# REPORT DOCUMENTATION PAGE

*Form Approved*
**OMB No. 0704-0188**



| **1. REPORT DATE** *(DD-MM-YYYY)* | **2. REPORT TYPE** | **3. DATES COVERED** *(From - To)* |
|---|---|---|
| 06-01-2017 | Technical Report | |

**4. TITLE AND SUBTITLE**
Shot Group Statistics for Small Arms Applications

**5a. CONTRACT NUMBER**
N/A

**5b. GRANT NUMBER**
N/A

**5c. PROGRAM ELEMENT NUMBER**
N/A

**6. AUTHOR(S)**
W. J. Gammon, Ph.D.

**5d. PROJECT NUMBER**
N/A

**5e. TASK NUMBER**
N/A

**5f. WORK UNIT NUMBER**
N/A

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**
Product Manager Infantry Weapons
Program Manager Infantry Weapon Systems
Marine Corps Systems Command
Quantico, VA 22134

**8. PERFORMING ORGANIZATION REPORT NUMBER**

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**
Product Manager Infantry Weapons
Program Manager Infantry Weapon Systems
Marine Corps Systems Command
Quantico, VA 22134

**10. SPONSOR/MONITOR'S ACRONYM(S)**
MARCORSYSCOM

**11. SPONSORING/MONITORING AGENCY REPORT NUMBER**

**12. DISTRIBUTION AVAILABILITY STATEMENT**
Approved for public release; distribution is unlimited.

**13. SUPPLEMENTARY NOTES**

**14. ABSTRACT**
This report provides a systematic analysis of dispersion measures of shot groups from firearms. A dispersion measure is a random variable, and if its probability distribution is known with sufficient accuracy, then it can be used to make a sound statistical inference on the unknown, population standard deviations of the x and y impact-point positions. The dispersion measures treated in this report are the univariate measures: range, mean deviation, and standard deviation; and the bivariate measures: extreme spread, mean radius, and radial standard deviation. Analysis is presented as applied to one, n-round shot group and then is extended to treat multiple, n-round shot groups. A dispersion measure for multiple, n-round shot groups can be constructed by selecting one of the dispersion measures listed above, measuring the dispersion of each group, and averaging the dispersion over groups. This procedure was followed except for the construction of dispersion measures based on the standard deviation and radial standard deviation of multiple n-round shot groups, for which we used the root mean square of the dispersion measure over groups. For each dispersion measure, tables of means, standard deviations, and percentiles were computed for single, n-round shot groups (n = 2-30), as well as for treatments of various k multiples (k = 2-5, 10) of n-round shot groups (n = 2-5, 10, 25). The percentile tables provided in this report are to a considerable extent unavailable elsewhere, and their inclusion is of practical interest to those developing requirements or tests on firearms precision.

**15. SUBJECT TERMS**
shot group statistics, dispersion, precision, percentile tables, range, extreme spread, mean radius, mean deviation, standard deviation, radial standard deviation, ballistics, firearms, small arms, marksmanship

| **16. SECURITY CLASSIFICATION OF:** | | | **17. LIMITATION OF ABSTRACT** | **18. NUMBER OF PAGES** | **19a. NAME OF RESPONSIBLE PERSON** |
|---|---|---|---|---|---|
| **a. REPORT** | **b. ABSTRACT** | **c. THIS PAGE** | | | W. J. Gammon, Ph.D. |
| Unclassified | Unclassified | Unclassified | SAR | 74 | **19b. TELEPHONE NUMBER (*Include area code*)** 703-784-3759 |

# Shot Group Statistics for Small Arms Applications

6/1/2017
Marine Corps Systems Command
W. J. Gammon, Ph.D.

# SHOT GROUP STATISTICS FOR SMALL ARMS APPLICATIONS


## Abstract

This report provides a systematic analysis of dispersion measures of shot groups from firearms. A dispersion measure is a random variable, and if its probability distribution is known with sufficient accuracy, then it can be used to make a sound statistical inference on the unknown, population standard deviations of the *x* and *y* impact-point positions. The dispersion measures treated in this report are the univariate measures: range, mean deviation, and standard deviation; and the bivariate measures: extreme spread, mean radius, and radial standard deviation. Analysis is presented as applied to one, *n*-round shot group and then is extended to treat multiple, *n*-round shot groups. A dispersion measure for multiple, *n*-round shot groups can be constructed by selecting one of the dispersion measures listed above, measuring the dispersion of each group, and averaging the dispersion over groups. This procedure was followed except for the construction of dispersion measures based on the standard deviation and radial standard deviation of multiple *n*-round shot groups, for which we used the root mean square of the dispersion measure over groups. For each dispersion measure, tables of means, standard deviations, and percentiles were computed for single, *n*-round shot groups ($n$ = 2-30), as well as for treatments of various *k* multiples ($k$ = 2-5, 10) of *n*-round shot groups ($n$ = 2-5, 10, 25). The percentile tables provided in this report are to a considerable extent unavailable elsewhere, and their inclusion is of practical interest to those developing requirements or tests on firearms precision.

## Contents

## List of Figures



## List of Tables

## 1. Introduction and Summary

This report provides a systematic analysis of dispersion measures of shot groups from firearms and was written to assist those developing requirements or tests on firearms precision. A shot group is a two-dimensional pattern generated by the impact points of the shots, which pierce a plane at some distance away from the firing line. For our purposes, this plane will be considered to be parallel to the firing line. As such, an *n*-round shot group consists of the impact points of the *n* shots, which have positions $x_i$ and $y_i$ along the horizontal and vertical directions, respectively. As an example, a 5-round shot group at 300 yards from an M40A5 sniper rifle is displayed in Figure 1, and its *x* and *y* impact points are listed in Table 1. The scatter observed in the shot group is due to the random nature of ballistic dispersion.

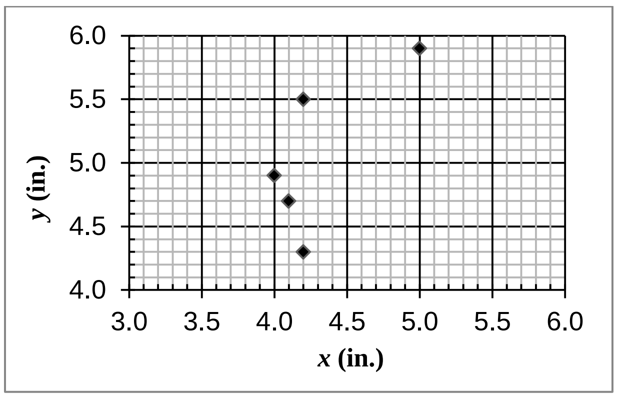


**Figure 1.** A 5-round shot group at 300 yards from an M40A5 sniper rifle fired from a test fixture.

**Table 1.** The impact points of the 5-round shot group displayed in Figure 1.

| Shot Number | *x*(in.) | *y*(in.) |
|---|---|---|
| 1 | 4.0 | 4.9 |
| 2 | 4.2 | 4.3 |
| 3 | 4.1 | 4.7 |
| 4 | 4.2 | 5.5 |
| 5 | 5.0 | 5.9 |

In the study of firearms precision, knowledge of the underlying, population standard deviations of the *x* and *y* impact-point positions is necessary, and dispersion measures are valuable tools to gain this knowledge. A dispersion measure is a random variable, and if its probability distribution is known with sufficient accuracy, then it can be used to make a sound statistical inference on the unknown, population standard deviations of the *x* and *y* impact-point positions. Throughout this report, it is assumed that the random variables *x* and *y* are normally distributed, and that the standard deviations of *x* and *y* are the same ($\sigma = \sigma_x = \sigma_y$). The dispersion measures treated in this report are the univariate measures (Section 3.1):

Range (*R*), Mean Deviation (*MD*), and Standard Deviation (*SD*); and the bivariate measures (Section 3.2): Extreme Spread (*ES*), Mean Radius (*MR*), and Radial Standard Deviation (*RSD*). Analysis of each dispersion measure is first presented as applied to one, *n*-round shot group and then is extended to treat multiple, *n*-round shot groups. Our analytical approach is described below.

First, an analysis of each dispersion measure is given as applied to one, *n*-round shot group ($n$ = 2-30). For each dispersion measure, percentiles and the first two moments were computed in units of the population standard deviation, $\sigma$. Percentiles of these dispersion measures are of principal interest for small sample sizes, especially the upper and lower $10^{th}$ percentiles. For large sample sizes, these probability distributions become nearly normally distributed. The probability distributions of *SD* and *RSD* are well known: $\sqrt{n}SD/\sigma$ and $\sqrt{n}RSD/\sigma$ are chi random variables with $(n - 1)$ and $2(n - 1)$ Degrees of Freedom (DoF), respectively. (For DoF $\geq$ 30, chi distributions are nearly normally distributed.) The percentiles of *SD* and *RSD* are obtained by taking the square root of the percentiles of the corresponding chi-square distribution, and are thus easily computed—we used the built-in functions in Microsoft Excel 2010. Similarly for $n = 2$, the probability distributions of *R*, *MD*, *ES*, and *MR* are scaled chi random variables with one DoF (univariate) or two DoF (bivariate) and were computed in a similar fashion. Conversely for $n > 2$, the probability distributions of *R*, *MD*, *ES*, and *MR* lack convenient functional forms, and in this case, Monte Carlo simulations were used to estimate their percentiles. In addition, we provide analytical formulas for the means and variances of these dispersion measures treating one shot group, with exception to *R* and *ES* with $n > 2$ for which the moments are Monte Carlo estimates. Details on the Monte Carlo simulations used in this work are found in Section 2.

Our analysis was then extended to treat various *k* multiples ($k$ = 2-5, 10) of *n*-round shot groups ($n$ = 2-30) where the multiples are independent, repeat firings of *n*-round groups from the same population. Repeated firings of groups allow for a more precise estimate of σ. A dispersion measure for *k*, *n*-round shot groups can be constructed by selecting a dispersion measure, measuring the dispersion of each of the *k* groups, and averaging the dispersion over groups. This procedure was used to construct a dispersion measure for multiple, *n*-round shot groups based on *R*, *MD*, *ES*, and *MR*, which when averaged over the sampling of groups, yields the corresponding dispersion measures: Average *R* (*AR*), Average *MD* (*AMD*), Average *ES* (*AES*), and Average *MR* (*AMR*), respectively. Similarly, such a dispersion measure based on *SD* or *RSD* for multiple groups may be constructed; however, an alternative procedure was selected for convenience. Instead of averaging the *SD* (*RSD*) over groups, our measure of dispersion for multiple groups is the Root Mean Square (*RMS*) of the *SD*'s (*RSD*'s) over groups, which is denoted as *RMS-SD* (*RMS-RSD*). The construction of *RMS-SD* and *RMS-RSD* was chosen, because their probability distributions are well known: $\sqrt{kn}RMS\text{-}SD/\sigma$ and $\sqrt{kn}\ RMS\text{-}RSD/\sigma$ are chi random variables with $k(n\text{-}1)$ and $2k(n\text{-}1)$ DoF, respectively.

Whereas, in general, the probability distributions of *AR*, *AMD*, *AES*, and *AMR* lack convenient functional forms; therefore, Monte Carlo simulations were used to estimate their percentiles. From the central limit theorem, the probability distribution of the average over a dispersion measure is asymptotically normal as the number of groups averaged over increases. Our Monte Carlo estimates of percentiles can be used to determine the error associated with approximating the probability distribution with a normal one. The largest errors committed in using this approximation occur in the tails of the distribution. Convergence to a normal distribution is quick, justifying the normal-distribution approximation for many cases; therefore, practical interest in the percentile tables of *AR*, *AMD*, *AES*, and *AMR* will typically be restricted to cases involving a few groups with small sample size. For $n \geq 10$ and $k > 1$, the error associated with this approximation is no more than 10% for any of the percentiles computed in this work. Finally, supplementing our results on percentiles, the first two moments were computed for the dispersion measures treating multiple shot groups. Wrapping up the description of our analytical approach, we present Table 2, which provides a listing of the dispersion measures examined in this report.

**Table 2.** Dispersion measures examined in this report.

| Dispersion Measures | | | |
|---|---|---|---|
| Univariate | | Bivariate | |
| One Shot Group | Multiple Shot Groups | One Shot Group | Multiple Shot Groups |
| *R* | *AR* | *ES* | *AES* |
| *MD* | *AMD* | *MR* | *AMR* |
| *SD* | *RMS-SD* | *RSD* | *RMS-RSD* |

Next, a review of the literature is provided as pertaining to the dispersion measures covered in this report. In Grubbs' monograph,[1] he provided the means, standard deviations, and 95th percentiles of *R*, *MD*, *SD*, *ES*, *MR*, and *RSD* for single, *n*-round shot groups ($n$ = 2-20). Taylor and Grubbs[2] also provided percentile tables of *ES* along with the first four moments ($n$ = 3-10, 15, 20, 25, 28, 30, 31, 34). In addition, Pearson and Hartley[3] published tables of the upper and lower 10th percentiles of *R* ($n$ = 2-20) and *MD* ($n$ = 2-10) as well as the means and standard deviations of *R* and *MD* for various sample sizes ($n$ = 2-20, 30, 60). The results presented in this work extend upon those reported in previous studies. To the author's knowledge, published percentile tables of *MR* are unavailable, and a comprehensive treatment of multiple shot groups is unavailable. The goal of this report is to fill this gap and provide a thorough treatment of dispersion measures of shot groups in a single source.

In summary, for each dispersion measure treated herein, tables of means, standard deviations, and percentiles are provided for single, *n*-round shot groups ($n$ = 2-30), as well as for treatments of various $k$ multiples ($k$ = 2-5, 10) of *n*-round shot groups ($n$ = 2-5, 10, 25), where the multiples are independent, repeat firings from the same population. When possible, a sketch of the derivation of analytical formulas for the first two moments is provided. This approach is perhaps a theoretical distraction to many, but to the interested reader, it provides the method to compute moments for ($n$, $k$) combinations not covered in this report. To facilitate use of these results, numerous examples are provided, which will prove indispensable to those developing requirements or tests on firearms precision.

## 2. Monte Carlo Simulations

Monte Carlo simulations were used to compute percentiles of the probability distributions of *R*, *MD*, *ES*, and *MR* for single, *n*-round shot groups, as well as the percentiles of Average *R*, Average *MD*, Average *ES*, and Average *MR* for $k$ multiples of *n*-round shot groups. Monte Carlo simulations were also used to estimate the means and variances of *R*, *ES*, *AR*, and *AES*, and their Monte Carlo estimates are reported in units of the population standard deviation, $\sigma$. Simulated moments and percentiles are based on $N$ Monte Carlo samples (here, $N$ = 20,000) of each $n$ impact-point position in each of the $k$ sampled, *n*-round shot groups. Thus, $knN$ and $2knN$ random samples were generated for one Monte Carlo run to simulate moments and percentiles of the univariate and bivariate distributions, respectively. All Monte Carlo simulations were repeated for a total of 9 runs, and Monte Carlo estimates of the simulated moments and percentiles are reported as the sample mean ($\bar{Z}$) over the replications ($r$ = 9). In addition, we report the Monte Carlo Error (*MCE*) associated with each Monte Carlo estimate, which provides a measure of the variability between simulations.  The *MCE* is defined as the sample standard deviation of the Monte Carlo estimates,[4] and the 1-2α confidence interval of the population mean can be estimated by:

$$\left(\bar{Z} - \frac{t_{1-\alpha}MCE}{\sqrt{r}},\ \bar{Z} + \frac{t_{1-\alpha}MCE}{\sqrt{r}}\right),$$

where $t_{1-\alpha}$ is the 100(1– α) percentile of the Student's t-distribution with (*r* – 1) DoF.

Random samples of the dispersion measures were constructed by sampling the normal distributions of the *x* and *y* impact-point positions.  Microsoft Excel 2010 was used for all Monte Carlo simulations, and random samples from a normal distribution were generated by the inverse transform technique using the NORM.INV(RAND(),0,1) function, which returns a random sample from a normal distribution with zero mean and standard deviation of one.  These random samples were used to construct *N* samples of the dispersion measure, whose percentiles were computed using the PERCENTILE function.

## 3.  Results and Discussion

### 3.1.  Univariate Measures of Dispersion

#### 3.1.1.  Range, One Shot Group

The univariate measure of dispersion known as the Range (*R*) is given by:

$$R = (x_n - x_1),\ \text{here } x_1 \le x_2 \le x_3 \cdots \le x_n. \tag{1}$$

*R* is measured by projecting the *n* shot positions on the *x*-axis and then measuring the distance between the two furthest shots along that projection.  If the *x*-components are horizontal distances, then *R* is referred to as the Extreme Horizontal Dispersion (*EHD*), and if these are vertical distances, then it is referred to as the Extreme Vertical Dispersion (*EVD*).[1]  For convenience, recast (1) as:

$$R = max_{i,j}\left\{\sqrt{\left(x_i - x_j\right)^2}\right\}; \tag{2}$$

that is, *R* is the maximum of the distances between all possible pairs of $x_i$ and $x_j$.  Dividing both sides by σ:

$$R/\sigma = max_{i,j}\left\{\sqrt{\left((x_i - x_j)/\sigma\right)^2}\right\}, \tag{3}$$

where σ is the population standard deviation of the *x*-components of impact points.  Next, consider the special case of (3) with *n* = 2:

$$R/\sigma = |(x_2 - x_1)/\sigma|$$

$$R/\sigma = \sqrt{2}\left|(x_2 - x_1)/\sqrt{2}\sigma\right|.$$

Since the $x_i$ are normally distributed and independent, $(x_2 - x_1)/\sqrt{2}\sigma$ is normally distributed with zero mean and unity variance; therefore,

$$R/\sigma = \sqrt{2}\chi(1) \quad (n = 2), \tag{4}$$

where $\chi(1)$ is a chi-distributed random variable with one degree of freedom. Continuing with $n = 2$, the mean of $R/\sigma$ is given by:

$$\langle R/\sigma \rangle = \sqrt{2}\langle \chi(1) \rangle$$

$$\langle R/\sigma \rangle = 2/\Gamma\left(\tfrac{1}{2}\right),$$

where Γ is the gamma function, and inserting the value of $\Gamma\left(\frac{1}{2}\right)$ gives:

$$\langle R/\sigma \rangle = 2/\sqrt{\pi} \quad (n = 2), \tag{5}$$

and its variance is computed from:

$$Var(R/\sigma) = \langle (R/\sigma)^2 \rangle - \langle R/\sigma \rangle^2$$

$$= 2[\langle \chi^2(1) \rangle - 2/\pi]$$

$$Var(R/\sigma) = 2(1 - 2/\pi) \quad (n = 2). \tag{6}$$

Taking the square root of (6) gives the standard deviation:

$$Std(R/\sigma) = \sqrt{2(1 - 2/\pi)} \quad (n = 2). \tag{7}$$

For $n = 2$, simple analytical results are provided for the probability distribution and first two moments of $R/\sigma$. If $n > 2$, an extension of these analytical results is unavailable, because the exact probability distributions are unknown. As such, computation of the percentiles and moments would require methods such as numerical integration techniques[3] or Monte Carlo. For $n > 2$, Monte Carlo simulations were chosen to compute the percentiles and moments, because these methods are easy to implement and obtain results with sufficient accuracy for practical firearms applications. This approach was taken throughout this report for other dispersion measures with probability distributions not easily expressed in a convenient form.

In Table 3, results are presented for the means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of $R/\sigma$ for sample sizes of $n$ = 2-30. For $n = 2$, the means and standard deviations displayed in the table are from (5) and (7), and the percentiles were determined from those of a chi-distributed random variable (one degree of freedom) scaled by $\sqrt{2}$ as shown in (4). For $n > 2$, all tabulated results are Monte Carlo estimates. A listing of percentiles of $R/\sigma$ is found in Appendix 1-1, along with the *MCE*'s of the estimated percentiles.

In earlier work, Pearson and Hartley[3] published means and standard deviations of $R/\sigma$ ($n$ = 2-20, 30, 60), and the means and standard deviations in Table 3 agree with their results within 0.5%. Pearson and Hartley[3] also published values of the upper and lower 10$^{th}$ percentiles of $R/\sigma$ ($n$ = 2-20) to two decimal places. Comparison of their results with the percentiles in Appendix 1-1 is generally in very good agreement. However, we must remark on the 99.9 percentiles listed in Appendix 1-1. We note that as the sample size increases, these percentiles are expected to monotonically increase and eventually reach an asymptotic value. Up to $n = 14$, these percentiles monotonically increase; however, as the sample size further increases, small fluctuations disrupting this behavior are observed. This is perhaps not surprising, since only 20 samples per Monte Carlo replication are expected above the 99.9 percentile (see Section 2).

More Monte Carlo replications would be required to smooth out this noisy behavior to attain better accuracy. This effort was not pursued—we only make a cautionary note, which indicates caution should be taken when estimating a confidence interval around the Monte Carlo estimates of these percentiles as prescribed in Section 2. Similar behavior may be observed in other results in this report, but any such behavior is less pronounced and will not be considered further.

**Table 3**. Table of means, standard deviations, and 5th and 95th percentiles of $R/\sigma$ for sample sizes of $n$ = 2-30. Numbers reported in parentheses are the *MCE*'s of the value of the mean or standard deviation preceding it.

| n | Mean | | Std | | *CV* | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.1284 | | 0.8525 | | 0.7555 | 0.0887 | 2.7718 | 0.0786 | 2.4565 |
| 3 | 1.695 | (0.004) | 0.890 | (0.004) | 0.525 | 0.431 | 3.317 | 0.254 | 1.957 |
| 4 | 2.060 | (0.006) | 0.879 | (0.006) | 0.426 | 0.761 | 3.625 | 0.369 | 1.759 |
| 5 | 2.325 | (0.007) | 0.863 | (0.005) | 0.371 | 1.032 | 3.852 | 0.444 | 1.657 |
| 6 | 2.534 | (0.003) | 0.848 | (0.004) | 0.335 | 1.253 | 4.032 | 0.495 | 1.591 |
| 7 | 2.704 | (0.006) | 0.833 | (0.005) | 0.308 | 1.443 | 4.172 | 0.534 | 1.543 |
| 8 | 2.848 | (0.006) | 0.818 | (0.003) | 0.287 | 1.602 | 4.286 | 0.563 | 1.505 |
| 9 | 2.971 | (0.007) | 0.808 | (0.005) | 0.272 | 1.739 | 4.384 | 0.585 | 1.476 |
| 10 | 3.079 | (0.005) | 0.796 | (0.003) | 0.259 | 1.866 | 4.475 | 0.606 | 1.453 |
| 11 | 3.175 | (0.003) | 0.788 | (0.004) | 0.248 | 1.975 | 4.554 | 0.622 | 1.434 |
| 12 | 3.260 | (0.005) | 0.780 | (0.006) | 0.239 | 2.069 | 4.624 | 0.635 | 1.419 |
| 13 | 3.335 | (0.005) | 0.772 | (0.003) | 0.231 | 2.164 | 4.686 | 0.649 | 1.405 |
| 14 | 3.407 | (0.005) | 0.763 | (0.004) | 0.224 | 2.241 | 4.742 | 0.658 | 1.392 |
| 15 | 3.473 | (0.006) | 0.756 | (0.003) | 0.218 | 2.323 | 4.795 | 0.669 | 1.381 |
| 16 | 3.531 | (0.005) | 0.750 | (0.006) | 0.212 | 2.386 | 4.845 | 0.676 | 1.372 |
| 17 | 3.589 | (0.004) | 0.747 | (0.005) | 0.208 | 2.450 | 4.898 | 0.683 | 1.365 |
| 18 | 3.642 | (0.005) | 0.739 | (0.004) | 0.203 | 2.517 | 4.939 | 0.691 | 1.356 |
| 19 | 3.688 | (0.006) | 0.733 | (0.003) | 0.199 | 2.571 | 4.971 | 0.697 | 1.348 |
| 20 | 3.734 | (0.005) | 0.729 | (0.003) | 0.195 | 2.622 | 5.015 | 0.702 | 1.343 |
| 21 | 3.776 | (0.004) | 0.727 | (0.003) | 0.193 | 2.669 | 5.048 | 0.707 | 1.337 |
| 22 | 3.822 | (0.003) | 0.722 | (0.003) | 0.189 | 2.724 | 5.088 | 0.713 | 1.331 |
| 23 | 3.859 | (0.003) | 0.716 | (0.003) | 0.186 | 2.769 | 5.107 | 0.717 | 1.323 |
| 24 | 3.895 | (0.007) | 0.711 | (0.003) | 0.183 | 2.813 | 5.143 | 0.722 | 1.320 |
| 25 | 3.928 | (0.005) | 0.710 | (0.006) | 0.181 | 2.853 | 5.179 | 0.726 | 1.319 |
| 26 | 3.963 | (0.006) | 0.707 | (0.004) | 0.178 | 2.887 | 5.202 | 0.728 | 1.313 |
| 27 | 3.999 | (0.004) | 0.702 | (0.003) | 0.176 | 2.932 | 5.228 | 0.733 | 1.307 |
| 28 | 4.026 | (0.005) | 0.700 | (0.003) | 0.174 | 2.964 | 5.254 | 0.736 | 1.305 |
| 29 | 4.058 | (0.006) | 0.695 | (0.004) | 0.171 | 3.002 | 5.275 | 0.740 | 1.300 |
| 30 | 4.085 | (0.005) | 0.692 | (0.004) | 0.169 | 3.035 | 5.300 | 0.743 | 1.297 |

**Notes Table 3**: The first column displays the number of shots ($n$) in the group. The second and third columns display the means and standard deviations. The fourth column displays the Coefficient of Variation (*CV*), which is computed by dividing the standard deviation (third column) by the mean (second column). The *CV* (see (9)) is the standard deviation of the ratio of the unbiased estimate of the population standard deviation ($\hat{\sigma}$) to σ (i.e., $\hat{\sigma}/\sigma$). The fifth and sixth columns display the 5th and 95th percentiles, respectively. The seventh and eighth columns show the 5% and 95% Confidence Levels (CL) of the ratio of $\hat{\sigma}/\sigma$ (see (17)), respectively, and values in these columns were computed by dividing the 5th and 95th percentiles by the corresponding mean. Taken together, the values in the seventh and eighth columns provide the 90% confidence interval of $\hat{\sigma}/\sigma$. Values in the table reported to 3 decimal places indicate a result based on Monte Carlo estimates.

**Example 1.** Compute an unbiased estimate of σ using the Extreme Vertical Dispersion of the 5-round shot group in Table 1.

Using the values in Table 1, *EVD* = 1.6” as computed from (2). An unbiased estimate of the population standard deviation ($\hat{\sigma}$) is computed from:

$$\hat{\sigma} = M/\langle M/\sigma \rangle, \quad (8)$$

where *M* is the dispersion measure—in this case *M* = *R*. (Computing the mean of (8) demonstrates $\hat{\sigma}$ is an unbiased estimate.) Substituting the computed *EVD* and mean of *R*/σ for one, 5-round group (Table 3):

$$\hat{\sigma} = 1.6''/2.325 = 0.69''.$$

**Example 2.** Select the Range as the dispersion measure, and determine the Coefficient of Variation for a 5-round group.

The Coefficient of Variation is defined by:

$$CV \equiv Std(\hat{\sigma}/\sigma).$$

Using (8) rewrite the above:

$$CV = Std(M/\sigma)/\langle M/\sigma \rangle, \quad (9)$$

and consulting Table 3, the *CV* for a 5-round group is:

$$CV = 0.371.$$

Note multiplying *CV* by σ gives the standard deviation of the unbiased estimate $\hat{\sigma}$.

**Example 3.** Using the Extreme Vertical Dispersion, find the 90% Confidence Interval (CI) of the population standard deviation of the 5-round group listed in Table 1.

Write the probability distribution of the dispersion measure *M* in units of the population standard deviation:

$$T \equiv M/\sigma. \quad (10)$$

The 90% CI of *T* is:

$$T_{0.05} \leq M/\sigma \leq T_{0.95}, \quad (11)$$

from which the 90% CI of σ is determined:

$$M/T_{0.95} \leq \sigma \leq M/T_{0.05} \quad \text{(90\% CI)}, \quad (12)$$

where *M* = *EVD*. Inserting the *EVD* from Example 1 with the needed percentiles for a 5-round group (Table 3):

$$1.6''/3.852 \leq \sigma \leq 1.6''/1.032$$

$$0.42'' \leq \sigma \leq 1.55'' \quad \text{(90\% CI)}.$$

**Example 4.** Repeat the calculation in Example 3 for a 95% Confidence Interval of the population standard deviation.

Referring to Appendix 1-1, the 97.5 and 2.5 percentiles of *R*/σ for a 5-round shot group are 4.190 and 0.849, respectively. Analogous to (12):

$$M/T_{0.975} \leq \sigma \leq M/T_{0.025} \quad \text{(95\% CI)}.$$

Computation gives:

$$1.6''/4.190 \leq \sigma \leq 1.6''/0.849$$

$$0.38'' \leq \sigma \leq 1.88'' \quad \text{(95\% CI)}.$$

### 3.1.2. Average Range, Multiple Shot Groups

For multiple, *n*-round shot groups, the range is averaged over all groups to provide a measure of dispersion:

$$AR/\sigma = \frac{1}{k}\sum_{i=1}^{k} R_i/\sigma, \tag{13}$$

where *AR* is the Average Range, $R_i$ is the range of the $i^{th}$ *n*-round shot group, and *k* is the total number of groups. The mean of (13) is:

$$\langle AR/\sigma \rangle = \langle R/\sigma \rangle, \tag{14}$$

since the mean of *R* for one, *n*-round shot group is the same as the mean of an average over multiple, *n*-round groups. The standard deviation of $AR/\sigma$ is:

$$Std(AR/\sigma) = \frac{1}{\sqrt{k}} Std(R/\sigma), \tag{15}$$

as expected for an average over *k* groups. The Coefficient of Variation of *AR* is defined by its standard deviation (15) divided by its mean (14):

$$CV(AR) = \frac{1}{\sqrt{k}} CV(R), \tag{16}$$

which indicates that its *CV* is the same as the *CV* of *R* divided by $\sqrt{k}$.

In general the probability distributions of *AR*/σ do not have handy functional forms for easy computations. However, we note that for $k = 2$ and $n = 2$, *AR*/σ is the average of the sum of two scaled chi random variables each with one DoF (see (4)). The joint density function of the sum of two independent random variables is given by the convolution of their probability density functions, and the joint density function can be integrated to determine its percentiles. This is a fairly easy exercise for $k = 2$ and $n = 2$, but is laborious for larger values of *k*. As such, Monte Carlo simulations were selected to compute all percentiles of *AR*/σ. Similarly, this remark applies to the percentile computations of *AMD/*σ, *AES/*σ, and *AMR/*σ which are presented in later sections.

In Table 4, results are presented for the means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of $AR/\sigma$ for *n*-round shot groups ($n$ = 2-5, 10, 25) fired for a total of *k* groups ($k$ = 1-5, 10 for each of the listed *n*). For $n = 2$, the listed mean is $2/\sqrt{\pi}$ via (5), and the standard deviations were computed from (7) divided by $\sqrt{k}$. For $n = 2$ and $k > 1$, the percentiles are Monte Carlo estimates. For $n > 2$, all tabulated results are from Monte Carlo simulations, and estimates of the means are from simulations with $k = 10$. A listing of the percentiles of *AR*/σ for the various *n* and *k* is found in Appendix 1-2, along with the *MCE*'s of the estimated percentiles.

We know from the central limit theorem that as *k* increases, *AR*/σ is asymptotically normal. The largest errors committed in using this approximation are, of course, in the tails of the distribution. If *k* is large enough, *AR*/σ is well approximated by a normal distribution with mean $\langle R/\sigma \rangle$ and standard deviation $Std(R/\sigma)/\sqrt{k}$. A similar argument obviously applies to *AMD*/σ, *AES*/σ, and *AMR*/σ (Sections 3.1.4, 3.2.2 and 3.2.4, respectively). The Monte Carlo estimates of percentiles given in Appendix 1-2 allow the error to be determined when making this approximation to $AR/\sigma$. As an example, consider the expected mean and standard deviation of the Range for one, 5-round group, which from Table 3 are 2.325 and 0.863, respectively. The normal approximation to $AR/\sigma$ for 3, 5-round shot groups would be a normal distribution with a mean of 2.325 and standard deviation of $0.863/\sqrt{3}$. The 5$^{th}$ percentile of this normal distribution is 1.505, which can be compared to 1.545 (the corresponding 5$^{th}$ percentile reported in Table 4). For this example, the error is only 2.6%. As demonstrated, our results can be used to quickly determine the error in assuming this approximation. Error within a few percent is suitable for most firearms applications, and primary concerns with using this approximation are for dispersion measures involving averages over a few groups with a small sample size per group.

**Table 4**. Table of means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of $AR/\sigma$ for $n$-round shot groups ($n$ = 2-5, 10, 25) fired for a total of $k$ groups ($k$ = 1-5, 10). Numbers reported in parentheses are the *MCE*'s of the value of the mean or standard deviation preceding it.

| k | n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1.1284 | 0.8525 | 0.7555 | 0.0887 | 2.7718 | 0.0786 | 2.4565 |
| 2 | | | 0.6028 | 0.5342 | 0.286 | 2.239 | 0.253 | 1.984 |
| 3 | | | 0.4922 | 0.4362 | 0.411 | 2.015 | 0.364 | 1.786 |
| 4 | | | 0.4263 | 0.3778 | 0.494 | 1.892 | 0.437 | 1.676 |
| 5 | | | 0.3813 | 0.3379 | 0.554 | 1.804 | 0.491 | 1.599 |
| 10 | | | 0.2696 | 0.2389 | 0.709 | 1.595 | 0.629 | 1.414 |
| 1 | 3 | 1.693 (0.002) | 0.890 (0.004) | 0.526 | 0.431 | 3.317 | 0.255 | 1.960 |
| 2 | | | 0.628 (0.005) | 0.371 | 0.754 | 2.809 | 0.445 | 1.660 |
| 3 | | | 0.512 (0.003) | 0.302 | 0.907 | 2.586 | 0.536 | 1.527 |
| 4 | | | 0.444 (0.0016) | 0.262 | 1.004 | 2.464 | 0.593 | 1.456 |
| 5 | | | 0.396 (0.003) | 0.234 | 1.073 | 2.374 | 0.634 | 1.402 |
| 10 | | | 0.281 (0.0013) | 0.166 | 1.249 | 2.171 | 0.738 | 1.282 |
| 1 | 4 | 2.058 (0.002) | 0.879 (0.006) | 0.427 | 0.761 | 3.625 | 0.370 | 1.761 |
| 2 | | | 0.622 (0.003) | 0.302 | 1.109 | 3.146 | 0.539 | 1.528 |
| 3 | | | 0.508 (0.003) | 0.247 | 1.269 | 2.939 | 0.617 | 1.428 |
| 4 | | | 0.439 (0.002) | 0.213 | 1.369 | 2.809 | 0.665 | 1.365 |
| 5 | | | 0.393 (0.002) | 0.191 | 1.442 | 2.734 | 0.701 | 1.328 |
| 10 | | | 0.278 (0.0009) | 0.135 | 1.614 | 2.529 | 0.784 | 1.229 |
| 1 | 5 | 2.326 (0.0015) | 0.863 (0.005) | 0.371 | 1.032 | 3.852 | 0.444 | 1.656 |
| 2 | | | 0.612 (0.003) | 0.263 | 1.379 | 3.384 | 0.593 | 1.455 |
| 3 | | | 0.499 (0.003) | 0.214 | 1.545 | 3.184 | 0.664 | 1.369 |
| 4 | | | 0.434 (0.002) | 0.186 | 1.642 | 3.069 | 0.706 | 1.319 |
| 5 | | | 0.385 (0.0009) | 0.166 | 1.715 | 2.982 | 0.737 | 1.282 |
| 10 | | | 0.273 (0.0016) | 0.117 | 1.890 | 2.787 | 0.812 | 1.198 |
| 1 | 10 | 3.077 (0.0013) | 0.796 (0.003) | 0.259 | 1.866 | 4.475 | 0.606 | 1.454 |
| 2 | | | 0.562 (0.004) | 0.183 | 2.199 | 4.042 | 0.714 | 1.314 |
| 3 | | | 0.461 (0.0019) | 0.150 | 2.349 | 3.869 | 0.763 | 1.257 |
| 4 | | | 0.398 (0.003) | 0.129 | 2.444 | 3.754 | 0.794 | 1.220 |
| 5 | | | 0.357 (0.003) | 0.116 | 2.509 | 3.681 | 0.815 | 1.196 |
| 10 | | | 0.252 (0.002) | 0.082 | 2.672 | 3.499 | 0.868 | 1.137 |
| 1 | 25 | 3.931 (0.0015) | 0.710 (0.006) | 0.181 | 2.853 | 5.179 | 0.726 | 1.318 |
| 2 | | | 0.500 (0.002) | 0.127 | 3.150 | 4.793 | 0.801 | 1.219 |
| 3 | | | 0.409 (0.003) | 0.104 | 3.285 | 4.628 | 0.836 | 1.177 |
| 4 | | | 0.354 (0.005) | 0.090 | 3.369 | 4.531 | 0.857 | 1.153 |
| 5 | | | 0.316 (0.0016) | 0.080 | 3.429 | 4.467 | 0.872 | 1.136 |
| 10 | | | 0.225 (0.0012) | 0.057 | 3.571 | 4.309 | 0.908 | 1.096 |

**Notes Table 4**: The first and second columns display the number of groups ($k$) and the number of shots ($n$) in the group, respectively. The third and fourth columns display the means and standard deviations. The fifth column displays the Coefficient of Variation (*CV*), which was computed by dividing the standard deviation (fourth column) by the mean (third column). The *CV* (see (9)) is the standard deviation of the ratio of the unbiased estimate of the population standard deviation ($\hat{\sigma}$) to σ (i.e., $\hat{\sigma}/\sigma$). The sixth and seventh columns provide the 5$^{th}$ and 95$^{th}$ percentiles, respectively. The eighth and ninth columns provide the 5% and 95% Confidence Levels (CL) of the ratio of $\hat{\sigma}/\sigma$ (see (17)), respectively, and these values were computed by dividing the 5$^{th}$ and 95$^{th}$ percentiles by the corresponding mean. Taken together the values in the eighth and ninth columns provide the 90% confidence interval of $\hat{\sigma}/\sigma$. Values in the table reported to 3 decimal places indicate a result based on Monte Carlo estimates.

---

**Example 5.** If the Average Range is used as the dispersion measure, how many 3-round shot groups need to be fired to get at least the same Coefficient of Variation as from firing 5, 10-round shot groups?

Equating *CV*'s gives:

$$CV\big(AR(k,3)\big) = CV(AR(5{,}10)),$$

where $AR(k,n)$ denotes the average range for $k$, $n$-round shot groups. Using (16), this is rewritten as:

$$CV\big(R(3)\big)/\sqrt{k} = CV(R(10))/\sqrt{5},$$

where $R(n)$ denotes the range for one, $n$-round shot group. Substituting the *CV*'s for one, 3-round shot group and one, 10-round shot group (*CV*'s are listed in Table 3), and solving for $k$:

$$k = 5(0.525/0.259)^2 = 20.5;$$

thus, at least 21 shot groups are needed at a cost of expending 26% more rounds.

---

**Example 6.** Select the Average Range as the dispersion measure, and compute the 90% Confidence Interval of the ratio of the unbiased estimate of the population standard deviation ($\hat{\sigma}$) to σ (i.e., $\hat{\sigma}/\sigma$) for 3, 5-round shot groups.

Recall (11):

$$T_{0.05} \leq M/\sigma \leq T_{0.95}.$$

Dividing through by $\langle T \rangle$ and using (8):

$$T_{0.05}/\langle T \rangle \leq \hat{\sigma}/\sigma \leq T_{0.95}/\langle T \rangle. \quad (17)$$

Substituting the values in the last two columns of the Table 4 for 3, 5-round shot groups gives:

$$0.664 \leq \hat{\sigma}/\sigma \leq 1.369 \quad \text{(90\% CI)}.$$

---

**Example 7.** Repeat the computation in Example 6 for 5, 3-round shot groups. (Note the total number of rounds is the same as in the previous example.)

Following the procedure used in Example 6 and using the values in the last two columns of Table 4 for 5, 3-round shot groups:

$$0.634 \leq \hat{\sigma}/\sigma \leq 1.402 \quad \text{(90\% CI)}.$$

Comparing with Example 6, the overall 90% CI is ~9% larger if 5, 3-round groups are fired.

**Example 8.** Select the Average Range as the dispersion measure, and compute the 80% Confidence Interval of the ratio of $\hat{\sigma}/\sigma$ for 3, 5-round shot groups.

