4,-2,0,2,4,\ldots\}$. Let $g = 1$. Then $gH = 1 + 2\mathbb{Z} = \{\ldots,-3,-1,1,3,5,\ldots\}$ is the left coset.

We now introduce one more property a measure can have.

Definition 8.8. Let G be a group and μ a measure on G. We say μ is *left-invariant* if for all subgroups $H \subseteq G$, and every $g \in G$, we have

$$\mu(gH) = \mu(H)$$

We are now ready to define amenability.

Definition 8.9. Let G be a group. We say G is *amenable* if it admits a left-invariant finitely additive probability measure.

Some basic results are that every finite group is amenable, and every abelian group is amenable. However, the free group is not amenable.

Proposition 8.10. *$\mathbb{F}_2$ is not amenable.*

Proof. Suppose $\mathbb{F}_2$ is amenable, i.e. there exists a left-invariant finitely additive probability measure μ on $\mathbb{F}_2$. As above, we note that

$$\mathbb{F}_2 = \{\epsilon\} \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$$

And also

$$\mathbb{F}_2 = aS(a^{-1}) \cup S(a)$$

Applying μ, and noting that $\mu(\{\epsilon\}) = 0$, we have

$$\mu(\mathbb{F}_2) = \mu(S(a)) + \mu(S(b)) + \mu(S(a^{-1})) + \mu(S(b^{-1}))$$

On the other hand, we have

$$\mu(\mathbb{F}_2) = \mu(aS(a^{-1})) + \mu(S(a))$$

But $\mu(aS(a^{-1})) = \mu(S(a^{-1})$ since μ is left-invariant. So we have

$$\mu(S(a)) + \mu(S(b)) + \mu(S(a^{-1})) + \mu(S(b^{-1})) = \mu(S(a^{-1})) + \mu(S(a))$$
$$\therefore \mu(S(b)) + \mu(S(b^{-1})) = 0$$

which is a contradiction. So no such measure can exist, therefore $\mathbb{F}_2$ is not amenable. □

Example 8.11. The measure defined in Example 8.5 is left-invariant, finitely additive and is a probability measure, and so the group $G = \{0, 1, 2\}$ with the addition operation is amenable.

Amenability is a nice property for groups which makes them easy to work with. Some other nice properties of groups can guarantee amenability. For example, all finite groups are amenable.

Proposition 8.12. *All finite groups are amenable.*

Proof. Let G be a finite group and $|G| = k$. For any $X \subseteq G$, define

$$\mu(X) = \frac{|X|}{|G|}$$

We note that this is indeed a probability measure since $0 \leq \mu(X) \leq 1$ for all $X \subseteq G$. Let $\{X_i\}_{i=0}^n$ be a family of disjoint subsets of G. Then

$$\left| \bigcup_{i=0}^{n} X_i \right| = \sum_{i=0}^{n} |X_i|$$

and finite additivity immediately follows. Let $X \subseteq G$ be a subgroup and $g \in G$. Then $|gX| = |X|$ and so this measure is left-invariant. Hence G admits a left-invariant finitely additive probability measure, and so it is abelian. □

Note the measure constructed in the proof of Proposition 8.12 is exactly the measure used in Example 8.5.

We now provide another definition of amenability which is equivalent to the first definition. This definition is due to Følner in 1955 in [17].

Definition 8.13 (Følner Sequence)**.** Let G be a countable group. A *Følner Sequence* is a sequence of subsets $\{F_i\} \subseteq G$ such that for all $x \in G$, there exists i such that for all $j > i$ $x \in F_j$ and

$$\lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0$$

for all $g \in G$.

Here, $gF_i \triangle F_i$ is the *symmetric difference* between gF_i and F_i, i.e.

$$gF_i \triangle F_i = (gF_i - F_i) \cup (F_i - gF_i)$$

Intuitively, this definition tells us when a group acting on itself doesn't move the subsets around "too much". This is far from rigorous, but gives some intuition as to why this definition is equivalent to the invariant measure definition. Essentially, in this definition, we have sets that, when disturbed on the left by a group element remain similar. In the invariant measures definition, this disturbance doesn't change the measure of those sets, so there is kind of a philosophical connection between these two definitions. Of course, we will prove this in full mathematical rigor.

Proposition 8.14. *A group G is amenable if and only if it admits a Følner Sequence.*

Example 8.15. We will show, using a Følner Sequence, that $\mathbb{Z}$ is amenable.

Let $X = \mathbb{Z}$. Let $F_i = [-i, i]$ be closed intervals. Then for $g \in \mathbb{Z}$, $gF_i = [-i+g, i+g]$ and so $gF_i \triangle F_i = [-i, -i+g] \cup [i, i+g]$. So $|gF_i \triangle F_i| = 2g$ and $|F_i| = 2i+1$. So,

$$\begin{aligned} \lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} &= \lim_{i\to\infty} \frac{2g}{2i+1} \\ &= 0 \ \forall\ g \in G \end{aligned}$$

Therefore $\mathbb{Z}$ admits a Følner Sequence, and so it is amenable.

Ultrafilters and Ultralimits. In order to prove this equivalence between these two definitions of amenability, we need to develop the idea of *ultrafilters* and *ultralimits*. Ultrafilters give us an idea of which subsets of a set are, in a sense, large, and which ones are small. And using this notion of ultrafilters, we are able to expand the notion of a usual limit, to an ultralimit.

Definition 8.16. Let X be a set, and $\mathcal{F}$ be a nonempty collection of subsets of X. We say $\mathcal{F}$ is a *filter* if

- $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

If we also have $\forall A \subseteq X$, $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$, then we call $\mathcal{F}$ an *ultrafilter*.

Intuitively speaking, a set is in the filter if it is "big", and it is not if it is not "big".

Example 8.17. Consider the filter on $\mathbb{N}$, known as the cofinite filter defined as

$$\mathcal{F} = \{A \subseteq \mathbb{N} \mid \mathbb{N} - A \text{ is finite.}\}$$

We can check all the above conditions to confirm that this is indeed a filter.

