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14. ABSTRACT This report describes a brief research project on foundational aspects of systems-of-systems design and operation. The overarching goal of the research was a design approach for composing structures and behaviors such that the resulting systems will be able to function and adapt using an available but a priori unknown mix of sensing and communication modalities. The research reported herein was focused on a model problem of deploying and operating fielded medical treatment facilities (MTFs). The solution to the problem was found to have two coupled components: the optimal location of treatment facilities of prescribed types and the optimal routing and treatment protocols for casualties moving through the facility network. Several approaches to the problem were explored, including: (1) representing the problem as a resource selection problem and formulating this as a nonlinear mixed integer optimization problem; (2) using an abstraction of the problem that interpret system "compositionality" in terms of dynamical systems on function semigroups; and (3) modeling the operation of the MTF network using switched mode ordinary differential equations. The report concludes with a discussion of model sensitivity based on a combinatorial witness complex.						
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Topological Methods for Design and Control of Adaptive Stochastic Complex Systems

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1 Executive Summary

This report describes a brief research project on foundational aspects of systems-of-systems design and operation. The overarching goal of the research is a design approach for composing structures and behaviors such that the resulting systems will be able to function and adapt using an available but a priori unknown mix of sensing and communication modalities. The research reported herein was focused on a model problem of deploying and operating fielded medical treatment facilities (MTFs). The solution to the problem was found to have two coupled components: the optimal location of treatment facilities of prescribed types and the optimal routing and treatment protocols for casualties moving through the facility network. Several approaches to the problem were explored, including: (1) representing the problem as a resource selection problem and formulating this as a nonlinear mixed integer optimization problem; (2) using an abstraction of the problem that interpret system "compositionality" in terms of dynamical systems on function semigroups; and (3) modeling the operation of the MTF network using switched mode ordinary differential equation systems.

2 Introduction and Issues to Be Addressed

Among the most difficult of real-world Systems-of-Systems challenges is the design and operational control of medical treatment networks that support forces operating in austere environments. Because the effectiveness of medical operations in the field is crucially dependent on access to diagnostic technology, availability of medicines and trained medical personnel, and dependable transport services for bringing those in need of treatment to the facility, the realization of decision support systems for design and operation must account for interactions within a hierarchy of qualitatively dissimilar kinds of networks including the treatment facility networks themselves (the top of the hierarchy) and the logistical support networks, communication networks, and local human resource (talent-pool) networks. Clearly changes in the operation of any of the networks playing a supporting role will impact the operation of medical treatment facilities (MTFs). Because of the complexity of design and operation of fielded MTFs network is high, DARPA has proposed a "toy problem" that has essential qualitative features of the general problem while being simple enough to explore mathematical model alternatives for decision support systems. The report that follows provides results and concepts that address some of the main problem elements within the illuminating context of the toy problem. The goal is to understand the problem in enough detail that it will be possible to develop techniques that can be both carefully validated by provably correct solutions to the toy problem while being extensible to problems of realistic scale.

The report is organized as follows. The next section recaps the problem statement as formulated by DARPA. We then formulate an optimization problem that seeks to optimize the expected positive outcomes of treatment in the MTF network while being constrained by the available resources. The solution to the problem is seen to have two coupled components: the optimal location of treatment facilities of the different types that have been prescribed and the optimal routing of casualties from the points at which injuries occur through the network and toward the final treatment outcome states. The problem is represented as a resource selection problem and modeled as a nonlinear mixed integer optimization problem. Several versions of the problem are discussed - each having a different level of complexity. A number of general approaches to solution of the optimization problem are proposed, and a very detailed analysis at the prescribed *toy scale* is carried out in order to provide "ground truth" for the general solution approaches that are now being studied. These solutions are based of relaxation and are discussed in Section 5.2. The techniques appear to be very fast when applied to the toy problem parameters, and this suggests that the algorithms will be useful in highly dynamic environments where problem parameters and constraints change over very short time scales. A parallel model of the dynamic operation of the MTF network involving switched mode differential equations is given in Section 7.