Referring to Appendix 1-2, the $10^{th}$ and $90^{th}$ percentiles of $R/\sigma$ for 3, 5-round shot groups are 1.701 and 2.979, respectively. Analogous to (17):

$$T_{0.10}/\langle T\rangle \le \hat{\sigma}/\sigma \le T_{0.90}/\langle T\rangle.$$

Inserting the mean for 3, 5-round shot groups (Table 4) and the needed percentiles into the above inequality:

$$1.701/2.326 \le \hat{\sigma}/\sigma \le 2.979/2.326$$

$$0.731 \le \hat{\sigma}/\sigma \le 1.281 \quad \text{(80\% CI)}.$$

**Example 9.** If a total of *N* rounds is available to fire, what group size (*n*) gives the smallest Coefficient of Variation of the Average Range? (This example was inspired by Ref. 5.)

The number of groups is $k = N/n$, and the *CV* of *AR* for *k*, *n*-round groups is:

$$CV\big(AR(k,n)\big) = CV(R(n))/\sqrt{k}$$

$$CV\big(AR(k,n)\big) = \sqrt{\frac{n}{N}}\,CV(R(n)).$$

(See Example 5 for an explanation of the notation used above.) The value of *n* that minimizes $\sqrt{n}CV(R(n))$ gives the optimal group size, and computation of the optimal group size is illustrated in Table 5. The first two columns display the group size (*n*) and the *CV* (values are from Table 3). The third column shows the product of the value in the second column with the square root of the value in the first column, and the value of *n* for optimal group size corresponds to the minimum value in the third column. Finally, the fourth column shows an estimate of the Standard Error (SE) of the corresponding values in the third column. Figure 2 displays a plot of $\sqrt{n}CV$ versus *n*, and inspection shows a shallow minimum in the interval $6 < n < 11$, which indicates an optimal group size of about 8 for the univariate Range. From a practical standpoint, 10-round groups should serve just as well as 8-round groups to minimize *CV*.

**Table 5**. Table illustrating the computation of the optimal group size for the univariate Range.

| $n$ | $CV$ | $\sqrt{n}CV$ | SE |
|---|---|---|---|
| 3 | 0.525 | 0.909 | 0.002 |
| 4 | 0.426 | 0.853 | 0.002 |
| 5 | 0.371 | 0.830 | 0.002 |
| 6 | 0.335 | 0.820 | 0.001 |
| 7 | 0.308 | 0.815 | 0.002 |
| 8 | 0.287 | 0.813 | 0.001 |
| 9 | 0.272 | 0.816 | 0.002 |
| 10 | 0.259 | 0.818 | 0.001 |
| 11 | 0.248 | 0.823 | 0.001 |
| 12 | 0.239 | 0.829 | 0.002 |
| 13 | 0.231 | 0.834 | 0.001 |
| 14 | 0.224 | 0.838 | 0.002 |
| 15 | 0.218 | 0.842 | 0.001 |

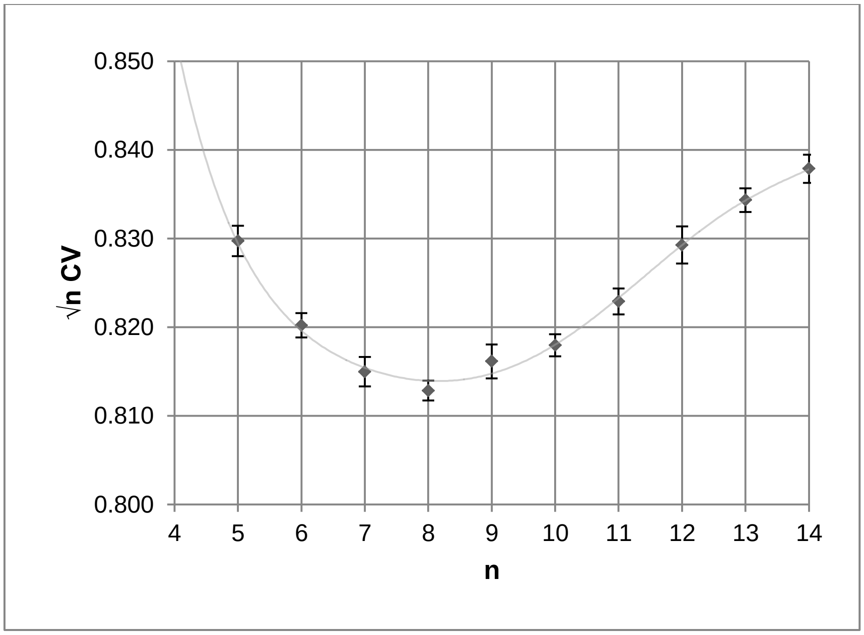


**Figure 2.** The optimal group size for the univariate Range is determined from the minimum observed in the plot of $\sqrt{n}CV$ versus $n$. The solid curve is a 6th order polynomial fit to guide the eye.

### 3.1.3. Mean Deviation, One Shot Group

The measure of dispersion known as the Mean Deviation is given by:

$$MD = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|, \tag{18}$$

which is the average of the distances of the *x*-components of the *n* impact points from the sample mean of *x*-components ($\bar{x}$). If the *x*-components are horizontal distances, then the *MD* is referred to as the Mean Horizontal Deviation (*MHD*), and if these are vertical distances, then it is referred to as the Mean Vertical Deviation (*MVD*).[1] Dividing (18) through by σ and rewriting:

$$MD/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}\sqrt{\left(\frac{\sqrt{n}(x_i-\bar{x})}{\sqrt{(n-1)}\sigma}\right)^2}. \tag{19}$$

Next, define:

$$u_i \equiv \frac{\sqrt{n}(x_i-\bar{x})}{\sqrt{(n-1)}\sigma}, \tag{20}$$

which is normally distributed with zero mean and unity variance. Making use of (20), (19) is rewritten as:

$$MD/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}|u_i|$$

$$MD/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}\chi_i\,(1), \tag{21}$$

where $\chi_i(1)$ is a chi-distributed random variable with one degree of freedom. For *n* = 2, we can write (21) as:

$$MD/\sigma = \tfrac{1}{\sqrt{2}}\chi(1) \quad (n = 2). \tag{22}$$

From (21), the mean of *MD*/σ is:

$$\langle MD/\sigma \rangle = \tfrac{\sqrt{(n-1)}}{n\sqrt{n}} \textstyle\sum_{i=1}^{n} \langle \chi_i(1) \rangle$$

$$= \sqrt{\tfrac{(n-1)}{n}} \langle \chi(1) \rangle$$

$$= \sqrt{\tfrac{(n-1)}{n}} \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}$$

$$\langle MD/\sigma \rangle = \sqrt{\tfrac{2(n-1)}{n\pi}}. \tag{23}$$

In the limit of large *n*, $\langle MD/\sigma \rangle \rightarrow \sqrt{2/\pi}$, and for *n* = 2, (23) becomes:

$$\langle MD/\sigma \rangle = 1/\sqrt{\pi} \quad (n = 2). \tag{24}$$

Note (24) is in agreement with (5), since for this case $MD = \tfrac{1}{2}R$. Finally, we seek an analytical expression for the variance of *MD*/σ. Pearson and Hartley[3] have provided such an expression, which is given by:

$$Var(MD/\sigma) = \tfrac{2(n-1)}{\pi n^2}\left[\tfrac{\pi}{2} + \sqrt{n(n-2)} - n + \sin^{-1}\tfrac{1}{n-1}\right]. \tag{25}$$

Recapping, analytical formulas as a function of sample size are provided for the first two moments of *MD*/σ. If *n* = 2, the probability distribution of *MD*/σ has a convenient analytical form (22); however, a simple extension of these results is unknown for *n* > 2, as evident from the fact that *MD*/σ is proportional to a sum of correlated chi random variables (21), which are correlated through their dependence on the sample mean (the $u_{\mathrm{i}}$ and $u_{\mathrm{j}}$ pairs (see (20)) are correlated). As such, Monte Carlo simulations were used to estimate the percentiles for *n* > 2.

In Table 6, results are presented for the means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of *MD*/σ for sample sizes of *n* = 2-30. The means were computed using (23), and standard deviations were computed from the square root of the variance (25). For *n* = 2, the percentiles were determined from those of a chi-distributed random variable (one degree of freedom) scaled by $\frac{1}{\sqrt{2}}$ as shown in (22). For *n* > 2, the percentiles are Monte Carlo estimates. A listing of percentiles of *MD*/σ is found in Appendix 1-3, along with the *MCE*'s of the estimated percentiles. In earlier work, Pearson and Hartley[3] published the upper and lower 10$^{th}$ percentiles of *MD*/σ (*n* = 2-10) to three decimal places. Comparison of their results with the percentiles in Appendix 1-3 are in good agreement.

**Table 6**. Table of means, standard deviations, and 5th and 95th percentiles of *MD*/σ for sample sizes of $n$ = 2-30.

| n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|
| 2 | 0.5642 | 0.4263 | 0.7555 | 0.0443 | 1.3859 | 0.0786 | 2.4565 |
| 3 | 0.6515 | 0.3419 | 0.5249 | 0.165 | 1.275 | 0.254 | 1.958 |
| 4 | 0.6910 | 0.2970 | 0.4298 | 0.256 | 1.221 | 0.370 | 1.767 |
| 5 | 0.7136 | 0.2663 | 0.3732 | 0.316 | 1.185 | 0.443 | 1.661 |
| 6 | 0.7284 | 0.2436 | 0.3344 | 0.360 | 1.157 | 0.494 | 1.588 |
| 7 | 0.7387 | 0.2258 | 0.3057 | 0.394 | 1.133 | 0.534 | 1.533 |
| 8 | 0.7464 | 0.2115 | 0.2834 | 0.423 | 1.115 | 0.566 | 1.494 |
| 9 | 0.7523 | 0.1996 | 0.2653 | 0.446 | 1.100 | 0.593 | 1.463 |
| 10 | 0.7569 | 0.1894 | 0.2503 | 0.464 | 1.087 | 0.613 | 1.436 |
| 11 | 0.7608 | 0.1807 | 0.2376 | 0.481 | 1.074 | 0.632 | 1.411 |
| 12 | 0.7639 | 0.1731 | 0.2266 | 0.495 | 1.063 | 0.647 | 1.391 |
| 13 | 0.7666 | 0.1664 | 0.2171 | 0.507 | 1.054 | 0.661 | 1.376 |
| 14 | 0.7689 | 0.1604 | 0.2086 | 0.519 | 1.045 | 0.675 | 1.360 |
| 15 | 0.7708 | 0.1550 | 0.2011 | 0.529 | 1.036 | 0.686 | 1.344 |
| 16 | 0.7725 | 0.1501 | 0.1943 | 0.537 | 1.030 | 0.696 | 1.333 |
| 17 | 0.7741 | 0.1457 | 0.1882 | 0.545 | 1.024 | 0.705 | 1.322 |
| 18 | 0.7754 | 0.1416 | 0.1826 | 0.552 | 1.018 | 0.712 | 1.312 |
| 19 | 0.7766 | 0.1378 | 0.1775 | 0.560 | 1.012 | 0.721 | 1.303 |
| 20 | 0.7777 | 0.1344 | 0.1728 | 0.565 | 1.007 | 0.727 | 1.295 |
| 21 | 0.7787 | 0.1312 | 0.1684 | 0.571 | 1.003 | 0.734 | 1.288 |
| 22 | 0.7795 | 0.1282 | 0.1644 | 0.577 | 0.998 | 0.741 | 1.280 |
| 23 | 0.7803 | 0.1254 | 0.1606 | 0.582 | 0.993 | 0.746 | 1.272 |
| 24 | 0.7811 | 0.1227 | 0.1571 | 0.586 | 0.989 | 0.750 | 1.267 |
| 25 | 0.7818 | 0.1203 | 0.1538 | 0.591 | 0.987 | 0.756 | 1.262 |
| 26 | 0.7824 | 0.1179 | 0.1507 | 0.596 | 0.983 | 0.761 | 1.256 |
| 27 | 0.7830 | 0.1157 | 0.1478 | 0.599 | 0.978 | 0.765 | 1.250 |
| 28 | 0.7835 | 0.1137 | 0.1451 | 0.603 | 0.977 | 0.769 | 1.247 |
| 29 | 0.7840 | 0.1117 | 0.1425 | 0.607 | 0.975 | 0.774 | 1.243 |
| 30 | 0.7845 | 0.1098 | 0.1400 | 0.610 | 0.970 | 0.778 | 1.237 |

**Notes Table 6**: For more details, refer to the notes at the bottom of Table 3.

### 3.1.4. Average Mean Deviation, Multiple Shot Groups

For multiple, *n*-round shot groups, the Mean Deviation is averaged over all groups to provide a measure of dispersion:

$$AMD/\sigma = \frac{1}{k}\sum_{i=1}^{k} MD_i/\sigma, \tag{26}$$

where *AMD* is the Average Mean Deviation, $MD_i$ is the mean deviation of the $i^{th}$ *n*-round shot group, and *k* is the total number of groups. Computing the mean of (26) gives:

$$\langle AMD/\sigma\rangle = \langle MD/\sigma\rangle; \tag{27}$$

i.e., the mean of an average over groups is the same as the mean of one shot group. The standard deviation of $AMD/\sigma$ is:

$$Std(AMD/\sigma) = \frac{1}{\sqrt{k}} Std(MD/\sigma). \tag{28}$$

Analogous to (16), the *CV* is:

$$CV(AMD) = \frac{1}{\sqrt{k}} CV(MD). \tag{29}$$

In Table 7, results are presented for the means, standard deviations, and 5th and 95th percentiles of *AMD*/σ for *n*-round shot groups (*n* = 2-5, 10, 25) fired for a total of *k* groups (*k* = 1-5, 10 for each of the listed *n*). The means were computed using (23), and the standard deviations were computed from the square root of the variance (25) and dividing by $\sqrt{k}$ . For *n* = 2 with *k* > 1 or if *n* > 2, the percentiles are Monte Carlo estimates. A listing of the percentiles of *AMD*/σ for the various *n* and *k* is found in Appendix 1-4, along with the *MCE*'s of the estimated percentiles. As *k* increases, *AMD*/σ is asymptotically normal in accordance with the central limit theorem. As mentioned in Section 3.1.2, our Monte Carlo results on percentiles may be used to provide the error in approximating the probability distribution of *AMD*/σ by a normal distribution. We will not belabor this point any further in Section 3.2.2 on *AES* or Section 3.2.4 on *AMR*, but note that this approximation improves for these bivariate distributions.

**Table 7**. Table of means, standard deviations, and 5th and 95th percentiles of *AMD*/σ for *n*-round shot groups (*n* = 2-5, 10, 25) fired for a total of *k* groups (*k* = 1-5, 10).

| k | n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.5642 | 0.4263 | 0.7555 | 0.0443 | 1.3859 | 0.0786 | 2.4565 |
| 2 | | | 0.3014 | 0.5342 | 0.142 | 1.120 | 0.253 | 1.985 |
| 3 | | | 0.2461 | 0.4362 | 0.205 | 1.007 | 0.363 | 1.784 |
| 4 | | | 0.2131 | 0.3778 | 0.246 | 0.943 | 0.435 | 1.672 |
| 5 | | | 0.1906 | 0.3379 | 0.277 | 0.902 | 0.491 | 1.599 |
| 10 | | | 0.1348 | 0.2389 | 0.355 | 0.798 | 0.630 | 1.414 |
| 1 | 3 | 0.6515 | 0.3419 | 0.5249 | 0.165 | 1.275 | 0.254 | 1.958 |
| 2 | | | 0.2418 | 0.3711 | 0.288 | 1.079 | 0.443 | 1.656 |
| 3 | | | 0.1974 | 0.3030 | 0.349 | 1.000 | 0.535 | 1.534 |
| 4 | | | 0.1710 | 0.2624 | 0.387 | 0.948 | 0.594 | 1.455 |
| 5 | | | 0.1529 | 0.2347 | 0.413 | 0.915 | 0.634 | 1.405 |
| 10 | | | 0.1081 | 0.1660 | 0.480 | 0.835 | 0.737 | 1.282 |
| 1 | 4 | 0.6910 | 0.2970 | 0.4298 | 0.256 | 1.221 | 0.370 | 1.767 |
| 2 | | | 0.2100 | 0.3039 | 0.369 | 1.060 | 0.535 | 1.535 |
| 3 | | | 0.1715 | 0.2482 | 0.426 | 0.989 | 0.616 | 1.431 |
| 4 | | | 0.1485 | 0.2149 | 0.459 | 0.947 | 0.664 | 1.370 |
| 5 | | | 0.1328 | 0.1922 | 0.482 | 0.919 | 0.698 | 1.329 |
| 10 | | | 0.0939 | 0.1359 | 0.541 | 0.850 | 0.783 | 1.230 |
| 1 | 5 | 0.7136 | 0.2663 | 0.3732 | 0.316 | 1.185 | 0.443 | 1.661 |
| 2 | | | 0.1883 | 0.2639 | 0.422 | 1.042 | 0.591 | 1.459 |
| 3 | | | 0.1538 | 0.2155 | 0.473 | 0.976 | 0.663 | 1.367 |
| 4 | | | 0.1332 | 0.1866 | 0.504 | 0.940 | 0.706 | 1.318 |
| 5 | | | 0.1191 | 0.1669 | 0.525 | 0.916 | 0.736 | 1.283 |
| 10 | | | 0.0842 | 0.1180 | 0.579 | 0.855 | 0.812 | 1.199 |
| 1 | 10 | 0.7569 | 0.1894 | 0.2503 | 0.464 | 1.087 | 0.613 | 1.436 |
| 2 | | | 0.1340 | 0.1770 | 0.546 | 0.985 | 0.721 | 1.302 |
| 3 | | | 0.1094 | 0.1445 | 0.583 | 0.943 | 0.770 | 1.246 |
| 4 | | | 0.0947 | 0.1251 | 0.606 | 0.917 | 0.801 | 1.211 |
| 5 | | | 0.0847 | 0.1119 | 0.621 | 0.900 | 0.821 | 1.189 |
| 10 | | | 0.0599 | 0.0791 | 0.660 | 0.857 | 0.872 | 1.133 |
| 1 | 25 | 0.7818 | 0.1203 | 0.1538 | 0.591 | 0.987 | 0.756 | 1.262 |
| 2 | | | 0.0850 | 0.1088 | 0.645 | 0.924 | 0.826 | 1.182 |
| 3 | | | 0.0694 | 0.0888 | 0.670 | 0.898 | 0.857 | 1.149 |
| 4 | | | 0.0601 | 0.0769 | 0.685 | 0.882 | 0.876 | 1.128 |
| 5 | | | 0.0538 | 0.0688 | 0.694 | 0.872 | 0.888 | 1.115 |
| 10 | | | 0.0380 | 0.0486 | 0.720 | 0.845 | 0.921 | 1.081 |

**Notes Table 7**: For more details, refer to the notes at the bottom of Table 4.

**Example 10.** Determine the power curve of the Average Mean Deviation for 3, 5-round shot groups with α = 0.05.

The power curve is generated by plotting the probability of rejection ($P_{rej}(T > \left(\frac{\sigma^*}{\sigma}\right) T_{1-\alpha})$) versus the ratio of $\sigma/\sigma^*$, where $\sigma^*$ is the value of the population standard deviation that we do not wish to exceed for accepted product (see Appendix 2-1 for additional details).

For notational convenience, the probability distribution of *AMD* for 3, 5-round groups is denoted as *AMD*(3,5), and here $T$ (see (10)) is its probability distribution in units of σ (i.e., $T = AMD(3{,}5)/\sigma$). In Table 8, a tabular method is used to construct the power curve (α = 0.05) of *AMD*(3,5). The percentiles of $AMD(3{,}5)/\sigma$ are from Appendix 1-4, and the ratios of $\sigma/\sigma^*$ (third column) are obtained by computing the ratio of the 95th percentile ( $T_{1-\alpha}$ ) with each listed $T_p$. The fourth column displays the probability of rejection for a particular value of $\sigma/\sigma^*$ and is computed by subtracting the values in the first column from one. Finally, the power curve is obtained by plotting the 4th versus 3rd column (see Figure 3).

**Table 8**. The tabular method used to construct the power curve (α = 0.05) of *AMD* for 3, 5-round groups. Plotting the 4th versus the 3rd column generates the power curve.

| $T = AMD(3,5)/\sigma$ | | | |
|---|---|---|---|
| $p$ | $T_p$ | $\sigma/\sigma^*$ | $P_{rej}$ |
| 0.999 | 1.247 | 0.782 | 0.001 |
| 0.995 | 1.148 | 0.850 | 0.005 |
| 0.99 | 1.101 | 0.887 | 0.01 |
| 0.975 | 1.032 | 0.946 | 0.025 |
| 0.95 | 0.976 | 1.000 | 0.05 |
| 0.9 | 0.914 | 1.068 | 0.1 |
| 0.8 | 0.841 | 1.161 | 0.2 |
| 0.75 | 0.813 | 1.201 | 0.25 |
| 0.5 | 0.707 | 1.381 | 0.5 |
| 0.25 | 0.606 | 1.611 | 0.75 |
| 0.2 | 0.582 | 1.678 | 0.8 |
| 0.1 | 0.521 | 1.874 | 0.9 |
| 0.05 | 0.473 | 2.063 | 0.95 |
| 0.025 | 0.433 | 2.253 | 0.975 |
| 0.01 | 0.389 | 2.506 | 0.99 |
| 0.005 | 0.361 | 2.704 | 0.995 |

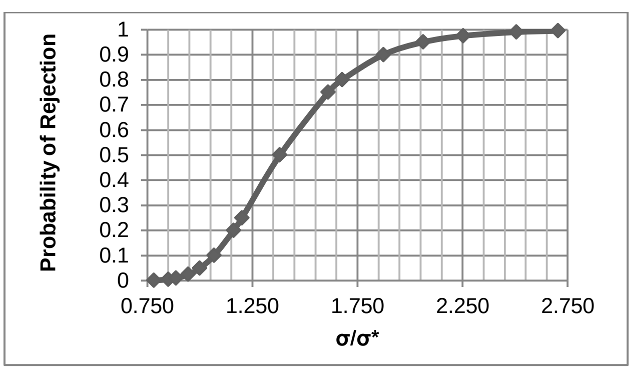


**Figure 3**. The power curve of the Average Mean Deviation for 3, 5-round shot groups with α = 0.05.

**Example 11**. Using the results in Table 8, determine the probability of rejecting 3, 5-round groups when $\sigma = 2.5\ \sigma^*$.

Inspection of Table 8 shows the probability of rejection is approximately 0.99.

**Example 12.** Suppose we shoot 10, 5-round groups and use the Average Mean Deviation as the dispersion measure. What is $\sigma/\sigma^*$ for a rejection rate (power = 1- β) of 0.80 with α = 0.05?

Following the method in Example 10, we seek the value of $\sigma/\sigma^*$ for which:

$$\frac{\sigma}{\sigma^*} = \frac{T_{1-\alpha}}{T_\beta}$$

$$= \frac{T_{0.95}}{T_{0.20}},$$

where *T* = *AMD/σ*. Using the percentile values of *AMD/σ* for 10, 5-round shot groups (Appendix 1-4):

$$\frac{\sigma}{\sigma^*} = \frac{0.855}{0.642} = 1.332,$$

which demonstrates if $\sigma = 1.332\ \sigma^*$, then the probability of rejection is 80%.

### 3.1.5. Standard Deviation, One Shot Group

The sample variance of the random variable *x* is given by:

$$SD^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2, \tag{30}$$

which is the average of the sum of squares of the deviations of the *x*-components of the impact points from the sample mean of *x*-components ($\bar{x}$). Taking the square root of both sides of (30) gives the univariate measure of dispersion known as the Standard Deviation:

$$SD = \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}. \tag{31}$$

If the *x*-components are horizontal distances, then *SD* is referred to as the Horizontal Standard Deviation (*HSD*), and if these are vertical distances, then it is referred to as the Vertical Standard Deviation (*VSD*). Dividing (31) by σ and rewriting:

$$SD/\sigma = \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2/\sigma^2}$$

$$SD/\sigma = \frac{\chi(n-1)}{\sqrt{n}}, \tag{32}$$

where $\chi(n-1)$ is a chi random variable with ($n$ - 1) degrees of freedom. The mean of $SD/\sigma$ is given by:

$$\langle SD/\sigma\rangle = \langle\chi(n-1)\rangle/\sqrt{n}$$

$$\langle SD/\sigma\rangle = \sqrt{\frac{2}{n}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}, \tag{33}$$

and its variance by:

$$Var(SD/\sigma) = \frac{1}{n}[\langle \chi^2(n-1) \rangle - \langle \chi(n-1) \rangle^2]$$

$$Var(SD/\sigma) = \frac{1}{n}\left[(n-1) - 2\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right)^2\right]. \quad (34)$$

The above results demonstrate the advantage of using *SD* as a dispersion measure. The probability distribution of $\sqrt{n}SD/\sigma$ is a chi distribution whose percentiles are easily computed by taking the square root of the percentiles from the corresponding chi-square distribution. In addition, formulas for the first two moments are provided, which are easy to compute using the built-in functions in Microsoft Excel 2010.

In Table 9, results are presented for the means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of $SD/\sigma$ for sample sizes of $n$ = 2-30. The means and standard deviations were computed using (33) and (34). The percentiles were computed from those of a chi-distributed random variable (($n$ - 1) DoF) scaled by $1/\sqrt{n}$ as shown in (32). A listing of additional percentiles of $SD/\sigma$ is found in Appendix 1-5.

**Table 9**. Table of means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of *SD*/σ for sample sizes $n$ = 2-30.

| n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|
| 2 | 0.5642 | 0.4263 | 0.7555 | 0.0443 | 1.3859 | 0.0786 | 2.4565 |
| 3 | 0.7236 | 0.3782 | 0.5227 | 0.1849 | 1.4132 | 0.2556 | 1.9530 |
| 4 | 0.7979 | 0.3367 | 0.4220 | 0.2966 | 1.3977 | 0.3717 | 1.7518 |
| 5 | 0.8407 | 0.3052 | 0.3630 | 0.3770 | 1.3775 | 0.4484 | 1.6384 |
| 6 | 0.8686 | 0.2808 | 0.3232 | 0.4369 | 1.3583 | 0.5030 | 1.5638 |
| 7 | 0.8882 | 0.2612 | 0.2941 | 0.4833 | 1.3412 | 0.5442 | 1.5100 |
| 8 | 0.9027 | 0.2452 | 0.2716 | 0.5205 | 1.3260 | 0.5766 | 1.4690 |
| 9 | 0.9139 | 0.2318 | 0.2536 | 0.5510 | 1.3126 | 0.6030 | 1.4363 |
| 10 | 0.9227 | 0.2203 | 0.2388 | 0.5766 | 1.3007 | 0.6249 | 1.4096 |
| 11 | 0.9300 | 0.2104 | 0.2262 | 0.5985 | 1.2901 | 0.6436 | 1.3872 |
| 12 | 0.9359 | 0.2017 | 0.2155 | 0.6174 | 1.2805 | 0.6597 | 1.3681 |
| 13 | 0.9410 | 0.1940 | 0.2061 | 0.6340 | 1.2718 | 0.6738 | 1.3515 |
| 14 | 0.9453 | 0.1871 | 0.1979 | 0.6487 | 1.2638 | 0.6863 | 1.3370 |
| 15 | 0.9490 | 0.1809 | 0.1906 | 0.6618 | 1.2566 | 0.6974 | 1.3241 |
| 16 | 0.9523 | 0.1753 | 0.1840 | 0.6737 | 1.2499 | 0.7074 | 1.3126 |
| 17 | 0.9551 | 0.1701 | 0.1781 | 0.6843 | 1.2437 | 0.7165 | 1.3022 |
| 18 | 0.9576 | 0.1654 | 0.1727 | 0.6941 | 1.2380 | 0.7248 | 1.2927 |
| 19 | 0.9599 | 0.1611 | 0.1678 | 0.7030 | 1.2327 | 0.7324 | 1.2841 |
| 20 | 0.9619 | 0.1570 | 0.1633 | 0.7112 | 1.2277 | 0.7394 | 1.2762 |
| 21 | 0.9638 | 0.1533 | 0.1591 | 0.7188 | 1.2230 | 0.7458 | 1.2690 |
| 22 | 0.9655 | 0.1498 | 0.1552 | 0.7259 | 1.2186 | 0.7518 | 1.2622 |
| 23 | 0.9670 | 0.1466 | 0.1516 | 0.7324 | 1.2145 | 0.7574 | 1.2560 |
| 24 | 0.9684 | 0.1435 | 0.1482 | 0.7385 | 1.2106 | 0.7627 | 1.2501 |
| 25 | 0.9696 | 0.1407 | 0.1451 | 0.7443 | 1.2069 | 0.7676 | 1.2447 |
| 26 | 0.9708 | 0.1380 | 0.1421 | 0.7497 | 1.2034 | 0.7722 | 1.2396 |
| 27 | 0.9719 | 0.1354 | 0.1393 | 0.7547 | 1.2001 | 0.7765 | 1.2348 |
| 28 | 0.9729 | 0.1330 | 0.1367 | 0.7595 | 1.1969 | 0.7806 | 1.2302 |
| 29 | 0.9739 | 0.1307 | 0.1342 | 0.7640 | 1.1939 | 0.7845 | 1.2259 |
| 30 | 0.9748 | 0.1285 | 0.1319 | 0.7683 | 1.1910 | 0.7882 | 1.2219 |

**Notes Table 9**: For more details, refer to the notes at the bottom of Table 3.

### 3.1.6. Standard Deviation, Multiple Shot Groups

Previously, we presented dispersion measures for multiple shot groups based on averaging the dispersion over groups. Similarly, such a dispersion measure may be developed for *SD* for multiple groups. However, we will depart from this usual procedure, and our dispersion measure for multiple groups will be the Root Mean Square (*RMS*) of the *SD*'s over groups. This departure is advantageous as explained below.

First, we present an analogous treatment to that given in Sections 3.1.2 and 3.1.4. Consider the dispersion measure given by the standard deviation averaged over all groups:

$$ASD/\sigma = \frac{1}{k}\sum_{i=1}^{k} SD_i\,/\sigma, \tag{35}$$

where *ASD* denotes the Average Standard Deviation, $SD_i$ is the standard deviation of the $i^{th}$ *n*-round shot group, and *k* is the total number of groups. Using (32), (35) is expressed as:

$$ASD/\sigma = \frac{1}{k\sqrt{n}}\sum_{i=1}^{k} \chi_i\,(n-1), \tag{36}$$

which shows that $\sqrt{n}\,ASD/\sigma$ is a group average over chi-distributed, independent random variables each with (*n* - 1) degrees of freedom. For $k > 1$, the probability distribution of *ASD*/σ is determined from the sum over chi-distributed variables and is not in the best form for easy computation of percentiles. However, we do acknowledge that the probability distribution of *ASD*/σ is asymptotically normal with increasing *k*, and its convergence to a normal distribution is expected to be fast. However, we depart from use of this dispersion measure in favor of $RMS\text{-}SD$, which is discussed below.

Next, we consider an alternative procedure for constructing a dispersion measure for multiple groups of *SD*. Instead of averaging over groups, a measure of dispersion for multiple, *n*-round, shot groups is constructed from the *RMS* of the *SD*'s over the *k* groups:

$$RMS\text{-}SD = \sqrt{\frac{1}{k}\sum_{i=1}^{k} SD_i^2}, \tag{37}$$

Dividing (37) by σ and making use of (32):

$$RMS\text{-}SD/\sigma = \frac{1}{\sqrt{kn}}\sqrt{\sum_{i=1}^{k} \chi_i^2(n-1)}. \tag{38}$$

Note the sum of *k*, chi-square, independent variables each with (*n* - 1) degrees of freedom is a chi-square variable with *k*(*n* - 1) degrees of freedom; therefore (38) is rewritten:

$$RMS\text{-}SD/\sigma = \frac{1}{\sqrt{kn}}\sqrt{\chi^2(k(n-1))}$$

$$RMS\text{-}SD/\sigma = \frac{1}{\sqrt{kn}}\chi(k(n-1)). \tag{39}$$

Percentiles of $\sqrt{kn} \times RMS\text{-}SD/\sigma$ are determined from a well-known probability distribution (chi-distributed random variable with *k*(*n* - 1) degrees of freedom), which is why *RMS-SD* is preferred to *ASD* (see (36)) as a dispersion measure. The percentile tables for $RMS\text{-}SD/\sigma$ for various values of *k* and *n* were easily computed using the built-in functions in Microsoft Excel. Continuing, the mean of $RMS\text{-}SD/\sigma$ is given by:

$$\langle RMS\text{-}SD/\sigma\rangle = \frac{1}{\sqrt{kn}}\langle\chi(k(n-1))\rangle$$

$$\langle RMS\text{-}SD/\sigma\rangle = \sqrt{\frac{2}{kn}}\frac{\Gamma(\frac{k(n-1)+1}{2})}{\Gamma(\frac{k(n-1)}{2})}, \qquad (40)$$

and its variance is given by:

$$Var(RMS\text{-}SD/\sigma) = \frac{1}{kn}Var(\chi(k(n-1)))$$

$$Var(RMS\text{-}SD/\sigma) = \frac{1}{kn}\left\{k(n-1) - 2\left(\frac{\Gamma(\frac{k(n-1)+1}{2})}{\Gamma(\frac{k(n-1)}{2})}\right)^2\right\}. \qquad (41)$$

The *CV* of *RMS-SD* is computed from:

$$CV(RMS\text{-}SD) = \frac{\sqrt{k(n-1)-\langle\chi(k(n-1))\rangle^2}}{\langle\chi(k(n-1))\rangle}, \qquad (42)$$

which can be compared to the Coefficient of Variation of *ASD*:

$$CV(ASD) = \frac{1}{\sqrt{k}}\frac{\sqrt{(n-1)-\langle\chi((n-1))\rangle^2}}{\langle\chi((n-1))\rangle} \qquad (43)$$

Comparison of the *CV*'s given by (42) and (43) for the same *k* and *n* would show the *CV* of *RMS-SD* is less than that of *ASD*, which demonstrates the *CV* of *RMS-SD* is a more efficient estimator of σ. Albeit, the difference between *CV*'s is small for the various (*k*, *n*) pairs.

Next in Table 10, results are presented for the means, standard deviations, and 5th and 95th percentiles of $RMS\text{-SD}/\sigma$ for *n*-round shot groups (*n* = 2-5, 10, 25) fired for a total of *k* groups (*k* = 1-5, 10 for each of the listed *n*). The means and standard deviations were computed from (40) and the square root of (41), and the percentiles were computed from a chi distribution (*k*(*n* - 1)) degrees of freedom) scaled by $1/\sqrt{kn}$ per (39). A listing of additional percentiles of $RMS\text{-}SD/\sigma$ for the various *k* and *n* is found in Appendix 1-6.

**Table 10**. Table of means, standard deviations, and 5th and 95th percentiles of *RMS-SD*/σ for *n*-round shot groups ($n$ = 2-5, 10, 25) fired for a total of *k* groups ($k$ = 1-5, 10).

| k | n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.5642 | 0.4263 | 0.7555 | 0.0443 | 1.3859 | 0.0786 | 2.4565 |
| 2 | | 0.6267 | 0.3276 | 0.5227 | 0.1601 | 1.2239 | 0.2556 | 1.9530 |
| 3 | | 0.6515 | 0.2749 | 0.4220 | 0.2422 | 1.1413 | 0.3717 | 1.7518 |
| 4 | | 0.6647 | 0.2413 | 0.3630 | 0.2981 | 1.0890 | 0.4484 | 1.6384 |
| 5 | | 0.6728 | 0.2175 | 0.3232 | 0.3384 | 1.0522 | 0.5030 | 1.5638 |
| 10 | | 0.6897 | 0.1560 | 0.2262 | 0.4439 | 0.9567 | 0.6436 | 1.3872 |
| 1 | 3 | 0.7236 | 0.3782 | 0.5227 | 0.1849 | 1.4132 | 0.2556 | 1.9530 |
| 2 | | 0.7675 | 0.2786 | 0.3630 | 0.3442 | 1.2575 | 0.4484 | 1.6384 |
| 3 | | 0.7833 | 0.2304 | 0.2941 | 0.4263 | 1.1828 | 0.5442 | 1.5100 |
| 4 | | 0.7914 | 0.2007 | 0.2536 | 0.4772 | 1.1368 | 0.6030 | 1.4363 |
| 5 | | 0.7964 | 0.1802 | 0.2262 | 0.5125 | 1.1047 | 0.6436 | 1.3872 |
| 10 | | 0.8064 | 0.1283 | 0.1591 | 0.6014 | 1.0232 | 0.7458 | 1.2690 |
| 1 | 4 | 0.7979 | 0.3367 | 0.4220 | 0.2966 | 1.3977 | 0.3717 | 1.7518 |
| 2 | | 0.8308 | 0.2444 | 0.2941 | 0.4521 | 1.2546 | 0.5442 | 1.5100 |
| 3 | | 0.8423 | 0.2011 | 0.2388 | 0.5264 | 1.1874 | 0.6249 | 1.4096 |
| 4 | | 0.8482 | 0.1749 | 0.2061 | 0.5715 | 1.1464 | 0.6738 | 1.3515 |
| 5 | | 0.8517 | 0.1567 | 0.1840 | 0.6025 | 1.1179 | 0.7074 | 1.3126 |
| 10 | | 0.8588 | 0.1113 | 0.1296 | 0.6799 | 1.0461 | 0.7917 | 1.2180 |
| 1 | 5 | 0.8407 | 0.3052 | 0.3630 | 0.3770 | 1.3775 | 0.4484 | 1.6384 |
| 2 | | 0.8670 | 0.2199 | 0.2536 | 0.5227 | 1.2453 | 0.6030 | 1.4363 |
| 3 | | 0.8760 | 0.1806 | 0.2061 | 0.5903 | 1.1840 | 0.6738 | 1.3515 |
| 4 | | 0.8806 | 0.1568 | 0.1781 | 0.6309 | 1.1467 | 0.7165 | 1.3022 |
| 5 | | 0.8833 | 0.1405 | 0.1591 | 0.6588 | 1.1209 | 0.7458 | 1.2690 |
| 10 | | 0.8889 | 0.0997 | 0.1121 | 0.7281 | 1.0560 | 0.8192 | 1.1881 |
| 1 | 10 | 0.9227 | 0.2203 | 0.2388 | 0.5766 | 1.3007 | 0.6249 | 1.4096 |
| 2 | | 0.9356 | 0.1570 | 0.1678 | 0.6852 | 1.2014 | 0.7324 | 1.2841 |
| 3 | | 0.9399 | 0.1285 | 0.1367 | 0.7337 | 1.1563 | 0.7806 | 1.2302 |
| 4 | | 0.9421 | 0.1114 | 0.1183 | 0.7627 | 1.1291 | 0.8096 | 1.1985 |
| 5 | | 0.9434 | 0.0997 | 0.1057 | 0.7825 | 1.1105 | 0.8294 | 1.1770 |
| 10 | | 0.9461 | 0.0706 | 0.0746 | 0.8314 | 1.0637 | 0.8788 | 1.1244 |
| 1 | 25 | 0.9696 | 0.1407 | 0.1451 | 0.7443 | 1.2069 | 0.7676 | 1.2447 |
| 2 | | 0.9747 | 0.0997 | 0.1023 | 0.8136 | 1.1417 | 0.8347 | 1.1713 |
| 3 | | 0.9764 | 0.0815 | 0.0835 | 0.8443 | 1.1124 | 0.8647 | 1.1393 |
| 4 | | 0.9772 | 0.0706 | 0.0723 | 0.8626 | 1.0949 | 0.8826 | 1.1203 |
| 5 | | 0.9778 | 0.0632 | 0.0646 | 0.8750 | 1.0828 | 0.8949 | 1.1075 |
| 10 | | 0.9788 | 0.0447 | 0.0457 | 0.9058 | 1.0529 | 0.9255 | 1.0757 |

**Notes Table 10**: For more details, refer to the notes at the bottom of Table 4.

## 3.2. Bivariate Dispersion Measures

### 3.2.1. Extreme Spread, One Shot Group

The bivariate measure of dispersion known as the Extreme Spread is given by:

$$ES = max_{i,j}\left\{\sqrt{\left(\vec{x}_i - \vec{x}_j\right)^2}\right\}, \tag{44}$$

where the vector $\vec{x}_i = (x_i, y_i)$ is the $i^{\text{th}}$ impact-point position. *ES* is the maximum of the distances between all possible pairs of impact points. Dividing through by σ:

$$ES/\sigma = max_{i,j}\left\{\sqrt{\left((\vec{x}_i - \vec{x}_j)/\sigma\right)^2}\right\}, \quad (45)$$

where σ is the population standard deviation of *x* and *y* of the impact-point positions. For *n* = 2, (45) becomes:

$$ES/\sigma = |\vec{x}_2 - \vec{x}_1|/\sigma,$$

and is rewritten as:

$$ES/\sigma = \sqrt{2}\sqrt{\left(\frac{x_2-x_1}{\sqrt{2}\sigma}\right)^2 + \left(\frac{y_2-y_1}{\sqrt{2}\sigma}\right)^2} \quad (n = 2). \quad (46)$$

Since $\frac{x_2-x_1}{\sqrt{2}\sigma}$ and $\frac{y_2-y_1}{\sqrt{2}\sigma}$ are normally distributed with zero mean and unity variance, (46) is expressed as:

$$ES/\sigma = \sqrt{2}\chi(2) \quad (n = 2), \quad (47)$$

where $\chi(2)$ is a chi-distributed random variable with two degrees of freedom. Continuing with *n* = 2, the mean of $ES/\sigma$ is given by:

$$\langle ES/\sigma\rangle = \sqrt{2}\langle\chi(2)\rangle$$

$$= 2\Gamma\left(\frac{3}{2}\right)$$

$$\langle ES/\sigma\rangle = \sqrt{\pi} \quad (n = 2), \quad (48)$$

and its variance is computed from:

$$Var(ES/\sigma) = \langle(ES/\sigma)^2\rangle - \langle ES/\sigma\rangle^2$$

$$= 2[\langle\chi^2(2)\rangle - \pi/2]$$

$$Var(ES/\sigma) = (4-\pi) \quad (n = 2). \quad (49)$$

Taking the square root of (49) gives the standard deviation:

$$Std(ES/\sigma) = \sqrt{(4-\pi)} \quad (n = 2). \quad (50)$$

For *n* = 2, simple analytical results are provided for the probability distribution and the first two moments of $ES/\sigma$. If *n* > 2, an extension of these analytic results is unavailable, because the exact probability distributions of *ES* are unknown. In this case, Monte Carlo methods are an obvious approach to compute the percentiles and moments.[2]

In Table 11, results are presented for the means, standard deviations, and 5$^{th}$ and 95$^{th}$ percentiles of $ES/\sigma$ for sample sizes of *n* = 2-30. For *n* = 2, the means and standard deviations displayed in the table are from (48) and (50), and the percentiles were determined from those of a chi-distributed random variable (two degrees of freedom) scaled by $\sqrt{2}$ as shown in (47). For *n* > 2, all tabulated results are from Monte Carlo estimates. A listing of additional percentiles of *ES*/σ is found in Appendix 1-7, along with the *MCE*'s of the estimated percentiles. In previous work, Taylor and Grubbs[2] examined the probability distribution of the Extreme Spread for various sample sizes ($n$ = 3-15, 20, 25, 28, 30, 31, 34) using Monte Carlo methods and published percentiles of *ES* along with its first four moments. Our results presented in Table 11 and Appendix 1-7 are in excellent agreement with theirs—our published values of percentiles, means, and standard deviations agree within 1%.