Example 8.18. The easiest construction, and, in fact, the only explicit construction of an ultrafilter is the *principal ultrafilter*. Let X be an infinite set and let $x \in X$. Define

$$\mathcal{F} = \{A \subseteq X \mid x \in A\}$$

We now prove a basic fact about filters.

Lemma 8.19. *If $\mathcal{F}$ is a filter then every finite subfamily of $\mathcal{F}$ has nonempty intersection. This is called the finite intersection property.*

Proof. Let $\mathcal{F}$ be a filter on a set X. We note that any finite intersection of members of $\mathcal{F}$ must also be a member of $\mathcal{F}$ and so it cannot be empty. $\square$

We now show how ultrafilters can be used to expand our notion of limits. In particular, we use ultrafilters to define *ultralimits*.

Definition 8.20 (Convergence of an ultrafilter in a topological space)**.** Let Y be a compact topological space and ω an ultrafilter on Y. We say ω *converges* to $y \in Y$ if for every open set U containing y, $U \in \omega$.

Example 8.21. Let $Y = \mathbb{R}$ with the usual topology. Let ω be the principal ultrafilter defined by $\omega = \{A \subseteq \mathbb{R} \mid 0 \in A\}$. Then $\omega \to 0$.

In the following proposition, we use an ultrafilter in one space to construct an ultrafilter in another space, connected to the first by a function. This will allow us to define a general ultralimit.

Proposition 8.22. *Let ω be an ultrafilter on an infinite set X , Y be a topological space and $f : X \to Y$ be a function. Then $\omega_f = \{A \subseteq Y \mid f^{-1}(A) \in \omega\}$ is an ultrafilter on Y.*

Proof. Note $f^{-1}(Y) = X \in \omega \Rightarrow Y \in \omega_f$.

Let $A, B \in \omega_f$. Then $f^{-1}(A) \in \omega$ and $f^{-1}(B) \in \omega$. So $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in \omega_f \Rightarrow A \cap B \in \omega_f$.

Let $A \in \omega_f$ and $A \subseteq B$. Then $f^{-1}(A) \in \omega$. Also, $f^{-1}(A) \subseteq f^{-1}(B) \Rightarrow f^{-1}(B) \in \omega \Rightarrow B \in \omega_f$.

Finally, let $A \in Y$ and suppose $A \notin \omega_f$. Then $f^{-1}(A) \notin \omega \Rightarrow X - f^{-1}(A) \in \omega$, since ω is an ultrafilter.

Note $f^{-1}(Y - A) = X - f^{-1}(A) \in \omega \Rightarrow Y - A \in \omega_f$. And so ω_f is an ultrafilter on Y. $\square$

Now we can introduce ultralimits:

Definition 8.23 (Ultralimit)**.** Let X be an infinite set, ω an ultrafilter on X, Y a topological space and $f : X \to Y$ a function. The the *ultralimit* of f with respect to ω is

$$\lim_{\omega} f = y \in Y$$

if ω_f converges to y.

In Example 8.21, we see the importance of non-principal ultrafilters. Specifically, the ultralimit taken over any principal ultrafilter will always converge to the same point. Because of this, we will show the existence of a non-principal ultrafilter. The proof is not constructive, meaning that an explicit non-principal ultrafilter is not constructed. Rather we show that one must exist, but we do not know what it actually looks like.

In order to prove the existence of a non-principal ultrafilter, we first show that ultrafilters are maximal filters.

Definition 8.24. Let $\mathcal{F}$ be a filter on a set X. We say that $\mathcal{F}$ is *maximal* if for any other filter $\mathcal{F}'$ of X, $\mathcal{F}' \subset \mathcal{F}$.

We will show that maximal filters are exactly the ultrafilters.

Proposition 8.25. *Let $\mathcal{F}$ be a filter on a set X. Then $\mathcal{F}$ is an ultrafilter if and only if it is maximal.*

Proof. ($\Rightarrow$) Suppose $\mathcal{F}$ is an ultrafilter on X which is not maximal. Then $\mathcal{F} \subset \mathcal{F}'$ where $\mathcal{F}'$ is some filter on X. Note $\mathcal{F}' - \mathcal{F}$ contains some set $B \in \mathcal{F}'$ with $B \notin \mathcal{F}$. Then, since $\mathcal{F}$ is an ultrafilter, $X - B \in \mathcal{F}$. But since $\mathcal{F} \subset \mathcal{F}'$, we have $X - B \in \mathcal{F}'$. And so $B \cap (X - B) \in \mathcal{F}' \Rightarrow \emptyset \in \mathcal{F}'$. This is a contradiction, so $\mathcal{F}$ is maximal.

($\Leftarrow$) Suppose $\mathcal{F}$ is a maximal filter on X and there exists $A \subseteq X$ such that $A \notin \mathcal{F}$. Since $\mathcal{F}$ is maximal, $\mathcal{F} \cup \{A\}$ is not a filter, therefore, by Lemma 8.19 it does not have the finite intersection property. So $\exists B \in \mathcal{F}$ such that $B \cap A = \emptyset$ so $B \subseteq (X - A) \Rightarrow (X - A) \in \mathcal{F}$. So $\mathcal{F}$ is an ultrafilter.

□

We will now show that non-principal ultrafilters exist. In doing so, we will make use of Zorn's Lemma. Zorn's Lemma was actually first described in 1922 by Casimir Kuratowski [26] and then independently described by Max Zorn [35] in 1935. Curiously, it is named after Zorn.

Lemma 8.26 (Zorn's Lemma)**.** *Let X be a partially ordered set such that every non-empty chain has an upper bound. Then X admits a maximal element.*

This lemma is equivalent to the once controversial Axiom of Choice, but proof of that equivalence, and a proof of the lemma are beyond the scope of this paper.

Proposition 8.27. *Non-principal ultrafilters exist.*

To prove this, we will make use of free filters.

Definition 8.28. A filter $\mathcal{F}$ is free if $\bigcap_{A \in \mathcal{F}} A = \emptyset$.

We note that a principal ultrafilter is not free, so in proving the existence of non-principal ultrafilters, it suffices to prove the existence of a free ultrafilter. We also note that the cofinite filter from Example 8.17 is free, so every infinite set has a free filter.