3 A Stylized Problem in Agile Field Medicine

In a battlefield theater, there are two deployed *Role 2* facilities that receive battlefield casualties. From these facilities casualties are shipped to *Role 3* field hospitals, MTF's. There are three types:

110 – bed MTF	30 – bed MTF	10 – bed MTF
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Role 3 Alternative	Cost per MTF	DOW Percentage (per 24 hours)	RTD Percentage (per 24 hours)	MEDEVAC Rate (3 -> 4)
100-bed Theater Hospital	\$1.5M	10%	10%	20 / day
30-bed MTFs	\$500K	15%	10%	10 / day
10-bed MTFs	\$100K	25%	5%	10 / day

Figure 1: Characteristics of MTF Facility Alternatives.

Essential features of the Role 3 treatment facilities of each type are listed in Fig. 1. Further assumptions that define the stylized problem are as follows:

- **Role 2:** Casualties enter the system via two Role 2 casualty streams, with each Role 2 facility producing 15 casualties per day. To establish a baseline solution, this number of casualties is assumed to be a constant, and moreover, it is assumed that the injuries of all casualties are the same. Clearly these assumptions would be unlikely to hold in an actual theater of operation.
- **MEDEVAC and patient outcomes:** It is assumed that casualties are moved through the network of medical treatment facilities (MTS) and eventually leave the network in one of three states: DOW (the patient died of injuries), RTD (the patient has been returned to duty), or “Moved to Air Base” – i.e. the patient has gone to a long term care facility and is no longer considered part of the system being modeled. The actual spatial movement of the casualties is by MEDEVAC vehicles that could include ambulance vehicles or flight vehicles —primarily helicopters. For the purpose of the stylized problem, all MEDEVAC vehicles are regarded as equivalent.
- **MEDEVAC from Role 2 to Role 3:** The movement from Role 2 facilities is dangerous and the following assumptions are made: MEDEVAC capacity is always available, and the DOW rate on this leg of transport is 1.5% per km (Bernoulli).
- **MEDEVAC from Role 3 to Role 4:** Air Base: Here it is assumed that patients have been stabilized so that DOW rate is reduced to .5% per km.
- **Geometry:**
 - There are two Role 2 facilities which are 40km apart;
 - Each Role 2 facility is 100km from the Air Base;
 - Each Role 3 facility must be a minimum of 25km from the Role 2 facilities to be in the “safe zone”;
 - Each Role 3 facility must be at least 5km from any other Role 3 facility.
- **Daily Update Process:**
 - Patient outcomes for the previous 24 hours in Role 3 facilities are resolved (DOW - RTD - Moved to Air Base). We assume for now that the Role 3 facilities discharge all their casualties at the end of each day. This simplification will be relaxed later.
 - MEDEVAC 3 → Air Base is planned and then resolved;
 - MEDEVAC 2 → 3 is planned and then resolved.
- Time is not explicitly modeled – transportation is considered to be instantaneous. Section 6 below relaxes this assumption and represents one of several relaxation approaches that will be discussed.
- Patients can be transported directly from Role 2 to the Air Base. This transportation is unlimited but with a DOW rate of 1.5% per km or $1 - (0.985^{100})$ for the full trip..
- It is assumed that all parties have complete information about the outcomes of every previous step.

- It is temporarily assumed that all patients who arrive at an MTF stay exactly 24 hours. This will later be relaxed to assume that patients who arrive at an MTF will stay a minimum of 24 hours.

With this setup, the following problem is considered. **Problem:** What combination of the three types of Role 3 MTFs provides the best functional capability assuming a deployment cost constraint that the maximum budget for deployment of the MTFs is \$2M.

The **solution** we seek consists of:

- A list of which Role 3 MTF's to deploy,
- Their deployment geometry in the field, and
- A policy that determines how to route casualties from the two Role 2 facilities..

Some possible metrics for “best capability” include

$$\eta = \lambda_1 DOW + \lambda_2 DOW * \$\$ + \lambda_3 \$\$, \quad (1)$$

where the λ_k 's tradeoff the cost of deaths from wounds against the cost of providing the facilities. We note that this cost function will be nonconvex in general.