**Table 11.** Table of means, standard deviations, 5th and 95th percentiles of $ES/\sigma$ for sample sizes of $n$ = 2-30. Numbers reported in parentheses are the *MCE*'s of the value of the mean or standard deviation preceding it.

| n | Mean | | Std | | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.7725 | | 0.9265 | | 0.5227 | 0.4530 | 3.4616 | 0.2556 | 1.9530 |
| 3 | 2.410 | (0.008) | 0.893 | (0.006) | 0.370 | 1.067 | 3.986 | 0.443 | 1.654 |
| 4 | 2.795 | (0.004) | 0.855 | (0.004) | 0.306 | 1.497 | 4.291 | 0.535 | 1.535 |
| 5 | 3.062 | (0.006) | 0.828 | (0.004) | 0.270 | 1.804 | 4.515 | 0.589 | 1.474 |
| 6 | 3.275 | (0.005) | 0.803 | (0.002) | 0.245 | 2.048 | 4.679 | 0.625 | 1.429 |
| 7 | 3.443 | (0.005) | 0.782 | (0.003) | 0.227 | 2.246 | 4.811 | 0.652 | 1.397 |
| 8 | 3.584 | (0.004) | 0.766 | (0.004) | 0.214 | 2.411 | 4.924 | 0.673 | 1.374 |
| 9 | 3.705 | (0.004) | 0.752 | (0.002) | 0.203 | 2.556 | 5.015 | 0.690 | 1.354 |
| 10 | 3.815 | (0.005) | 0.741 | (0.004) | 0.194 | 2.686 | 5.106 | 0.704 | 1.339 |
| 11 | 3.904 | (0.007) | 0.729 | (0.003) | 0.187 | 2.791 | 5.176 | 0.715 | 1.326 |
| 12 | 3.989 | (0.005) | 0.719 | (0.004) | 0.180 | 2.893 | 5.250 | 0.725 | 1.316 |
| 13 | 4.064 | (0.006) | 0.711 | (0.004) | 0.175 | 2.983 | 5.312 | 0.734 | 1.307 |
| 14 | 4.131 | (0.005) | 0.704 | (0.004) | 0.170 | 3.062 | 5.360 | 0.741 | 1.297 |
| 15 | 4.197 | (0.005) | 0.698 | (0.003) | 0.166 | 3.133 | 5.424 | 0.746 | 1.292 |
| 16 | 4.257 | (0.003) | 0.692 | (0.003) | 0.163 | 3.204 | 5.474 | 0.753 | 1.286 |
| 17 | 4.310 | (0.005) | 0.685 | (0.005) | 0.159 | 3.266 | 5.505 | 0.758 | 1.277 |
| 18 | 4.360 | (0.006) | 0.679 | (0.004) | 0.156 | 3.327 | 5.554 | 0.763 | 1.274 |
| 19 | 4.407 | (0.005) | 0.675 | (0.003) | 0.153 | 3.383 | 5.592 | 0.768 | 1.269 |
| 20 | 4.451 | (0.004) | 0.670 | (0.003) | 0.150 | 3.438 | 5.623 | 0.772 | 1.263 |
| 21 | 4.495 | (0.007) | 0.665 | (0.003) | 0.148 | 3.488 | 5.658 | 0.776 | 1.259 |
| 22 | 4.533 | (0.005) | 0.660 | (0.004) | 0.146 | 3.533 | 5.696 | 0.780 | 1.257 |
| 23 | 4.571 | (0.005) | 0.656 | (0.004) | 0.144 | 3.578 | 5.720 | 0.783 | 1.251 |
| 24 | 4.606 | (0.005) | 0.654 | (0.004) | 0.142 | 3.615 | 5.757 | 0.785 | 1.250 |
| 25 | 4.640 | (0.004) | 0.650 | (0.003) | 0.140 | 3.657 | 5.784 | 0.788 | 1.247 |
| 26 | 4.672 | (0.004) | 0.645 | (0.002) | 0.138 | 3.697 | 5.809 | 0.791 | 1.243 |
| 27 | 4.704 | (0.007) | 0.642 | (0.004) | 0.137 | 3.732 | 5.831 | 0.793 | 1.240 |
| 28 | 4.733 | (0.006) | 0.639 | (0.004) | 0.135 | 3.767 | 5.859 | 0.796 | 1.238 |
| 29 | 4.763 | (0.005) | 0.637 | (0.005) | 0.134 | 3.802 | 5.884 | 0.798 | 1.235 |
| 30 | 4.790 | (0.004) | 0.634 | (0.003) | 0.132 | 3.837 | 5.908 | 0.801 | 1.233 |

**Notes Table 11**: For more details, refer to the notes at the bottom of Table 3.

**Example 13.** Compute an unbiased estimate of σ using the Extreme Spread of the 5-round shot group in Table 1.

First compute the *ES* by substituting the $x$ and $y$ values of the impact points (Table 1) into (44), which gives $ES$ = 1.79". An unbiased estimate is computed from:

$$\hat{\sigma} = ES/\langle ES/\sigma \rangle.$$

Substituting the needed mean (Table 11) and computed *ES* :

$$\hat{\sigma} = \frac{1.79''}{3.062} = 0.58''.$$

Comparison with Example 1 shows a 16% reduction in $\hat{\sigma}$ when the bivariate measure is used.

---

**Example 14.** Using the Extreme Spread computed in Example 13, find the 90% Confidence Interval of σ from this 5-round group.

Recall (12):

$$M/T_{0.95} \leq \sigma \leq M/T_{0.05}.$$

Substituting the *ES* of the group and the needed values for a 5-round group (Table 11):

$$1.79''/4.515 \leq \sigma \leq 1.79''/1.804$$

$$0.40'' \leq \sigma \leq 0.99'' \quad \text{(90\% CI).}$$

Compare this result with Example 3, where the 90% CI was computed using the univariate Range. Note the 95% upper CL is reduced by ~36% when the bivariate measure of dispersion is used.

### 3.2.2. Extreme Spread, Multiple Shot Groups

For multiple, *n*-round shot groups, the Extreme Spread is averaged over groups to provide a measure of dispersion:

$$AES/\sigma = \frac{1}{k}\sum_{i=1}^{k} ES_i\,/\sigma, \tag{51}$$

where *AES* is the Average Extreme Spread, $ES_i$ is the extreme spread of the $i^{th}$ *n*-round shot group, and *k* is the total number of groups. The mean of (51) is:

$$\langle AES/\sigma\rangle = \langle ES/\sigma\rangle, \tag{52}$$

and its standard deviation is:

$$Std(AES/\sigma) = \frac{1}{\sqrt{k}}Std(ES/\sigma), \tag{53}$$

and its Coefficient of Variation is:

$$CV(AES) = \frac{1}{\sqrt{k}}CV(ES). \tag{54}$$

In Table 12, results are presented for the means, standard deviations, and $5^{th}$ and $95^{th}$ percentiles of $AES/\sigma$ for *n*-round shot groups (*n* = 2-5, 10, 25) fired for a total of *k* groups (*k* =1-5, 10 for each of the listed *n*). For *n* = 2, the listed mean equals $\sqrt{\pi}$ via (48), and the standard deviations are computed from (50) divided by $\sqrt{k}$. For *n* = 2 and *k* > 1, the percentiles are Monte Carlo estimates. For *n* > 2, all tabulated results are Monte Carlo estimates, and estimates of the means are from simulations with *k* = 10. A listing of the percentiles of *AES*/σ for the various *n* and *k* is found in Appendix 1-8, along with the *MCE*'s of the estimated percentiles.

**Table 12**. Table of means, standard deviations, and 5th and 95th percentiles of *AES*/σ for *n*-round shot groups (*n* = 2-5, 10, 25) fired for a total of *k* groups (*k* = 1-5, 10). Numbers reported in parentheses are the *MCE*'s of the value of the mean or standard deviation preceding it.

| k | n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1.7725 | 0.9265 | 0.5227 | 0.4530 | 3.4616 | 0.2556 | 1.9530 |
| 2 | | | 0.6551 | 0.3696 | 0.788 | 2.934 | 0.445 | 1.656 |
| 3 | | | 0.5349 | 0.3018 | 0.952 | 2.711 | 0.537 | 1.529 |
| 4 | | | 0.4633 | 0.2614 | 1.056 | 2.579 | 0.596 | 1.455 |
| 5 | | | 0.4143 | 0.2338 | 1.126 | 2.487 | 0.635 | 1.403 |
| 10 | | | 0.2930 | 0.1653 | 1.307 | 2.269 | 0.738 | 1.280 |
| 1 | 3 | 2.409 (0.003) | 0.893 (0.006) | 0.371 | 1.067 | 3.986 | 0.443 | 1.655 |
| 2 | | | 0.632 (0.004) | 0.262 | 1.428 | 3.502 | 0.593 | 1.454 |
| 3 | | | 0.515 (0.002) | 0.214 | 1.602 | 3.291 | 0.665 | 1.367 |
| 4 | | | 0.446 (0.002) | 0.185 | 1.706 | 3.173 | 0.708 | 1.317 |
| 5 | | | 0.400 (0.0019) | 0.166 | 1.775 | 3.092 | 0.737 | 1.284 |
| 10 | | | 0.282 (0.0008) | 0.117 | 1.957 | 2.884 | 0.812 | 1.197 |
| 1 | 4 | 2.794 (0.0013) | 0.855 (0.004) | 0.306 | 1.497 | 4.291 | 0.536 | 1.536 |
| 2 | | | 0.605 (0.004) | 0.217 | 1.850 | 3.837 | 0.662 | 1.373 |
| 3 | | | 0.494 (0.002) | 0.177 | 2.016 | 3.638 | 0.722 | 1.302 |
| 4 | | | 0.428 (0.003) | 0.153 | 2.113 | 3.520 | 0.756 | 1.260 |
| 5 | | | 0.383 (0.0018) | 0.137 | 2.183 | 3.443 | 0.781 | 1.232 |
| 10 | | | 0.271 (0.0016) | 0.097 | 2.359 | 3.250 | 0.845 | 1.163 |
| 1 | 5 | 3.066 (0.002) | 0.828 (0.004) | 0.270 | 1.804 | 4.515 | 0.588 | 1.473 |
| 2 | | | 0.586 (0.003) | 0.191 | 2.154 | 4.077 | 0.703 | 1.330 |
| 3 | | | 0.477 (0.002) | 0.156 | 2.311 | 3.881 | 0.754 | 1.266 |
| 4 | | | 0.413 (0.0015) | 0.135 | 2.409 | 3.765 | 0.786 | 1.228 |
| 5 | | | 0.370 (0.0014) | 0.121 | 2.477 | 3.691 | 0.808 | 1.204 |
| 10 | | | 0.262 (0.0012) | 0.086 | 2.644 | 3.505 | 0.862 | 1.143 |
| 1 | 10 | 3.811 (0.0014) | 0.741 (0.004) | 0.194 | 2.686 | 5.106 | 0.705 | 1.340 |
| 2 | | | 0.524 (0.004) | 0.137 | 2.994 | 4.710 | 0.786 | 1.236 |
| 3 | | | 0.428 (0.002) | 0.112 | 3.136 | 4.542 | 0.823 | 1.192 |
| 4 | | | 0.371 (0.0012) | 0.097 | 3.224 | 4.442 | 0.846 | 1.166 |
| 5 | | | 0.331 (0.002) | 0.087 | 3.285 | 4.373 | 0.862 | 1.148 |
| 10 | | | 0.234 (0.0011) | 0.061 | 3.434 | 4.204 | 0.901 | 1.103 |
| 1 | 25 | 4.641 (0.0018) | 0.650 (0.003) | 0.140 | 3.657 | 5.784 | 0.788 | 1.246 |
| 2 | | | 0.458 (0.0012) | 0.099 | 3.927 | 5.430 | 0.846 | 1.170 |
| 3 | | | 0.375 (0.003) | 0.081 | 4.051 | 5.283 | 0.873 | 1.138 |
| 4 | | | 0.324 (0.0017) | 0.070 | 4.130 | 5.192 | 0.890 | 1.119 |
| 5 | | | 0.291 (0.0016) | 0.063 | 4.179 | 5.133 | 0.900 | 1.106 |
| 10 | | | 0.206 (0.0013) | 0.0443 | 4.311 | 4.987 | 0.929 | 1.075 |

**Notes Table 12**: For more details, refer to the notes at the bottom of Table 4.

**Example 15.** If a total of *N* rounds is available to fire, what group size (*n*) gives the smallest Coefficient of Variation of the Average Extreme Spread? (This example was inspired by Ref. 5.)

Following Example 9, the number of groups is $k = N/n$, and the *CV* of *AES* for *k*, *n*-round groups is:

$$CV\big(AES(k,n)\big) = CV(ES(n))/\sqrt{k}$$

$$CV\big(AES(k,n)\big) = \sqrt{\frac{n}{N}}\,CV(ES(n)).$$

The value of *n* that minimizes $\sqrt{n}CV(ES(n))$ gives the optimal group size, and computation of the optimal group size is illustrated in Table 13. The value of *n* for optimal group size corresponds to the minimum value in the third column, and the fourth column shows an estimate of the Standard Error (SE) of the corresponding values in the third column. Figure 4 shows a plot of $\sqrt{n}CV$ versus *n*, and inspection of the figure shows a shallow minimum in the interval 5 < *n* < 8 indicating an optimal group size for the Extreme Spread of about *n* = 7. From a practical standpoint, 5-round groups should serve just as well as 7-round groups to minimize *CV*.

**Table 13**. Table illustrating the computation of the optimal group size for the Extreme Spread.

| *n* | *CV* | $\sqrt{n}CV$ | SE |
|---|---|---|---|
| 3 | 0.370 | 0.642 | 0.002 |
| 4 | 0.306 | 0.612 | 0.001 |
| 5 | 0.270 | 0.604 | 0.001 |
| 6 | 0.245 | 0.600 | 0.001 |
| 7 | 0.227 | 0.600 | 0.001 |
| 8 | 0.214 | 0.605 | 0.001 |
| 9 | 0.203 | 0.609 | 0.001 |
| 10 | 0.194 | 0.614 | 0.001 |
| 11 | 0.187 | 0.619 | 0.001 |
| 12 | 0.180 | 0.625 | 0.001 |
| 13 | 0.175 | 0.630 | 0.001 |
| 14 | 0.170 | 0.637 | 0.001 |
| 15 | 0.166 | 0.644 | 0.001 |

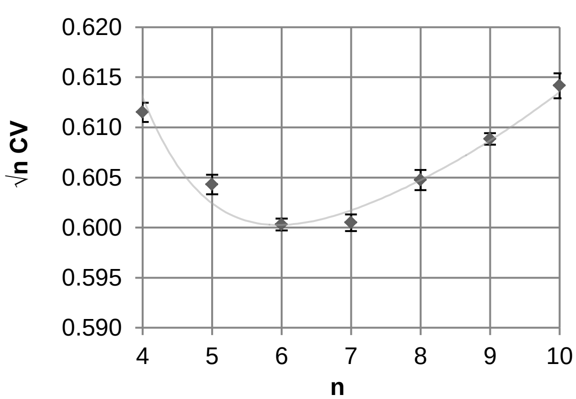


**Figure 4**. The optimal group size for the Extreme Spread is determined from the minimum observed in the plot of $\sqrt{n}$CV versus *n*. The solid curve is a 6th order polynomial fit to guide the eye.

**Example 16.** Suppose 3, 5-round shot groups are subjected to the requirement that the Average Extreme Spread over the three groups shall be less than 1 MOA ($AES \leq 1$ MOA). What is the upper limit on σ ($\sigma_{upper}$) at the 95% CL for acceptance? What is the Circular Error Probability (*CEP*) at the 95% CL?

Inspection of (12) shows:

$$\sigma_{upper} = 1\text{ MOA}/T_{0.05} \quad \text{(95\% CL)}.$$

Here $T_{0.05}$ represents the 5th percentile of *AES*/σ. Substituting the value of this percentile for 3, 5-round groups (Table 12):

$$\sigma_{upper} = 1\text{ MOA}/2.311 = 0.433\text{ MOA} \quad \text{(95\% CL)}. \qquad (55)$$

The *CEP* is defined as:

$$CEP \equiv \sqrt{2\ln 2}\,\sigma_{upper}, \qquad (56)$$

which is the radius of the circle in which 50% of the shots will fall at the CL of $\sigma_{upper}$. Substituting (55) into (56):

$$CEP = 0.509\ MOA \quad \text{(95\% CL)}.$$

---

**Example 17.** Another test agency decides to shoot 3, 5-round groups and use the maximum Extreme Spread of the three groups as their dispersion measure:

$$ES_{max} = Max(ES_1, ES_2, ES_3).$$

What should the requirement on $ES_{max}$ be to ensure $\sigma_{upper}$ is the same as given in Example 16?

Our requirement is:

$$ES_{max} \leq Q,$$

where *Q* is the quantity that we are seeking to determine. Accordingly, we have:

$$\sigma_{upper} = Q/T'_{0.05} \quad \text{(95\% CL)}, \qquad (57)$$

where $T'_{0.05}$ represents the 5th percentile of $ES_{max}/\sigma$. This percentile needs to be computed, and we note that it is equivalent to some $p \times 100$ percentile of $ES/\sigma$ for one, 5-round group. Since there are 3 groups from which the maximum *ES* is selected:

$$p = (0.05)^{\frac{1}{3}} = 0.368.$$

So, we are seeking the value of $T_{0.368}$ for $ES/\sigma$ for one, 5-round group, which we determine by linear interpolation between the 25th and 50th percentiles given in Appendix 1-7. Interpolation gives:

$$T'_{0.05} = 2.725.$$

To satisfy (57):

$$Q = T'_{0.05} \times \sigma_{upper} = 2.725 \times 0.433\ MOA = 1.18\ MOA;$$

therefore, our requirement is:

$$ES_{max} \leq 1.18\ MOA.$$

**Example 18.** Compare the power curves of $ES_{max}$ (defined in Example 17) and the Average Extreme Spread for 3, 5-round shot groups with $\alpha = 0.05$.

Following the procedure in Example 10, construct the power curves ($\alpha = 0.05$) of $AES$ and $ES_{max}$ for 3, 5-round groups. The power curves are plotted in Figure 5, which shows that $\sigma/\sigma^*$ at a rejection rate of 0.80 is ~9% smaller for $AES/\sigma$. For clarity, Table 14 illustrates the tabular method used to construct the power curve of $ES_{max}(3,5)$. The percentiles of $ES_{max}(3,5)/\sigma$ were computed using the percentiles of $ES/\sigma$ for one, 5-round group $ES(5)/\sigma$. For example, the value of 6.04 is the 99.9 percentile of $ES(5)/\sigma$ (Appendix 1-7), which is thus the value of the $(0.999)^{\frac{1}{3}} \times 100 = 99.7$ percentile of $ES_{max}(3,5)/\sigma$. The other percentiles were similarly computed with exception to the 20th and 95th percentiles, which were computed by linear interpolation using the corresponding adjacent percentiles computed for $ES_{max}(3,5)/\sigma$ in Table 14.

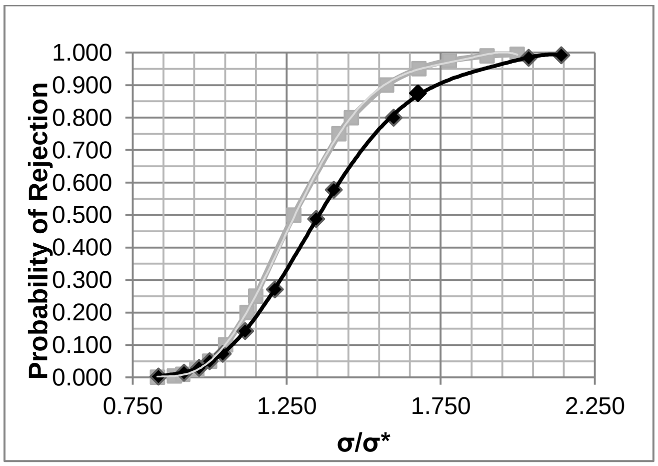


**Figure 5**. A comparison of the power curves (α = 0.05) of *AES* (gray squares) and $ES_{max}$ (black diamonds) for 3, 5-round groups.

**Table 14**. The tabular method used to construct the power curve (α = 0.05) of $ES_{max}$ for 3, 5-round groups. Plotting the 4th versus the 3rd column generates the power curve.

| $T' = ES_{max}(3,5)/\sigma$ | | | |
|---|---|---|---|
| **p** | $T'_p$ | **σ/σ*** | $P_{rej}$ |
| 0.997 | 6.04 | 0.83 | 0.003 |
| 0.985 | 5.49 | 0.92 | 0.015 |
| 0.970 | 5.22 | 0.97 | 0.030 |
| 0.950 | 5.04 | 1.00 | 0.050 |
| 0.927 | 4.83 | 1.04 | 0.073 |
| 0.857 | 4.515 | 1.12 | 0.143 |
| 0.729 | 4.156 | 1.21 | 0.271 |
| 0.512 | 3.743 | 1.35 | 0.488 |
| 0.422 | 3.589 | 1.40 | 0.578 |
| 0.200 | 3.15 | 1.60 | 0.800 |
| 0.125 | 3.005 | 1.68 | 0.875 |
| 0.016 | 2.474 | 2.04 | 0.984 |
| 0.008 | 2.352 | 2.14 | 0.992 |

### 3.2.3. Mean Radius, One Shot Group

The measure of dispersion known as the Mean Radius is given by:

$$MR = \frac{1}{n}\sum_{i=1}^{n}\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}, \tag{58}$$

which is the average of the radial distances of the *n* impact points from the observed center of impact ($\bar{x}$, $\bar{y}$). Dividing both sides of (58) by σ and rewriting:

$$MR/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}\sqrt{\left(\frac{\sqrt{n}(x_i-\bar{x})}{\sqrt{(n-1)}\sigma}\right)^2 + \left(\frac{\sqrt{n}(y_i-\bar{y})}{\sqrt{(n-1)}\sigma}\right)^2}. \tag{59}$$

Recall the definition given in (20):

$$u_i \equiv \frac{\sqrt{n}(x_i-\bar{x})}{\sqrt{(n-1)}\sigma}. \tag{60}$$

Similarly, we define:

$$v_i \equiv \frac{\sqrt{n}(y_i-\bar{y})}{\sqrt{(n-1)}\sigma}. \tag{61}$$

The $u_i$ and $v_i$ are normally distributed with zero mean and unity variance; therefore, substituting (60) and (61) into (59) allows us to write:

$$MR/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}\sqrt{\chi_i^2(2)},$$

where we have used the fact that the sum of two independent chi-square random variables with one degree of freedom is a chi-square variable with two degrees of freedom. Next, rewrite the above as:

$$MR/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}\chi_i(2). \tag{62}$$

For *n* = 2, we can write (62) as:

$$MR/\sigma = \frac{1}{\sqrt{2}}\chi(2) \quad (n = 2). \tag{63}$$

From (62), the mean of $MR/\sigma$ is:

$$\langle MR/\sigma\rangle = \frac{\sqrt{(n-1)}}{n\sqrt{n}}\sum_{i=1}^{n}\langle\chi_i(2)\rangle$$

$$= \sqrt{\frac{n-1}{n}}\langle\chi(2)\rangle$$

$$= \sqrt{\frac{2(n-1)}{n}}\,\Gamma\left(\frac{3}{2}\right)$$

$$\langle MR/\sigma\rangle = \sqrt{\frac{(n-1)\pi}{2n}}\,. \tag{64}$$

In the limit of large *n*, $\langle MR/\sigma\rangle \to \sqrt{\frac{\pi}{2}}$, and for *n* = 2, (64) becomes:

$$\langle MR/\sigma\rangle = \frac{\sqrt{\pi}}{2} \quad (n = 2), \tag{65}$$

which agrees with (48), since $MR = \frac{1}{2}ES$ for this special case. Next, we seek an analytical expression for the variance of $MR/\sigma$. Grubbs[1] has provided such an expression for the variance of $MR/\sigma$ for *n* > 2:

$$Var(MR/\sigma) = \frac{2(n-1)}{n^2}\left\{1 - \frac{n\pi}{4} + \frac{(n-1)}{2}\Psi\right\} \quad (n > 2), \qquad (66)$$

where:

$$\Psi = \left[1 - \left(\frac{1}{n-1}\right)^2\right]^2 \sum_{j=0}^{\infty} \frac{\left(\frac{1}{n-1}\right)^j 2^{j+1}\left[\Gamma\left(\frac{j+3}{2}\right)\right]^2}{j!} \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \text{ is odd} \\ \frac{(j-1)!!}{j!!} & \text{if } j \text{ is even} \end{cases}. \qquad (67)$$

For $n = 2$, the variance of $MR/\sigma$ is computed directly from (63):

$$Var(MR/\sigma) = \frac{1}{2}Var(\chi(2))$$

$$Var(MR/\sigma) = \left(1 - \frac{\pi}{4}\right) \quad (n = 2). \qquad (68)$$

**Table 15**. Table of means, standard deviations, and 5th and 95th percentiles of *MR/σ* for sample sizes of $n$ = 2-30.

| n | Mean | Std | *CV* | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|
| 2 | 0.8862 | 0.4633 | 0.5227 | 0.2265 | 1.7308 | 0.2556 | 1.9530 |
| 3 | 1.0233 | 0.3738 | 0.3653 | 0.459 | 1.683 | 0.448 | 1.645 |
| 4 | 1.0854 | 0.3243 | 0.2988 | 0.586 | 1.651 | 0.539 | 1.521 |
| 5 | 1.1210 | 0.2906 | 0.2592 | 0.669 | 1.623 | 0.597 | 1.448 |
| 6 | 1.1441 | 0.2656 | 0.2322 | 0.729 | 1.602 | 0.637 | 1.400 |
| 7 | 1.1603 | 0.2461 | 0.2121 | 0.772 | 1.583 | 0.666 | 1.364 |
| 8 | 1.1724 | 0.2304 | 0.1965 | 0.809 | 1.565 | 0.690 | 1.335 |
| 9 | 1.1816 | 0.2174 | 0.1839 | 0.839 | 1.552 | 0.710 | 1.314 |
| 10 | 1.1890 | 0.2063 | 0.1735 | 0.862 | 1.541 | 0.725 | 1.296 |
| 11 | 1.1950 | 0.1968 | 0.1647 | 0.882 | 1.531 | 0.738 | 1.281 |
| 12 | 1.2000 | 0.1885 | 0.1571 | 0.900 | 1.519 | 0.750 | 1.266 |
| 13 | 1.2041 | 0.1811 | 0.1504 | 0.916 | 1.511 | 0.761 | 1.255 |
| 14 | 1.2077 | 0.1746 | 0.1445 | 0.930 | 1.505 | 0.770 | 1.246 |
| 15 | 1.2108 | 0.1687 | 0.1393 | 0.941 | 1.496 | 0.778 | 1.235 |
| 16 | 1.2135 | 0.1633 | 0.1346 | 0.954 | 1.489 | 0.786 | 1.227 |
| 17 | 1.2159 | 0.1585 | 0.1304 | 0.962 | 1.484 | 0.791 | 1.220 |
| 18 | 1.2180 | 0.1541 | 0.1265 | 0.972 | 1.477 | 0.798 | 1.213 |
| 19 | 1.2199 | 0.1500 | 0.1229 | 0.980 | 1.472 | 0.803 | 1.207 |
| 20 | 1.2216 | 0.1462 | 0.1197 | 0.986 | 1.468 | 0.807 | 1.202 |
| 21 | 1.2231 | 0.1427 | 0.1166 | 0.994 | 1.464 | 0.812 | 1.197 |
| 22 | 1.2245 | 0.1394 | 0.1138 | 1.002 | 1.461 | 0.818 | 1.193 |
| 23 | 1.2258 | 0.1364 | 0.1112 | 1.007 | 1.455 | 0.821 | 1.187 |
| 24 | 1.2269 | 0.1335 | 0.1088 | 1.011 | 1.451 | 0.824 | 1.183 |
| 25 | 1.2280 | 0.1308 | 0.1065 | 1.018 | 1.449 | 0.829 | 1.180 |
| 26 | 1.2290 | 0.1283 | 0.1044 | 1.023 | 1.444 | 0.833 | 1.175 |
| 27 | 1.2299 | 0.1259 | 0.1024 | 1.028 | 1.442 | 0.836 | 1.173 |
| 28 | 1.2307 | 0.1236 | 0.1004 | 1.032 | 1.438 | 0.838 | 1.168 |
| 29 | 1.2315 | 0.1215 | 0.0986 | 1.036 | 1.436 | 0.841 | 1.166 |
| 30 | 1.2322 | 0.1194 | 0.0969 | 1.039 | 1.432 | 0.843 | 1.162 |

**Notes Table 15**: For more details, refer to the notes at the bottom of Table 3.

From (68), the standard deviation is:

$$Std(MR/\sigma) = \sqrt{\left(1 - \frac{\pi}{4}\right)} \quad (n = 2). \tag{69}$$

In the above, analytical formulas as a function of sample size are provided for the first two moments of $MR/\sigma$. If $n$ = 2, the probability distribution of $MR/\sigma$ has a convenient analytical form (63); however, a simple extension of this analytical result is unavailable for $n > 2$, as evident from, the fact that $MR/\sigma$ is proportional to a sum of correlated chi random variables[1] (the $u_i$ and $u_j$ pairs (see (60)) are correlated, and the $v_i$ and $v_j$ pairs (see (61)) are correlated). As such, Monte Carlo simulations were used to estimate the percentiles for $n > 2$.

In Table 15, results are presented for the means, standard deviations, and 5th and 95th percentiles of $MR/\sigma$ for sample sizes of $n$ = 2-30. Means were computed from (64), and standard deviations were computed from (69) for $n$ = 2 and from the square root of the variance (see (66) and (67)) for $n > 2$. For $n$ = 2, the percentiles were determined from those of a chi-distributed random variable (two degrees of freedom) scaled by $\frac{1}{\sqrt{2}}$ as shown in (63). However, for sample sizes $n > 2$, the percentiles are Monte Carlo estimates. A listing of percentiles of $MR/\sigma$ is found in Appendix 1-9, along with the *MCE*'s of the estimated percentiles.

### 3.2.4. Average Mean Radius, Multiple Shot Groups

For multiple, *n*-round shot groups, the Mean Radius is averaged over all groups to provide a measure of dispersion:

$$AMR/\sigma = \frac{1}{k}\sum_1^k MR_i/\sigma \tag{70}$$

where *AMR* is the Average Mean Radius, $MR_i$ is the mean radius of the *i*th *n*-round shot group, and *k* is the total number of groups. The mean of (70) is:

$$\langle AMR/\sigma \rangle = \langle MR/\sigma \rangle, \tag{71}$$

and its standard deviation is:

$$Std(AMR/\sigma) = \frac{1}{\sqrt{k}} Std(MR/\sigma), \tag{72}$$

and its *CV* is:

$$CV(AMR) = \frac{1}{\sqrt{k}} CV(MR). \tag{73}$$

In Table 16, results are presented for the means, standard deviations, and 5th and 95th percentiles of $AMR/\sigma$ for *n*-round shot groups ($n$ = 2-5, 10, 25) fired for a total of *k* groups ($k$ = 1-5, 10 for each of the listed *n*). For $n$ = 2, the listed mean equals $\sqrt{\pi}/2$ via (65), and the standard deviations were computed from (69) divided by $\sqrt{k}$. For $n > 2$, the means were computed using (64), and the standard deviations were computed from the square root of the variance (66) and dividing by $\sqrt{k}$. If $n$ = 2 with $k > 1$ or for $n > 2$, the percentiles are Monte Carlo estimates. A listing of percentiles of $AMR/\sigma$ for the various *k* and *n* is found in Appendix 1-10, along with *MCE*'s of the estimated percentiles.

---

**Example 19.** A test agency decides to shoot 3, 5-round groups and use the Average Mean Radius as their dispersion measure. What should the requirement on *AMR* be to ensure that the 95% CL of σ ($\sigma_{upper}$) is the same as in Example 16.

We require: $$AMR \leq Q,$$

where Q is the quantity that we are seeking to determine. By following the procedure outlined in Example 17:

$$Q = T_{0.05} \times \sigma_{upper}.$$

Here $T_{0.05}$ is the 5th percentile of *AMR*/σ for 3, 5-round groups. Substituting the required percentile (Table 16) and $\sigma_{upper}$ from Example 16, we find:

$$Q = 0.855 \times 0.433\ MOA = 0.370\ MOA,$$

so:

$$AMR \leq 0.370\ MOA.$$

**Table 16**. Table of means, standard deviations, and 5th and 95th percentiles of *AMR/σ* for *n*-round shot groups (*n* =2-5, 10, 25) fired for a total of *k* groups (*k* = 1-5, 10).

| *k* | *n* | *Mean* | *Std* | *CV* | *P 0.05* | *P 0.95* | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.8862 | 0.4633 | 0.5227 | 0.2265 | 1.7308 | 0.2556 | 1.9530 |
| 2 | | | 0.3276 | 0.3696 | 0.393 | 1.466 | 0.443 | 1.654 |
| 3 | | | 0.2675 | 0.3018 | 0.476 | 1.351 | 0.537 | 1.525 |
| 4 | | | 0.2316 | 0.2614 | 0.526 | 1.286 | 0.593 | 1.451 |
| 5 | | | 0.2072 | 0.2338 | 0.561 | 1.241 | 0.633 | 1.401 |
| 10 | | | 0.1465 | 0.1653 | 0.654 | 1.135 | 0.738 | 1.281 |
| 1 | 3 | 1.0233 | 0.3738 | 0.3653 | 0.459 | 1.683 | 0.448 | 1.645 |
| 2 | | | 0.2643 | 0.2583 | 0.611 | 1.479 | 0.597 | 1.446 |
| 3 | | | 0.2158 | 0.2109 | 0.684 | 1.394 | 0.669 | 1.362 |
| 4 | | | 0.1869 | 0.1826 | 0.728 | 1.343 | 0.712 | 1.313 |
| 5 | | | 0.1672 | 0.1634 | 0.757 | 1.307 | 0.740 | 1.278 |
| 10 | | | 0.1182 | 0.1155 | 0.833 | 1.222 | 0.814 | 1.194 |
| 1 | 4 | 1.0854 | 0.3243 | 0.2988 | 0.586 | 1.651 | 0.539 | 1.521 |
| 2 | | | 0.2293 | 0.2113 | 0.725 | 1.480 | 0.668 | 1.364 |
| 3 | | | 0.1873 | 0.1725 | 0.788 | 1.402 | 0.726 | 1.292 |
| 4 | | | 0.1622 | 0.1494 | 0.827 | 1.362 | 0.762 | 1.255 |
| 5 | | | 0.1450 | 0.1336 | 0.852 | 1.332 | 0.785 | 1.227 |
| 10 | | | 0.1026 | 0.0945 | 0.921 | 1.257 | 0.848 | 1.158 |
| 1 | 5 | 1.1210 | 0.2906 | 0.2592 | 0.669 | 1.623 | 0.597 | 1.448 |
| 2 | | | 0.2055 | 0.1833 | 0.797 | 1.472 | 0.711 | 1.313 |
| 3 | | | 0.1678 | 0.1497 | 0.855 | 1.404 | 0.762 | 1.253 |
| 4 | | | 0.1453 | 0.1296 | 0.889 | 1.365 | 0.793 | 1.218 |
| 5 | | | 0.1300 | 0.1159 | 0.912 | 1.340 | 0.814 | 1.195 |
| 10 | | | 0.0919 | 0.0820 | 0.973 | 1.275 | 0.868 | 1.137 |
| 1 | 10 | 1.1890 | 0.2063 | 0.1735 | 0.862 | 1.541 | 0.725 | 1.296 |
| 2 | | | 0.1459 | 0.1227 | 0.956 | 1.435 | 0.804 | 1.207 |
| 3 | | | 0.1191 | 0.1002 | 0.997 | 1.388 | 0.838 | 1.168 |
| 4 | | | 0.1031 | 0.0868 | 1.023 | 1.362 | 0.860 | 1.145 |
| 5 | | | 0.0923 | 0.0776 | 1.040 | 1.343 | 0.874 | 1.129 |
| 10 | | | 0.0652 | 0.0549 | 1.083 | 1.297 | 0.911 | 1.091 |
| 1 | 25 | 1.2280 | 0.1308 | 0.1065 | 1.018 | 1.449 | 0.829 | 1.180 |
| 2 | | | 0.0925 | 0.0753 | 1.079 | 1.382 | 0.878 | 1.126 |
| 3 | | | 0.0755 | 0.0615 | 1.105 | 1.354 | 0.900 | 1.103 |
| 4 | | | 0.0654 | 0.0533 | 1.121 | 1.336 | 0.913 | 1.088 |
| 5 | | | 0.0585 | 0.0476 | 1.132 | 1.325 | 0.922 | 1.079 |
| 10 | | | 0.0414 | 0.0337 | 1.160 | 1.296 | 0.945 | 1.056 |

**Notes Table 16**: For more details, refer to the notes at the bottom of Table 4.

**Example 20.** Suppose we shoot 3, 5-round groups and use the Average Mean Radius as our dispersion measure. What is $\sigma/\sigma^*$ for a rejection rate of 0.80 with α = 0.05? If the Average Extreme Spread is used instead, how does it compare?

Following Example 12, we seek the value of $\sigma/\sigma^*$ for which:

$$\frac{\sigma}{\sigma^*} = \frac{T_{0.95}}{T_{0.20}}.$$

Using the percentiles of $AMR/\sigma$ for 3, 5-round shot groups (Appendix 1-10):

$$\sigma/\sigma^* = 1.404/0.979 = 1.434$$

Similarly, using the percentiles of $AES/\sigma$ for 3, 5-round shot groups (Appendix 1-8):

$$\sigma/\sigma^* = 3.881/2.658 = 1.460.$$

The rejection rates are nearly the same, indicating the power curves are similar as well.

**Example 21.** If the Average Mean Radius is used as the dispersion measure, how many *k*, 5-round groups need to be fired to ensure if $\sigma/\sigma^* \geq 1.5$, then the test article is rejected at least 80% of the time?