Proof. Proposition 8.27.
Let $\mathcal{F}_0$ be a free filter on an infinite set X and let S be the set of free filters on X containing $\mathcal{F}_)$, i.e. $S = \{\mathcal{F} \mid \mathcal{F}_0 \subseteq \mathcal{F} \text{ and } \mathcal{F} \text{ is a free filter}\}$. We note S is nonempty because $\mathcal{F} \in S$. We now partially order S by set inclusion. Let C be a chain in S such that $\forall \mathcal{F}_i, \mathcal{F}_j \in C$ either $\mathcal{F}_i \subseteq \mathcal{F}_j$ or $\mathcal{F}_j \subseteq \mathcal{F}_i$. Let $\Gamma = \bigcup_{\mathcal{F} \in C} \mathcal{F}$.

Lemma 8.29. $\Gamma \in S$.

To prove this lemma, we need to show that Γ is a free filter on X containing $\mathcal{F}_0$. First we show that Γ is a filter:

Suppose $\emptyset \in \Gamma$. Since $\Gamma = \bigcup_{\mathcal{F} \in C} \mathcal{F}$, this implies that $\emptyset \in \mathcal{F}$, for some $\mathcal{F} \in C$ which is a contradiction.

Let $A, B \in \Gamma$. Then $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ for some i, j. Note $A \cap B \in \mathcal{F}_i \cup \mathcal{F}_j$ and so $A \cap B \in \mathcal{F}$ for some $\mathcal{F} \in C \Rightarrow A \cap B \in \Gamma$.

Let $A \in \Gamma$ and $A \subseteq B$. So $A \in \mathcal{F}$ for some $\mathcal{F} \in C$ which implies $B \in \mathcal{F}$ and so $B \in \Gamma$. Hence we have that Γ is a filter on X.

Now suppose Γ is not free, i.e. $\exists m \in X$ such that

$$m \in \bigcap_{A \in \Gamma} A$$

Then $\forall A \in \Gamma$, $m \in A$ which implies that for some $\mathcal{F} \in C$, $m \in \bigcap_{A \in \mathcal{F}} A$. This contradicts the fact that $\mathcal{F}$ is free. So we have proved our claim. In particular, we have that $\Gamma \in S$, thus proving Lemma 8.29.

Clearly Γ is an upper bound for C. So any chain in S has an upper bound. Invoking Zorn's Lemma we know that S admits a maximal element, $\mathcal{F}'$. This is a free maximal filter, i.e. it is a non-principal ultrafilter. □

Corollary 8.30. *As a corollary to Proposition 8.27 we note that any non-principal ultrafilter on an infinite set contains the cofinite filter.*

Now, (finally), we can prove the equivalence between the two definitions of amenability.

Proof. Proposition 8.14.
($\Leftarrow$) Let $F_i \subseteq G$ be such that

$$\lim_{i \to \infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0$$

for all $g \in G$.

For $A \subseteq G$ define

$$\mu(A) = \lim_{i \to \infty} \frac{|A \cap F_i|}{|F_i|}$$

This limit may not exist, so we take a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. We note that $\mathcal{F}$ contains the cofinite filter. So, we have

$$\mu(A) = \lim_{\mathcal{F}} \frac{|A \cap F_i|}{|F_i|}$$

Note that $0 \leq \mu(A) \leq 1$ for all $A \subseteq G$.

Now

$$\begin{aligned} \mu(G) &= \lim_{\mathcal{F}} \frac{|G \cap F_i|}{|F_i|} \\ &= \lim_{\mathcal{F}} \frac{|F_i|}{|F_i|} = 1 \end{aligned}$$

as desired.

Let $A, B \subseteq G$ be disjoint. Then

$$\begin{aligned} \mu(A \cup B) &= \lim_{\mathcal{F}} \frac{|(A \cup B) \cap F_1|}{|F_i|} \\ &= \lim_{\mathcal{F}} \frac{|(A \cap F_i) \cup (B \cap F_1)|}{|F_i|} \\ &= \lim_{\mathcal{F}} \left(\frac{|A \cap F_i|}{|F_i|} + \frac{|B \cap F_i|}{|F_i|} \right) \\ &= \mu(A) + \mu(B) \end{aligned}$$

and so μ is finitely additive.

Now, let $A \subseteq G$ and $g \in G$. Then

$$\begin{aligned} \left| \frac{|gA \cap F_i|}{|F_i|} - \frac{|A \cap F_i|}{|F_i|} \right| &= \left| \frac{|A \cap g^{-1}F_i| - |A \cap F_i|}{|F_i|} \right| \\ &\leq \left| \frac{|A \cap (g^{-1}F_i \triangle F_i)|}{|F_i|} \right| \to 0 \end{aligned}$$

So $\mu(gA) - \mu(A) = 0$ and so μ is left-invariant. Therefore G is amenable.

□

Other Characterizations of Amenability. Sometimes, the definitions above are somewhat cumbersome to work with. In practice, it is not always easy or possible to find an explicit invariant measure or Følner sequence on a group. Accordingly, mathematicians have developed other constructs through which we can check the amenability of a group. One we will present here, known as *Grigorchuk's Cogrowth Criteria* allows us to check amenability of groups essentially by counting how many words are allowable in the group. This criteria was first described by Grigorchuk [18] in 1977, and was further described by Joel Cohen [9].

We start with a group G with a presentation $\langle x_1, x_2, \ldots, x_t \mid r_1, r_2, \ldots \rangle$. That is, G is generated by $x_1, x_2, \ldots, x_t$ subject to the relations $r_1, r_2, \ldots$. For example, the group $\mathbb{Z}^2$ can be presented as $\mathbb{Z}^2 = \langle e_1, e_2 \mid e_1 e_2 = e_2 e_1 \rangle$, where e_1 and e_2 are the standard coordinate vectors.

Let $\mathbb{F}$ be the free group on the generators $\{x_1, x_2, \ldots, x_t\}$ and N be the normal subgroup of $\mathbb{F}$ generated by $r_1, r_2, \ldots$. Then $G = \mathbb{F}/N$.