4 Modeling

The problem set-up is depicted in Fig. 2. There are three types of Role 3 MTF facilities, per the above problem statement. We assume that there are M_1 100-bed units, M_2 30-bed units, and M_3 10-bed units. These have costs as listed in the table in Fig. 1. They receive casualties from the two Role 2 facilities. The total number of Role 3 MTFs is $M = M_1 + M_2 + M_3$, and we shall label them $1, \dots, M$. For each facility, let T_{ij} denote the number of casualties arriving at the j -th MTF from the i -th Role 2 facility. ($i = 1, 2$; $j = 1, \dots, M$), and let D_i denote the number of casualties that are directly sent to the air base from the i -th Role 2 facility ($i = 1, 2$). There are thus two routes of entry to each Role 3 MTF, one from each Role 2. There are three departure routes from each Role 3 MTF—one to each of the terminal states DOW, RTD, and Moved to Air Base. Denote the number of casualties being transported from the j -th Role 3 facility to the Role 4 facility (Air Base) by S_j , $j = 1, \dots, M$.

We define *siting parameters* d_{ij} ($i, 1, 2$, $j = 1, \dots, M$) and e_j ($j = 1, \dots, M$) as follows:

$$\begin{aligned} d_{ij} &= \text{distance of the } j\text{-th Role 3 MTF from the } i\text{-th Role 2 unit } (\geq 25 \text{ km}), \\ e_j &= \text{distance of the } j\text{-th Role 3 MTF from the Air Base.} \end{aligned}$$

With the routes and facilities thus labeled, we look at state transitions in the model. We note that the routes of transit must be considered to have the same status in the finite state model as the medical facilities, since from each of these routes transitions to the DOW state are possible. In each 24 hour period, there will be transitions to DOW from the boxes labeled T_{ij} , S_j and each MTF with the expected numbers being:

$$\begin{aligned} &1 - 0.985^{d_{ij}}, \quad i = 1, 2; \quad j = 1, \dots, M, \\ &1 - 0.995^{e_j} \quad j = 1, \dots, M, \\ &1 - 0.985^{100} \quad \text{for } D_i \quad i = 1, 2, \\ &0.1/\text{day} \quad \text{for 100 bed unit,} \\ &0.15/\text{day} \quad \text{for 30 bed unit, and} \\ &0.25/\text{day} \quad \text{for 10 bed unit.} \end{aligned} \quad (2)$$

We assume that the system is in steady state, which means T_{ij} and D_i do not vary with the time. Then there is an upper limit $\bar{T}_{ij}$ for the number of patients that can be sent from the i -th Role 2 facility to the j -th Role 3 facility due to the limited number of beds and the limited MEDEVAC rate, and it is given by

$$\sum_{i=1}^2 \bar{T}_{ij} \times 0.985^{d_{ij}} = \# \text{ of beds in } j \times (\text{DOW rate in } j + \text{RTD rate in } j) + \text{MEDEVAC capacity of } j. \quad (3)$$