Following Example 12, choose $\alpha = 0.05$, and we know if:

$$\sigma/\sigma^* \geq T_{0.95}/T_{0.20},$$

then the test article will be rejected at least 80% of the time. Using the percentile table for multiple, 5-round shot groups (Appendix 1-10), construct the table:

| k | P 0.95 | P 0.20 | $\sigma/\sigma^*$ |
|---|---|---|---|
| 1 | 1.623 | 0.872 | 1.862 |
| 2 | 1.472 | 0.945 | 1.557 |
| 3 | 1.404 | 0.979 | 1.434 |
| 4 | 1.365 | 0.998 | 1.369 |
| 5 | 1.340 | 1.011 | 1.326 |
| 10 | 1.275 | 1.043 | 1.222 |

Inspection of the table shows at least 3, 5-round groups need to be fired.

### 3.2.5. Radial Standard Deviation, One Shot Group

The sum of the sample variances of *x* and *y* is given by:

$$RSD^2 = \frac{1}{n}\sum_{i=1}^{n}[(x_i - \bar{x})^2 + (y_i - \bar{y})^2], \tag{74}$$

which is the average of the sum of squares of the distances of the impact points (*x*, *y*) from the observed center of impact ($\bar{x}$, $\bar{y}$). Taking the square root of (74) gives the bivariate measure of dispersion known as the Radial Standard Deviation (*RSD*):

$$RSD = \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}[(x_i - \bar{x})^2 + (y_i - \bar{y})^2]}. \tag{75}$$

Dividing (75) by $\sigma$:

$$RSD/\sigma = \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}[(x_i - \bar{x})^2/\sigma^2 + (y_i - \bar{y})^2/\sigma^2]}$$

$$RSD/\sigma = \chi(2(n-1))/\sqrt{n}, \tag{76}$$

where $\chi(2(n-1))$ is a chi random variable with $2(n-1)$ degrees of freedom.

The mean of $RSD/\sigma$ is given by:

$$\langle RSD/\sigma \rangle = \langle \chi(2(n-1)) \rangle/\sqrt{n}$$

$$\langle RSD/\sigma \rangle = \sqrt{\frac{2}{n}}\frac{\Gamma((n-1)+\frac{1}{2})}{\Gamma(n-1)}, \tag{77}$$

and its variance is:

$$Var(RSD/\sigma) = \frac{1}{n}Var(\chi(2(n-1)))$$

$$= \frac{1}{n}[\langle \chi^2(2(n-1)) \rangle - \langle \chi(2(n-1)) \rangle^2]$$

$$Var(RSD/\sigma) = \frac{1}{n}\left[2(n-1) - 2\left(\frac{\Gamma((n-1)+\frac{1}{2})}{\Gamma(n-1)}\right)^2\right]. \tag{78}$$

The above results demonstrate the advantage of using *RSD* as a dispersion measure. The remarks made here about *RSD* parallel those made about *SD* in Section 3.1.5. The probability distribution of $\sqrt{n}RSD/\sigma$ is a chi distribution and analytic formulas for the first two moments are given. All percentiles and moments were computed using the built-in functions provided by Microsoft Excel 2010.

In Table 17, results are presented for the means, standard deviations, and 5th and 95th percentiles of $RSD/\sigma$ for sample sizes of *n* = 2-30. The means and standard deviations were computed using (77) and the square root of the variance (78), respectively. The percentiles were determined from those of a chi-distributed random variable with $2(n-1)$ degrees of freedom) scaled by $1/\sqrt{n}$ as shown in (76). A listing of percentiles for $RSD/\sigma$ is found in Appendix 1-11.

**Table 17**. Table of means, standard deviations, and 5th and 95th percentiles for *RSD/σ* for sample sizes *n* = 2-30.

| n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% CL) |
|---|---|---|---|---|---|---|---|
| 2 | 0.8862 | 0.4633 | 0.5227 | 0.2265 | 1.7308 | 0.2556 | 1.9530 |
| 3 | 1.0854 | 0.3940 | 0.3630 | 0.4867 | 1.7784 | 0.4484 | 1.6384 |
| 4 | 1.1750 | 0.3456 | 0.2941 | 0.6394 | 1.7742 | 0.5442 | 1.5100 |
| 5 | 1.2261 | 0.3110 | 0.2536 | 0.7393 | 1.7611 | 0.6030 | 1.4363 |
| 6 | 1.2592 | 0.2849 | 0.2262 | 0.8104 | 1.7468 | 0.6436 | 1.3872 |
| 7 | 1.2823 | 0.2644 | 0.2061 | 0.8640 | 1.7331 | 0.6738 | 1.3515 |
| 8 | 1.2995 | 0.2477 | 0.1906 | 0.9063 | 1.7206 | 0.6974 | 1.3241 |
| 9 | 1.3127 | 0.2338 | 0.1781 | 0.9405 | 1.7093 | 0.7165 | 1.3022 |
| 10 | 1.3231 | 0.2220 | 0.1678 | 0.9690 | 1.6991 | 0.7324 | 1.2841 |
| 11 | 1.3317 | 0.2118 | 0.1591 | 0.9932 | 1.6898 | 0.7458 | 1.2690 |
| 12 | 1.3387 | 0.2029 | 0.1516 | 1.0140 | 1.6814 | 0.7574 | 1.2560 |
| 13 | 1.3447 | 0.1951 | 0.1451 | 1.0321 | 1.6737 | 0.7676 | 1.2447 |
| 14 | 1.3497 | 0.1881 | 0.1393 | 1.0481 | 1.6666 | 0.7765 | 1.2348 |
| 15 | 1.3541 | 0.1817 | 0.1342 | 1.0623 | 1.6601 | 0.7845 | 1.2259 |
| 16 | 1.3579 | 0.1760 | 0.1296 | 1.0751 | 1.6540 | 0.7917 | 1.2180 |
| 17 | 1.3613 | 0.1708 | 0.1255 | 1.0866 | 1.6484 | 0.7982 | 1.2109 |
| 18 | 1.3643 | 0.1660 | 0.1217 | 1.0971 | 1.6432 | 0.8041 | 1.2044 |
| 19 | 1.3670 | 0.1616 | 0.1183 | 1.1066 | 1.6383 | 0.8096 | 1.1985 |
| 20 | 1.3694 | 0.1576 | 0.1151 | 1.1154 | 1.6338 | 0.8146 | 1.1931 |
| 21 | 1.3715 | 0.1538 | 0.1121 | 1.1235 | 1.6295 | 0.8192 | 1.1881 |
| 22 | 1.3735 | 0.1503 | 0.1094 | 1.1311 | 1.6254 | 0.8235 | 1.1834 |
| 23 | 1.3753 | 0.1470 | 0.1069 | 1.1380 | 1.6216 | 0.8275 | 1.1791 |
| 24 | 1.3769 | 0.1439 | 0.1045 | 1.1445 | 1.6180 | 0.8312 | 1.1751 |
| 25 | 1.3784 | 0.1410 | 0.1023 | 1.1506 | 1.6146 | 0.8347 | 1.1713 |
| 26 | 1.3798 | 0.1383 | 0.1002 | 1.1563 | 1.6113 | 0.8380 | 1.1678 |
| 27 | 1.3811 | 0.1358 | 0.0983 | 1.1617 | 1.6082 | 0.8411 | 1.1644 |
| 28 | 1.3823 | 0.1333 | 0.0964 | 1.1667 | 1.6053 | 0.8441 | 1.1613 |
| 29 | 1.3834 | 0.1310 | 0.0947 | 1.1715 | 1.6025 | 0.8468 | 1.1583 |
| 30 | 1.3845 | 0.1288 | 0.0930 | 1.1760 | 1.5998 | 0.8495 | 1.1555 |

**Notes Table 17**: For more details, refer to the notes at the bottom of Table 3.

### 3.2.6. Radial Standard Deviation, Multiple Shot Groups

Following a similar development as given in Section 3.1.6, a measurement of dispersion for *k*, *n*-round shot groups is constructed from the Root Mean Square (*RMS*) of the *RSD*'s of the *k* groups:

$$RMS\text{-}RSD = \sqrt{\frac{1}{k}\sum_{i=1}^{k} RSD_i^2}. \qquad (79)$$

Dividing (79) by σ, and making use of (76):

$$RMS\text{-}RSD/\sigma = \frac{1}{\sqrt{kn}}\sqrt{\sum_{i=1}^{k}\chi_i^2(2(n-1))}. \qquad (80)$$

Note that the sum of *k*, chi-square independent variables each with 2(*n*-1) degrees of freedom is a chi-square variable with 2*k*(*n*-1) degrees of freedom; therefore:

$$RMS\text{-}RSD/\sigma = \frac{1}{\sqrt{kn}}\sqrt{\chi^2(2k(n-1))}$$

$$RMS\text{-}RSD/\sigma = \frac{1}{\sqrt{kn}}\chi(2k(n-1)). \qquad (81)$$

Continuing, the mean of $RMS\text{-}RSD/\sigma$ is given by:

$$\langle RMS\text{-}RSD/\sigma\rangle = \frac{1}{\sqrt{kn}}\langle\chi(2k(n-1))\rangle$$

$$\langle RMS\text{-}RSD/\sigma\rangle = \sqrt{\frac{2}{kn}}\frac{\Gamma(k(n-1)+\frac{1}{2})}{\Gamma(k(n-1))}, \qquad (82)$$

and its variance by:

$$Var(RMS\text{-}RSD/\sigma) = \frac{1}{kn}Var(\chi(2k(n-1)))$$

$$Var(RMS\text{-}RSD/\sigma) = \frac{1}{kn}\left\{2k(n-1) - 2\left(\frac{\Gamma(k(n-1)+\frac{1}{2})}{\Gamma(k(n-1))}\right)^2\right\}, \qquad (83)$$

and its *CV* is computed from:

$$CV(RMS\text{-}RSD) = \frac{\sqrt{2k(n-1)-\langle\chi(2k(n-1))\rangle^2}}{\langle\chi(2k(n-1))\rangle}. \qquad (84)$$

A comparison of the *CV*'s given by (84) with that given for a group average over *RSD*'s would show that the *CV* of *RMS-RSD* is slightly smaller, which means that *RMS-RSD* is a slightly more efficient estimator of σ.

In Table 18, results are presented for the means, standard deviations, and 5th and 95th percentiles of $RMS\text{-}RSD/\sigma$ for *n*-round shot groups (*n* = 2-5, 10, 25) fired for a total of *k* groups (*k* = 1-5, 10 for each of the listed *n*). The means are computed from (82), and the standard deviations from the square root of the variance (83), and the percentiles are computed from a chi random variable (2*k*(*n* - 1) degrees of freedom) scaled by $1/\sqrt{kn}$ in accordance with (81). A listing of additional percentiles of $RMS\text{-}RSD/\sigma$ for the various *n* and *k* is found in Appendix 1-12.

**Table 18**. Table of means, standard deviations, and 5th and 95th percentiles of *RMS-RSD/σ* for *n*-round shot groups ($n$ = 2-5, 10, 25) fired for a total of $k$ groups ($k$ = 1-5, 10).

| k | n | Mean | Std | CV | P 0.05 | P 0.95 | $\hat{\sigma}/\sigma$ (5% CL) | $\hat{\sigma}/\sigma$ (95% |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.8862 | 0.4633 | 0.5227 | 0.2265 | 1.7308 | 0.2556 | 1.9530 |
| 2 | | 0.9400 | 0.3412 | 0.3630 | 0.4215 | 1.5401 | 0.4484 | 1.6384 |
| 3 | | 0.9594 | 0.2822 | 0.2941 | 0.5221 | 1.4487 | 0.5442 | 1.5100 |
| 4 | | 0.9693 | 0.2458 | 0.2536 | 0.5844 | 1.3923 | 0.6030 | 1.4363 |
| 5 | | 0.9754 | 0.2207 | 0.2262 | 0.6277 | 1.3530 | 0.6436 | 1.3872 |
| 10 | | 0.9876 | 0.1571 | 0.1591 | 0.7366 | 1.2532 | 0.7458 | 1.2690 |
| 1 | 3 | 1.0854 | 0.3940 | 0.3630 | 0.4867 | 1.7784 | 0.4484 | 1.6384 |
| 2 | | 1.1193 | 0.2839 | 0.2536 | 0.6749 | 1.6077 | 0.6030 | 1.4363 |
| 3 | | 1.1309 | 0.2331 | 0.2061 | 0.7620 | 1.5285 | 0.6738 | 1.3515 |
| 4 | | 1.1368 | 0.2025 | 0.1781 | 0.8145 | 1.4803 | 0.7165 | 1.3022 |
| 5 | | 1.1404 | 0.1814 | 0.1591 | 0.8505 | 1.4471 | 0.7458 | 1.2690 |
| 10 | | 1.1475 | 0.1287 | 0.1121 | 0.9400 | 1.3633 | 0.8192 | 1.1881 |
| 1 | 4 | 1.1750 | 0.3456 | 0.2941 | 0.6394 | 1.7742 | 0.5442 | 1.5100 |
| 2 | | 1.1995 | 0.2473 | 0.2061 | 0.8082 | 1.6212 | 0.6738 | 1.3515 |
| 3 | | 1.2079 | 0.2027 | 0.1678 | 0.8846 | 1.5511 | 0.7324 | 1.2841 |
| 4 | | 1.2121 | 0.1758 | 0.1451 | 0.9303 | 1.5086 | 0.7676 | 1.2447 |
| 5 | | 1.2146 | 0.1574 | 0.1296 | 0.9616 | 1.4794 | 0.7917 | 1.2180 |
| 10 | | 1.2197 | 0.1116 | 0.0915 | 1.0391 | 1.4061 | 0.8520 | 1.1528 |
| 1 | 5 | 1.2261 | 0.3110 | 0.2536 | 0.7393 | 1.7611 | 0.6030 | 1.4363 |
| 2 | | 1.2453 | 0.2218 | 0.1781 | 0.8923 | 1.6216 | 0.7165 | 1.3022 |
| 3 | | 1.2518 | 0.1816 | 0.1451 | 0.9608 | 1.5581 | 0.7676 | 1.2447 |
| 4 | | 1.2551 | 0.1575 | 0.1255 | 1.0018 | 1.5198 | 0.7982 | 1.2109 |
| 5 | | 1.2570 | 0.1410 | 0.1121 | 1.0297 | 1.4934 | 0.8192 | 1.1881 |
| 10 | | 1.2610 | 0.0998 | 0.0792 | 1.0990 | 1.4274 | 0.8716 | 1.1320 |
| 1 | 10 | 1.3231 | 0.2220 | 0.1678 | 0.9690 | 1.6991 | 0.7324 | 1.2841 |
| 2 | | 1.3324 | 0.1576 | 0.1183 | 1.0786 | 1.5968 | 0.8096 | 1.1985 |
| 3 | | 1.3354 | 0.1288 | 0.0964 | 1.1272 | 1.5508 | 0.8441 | 1.1613 |
| 4 | | 1.3370 | 0.1116 | 0.0835 | 1.1561 | 1.5232 | 0.8647 | 1.1393 |
| 5 | | 1.3379 | 0.0999 | 0.0746 | 1.1758 | 1.5043 | 0.8788 | 1.1244 |
| 10 | | 1.3398 | 0.0707 | 0.0527 | 1.2246 | 1.4571 | 0.9140 | 1.0875 |
| 1 | 25 | 1.3784 | 0.1410 | 0.1023 | 1.1506 | 1.6146 | 0.8347 | 1.1713 |
| 2 | | 1.3820 | 0.0999 | 0.0723 | 1.2198 | 1.5484 | 0.8826 | 1.1203 |
| 3 | | 1.3832 | 0.0816 | 0.0590 | 1.2504 | 1.5188 | 0.9040 | 1.0980 |
| 4 | | 1.3838 | 0.0707 | 0.0511 | 1.2686 | 1.5011 | 0.9168 | 1.0847 |
| 5 | | 1.3842 | 0.0632 | 0.0457 | 1.2810 | 1.4890 | 0.9255 | 1.0757 |
| 10 | | 1.3849 | 0.0447 | 0.0323 | 1.3118 | 1.4589 | 0.9472 | 1.0534 |

**Notes Table 18**: For more details, refer to the notes at the bottom of Table 4.

**Example 22.** If the Average Extreme Spread is used as the dispersion measure, how many *k*, 5-round shot groups need to be fired to get at least the same *CV* of *RMS-RSD* for 10, 5-round shot groups?

Equating *CV*'s gives:

$$CV\big(AES(k,5)\big) = CV(RMS\text{-}RSD(10{,}5))$$

$$CV\big(ES(5)\big)/\sqrt{k} = CV(RMS\text{-}RSD(10{,}5))$$

(See Example 5 for an explanation of the notation.) Solving for *k* and substituting for the *CV*'s (Tables 11 and 18):

$$k = (0.270/0.0792)^2 = 11.6,$$

which shows at least 12, 5-round groups need to be fired at a cost of expending 20% more rounds.

**Example 23.** Compare the power curves (α = 0.05) of *RMS-SD* and *RMS-RSD* for 3, 5-round shot groups.

Following the procedure outlined in Example 10, the power curves of *RMS-SD* and *RMS-RSD* are generated (Figure 6). The value of $\sigma/\sigma^*$ at a rejection rate of 0.80 is 13.5% smaller for *RMS-RSD*, which reflects the improved power of an acceptance test for which the bivariate measure can be used in lieu of the univariate measure.

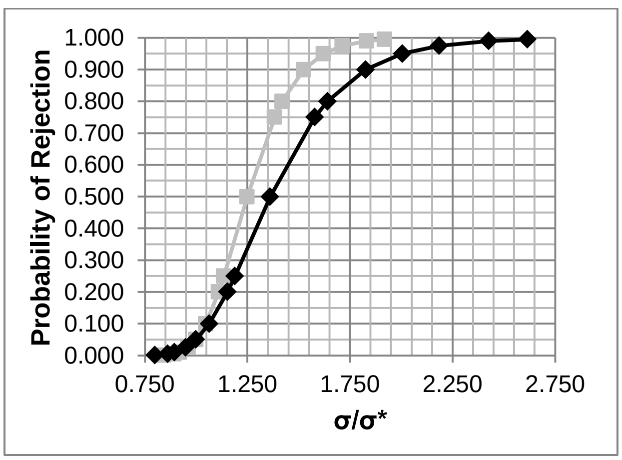


**Figure 6**. A comparison between the power curves (α = 0.05) of *RMS-SD* (black diamonds) and *RMS-RSD* (gray squares) for 3, 5-round groups.

## 4. Conclusions

In this report, we have examined the univariate dispersion measures (Section 3.1): Range (*R*), Mean Deviation (*MD*), and Standard Deviation (*SD*); and bivariate measures (Section 3.2): Extreme Spread (*ES*), Mean Radius (*MR*), and Radial Standard Deviation (*RSD*). First, an analysis of each dispersion measure was given as applied to one, *n*-round shot group (*n* = 2-30). Percentiles and the first two moments for these measures were computed in units on the population standard deviation, σ. The probability distributions of $\sqrt{n}SD/\sigma$ and $\sqrt{n}RSD/\sigma$ are chi random variables and their percentiles are easily computed by taking the square root of the percentiles from the corresponding chi-square distribution. Conversely, for the other dispersion measures, Monte Carlo simulations were used to estimate percentiles (except for the case with $n = 2$ as discussed in the text).

Next, we extended our analysis to treat various $k$ multiples of $n$-round shot groups, where the multiples are independent, repeat firings of $n$-round groups from the same population. We examined the *RMS* of the *SD*'s and similarly for the *RSD*'s. These dispersion measures were denoted as *RMS-SD* and *RMS-RSD*, respectively. We note that $\sqrt{kn}$ *RMS-SD*/σ and $\sqrt{kn}$ *RMS-RSD*/σ are chi distributed, and their percentiles are straightforward to compute from the chi-square distribution. We also examined the Average Range (*AR*), Average Mean Deviation (*AMD*), Average Extreme Spread (*AES*) and the Average Mean Radius (*AMR*). These dispersion measures were constructed by averaging the dispersion measure of the $n$-round group over all $k$ groups. For these cases, more effort was required to compute percentiles, and Monte Carlo simulations were used to estimate the percentiles of the dispersion measures given by averages over groups. From the central limit theorem, the probability distribution of the average over a dispersion measure is asymptotically normal as the number of groups averaged over increases. The convergence to a normal distribution is rapid, and for many applications using a group average to measure dispersion, the normal-distributed approximation will be sufficient. Our Monte Carlo estimates of percentiles can be used to determine the error in using this approximation to justify that it is within one's error budget (see remarks in Section 3.1.2). Inclusion of percentile tables for dispersion measures treating averages over multiple groups is instructive, but the predominant utility in these results is for cases involving a few groups with a small sample size per group.

Finally, we provide some thoughts on choosing a dispersion measure. A metric for selecting a dispersion measure is the Coefficient of Variation, since measures with smaller *CV* provide a more precise estimate of σ. It is surprising that *SD* and *RSD* are generally not used to measure dispersion in shot groups (especially considering the ease with which the percentiles of *RMS-SD* and *RMS-RSD* can be computed). For $n > 2$, both *SD* and *RSD* have the best performance in terms of Coefficient of Variation[1]—keep in mind we are comparing *SD* with other univariate measures with the same sample size and accordingly, *RSD* with other bivariate measures. When measuring the dispersion on paper targets, one cannot match the simplicity of the univariate Range or *ES*; for example, the *ES* is easily obtained by measuring the largest distance between two shots. However, with the advent of electronic targeting systems, computation of any of the dispersion measures poses no difficulty. The danger in using the Range or *ES* is failing to reject a bad round, an outlier which contributes to the computation of Range or *ES*. In this regard, the other dispersion measures are more forgiving. In addition, Grubbs suggests using the Relative Efficiency[1] as a metric to compare two dispersion measures, where the Relative Efficiency (*RE*) is defined as the square of the ratio of Coefficients of Variation, where the ratio is formed by the smaller *CV* divided by the larger *CV* so that the $RE \leq 1$. In Example 5, we have already encountered this ratio and put it to use. Considerations of *CV* and *RE* are useful tools for justifying the choice of dispersion measure. Regardless of which dispersion measure is selected, the aim of the measurement is to obtain an estimate of the unknown, population standard deviation, σ to within the precision required.

In closing, we hope the results presented in this report will prove useful to those developing requirements or tests on firearms precision. It is with this purpose that this report was prepared.

## Appendix 1. Percentile Tables

All percentiles are reported in units of the population standard deviation, σ. Percentiles reported with 3 or less decimal places are Monte Carlo estimates. Monte Carlo simulations were repeated for a total of 9 runs, and the reported percentiles are the sample average over the 9 runs. The *MCE*'s were computed for each estimated percentile, and *MCE*'s are reported in accompanying tables. The standard error of each estimated percentile is one third the *MCE*, since 9 runs were averaged over. Consult Section 2 for additional details.

## Appendix 1-1. Range, One Shot Group

**Table A1-1.1**. Percentiles of the Range (*R*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|
| 2 | 4.6535 | 3.9697 | 3.6428 | 3.1698 | 2.7718 | 2.3262 | 1.8124 | 1.6268 |
| 3 | 5.08 | 4.44 | 4.12 | 3.685 | 3.317 | 2.907 | 2.427 | 2.252 |
| 4 | 5.30 | 4.69 | 4.39 | 3.976 | 3.625 | 3.238 | 2.783 | 2.620 |
| 5 | 5.47 | 4.86 | 4.59 | 4.190 | 3.852 | 3.473 | 3.039 | 2.873 |
| 6 | 5.61 | 5.04 | 4.76 | 4.367 | 4.032 | 3.658 | 3.231 | 3.074 |
| 7 | 5.71 | 5.15 | 4.89 | 4.496 | 4.172 | 3.808 | 3.387 | 3.234 |
| 8 | 5.82 | 5.25 | 4.985 | 4.605 | 4.286 | 3.930 | 3.519 | 3.368 |
| 9 | 5.91 | 5.34 | 5.07 | 4.698 | 4.384 | 4.038 | 3.636 | 3.486 |
| 10 | 5.96 | 5.41 | 5.16 | 4.785 | 4.475 | 4.131 | 3.731 | 3.584 |
| 11 | 6.03 | 5.475 | 5.229 | 4.861 | 4.554 | 4.213 | 3.822 | 3.676 |
| 12 | 6.12 | 5.56 | 5.30 | 4.934 | 4.624 | 4.287 | 3.896 | 3.750 |
| 13 | 6.16 | 5.59 | 5.350 | 4.989 | 4.686 | 4.349 | 3.967 | 3.827 |
| 14 | 6.22 | 5.64 | 5.397 | 5.037 | 4.742 | 4.414 | 4.029 | 3.891 |
| 15 | 6.21 | 5.70 | 5.45 | 5.094 | 4.795 | 4.466 | 4.090 | 3.952 |
| 16 | 6.24 | 5.72 | 5.48 | 5.137 | 4.845 | 4.523 | 4.146 | 4.009 |
| 17 | 6.36 | 5.80 | 5.56 | 5.194 | 4.898 | 4.570 | 4.194 | 4.058 |
| 18 | 6.29 | 5.81 | 5.57 | 5.229 | 4.939 | 4.617 | 4.243 | 4.109 |
| 19 | 6.36 | 5.87 | 5.618 | 5.264 | 4.971 | 4.651 | 4.285 | 4.150 |
| 20 | 6.36 | 5.88 | 5.64 | 5.300 | 5.015 | 4.695 | 4.329 | 4.195 |
| 21 | 6.44 | 5.92 | 5.688 | 5.344 | 5.048 | 4.732 | 4.367 | 4.234 |
| 22 | 6.45 | 5.95 | 5.71 | 5.375 | 5.088 | 4.771 | 4.410 | 4.278 |
| 23 | 6.50 | 5.98 | 5.729 | 5.384 | 5.107 | 4.801 | 4.444 | 4.314 |
| 24 | 6.51 | 6.00 | 5.76 | 5.421 | 5.143 | 4.833 | 4.475 | 4.343 |
| 25 | 6.57 | 6.03 | 5.79 | 5.457 | 5.179 | 4.865 | 4.506 | 4.377 |
| 26 | 6.57 | 6.07 | 5.82 | 5.479 | 5.202 | 4.893 | 4.541 | 4.409 |
| 27 | 6.59 | 6.086 | 5.852 | 5.509 | 5.228 | 4.923 | 4.569 | 4.438 |
| 28 | 6.65 | 6.13 | 5.88 | 5.532 | 5.254 | 4.947 | 4.595 | 4.465 |
| 29 | 6.58 | 6.13 | 5.89 | 5.555 | 5.275 | 4.969 | 4.622 | 4.493 |
| 30 | 6.62 | 6.13 | 5.911 | 5.569 | 5.300 | 4.994 | 4.648 | 4.520 |

**Table A1-1.1.** Percentiles of the Range (*R*). (CONTINUED)

| n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.9539 | 0.4506 | 0.3583 | 0.1777 | 0.0887 | 0.0443 | 0.0177 | 0.0089 |
| 3 | 1.590 | 1.026 | 0.902 | 0.619 | 0.431 | 0.306 | 0.192 | 0.134 |
| 4 | 1.980 | 1.416 | 1.285 | 0.980 | 0.761 | 0.598 | 0.434 | 0.342 |
| 5 | 2.257 | 1.699 | 1.572 | 1.261 | 1.032 | 0.849 | 0.667 | 0.560 |
| 6 | 2.473 | 1.924 | 1.798 | 1.487 | 1.253 | 1.064 | 0.873 | 0.751 |
| 7 | 2.643 | 2.109 | 1.985 | 1.676 | 1.443 | 1.254 | 1.054 | 0.927 |
| 8 | 2.791 | 2.264 | 2.143 | 1.840 | 1.602 | 1.415 | 1.218 | 1.090 |
| 9 | 2.914 | 2.397 | 2.276 | 1.975 | 1.739 | 1.550 | 1.344 | 1.214 |
| 10 | 3.026 | 2.517 | 2.399 | 2.095 | 1.866 | 1.674 | 1.467 | 1.334 |
| 11 | 3.125 | 2.618 | 2.501 | 2.204 | 1.975 | 1.784 | 1.577 | 1.443 |
| 12 | 3.208 | 2.711 | 2.593 | 2.298 | 2.069 | 1.885 | 1.673 | 1.540 |
| 13 | 3.283 | 2.787 | 2.673 | 2.387 | 2.164 | 1.976 | 1.767 | 1.630 |
| 14 | 3.358 | 2.868 | 2.754 | 2.468 | 2.241 | 2.061 | 1.862 | 1.72 |
| 15 | 3.424 | 2.942 | 2.829 | 2.543 | 2.323 | 2.137 | 1.931 | 1.796 |
| 16 | 3.479 | 3.003 | 2.891 | 2.609 | 2.386 | 2.201 | 2.007 | 1.876 |
| 17 | 3.539 | 3.064 | 2.951 | 2.669 | 2.450 | 2.273 | 2.074 | 1.947 |
| 18 | 3.593 | 3.119 | 3.007 | 2.733 | 2.517 | 2.334 | 2.138 | 2.019 |
| 19 | 3.639 | 3.174 | 3.063 | 2.786 | 2.571 | 2.398 | 2.202 | 2.069 |
| 20 | 3.686 | 3.221 | 3.111 | 2.836 | 2.622 | 2.444 | 2.245 | 2.130 |
| 21 | 3.727 | 3.264 | 3.155 | 2.882 | 2.669 | 2.490 | 2.296 | 2.173 |
| 22 | 3.773 | 3.315 | 3.207 | 2.936 | 2.724 | 2.547 | 2.356 | 2.221 |
| 23 | 3.812 | 3.355 | 3.246 | 2.980 | 2.769 | 2.595 | 2.401 | 2.271 |
| 24 | 3.847 | 3.394 | 3.289 | 3.024 | 2.813 | 2.644 | 2.452 | 2.325 |
| 25 | 3.878 | 3.427 | 3.321 | 3.059 | 2.853 | 2.682 | 2.495 | 2.369 |
| 26 | 3.914 | 3.465 | 3.362 | 3.096 | 2.887 | 2.717 | 2.530 | 2.405 |
| 27 | 3.952 | 3.506 | 3.401 | 3.138 | 2.932 | 2.763 | 2.573 | 2.453 |
| 28 | 3.975 | 3.534 | 3.430 | 3.169 | 2.964 | 2.798 | 2.609 | 2.488 |
| 29 | 4.010 | 3.570 | 3.466 | 3.205 | 3.002 | 2.833 | 2.650 | 2.524 |
| 30 | 4.037 | 3.599 | 3.496 | 3.236 | 3.035 | 2.868 | 2.685 | 2.564 |

**Table A1-1.2**. Monte Carlo Errors of the Percentiles of the Range ($R$).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | | | | | |
| 3 | 0.07 | 0.04 | 0.03 | 0.017 | 0.015 | 0.013 | 0.005 | 0.006 | 0.005 | 0.007 | 0.007 | 0.006 | 0.008 | 0.007 | 0.007 | 0.010 |
| 4 | 0.12 | 0.05 | 0.03 | 0.018 | 0.019 | 0.018 | 0.011 | 0.010 | 0.007 | 0.006 | 0.005 | 0.007 | 0.007 | 0.006 | 0.012 | 0.013 |
| 5 | 0.10 | 0.04 | 0.03 | 0.012 | 0.009 | 0.016 | 0.013 | 0.012 | 0.009 | 0.008 | 0.008 | 0.009 | 0.010 | 0.015 | 0.02 | 0.018 |
| 6 | 0.07 | 0.05 | 0.04 | 0.02 | 0.013 | 0.011 | 0.008 | 0.006 | 0.006 | 0.005 | 0.006 | 0.008 | 0.011 | 0.013 | 0.014 | 0.016 |
| 7 | 0.04 | 0.03 | 0.03 | 0.02 | 0.017 | 0.014 | 0.010 | 0.008 | 0.009 | 0.007 | 0.008 | 0.008 | 0.010 | 0.013 | 0.016 | 0.018 |
| 8 | 0.08 | 0.04 | 0.02 | 0.02 | 0.009 | 0.008 | 0.006 | 0.006 | 0.008 | 0.010 | 0.010 | 0.013 | 0.012 | 0.015 | 0.013 | 0.015 |
| 9 | 0.10 | 0.04 | 0.03 | 0.02 | 0.013 | 0.011 | 0.012 | 0.010 | 0.009 | 0.008 | 0.009 | 0.008 | 0.009 | 0.012 | 0.019 | 0.02 |
| 10 | 0.05 | 0.04 | 0.03 | 0.011 | 0.009 | 0.008 | 0.007 | 0.008 | 0.006 | 0.006 | 0.008 | 0.014 | 0.014 | 0.013 | 0.015 | 0.02 |
| 11 | 0.07 | 0.02 | 0.02 | 0.018 | 0.014 | 0.010 | 0.006 | 0.006 | 0.006 | 0.007 | 0.004 | 0.007 | 0.006 | 0.010 | 0.013 | 0.017 |
| 12 | 0.08 | 0.04 | 0.03 | 0.019 | 0.017 | 0.012 | 0.011 | 0.012 | 0.003 | 0.007 | 0.005 | 0.010 | 0.013 | 0.014 | 0.016 | 0.017 |
| 13 | 0.08 | 0.04 | 0.02 | 0.012 | 0.012 | 0.008 | 0.011 | 0.010 | 0.007 | 0.005 | 0.005 | 0.004 | 0.006 | 0.007 | 0.011 | 0.009 |
| 14 | 0.05 | 0.04 | 0.02 | 0.02 | 0.02 | 0.009 | 0.008 | 0.008 | 0.003 | 0.005 | 0.006 | 0.010 | 0.008 | 0.010 | 0.02 | 0.03 |
| 15 | 0.06 | 0.04 | 0.03 | 0.017 | 0.014 | 0.010 | 0.008 | 0.006 | 0.008 | 0.006 | 0.007 | 0.006 | 0.008 | 0.016 | 0.010 | 0.019 |
| 16 | 0.04 | 0.03 | 0.04 | 0.02 | 0.015 | 0.010 | 0.008 | 0.007 | 0.008 | 0.007 | 0.008 | 0.008 | 0.008 | 0.014 | 0.019 | 0.013 |
| 17 | 0.09 | 0.04 | 0.03 | 0.018 | 0.013 | 0.010 | 0.007 | 0.006 | 0.006 | 0.007 | 0.009 | 0.013 | 0.012 | 0.013 | 0.016 | 0.015 |
| 18 | 0.06 | 0.04 | 0.03 | 0.02 | 0.014 | 0.009 | 0.008 | 0.009 | 0.005 | 0.009 | 0.010 | 0.011 | 0.014 | 0.013 | 0.017 | 0.019 |
| 19 | 0.06 | 0.04 | 0.02 | 0.018 | 0.016 | 0.010 | 0.007 | 0.009 | 0.008 | 0.007 | 0.006 | 0.006 | 0.008 | 0.013 | 0.014 | 0.02 |
| 20 | 0.05 | 0.05 | 0.03 | 0.015 | 0.016 | 0.012 | 0.006 | 0.008 | 0.007 | 0.005 | 0.006 | 0.007 | 0.011 | 0.010 | 0.017 | 0.016 |
| 21 | 0.06 | 0.03 | 0.02 | 0.010 | 0.012 | 0.012 | 0.008 | 0.006 | 0.005 | 0.005 | 0.003 | 0.003 | 0.006 | 0.006 | 0.013 | 0.02 |
| 22 | 0.08 | 0.03 | 0.03 | 0.008 | 0.008 | 0.010 | 0.006 | 0.005 | 0.006 | 0.004 | 0.005 | 0.008 | 0.008 | 0.009 | 0.015 | 0.02 |
| 23 | 0.05 | 0.04 | 0.02 | 0.02 | 0.010 | 0.008 | 0.005 | 0.006 | 0.005 | 0.005 | 0.005 | 0.007 | 0.008 | 0.007 | 0.009 | 0.015 |
| 24 | 0.04 | 0.03 | 0.03 | 0.017 | 0.013 | 0.012 | 0.010 | 0.009 | 0.007 | 0.007 | 0.007 | 0.006 | 0.002 | 0.008 | 0.009 | 0.017 |
| 25 | 0.08 | 0.04 | 0.04 | 0.02 | 0.007 | 0.009 | 0.009 | 0.009 | 0.007 | 0.004 | 0.006 | 0.005 | 0.009 | 0.011 | 0.015 | 0.013 |
| 26 | 0.06 | 0.04 | 0.03 | 0.018 | 0.012 | 0.008 | 0.009 | 0.008 | 0.007 | 0.005 | 0.006 | 0.005 | 0.008 | 0.009 | 0.015 | 0.018 |
| 27 | 0.05 | 0.02 | 0.02 | 0.011 | 0.007 | 0.008 | 0.005 | 0.005 | 0.005 | 0.008 | 0.007 | 0.009 | 0.009 | 0.009 | 0.011 | 0.014 |
| 28 | 0.08 | 0.04 | 0.03 | 0.014 | 0.012 | 0.008 | 0.007 | 0.006 | 0.005 | 0.006 | 0.007 | 0.007 | 0.008 | 0.013 | 0.016 | 0.02 |
| 29 | 0.07 | 0.03 | 0.04 | 0.02 | 0.014 | 0.009 | 0.006 | 0.007 | 0.006 | 0.006 | 0.006 | 0.006 | 0.004 | 0.010 | 0.010 | 0.018 |
| 30 | 0.06 | 0.03 | 0.02 | 0.012 | 0.012 | 0.009 | 0.008 | 0.007 | 0.006 | 0.005 | 0.006 | 0.006 | 0.007 | 0.009 | 0.012 | 0.014 |

## Appendix 1-2.  Average Range, Multiple Shot Groups

**Table A1-2.1**.  Percentiles of the Average Range (*AR*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4.6535 | 3.9697 | 3.6428 | 3.1698 | 2.7718 | 2.3262 | 1.8124 | 1.6268 |
| 2 | | 3.44 | 3.017 | 2.802 | 2.499 | 2.239 | 1.947 | 1.615 | 1.495 |
| 3 | | 2.97 | 2.622 | 2.457 | 2.221 | 2.015 | 1.791 | 1.531 | 1.436 |
| 4 | | 2.70 | 2.408 | 2.264 | 2.061 | 1.892 | 1.697 | 1.477 | 1.395 |
| 5 | | 2.51 | 2.253 | 2.132 | 1.950 | 1.804 | 1.636 | 1.444 | 1.372 |
| 10 | | 2.070 | 1.899 | 1.815 | 1.695 | 1.595 | 1.483 | 1.352 | 1.303 |
| 1 | 3 | 5.08 | 4.44 | 4.12 | 3.685 | 3.317 | 2.907 | 2.427 | 2.252 |
| 2 | | 3.96 | 3.539 | 3.343 | 3.048 | 2.809 | 2.534 | 2.215 | 2.096 |
| 3 | | 3.49 | 3.159 | 3.008 | 2.779 | 2.586 | 2.370 | 2.116 | 2.021 |
| 4 | | 3.26 | 2.958 | 2.826 | 2.628 | 2.464 | 2.276 | 2.061 | 1.979 |
| 5 | | 3.07 | 2.804 | 2.690 | 2.515 | 2.374 | 2.212 | 2.020 | 1.949 |
| 10 | | 2.639 | 2.470 | 2.388 | 2.270 | 2.171 | 2.059 | 1.927 | 1.877 |
| 1 | 4 | 5.30 | 4.69 | 4.39 | 3.976 | 3.625 | 3.238 | 2.783 | 2.620 |
| 2 | | 4.27 | 3.859 | 3.66 | 3.382 | 3.146 | 2.882 | 2.573 | 2.460 |
| 3 | | 3.82 | 3.501 | 3.344 | 3.123 | 2.939 | 2.725 | 2.479 | 2.389 |
| 4 | | 3.55 | 3.285 | 3.154 | 2.968 | 2.809 | 2.633 | 2.425 | 2.346 |
| 5 | | 3.39 | 3.145 | 3.032 | 2.871 | 2.734 | 2.574 | 2.386 | 2.317 |
| 10 | | 2.975 | 2.815 | 2.738 | 2.624 | 2.529 | 2.419 | 2.291 | 2.242 |
| 1 | 5 | 5.47 | 4.86 | 4.59 | 4.190 | 3.852 | 3.473 | 3.039 | 2.873 |
| 2 | | 4.46 | 4.079 | 3.889 | 3.615 | 3.384 | 3.132 | 2.833 | 2.724 |
| 3 | | 4.04 | 3.719 | 3.575 | 3.361 | 3.184 | 2.979 | 2.741 | 2.650 |
| 4 | | 3.79 | 3.53 | 3.403 | 3.223 | 3.069 | 2.893 | 2.685 | 2.609 |
| 5 | | 3.63 | 3.390 | 3.280 | 3.119 | 2.982 | 2.828 | 2.647 | 2.579 |
| 10 | | 3.222 | 3.071 | 2.994 | 2.881 | 2.787 | 2.681 | 2.553 | 2.506 |
| 1 | 10 | 5.96 | 5.41 | 5.16 | 4.785 | 4.475 | 4.131 | 3.731 | 3.584 |
| 2 | | 5.00 | 4.67 | 4.500 | 4.253 | 4.042 | 3.811 | 3.541 | 3.44 |
| 3 | | 4.636 | 4.349 | 4.222 | 4.033 | 3.869 | 3.681 | 3.460 | 3.378 |
| 4 | | 4.41 | 4.176 | 4.060 | 3.893 | 3.754 | 3.595 | 3.409 | 3.339 |
| 5 | | 4.249 | 4.051 | 3.950 | 3.804 | 3.681 | 3.541 | 3.375 | 3.314 |
| 10 | | 3.897 | 3.756 | 3.689 | 3.585 | 3.499 | 3.404 | 3.287 | 3.244 |
| 1 | 25 | 6.57 | 6.03 | 5.79 | 5.457 | 5.179 | 4.865 | 4.506 | 4.377 |
| 2 | | 5.67 | 5.36 | 5.204 | 4.983 | 4.793 | 4.582 | 4.341 | 4.252 |
| 3 | | 5.335 | 5.077 | 4.954 | 4.776 | 4.628 | 4.464 | 4.269 | 4.197 |
| 4 | | 5.11 | 4.903 | 4.804 | 4.658 | 4.531 | 4.389 | 4.221 | 4.159 |
| 5 | | 5.00 | 4.807 | 4.708 | 4.579 | 4.467 | 4.340 | 4.193 | 4.137 |
| 10 | | 4.666 | 4.538 | 4.479 | 4.386 | 4.309 | 4.221 | 4.118 | 4.080 |