A reduced word is a word in which a symbol is not immediately followed by its inverse. If this were the case, then we can *reduce* these two symbols to the identity. Let $E_n = \{w \in \mathbb{F} \mid |w| = n\}$; that is, E_n is the number of reduced words in $\mathbb{F}$ of length n. Let $N_n = N \cap E_n$. So N_n is the set of reduced words of length n in G which are equal to the identity. Let $\gamma_n = |N_n|$. In particular, γ_n is the number of reduced words of length n which are equal to the identity. Let $\overline{\gamma_n} = \sum_{i=0}^{n} \gamma_i$ which is the number of reduced words of length at most n equal to the identity.

Theorem 8.31 (Grigorchuk's Cogrowth Criteria)**.** *Let*

$$\gamma = \lim_{n\to\infty} (\overline{\gamma_n})^{1/n}$$

Then $\gamma = 1$ if and only if G is the free group, and $\gamma = 2t - 1$ if and only if G is amenable.

A proof of this is beyond the scope of this report, but we will provide examples to demonstrate the use of this criterion. However, we will prove that this limit always exists.

Lemma 8.32. *Let γ_n be as defined above. Then $\gamma_n\gamma_m \leq \gamma_{n+m+2}$ for all $n, m \geq 0$.*

Proof. Let $\alpha \in N_m$, $\beta \in N_n$. Suppose α ends with the letter $v_0 \in \{x_1, x_2, \ldots, x_t\}$ and β begins with u_0 and ends with u_1. Pick any letter $u \in E_1 - \{v_0^{-1}, u_0^{-1}, u_1\}$. Then $\alpha u \beta u^{-1} \in N_{m+n+2}$. So, $N_n \times N_m \subseteq N_{m+n+2} \Rightarrow \gamma_n\gamma_m \leq \gamma_{m+n+2}$. □

We now prove something very closely related to the existence of γ. Note that in the following lemma, we are proving the convergence of the limit of $(\gamma_n)^{1/n}$ as opposed to the convergence of the limit of $(\overline{\gamma_n})^{1/n}$. The equivalence of this to the existence of γ is then proven in Proposition 8.34.

Lemma 8.33. *The limit $\lim_{n\to\infty}(\gamma_n)^{1/n}$ exists when taken over n such that $\gamma_n \neq 0$.*

Proof. Let $a_n = \log(\gamma_{n-2}) \geq 0$. So $a_n + a_m \leq a_{n+m}$, by Lemma 8.32. Let $b_n = -a_n$. Then $b_{n+m} \leq b_n + b_m$. So, $\{b_n\}$ is a subadditive sequence and hence $\lim_{n\to\infty} \frac{b_n}{n}$ exists. Therefore, $\lim_{n\to\infty} \frac{a_n}{n}$ exists.

Note

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{n} &= \lim_{n\to\infty} \frac{\log(\gamma_{n-2})}{n} \\ &= \lim_{n\to\infty} \log(\gamma_{n-2})^{1/n} = \lim_{n\to\infty} (\gamma_n)^{1/n}\end{aligned}$$

So $\lim_{n\to\infty}(\gamma_n)^{1/n}$ exists. □

Proposition 8.34. *The limit $\lim_{n\to\infty}(\overline{\gamma_n})^{1/n} = \lim_{n\to\infty}(\gamma_n)^{1/n}$. In particular, $\gamma = \lim_{n\to\infty}(\overline{\gamma_n})^{1/n}$ exists.*

Proof. We note $\gamma_{n-1} + \gamma_n \leq \overline{\gamma_n}$ for each $n \geq 1$ since $\gamma_n \geq 0$ for all $n \geq 0$. Also $\gamma_m \leq \gamma_{m+2}$, by Lemma 8.32. So,

$$\begin{aligned}
\gamma_{n-1} + \gamma_n &\leq \gamma_{n-1} + \gamma_n \\
\gamma_{n-3} + \gamma_{n-2} &\leq \gamma_{n-1} + \gamma_n \\
\gamma_{n-5} + \gamma_{n-4} &\leq \gamma_{n-3} + \gamma_{n-2} \leq \gamma_{n-1} + \gamma_n \\
&\vdots \\
\gamma_1 + \gamma_0 &\leq \gamma_{n-1} + \gamma_n
\end{aligned}$$

Summing, we have

$$\overline{\gamma_n} = \sum_{i=0}^{n} \gamma_i \leq n(\gamma_{n-1} + \gamma_n)$$

So,

$$\gamma_{n-1} + \gamma_n \leq \overline{\gamma_n} \leq n(\gamma_{n-1} + \gamma_n)$$

We note that either γ_n or γ_{n-1} is equal to 0, so

$$\lim_{n\to\infty} (\gamma_n)^{1/n} \leq \lim_{n\to\infty} (\overline{\gamma_n})^{1/n} \leq \lim_{n\to\infty} (n\gamma_n)^{1/n} = \lim_{n\to\infty} (\gamma_n)^{1/n}$$

So, by the squeeze theorem, $\lim_{n\to\infty}(\overline{\gamma_n})^{1/n}$ exists, and is equal to $\lim_{n\to\infty}(\gamma_n)^{1/n}$. $\square$

We now provide some examples of what Grigorchuk's criteria can tell us about the amenability of groups.

Example 8.35. Let $G = \mathbb{F}_2$, i.e. the free group on two generators. We need to calculate γ_n, the number of reduced words of length n equal to the identity in this group. Clearly $\gamma_0 = 1$. Note that, since there are no relations in the free group, there is no way to have a reduced word that is equal to the identity. So, $\gamma_n = 0$ for $n \geq 1$. Therefore, $\overline{\gamma_n} = 1$ for all n, and so $\gamma = \lim_{n\to\infty}(\overline{\gamma_n})^{1/n} = \lim_{n\to\infty}(1)^{1/n} = 1$. Therefore, G is not amenable, and we have in fact confirmed that it is free.

We will now show an example of a group which is amenable.

Example 8.36. Let $G = \mathbb{Z}^2$. We note that $\mathbb{Z}^2 = \langle e_1, e_2 \mid e_1 e_2 = e_2 e_1 \rangle$. This is generated by two generators, so $t = 2$. Since we know $\mathbb{Z}^2$ to be abelian and hence amenable, we expect $\gamma = 2t - 1 = 3$.