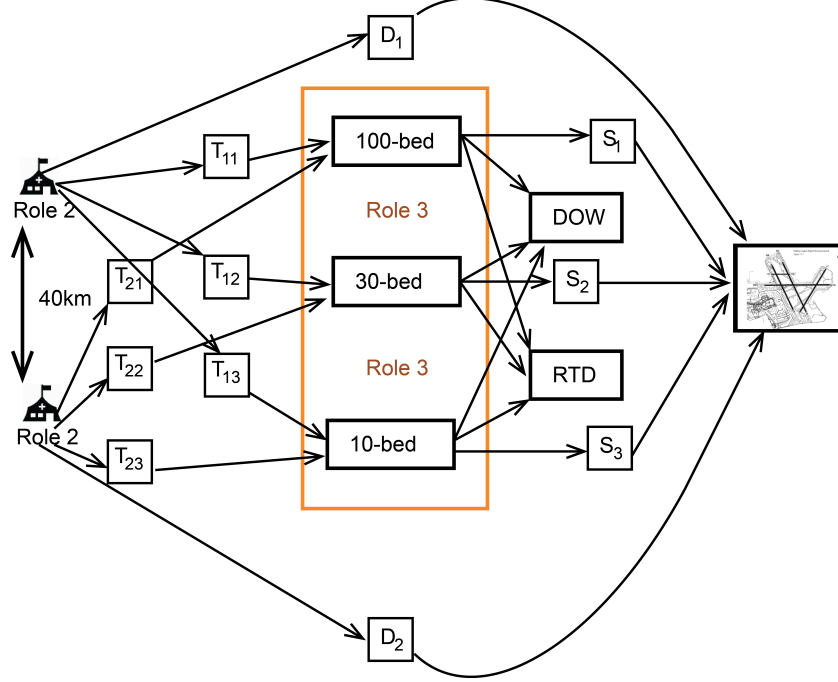


Figure 2: Each deployment strategy of Role 3 MTF's is represented by a finite state system with two sources (Role 2 care facilities on the left) and three sinks or terminal states (states labeled DOW, RTD, and Moved to Air Base).

To be more specific, if the j -th Role 3 facility is a 100-bed unit, then

$$\sum_{i=1}^2 \bar{T}_{ij} \times 0.985^{d_{ij}} = 100 \times (10\% + 10\%) + 20 = 40;$$

if the j -th Role 3 facility is 30-bed unit, then

$$\sum_{i=1}^2 \bar{T}_{ij} \times 0.985^{d_{ij}} = 30 \times (15\% + 10\%) + 10 = 17.5;$$

if the j -th Role 3 facility is 10-bed unit, then

$$\sum_{i=1}^2 \bar{T}_{ij} \times 0.985^{d_{ij}} = 10,$$

since the MEDEVAC capacity is large relative to the number of the beds in that unit. We can now compute the average DOW rate of the patients that are sent from the i -th Role 2 facility to the j -th Role 3 facility. Here we need to consider two cases.

The first case is that the length of stay of all patients in the j -th Role 3 facility is exactly 24 hours which means

$$\sum_{i=1}^2 T_{ij} \times 0.985^{d_{ij}} \leq \frac{\text{MEDEVAC capacity of } j}{1 - \text{DOW rate in } j - \text{RTD rate in } j}.$$

To be specific, the constraints on patients entering each type of facility are

$$\begin{aligned} \sum_{i=1}^2 T_{ij} \times 0.985^{d_{ij}} &\leq \frac{20}{1 - 10\% - 10\%} = 25 \quad \text{for the 100 bed unit;} \\ \sum_{i=1}^2 T_{ij} \times 0.985^{d_{ij}} &\leq \frac{10}{1 - 15\% - 10\%} = 13.3 \quad \text{for the 30 bed unit.} \end{aligned}$$

Clearly, the patients going to the 10-bed unit won't stay for more than 24 hours due to its relatively large MEDEVAC rate availability. The constraint in this case is thus

$$\sum_{i=1}^2 T_{ij} \times 0.985^{d_{ij}} \leq 10 \quad \text{for the 10 bed unit.}$$

In the case of complete patient turnover in each 24 hour period, the DOW rate of the patients that are sent from the i -th Role 2 facility to the j -th Role 3 facility is given by

$$DOW_{ij} = 1 - \frac{\# \text{ of RTD patients from } j + \# \text{ of patients from } j \text{ arriving at air base}}{T_{ij}}, \quad (4)$$

where the *numbers of RTD patients from the Role 3 facilities* are given by

$$\begin{aligned} T_{ij} \times .985^{d_{ij}} \times 10\% & \quad \text{for the 100 bed unit,} \\ T_{ij} \times .985^{d_{ij}} \times 10\% & \quad \text{for 30 the bed unit, and} \\ T_{ij} \times .985^{d_{ij}} \times 5\% & \quad \text{for 10 the bed unit,} \end{aligned}$$

respectively and *numbers of patients arriving at the Role 4 (Air Base)* are given by

$$\begin{aligned} T_{ij} \times .985^{d_{ij}} \times 80\% \times .995^{e_j} & \quad \text{for the 100 bed unit,} \\ T_{ij} \times .985^{d_{ij}} \times 75\% \times .995^{e_j} & \quad \text{for the 30 bed unit,} \\ T_{ij} \times .985^{d_{ij}} \times 70\% \times .995^{e_j} & \quad \text{for the 10 bed unit.} \end{aligned}$$