**Table A1-2.1.** Percentiles of the Average Range (*AR*). (CONTINUED)

| k | n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.9539 | 0.4506 | 0.3583 | 0.1777 | 0.0887 | 0.0443 | 0.0177 | 0.0089 |
| 2 | | 1.048 | 0.673 | 0.592 | 0.408 | 0.286 | 0.198 | 0.127 | 0.091 |
| 3 | | 1.078 | 0.765 | 0.696 | 0.529 | 0.411 | 0.321 | 0.234 | 0.185 |
| 4 | | 1.091 | 0.820 | 0.758 | 0.605 | 0.494 | 0.408 | 0.319 | 0.268 |
| 5 | | 1.100 | 0.853 | 0.797 | 0.658 | 0.554 | 0.472 | 0.387 | 0.334 |
| 10 | | 1.115 | 0.939 | 0.898 | 0.792 | 0.709 | 0.642 | 0.568 | 0.517 |
| 1 | 3 | 1.590 | 1.026 | 0.902 | 0.619 | 0.431 | 0.306 | 0.192 | 0.134 |
| 2 | | 1.646 | 1.241 | 1.149 | 0.920 | 0.754 | 0.620 | 0.483 | 0.400 |
| 3 | | 1.661 | 1.328 | 1.249 | 1.056 | 0.907 | 0.788 | 0.660 | 0.580 |
| 4 | | 1.668 | 1.378 | 1.309 | 1.137 | 1.004 | 0.898 | 0.781 | 0.709 |
| 5 | | 1.672 | 1.412 | 1.352 | 1.197 | 1.073 | 0.971 | 0.865 | 0.791 |
| 10 | | 1.683 | 1.498 | 1.453 | 1.339 | 1.249 | 1.172 | 1.089 | 1.030 |
| 1 | 4 | 1.980 | 1.416 | 1.285 | 0.980 | 0.761 | 0.598 | 0.434 | 0.342 |
| 2 | | 2.017 | 1.615 | 1.522 | 1.287 | 1.109 | 0.962 | 0.804 | 0.711 |
| 3 | | 2.033 | 1.701 | 1.623 | 1.424 | 1.269 | 1.146 | 1.007 | 0.917 |
| 4 | | 2.041 | 1.750 | 1.682 | 1.506 | 1.369 | 1.253 | 1.124 | 1.041 |
| 5 | | 2.045 | 1.786 | 1.724 | 1.566 | 1.442 | 1.336 | 1.220 | 1.144 |
| 10 | | 2.051 | 1.866 | 1.821 | 1.707 | 1.614 | 1.536 | 1.448 | 1.393 |
| 1 | 5 | 2.257 | 1.699 | 1.572 | 1.261 | 1.032 | 0.849 | 0.667 | 0.560 |
| 2 | | 2.296 | 1.896 | 1.803 | 1.564 | 1.379 | 1.229 | 1.069 | 0.964 |
| 3 | | 2.302 | 1.975 | 1.897 | 1.701 | 1.545 | 1.416 | 1.273 | 1.181 |
| 4 | | 2.309 | 2.024 | 1.956 | 1.780 | 1.642 | 1.525 | 1.398 | 1.311 |
| 5 | | 2.312 | 2.058 | 1.998 | 1.840 | 1.715 | 1.613 | 1.496 | 1.421 |
| 10 | | 2.318 | 2.138 | 2.095 | 1.981 | 1.890 | 1.814 | 1.724 | 1.665 |
| 1 | 10 | 3.026 | 2.517 | 2.399 | 2.095 | 1.866 | 1.674 | 1.467 | 1.334 |
| 2 | | 3.052 | 2.684 | 2.597 | 2.374 | 2.199 | 2.047 | 1.886 | 1.77 |
| 3 | | 3.061 | 2.757 | 2.684 | 2.499 | 2.349 | 2.225 | 2.087 | 1.991 |
| 4 | | 3.064 | 2.802 | 2.739 | 2.575 | 2.444 | 2.336 | 2.209 | 2.123 |
| 5 | | 3.067 | 2.832 | 2.774 | 2.627 | 2.509 | 2.409 | 2.295 | 2.222 |
| 10 | | 3.072 | 2.904 | 2.863 | 2.758 | 2.672 | 2.599 | 2.516 | 2.459 |
| 1 | 25 | 3.878 | 3.427 | 3.321 | 3.059 | 2.853 | 2.682 | 2.495 | 2.369 |
| 2 | | 3.906 | 3.582 | 3.505 | 3.307 | 3.150 | 3.019 | 2.873 | 2.777 |
| 3 | | 3.913 | 3.646 | 3.581 | 3.417 | 3.285 | 3.178 | 3.055 | 2.972 |
| 4 | | 3.916 | 3.684 | 3.628 | 3.484 | 3.369 | 3.268 | 3.155 | 3.082 |
| 5 | | 3.920 | 3.713 | 3.663 | 3.533 | 3.429 | 3.339 | 3.239 | 3.171 |
| 10 | | 3.926 | 3.776 | 3.739 | 3.646 | 3.571 | 3.506 | 3.430 | 3.379 |

**Table A1-2.2**. Monte Carlo Errors of the Percentiles of the Average Range (*AR*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | | | | | | | | | | | | | | | | |
| 2 | 2 | 0.04 | 0.016 | 0.018 | 0.011 | 0.012 | 0.003 | 0.006 | 0.007 | 0.005 | 0.004 | 0.005 | 0.004 | 0.005 | 0.006 | 0.005 | 0.005 |
| 3 | 2 | 0.04 | 0.019 | 0.012 | 0.008 | 0.007 | 0.005 | 0.004 | 0.006 | 0.004 | 0.004 | 0.003 | 0.003 | 0.003 | 0.005 | 0.007 | 0.006 |
| 4 | 2 | 0.03 | 0.02 | 0.012 | 0.009 | 0.010 | 0.007 | 0.004 | 0.005 | 0.004 | 0.003 | 0.004 | 0.005 | 0.005 | 0.006 | 0.007 | 0.007 |
| 5 | 2 | 0.03 | 0.013 | 0.012 | 0.007 | 0.005 | 0.005 | 0.005 | 0.005 | 0.003 | 0.003 | 0.004 | 0.003 | 0.004 | 0.004 | 0.006 | 0.004 |
| 10 | 2 | 0.02 | 0.016 | 0.007 | 0.003 | 0.004 | 0.003 | 0.003 | 0.003 | 0.0012 | 0.0018 | 0.002 | 0.0017 | 0.003 | 0.003 | 0.006 | 0.009 |
| 1 | 3 | 0.07 | 0.04 | 0.03 | 0.017 | 0.015 | 0.013 | 0.005 | 0.006 | 0.005 | 0.007 | 0.007 | 0.006 | 0.008 | 0.007 | 0.007 | 0.010 |
| 2 | 3 | 0.06 | 0.02 | 0.019 | 0.017 | 0.015 | 0.010 | 0.006 | 0.007 | 0.005 | 0.004 | 0.005 | 0.006 | 0.005 | 0.006 | 0.007 | 0.011 |
| 3 | 3 | 0.05 | 0.014 | 0.011 | 0.011 | 0.010 | 0.005 | 0.007 | 0.006 | 0.005 | 0.005 | 0.006 | 0.005 | 0.003 | 0.008 | 0.009 | 0.011 |
| 4 | 3 | 0.03 | 0.02 | 0.016 | 0.008 | 0.009 | 0.008 | 0.006 | 0.006 | 0.004 | 0.003 | 0.002 | 0.004 | 0.003 | 0.005 | 0.006 | 0.008 |
| 5 | 3 | 0.03 | 0.012 | 0.013 | 0.009 | 0.007 | 0.006 | 0.004 | 0.003 | 0.002 | 0.004 | 0.004 | 0.005 | 0.005 | 0.006 | 0.009 | 0.015 |
| 10 | 3 | 0.014 | 0.015 | 0.011 | 0.006 | 0.004 | 0.003 | 0.0019 | 0.003 | 0.003 | 0.004 | 0.003 | 0.003 | 0.005 | 0.006 | 0.008 | 0.009 |
| 1 | 4 | 0.12 | 0.05 | 0.03 | 0.018 | 0.019 | 0.018 | 0.011 | 0.010 | 0.007 | 0.006 | 0.005 | 0.007 | 0.007 | 0.006 | 0.012 | 0.013 |
| 2 | 4 | 0.06 | 0.02 | 0.03 | 0.013 | 0.008 | 0.007 | 0.005 | 0.004 | 0.005 | 0.003 | 0.004 | 0.006 | 0.007 | 0.007 | 0.008 | 0.007 |
| 3 | 4 | 0.05 | 0.018 | 0.02 | 0.013 | 0.007 | 0.009 | 0.006 | 0.006 | 0.005 | 0.002 | 0.003 | 0.004 | 0.003 | 0.008 | 0.010 | 0.011 |
| 4 | 4 | 0.03 | 0.02 | 0.019 | 0.007 | 0.010 | 0.008 | 0.006 | 0.005 | 0.003 | 0.003 | 0.003 | 0.004 | 0.003 | 0.004 | 0.008 | 0.008 |
| 5 | 4 | 0.05 | 0.018 | 0.011 | 0.010 | 0.009 | 0.006 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.007 | 0.008 | 0.011 |
| 10 | 4 | 0.02 | 0.013 | 0.009 | 0.005 | 0.003 | 0.004 | 0.003 | 0.003 | 0.002 | 0.003 | 0.004 | 0.003 | 0.004 | 0.004 | 0.007 | 0.007 |
| 1 | 5 | 0.10 | 0.04 | 0.03 | 0.012 | 0.009 | 0.016 | 0.013 | 0.012 | 0.009 | 0.008 | 0.008 | 0.009 | 0.010 | 0.015 | 0.02 | 0.018 |
| 2 | 5 | 0.03 | 0.02 | 0.013 | 0.014 | 0.010 | 0.007 | 0.007 | 0.006 | 0.006 | 0.007 | 0.006 | 0.007 | 0.007 | 0.007 | 0.013 | 0.015 |
| 3 | 5 | 0.05 | 0.019 | 0.019 | 0.012 | 0.008 | 0.008 | 0.005 | 0.004 | 0.003 | 0.003 | 0.002 | 0.004 | 0.004 | 0.005 | 0.004 | 0.008 |
| 4 | 5 | 0.03 | 0.03 | 0.02 | 0.009 | 0.005 | 0.006 | 0.007 | 0.005 | 0.004 | 0.004 | 0.005 | 0.005 | 0.004 | 0.006 | 0.005 | 0.008 |
| 5 | 5 | 0.03 | 0.013 | 0.010 | 0.008 | 0.005 | 0.003 | 0.002 | 0.002 | 0.003 | 0.002 | 0.003 | 0.004 | 0.003 | 0.004 | 0.010 | 0.010 |
| 10 | 5 | 0.013 | 0.007 | 0.006 | 0.005 | 0.005 | 0.003 | 0.002 | 0.002 | 0.002 | 0.0014 | 0.0014 | 0.004 | 0.003 | 0.002 | 0.008 | 0.009 |
| 1 | 10 | 0.05 | 0.04 | 0.03 | 0.011 | 0.009 | 0.008 | 0.007 | 0.008 | 0.006 | 0.006 | 0.008 | 0.014 | 0.014 | 0.013 | 0.015 | 0.02 |
| 2 | 10 | 0.03 | 0.03 | 0.019 | 0.010 | 0.008 | 0.005 | 0.006 | 0.004 | 0.003 | 0.005 | 0.006 | 0.008 | 0.010 | 0.013 | 0.015 | 0.03 |
| 3 | 10 | 0.02 | 0.010 | 0.012 | 0.007 | 0.008 | 0.007 | 0.006 | 0.005 | 0.004 | 0.003 | 0.004 | 0.004 | 0.006 | 0.006 | 0.008 | 0.016 |
| 4 | 10 | 0.03 | 0.02 | 0.015 | 0.010 | 0.008 | 0.006 | 0.007 | 0.005 | 0.006 | 0.006 | 0.006 | 0.007 | 0.005 | 0.004 | 0.008 | 0.007 |
| 5 | 10 | 0.02 | 0.016 | 0.012 | 0.008 | 0.007 | 0.006 | 0.005 | 0.003 | 0.004 | 0.005 | 0.005 | 0.007 | 0.007 | 0.006 | 0.009 | 0.009 |
| 10 | 10 | 0.02 | 0.009 | 0.009 | 0.004 | 0.004 | 0.004 | 0.003 | 0.003 | 0.0016 | 0.003 | 0.003 | 0.004 | 0.004 | 0.004 | 0.006 | 0.009 |
| 1 | 25 | 0.08 | 0.04 | 0.04 | 0.02 | 0.007 | 0.009 | 0.009 | 0.009 | 0.007 | 0.004 | 0.006 | 0.005 | 0.009 | 0.011 | 0.015 | 0.013 |
| 2 | 25 | 0.04 | 0.03 | 0.016 | 0.011 | 0.009 | 0.004 | 0.003 | 0.004 | 0.005 | 0.004 | 0.004 | 0.004 | 0.004 | 0.005 | 0.007 | 0.013 |
| 3 | 25 | 0.02 | 0.014 | 0.009 | 0.010 | 0.007 | 0.007 | 0.007 | 0.006 | 0.004 | 0.002 | 0.003 | 0.004 | 0.004 | 0.007 | 0.010 | 0.009 |
| 4 | 25 | 0.03 | 0.015 | 0.012 | 0.010 | 0.008 | 0.007 | 0.004 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 | 0.004 | 0.005 | 0.004 | 0.006 |
| 5 | 25 | 0.04 | 0.014 | 0.005 | 0.009 | 0.008 | 0.004 | 0.0015 | 0.002 | 0.0018 | 0.003 | 0.003 | 0.003 | 0.004 | 0.005 | 0.008 | 0.006 |
| 10 | 25 | 0.02 | 0.010 | 0.008 | 0.005 | 0.003 | 0.003 | 0.002 | 0.003 | 0.002 | 0.0019 | 0.002 | 0.004 | 0.004 | 0.003 | 0.007 | 0.007 |

## Appendix 1-3. Mean Deviation, One Shot group

**Table A1-3.1**. Percentiles of the Mean Deviation (*MD*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|
| 2 | 2.3268 | 1.9849 | 1.8214 | 1.5849 | 1.3859 | 1.1631 | 0.9062 | 0.8134 |
| 3 | 1.93 | 1.702 | 1.583 | 1.417 | 1.275 | 1.116 | 0.931 | 0.865 |
| 4 | 1.80 | 1.586 | 1.487 | 1.342 | 1.221 | 1.087 | 0.932 | 0.876 |
| 5 | 1.694 | 1.500 | 1.416 | 1.290 | 1.185 | 1.068 | 0.932 | 0.882 |
| 6 | 1.618 | 1.445 | 1.369 | 1.252 | 1.157 | 1.051 | 0.929 | 0.884 |
| 7 | 1.544 | 1.399 | 1.327 | 1.220 | 1.133 | 1.037 | 0.924 | 0.882 |
| 8 | 1.511 | 1.364 | 1.294 | 1.195 | 1.115 | 1.025 | 0.920 | 0.881 |
| 9 | 1.46 | 1.327 | 1.267 | 1.175 | 1.100 | 1.016 | 0.916 | 0.881 |
| 10 | 1.426 | 1.302 | 1.240 | 1.157 | 1.087 | 1.008 | 0.913 | 0.879 |
| 11 | 1.389 | 1.270 | 1.217 | 1.137 | 1.074 | 0.998 | 0.910 | 0.877 |
| 12 | 1.367 | 1.256 | 1.202 | 1.125 | 1.063 | 0.991 | 0.908 | 0.877 |
| 13 | 1.342 | 1.241 | 1.189 | 1.116 | 1.054 | 0.986 | 0.904 | 0.874 |
| 14 | 1.316 | 1.219 | 1.172 | 1.104 | 1.045 | 0.979 | 0.901 | 0.873 |
| 15 | 1.300 | 1.202 | 1.159 | 1.092 | 1.036 | 0.973 | 0.899 | 0.871 |
| 16 | 1.286 | 1.194 | 1.150 | 1.084 | 1.030 | 0.969 | 0.897 | 0.870 |
| 17 | 1.272 | 1.183 | 1.141 | 1.077 | 1.024 | 0.964 | 0.895 | 0.869 |
| 18 | 1.260 | 1.172 | 1.129 | 1.068 | 1.018 | 0.961 | 0.893 | 0.868 |
| 19 | 1.244 | 1.158 | 1.119 | 1.060 | 1.012 | 0.957 | 0.892 | 0.868 |
| 20 | 1.230 | 1.148 | 1.110 | 1.055 | 1.007 | 0.952 | 0.889 | 0.865 |
| 21 | 1.219 | 1.140 | 1.104 | 1.049 | 1.003 | 0.949 | 0.887 | 0.864 |
| 22 | 1.209 | 1.134 | 1.098 | 1.044 | 0.998 | 0.947 | 0.886 | 0.864 |
| 23 | 1.207 | 1.126 | 1.091 | 1.038 | 0.993 | 0.944 | 0.885 | 0.862 |
| 24 | 1.190 | 1.119 | 1.084 | 1.033 | 0.989 | 0.941 | 0.883 | 0.862 |
| 25 | 1.186 | 1.116 | 1.080 | 1.030 | 0.987 | 0.939 | 0.882 | 0.860 |
| 26 | 1.179 | 1.109 | 1.075 | 1.025 | 0.983 | 0.936 | 0.881 | 0.860 |
| 27 | 1.175 | 1.102 | 1.069 | 1.019 | 0.978 | 0.933 | 0.879 | 0.859 |
| 28 | 1.163 | 1.096 | 1.062 | 1.016 | 0.977 | 0.932 | 0.878 | 0.858 |
| 29 | 1.158 | 1.089 | 1.059 | 1.013 | 0.975 | 0.930 | 0.877 | 0.857 |
| 30 | 1.149 | 1.087 | 1.055 | 1.009 | 0.970 | 0.927 | 0.876 | 0.857 |

**Table A1-3.1.** Percentiles of the Mean Deviation (*MD*). (CONTINUED)

| n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.4769 | 0.2253 | 0.1791 | 0.0889 | 0.0443 | 0.0222 | 0.0089 | 0.0044 |
| 3 | 0.613 | 0.396 | 0.348 | 0.238 | 0.165 | 0.116 | 0.073 | 0.052 |
| 4 | 0.662 | 0.473 | 0.431 | 0.329 | 0.256 | 0.200 | 0.145 | 0.114 |
| 5 | 0.691 | 0.521 | 0.482 | 0.387 | 0.316 | 0.262 | 0.205 | 0.171 |
| 6 | 0.710 | 0.554 | 0.517 | 0.427 | 0.360 | 0.305 | 0.250 | 0.217 |
| 7 | 0.723 | 0.578 | 0.543 | 0.459 | 0.394 | 0.342 | 0.286 | 0.252 |
| 8 | 0.733 | 0.596 | 0.564 | 0.484 | 0.423 | 0.373 | 0.319 | 0.284 |
| 9 | 0.741 | 0.611 | 0.581 | 0.504 | 0.446 | 0.398 | 0.344 | 0.310 |
| 10 | 0.746 | 0.623 | 0.594 | 0.521 | 0.464 | 0.416 | 0.366 | 0.331 |
| 11 | 0.751 | 0.634 | 0.606 | 0.536 | 0.481 | 0.436 | 0.385 | 0.352 |
| 12 | 0.755 | 0.642 | 0.615 | 0.547 | 0.495 | 0.450 | 0.402 | 0.372 |
| 13 | 0.758 | 0.649 | 0.624 | 0.559 | 0.507 | 0.464 | 0.418 | 0.388 |
| 14 | 0.761 | 0.657 | 0.632 | 0.568 | 0.519 | 0.477 | 0.431 | 0.403 |
| 15 | 0.764 | 0.662 | 0.638 | 0.578 | 0.529 | 0.488 | 0.444 | 0.414 |
| 16 | 0.766 | 0.668 | 0.644 | 0.585 | 0.537 | 0.498 | 0.453 | 0.424 |
| 17 | 0.768 | 0.673 | 0.650 | 0.591 | 0.545 | 0.507 | 0.464 | 0.434 |
| 18 | 0.769 | 0.676 | 0.654 | 0.598 | 0.552 | 0.515 | 0.473 | 0.443 |
| 19 | 0.772 | 0.681 | 0.659 | 0.604 | 0.560 | 0.522 | 0.481 | 0.455 |
| 20 | 0.772 | 0.685 | 0.663 | 0.609 | 0.565 | 0.529 | 0.490 | 0.463 |
| 21 | 0.774 | 0.688 | 0.667 | 0.613 | 0.571 | 0.536 | 0.497 | 0.471 |
| 22 | 0.775 | 0.691 | 0.670 | 0.619 | 0.577 | 0.543 | 0.504 | 0.478 |
| 23 | 0.775 | 0.693 | 0.673 | 0.622 | 0.582 | 0.548 | 0.511 | 0.486 |
| 24 | 0.776 | 0.695 | 0.676 | 0.626 | 0.586 | 0.553 | 0.516 | 0.490 |
| 25 | 0.777 | 0.698 | 0.679 | 0.630 | 0.591 | 0.557 | 0.520 | 0.496 |
| 26 | 0.778 | 0.701 | 0.682 | 0.634 | 0.596 | 0.562 | 0.525 | 0.501 |
| 27 | 0.779 | 0.703 | 0.685 | 0.637 | 0.599 | 0.567 | 0.531 | 0.508 |
| 28 | 0.780 | 0.705 | 0.687 | 0.640 | 0.603 | 0.572 | 0.537 | 0.513 |
| 29 | 0.781 | 0.707 | 0.689 | 0.644 | 0.607 | 0.576 | 0.540 | 0.518 |
| 30 | 0.781 | 0.708 | 0.690 | 0.646 | 0.610 | 0.580 | 0.545 | 0.523 |

**Table A1-3.2**. Monte Carlo Errors of the Percentiles of the Mean Deviation (*MD*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | | | | | |
| 3 | 0.03 | 0.016 | 0.015 | 0.009 | 0.009 | 0.007 | 0.005 | 0.005 | 0.005 | 0.003 | 0.0018 | 0.002 | 0.003 | 0.003 | 0.003 | 0.003 |
| 4 | 0.03 | 0.017 | 0.012 | 0.005 | 0.005 | 0.005 | 0.003 | 0.003 | 0.0010 | 0.0017 | 0.0018 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 |
| 5 | 0.02 | 0.008 | 0.005 | 0.004 | 0.006 | 0.005 | 0.003 | 0.003 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.003 | 0.004 | 0.005 |
| 6 | 0.02 | 0.014 | 0.008 | 0.005 | 0.005 | 0.0019 | 0.0013 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.003 | 0.004 | 0.004 | 0.004 |
| 7 | 0.02 | 0.006 | 0.006 | 0.006 | 0.005 | 0.004 | 0.0016 | 0.003 | 0.0015 | 0.0017 | 0.002 | 0.003 | 0.004 | 0.004 | 0.005 | 0.006 |
| 8 | 0.01 | 0.005 | 0.004 | 0.003 | 0.004 | 0.004 | 0.0018 | 0.002 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.003 | 0.005 | 0.006 |
| 9 | 0.03 | 0.012 | 0.008 | 0.005 | 0.005 | 0.003 | 0.003 | 0.002 | 0.0014 | 0.0013 | 0.0017 | 0.002 | 0.002 | 0.002 | 0.005 | 0.005 |
| 10 | 0.015 | 0.008 | 0.004 | 0.004 | 0.004 | 0.0018 | 0.003 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.0014 | 0.002 | 0.002 | 0.004 | 0.004 |
| 11 | 0.010 | 0.008 | 0.006 | 0.004 | 0.003 | 0.0019 | 0.0017 | 0.0013 | 0.0018 | 0.0016 | 0.002 | 0.002 | 0.002 | 0.003 | 0.004 | 0.003 |
| 12 | 0.010 | 0.011 | 0.008 | 0.004 | 0.004 | 0.002 | 0.003 | 0.002 | 0.0018 | 0.0015 | 0.0016 | 0.002 | 0.0013 | 0.0016 | 0.003 | 0.003 |
| 13 | 0.011 | 0.006 | 0.006 | 0.005 | 0.003 | 0.003 | 0.0015 | 0.0013 | 0.0016 | 0.0014 | 0.0018 | 0.002 | 0.002 | 0.0014 | 0.003 | 0.0015 |
| 14 | 0.015 | 0.006 | 0.003 | 0.002 | 0.003 | 0.0018 | 0.0018 | 0.002 | 0.0018 | 0.0018 | 0.0017 | 0.0014 | 0.0018 | 0.002 | 0.003 | 0.004 |
| 15 | 0.017 | 0.006 | 0.002 | 0.003 | 0.0013 | 0.002 | 0.002 | 0.002 | 0.0017 | 0.0013 | 0.0013 | 0.0012 | 0.0019 | 0.0019 | 0.0017 | 0.003 |
| 16 | 0.016 | 0.008 | 0.004 | 0.002 | 0.0016 | 0.0016 | 0.0012 | 0.0010 | 0.0014 | 0.0013 | 0.0016 | 0.0013 | 0.0012 | 0.0014 | 0.003 | 0.004 |
| 17 | 0.010 | 0.005 | 0.004 | 0.003 | 0.002 | 0.0015 | 0.0018 | 0.002 | 0.0012 | 0.0013 | 0.0017 | 0.002 | 0.0014 | 0.003 | 0.003 | 0.005 |
| 18 | 0.012 | 0.008 | 0.005 | 0.002 | 0.003 | 0.0017 | 0.0014 | 0.0013 | 0.0014 | 0.0012 | 0.0012 | 0.0013 | 0.0018 | 0.002 | 0.003 | 0.003 |
| 19 | 0.013 | 0.005 | 0.005 | 0.003 | 0.002 | 0.0016 | 0.0015 | 0.0014 | 0.0015 | 0.0010 | 0.0010 | 0.0013 | 0.0019 | 0.002 | 0.003 | 0.004 |
| 20 | 0.007 | 0.005 | 0.004 | 0.0007 | 0.002 | 0.0017 | 0.0013 | 0.0012 | 0.0012 | 0.0014 | 0.0011 | 0.0012 | 0.0019 | 0.0018 | 0.003 | 0.002 |
| 21 | 0.005 | 0.004 | 0.003 | 0.002 | 0.002 | 0.0017 | 0.0011 | 0.0011 | 0.0009 | 0.0011 | 0.0012 | 0.0012 | 0.0017 | 0.0019 | 0.003 | 0.004 |
| 22 | 0.011 | 0.004 | 0.004 | 0.002 | 0.0016 | 0.0012 | 0.0014 | 0.0014 | 0.0012 | 0.0011 | 0.0015 | 0.0015 | 0.0017 | 0.003 | 0.003 | 0.003 |
| 23 | 0.008 | 0.004 | 0.003 | 0.004 | 0.003 | 0.003 | 0.0018 | 0.0016 | 0.0010 | 0.0010 | 0.0010 | 0.0017 | 0.0013 | 0.0015 | 0.003 | 0.002 |
| 24 | 0.011 | 0.005 | 0.004 | 0.004 | 0.002 | 0.002 | 0.0016 | 0.0012 | 0.0013 | 0.0012 | 0.0010 | 0.0017 | 0.0019 | 0.0019 | 0.003 | 0.004 |
| 25 | 0.007 | 0.0013 | 0.002 | 0.002 | 0.002 | 0.002 | 0.0015 | 0.0017 | 0.0012 | 0.0009 | 0.0009 | 0.0016 | 0.0018 | 0.0014 | 0.0013 | 0.003 |
| 26 | 0.012 | 0.006 | 0.004 | 0.002 | 0.0015 | 0.0016 | 0.0015 | 0.0016 | 0.0008 | 0.0005 | 0.0007 | 0.0013 | 0.0010 | 0.0008 | 0.0019 | 0.002 |
| 27 | 0.008 | 0.004 | 0.003 | 0.0016 | 0.0014 | 0.0018 | 0.0015 | 0.0016 | 0.0011 | 0.0013 | 0.0013 | 0.0017 | 0.0014 | 0.002 | 0.003 | 0.003 |
| 28 | 0.004 | 0.003 | 0.004 | 0.003 | 0.002 | 0.0014 | 0.0006 | 0.0008 | 0.0011 | 0.0014 | 0.0013 | 0.0016 | 0.0013 | 0.0017 | 0.003 | 0.004 |
| 29 | 0.003 | 0.005 | 0.004 | 0.002 | 0.003 | 0.0016 | 0.0012 | 0.0010 | 0.0007 | 0.0012 | 0.0013 | 0.0017 | 0.0016 | 0.002 | 0.003 | 0.002 |
| 30 | 0.009 | 0.005 | 0.004 | 0.002 | 0.0014 | 0.0019 | 0.0016 | 0.0014 | 0.0010 | 0.0007 | 0.0007 | 0.0011 | 0.0015 | 0.0018 | 0.0017 | 0.0018 |

## Appendix 1-4. Average Mean Deviation, Multiple Shot Groups

**Table A1-4.1**. Percentiles of the Average Mean Deviation (*AMD*).

| k | n | P 0.999 | P | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2.3268 | 1.9849 | 1.8214 | 1.5849 | 1.3859 | 1.1631 | 0.9062 | 0.8134 |
| 2 | | 1.72 | 1.508 | 1.401 | 1.246 | 1.120 | 0.976 | 0.811 | 0.750 |
| 3 | | 1.48 | 1.315 | 1.228 | 1.108 | 1.007 | 0.895 | 0.764 | 0.716 |
| 4 | | 1.347 | 1.204 | 1.133 | 1.030 | 0.943 | 0.848 | 0.737 | 0.697 |
| 5 | | 1.263 | 1.126 | 1.062 | 0.977 | 0.902 | 0.819 | 0.722 | 0.686 |
| 10 | | 1.042 | 0.949 | 0.907 | 0.849 | 0.798 | 0.742 | 0.676 | 0.652 |
| 1 | 3 | 1.93 | 1.702 | 1.583 | 1.417 | 1.275 | 1.116 | 0.931 | 0.865 |
| 2 | | 1.525 | 1.368 | 1.287 | 1.176 | 1.079 | 0.976 | 0.851 | 0.806 |
| 3 | | 1.340 | 1.222 | 1.164 | 1.074 | 1.000 | 0.914 | 0.815 | 0.780 |
| 4 | | 1.252 | 1.139 | 1.085 | 1.010 | 0.948 | 0.878 | 0.794 | 0.763 |
| 5 | | 1.180 | 1.085 | 1.039 | 0.972 | 0.915 | 0.853 | 0.778 | 0.751 |
| 10 | | 1.008 | 0.945 | 0.916 | 0.872 | 0.835 | 0.792 | 0.741 | 0.722 |
| 1 | 4 | 1.80 | 1.586 | 1.487 | 1.342 | 1.221 | 1.087 | 0.932 | 0.876 |
| 2 | | 1.438 | 1.299 | 1.235 | 1.140 | 1.060 | 0.970 | 0.866 | 0.827 |
| 3 | | 1.290 | 1.182 | 1.132 | 1.054 | 0.989 | 0.916 | 0.832 | 0.802 |
| 4 | | 1.204 | 1.110 | 1.066 | 1.002 | 0.947 | 0.886 | 0.815 | 0.788 |
| 5 | | 1.145 | 1.063 | 1.023 | 0.966 | 0.919 | 0.865 | 0.801 | 0.778 |
| 10 | | 1.002 | 0.948 | 0.921 | 0.882 | 0.850 | 0.813 | 0.770 | 0.753 |
| 1 | 5 | 1.694 | 1.500 | 1.416 | 1.290 | 1.185 | 1.068 | 0.932 | 0.882 |
| 2 | | 1.380 | 1.260 | 1.197 | 1.111 | 1.042 | 0.962 | 0.869 | 0.835 |
| 3 | | 1.247 | 1.148 | 1.101 | 1.032 | 0.976 | 0.914 | 0.841 | 0.813 |
| 4 | | 1.162 | 1.083 | 1.044 | 0.988 | 0.940 | 0.887 | 0.824 | 0.800 |
| 5 | | 1.111 | 1.041 | 1.009 | 0.957 | 0.916 | 0.868 | 0.813 | 0.792 |
| 10 | | 0.987 | 0.943 | 0.918 | 0.884 | 0.855 | 0.823 | 0.784 | 0.769 |
| 1 | 10 | 1.426 | 1.302 | 1.240 | 1.157 | 1.087 | 1.008 | 0.913 | 0.879 |
| 2 | | 1.212 | 1.129 | 1.090 | 1.033 | 0.985 | 0.932 | 0.868 | 0.845 |
| 3 | | 1.123 | 1.058 | 1.026 | 0.981 | 0.943 | 0.899 | 0.848 | 0.829 |
| 4 | | 1.067 | 1.014 | 0.988 | 0.949 | 0.917 | 0.880 | 0.836 | 0.820 |
| 5 | | 1.037 | 0.987 | 0.963 | 0.929 | 0.900 | 0.867 | 0.827 | 0.813 |
| 10 | | 0.949 | 0.916 | 0.900 | 0.877 | 0.857 | 0.834 | 0.807 | 0.796 |
| 1 | 25 | 1.186 | 1.116 | 1.080 | 1.030 | 0.987 | 0.939 | 0.882 | 0.860 |
| 2 | | 1.062 | 1.012 | 0.988 | 0.954 | 0.924 | 0.891 | 0.853 | 0.838 |
| 3 | | 1.009 | 0.968 | 0.949 | 0.922 | 0.898 | 0.872 | 0.840 | 0.828 |
| 4 | | 0.973 | 0.942 | 0.925 | 0.902 | 0.882 | 0.859 | 0.832 | 0.822 |
| 5 | | 0.956 | 0.926 | 0.911 | 0.890 | 0.872 | 0.851 | 0.827 | 0.818 |
| 10 | | 0.901 | 0.882 | 0.872 | 0.857 | 0.845 | 0.831 | 0.814 | 0.807 |

**Table A1-4.1.** Percentiles of the Average Mean Deviation (*AMD*). (CONTINUED)

| k | n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.4769 | 0.2253 | 0.1791 | 0.0889 | 0.0443 | 0.0222 | 0.0089 | 0.0044 |
| 2 | | 0.527 | 0.338 | 0.298 | 0.205 | 0.142 | 0.100 | 0.064 | 0.045 |
| 3 | | 0.540 | 0.384 | 0.348 | 0.265 | 0.205 | 0.162 | 0.118 | 0.094 |
| 4 | | 0.545 | 0.408 | 0.378 | 0.302 | 0.246 | 0.203 | 0.159 | 0.132 |
| 5 | | 0.549 | 0.426 | 0.398 | 0.329 | 0.277 | 0.235 | 0.191 | 0.164 |
| 10 | | 0.557 | 0.469 | 0.449 | 0.396 | 0.355 | 0.322 | 0.285 | 0.261 |
| 1 | 3 | 0.613 | 0.396 | 0.348 | 0.238 | 0.165 | 0.116 | 0.073 | 0.052 |
| 2 | | 0.632 | 0.476 | 0.441 | 0.353 | 0.288 | 0.238 | 0.187 | 0.157 |
| 3 | | 0.640 | 0.512 | 0.482 | 0.406 | 0.349 | 0.302 | 0.254 | 0.222 |
| 4 | | 0.643 | 0.531 | 0.504 | 0.438 | 0.387 | 0.344 | 0.297 | 0.269 |
| 5 | | 0.644 | 0.544 | 0.520 | 0.460 | 0.413 | 0.375 | 0.333 | 0.304 |
| 10 | | 0.647 | 0.576 | 0.559 | 0.515 | 0.480 | 0.451 | 0.418 | 0.396 |
| 1 | 4 | 0.662 | 0.473 | 0.431 | 0.329 | 0.256 | 0.200 | 0.145 | 0.114 |
| 2 | | 0.678 | 0.542 | 0.510 | 0.430 | 0.369 | 0.321 | 0.269 | 0.236 |
| 3 | | 0.682 | 0.570 | 0.544 | 0.477 | 0.426 | 0.383 | 0.338 | 0.309 |
| 4 | | 0.684 | 0.587 | 0.564 | 0.505 | 0.459 | 0.420 | 0.378 | 0.351 |
| 5 | | 0.685 | 0.599 | 0.578 | 0.524 | 0.482 | 0.447 | 0.409 | 0.384 |
| 10 | | 0.689 | 0.626 | 0.611 | 0.572 | 0.541 | 0.515 | 0.485 | 0.466 |
| 1 | 5 | 0.691 | 0.521 | 0.482 | 0.387 | 0.316 | 0.262 | 0.205 | 0.171 |
| 2 | | 0.703 | 0.580 | 0.551 | 0.478 | 0.422 | 0.375 | 0.325 | 0.293 |
| 3 | | 0.707 | 0.606 | 0.582 | 0.521 | 0.473 | 0.433 | 0.389 | 0.361 |
| 4 | | 0.707 | 0.620 | 0.599 | 0.547 | 0.504 | 0.468 | 0.428 | 0.403 |
| 5 | | 0.709 | 0.631 | 0.612 | 0.564 | 0.525 | 0.493 | 0.458 | 0.435 |
| 10 | | 0.711 | 0.655 | 0.642 | 0.607 | 0.579 | 0.556 | 0.528 | 0.510 |
| 1 | 10 | 0.746 | 0.623 | 0.594 | 0.521 | 0.464 | 0.416 | 0.366 | 0.331 |
| 2 | | 0.751 | 0.664 | 0.643 | 0.589 | 0.546 | 0.510 | 0.469 | 0.443 |
| 3 | | 0.753 | 0.682 | 0.664 | 0.619 | 0.583 | 0.552 | 0.517 | 0.494 |
| 4 | | 0.754 | 0.692 | 0.676 | 0.637 | 0.606 | 0.579 | 0.548 | 0.529 |
| 5 | | 0.755 | 0.699 | 0.685 | 0.650 | 0.621 | 0.597 | 0.570 | 0.551 |
| 10 | | 0.756 | 0.716 | 0.706 | 0.681 | 0.660 | 0.643 | 0.623 | 0.609 |
| 1 | 25 | 0.777 | 0.698 | 0.679 | 0.630 | 0.591 | 0.557 | 0.520 | 0.496 |
| 2 | | 0.780 | 0.723 | 0.710 | 0.674 | 0.645 | 0.622 | 0.593 | 0.573 |
| 3 | | 0.781 | 0.734 | 0.723 | 0.694 | 0.670 | 0.650 | 0.627 | 0.612 |
| 4 | | 0.781 | 0.740 | 0.731 | 0.706 | 0.685 | 0.667 | 0.647 | 0.633 |
| 5 | | 0.781 | 0.745 | 0.736 | 0.713 | 0.694 | 0.678 | 0.661 | 0.649 |
| 10 | | 0.782 | 0.756 | 0.750 | 0.733 | 0.720 | 0.708 | 0.695 | 0.686 |