Finding the number of reduced words of length n in this group equal to the identity turns out to be a non-trivial combinatorics problem. We can find the result as Lemma 4.1 due to Kempton [23] on page 8 in which we learn that

$$\gamma_{2n} = \sum_{i=0}^{n} (-3)^i \binom{2n-i}{i} \binom{2n-2i}{n-1}^2 - \sum_{i=0}^{n-1} (-3)^i \binom{2n-i-2}{i} \binom{2n-2i-2}{n-i-2}^2$$

Asymptotically, this behaves like

$$\gamma_{2n} \sim \frac{2}{\pi n} 3^{2n-1}$$

which is Corollary 4.2 from Kempton [23] on page 8.

Calculating

$$\begin{aligned}
\gamma &= \lim_{n\to\infty} \left(\overline{\gamma_{2n}}\right)^{1/2n} \\
&= \lim_{n\to\infty} \left(\frac{2}{\pi n} 3^{2n-1}\right)^{1/2n} \\
&= \lim_{n\to\infty} \left(\frac{2}{\pi n}\right)^{1/2n} \lim_{n\to\infty} \left(3^{2n-1}\right)^{1/2n} \\
&= \lim_{n\to\infty} \left(\frac{2}{\pi}\right)^{1/2n} \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/2n} \lim_{n\to\infty} \left(3^{1-\frac{1}{2n}}\right) \\
&= 1 \cdot 1 \cdot 3 = 3
\end{aligned}$$

And so $\gamma = 3$, as expected and we have shown that $\mathbb{Z}^2$ is in fact amenable.

Admittedly this proof of amenability of $\mathbb{Z}^2$ is more cumbersome than proving it using Følner sequences or invariant measures, but it is helpful as an illustrative example. Sometimes actually working with the groups on which we are trying to prove amenability can be extremely difficult. In these cases it is sometimes far more helpful and enlightening to work with word growth in these groups and invoke Grigorchuk's criteria.

9. Topological Full Group

In this section, we introduce another group associated with a dynamical system. Recall the centralizer of a dynamical system is a group which captures the symmetries of that system. The topological full group of a dynamical system has been shown to contain all the information about a dynamical system, up to a reversal in time. See Medynets, Bezuglyi [3] for details. Principally, in this section we are concerned with determining when the topological full group of dynamical systems is amenable. The motivation for this is trying to find an example of an amenable group which is simple, finitely generated and finitely presented.

Let $X \subseteq A^{\mathbb{Z}}$ be a subshift on some finite alphabet A. Let $T : X \to X$ be single homeomorphism which is the shift action on X. We assume that T is minimal, that is there are no nonempty, closed T-invariant proper subsets of X. Alternatively, T is minimal if every orbit is dense in X. Let $L_n(X)$ be all the words of length n which appear in points in X. Furthermore, we want to restrict $L_n(X)$ to be such that for all $w \in L_n(X)$, $w_i \neq w_{i+1}$. That is, the words of length n do not contain two consecutive letters which are the same. If X does not have this property, we can always shift to a higher block code. This means we can take our letters to be elements of A^k for some $k > 1$. It is known that, as long as X is infinite, such a (finite) k will always exist (Lind and Marcus[27]). Let the minimum such number be k and let B be our new alphabet consisting of words from A^k.

For $x \in X$ and $b \in B$, define

$$\sigma_b(x) = \begin{cases} Tx & \text{if } x_0 = b \\ T^{-1}x & \text{if } x_{-1} = b \\ x & \text{otherwise} \end{cases}$$

Note that, the condition of $w_i \neq w_{i+1}$ is important so that this function is well defined. Otherwise, if two of the same letters appeared consecutively there would be ambiguity over whether the first or second case applied.

The function σ_b is defined for letters $b \in B$. We can similarly define σ_w where w is a word containing letters in B. The word w does not necessarily need to be a word in X. Say $w = b_0 b_1 \dots b_n$. Then, $\sigma_w = \sigma_{b_0}\sigma_{b_1}\dots\sigma_{b_n}$.

So, σ_w acts on a point $x \in X$ by either shifting the point left or right one place for each letter it sees in the central window. Note that, if $|w| = n$, then we need only to look at the the central window of x of length $2n+1$.

Example 9.1. Let $B = \{0,1,2,3,4,5,6,7\}$, and let $w = 3342$. Suppose $x = \dots 6234|71034\dots$, where the first entry to the right of the vertical bar is x_0. Applying σ_w, we first apply σ_2. Note

$$\begin{aligned} \sigma_2(x) &= \dots 6234|71034\dots \text{since 2 doesn't appear at the } 0^{\text{th}} \text{ or } -1^{\text{st}} \text{ spot} \\ \sigma_4\sigma_2(x) &= \dots 623|471034\dots \\ \sigma_3\sigma_4\sigma_2(x) &= \dots 62|3471034 \\ \sigma_w(x) = \sigma_3\sigma_3\sigma_4\sigma_2(x) &= \dots 623|471034 \end{aligned}$$

Here we see that applying σ essentially is equivalent to shifting the vertical bar separating positive and negative time back and forth.

We are interested in finding when applying σ_w reduces to the identity, i.e. which words have the property so that when we are done shifting the vertical bar back and forth, we are left with what we began with.

Let $G = \langle \sigma_b \mid b \in B\rangle$. Let $B_n = \{w \in B^n \mid \sigma_w = 1\}$. In other words, B_n is all the words w of length n containing letters from B which when σ_w is applied to any point $x \in X$ acts as identity, i.e. $\sigma_w(x) = x$. We then note that, using the notation from Theorem 8.31, $|B_n| = \gamma_n$. This observation gives us a way to check the amenability of G. In particular, we want to calculate

$$\lim_{n\to\infty} |B_n|^{1/n}$$

If G is amenable, we would expect this limit to be equal to $2|B|-1$, by Grigorchuk's Criteria.

The set G is not exactly the topological full group. The definition of the full group is a little more complicated.

Note that, for each word $w \in B^n$ and point $x \in X$, $\sigma_w(x) = T^{n(x)}x$. That is, every function σ_w acts as some power of the shift on x.