With the above notational conventions and constraints, the baseline optimization problem is to minimize the total expected casualties entering the DOW state

$$\begin{aligned} \eta = & \sum_{j=1}^{M_1} \sum_{i=1}^2 [T_{ij}(1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 80\% \times .995^{e_j})] \\ & + \sum_{j=M_1+1}^{M_1+M_2} \sum_{i=1}^2 [T_{ij}(1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 75\% \times .995^{e_j})] \\ & + \sum_{j=M_1+M_2+1}^M \sum_{i=1}^2 [T_{ij}(1 - .985^{d_{ij}} \times 5\% - .985^{d_{ij}} \times 70\% \times .995^{e_j})] \\ & + \sum_{i=1}^2 (1 - .985^{100}) D_i \end{aligned} \quad (5)$$

subject to

$$\begin{aligned} \sum_{i=1}^2 T_{ij} \times .985^{d_{ij}} & \leq 25 \quad \forall j \in \{1, \dots, M_1\}, \\ \sum_{i=1}^2 T_{ij} \times 0.985^{d_{ij}} & \leq 13.3 \quad \forall j \in \{M_1, \dots, M_1 + M_2\}, \\ \sum_{i=1}^2 T_{ij} \times 0.985^{d_{ij}} & \leq 10 \quad \forall j \in \{M_1 + M_2, \dots, M_1 + M_2 + M_3\}, \\ D_i + \sum_{j=1}^M T_{ij} & = 15 \quad \forall i, \\ T_{ij}, D_i & \geq 0 \quad \forall i, j, \\ T_{ij}, D_i & \text{ are integers} \quad \forall i, j, \text{ and} \\ & \text{All geometry constraints.} \end{aligned} \quad (6)$$

Remarks: (1) This stylized problem setup is somewhat special, and will be subject to generalization and extensions of many types. For the particular case of precisely two Role 2 facilities through which all casualties are processed, and with the Role 4 facility having its location determined as 100 km from each of the Role 2 facilities, the distances e_j are determined as nonlinear functions of the distances d_{1j}, d_{2j} .

To carry this out explicitly, we choose a Euclidean coordinate system for the theater of operation, Fig. 3. The y -axis passes through the two Role 2 facilities, and the x -axis is the perpendicular bisector of the 40 km line segment between these facilities as illustrated. If both the Role 2 facilities are 100 km from the Role 4 facility (Air Base) as depicted, then the Air Base is located on the x -axis $\sqrt{9600}$ km from the origin of the coordinate system. Given that d_1 and d_2 denote the distances of the Role 3 MTF from the respective Role

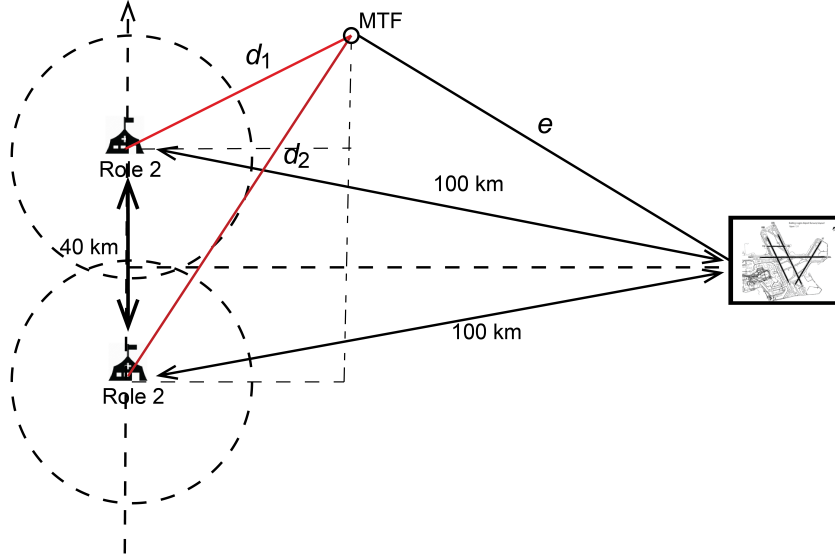


Figure 3: The distances d_{1j}, d_{2j} of the MTF facilities from the Role 2 medical facilities provide a local coordinate system for the problem. These coordinates can be easily transformed to Euclidean coordinates as discussed in the text.