**Table A1-4.2**. Monte Carlo Errors of the Percentiles of the Average Mean Deviation (*AMD*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | | | | | | | | | | | | | | | | |
| 2 | 2 | 0.03 | 0.018 | 0.010 | 0.008 | 0.007 | 0.006 | 0.004 | 0.003 | 0.003 | 0.002 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.003 | 0.0018 |
| 3 | 2 | 0.03 | 0.011 | 0.007 | 0.005 | 0.005 | 0.003 | 0.003 | 0.002 | 0.0018 | 0.003 | 0.003 | 0.002 | 0.003 | 0.003 | 0.003 | 0.003 |
| 4 | 2 | 0.02 | 0.010 | 0.009 | 0.003 | 0.003 | 0.003 | 0.002 | 0.002 | 0.0015 | 0.0014 | 0.0017 | 0.002 | 0.001 | 0.002 | 0.003 | 0.003 |
| 5 | 2 | 0.016 | 0.009 | 0.006 | 0.005 | 0.004 | 0.003 | 0.0016 | 0.0018 | 0.0016 | 0.0012 | 0.0010 | 0.002 | 0.002 | 0.0017 | 0.003 | 0.003 |
| 10 | 2 | 0.006 | 0.004 | 0.004 | 0.003 | 0.0019 | 0.0014 | 0.0016 | 0.0012 | 0.0013 | 0.0014 | 0.0012 | 0.0012 | 0.0019 | 0.0011 | 0.002 | 0.003 |
| 1 | 3 | 0.03 | 0.016 | 0.015 | 0.009 | 0.009 | 0.007 | 0.005 | 0.005 | 0.005 | 0.003 | 0.0018 | 0.002 | 0.003 | 0.003 | 0.003 | 0.003 |
| 2 | 3 | 0.02 | 0.010 | 0.008 | 0.006 | 0.006 | 0.005 | 0.0019 | 0.0017 | 0.0015 | 0.003 | 0.003 | 0.002 | 0.003 | 0.002 | 0.004 | 0.004 |
| 3 | 3 | 0.016 | 0.009 | 0.005 | 0.004 | 0.003 | 0.004 | 0.003 | 0.002 | 0.0011 | 0.0007 | 0.0007 | 0.0013 | 0.0013 | 0.0012 | 0.002 | 0.005 |
| 4 | 3 | 0.013 | 0.008 | 0.005 | 0.005 | 0.002 | 0.002 | 0.0013 | 0.0011 | 0.0009 | 0.0008 | 0.0014 | 0.002 | 0.003 | 0.002 | 0.002 | 0.003 |
| 5 | 3 | 0.008 | 0.007 | 0.004 | 0.003 | 0.0013 | 0.0016 | 0.0012 | 0.0013 | 0.0008 | 0.0015 | 0.0013 | 0.0011 | 0.002 | 0.0019 | 0.003 | 0.004 |
| 10 | 3 | 0.005 | 0.005 | 0.004 | 0.0019 | 0.0012 | 0.0012 | 0.0007 | 0.0005 | 0.0010 | 0.0004 | 0.0007 | 0.0011 | 0.0009 | 0.0017 | 0.0015 | 0.002 |
| 1 | 4 | 0.03 | 0.017 | 0.012 | 0.005 | 0.005 | 0.005 | 0.003 | 0.003 | 0.0010 | 0.0017 | 0.0018 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 |
| 2 | 4 | 0.02 | 0.007 | 0.004 | 0.004 | 0.003 | 0.003 | 0.002 | 0.002 | 0.0017 | 0.0005 | 0.0009 | 0.0013 | 0.0018 | 0.002 | 0.004 | 0.005 |
| 3 | 4 | 0.014 | 0.005 | 0.003 | 0.004 | 0.003 | 0.002 | 0.0016 | 0.0014 | 0.0017 | 0.0011 | 0.0012 | 0.0013 | 0.002 | 0.0013 | 0.003 | 0.004 |
| 4 | 4 | 0.014 | 0.005 | 0.004 | 0.002 | 0.0011 | 0.0015 | 0.0012 | 0.0011 | 0.0007 | 0.0013 | 0.0012 | 0.0014 | 0.0019 | 0.002 | 0.0019 | 0.004 |
| 5 | 4 | 0.011 | 0.007 | 0.005 | 0.002 | 0.002 | 0.0010 | 0.0014 | 0.0012 | 0.0013 | 0.0012 | 0.0014 | 0.0016 | 0.0014 | 0.0014 | 0.0019 | 0.002 |
| 10 | 4 | 0.004 | 0.003 | 0.003 | 0.002 | 0.0017 | 0.0011 | 0.0011 | 0.0009 | 0.0006 | 0.0013 | 0.0013 | 0.0013 | 0.0011 | 0.0012 | 0.0013 | 0.002 |
| 1 | 5 | 0.02 | 0.008 | 0.005 | 0.004 | 0.006 | 0.005 | 0.003 | 0.003 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.003 | 0.004 | 0.005 |
| 2 | 5 | 0.02 | 0.010 | 0.008 | 0.005 | 0.003 | 0.002 | 0.003 | 0.002 | 0.0016 | 0.0019 | 0.0016 | 0.002 | 0.003 | 0.002 | 0.003 | 0.003 |
| 3 | 5 | 0.008 | 0.005 | 0.004 | 0.004 | 0.002 | 0.0017 | 0.0016 | 0.0015 | 0.0014 | 0.0009 | 0.0012 | 0.0017 | 0.003 | 0.003 | 0.004 | 0.004 |
| 4 | 5 | 0.010 | 0.007 | 0.004 | 0.0016 | 0.003 | 0.002 | 0.0015 | 0.0014 | 0.0013 | 0.0016 | 0.0013 | 0.0014 | 0.0015 | 0.0014 | 0.003 | 0.004 |
| 5 | 5 | 0.011 | 0.004 | 0.003 | 0.0019 | 0.002 | 0.0017 | 0.0016 | 0.0014 | 0.0011 | 0.0008 | 0.0007 | 0.0012 | 0.0010 | 0.0009 | 0.0016 | 0.0017 |
| 10 | 5 | 0.006 | 0.002 | 0.0013 | 0.0007 | 0.0010 | 0.0014 | 0.0015 | 0.0012 | 0.0005 | 0.0006 | 0.0006 | 0.0010 | 0.0007 | 0.0015 | 0.0015 | 0.0019 |
| 1 | 10 | 0.015 | 0.008 | 0.004 | 0.004 | 0.004 | 0.0018 | 0.003 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.0014 | 0.002 | 0.002 | 0.004 | 0.004 |
| 2 | 10 | 0.005 | 0.003 | 0.003 | 0.003 | 0.0019 | 0.0012 | 0.0018 | 0.0015 | 0.0015 | 0.0017 | 0.0019 | 0.0016 | 0.003 | 0.003 | 0.003 | 0.005 |
| 3 | 10 | 0.010 | 0.003 | 0.003 | 0.002 | 0.0015 | 0.0012 | 0.0008 | 0.0007 | 0.0008 | 0.0007 | 0.0008 | 0.0010 | 0.0013 | 0.0011 | 0.003 | 0.003 |
| 4 | 10 | 0.009 | 0.003 | 0.003 | 0.002 | 0.0012 | 0.0011 | 0.0008 | 0.0010 | 0.0010 | 0.0009 | 0.0008 | 0.0008 | 0.0013 | 0.0017 | 0.003 | 0.002 |
| 5 | 10 | 0.007 | 0.004 | 0.003 | 0.003 | 0.0012 | 0.0008 | 0.0009 | 0.0009 | 0.0013 | 0.0011 | 0.0013 | 0.0011 | 0.0011 | 0.0017 | 0.0018 | 0.0019 |
| 10 | 10 | 0.003 | 0.0017 | 0.0010 | 0.0010 | 0.0009 | 0.0007 | 0.0007 | 0.0006 | 0.0004 | 0.0005 | 0.0004 | 0.0004 | 0.0008 | 0.0011 | 0.002 | 0.002 |
| 1 | 25 | 0.007 | 0.0013 | 0.002 | 0.002 | 0.002 | 0.002 | 0.0015 | 0.0017 | 0.0012 | 0.0009 | 0.0009 | 0.0016 | 0.0018 | 0.0014 | 0.0013 | 0.003 |
| 2 | 25 | 0.006 | 0.0017 | 0.0017 | 0.0015 | 0.002 | 0.0010 | 0.0008 | 0.0009 | 0.0006 | 0.0008 | 0.0006 | 0.0009 | 0.0012 | 0.0017 | 0.003 | 0.003 |
| 3 | 25 | 0.006 | 0.002 | 0.0017 | 0.0010 | 0.0006 | 0.0006 | 0.0005 | 0.0007 | 0.0010 | 0.0009 | 0.0010 | 0.0010 | 0.0012 | 0.0015 | 0.0015 | 0.002 |
| 4 | 25 | 0.003 | 0.0019 | 0.0015 | 0.0011 | 0.0008 | 0.0008 | 0.0005 | 0.0005 | 0.0004 | 0.0003 | 0.0004 | 0.0005 | 0.0007 | 0.0006 | 0.0014 | 0.0012 |
| 5 | 25 | 0.005 | 0.002 | 0.0018 | 0.0011 | 0.0007 | 0.0007 | 0.0005 | 0.0007 | 0.0006 | 0.0008 | 0.0007 | 0.0007 | 0.0007 | 0.0009 | 0.0014 | 0.0012 |
| 10 | 25 | 0.002 | 0.0017 | 0.0010 | 0.0008 | 0.0004 | 0.0004 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0004 | 0.0005 | 0.0006 | 0.0007 | 0.0011 | 0.0014 |

## Appendix 1-5.  Standard Deviation, One Shot Group

**Table A1-5.1**.  Percentiles of the Standard Deviation (*SD*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|
| 2 | 2.3268 | 1.9849 | 1.8214 | 1.5849 | 1.3859 | 1.1631 | 0.9062 | 0.8134 |
| 3 | 2.1460 | 1.8794 | 1.7522 | 1.5682 | 1.4132 | 1.2390 | 1.0358 | 0.9614 |
| 4 | 2.0166 | 1.7915 | 1.6841 | 1.5288 | 1.3977 | 1.2501 | 1.0772 | 1.0135 |
| 5 | 1.9218 | 1.7240 | 1.6295 | 1.4929 | 1.3775 | 1.2474 | 1.0944 | 1.0378 |
| 6 | 1.8491 | 1.6708 | 1.5857 | 1.4624 | 1.3583 | 1.2407 | 1.1022 | 1.0508 |
| 7 | 1.7912 | 1.6278 | 1.5497 | 1.4367 | 1.3412 | 1.2332 | 1.1057 | 1.0584 |
| 8 | 1.7436 | 1.5921 | 1.5197 | 1.4148 | 1.3260 | 1.2256 | 1.1070 | 1.0628 |
| 9 | 1.7037 | 1.5619 | 1.4941 | 1.3958 | 1.3126 | 1.2184 | 1.1071 | 1.0656 |
| 10 | 1.6696 | 1.5359 | 1.4719 | 1.3792 | 1.3007 | 1.2118 | 1.1064 | 1.0672 |
| 11 | 1.6401 | 1.5132 | 1.4526 | 1.3646 | 1.2901 | 1.2056 | 1.1054 | 1.0681 |
| 12 | 1.6141 | 1.4932 | 1.4354 | 1.3515 | 1.2805 | 1.1998 | 1.1042 | 1.0685 |
| 13 | 1.5911 | 1.4754 | 1.4201 | 1.3398 | 1.2718 | 1.1945 | 1.1029 | 1.0686 |
| 14 | 1.5704 | 1.4594 | 1.4063 | 1.3292 | 1.2638 | 1.1896 | 1.1015 | 1.0685 |
| 15 | 1.5518 | 1.4450 | 1.3938 | 1.3196 | 1.2566 | 1.1850 | 1.1000 | 1.0682 |
| 16 | 1.5350 | 1.4318 | 1.3824 | 1.3107 | 1.2499 | 1.1808 | 1.0986 | 1.0679 |
| 17 | 1.5195 | 1.4198 | 1.3720 | 1.3026 | 1.2437 | 1.1768 | 1.0972 | 1.0674 |
| 18 | 1.5054 | 1.4087 | 1.3624 | 1.2951 | 1.2380 | 1.1731 | 1.0958 | 1.0669 |
| 19 | 1.4923 | 1.3984 | 1.3535 | 1.2881 | 1.2327 | 1.1696 | 1.0945 | 1.0663 |
| 20 | 1.4802 | 1.3889 | 1.3452 | 1.2816 | 1.2277 | 1.1663 | 1.0932 | 1.0658 |
| 21 | 1.4690 | 1.3801 | 1.3375 | 1.2756 | 1.2230 | 1.1632 | 1.0919 | 1.0652 |
| 22 | 1.4585 | 1.3718 | 1.3303 | 1.2699 | 1.2186 | 1.1602 | 1.0907 | 1.0646 |
| 23 | 1.4487 | 1.3641 | 1.3235 | 1.2646 | 1.2145 | 1.1575 | 1.0895 | 1.0640 |
| 24 | 1.4394 | 1.3568 | 1.3172 | 1.2596 | 1.2106 | 1.1548 | 1.0884 | 1.0634 |
| 25 | 1.4308 | 1.3499 | 1.3112 | 1.2548 | 1.2069 | 1.1523 | 1.0873 | 1.0628 |
| 26 | 1.4226 | 1.3435 | 1.3055 | 1.2503 | 1.2034 | 1.1499 | 1.0862 | 1.0623 |
| 27 | 1.4149 | 1.3374 | 1.3002 | 1.2461 | 1.2001 | 1.1477 | 1.0852 | 1.0617 |
| 28 | 1.4076 | 1.3316 | 1.2951 | 1.2420 | 1.1969 | 1.1455 | 1.0842 | 1.0611 |
| 29 | 1.4006 | 1.3260 | 1.2903 | 1.2382 | 1.1939 | 1.1434 | 1.0832 | 1.0606 |
| 30 | 1.3940 | 1.3208 | 1.2857 | 1.2345 | 1.1910 | 1.1415 | 1.0823 | 1.0600 |

**Table A1-5.1.** Percentiles of the Standard Deviation (*SD*). (CONTINUED)

| n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.4769 | 0.2253 | 0.1791 | 0.0889 | 0.0443 | 0.0222 | 0.0089 | 0.0044 |
| 3 | 0.6798 | 0.4379 | 0.3857 | 0.2650 | 0.1849 | 0.1299 | 0.0819 | 0.0578 |
| 4 | 0.7691 | 0.5506 | 0.5013 | 0.3822 | 0.2966 | 0.2323 | 0.1694 | 0.1339 |
| 5 | 0.8194 | 0.6201 | 0.5742 | 0.4612 | 0.3770 | 0.3113 | 0.2438 | 0.2035 |
| 6 | 0.8516 | 0.6677 | 0.6248 | 0.5181 | 0.4369 | 0.3722 | 0.3039 | 0.2620 |
| 7 | 0.8741 | 0.7025 | 0.6623 | 0.5611 | 0.4833 | 0.4204 | 0.3530 | 0.3107 |
| 8 | 0.8906 | 0.7293 | 0.6912 | 0.5951 | 0.5205 | 0.4596 | 0.3935 | 0.3516 |
| 9 | 0.9033 | 0.7506 | 0.7144 | 0.6227 | 0.5510 | 0.4921 | 0.4277 | 0.3865 |
| 10 | 0.9134 | 0.7680 | 0.7335 | 0.6456 | 0.5766 | 0.5197 | 0.4569 | 0.4165 |
| 11 | 0.9216 | 0.7826 | 0.7495 | 0.6650 | 0.5985 | 0.5433 | 0.4822 | 0.4427 |
| 12 | 0.9283 | 0.7950 | 0.7631 | 0.6818 | 0.6174 | 0.5639 | 0.5044 | 0.4658 |
| 13 | 0.9340 | 0.8057 | 0.7750 | 0.6964 | 0.6340 | 0.5820 | 0.5241 | 0.4863 |
| 14 | 0.9388 | 0.8150 | 0.7853 | 0.7092 | 0.6487 | 0.5981 | 0.5416 | 0.5046 |
| 15 | 0.9430 | 0.8232 | 0.7945 | 0.7206 | 0.6618 | 0.6126 | 0.5574 | 0.5212 |
| 16 | 0.9467 | 0.8305 | 0.8026 | 0.7309 | 0.6737 | 0.6256 | 0.5717 | 0.5362 |
| 17 | 0.9499 | 0.8371 | 0.8099 | 0.7401 | 0.6843 | 0.6374 | 0.5847 | 0.5500 |
| 18 | 0.9527 | 0.8430 | 0.8166 | 0.7485 | 0.6941 | 0.6483 | 0.5966 | 0.5626 |
| 19 | 0.9553 | 0.8484 | 0.8226 | 0.7562 | 0.7030 | 0.6582 | 0.6076 | 0.5742 |
| 20 | 0.9575 | 0.8533 | 0.8281 | 0.7632 | 0.7112 | 0.6673 | 0.6178 | 0.5850 |
| 21 | 0.9596 | 0.8578 | 0.8332 | 0.7697 | 0.7188 | 0.6758 | 0.6272 | 0.5950 |
| 22 | 0.9615 | 0.8619 | 0.8379 | 0.7758 | 0.7259 | 0.6837 | 0.6359 | 0.6043 |
| 23 | 0.9632 | 0.8658 | 0.8422 | 0.7813 | 0.7324 | 0.6910 | 0.6441 | 0.6130 |
| 24 | 0.9647 | 0.8693 | 0.8462 | 0.7866 | 0.7385 | 0.6979 | 0.6518 | 0.6212 |
| 25 | 0.9662 | 0.8726 | 0.8500 | 0.7914 | 0.7443 | 0.7043 | 0.6590 | 0.6288 |
| 26 | 0.9675 | 0.8757 | 0.8535 | 0.7960 | 0.7497 | 0.7104 | 0.6658 | 0.6361 |
| 27 | 0.9687 | 0.8786 | 0.8568 | 0.8003 | 0.7547 | 0.7161 | 0.6721 | 0.6429 |
| 28 | 0.9698 | 0.8813 | 0.8599 | 0.8043 | 0.7595 | 0.7214 | 0.6782 | 0.6494 |
| 29 | 0.9709 | 0.8839 | 0.8628 | 0.8081 | 0.7640 | 0.7265 | 0.6839 | 0.6555 |
| 30 | 0.9719 | 0.8863 | 0.8655 | 0.8117 | 0.7683 | 0.7314 | 0.6894 | 0.6613 |

## Appendix 1-6. RMS of the Standard Deviations, Multiple Shot Groups

**Table A1-6.1.** Percentiles of the RMS of the Standard Deviations (*RMS-SD*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2.3268 | 1.9849 | 1.8214 | 1.5849 | 1.3859 | 1.1631 | 0.9062 | 0.8134 |
| 2 | | 1.8585 | 1.6276 | 1.5174 | 1.3581 | 1.2239 | 1.0730 | 0.8971 | 0.8326 |
| 3 | | 1.6465 | 1.4628 | 1.3751 | 1.2482 | 1.1413 | 1.0207 | 0.8795 | 0.8275 |
| 4 | | 1.5193 | 1.3629 | 1.2882 | 1.1802 | 1.0890 | 0.9861 | 0.8652 | 0.8205 |
| 5 | | 1.4323 | 1.2942 | 1.2283 | 1.1328 | 1.0522 | 0.9611 | 0.8538 | 0.8140 |
| 10 | | 1.2163 | 1.1222 | 1.0772 | 1.0120 | 0.9567 | 0.8941 | 0.8198 | 0.7921 |
| 1 | 3 | 2.1460 | 1.8794 | 1.7522 | 1.5682 | 1.4132 | 1.2390 | 1.0358 | 0.9614 |
| 2 | | 1.7544 | 1.5738 | 1.4875 | 1.3628 | 1.2575 | 1.1387 | 0.9991 | 0.9474 |
| 3 | | 1.5797 | 1.4356 | 1.3667 | 1.2671 | 1.1828 | 1.0875 | 0.9751 | 0.9334 |
| 4 | | 1.4755 | 1.3526 | 1.2939 | 1.2088 | 1.1368 | 1.0552 | 0.9587 | 0.9228 |
| 5 | | 1.4045 | 1.2958 | 1.2439 | 1.1686 | 1.1047 | 1.0324 | 0.9466 | 0.9147 |
| 10 | | 1.2290 | 1.1547 | 1.1190 | 1.0672 | 1.0232 | 0.9732 | 0.9136 | 0.8912 |
| 1 | 4 | 2.0166 | 1.7915 | 1.6841 | 1.5288 | 1.3977 | 1.2501 | 1.0772 | 1.0135 |
| 2 | | 1.6755 | 1.5226 | 1.4497 | 1.3439 | 1.2546 | 1.1535 | 1.0343 | 0.9900 |
| 3 | | 1.5242 | 1.4021 | 1.3437 | 1.2591 | 1.1874 | 1.1062 | 1.0100 | 0.9742 |
| 4 | | 1.4342 | 1.3299 | 1.2801 | 1.2077 | 1.1464 | 1.0767 | 0.9941 | 0.9632 |
| 5 | | 1.3729 | 1.2807 | 1.2365 | 1.1724 | 1.1179 | 1.0561 | 0.9826 | 0.9551 |
| 10 | | 1.2217 | 1.1584 | 1.1280 | 1.0837 | 1.0461 | 1.0032 | 0.9520 | 0.9327 |
| 1 | 5 | 1.9218 | 1.7240 | 1.6295 | 1.4929 | 1.3775 | 1.2474 | 1.0944 | 1.0378 |
| 2 | | 1.6163 | 1.4817 | 1.4174 | 1.3242 | 1.2453 | 1.1559 | 1.0502 | 1.0109 |
| 3 | | 1.4812 | 1.3735 | 1.3220 | 1.2473 | 1.1840 | 1.1120 | 1.0267 | 0.9948 |
| 4 | | 1.4009 | 1.3090 | 1.2649 | 1.2009 | 1.1467 | 1.0849 | 1.0116 | 0.9841 |
| 5 | | 1.3463 | 1.2649 | 1.2258 | 1.1691 | 1.1209 | 1.0661 | 1.0007 | 0.9763 |
| 10 | | 1.2116 | 1.1556 | 1.1286 | 1.0894 | 1.0560 | 1.0179 | 0.9723 | 0.9552 |
| 1 | 10 | 1.6696 | 1.5359 | 1.4719 | 1.3792 | 1.3007 | 1.2118 | 1.1064 | 1.0672 |
| 2 | | 1.4545 | 1.3630 | 1.3192 | 1.2555 | 1.2014 | 1.1399 | 1.0668 | 1.0393 |
| 3 | | 1.3599 | 1.2864 | 1.2512 | 1.1999 | 1.1563 | 1.1067 | 1.0474 | 1.0252 |
| 4 | | 1.3037 | 1.2408 | 1.2106 | 1.1666 | 1.1291 | 1.0864 | 1.0354 | 1.0162 |
| 5 | | 1.2655 | 1.2097 | 1.1829 | 1.1438 | 1.1105 | 1.0724 | 1.0269 | 1.0098 |
| 10 | | 1.1714 | 1.1327 | 1.1141 | 1.0869 | 1.0637 | 1.0371 | 1.0053 | 0.9932 |
| 1 | 25 | 1.4308 | 1.3499 | 1.3112 | 1.2548 | 1.2069 | 1.1523 | 1.0873 | 1.0628 |
| 2 | | 1.2964 | 1.2407 | 1.2139 | 1.1749 | 1.1417 | 1.1037 | 1.0582 | 1.0411 |
| 3 | | 1.2374 | 1.1925 | 1.1708 | 1.1393 | 1.1124 | 1.0816 | 1.0447 | 1.0308 |
| 4 | | 1.2024 | 1.1638 | 1.1452 | 1.1180 | 1.0949 | 1.0683 | 1.0365 | 1.0244 |
| 5 | | 1.1785 | 1.1442 | 1.1277 | 1.1035 | 1.0828 | 1.0592 | 1.0308 | 1.0200 |
| 10 | | 1.1197 | 1.0958 | 1.0842 | 1.0673 | 1.0529 | 1.0363 | 1.0163 | 1.0087 |

**Table A1-6.1.** Percentiles of the RMS of the Standard Deviations (*RMS-SD*). (CONTINUED)

| k | n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.4769 | 0.2253 | 0.1791 | 0.0889 | 0.0443 | 0.0222 | 0.0089 | 0.0044 |
| 2 | | 0.5887 | 0.3793 | 0.3340 | 0.2295 | 0.1601 | 0.1125 | 0.0709 | 0.0501 |
| 3 | | 0.6280 | 0.4495 | 0.4093 | 0.3121 | 0.2422 | 0.1896 | 0.1383 | 0.1093 |
| 4 | | 0.6478 | 0.4902 | 0.4540 | 0.3646 | 0.2981 | 0.2461 | 0.1927 | 0.1609 |
| 5 | | 0.6597 | 0.5172 | 0.4840 | 0.4013 | 0.3384 | 0.2883 | 0.2354 | 0.2029 |
| 10 | | 0.6834 | 0.5804 | 0.5558 | 0.4932 | 0.4439 | 0.4029 | 0.3576 | 0.3283 |
| 1 | 3 | 0.6798 | 0.4379 | 0.3857 | 0.2650 | 0.1849 | 0.1299 | 0.0819 | 0.0578 |
| 2 | | 0.7480 | 0.5661 | 0.5242 | 0.4210 | 0.3442 | 0.2841 | 0.2225 | 0.1857 |
| 3 | | 0.7709 | 0.6196 | 0.5841 | 0.4949 | 0.4263 | 0.3708 | 0.3113 | 0.2740 |
| 4 | | 0.7823 | 0.6500 | 0.6187 | 0.5393 | 0.4772 | 0.4262 | 0.3704 | 0.3347 |
| 5 | | 0.7892 | 0.6702 | 0.6418 | 0.5695 | 0.5125 | 0.4653 | 0.4130 | 0.3791 |
| 10 | | 0.8029 | 0.7177 | 0.6971 | 0.6440 | 0.6014 | 0.5654 | 0.5247 | 0.4978 |
| 1 | 4 | 0.7691 | 0.5506 | 0.5013 | 0.3822 | 0.2966 | 0.2323 | 0.1694 | 0.1339 |
| 2 | | 0.8176 | 0.6571 | 0.6195 | 0.5249 | 0.4521 | 0.3933 | 0.3302 | 0.2906 |
| 3 | | 0.8338 | 0.7011 | 0.6696 | 0.5894 | 0.5264 | 0.4744 | 0.4171 | 0.3802 |
| 4 | | 0.8419 | 0.7262 | 0.6985 | 0.6277 | 0.5715 | 0.5246 | 0.4724 | 0.4383 |
| 5 | | 0.8467 | 0.7429 | 0.7179 | 0.6537 | 0.6025 | 0.5596 | 0.5113 | 0.4796 |
| 10 | | 0.8564 | 0.7823 | 0.7643 | 0.7176 | 0.6799 | 0.6479 | 0.6114 | 0.5871 |
| 1 | 5 | 0.8194 | 0.6201 | 0.5742 | 0.4612 | 0.3770 | 0.3113 | 0.2438 | 0.2035 |
| 2 | | 0.8570 | 0.7121 | 0.6778 | 0.5907 | 0.5227 | 0.4669 | 0.4058 | 0.3667 |
| 3 | | 0.8695 | 0.7500 | 0.7214 | 0.6483 | 0.5903 | 0.5418 | 0.4879 | 0.4527 |
| 4 | | 0.8757 | 0.7718 | 0.7467 | 0.6824 | 0.6309 | 0.5877 | 0.5391 | 0.5071 |
| 5 | | 0.8795 | 0.7862 | 0.7636 | 0.7055 | 0.6588 | 0.6194 | 0.5748 | 0.5453 |
| 10 | | 0.8870 | 0.8205 | 0.8043 | 0.7622 | 0.7281 | 0.6990 | 0.6658 | 0.6435 |
| 1 | 10 | 0.9134 | 0.7680 | 0.7335 | 0.6456 | 0.5766 | 0.5197 | 0.4569 | 0.4165 |
| 2 | | 0.9311 | 0.8269 | 0.8018 | 0.7371 | 0.6852 | 0.6415 | 0.5922 | 0.5597 |
| 3 | | 0.9370 | 0.8515 | 0.8307 | 0.7770 | 0.7337 | 0.6970 | 0.6552 | 0.6274 |
| 4 | | 0.9399 | 0.8656 | 0.8476 | 0.8007 | 0.7627 | 0.7303 | 0.6934 | 0.6687 |
| 5 | | 0.9416 | 0.8751 | 0.8589 | 0.8167 | 0.7825 | 0.7532 | 0.7197 | 0.6973 |
| 10 | | 0.9452 | 0.8979 | 0.8863 | 0.8561 | 0.8314 | 0.8102 | 0.7858 | 0.7694 |
| 1 | 25 | 0.9662 | 0.8726 | 0.8500 | 0.7914 | 0.7443 | 0.7043 | 0.6590 | 0.6288 |
| 2 | | 0.9730 | 0.9064 | 0.8902 | 0.8479 | 0.8136 | 0.7843 | 0.7507 | 0.7282 |
| 3 | | 0.9753 | 0.9208 | 0.9074 | 0.8726 | 0.8443 | 0.8200 | 0.7921 | 0.7732 |
| 4 | | 0.9764 | 0.9291 | 0.9175 | 0.8873 | 0.8626 | 0.8413 | 0.8169 | 0.8004 |
| 5 | | 0.9771 | 0.9347 | 0.9244 | 0.8972 | 0.8750 | 0.8559 | 0.8339 | 0.8190 |
| 10 | | 0.9784 | 0.9484 | 0.9410 | 0.9217 | 0.9058 | 0.8922 | 0.8763 | 0.8656 |

## Appendix 1-7.  Extreme Spread, One Shot Group

**Table A1-7.1**.  Percentiles of the Extreme Spread (*ES*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|
| 2 | 5.2565 | 4.6036 | 4.2919 | 3.8413 | 3.4616 | 3.0349 | 2.5373 | 2.3548 |
| 3 | 5.66 | 5.05 | 4.74 | 4.329 | 3.986 | 3.599 | 3.147 | 2.981 |
| 4 | 5.90 | 5.30 | 5.01 | 4.625 | 4.291 | 3.927 | 3.497 | 3.342 |
| 5 | 6.04 | 5.49 | 5.22 | 4.83 | 4.515 | 4.156 | 3.743 | 3.589 |
| 6 | 6.17 | 5.628 | 5.36 | 4.987 | 4.679 | 4.332 | 3.935 | 3.786 |
| 7 | 6.26 | 5.73 | 5.480 | 5.111 | 4.811 | 4.472 | 4.083 | 3.938 |
| 8 | 6.34 | 5.82 | 5.569 | 5.222 | 4.924 | 4.595 | 4.211 | 4.072 |
| 9 | 6.47 | 5.91 | 5.665 | 5.314 | 5.015 | 4.692 | 4.319 | 4.180 |
| 10 | 6.49 | 6.00 | 5.75 | 5.403 | 5.106 | 4.789 | 4.419 | 4.285 |
| 11 | 6.56 | 6.06 | 5.81 | 5.462 | 5.176 | 4.862 | 4.499 | 4.366 |
| 12 | 6.60 | 6.11 | 5.878 | 5.532 | 5.250 | 4.935 | 4.573 | 4.441 |
| 13 | 6.67 | 6.16 | 5.92 | 5.591 | 5.312 | 4.999 | 4.640 | 4.510 |
| 14 | 6.71 | 6.22 | 5.986 | 5.648 | 5.360 | 5.055 | 4.701 | 4.574 |
| 15 | 6.76 | 6.27 | 6.037 | 5.695 | 5.424 | 5.115 | 4.763 | 4.636 |
| 16 | 6.81 | 6.30 | 6.07 | 5.743 | 5.474 | 5.167 | 4.819 | 4.690 |
| 17 | 6.86 | 6.35 | 6.110 | 5.774 | 5.505 | 5.204 | 4.861 | 4.736 |
| 18 | 6.83 | 6.38 | 6.15 | 5.824 | 5.554 | 5.252 | 4.910 | 4.785 |
| 19 | 6.93 | 6.42 | 6.199 | 5.862 | 5.592 | 5.293 | 4.952 | 4.829 |
| 20 | 6.96 | 6.453 | 6.226 | 5.897 | 5.623 | 5.330 | 4.994 | 4.872 |
| 21 | 6.99 | 6.49 | 6.26 | 5.936 | 5.658 | 5.365 | 5.032 | 4.910 |
| 22 | 7.00 | 6.50 | 6.270 | 5.957 | 5.696 | 5.401 | 5.066 | 4.946 |
| 23 | 7.01 | 6.54 | 6.31 | 5.994 | 5.720 | 5.433 | 5.101 | 4.979 |
| 24 | 7.07 | 6.56 | 6.34 | 6.019 | 5.757 | 5.466 | 5.134 | 5.014 |
| 25 | 7.09 | 6.60 | 6.371 | 6.051 | 5.784 | 5.496 | 5.162 | 5.044 |
| 26 | 7.10 | 6.61 | 6.384 | 6.067 | 5.809 | 5.521 | 5.193 | 5.075 |
| 27 | 7.09 | 6.62 | 6.401 | 6.09 | 5.831 | 5.553 | 5.224 | 5.105 |
| 28 | 7.12 | 6.66 | 6.43 | 6.120 | 5.859 | 5.569 | 5.247 | 5.129 |
| 29 | 7.16 | 6.69 | 6.46 | 6.14 | 5.884 | 5.598 | 5.276 | 5.159 |
| 30 | 7.18 | 6.71 | 6.49 | 6.167 | 5.908 | 5.624 | 5.300 | 5.184 |

**Table A1-7.1.** Percentiles of the Extreme Spread (*ES*). (CONTINUED)

| n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|
| 2 | 1.6651 | 1.0727 | 0.9448 | 0.6492 | 0.4530 | 0.3182 | 0.2005 | 0.1416 |
| 3 | 2.341 | 1.764 | 1.631 | 1.306 | 1.067 | 0.881 | 0.691 | 0.573 |
| 4 | 2.735 | 2.186 | 2.057 | 1.739 | 1.497 | 1.302 | 1.090 | 0.958 |
| 5 | 3.005 | 2.474 | 2.352 | 2.043 | 1.804 | 1.604 | 1.390 | 1.255 |
| 6 | 3.222 | 2.708 | 2.589 | 2.289 | 2.048 | 1.855 | 1.634 | 1.500 |
| 7 | 3.393 | 2.894 | 2.776 | 2.478 | 2.246 | 2.057 | 1.847 | 1.716 |
| 8 | 3.532 | 3.045 | 2.930 | 2.641 | 2.411 | 2.227 | 2.012 | 1.878 |
| 9 | 3.657 | 3.177 | 3.064 | 2.779 | 2.556 | 2.372 | 2.164 | 2.030 |
| 10 | 3.766 | 3.291 | 3.181 | 2.903 | 2.686 | 2.502 | 2.300 | 2.173 |
| 11 | 3.855 | 3.392 | 3.285 | 3.009 | 2.791 | 2.611 | 2.415 | 2.286 |
| 12 | 3.942 | 3.485 | 3.377 | 3.103 | 2.893 | 2.717 | 2.520 | 2.391 |
| 13 | 4.014 | 3.566 | 3.459 | 3.192 | 2.983 | 2.805 | 2.615 | 2.49 |
| 14 | 4.085 | 3.636 | 3.532 | 3.268 | 3.062 | 2.888 | 2.693 | 2.569 |
| 15 | 4.150 | 3.707 | 3.606 | 3.345 | 3.133 | 2.959 | 2.768 | 2.645 |
| 16 | 4.210 | 3.771 | 3.668 | 3.407 | 3.204 | 3.037 | 2.850 | 2.726 |
| 17 | 4.261 | 3.827 | 3.725 | 3.468 | 3.266 | 3.097 | 2.907 | 2.784 |
| 18 | 4.314 | 3.883 | 3.784 | 3.527 | 3.327 | 3.158 | 2.969 | 2.844 |
| 19 | 4.359 | 3.935 | 3.835 | 3.581 | 3.383 | 3.220 | 3.035 | 2.907 |
| 20 | 4.405 | 3.981 | 3.883 | 3.632 | 3.438 | 3.274 | 3.088 | 2.971 |
| 21 | 4.449 | 4.027 | 3.930 | 3.683 | 3.488 | 3.328 | 3.142 | 3.019 |
| 22 | 4.486 | 4.070 | 3.973 | 3.726 | 3.533 | 3.374 | 3.198 | 3.078 |
| 23 | 4.525 | 4.111 | 4.013 | 3.768 | 3.578 | 3.418 | 3.246 | 3.126 |
| 24 | 4.561 | 4.146 | 4.051 | 3.808 | 3.615 | 3.459 | 3.283 | 3.170 |
| 25 | 4.592 | 4.184 | 4.089 | 3.848 | 3.657 | 3.500 | 3.323 | 3.212 |
| 26 | 4.625 | 4.219 | 4.123 | 3.886 | 3.697 | 3.540 | 3.369 | 3.257 |
| 27 | 4.657 | 4.252 | 4.157 | 3.920 | 3.732 | 3.576 | 3.407 | 3.301 |
| 28 | 4.688 | 4.287 | 4.192 | 3.954 | 3.767 | 3.613 | 3.442 | 3.329 |
| 29 | 4.715 | 4.316 | 4.222 | 3.988 | 3.802 | 3.648 | 3.48 | 3.36 |
| 30 | 4.744 | 4.347 | 4.253 | 4.017 | 3.837 | 3.683 | 3.515 | 3.405 |

**Table A1-7.2**. Monte Carlo Errors of the Percentiles of the Extreme Spread (*ES*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | | | | | |
| 3 | 0.09 | 0.05 | 0.04 | 0.02 | 0.02 | 0.014 | 0.012 | 0.011 | 0.010 | 0.008 | 0.007 | 0.009 | 0.006 | 0.010 | 0.014 | 0.018 |
| 4 | 0.09 | 0.06 | 0.03 | 0.014 | 0.010 | 0.007 | 0.007 | 0.006 | 0.007 | 0.008 | 0.006 | 0.007 | 0.008 | 0.009 | 0.009 | 0.013 |
| 5 | 0.06 | 0.04 | 0.03 | 0.03 | 0.02 | 0.011 | 0.009 | 0.006 | 0.008 | 0.007 | 0.006 | 0.008 | 0.012 | 0.010 | 0.02 | 0.02 |
| 6 | 0.07 | 0.02 | 0.03 | 0.015 | 0.007 | 0.007 | 0.009 | 0.009 | 0.005 | 0.007 | 0.007 | 0.006 | 0.010 | 0.011 | 0.017 | 0.02 |
| 7 | 0.06 | 0.04 | 0.02 | 0.02 | 0.008 | 0.010 | 0.011 | 0.008 | 0.007 | 0.007 | 0.008 | 0.008 | 0.005 | 0.009 | 0.013 | 0.014 |
| 8 | 0.07 | 0.04 | 0.02 | 0.013 | 0.008 | 0.005 | 0.006 | 0.006 | 0.006 | 0.008 | 0.009 | 0.009 | 0.009 | 0.011 | 0.014 | 0.014 |
| 9 | 0.04 | 0.03 | 0.013 | 0.017 | 0.013 | 0.008 | 0.011 | 0.009 | 0.004 | 0.005 | 0.005 | 0.004 | 0.008 | 0.011 | 0.015 | 0.02 |
| 10 | 0.08 | 0.03 | 0.03 | 0.018 | 0.012 | 0.010 | 0.009 | 0.008 | 0.006 | 0.008 | 0.007 | 0.008 | 0.010 | 0.014 | 0.015 | 0.013 |
| 11 | 0.06 | 0.04 | 0.03 | 0.016 | 0.014 | 0.011 | 0.009 | 0.009 | 0.009 | 0.007 | 0.007 | 0.010 | 0.009 | 0.009 | 0.017 | 0.02 |
| 12 | 0.06 | 0.03 | 0.018 | 0.02 | 0.015 | 0.007 | 0.006 | 0.007 | 0.006 | 0.008 | 0.008 | 0.009 | 0.011 | 0.013 | 0.018 | 0.02 |
| 13 | 0.05 | 0.04 | 0.03 | 0.02 | 0.014 | 0.012 | 0.008 | 0.007 | 0.006 | 0.006 | 0.007 | 0.009 | 0.012 | 0.009 | 0.013 | 0.03 |
| 14 | 0.05 | 0.03 | 0.02 | 0.014 | 0.01 | 0.012 | 0.007 | 0.007 | 0.006 | 0.005 | 0.006 | 0.006 | 0.006 | 0.012 | 0.01 | 0.015 |
| 15 | 0.08 | 0.03 | 0.019 | 0.02 | 0.014 | 0.008 | 0.006 | 0.006 | 0.005 | 0.007 | 0.008 | 0.009 | 0.007 | 0.009 | 0.014 | 0.016 |
| 16 | 0.04 | 0.03 | 0.03 | 0.02 | 0.016 | 0.007 | 0.004 | 0.006 | 0.006 | 0.004 | 0.005 | 0.007 | 0.008 | 0.009 | 0.009 | 0.012 |
| 17 | 0.06 | 0.04 | 0.02 | 0.015 | 0.006 | 0.010 | 0.009 | 0.008 | 0.006 | 0.006 | 0.007 | 0.010 | 0.011 | 0.012 | 0.011 | 0.019 |
| 18 | 0.08 | 0.04 | 0.03 | 0.018 | 0.016 | 0.012 | 0.008 | 0.008 | 0.007 | 0.007 | 0.007 | 0.007 | 0.007 | 0.011 | 0.016 | 0.017 |
| 19 | 0.07 | 0.03 | 0.02 | 0.019 | 0.013 | 0.007 | 0.008 | 0.007 | 0.004 | 0.007 | 0.006 | 0.009 | 0.008 | 0.011 | 0.011 | 0.02 |
| 20 | 0.08 | 0.02 | 0.018 | 0.017 | 0.014 | 0.010 | 0.008 | 0.006 | 0.005 | 0.006 | 0.004 | 0.005 | 0.010 | 0.011 | 0.011 | 0.015 |
| 21 | 0.08 | 0.04 | 0.03 | 0.016 | 0.013 | 0.014 | 0.006 | 0.007 | 0.006 | 0.009 | 0.009 | 0.009 | 0.009 | 0.012 | 0.005 | 0.014 |
| 22 | 0.06 | 0.03 | 0.02 | 0.02 | 0.011 | 0.008 | 0.003 | 0.005 | 0.005 | 0.006 | 0.006 | 0.008 | 0.008 | 0.012 | 0.013 | 0.012 |
| 23 | 0.08 | 0.04 | 0.03 | 0.019 | 0.014 | 0.008 | 0.006 | 0.008 | 0.006 | 0.005 | 0.006 | 0.007 | 0.009 | 0.013 | 0.012 | 0.019 |
| 24 | 0.08 | 0.04 | 0.03 | 0.015 | 0.015 | 0.009 | 0.006 | 0.006 | 0.008 | 0.006 | 0.005 | 0.007 | 0.008 | 0.011 | 0.013 | 0.006 |
| 25 | 0.07 | 0.04 | 0.017 | 0.02 | 0.014 | 0.006 | 0.005 | 0.006 | 0.007 | 0.005 | 0.005 | 0.008 | 0.012 | 0.013 | 0.015 | 0.018 |
| 26 | 0.04 | 0.03 | 0.02 | 0.016 | 0.012 | 0.012 | 0.006 | 0.006 | 0.005 | 0.004 | 0.005 | 0.005 | 0.006 | 0.009 | 0.013 | 0.011 |
| 27 | 0.06 | 0.03 | 0.02 | 0.03 | 0.02 | 0.012 | 0.008 | 0.007 | 0.005 | 0.006 | 0.007 | 0.011 | 0.011 | 0.014 | 0.017 | 0.02 |
| 28 | 0.05 | 0.03 | 0.03 | 0.019 | 0.015 | 0.013 | 0.010 | 0.007 | 0.007 | 0.007 | 0.008 | 0.004 | 0.006 | 0.009 | 0.012 | 0.011 |
| 29 | 0.04 | 0.04 | 0.03 | 0.03 | 0.019 | 0.012 | 0.007 | 0.005 | 0.007 | 0.003 | 0.003 | 0.006 | 0.007 | 0.005 | 0.03 | 0.050 |
| 30 | 0.05 | 0.03 | 0.03 | 0.013 | 0.011 | 0.007 | 0.003 | 0.003 | 0.006 | 0.006 | 0.006 | 0.005 | 0.007 | 0.009 | 0.009 | 0.02 |