Definition 9.2 (Cocycle)**.** Let $w \in B^n$ be a word and $x \in X$. The value $n(x)$ such that $\sigma_w(x) = T^{n(x)}$ is the *cocycle*. Note that the cocycle depends on the point x.

Definition 9.3. The topological full group G_X is defined as $G_X = \{S : X \to X \mid S(x) = T^{n(x)}(x)\ \forall x \in X$ and all continuous functions $n : X \to \mathbb{Z}\}$.

Definition 9.4. Let G be a group. Then its *commutator subgroup* is $G' =< [g_1, g_2] \mid g_1, g_2 \in G >$, where $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. It can be checked that $G' \subseteq G$ is a subgroup.

Loosely speaking, this subgroup tells us how far a group is from being abelian. This is characterized in the following proposition.

Proposition 9.5. *Let G be a group and G' its commutator subgroup. Then G/G' is abelian.*

Proof. Let $h_1, h_2 \in G/G'$. Then $h_1 = g_1G'$ for some $g_1 \in G$ and $h_2 = g_2G'$ for some $g_2 \in G$. So $h_1h_2 = g_1g_2G'$ and $h_2h_1 = g_2g_1G'$. Note $g_1g_2g_1^{-1}g_2^{-1}G' = G'$ since $g_1g_2g_1^{-1}g_2^{-1} \in G'$. So $g_1g_2G' = g_2g_1G' \Rightarrow h_1h_2 = h_2h_1$. And so G/G' is abelian. □

Now, let G_X be the full group of a subshift X and G be defined as above. Then $G'_X = G' \subseteq G \subseteq G_X$. Furthermore, it is a basic fact (Bogopolskij [4]) that the commutator subgroup is a normal subgroup. Additionally, it is known (Paterson [31]) that if H is a normal subgroup of G and G is amenable then H is amenable. So, G'_X is a normal subgroup of G and so if we can show that G is amenable, then we can show that G'_X is amenable. As such, we seek the amenability of the group G in order to learn about the amenability of the topological full group.

Fibonacci Substitution. We studied the full group on one particular dynamical system called the *Fibonacci Substitution*. It is defined by a simple substitution rule.

Definition 9.6 (Fibonacci Substitution)**.** Let $A = \{0, 1\}$ be the alphabet. Define α as $0 \mapsto 01$ and $1 \mapsto 0$. Then $\lim_{k\to\infty} \alpha^k(0)$ defines a unique point ω, known as the *Fibonacci Word*. Let X be the system defined by this point, i.e. $X = (\overline{O(\omega)}, T)$.

This is called a substitution system because the rule that defines it is a substitution rule. In particular, whenever we see 0 we substitute it with 01, and whenever we see 1 we substitute it with 0.

We will compute the first few values of the Fibonacci word:

$$0 \to 01 \to 010 \to 01001 \to 01001010 \to 0100101001001 \to 010010100100101001010\ldots$$

Definition 9.7. The point ω which is the limiting point of this process is known as the *fixed point* in this system because it is fixed under the substitution rule.

This substitution rule gives rise to a number of interesting properties. We note that any finite block which could occur in the orbit closure of the fixed point must occur in the fixed point itself. We analyze the number of words of length n.

Let $w(n)$ be the allowable words in the Fibonacci substitution of length n. We will list out $w(n)$ for some small numbers.

$$w(1) = 0, 1$$
$$w(2) = 01, 00, 10$$
$$w(3) = 010, 001, 101, 100$$
$$w(4) = 0100, 0101, 0010, 1010, 1001$$
$$w(5) = 01001, 01010, 00101, 00100, 10100, 10010$$
$$w(6) = 010010, 010100, 001010, 001001, 101001, 100101, 100100$$
$$w(7) = 0100101, 0100100, 0101001, 0010100, 0010010, 1010010, 1001010, 1001001$$

There are a few things to note here, Firstly, $|w(n)| = n + 1$. Additionally, because of this relationship, each word in $w(n)$ either makes one word in $w(n+1)$, and one of these words makes two words in $w(n+1)$. Understanding this relationship helps us in analyzing and computing the cogrowth of the topological full group.

The numerical analysis was performed in Mathematica, and is contained in Appendix A. Results are shown in Table 1.

In the first column of this table we have the computation time in seconds that it required us to find the number of words that reduced to the identity. We note that this time grows exponentially with n. As such, in order to compute the number of words that reduce to the identity for larger n, we require greater computational limits, or we must organically improve the algorithm.

The second column of the table indicates the word length. In particular, the algorithm looks for words which reduce to the identity one word at a time, increasing the word length after it has exhausted all the words of a specific length. We note that we only use even values for the word length, because it is not possible that an odd length word reduces to the identity.

The third and fourth column show how many words in the Fibonacci Substitution reduce to the identity. We call these *trivial reduced elements* because they are trivial when reduced.

Computation Time (sec)	Word Length	– Trivial Reduced Elements	– Trivial Reduced Elements w Multiplicty	Co-Growth
0.079	2	0	0	0.
0.142	4	38	608	24.6577
0.233	6	756	48 384	36.5911
1.762	8	18 122	4 639 232	46.532
77.13	10	484 716	496 349 184	54.9508
4936	12	13 806 822	56 552 742 912	62.046

TABLE 1. Cogrowth Calculations for the Full Group of the Fibonacci Substitution Performed in Mathematica

Finally, in the last column of the table, we have the cogrowth of this group. We expect that this limit should converge to $8^2 = 64$ which would mean that this group is amenable.

We expect that this criterion can be used to study the topological full group of multidimensional systems. In particular, a still open problem in mathematics is about the existence of a dimple finitely presented infinite amenable group. An example of such a group has eluded mathematicians, however we suspect that such a group may arise as the commutator subgroup of the topological full group of a multidimensional subshift. This would be a unique strategy in trying to solve problems in the field of algebra and group theory with the field of dynamical systems. This would work to further establish and develop the intricate connection between algebra and dynamical systems.