2 facilities, the Euclidean coordinates of the facility are easily determined by the *cosine law*. Specifically, elementary trigonometry yields facility coordinates

$$y = y(d_1, d_2) = \frac{d_2^2 - d_1^2}{80}, \quad x = x(d_1, d_2) = \sqrt{d_2^2 - (y + 20)^2}.$$

We have chosen the positive square root; the negative square root identifies a location at distances d_1, d_2 on the opposite side of the y -axis. Given the chosen Euclidean coordinates, the distance e from the Role 3 MTF to the Air Base is simply the Euclidean distance $\|(x(d_1, d_2), y(d_1, d_2)) - (\sqrt{9600}, 0)\|$. For the present problem parameters, this is given as

$$e(d_1, d_2) = \frac{\sqrt{d_1^2 + d_2^2 - 2\sqrt{6}\sqrt{-d_1^4 + 2d_2^2d_1^2 + 3200d_1^2 - d_2^4 + 3200d_2^2 - 2560000} + 18400}}{\sqrt{2}}. \quad (7)$$

(2) The importance of noting this functional dependency is that for more general versions of the problem it will be necessary to classify variables as to whether they are free or determined. Finding functional relationships of this kind will be key to both the problem solution and to complexity reduction as well. By way of a further example, if the j -th Role 3 MTF is to be optimally sited with respect to three (or more) Role 2 facilities, the respective distances $d_{1j}, d_{2j}, d_{3j}, \dots$ cannot all be free variables. Indeed, specifying any two will determine the others. This point is developed further in Section 5.2 below in the context of relaxation methods.

(3) Transforming the d_{ij} variables to Euclidean coordinates allows us to express the constraint that no pair of MTFs can be closer than 5 km to one another. This requirement is expressed by the nonlinear inequality constraint

$$(x(d_{1,j}, d_{2j}) - x(d_{1,k}, d_{2k}))^2 + (y(d_{1,j}, d_{2j}) - y(d_{1,k}, d_{2k}))^2 \geq 25.$$

One of the challenges to be addressed by the research going forward is how to relax such nonlinear relationships in our computational approaches.

The above model will serve as the basis of analysis in several of the sections that follow. A model that captures the effect of facility congestion is presented in the Section 7. In the next section, we discuss explicit solution of the stylized problem as formulated above in Section 4. Assuming the total cost limit of \$2M for Role 3 medical facilities is binding and that the cost of each facility type is as given in Fig. 1, there are

exactly seven possible mixtures of such facilities with collective cost of exactly \$2M:

$$(M1, M2, M3) = (1, 1, 0), (1, 0, 5), (0, 1, 15), (0, 2, 10), (0, 3, 5), (0, 4, 0), (0, 0, 20).$$

If we include all feasible mixtures of facilities (most mixtures costing strictly less than \$2M), then it is not difficult to verify that there is a total of 59 feasible solutions. We shall show that many of these solutions can be discarded, and the choice among those that remain will depend on trading off quality of patient care against the cost of deploying the MTFs. An unmodeled utility of the field commander would be the basis of the deployment choice.

5 The Mathematics of Resource Selection and Operation

5.1 Optimal Provision of Mobile Medical Units and Casualty Routing Policies

We begin the discussion of the stylized problem solution by considering the case in which all casualty treatments are completed within 24 hours. Following that, we shall take facility congestion into account. We begin by assuming that the system objective is only to minimize the rate at which casualties die of their wounds (i.e. $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$ in eq. (1)).