## Appendix 1-8. Average Extreme Spread, Multiple Shot Groups

**Table A1-8.1.1**. Percentiles of the Average Extreme Spread (*AES*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 5.2565 | 4.6036 | 4.2919 | 3.8413 | 3.4616 | 3.0349 | 2.5373 | 2.3548 |
| 2 | | 4.17 | 3.71 | 3.500 | 3.192 | 2.934 | 2.650 | 2.313 | 2.192 |
| 3 | | 3.630 | 3.31 | 3.156 | 2.916 | 2.711 | 2.480 | 2.214 | 2.117 |
| 4 | | 3.38 | 3.090 | 2.952 | 2.750 | 2.579 | 2.382 | 2.155 | 2.071 |
| 5 | | 3.200 | 2.933 | 2.811 | 2.640 | 2.487 | 2.315 | 2.116 | 2.041 |
| 10 | | 2.748 | 2.571 | 2.490 | 2.370 | 2.269 | 2.152 | 2.016 | 1.965 |
| 1 | 3 | 5.66 | 5.05 | 4.74 | 4.329 | 3.986 | 3.599 | 3.147 | 2.981 |
| 2 | | 4.60 | 4.21 | 4.014 | 3.739 | 3.502 | 3.241 | 2.931 | 2.814 |
| 3 | | 4.18 | 3.85 | 3.698 | 3.481 | 3.291 | 3.081 | 2.835 | 2.745 |
| 4 | | 3.91 | 3.643 | 3.519 | 3.331 | 3.173 | 2.991 | 2.781 | 2.703 |
| 5 | | 3.75 | 3.513 | 3.396 | 3.232 | 3.092 | 2.930 | 2.742 | 2.671 |
| 10 | | 3.343 | 3.172 | 3.092 | 2.979 | 2.884 | 2.773 | 2.644 | 2.595 |
| 1 | 4 | 5.90 | 5.30 | 5.01 | 4.625 | 4.291 | 3.927 | 3.497 | 3.342 |
| 2 | | 4.90 | 4.50 | 4.32 | 4.057 | 3.837 | 3.588 | 3.295 | 3.187 |
| 3 | | 4.48 | 4.15 | 4.017 | 3.813 | 3.638 | 3.439 | 3.206 | 3.118 |
| 4 | | 4.221 | 3.972 | 3.853 | 3.673 | 3.520 | 3.353 | 3.149 | 3.075 |
| 5 | | 4.072 | 3.838 | 3.731 | 3.574 | 3.443 | 3.291 | 3.114 | 3.045 |
| 10 | | 3.678 | 3.523 | 3.451 | 3.342 | 3.250 | 3.147 | 3.021 | 2.974 |
| 1 | 5 | 6.04 | 5.49 | 5.22 | 4.83 | 4.515 | 4.156 | 3.743 | 3.589 |
| 2 | | 5.10 | 4.74 | 4.553 | 4.295 | 4.077 | 3.836 | 3.550 | 3.445 |
| 3 | | 4.67 | 4.38 | 4.25 | 4.048 | 3.881 | 3.688 | 3.462 | 3.377 |
| 4 | | 4.45 | 4.202 | 4.080 | 3.913 | 3.765 | 3.601 | 3.408 | 3.335 |
| 5 | | 4.30 | 4.075 | 3.972 | 3.821 | 3.691 | 3.548 | 3.374 | 3.311 |
| 10 | | 3.918 | 3.772 | 3.699 | 3.594 | 3.505 | 3.405 | 3.285 | 3.240 |
| 1 | 10 | 6.49 | 6.00 | 5.75 | 5.403 | 5.106 | 4.789 | 4.419 | 4.285 |
| 2 | | 5.65 | 5.298 | 5.140 | 4.905 | 4.710 | 4.495 | 4.243 | 4.151 |
| 3 | | 5.28 | 5.010 | 4.880 | 4.695 | 4.542 | 4.369 | 4.168 | 4.091 |
| 4 | | 5.063 | 4.838 | 4.730 | 4.572 | 4.442 | 4.294 | 4.120 | 4.055 |
| 5 | | 4.92 | 4.719 | 4.629 | 4.488 | 4.373 | 4.243 | 4.088 | 4.029 |
| 10 | | 4.574 | 4.438 | 4.376 | 4.283 | 4.204 | 4.113 | 4.006 | 3.965 |
| 1 | 25 | 7.09 | 6.60 | 6.371 | 6.051 | 5.784 | 5.496 | 5.162 | 5.044 |
| 2 | | 6.27 | 5.958 | 5.813 | 5.602 | 5.430 | 5.242 | 5.016 | 4.935 |
| 3 | | 5.931 | 5.690 | 5.581 | 5.419 | 5.283 | 5.130 | 4.949 | 4.883 |
| 4 | | 5.754 | 5.544 | 5.443 | 5.306 | 5.192 | 5.061 | 4.909 | 4.852 |
| 5 | | 5.618 | 5.442 | 5.358 | 5.237 | 5.133 | 5.019 | 4.883 | 4.831 |
| 10 | | 5.321 | 5.196 | 5.140 | 5.058 | 4.987 | 4.907 | 4.812 | 4.777 |

**Table A1-8.1.** Percentiles of the Average Extreme Spread (*AES*). (CONTINUED)

| k | n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1.6651 | 1.0727 | 0.9448 | 0.6492 | 0.4530 | 0.3182 | 0.2005 | 0.1416 |
| 2 | | 1.723 | 1.301 | 1.204 | 0.966 | 0.788 | 0.649 | 0.505 | 0.418 |
| 3 | | 1.740 | 1.392 | 1.310 | 1.106 | 0.952 | 0.826 | 0.692 | 0.610 |
| 4 | | 1.748 | 1.445 | 1.373 | 1.196 | 1.056 | 0.942 | 0.817 | 0.739 |
| 5 | | 1.752 | 1.482 | 1.417 | 1.256 | 1.126 | 1.021 | 0.904 | 0.831 |
| 10 | | 1.762 | 1.570 | 1.524 | 1.404 | 1.307 | 1.226 | 1.135 | 1.077 |
| 1 | 3 | 2.341 | 1.764 | 1.631 | 1.306 | 1.067 | 0.881 | 0.691 | 0.573 |
| 2 | | 2.373 | 1.962 | 1.865 | 1.621 | 1.428 | 1.274 | 1.106 | 0.999 |
| 3 | | 2.386 | 2.048 | 1.968 | 1.762 | 1.602 | 1.467 | 1.322 | 1.223 |
| 4 | | 2.393 | 2.097 | 2.027 | 1.848 | 1.706 | 1.587 | 1.457 | 1.367 |
| 5 | | 2.395 | 2.132 | 2.067 | 1.904 | 1.775 | 1.667 | 1.548 | 1.468 |
| 10 | | 2.401 | 2.215 | 2.169 | 2.051 | 1.957 | 1.876 | 1.785 | 1.725 |
| 1 | 4 | 2.735 | 2.186 | 2.057 | 1.739 | 1.497 | 1.302 | 1.090 | 0.958 |
| 2 | | 2.766 | 2.368 | 2.274 | 2.036 | 1.850 | 1.695 | 1.526 | 1.410 |
| 3 | | 2.775 | 2.450 | 2.372 | 2.172 | 2.016 | 1.882 | 1.727 | 1.632 |
| 4 | | 2.780 | 2.497 | 2.429 | 2.253 | 2.113 | 1.997 | 1.858 | 1.768 |
| 5 | | 2.782 | 2.530 | 2.469 | 2.310 | 2.183 | 2.077 | 1.954 | 1.875 |
| 10 | | 2.787 | 2.607 | 2.563 | 2.450 | 2.359 | 2.280 | 2.189 | 2.128 |
| 1 | 5 | 3.005 | 2.474 | 2.352 | 2.043 | 1.804 | 1.604 | 1.390 | 1.255 |
| 2 | | 3.040 | 2.657 | 2.566 | 2.336 | 2.154 | 2.003 | 1.832 | 1.716 |
| 3 | | 3.048 | 2.732 | 2.658 | 2.465 | 2.311 | 2.183 | 2.042 | 1.942 |
| 4 | | 3.049 | 2.778 | 2.712 | 2.543 | 2.409 | 2.293 | 2.165 | 2.076 |
| 5 | | 3.056 | 2.811 | 2.751 | 2.599 | 2.477 | 2.373 | 2.255 | 2.173 |
| 10 | | 3.061 | 2.886 | 2.844 | 2.733 | 2.644 | 2.569 | 2.478 | 2.419 |
| 1 | 10 | 3.766 | 3.291 | 3.181 | 2.903 | 2.686 | 2.502 | 2.300 | 2.173 |
| 2 | | 3.789 | 3.450 | 3.368 | 3.160 | 2.994 | 2.855 | 2.695 | 2.594 |
| 3 | | 3.796 | 3.515 | 3.447 | 3.275 | 3.136 | 3.017 | 2.884 | 2.796 |
| 4 | | 3.800 | 3.556 | 3.498 | 3.346 | 3.224 | 3.120 | 3.001 | 2.927 |
| 5 | | 3.802 | 3.582 | 3.530 | 3.394 | 3.285 | 3.192 | 3.085 | 3.016 |
| 10 | | 3.806 | 3.650 | 3.612 | 3.514 | 3.434 | 3.365 | 3.287 | 3.234 |
| 1 | 25 | 4.592 | 4.184 | 4.089 | 3.848 | 3.657 | 3.500 | 3.323 | 3.212 |
| 2 | | 4.618 | 4.321 | 4.252 | 4.072 | 3.927 | 3.808 | 3.672 | 3.582 |
| 3 | | 4.625 | 4.380 | 4.322 | 4.172 | 4.051 | 3.950 | 3.838 | 3.761 |
| 4 | | 4.629 | 4.416 | 4.365 | 4.234 | 4.130 | 4.041 | 3.940 | 3.872 |
| 5 | | 4.631 | 4.440 | 4.394 | 4.275 | 4.179 | 4.097 | 4.003 | 3.945 |
| 10 | | 4.636 | 4.501 | 4.467 | 4.381 | 4.311 | 4.251 | 4.184 | 4.138 |

**Table A1-8.2**. Monte Carlo Errors of Percentiles of the Average Extreme Spread (*AES*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | | | | | | | | | | | | | | | | |
| 2 | 2 | 0.06 | 0.03 | 0.02 | 0.007 | 0.008 | 0.008 | 0.005 | 0.005 | 0.005 | 0.004 | 0.006 | 0.006 | 0.009 | 0.008 | 0.008 | 0.009 |
| 3 | 2 | 0.02 | 0.03 | 0.018 | 0.019 | 0.010 | 0.009 | 0.005 | 0.005 | 0.004 | 0.004 | 0.004 | 0.004 | 0.007 | 0.006 | 0.010 | 0.016 |
| 4 | 2 | 0.04 | 0.013 | 0.010 | 0.012 | 0.009 | 0.007 | 0.005 | 0.004 | 0.005 | 0.004 | 0.005 | 0.004 | 0.005 | 0.004 | 0.006 | 0.009 |
| 5 | 2 | 0.017 | 0.02 | 0.016 | 0.012 | 0.008 | 0.006 | 0.005 | 0.004 | 0.004 | 0.003 | 0.004 | 0.004 | 0.003 | 0.004 | 0.008 | 0.012 |
| 10 | 2 | 0.019 | 0.009 | 0.010 | 0.005 | 0.005 | 0.003 | 0.003 | 0.003 | 0.003 | 0.0017 | 0.0017 | 0.0018 | 0.003 | 0.004 | 0.005 | 0.006 |
| 1 | 3 | 0.09 | 0.05 | 0.04 | 0.02 | 0.02 | 0.014 | 0.012 | 0.011 | 0.010 | 0.008 | 0.007 | 0.009 | 0.006 | 0.010 | 0.014 | 0.018 |
| 2 | 3 | 0.06 | 0.03 | 0.02 | 0.009 | 0.012 | 0.009 | 0.009 | 0.010 | 0.007 | 0.007 | 0.006 | 0.008 | 0.011 | 0.015 | 0.013 | 0.013 |
| 3 | 3 | 0.06 | 0.03 | 0.016 | 0.015 | 0.010 | 0.007 | 0.006 | 0.004 | 0.004 | 0.004 | 0.005 | 0.004 | 0.007 | 0.011 | 0.014 | 0.014 |
| 4 | 3 | 0.03 | 0.019 | 0.012 | 0.011 | 0.007 | 0.007 | 0.004 | 0.004 | 0.005 | 0.006 | 0.006 | 0.006 | 0.004 | 0.008 | 0.007 | 0.013 |
| 5 | 3 | 0.04 | 0.016 | 0.013 | 0.007 | 0.008 | 0.004 | 0.005 | 0.005 | 0.005 | 0.003 | 0.002 | 0.002 | 0.002 | 0.006 | 0.007 | 0.009 |
| 10 | 3 | 0.018 | 0.015 | 0.009 | 0.005 | 0.004 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 | 0.003 | 0.004 | 0.004 | 0.007 | 0.007 |
| 1 | 4 | 0.09 | 0.06 | 0.03 | 0.014 | 0.010 | 0.007 | 0.007 | 0.006 | 0.007 | 0.008 | 0.006 | 0.007 | 0.008 | 0.009 | 0.009 | 0.013 |
| 2 | 4 | 0.05 | 0.03 | 0.03 | 0.02 | 0.013 | 0.009 | 0.009 | 0.007 | 0.004 | 0.003 | 0.005 | 0.004 | 0.007 | 0.011 | 0.012 | 0.017 |
| 3 | 4 | 0.03 | 0.03 | 0.02 | 0.013 | 0.011 | 0.009 | 0.006 | 0.004 | 0.005 | 0.005 | 0.004 | 0.007 | 0.007 | 0.007 | 0.010 | 0.009 |
| 4 | 4 | 0.019 | 0.015 | 0.011 | 0.008 | 0.007 | 0.005 | 0.005 | 0.005 | 0.0019 | 0.003 | 0.003 | 0.005 | 0.005 | 0.009 | 0.012 | 0.010 |
| 5 | 4 | 0.02 | 0.018 | 0.012 | 0.007 | 0.007 | 0.003 | 0.002 | 0.003 | 0.003 | 0.004 | 0.003 | 0.004 | 0.004 | 0.006 | 0.009 | 0.012 |
| 10 | 4 | 0.013 | 0.009 | 0.007 | 0.007 | 0.006 | 0.003 | 0.003 | 0.002 | 0.0015 | 0.0017 | 0.0019 | 0.004 | 0.004 | 0.005 | 0.006 | 0.009 |
| 1 | 5 | 0.06 | 0.04 | 0.03 | 0.03 | 0.02 | 0.011 | 0.009 | 0.006 | 0.008 | 0.007 | 0.006 | 0.008 | 0.012 | 0.010 | 0.02 | 0.02 |
| 2 | 5 | 0.05 | 0.03 | 0.017 | 0.013 | 0.013 | 0.006 | 0.004 | 0.003 | 0.003 | 0.004 | 0.004 | 0.006 | 0.005 | 0.003 | 0.009 | 0.010 |
| 3 | 5 | 0.03 | 0.03 | 0.03 | 0.014 | 0.009 | 0.006 | 0.004 | 0.004 | 0.003 | 0.004 | 0.004 | 0.004 | 0.005 | 0.007 | 0.012 | 0.010 |
| 4 | 5 | 0.03 | 0.010 | 0.011 | 0.010 | 0.007 | 0.004 | 0.004 | 0.003 | 0.003 | 0.002 | 0.004 | 0.005 | 0.006 | 0.007 | 0.010 | 0.014 |
| 5 | 5 | 0.03 | 0.02 | 0.011 | 0.007 | 0.002 | 0.003 | 0.004 | 0.004 | 0.003 | 0.003 | 0.004 | 0.004 | 0.004 | 0.004 | 0.003 | 0.005 |
| 10 | 5 | 0.014 | 0.010 | 0.009 | 0.005 | 0.003 | 0.004 | 0.003 | 0.003 | 0.002 | 0.003 | 0.002 | 0.0019 | 0.003 | 0.007 | 0.008 | 0.009 |
| 1 | 10 | 0.08 | 0.03 | 0.03 | 0.018 | 0.012 | 0.010 | 0.009 | 0.008 | 0.006 | 0.008 | 0.007 | 0.008 | 0.010 | 0.014 | 0.015 | 0.013 |
| 2 | 10 | 0.06 | 0.015 | 0.013 | 0.015 | 0.012 | 0.010 | 0.005 | 0.004 | 0.004 | 0.005 | 0.006 | 0.004 | 0.005 | 0.008 | 0.011 | 0.013 |
| 3 | 10 | 0.05 | 0.017 | 0.018 | 0.011 | 0.007 | 0.004 | 0.004 | 0.005 | 0.004 | 0.004 | 0.004 | 0.004 | 0.005 | 0.007 | 0.004 | 0.010 |
| 4 | 10 | 0.02 | 0.010 | 0.010 | 0.008 | 0.005 | 0.004 | 0.004 | 0.004 | 0.003 | 0.002 | 0.004 | 0.004 | 0.005 | 0.006 | 0.010 | 0.012 |
| 5 | 10 | 0.03 | 0.016 | 0.016 | 0.008 | 0.007 | 0.006 | 0.004 | 0.003 | 0.005 | 0.0018 | 0.0012 | 0.0018 | 0.003 | 0.004 | 0.007 | 0.011 |
| 10 | 10 | 0.015 | 0.013 | 0.010 | 0.005 | 0.004 | 0.0017 | 0.0016 | 0.0015 | 0.002 | 0.0014 | 0.002 | 0.002 | 0.004 | 0.004 | 0.005 | 0.008 |
| 1 | 25 | 0.07 | 0.04 | 0.017 | 0.02 | 0.014 | 0.006 | 0.005 | 0.006 | 0.007 | 0.005 | 0.005 | 0.008 | 0.012 | 0.013 | 0.015 | 0.018 |
| 2 | 25 | 0.03 | 0.015 | 0.014 | 0.011 | 0.006 | 0.004 | 0.005 | 0.006 | 0.004 | 0.004 | 0.004 | 0.004 | 0.007 | 0.007 | 0.011 | 0.016 |
| 3 | 25 | 0.019 | 0.012 | 0.011 | 0.005 | 0.004 | 0.004 | 0.0016 | 0.002 | 0.004 | 0.003 | 0.005 | 0.005 | 0.005 | 0.005 | 0.007 | 0.013 |
| 4 | 25 | 0.011 | 0.013 | 0.009 | 0.005 | 0.005 | 0.005 | 0.005 | 0.004 | 0.005 | 0.003 | 0.003 | 0.004 | 0.005 | 0.005 | 0.006 | 0.009 |
| 5 | 25 | 0.019 | 0.011 | 0.009 | 0.007 | 0.004 | 0.004 | 0.002 | 0.003 | 0.003 | 0.003 | 0.002 | 0.003 | 0.003 | 0.003 | 0.007 | 0.013 |
| 10 | 25 | 0.014 | 0.008 | 0.006 | 0.004 | 0.003 | 0.002 | 0.0013 | 0.0016 | 0.002 | 0.003 | 0.002 | 0.004 | 0.004 | 0.004 | 0.004 | 0.005 |

## Appendix 1-9. Mean Radius, One Shot Group

**Table A1-9.1**. Percentiles of the Mean Radius (*MR*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|
| 2 | 2.6283 | 2.3018 | 2.1460 | 1.9206 | 1.7308 | 1.5174 | 1.2686 | 1.1774 |
| 3 | 2.35 | 2.106 | 1.990 | 1.824 | 1.683 | 1.522 | 1.335 | 1.266 |
| 4 | 2.23 | 2.013 | 1.917 | 1.772 | 1.651 | 1.516 | 1.354 | 1.296 |
| 5 | 2.117 | 1.944 | 1.855 | 1.728 | 1.623 | 1.501 | 1.360 | 1.308 |
| 6 | 2.044 | 1.885 | 1.808 | 1.696 | 1.602 | 1.493 | 1.364 | 1.317 |
| 7 | 1.991 | 1.847 | 1.776 | 1.672 | 1.583 | 1.483 | 1.364 | 1.320 |
| 8 | 1.956 | 1.815 | 1.744 | 1.645 | 1.565 | 1.474 | 1.364 | 1.324 |
| 9 | 1.909 | 1.783 | 1.720 | 1.631 | 1.552 | 1.466 | 1.363 | 1.324 |
| 10 | 1.881 | 1.755 | 1.697 | 1.612 | 1.541 | 1.458 | 1.360 | 1.324 |
| 11 | 1.857 | 1.739 | 1.683 | 1.600 | 1.531 | 1.452 | 1.359 | 1.324 |
| 12 | 1.821 | 1.720 | 1.665 | 1.586 | 1.519 | 1.445 | 1.356 | 1.324 |
| 13 | 1.805 | 1.695 | 1.647 | 1.573 | 1.511 | 1.440 | 1.354 | 1.323 |
| 14 | 1.785 | 1.684 | 1.634 | 1.565 | 1.505 | 1.435 | 1.354 | 1.323 |
| 15 | 1.762 | 1.670 | 1.623 | 1.553 | 1.496 | 1.429 | 1.351 | 1.322 |
| 16 | 1.755 | 1.657 | 1.612 | 1.546 | 1.489 | 1.425 | 1.349 | 1.321 |
| 17 | 1.729 | 1.643 | 1.602 | 1.537 | 1.484 | 1.422 | 1.349 | 1.321 |
| 18 | 1.720 | 1.633 | 1.591 | 1.529 | 1.477 | 1.417 | 1.347 | 1.320 |
| 19 | 1.706 | 1.624 | 1.584 | 1.524 | 1.472 | 1.414 | 1.345 | 1.319 |
| 20 | 1.702 | 1.615 | 1.574 | 1.518 | 1.468 | 1.412 | 1.344 | 1.318 |
| 21 | 1.686 | 1.610 | 1.570 | 1.512 | 1.464 | 1.407 | 1.342 | 1.318 |
| 22 | 1.683 | 1.604 | 1.565 | 1.509 | 1.461 | 1.406 | 1.342 | 1.317 |
| 23 | 1.673 | 1.594 | 1.556 | 1.502 | 1.455 | 1.403 | 1.340 | 1.317 |
| 24 | 1.660 | 1.585 | 1.549 | 1.496 | 1.451 | 1.400 | 1.339 | 1.316 |
| 25 | 1.655 | 1.581 | 1.545 | 1.494 | 1.449 | 1.399 | 1.339 | 1.316 |
| 26 | 1.651 | 1.575 | 1.540 | 1.489 | 1.444 | 1.395 | 1.336 | 1.314 |
| 27 | 1.634 | 1.567 | 1.532 | 1.484 | 1.442 | 1.394 | 1.336 | 1.314 |
| 28 | 1.628 | 1.563 | 1.528 | 1.480 | 1.438 | 1.391 | 1.335 | 1.314 |
| 29 | 1.630 | 1.563 | 1.528 | 1.477 | 1.436 | 1.388 | 1.333 | 1.312 |
| 30 | 1.625 | 1.552 | 1.519 | 1.472 | 1.432 | 1.386 | 1.332 | 1.311 |

**Table A1-9.1.** Percentiles of the Mean Radius (*MR*). (CONTINUED)

| n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.8326 | 0.5364 | 0.4724 | 0.3246 | 0.2265 | 0.1591 | 0.1003 | 0.0708 |
| 3 | 0.997 | 0.754 | 0.697 | 0.561 | 0.459 | 0.381 | 0.298 | 0.247 |
| 4 | 1.066 | 0.854 | 0.804 | 0.680 | 0.586 | 0.511 | 0.428 | 0.374 |
| 5 | 1.107 | 0.916 | 0.872 | 0.759 | 0.669 | 0.599 | 0.518 | 0.467 |
| 6 | 1.131 | 0.958 | 0.917 | 0.811 | 0.729 | 0.662 | 0.589 | 0.543 |
| 7 | 1.149 | 0.988 | 0.949 | 0.850 | 0.772 | 0.706 | 0.633 | 0.586 |
| 8 | 1.163 | 1.011 | 0.975 | 0.883 | 0.809 | 0.746 | 0.680 | 0.634 |
| 9 | 1.173 | 1.030 | 0.996 | 0.907 | 0.839 | 0.780 | 0.716 | 0.674 |
| 10 | 1.181 | 1.046 | 1.013 | 0.929 | 0.862 | 0.806 | 0.744 | 0.702 |
| 11 | 1.188 | 1.058 | 1.027 | 0.946 | 0.882 | 0.827 | 0.766 | 0.726 |
| 12 | 1.193 | 1.068 | 1.038 | 0.962 | 0.900 | 0.849 | 0.792 | 0.752 |
| 13 | 1.198 | 1.079 | 1.050 | 0.975 | 0.916 | 0.867 | 0.809 | 0.769 |
| 14 | 1.203 | 1.088 | 1.060 | 0.988 | 0.930 | 0.882 | 0.827 | 0.791 |
| 15 | 1.205 | 1.094 | 1.067 | 0.997 | 0.941 | 0.894 | 0.841 | 0.806 |
| 16 | 1.209 | 1.101 | 1.075 | 1.008 | 0.954 | 0.908 | 0.856 | 0.823 |
| 17 | 1.212 | 1.107 | 1.082 | 1.016 | 0.962 | 0.916 | 0.868 | 0.834 |
| 18 | 1.214 | 1.111 | 1.087 | 1.023 | 0.972 | 0.928 | 0.879 | 0.845 |
| 19 | 1.217 | 1.117 | 1.093 | 1.030 | 0.980 | 0.937 | 0.889 | 0.855 |
| 20 | 1.218 | 1.121 | 1.097 | 1.036 | 0.986 | 0.945 | 0.897 | 0.867 |
| 21 | 1.220 | 1.125 | 1.101 | 1.042 | 0.994 | 0.954 | 0.910 | 0.878 |
| 22 | 1.221 | 1.129 | 1.107 | 1.048 | 1.002 | 0.962 | 0.914 | 0.883 |
| 23 | 1.223 | 1.132 | 1.110 | 1.053 | 1.007 | 0.967 | 0.921 | 0.891 |
| 24 | 1.224 | 1.136 | 1.114 | 1.058 | 1.011 | 0.972 | 0.929 | 0.899 |
| 25 | 1.225 | 1.139 | 1.117 | 1.062 | 1.018 | 0.979 | 0.936 | 0.906 |
| 26 | 1.226 | 1.141 | 1.120 | 1.067 | 1.023 | 0.987 | 0.944 | 0.915 |
| 27 | 1.227 | 1.143 | 1.123 | 1.070 | 1.028 | 0.992 | 0.949 | 0.921 |
| 28 | 1.229 | 1.146 | 1.126 | 1.074 | 1.032 | 0.995 | 0.956 | 0.928 |
| 29 | 1.229 | 1.148 | 1.128 | 1.077 | 1.036 | 1.000 | 0.959 | 0.932 |
| 30 | 1.230 | 1.150 | 1.131 | 1.080 | 1.039 | 1.004 | 0.965 | 0.939 |

**Table A1-9.2**. Monte Carlo Errors of Percentiles of the Mean Radius (*MR*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | | | | | |
| 3 | 0.03 | 0.009 | 0.006 | 0.008 | 0.005 | 0.004 | 0.003 | 0.004 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 | 0.006 | 0.006 |
| 4 | 0.03 | 0.018 | 0.014 | 0.009 | 0.004 | 0.004 | 0.003 | 0.002 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 | 0.007 | 0.007 | 0.009 |
| 5 | 0.02 | 0.015 | 0.006 | 0.007 | 0.006 | 0.006 | 0.004 | 0.005 | 0.003 | 0.003 | 0.003 | 0.003 | 0.005 | 0.005 | 0.005 | 0.007 |
| 6 | 0.02 | 0.011 | 0.005 | 0.004 | 0.002 | 0.002 | 0.0017 | 0.0018 | 0.002 | 0.002 | 0.003 | 0.002 | 0.0016 | 0.003 | 0.004 | 0.005 |
| 7 | 0.018 | 0.009 | 0.005 | 0.003 | 0.003 | 0.002 | 0.002 | 0.0018 | 0.003 | 0.002 | 0.003 | 0.003 | 0.0017 | 0.003 | 0.006 | 0.007 |
| 8 | 0.019 | 0.011 | 0.008 | 0.005 | 0.005 | 0.004 | 0.003 | 0.002 | 0.0015 | 0.0013 | 0.0013 | 0.002 | 0.003 | 0.003 | 0.005 | 0.008 |
| 9 | 0.019 | 0.009 | 0.010 | 0.005 | 0.003 | 0.003 | 0.002 | 0.002 | 0.0009 | 0.002 | 0.002 | 0.0018 | 0.003 | 0.003 | 0.005 | 0.005 |
| 10 | 0.018 | 0.007 | 0.007 | 0.005 | 0.004 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.0012 | 0.0015 | 0.003 | 0.005 | 0.002 |
| 11 | 0.008 | 0.009 | 0.007 | 0.006 | 0.004 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.0017 | 0.0018 | 0.0011 | 0.002 | 0.004 | 0.004 | 0.004 |
| 12 | 0.015 | 0.007 | 0.006 | 0.004 | 0.003 | 0.003 | 0.002 | 0.0019 | 0.002 | 0.0016 | 0.0015 | 0.002 | 0.002 | 0.003 | 0.003 | 0.005 |
| 13 | 0.011 | 0.005 | 0.006 | 0.003 | 0.002 | 0.0016 | 0.0014 | 0.0015 | 0.0015 | 0.0016 | 0.0015 | 0.002 | 0.003 | 0.002 | 0.003 | 0.005 |
| 14 | 0.011 | 0.008 | 0.007 | 0.004 | 0.002 | 0.002 | 0.002 | 0.003 | 0.0019 | 0.0015 | 0.0018 | 0.0019 | 0.002 | 0.002 | 0.003 | 0.004 |
| 15 | 0.014 | 0.011 | 0.009 | 0.005 | 0.003 | 0.002 | 0.0019 | 0.0018 | 0.0011 | 0.0014 | 0.0019 | 0.0017 | 0.003 | 0.003 | 0.004 | 0.005 |
| 16 | 0.017 | 0.007 | 0.006 | 0.006 | 0.005 | 0.003 | 0.002 | 0.002 | 0.0015 | 0.0006 | 0.0008 | 0.0012 | 0.0017 | 0.003 | 0.004 | 0.005 |
| 17 | 0.009 | 0.006 | 0.004 | 0.002 | 0.002 | 0.0014 | 0.0016 | 0.002 | 0.0019 | 0.0015 | 0.0015 | 0.0017 | 0.002 | 0.004 | 0.005 | 0.007 |
| 18 | 0.014 | 0.005 | 0.005 | 0.002 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.0008 | 0.0009 | 0.0009 | 0.0015 | 0.0012 | 0.002 | 0.004 | 0.004 |
| 19 | 0.012 | 0.008 | 0.005 | 0.003 | 0.003 | 0.002 | 0.0018 | 0.002 | 0.0015 | 0.0015 | 0.0015 | 0.0016 | 0.002 | 0.002 | 0.004 | 0.003 |
| 20 | 0.011 | 0.005 | 0.004 | 0.002 | 0.0015 | 0.0015 | 0.0009 | 0.0011 | 0.0016 | 0.0015 | 0.0012 | 0.0013 | 0.0018 | 0.003 | 0.004 | 0.004 |
| 21 | 0.005 | 0.006 | 0.003 | 0.004 | 0.003 | 0.002 | 0.0017 | 0.0014 | 0.0012 | 0.0015 | 0.0009 | 0.0011 | 0.0019 | 0.0016 | 0.003 | 0.004 |
| 22 | 0.016 | 0.008 | 0.006 | 0.003 | 0.003 | 0.002 | 0.0019 | 0.002 | 0.0013 | 0.0011 | 0.0013 | 0.0015 | 0.003 | 0.002 | 0.002 | 0.004 |
| 23 | 0.012 | 0.003 | 0.004 | 0.003 | 0.002 | 0.0018 | 0.0010 | 0.0011 | 0.0009 | 0.0008 | 0.0012 | 0.0016 | 0.0016 | 0.0014 | 0.002 | 0.005 |
| 24 | 0.010 | 0.005 | 0.003 | 0.0015 | 0.0019 | 0.0016 | 0.0012 | 0.0012 | 0.0009 | 0.0011 | 0.0011 | 0.0017 | 0.0013 | 0.003 | 0.003 | 0.004 |
| 25 | 0.009 | 0.004 | 0.003 | 0.003 | 0.004 | 0.0019 | 0.0013 | 0.0009 | 0.0010 | 0.0014 | 0.0013 | 0.0016 | 0.0015 | 0.0012 | 0.003 | 0.004 |
| 26 | 0.010 | 0.005 | 0.003 | 0.002 | 0.0019 | 0.0012 | 0.0011 | 0.0014 | 0.0011 | 0.0009 | 0.0011 | 0.0009 | 0.0012 | 0.0015 | 0.003 | 0.003 |
| 27 | 0.009 | 0.006 | 0.004 | 0.003 | 0.003 | 0.002 | 0.002 | 0.0015 | 0.0013 | 0.0008 | 0.0009 | 0.0006 | 0.0014 | 0.0019 | 0.004 | 0.004 |
| 28 | 0.008 | 0.005 | 0.002 | 0.0013 | 0.0017 | 0.0013 | 0.0009 | 0.0007 | 0.0008 | 0.0014 | 0.0012 | 0.0011 | 0.0016 | 0.002 | 0.002 | 0.001 |
| 29 | 0.009 | 0.005 | 0.002 | 0.003 | 0.002 | 0.0017 | 0.0015 | 0.0014 | 0.0012 | 0.0014 | 0.0014 | 0.0010 | 0.0018 | 0.002 | 0.003 | 0.004 |
| 30 | 0.008 | 0.005 | 0.004 | 0.002 | 0.002 | 0.0019 | 0.0011 | 0.0008 | 0.0007 | 0.0008 | 0.0009 | 0.0015 | 0.002 | 0.002 | 0.003 | 0.003 |

## Appendix 1-10. Average Mean Radius, Multiple Shot Groups

**Table A1-10.1**. Percentiles of the Average Mean Radius (*AMR*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2.6283 | 2.3018 | 2.1460 | 1.9206 | 1.7308 | 1.5174 | 1.2686 | 1.1774 |
| 2 | | 2.08 | 1.852 | 1.743 | 1.592 | 1.466 | 1.326 | 1.160 | 1.098 |
| 3 | | 1.821 | 1.651 | 1.575 | 1.453 | 1.351 | 1.239 | 1.107 | 1.058 |
| 4 | | 1.685 | 1.544 | 1.473 | 1.371 | 1.286 | 1.191 | 1.079 | 1.037 |
| 5 | | 1.602 | 1.467 | 1.404 | 1.316 | 1.241 | 1.159 | 1.059 | 1.021 |
| 10 | | 1.375 | 1.291 | 1.248 | 1.186 | 1.135 | 1.077 | 1.008 | 0.982 |
| 1 | 3 | 2.35 | 2.106 | 1.990 | 1.824 | 1.683 | 1.522 | 1.335 | 1.266 |
| 2 | | 1.943 | 1.766 | 1.688 | 1.576 | 1.479 | 1.370 | 1.242 | 1.195 |
| 3 | | 1.748 | 1.624 | 1.562 | 1.472 | 1.394 | 1.307 | 1.204 | 1.165 |
| 4 | | 1.651 | 1.540 | 1.486 | 1.408 | 1.343 | 1.267 | 1.179 | 1.146 |
| 5 | | 1.582 | 1.484 | 1.436 | 1.366 | 1.307 | 1.242 | 1.163 | 1.134 |
| 10 | | 1.407 | 1.341 | 1.308 | 1.262 | 1.222 | 1.177 | 1.123 | 1.102 |
| 1 | 4 | 2.23 | 2.013 | 1.917 | 1.772 | 1.651 | 1.516 | 1.354 | 1.296 |
| 2 | | 1.869 | 1.728 | 1.660 | 1.563 | 1.480 | 1.387 | 1.277 | 1.236 |
| 3 | | 1.714 | 1.599 | 1.547 | 1.469 | 1.402 | 1.328 | 1.241 | 1.208 |
| 4 | | 1.625 | 1.532 | 1.484 | 1.419 | 1.362 | 1.297 | 1.221 | 1.192 |
| 5 | | 1.566 | 1.480 | 1.441 | 1.381 | 1.332 | 1.274 | 1.206 | 1.180 |
| 10 | | 1.417 | 1.359 | 1.331 | 1.292 | 1.257 | 1.218 | 1.171 | 1.154 |
| 1 | 5 | 2.117 | 1.944 | 1.855 | 1.728 | 1.623 | 1.501 | 1.360 | 1.308 |
| 2 | | 1.813 | 1.694 | 1.630 | 1.543 | 1.472 | 1.390 | 1.292 | 1.256 |
| 3 | | 1.677 | 1.578 | 1.530 | 1.463 | 1.404 | 1.339 | 1.262 | 1.232 |
| 4 | | 1.588 | 1.514 | 1.476 | 1.415 | 1.365 | 1.309 | 1.242 | 1.217 |
| 5 | | 1.550 | 1.473 | 1.436 | 1.385 | 1.340 | 1.290 | 1.230 | 1.207 |
| 10 | | 1.412 | 1.365 | 1.342 | 1.305 | 1.275 | 1.240 | 1.198 | 1.182 |
| 1 | 10 | 1.881 | 1.755 | 1.697 | 1.612 | 1.541 | 1.458 | 1.360 | 1.324 |
| 2 | | 1.662 | 1.583 | 1.544 | 1.484 | 1.435 | 1.378 | 1.311 | 1.286 |
| 3 | | 1.576 | 1.510 | 1.476 | 1.429 | 1.388 | 1.343 | 1.288 | 1.268 |
| 4 | | 1.521 | 1.465 | 1.436 | 1.396 | 1.362 | 1.322 | 1.275 | 1.258 |
| 5 | | 1.483 | 1.433 | 1.408 | 1.373 | 1.343 | 1.308 | 1.266 | 1.251 |
| 10 | | 1.396 | 1.361 | 1.344 | 1.319 | 1.297 | 1.273 | 1.244 | 1.233 |
| 1 | 25 | 1.655 | 1.581 | 1.545 | 1.494 | 1.449 | 1.399 | 1.339 | 1.316 |
| 2 | | 1.527 | 1.476 | 1.450 | 1.414 | 1.382 | 1.348 | 1.305 | 1.289 |
| 3 | | 1.468 | 1.427 | 1.408 | 1.379 | 1.354 | 1.325 | 1.291 | 1.278 |
| 4 | | 1.437 | 1.401 | 1.383 | 1.358 | 1.336 | 1.312 | 1.283 | 1.272 |
| 5 | | 1.413 | 1.382 | 1.366 | 1.344 | 1.325 | 1.303 | 1.277 | 1.267 |
| 10 | | 1.358 | 1.335 | 1.325 | 1.309 | 1.296 | 1.281 | 1.263 | 1.256 |