Appendix A

```
(* Execute this cell  *)

A0 = "0"; (* the first letter of the fixed point *)
(* Defines the substitution rule *)
substRule[letter_, iterations_] :=
 Last[SubstitutionSystem[{"0" -> "01", "1" -> "0"}, letter,
   iterations]]

(* Returns the prefix of length n for the fixed point starting at A0 *)


nPrefix[n_] :=
 Module[{word}, word = A0;
  While[StringLength[word] < n, word = substRule[word, 1]];
  StringTake[word, {1, n}]]

(*Returns the list of subwords of lengths n of the given word *)
nSubwords[word_, n_] :=
 Sort[DeleteDuplicates[
   StringJoin /@ Partition[Characters[word], n, 1]]]

(*
Returns the list of words of length n for a given substitution. \
Caveat: the algorithm simply takes the prefix of length 2*n and finds \
all subwords of length n in it, which is techniqualy incorrect.
TO-DO: Given n>0, compute a constant c=c(n), based on the \
substitution matrix M, such that the prefix of the fixed point of \
length c contains all words of length n. The primitivity of the system
(in fact, the minimality) guarantees that such a c exists. We've got \
to find an algorithm that would compute c.
 *)
nBlocks[n_] :=
 Module[{prefix, wordsOld, wordsNew}, prefix = nPrefix[2*n];
  wordsOld = nSubwords[prefix, n];
  wordsNew = nSubwords[substRule[prefix, 1]];
  While[! (wordsOld === wordsNew), wordsOld = wordsNew;
   prefix = substRule[prefix, 1]; wordsNew = nSubwords[prefix, n]];
  wordsNew]

In this section, we find the minimal length for n-words such
that for any w in L_n(X), the cylinder sets [w],T[w],...,T^4[w]
```

are disjoint. We then use the words in L_n(X) as a new alphabet
and we rewrite the original system in this new alphabet.
In other words, we switch to the n-fold subshift.
Set B = L_n(X). We then consider elements
sigma_b(x) = T^b(x)(x), where b(x) = 1 if x0=b,
b(x)=-1 if x(-1)=b and b(x)=0 elsewhere.

Thus, the topological full group is amenable iff the group
<sigma_b,b\in B> is amenable.

It is possible that it would suffice to use n=2 in the definition of B.
The homeomorphisms sigma_b are already well-defined. We would need to
verify that with this new definition the theorem above still holds.

The main rationale behind using the group <sigma_b,b\in B> is that this
group is algorithmically defined.
Our objective is to apply the co-growth criterion to <sigma_b,b\in B>.

```
(* Execute this cell *)

(* Finds all words of length n for the given substitution and creates \
a rule that assigns a unique letter for each n-block *)
rewritingSystem[n_] :=
 Table[nBlocks[n][[i]] -> Alphabet[][[i]], {i, Length[nBlocks[n]]}]
(* Returns the fixed point in the new alphabet of n-blocks. *)
rewriteFixedPoint[length_, nLetters_] :=
 StringJoin[
  SubstitutionSystem[rewritingSystem[nLetters],
   StringJoin /@ Partition[Characters[nPrefix[length]], nLetters, 1]]]
(* Find the minimal length for n-Block words such that in the \
conugate/rewritten system no block of length five has repeated \
letters. *)
nMin = 1;  While[
 Min[Length /@
    DeleteDuplicates /@
     Partition[Characters[rewriteFixedPoint[80 + nMin, nMin]], 5,
      1]] < 5, nMin++];
rewriteWord[word_, nLetters_] :=
 StringJoin[
  SubstitutionSystem[rewritingSystem[nLetters],
   StringJoin /@ Partition[Characters[word], nLetters, 1]]]
```

```
(* Returns blocks of length n for the rewritten/conjugate system *)
nNewBlocks[n_] :=
 Module[{words}, words = nBlocks[n + nMin - 1];
  Table[rewriteWord[words[[i]], nMin], {i, Length[words]}]]

Here we count the number of free group elements of length
n that vanish on every word from the language of the system.
In other words, the algorithm counts the number of elements
of length n in the free group representation that are trivial.

(* Execute this cell *)

(* Checks whether freeGen is in the reduced form. Returns True if \
freeGen is in the reduced form and False if otherwise.
  freeGen must be an array of integers.
*)
reducedQ = Compile[{{freeGen, _Integer, 1}},
   Module[{ret = True},
    Do[If[freeGen[[i]] == freeGen[[i + 1]], ret = False; Break[]], {i,
       Length[freeGen] - 1}];
    ret  (* return value *)
    ],
   CompilationTarget -> "C"
   ];

(* Computes the value of the cocycle of the element freeGen at the \
cylinder set [word] determined by "word".
*)
cocycleFun =
  Compile[{{freeGen, _Integer, 1}, {word, _Integer,
     1}, {freeGroupLength, _Integer}},
   Module[{pos},
    pos = freeGroupLength + 1;
    Fold[#1 +
        Which[word[[#1 + 1]] == #2, 1, word[[#1]] == #2, -1, True,
         0] &, pos, freeGen] - pos (* return value *)
    ],
   CompilationTarget -> "C"
   ];

   (* Execute this cell *)

(* The function "trivialElementCounter" returns 1 if the element \
```

```
grElem, given by an array of integers, vanishes on every word from \
the list "words" or returns 0 if otherwise.
words is an 2-dim array of integers. Rows represent b-ary encodings \
of words from Ln(X).
  *)
trivialElementCounter =
 Compile[{{grElem, _Integer, 1}, {words, _Integer,
    2}, {sizeAlph, _Integer}, {freeGroupLength, _Integer}, {nWords, \
_Integer}},
  Module[{val = 1, reducedQ = True, pos},
   (* Checking whether the element is in the reduced form *)
   Do[
    If[grElem[[i]] == grElem[[i + 1]],
     reducedQ = False;
     val = 0;
     Break[]
     ], (* change reducedQ to False, set val=0, and Break away*)
    {i, Length[grElem] - 1}];
   If[reducedQ,
        Do[
          (*
     checking whether the element grElem vanishes on each word from \
words. *)
        If[cocycleFun[grElem , words[[i]], freeGroupLength] != 0,
            val = 0; Break[]],
        {i, nWords}]];
      val (* return value *)
   ],
  CompilationTarget -> "C"
  ]

  (* Execute this cell *)

(* Optimized Compiled Function counting the number of elements \
vanishing on all cylinder sets *)
coGrowthCounter[freeGroupLength_] :=
  Module[{grElem  , val, nHits, sizeAlph, words, totalIter, nWords,
    maxInteger, nRuns, nSteps},
        sizeAlph = Length[nNewBlocks[1]];
   (* convert words in strings of integers *)
        words = (# - 1) & /@
     LetterNumber /@ Characters /@ nNewBlocks[2*freeGroupLength + 2];
        nWords = Length[words];
```

```
    totalIter = sizeAlph^freeGroupLength;
    nHits =
ParallelSum[
 trivialElementCounter[
  IntegerDigits[k, sizeAlph, freeGroupLength], words, sizeAlph,
  freeGroupLength, nWords],
  {k, 0, totalIter - 1}];

    {freeGroupLength, nHits} (* return *)];
```