**Table A1-10.1.** Percentiles of the Average Mean Radius (*AMR*). (CONTINUED)

| k | n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.8326 | 0.5364 | 0.4724 | 0.3246 | 0.2265 | 0.1591 | 0.1003 | 0.0708 |
| 2 | | 0.862 | 0.650 | 0.601 | 0.481 | 0.393 | 0.325 | 0.255 | 0.215 |
| 3 | | 0.869 | 0.696 | 0.655 | 0.553 | 0.476 | 0.414 | 0.347 | 0.308 |
| 4 | | 0.875 | 0.722 | 0.687 | 0.596 | 0.526 | 0.470 | 0.407 | 0.368 |
| 5 | | 0.876 | 0.739 | 0.707 | 0.626 | 0.561 | 0.508 | 0.450 | 0.412 |
| 10 | | 0.881 | 0.784 | 0.762 | 0.702 | 0.654 | 0.615 | 0.568 | 0.537 |
| 1 | 3 | 0.997 | 0.754 | 0.697 | 0.561 | 0.459 | 0.381 | 0.298 | 0.247 |
| 2 | | 1.011 | 0.837 | 0.796 | 0.692 | 0.611 | 0.544 | 0.473 | 0.428 |
| 3 | | 1.014 | 0.872 | 0.839 | 0.752 | 0.684 | 0.629 | 0.564 | 0.525 |
| 4 | | 1.016 | 0.893 | 0.864 | 0.788 | 0.728 | 0.678 | 0.620 | 0.582 |
| 5 | | 1.018 | 0.908 | 0.881 | 0.812 | 0.757 | 0.710 | 0.659 | 0.624 |
| 10 | | 1.021 | 0.942 | 0.923 | 0.874 | 0.833 | 0.799 | 0.762 | 0.736 |
| 1 | 4 | 1.066 | 0.854 | 0.804 | 0.680 | 0.586 | 0.511 | 0.428 | 0.374 |
| 2 | | 1.077 | 0.925 | 0.889 | 0.798 | 0.725 | 0.666 | 0.598 | 0.555 |
| 3 | | 1.079 | 0.956 | 0.926 | 0.849 | 0.788 | 0.739 | 0.681 | 0.643 |
| 4 | | 1.081 | 0.973 | 0.946 | 0.880 | 0.827 | 0.782 | 0.731 | 0.695 |
| 5 | | 1.081 | 0.986 | 0.962 | 0.902 | 0.852 | 0.810 | 0.763 | 0.733 |
| 10 | | 1.084 | 1.016 | 0.999 | 0.955 | 0.921 | 0.891 | 0.856 | 0.833 |
| 1 | 5 | 1.107 | 0.916 | 0.872 | 0.759 | 0.669 | 0.599 | 0.518 | 0.467 |
| 2 | | 1.114 | 0.978 | 0.945 | 0.862 | 0.797 | 0.742 | 0.678 | 0.640 |
| 3 | | 1.116 | 1.006 | 0.979 | 0.910 | 0.855 | 0.809 | 0.756 | 0.721 |
| 4 | | 1.118 | 1.021 | 0.998 | 0.937 | 0.889 | 0.849 | 0.802 | 0.771 |
| 5 | | 1.118 | 1.032 | 1.011 | 0.956 | 0.912 | 0.875 | 0.834 | 0.806 |
| 10 | | 1.119 | 1.058 | 1.043 | 1.004 | 0.973 | 0.946 | 0.915 | 0.894 |
| 1 | 10 | 1.181 | 1.046 | 1.013 | 0.929 | 0.862 | 0.806 | 0.744 | 0.702 |
| 2 | | 1.185 | 1.089 | 1.065 | 1.004 | 0.956 | 0.914 | 0.867 | 0.835 |
| 3 | | 1.187 | 1.108 | 1.088 | 1.038 | 0.997 | 0.963 | 0.923 | 0.897 |
| 4 | | 1.187 | 1.119 | 1.102 | 1.058 | 1.023 | 0.991 | 0.956 | 0.933 |
| 5 | | 1.188 | 1.126 | 1.111 | 1.072 | 1.040 | 1.012 | 0.981 | 0.959 |
| 10 | | 1.188 | 1.145 | 1.134 | 1.106 | 1.083 | 1.064 | 1.041 | 1.026 |
| 1 | 25 | 1.225 | 1.139 | 1.117 | 1.062 | 1.018 | 0.979 | 0.936 | 0.906 |
| 2 | | 1.226 | 1.165 | 1.150 | 1.111 | 1.079 | 1.052 | 1.020 | 0.998 |
| 3 | | 1.227 | 1.176 | 1.164 | 1.132 | 1.105 | 1.083 | 1.056 | 1.038 |
| 4 | | 1.227 | 1.184 | 1.173 | 1.144 | 1.121 | 1.101 | 1.079 | 1.063 |
| 5 | | 1.227 | 1.188 | 1.178 | 1.153 | 1.132 | 1.115 | 1.094 | 1.080 |
| 10 | | 1.228 | 1.200 | 1.193 | 1.175 | 1.160 | 1.148 | 1.133 | 1.123 |

**Table A1-10.2**. Monte Carlo Errors of the Percentiles of the Average Mean Radius (*AMR*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | | | | | | | | | | | | | | | | |
| 2 | 2 | 0.03 | 0.011 | 0.011 | 0.008 | 0.006 | 0.006 | 0.003 | 0.004 | 0.004 | 0.004 | 0.003 | 0.003 | 0.003 | 0.004 | 0.002 | 0.003 |
| 3 | 2 | 0.02 | 0.010 | 0.005 | 0.007 | 0.004 | 0.003 | 0.002 | 0.002 | 0.002 | 0.003 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.007 |
| 4 | 2 | 0.014 | 0.012 | 0.010 | 0.004 | 0.003 | 0.003 | 0.002 | 0.003 | 0.0011 | 0.0014 | 0.0016 | 0.002 | 0.003 | 0.003 | 0.003 | 0.003 |
| 5 | 2 | 0.014 | 0.011 | 0.007 | 0.004 | 0.004 | 0.003 | 0.002 | 0.003 | 0.0015 | 0.002 | 0.0016 | 0.003 | 0.003 | 0.004 | 0.006 | 0.005 |
| 10 | 2 | 0.015 | 0.008 | 0.006 | 0.002 | 0.0011 | 0.0012 | 0.0008 | 0.0012 | 0.0011 | 0.0011 | 0.0008 | 0.0011 | 0.0011 | 0.0016 | 0.003 | 0.004 |
| 1 | 3 | 0.03 | 0.009 | 0.006 | 0.008 | 0.005 | 0.004 | 0.003 | 0.004 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 | 0.006 | 0.006 |
| 2 | 3 | 0.019 | 0.009 | 0.007 | 0.007 | 0.005 | 0.003 | 0.0018 | 0.003 | 0.002 | 0.003 | 0.002 | 0.003 | 0.005 | 0.006 | 0.006 | 0.009 |
| 3 | 3 | 0.018 | 0.011 | 0.010 | 0.005 | 0.003 | 0.003 | 0.003 | 0.002 | 0.0016 | 0.0010 | 0.0015 | 0.0011 | 0.002 | 0.004 | 0.004 | 0.004 |
| 4 | 3 | 0.012 | 0.009 | 0.006 | 0.004 | 0.004 | 0.004 | 0.003 | 0.0019 | 0.0011 | 0.0009 | 0.0014 | 0.002 | 0.003 | 0.002 | 0.004 | 0.005 |
| 5 | 3 | 0.015 | 0.006 | 0.005 | 0.002 | 0.002 | 0.0013 | 0.0017 | 0.0018 | 0.0015 | 0.0019 | 0.0018 | 0.003 | 0.002 | 0.002 | 0.0019 | 0.004 |
| 10 | 3 | 0.007 | 0.007 | 0.003 | 0.002 | 0.0016 | 0.0012 | 0.0010 | 0.0008 | 0.0009 | 0.0011 | 0.0014 | 0.0016 | 0.0014 | 0.0016 | 0.002 | 0.003 |
| 1 | 4 | 0.03 | 0.018 | 0.014 | 0.009 | 0.004 | 0.004 | 0.003 | 0.002 | 0.003 | 0.003 | 0.003 | 0.003 | 0.004 | 0.007 | 0.007 | 0.009 |
| 2 | 4 | 0.010 | 0.009 | 0.006 | 0.005 | 0.003 | 0.004 | 0.003 | 0.003 | 0.002 | 0.0017 | 0.002 | 0.0010 | 0.003 | 0.004 | 0.005 | 0.005 |
| 3 | 4 | 0.012 | 0.006 | 0.004 | 0.002 | 0.0017 | 0.0013 | 0.0015 | 0.0016 | 0.002 | 0.0010 | 0.0013 | 0.0018 | 0.003 | 0.003 | 0.005 | 0.005 |
| 4 | 4 | 0.007 | 0.003 | 0.004 | 0.003 | 0.003 | 0.002 | 0.0014 | 0.0012 | 0.0014 | 0.002 | 0.0018 | 0.0014 | 0.0015 | 0.002 | 0.003 | 0.004 |
| 5 | 4 | 0.012 | 0.003 | 0.003 | 0.003 | 0.002 | 0.0019 | 0.0014 | 0.0011 | 0.0013 | 0.0015 | 0.0016 | 0.0016 | 0.002 | 0.002 | 0.003 | 0.004 |
| 10 | 4 | 0.007 | 0.003 | 0.002 | 0.0016 | 0.0019 | 0.0015 | 0.0012 | 0.0009 | 0.0009 | 0.0008 | 0.0009 | 0.0018 | 0.002 | 0.002 | 0.003 | 0.005 |
| 1 | 5 | 0.02 | 0.015 | 0.006 | 0.007 | 0.006 | 0.006 | 0.004 | 0.005 | 0.003 | 0.003 | 0.003 | 0.003 | 0.005 | 0.005 | 0.005 | 0.007 |
| 2 | 5 | 0.018 | 0.010 | 0.007 | 0.003 | 0.004 | 0.004 | 0.003 | 0.003 | 0.003 | 0.002 | 0.002 | 0.003 | 0.003 | 0.003 | 0.004 | 0.005 |
| 3 | 5 | 0.015 | 0.010 | 0.006 | 0.005 | 0.003 | 0.003 | 0.0019 | 0.0018 | 0.0014 | 0.0019 | 0.002 | 0.003 | 0.003 | 0.004 | 0.007 | 0.008 |
| 4 | 5 | 0.011 | 0.005 | 0.004 | 0.003 | 0.002 | 0.0017 | 0.0014 | 0.0013 | 0.0010 | 0.0014 | 0.0014 | 0.0018 | 0.0019 | 0.003 | 0.003 | 0.004 |
| 5 | 5 | 0.013 | 0.006 | 0.003 | 0.003 | 0.003 | 0.0018 | 0.0017 | 0.0012 | 0.0006 | 0.0011 | 0.0013 | 0.0015 | 0.003 | 0.002 | 0.004 | 0.003 |
| 10 | 5 | 0.005 | 0.002 | 0.0017 | 0.003 | 0.0014 | 0.0010 | 0.0007 | 0.0008 | 0.0011 | 0.0011 | 0.0008 | 0.0012 | 0.0011 | 0.0013 | 0.0018 | 0.002 |
| 1 | 10 | 0.018 | 0.007 | 0.007 | 0.005 | 0.004 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.0019 | 0.0012 | 0.0015 | 0.003 | 0.005 | 0.002 |
| 2 | 10 | 0.006 | 0.004 | 0.004 | 0.003 | 0.003 | 0.002 | 0.0017 | 0.0013 | 0.0011 | 0.0014 | 0.0012 | 0.0018 | 0.0010 | 0.0018 | 0.003 | 0.004 |
| 3 | 10 | 0.010 | 0.005 | 0.004 | 0.004 | 0.003 | 0.002 | 0.0017 | 0.0013 | 0.0015 | 0.0009 | 0.0009 | 0.0012 | 0.0014 | 0.002 | 0.003 | 0.005 |
| 4 | 10 | 0.009 | 0.004 | 0.002 | 0.003 | 0.0019 | 0.0008 | 0.0011 | 0.0010 | 0.0010 | 0.0009 | 0.0009 | 0.0012 | 0.0012 | 0.0018 | 0.0019 | 0.004 |
| 5 | 10 | 0.005 | 0.005 | 0.003 | 0.002 | 0.0015 | 0.0010 | 0.0008 | 0.0006 | 0.0005 | 0.0007 | 0.0008 | 0.0010 | 0.0010 | 0.0014 | 0.0017 | 0.002 |
| 10 | 10 | 0.002 | 0.002 | 0.0018 | 0.0014 | 0.0013 | 0.0013 | 0.0008 | 0.0006 | 0.0005 | 0.0008 | 0.0007 | 0.0010 | 0.0014 | 0.0018 | 0.002 | 0.004 |
| 1 | 25 | 0.009 | 0.004 | 0.003 | 0.003 | 0.004 | 0.0019 | 0.0013 | 0.0009 | 0.0010 | 0.0014 | 0.0013 | 0.0016 | 0.0015 | 0.0012 | 0.003 | 0.004 |
| 2 | 25 | 0.008 | 0.004 | 0.003 | 0.002 | 0.0013 | 0.0012 | 0.0008 | 0.0008 | 0.0013 | 0.0010 | 0.0009 | 0.0011 | 0.0012 | 0.0012 | 0.002 | 0.004 |
| 3 | 25 | 0.006 | 0.003 | 0.002 | 0.002 | 0.0013 | 0.0012 | 0.0011 | 0.0009 | 0.0008 | 0.0010 | 0.0009 | 0.0012 | 0.0015 | 0.002 | 0.0018 | 0.002 |
| 4 | 25 | 0.004 | 0.003 | 0.0018 | 0.0011 | 0.0014 | 0.0011 | 0.0009 | 0.0009 | 0.0006 | 0.0006 | 0.0007 | 0.0011 | 0.0006 | 0.0014 | 0.002 | 0.003 |
| 5 | 25 | 0.002 | 0.002 | 0.0008 | 0.0008 | 0.000 | 0.0004 | 0.0005 | 0.0006 | 0.0003 | 0.0005 | 0.0005 | 0.0004 | 0.0005 | 0.0012 | 0.0019 | 0.003 |
| 10 | 25 | 0.002 | 0.0013 | 0.0015 | 0.0007 | 0.000 | 0.0006 | 0.0004 | 0.0004 | 0.0003 | 0.0004 | 0.0004 | 0.0008 | 0.0009 | 0.0007 | 0.0013 | 0.0010 |

## Appendix 1-11. Radial Standard Deviation, One Shot Group

**Table A1-11.1**. Percentiles of the Radial Standard Deviation (*RSD*).

| n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|
| 2 | 2.6283 | 2.3018 | 2.1460 | 1.9206 | 1.7308 | 1.5174 | 1.2686 | 1.1774 |
| 3 | 2.4810 | 2.2256 | 2.1037 | 1.9273 | 1.7784 | 1.6103 | 1.4129 | 1.3398 |
| 4 | 2.3695 | 2.1533 | 2.0501 | 1.9006 | 1.7742 | 1.6313 | 1.4627 | 1.4001 |
| 5 | 2.2858 | 2.0955 | 2.0045 | 1.8727 | 1.7611 | 1.6347 | 1.4853 | 1.4296 |
| 6 | 2.2207 | 2.0489 | 1.9668 | 1.8477 | 1.7468 | 1.6323 | 1.4968 | 1.4462 |
| 7 | 2.1683 | 2.0107 | 1.9353 | 1.8259 | 1.7331 | 1.6279 | 1.5029 | 1.4563 |
| 8 | 2.1249 | 1.9786 | 1.9086 | 1.8069 | 1.7206 | 1.6227 | 1.5063 | 1.4627 |
| 9 | 2.0884 | 1.9513 | 1.8856 | 1.7903 | 1.7093 | 1.6173 | 1.5079 | 1.4670 |
| 10 | 2.0570 | 1.9276 | 1.8656 | 1.7756 | 1.6991 | 1.6121 | 1.5086 | 1.4699 |
| 11 | 2.0297 | 1.9069 | 1.8480 | 1.7625 | 1.6898 | 1.6071 | 1.5087 | 1.4718 |
| 12 | 2.0056 | 1.8885 | 1.8323 | 1.7507 | 1.6814 | 1.6024 | 1.5084 | 1.4731 |
| 13 | 1.9841 | 1.8720 | 1.8183 | 1.7401 | 1.6737 | 1.5980 | 1.5078 | 1.4739 |
| 14 | 1.9649 | 1.8572 | 1.8056 | 1.7305 | 1.6666 | 1.5938 | 1.5070 | 1.4744 |
| 15 | 1.9475 | 1.8438 | 1.7940 | 1.7216 | 1.6601 | 1.5899 | 1.5061 | 1.4747 |
| 16 | 1.9317 | 1.8315 | 1.7835 | 1.7135 | 1.6540 | 1.5862 | 1.5052 | 1.4748 |
| 17 | 1.9172 | 1.8203 | 1.7738 | 1.7061 | 1.6484 | 1.5827 | 1.5042 | 1.4747 |
| 18 | 1.9039 | 1.8099 | 1.7648 | 1.6991 | 1.6432 | 1.5794 | 1.5032 | 1.4746 |
| 19 | 1.8916 | 1.8003 | 1.7565 | 1.6927 | 1.6383 | 1.5763 | 1.5023 | 1.4744 |
| 20 | 1.8802 | 1.7914 | 1.7487 | 1.6866 | 1.6338 | 1.5734 | 1.5013 | 1.4741 |
| 21 | 1.8696 | 1.7831 | 1.7415 | 1.6810 | 1.6295 | 1.5706 | 1.5003 | 1.4738 |
| 22 | 1.8597 | 1.7753 | 1.7348 | 1.6757 | 1.6254 | 1.5680 | 1.4993 | 1.4735 |
| 23 | 1.8504 | 1.7680 | 1.7284 | 1.6707 | 1.6216 | 1.5655 | 1.4984 | 1.4731 |
| 24 | 1.8417 | 1.7611 | 1.7224 | 1.6660 | 1.6180 | 1.5631 | 1.4975 | 1.4728 |
| 25 | 1.8334 | 1.7546 | 1.7168 | 1.6616 | 1.6146 | 1.5609 | 1.4966 | 1.4724 |
| 26 | 1.8257 | 1.7485 | 1.7114 | 1.6574 | 1.6113 | 1.5587 | 1.4957 | 1.4720 |
| 27 | 1.8183 | 1.7427 | 1.7064 | 1.6534 | 1.6082 | 1.5566 | 1.4948 | 1.4716 |
| 28 | 1.8114 | 1.7372 | 1.7016 | 1.6496 | 1.6053 | 1.5546 | 1.4940 | 1.4712 |
| 29 | 1.8048 | 1.7320 | 1.6970 | 1.6460 | 1.6025 | 1.5527 | 1.4932 | 1.4707 |
| 30 | 1.7985 | 1.7270 | 1.6926 | 1.6425 | 1.5998 | 1.5509 | 1.4924 | 1.4703 |

**Table A1-11.1.** Percentiles of the Radial Standard Deviation (*RSD*). (CONTINUED)

| n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.8326 | 0.5364 | 0.4724 | 0.3246 | 0.2265 | 0.1591 | 0.1003 | 0.0708 |
| 3 | 1.0578 | 0.8005 | 0.7413 | 0.5954 | 0.4867 | 0.4018 | 0.3147 | 0.2627 |
| 4 | 1.1563 | 0.9293 | 0.8761 | 0.7423 | 0.6394 | 0.5562 | 0.4669 | 0.4110 |
| 5 | 1.2120 | 1.0070 | 0.9585 | 0.8354 | 0.7393 | 0.6603 | 0.5738 | 0.5185 |
| 6 | 1.2478 | 1.0597 | 1.0148 | 0.9005 | 0.8104 | 0.7356 | 0.6530 | 0.5994 |
| 7 | 1.2728 | 1.0979 | 1.0561 | 0.9490 | 0.8640 | 0.7932 | 0.7142 | 0.6627 |
| 8 | 1.2913 | 1.1272 | 1.0878 | 0.9868 | 0.9063 | 0.8388 | 0.7633 | 0.7137 |
| 9 | 1.3055 | 1.1505 | 1.1132 | 1.0172 | 0.9405 | 0.8761 | 0.8036 | 0.7559 |
| 10 | 1.3167 | 1.1694 | 1.1339 | 1.0424 | 0.9690 | 0.9072 | 0.8376 | 0.7915 |
| 11 | 1.3259 | 1.1852 | 1.1512 | 1.0636 | 0.9932 | 0.9337 | 0.8666 | 0.8221 |
| 12 | 1.3334 | 1.1986 | 1.1660 | 1.0817 | 1.0140 | 0.9567 | 0.8917 | 0.8487 |
| 13 | 1.3398 | 1.2101 | 1.1787 | 1.0975 | 1.0321 | 0.9767 | 0.9138 | 0.8721 |
| 14 | 1.3453 | 1.2202 | 1.1898 | 1.1114 | 1.0481 | 0.9944 | 0.9334 | 0.8928 |
| 15 | 1.3500 | 1.2290 | 1.1997 | 1.1237 | 1.0623 | 1.0102 | 0.9510 | 0.9115 |
| 16 | 1.3541 | 1.2369 | 1.2084 | 1.1347 | 1.0751 | 1.0244 | 0.9667 | 0.9283 |
| 17 | 1.3577 | 1.2439 | 1.2163 | 1.1446 | 1.0866 | 1.0373 | 0.9811 | 0.9435 |
| 18 | 1.3609 | 1.2502 | 1.2233 | 1.1536 | 1.0971 | 1.0490 | 0.9941 | 0.9575 |
| 19 | 1.3637 | 1.2560 | 1.2298 | 1.1617 | 1.1066 | 1.0597 | 1.0061 | 0.9703 |
| 20 | 1.3663 | 1.2612 | 1.2357 | 1.1693 | 1.1154 | 1.0695 | 1.0171 | 0.9821 |
| 21 | 1.3686 | 1.2660 | 1.2411 | 1.1762 | 1.1235 | 1.0786 | 1.0273 | 0.9930 |
| 22 | 1.3707 | 1.2705 | 1.2460 | 1.1826 | 1.1311 | 1.0871 | 1.0368 | 1.0031 |
| 23 | 1.3726 | 1.2746 | 1.2506 | 1.1885 | 1.1380 | 1.0949 | 1.0457 | 1.0126 |
| 24 | 1.3744 | 1.2783 | 1.2549 | 1.1940 | 1.1445 | 1.1023 | 1.0539 | 1.0215 |
| 25 | 1.3760 | 1.2819 | 1.2589 | 1.1992 | 1.1506 | 1.1091 | 1.0616 | 1.0298 |
| 26 | 1.3775 | 1.2852 | 1.2626 | 1.2040 | 1.1563 | 1.1156 | 1.0689 | 1.0376 |
| 27 | 1.3789 | 1.2882 | 1.2661 | 1.2085 | 1.1617 | 1.1216 | 1.0758 | 1.0449 |
| 28 | 1.3802 | 1.2911 | 1.2694 | 1.2128 | 1.1667 | 1.1274 | 1.0822 | 1.0519 |
| 29 | 1.3813 | 1.2938 | 1.2725 | 1.2168 | 1.1715 | 1.1328 | 1.0883 | 1.0585 |
| 30 | 1.3824 | 1.2964 | 1.2754 | 1.2206 | 1.1760 | 1.1379 | 1.0941 | 1.0647 |

## Appendix 1-12. RMS of the Radial Standard Deviations, Multiple Shot Groups

**Table A1-12.1**. Percentiles of the RMS of the Radial Standard Deviations (*RMS-RSD*).

| k | n | P 0.999 | P 0.995 | P 0.99 | P 0.975 | P 0.95 | P 0.9 | P 0.8 | P 0.75 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2.6283 | 2.3018 | 2.1460 | 1.9206 | 1.7308 | 1.5174 | 1.2686 | 1.1774 |
| 2 | | 2.1487 | 1.9275 | 1.8219 | 1.6691 | 1.5401 | 1.3946 | 1.2236 | 1.1603 |
| 3 | | 1.9347 | 1.7582 | 1.6739 | 1.5518 | 1.4487 | 1.3320 | 1.1943 | 1.1432 |
| 4 | | 1.8071 | 1.6566 | 1.5847 | 1.4805 | 1.3923 | 1.2924 | 1.1742 | 1.1302 |
| 5 | | 1.7201 | 1.5871 | 1.5235 | 1.4312 | 1.3530 | 1.2644 | 1.1594 | 1.1202 |
| 10 | | 1.5052 | 1.4142 | 1.3705 | 1.3071 | 1.2532 | 1.1919 | 1.1189 | 1.0915 |
| 1 | 3 | 2.4810 | 2.2256 | 2.1037 | 1.9273 | 1.7784 | 1.6103 | 1.4129 | 1.3398 |
| 2 | | 2.0866 | 1.9129 | 1.8299 | 1.7095 | 1.6077 | 1.4923 | 1.3559 | 1.3050 |
| 3 | | 1.9122 | 1.7732 | 1.7068 | 1.6103 | 1.5285 | 1.4356 | 1.3255 | 1.2843 |
| 4 | | 1.8086 | 1.6899 | 1.6330 | 1.5504 | 1.4803 | 1.4006 | 1.3059 | 1.2705 |
| 5 | | 1.7381 | 1.6329 | 1.5825 | 1.5093 | 1.4471 | 1.3763 | 1.2920 | 1.2604 |
| 10 | | 1.5642 | 1.4918 | 1.4571 | 1.4064 | 1.3633 | 1.3141 | 1.2552 | 1.2331 |
| 1 | 4 | 2.3695 | 2.1533 | 2.0501 | 1.9006 | 1.7742 | 1.6313 | 1.4627 | 1.4001 |
| 2 | | 2.0282 | 1.8808 | 1.8103 | 1.7079 | 1.6212 | 1.5227 | 1.4059 | 1.3622 |
| 3 | | 1.8778 | 1.7597 | 1.7031 | 1.6209 | 1.5511 | 1.4717 | 1.3772 | 1.3418 |
| 4 | | 1.7885 | 1.6874 | 1.6390 | 1.5685 | 1.5086 | 1.4404 | 1.3591 | 1.3286 |
| 5 | | 1.7278 | 1.6382 | 1.5952 | 1.5326 | 1.4794 | 1.4187 | 1.3463 | 1.3191 |
| 10 | | 1.5780 | 1.5162 | 1.4864 | 1.4431 | 1.4061 | 1.3638 | 1.3131 | 1.2940 |
| 1 | 5 | 2.2858 | 2.0955 | 2.0045 | 1.8727 | 1.7611 | 1.6347 | 1.4853 | 1.4296 |
| 2 | | 1.9812 | 1.8511 | 1.7889 | 1.6984 | 1.6216 | 1.5343 | 1.4306 | 1.3917 |
| 3 | | 1.8471 | 1.7428 | 1.6927 | 1.6200 | 1.5581 | 1.4876 | 1.4036 | 1.3721 |
| 4 | | 1.7676 | 1.6782 | 1.6353 | 1.5729 | 1.5198 | 1.4592 | 1.3868 | 1.3597 |
| 5 | | 1.7135 | 1.6342 | 1.5961 | 1.5407 | 1.4934 | 1.4395 | 1.3750 | 1.3508 |
| 10 | | 1.5801 | 1.5253 | 1.4989 | 1.4603 | 1.4274 | 1.3898 | 1.3447 | 1.3276 |
| 1 | 10 | 2.0570 | 1.9276 | 1.8656 | 1.7756 | 1.6991 | 1.6121 | 1.5086 | 1.4699 |
| 2 | | 1.8437 | 1.7547 | 1.7120 | 1.6498 | 1.5968 | 1.5364 | 1.4642 | 1.4371 |
| 3 | | 1.7500 | 1.6783 | 1.6439 | 1.5937 | 1.5508 | 1.5019 | 1.4433 | 1.4213 |
| 4 | | 1.6944 | 1.6328 | 1.6032 | 1.5601 | 1.5232 | 1.4811 | 1.4305 | 1.4115 |
| 5 | | 1.6566 | 1.6019 | 1.5755 | 1.5371 | 1.5043 | 1.4667 | 1.4216 | 1.4046 |
| 10 | | 1.5632 | 1.5252 | 1.5068 | 1.4800 | 1.4571 | 1.4307 | 1.3991 | 1.3871 |
| 1 | 25 | 1.8334 | 1.7546 | 1.7168 | 1.6616 | 1.6146 | 1.5609 | 1.4966 | 1.4724 |
| 2 | | 1.7004 | 1.6458 | 1.6195 | 1.5811 | 1.5484 | 1.5108 | 1.4658 | 1.4488 |
| 3 | | 1.6419 | 1.5977 | 1.5765 | 1.5454 | 1.5188 | 1.4883 | 1.4517 | 1.4378 |
| 4 | | 1.6071 | 1.5691 | 1.5508 | 1.5240 | 1.5011 | 1.4748 | 1.4431 | 1.4312 |
| 5 | | 1.5835 | 1.5497 | 1.5333 | 1.5094 | 1.4890 | 1.4655 | 1.4373 | 1.4266 |
| 10 | | 1.5251 | 1.5014 | 1.4900 | 1.4732 | 1.4589 | 1.4424 | 1.4225 | 1.4149 |

**Table A1-12.1.** Percentiles of the RMS of the Radial Standard Deviations (*RMS-RSD*). (CONTINUED)

| k | n | P 0.5 | P 0.25 | P 0.2 | P 0.1 | P 0.05 | P 0.025 | P 0.01 | P 0.005 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.8326 | 0.5364 | 0.4724 | 0.3246 | 0.2265 | 0.1591 | 0.1003 | 0.0708 |
| 2 | | 0.9161 | 0.6933 | 0.6420 | 0.5157 | 0.4215 | 0.3480 | 0.2725 | 0.2275 |
| 3 | | 0.9441 | 0.7588 | 0.7153 | 0.6061 | 0.5221 | 0.4541 | 0.3812 | 0.3356 |
| 4 | | 0.9581 | 0.7961 | 0.7578 | 0.6604 | 0.5844 | 0.5220 | 0.4537 | 0.4099 |
| 5 | | 0.9665 | 0.8208 | 0.7861 | 0.6975 | 0.6277 | 0.5698 | 0.5058 | 0.4643 |
| 10 | | 0.9833 | 0.8790 | 0.8538 | 0.7888 | 0.7366 | 0.6925 | 0.6427 | 0.6097 |
| 1 | 3 | 1.0578 | 0.8005 | 0.7413 | 0.5954 | 0.4867 | 0.4018 | 0.3147 | 0.2627 |
| 2 | | 1.1064 | 0.9193 | 0.8750 | 0.7626 | 0.6749 | 0.6027 | 0.5238 | 0.4734 |
| 3 | | 1.1225 | 0.9683 | 0.9314 | 0.8369 | 0.7620 | 0.6995 | 0.6299 | 0.5844 |
| 4 | | 1.1306 | 0.9963 | 0.9640 | 0.8809 | 0.8145 | 0.7587 | 0.6960 | 0.6546 |
| 5 | | 1.1354 | 1.0149 | 0.9858 | 0.9108 | 0.8505 | 0.7996 | 0.7421 | 0.7040 |
| 10 | | 1.1451 | 1.0592 | 1.0383 | 0.9840 | 0.9400 | 0.9025 | 0.8595 | 0.8308 |
| 1 | 4 | 1.1563 | 0.9293 | 0.8761 | 0.7423 | 0.6394 | 0.5562 | 0.4669 | 0.4110 |
| 2 | | 1.1906 | 1.0270 | 0.9879 | 0.8877 | 0.8082 | 0.7419 | 0.6681 | 0.6199 |
| 3 | | 1.2020 | 1.0675 | 1.0351 | 0.9515 | 0.8846 | 0.8282 | 0.7646 | 0.7225 |
| 4 | | 1.2077 | 1.0908 | 1.0625 | 0.9893 | 0.9303 | 0.8804 | 0.8237 | 0.7861 |
| 5 | | 1.2111 | 1.1063 | 1.0808 | 1.0149 | 0.9616 | 0.9163 | 0.8647 | 0.8303 |
| 10 | | 1.2179 | 1.1434 | 1.1252 | 1.0777 | 1.0391 | 1.0060 | 0.9681 | 0.9425 |
| 1 | 5 | 1.2120 | 1.0070 | 0.9585 | 0.8354 | 0.7393 | 0.6603 | 0.5738 | 0.5185 |
| 2 | | 1.2385 | 1.0914 | 1.0560 | 0.9650 | 0.8923 | 0.8311 | 0.7624 | 0.7171 |
| 3 | | 1.2473 | 1.1266 | 1.0973 | 1.0217 | 0.9608 | 0.9093 | 0.8507 | 0.8118 |
| 4 | | 1.2517 | 1.1468 | 1.1213 | 1.0552 | 1.0018 | 0.9563 | 0.9045 | 0.8699 |
| 5 | | 1.2544 | 1.1603 | 1.1375 | 1.0780 | 1.0297 | 0.9886 | 0.9416 | 0.9101 |
| 10 | | 1.2596 | 1.1928 | 1.1765 | 1.1338 | 1.0990 | 1.0691 | 1.0348 | 1.0117 |
| 1 | 10 | 1.3167 | 1.1694 | 1.1339 | 1.0424 | 0.9690 | 0.9072 | 0.8376 | 0.7915 |
| 2 | | 1.3292 | 1.2242 | 1.1986 | 1.1323 | 1.0786 | 1.0329 | 0.9806 | 0.9457 |
| 3 | | 1.3334 | 1.2473 | 1.2263 | 1.1717 | 1.1272 | 1.0891 | 1.0455 | 1.0162 |
| 4 | | 1.3354 | 1.2608 | 1.2425 | 1.1949 | 1.1561 | 1.1228 | 1.0846 | 1.0588 |
| 5 | | 1.3367 | 1.2698 | 1.2535 | 1.2107 | 1.1758 | 1.1458 | 1.1113 | 1.0881 |
| 10 | | 1.3392 | 1.2918 | 1.2801 | 1.2496 | 1.2246 | 1.2031 | 1.1782 | 1.1614 |
| 1 | 25 | 1.3760 | 1.2819 | 1.2589 | 1.1992 | 1.1506 | 1.1091 | 1.0616 | 1.0298 |
| 2 | | 1.3808 | 1.3140 | 1.2976 | 1.2548 | 1.2198 | 1.1898 | 1.1552 | 1.1319 |
| 3 | | 1.3824 | 1.3278 | 1.3143 | 1.2792 | 1.2504 | 1.2257 | 1.1971 | 1.1778 |
| 4 | | 1.3832 | 1.3358 | 1.3242 | 1.2937 | 1.2686 | 1.2471 | 1.2222 | 1.2053 |
| 5 | | 1.3837 | 1.3413 | 1.3308 | 1.3035 | 1.2810 | 1.2617 | 1.2393 | 1.2242 |
| 10 | | 1.3847 | 1.3546 | 1.3472 | 1.3278 | 1.3118 | 1.2980 | 1.2820 | 1.2711 |

## Appendix 2. Supplementary Details

### Appendix 2-1. Power Curves

A brief treatment of power curves is presented for the examples given in this report. Power curves are plots of the probability of rejection under various alternatives. Consider the null hypothesis ($H_0$) and the alternate hypothesis ($H_a$):

$$H_0: \sigma \leq \sigma^*$$

$$H_a: \sigma > \sigma^*.$$

Our desire is to accept product with a population σ less than or equal to $\sigma^*$, and reject it otherwise. These hypotheses are evaluated by measuring the dispersion measure, *M*. Let our test for acceptance be:

$$M \leq M^*.$$

If our measurement *M* exceeds $M^*$, then the product is rejected. The probability of rejection is given by:

$$P_{rej}(M > M^*) = P_{rej}(M/\sigma > M^*/\sigma)$$

$$= P_{rej}(M/\sigma > (\sigma^*/\sigma)(M^*/\sigma^*)).$$

For compactness of notation, $T \equiv M/\sigma$. The probability of rejection is now rewritten as:

$$P_{rej}(M > M^*) = P_{rej}(T > (\sigma^*/\sigma)(M^*/\sigma^*)).$$

$M^*$is chosen such that if $H_0$ is true then the null hypothesis is rejected at a type-I error rate of no more than α%. This is ensured by the choice:

$$(M^*/\sigma^*) = T_{1-\alpha},$$

where $T_{1-\alpha}$ is the $(1-\alpha) \times 100$ percentile of *T*. We now have:

$$P_{rej}(M > M^*) = P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha}).$$

If $H_0$ is true, the probability of rejection is:

$$P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha}) \leq \alpha,$$

where the equality holds for $\sigma = \sigma^*$, and if $H_a$ is true, the probability of rejection is:

$$P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha}) = 1 - \beta,$$

where β is the type-II error rate, and $(1-\beta)$ is the power. The Power Curve is constructed by choosing α and plotting the probability of rejection ($P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha})$) versus the ratio of $\sigma/\sigma^*$.

## References

1. Grubbs, Frank E., "Statistical Measures of Accuracy for Riflemen and Missile Engineers," 1991.
2. Taylor, M.S. and F. E. Grubbs, "Approximate Probability Distributions for the Extreme Spread," BRL Memo Report, 1975.
3. Pearson, E. S. and H. O. Hartley, “Biometrika Tables for Statisticians,” Third Edition, Biometrika Trust, 1984.
4. Koehler, E., E. Brown, and S. J.-P.A. Haneuse, “On the Assessment of Monte Carlo Error in Simulation-Based Statistical Analyses,” The American Statistician, 2009, 155-162.
5. Sitton, G. "Groups, Statistics and Practical Facts," Handloader the Journal of Ammunition Reloading, September-October, 1990, 42-44.

## Bibliography

1. Spiegel, Murray R., “Schaum's Outline Series of Theory and Problems of Probability and Statistics,” McGraw-Hill, 1992.
2. Weisstein, Eric W. "Chi Distribution," Accessed 15 May 2017, http://mathworld.wolfram.com/ChiDistribution.html
3. Weisstein, Eric W. "Gamma Function," Accessed 15 May 2017, http://mathworld.wolfram.com/GammaFunction.html
4. Weisstein, Eric W. "Chi-Squared Distribution," Accessed 15 May 2017, http://mathworld.wolfram.com/Chi-SquaredDistribution.html

## List of Abbreviations and Acronyms

AES........................................................Average Extreme Spread
AMD.......................................................Average Mean Deviation
AMR.......................................................Average Mean Radius
AR...........................................................Average Range
ASD........................................................Average Standard Deviation
CEP........................................................Circular Error Probability
CI............................................................Confidence Interval
CV..........................................................Coefficient of Variation
CL...........................................................Confidence Level
DoF.........................................................Degrees of Freedom
EHD........................................................Extreme Horizontal Dispersion
ES............................................................Extreme Spread
EVD........................................................Extreme Vertical Dispersion
HSD........................................................Horizontal Standard Deviation
MCE........................................................Monte Carlo Error
MD..........................................................Mean Deviation
MHD.......................................................Mean Horizontal Deviation
MVD.......................................................Mean Vertical Deviation
MR..........................................................Mean Radius
R..............................................................Range
RE...........................................................Relative Efficiency
RMS........................................................Root Mean Square
RMS-SD..................................................Root Mean Square of Standard Deviations
RMS-RSD...............................................Root Mean Square of Radial Standard Deviations
RSD.........................................................Radial Standard Deviation
SD...........................................................Standard Deviation
SE............................................................Standard Error
VSD........................................................Vertical Standard Deviation