References

[1] Joseph Auslander. *Minimal flows and their extensions*, volume 153. Elsevier, 1988.

[2] Stefan Banach and Alfred Tarski. Sur la décomposition des ensembles de points en parties respectivement congruentes. *Fund. math*, 6(1924):244–277, 1924.

[3] Sergey Bezuglyi and Konstantin Medynets. Full groups, flip conjugacy, and orbit equivalence of cantor minimal systems. *arXiv preprint math/0611173*, 2006.

[4] Oleg Vladimirovič Bogopolskij. *Introduction to group theory*, volume 6. European Mathematical Society, 2008.

[5] Mike Boyle, Douglas Lind, and Daniel Rudolph. The automorphism group of a shift of finite type. *Transactions of the American Mathematical Society*, 306(1):71–114, 1988.

[6] L.E.J Brouwer. On the structure of perfect sets of points. *KNAW, Proceedings*, 12:785–794, 1910.

[7] Wojciech Bulatek and Jan Kwiatkowski. The topological centralizers of toeplitz flows and their $\mathbb{Z}_2$-extensions. *Publicacions matematiques*, pages 45–65, 1990.

[8] Wojciech Bułatek and Jan Kwiatkowski. Strictly ergodic toeplitz flows with positive entropies and trivial centralizers. *Studia Mathematica*, 103(2):133–142, 1992.

[9] Joel M Cohen. Cogrowth and amenability of discrete groups. *Journal of Functional Analysis*, 48(3):301–309, 1982.

[10] Marıa Isabel Cortez. $\mathbb{Z}^d$ toeplitz arrays. *Discrete and Continuous Dynamical Systems*, 15(3):859, 2006.

[11] María Isabel Cortez and Samuel Petite. G-odometers and their almost one-to-one extensions. *Journal of the London Mathematical Society*, 78(1):1–20, 2008.

[12] Van Cyr and Bryna Kra. The automorphism group of a minimal shift of stretched exponential growth. *arXiv preprint arXiv:1509.08493*, 2015.

[13] Van Cyr and Bryna Kra. The automorphism group of a shift of linear growth: beyond transitivity. In *Forum Math. Sigma*, volume 3, page e5, 2015.

[14] Van Cyr and Bryna Kra. The automorphism group of a shift of subquadratic growth. *Proceedings of the American Mathematical Society*, 144(2):613–621, 2016.

[15] Sebastián Donoso, Fabien Durand, Alejandro Maass, and Samuel Petite. On automorphism groups of low complexity subshifts. *Ergodic Theory and Dynamical Systems*, 36(01):64–95, 2016.

[16] Tomasz Downarowicz et al. Survey of odometers and toeplitz flows. *Contemporary Mathematics*, 385:7–38, 2005.

[17] Erling Følner. On groups with full banach mean value. *Mathematica Scandinavica*, pages 243–254, 1956.

[18] Rostislav Ivanovich Grigorchuk. Symmetric random walks on discrete groups. *Uspekhi Matematicheskikh Nauk*, 32(6):217–218, 1977.

[19] Gustav Arnold Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Theory of computing systems*, 3(4):320–375, 1969.

[20] Edwin Hewitt and Kenneth A Ross. *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*, volume 115. Springer Science & Business Media, 2012.

[21] Michael Hochman. On the automorphism groups of multidimensional shifts of finite type. *Ergodic Theory and Dynamical Systems*, 30(03):809–840, 2010.

[22] Konrad Jacobs and Michael Keane. 0-1-sequences of toeplitz type. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 13(2):123–131, 1969.

[23] Mark Kempton. A non-backtracking polya's theorem. *arXiv preprint arXiv:1610.04672*, 2016.

[24] Bruce P Kitchens. *Symbolic dynamics: one-sided, two-sided and countable state Markov shifts*. Springer Science & Business Media, 2012.

[25] Fabrice Krieger. Toeplitz subshifts and odometers for residually finite groups. *École de Théorie Ergodique*, 20:147–161, 2010.

[26] Casimir Kuratowski. Une méthode d'élimination des nombres transfinis des raisonnements mathématiques. *Fundamenta Mathematicae*, 3(1):76–108, 1922.

[27] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge university press, 1995.

[28] Nelson G Markley. Substitution-like minimal sets. *Israel Journal of Mathematics*, 22(3-4):332–353, 1975.

[29] Nelson G Markley and Michael E Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel Journal of Mathematics*, 34(3):259–272, 1979.

[30] Jeanette Olli. Endomorphisms of sturmian systems and the discrete chair substitution tiling system. *Discrete and Continuous Dynamical Systems*, 33(9):4173–4186, 2013.

[31] Alan LT Paterson. *Amenability*. Number 29. American Mathematical Soc., 2000.

[32] Clifford Pickover. *Archimedes to Hawking: laws of science and the great minds behind them*. Oxford University Press, 2008.

[33] Henri Poincaré. Sur les équations de la dynamique et le probleme des trois corps. *Acta Math*, 13(1):270, 1890.

[34] William A Veech. Point-distal flows. *American Journal of Mathematics*, 92(1):205–242, 1970.

[35] Max Zorn. A remark on method in transfinite algebra. *Bulletin of the American Mathematical Society*, 41(10):667–670, 1935.