5.1.1 The optimal solution when DOW rate is the prime consideration

Lemma 1. If we have only one 100-bed unit, then it must be placed at the intersection of 25 km circles centered at the Role 2 facilities as indicated by the blue dot in Fig. 4. This point is the unique minimizer of the DOW rate under the assumption that the MTF resource is a single 100 bed facility.

Proof. Note that one 100-bed unit can receive all patients from the two Role 2 facilities and can transfer all patients to the air base in 24 hours when it is a minimum of 25km from the two Role 2 facilities. The lemma can be proved by noting that eq. (5), specializes to

$$\eta = \sum_{i=1}^2 15 \times (1 - .985^{d_{i1}} \times 10\% - .985^{d_{i1}} \times 80\% \times .995^{e_1})$$

and recalling that e_1 is an explicit function of d_{11} and d_{21} given by (7). The details of the computation are omitted here. $\square$

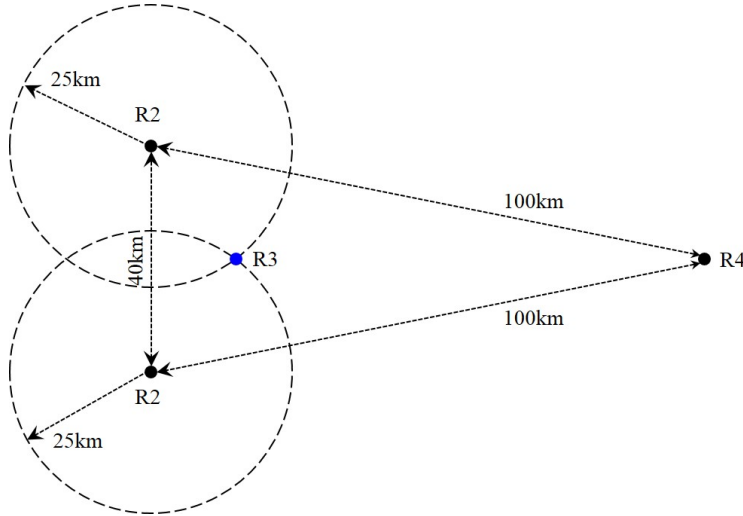


Figure 4: The optimal location for single *Role 3* facility that has large enough capacity.

Lemma 2. If we have two Role 3 facilities with both of them having enough beds and large enough MEDEVAC rates, and if both of them receive patients under an optimal routing protocol, then the optimal placement is indicated by the two blue dots in Fig. 6.

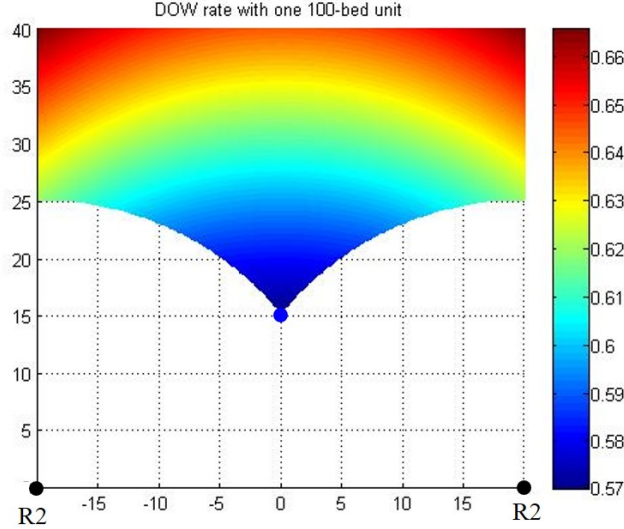


Figure 5: The map of DOW rate with single 100-bed unit being used and two Role 2 facilities being located at the lower corners.

Proof. There are two possible cases when we have two Role 3 facilities. The first case is that the two facilities have the same number of beds. Since we assume they have enough beds and large enough MEDEVAC rates, it is easy to prove that the optimal locations for the two facilities must be the two blue dots in Fig. 6.

The second case is that the two facilities have different number of beds. Without loss of generality, we assume that the first Role 3 facility is the more expensive one in terms of \$ cost of deployment, and the second Role 3 facility is the cheaper one. It is easy to prove that there must exist at least one path through the first Role 3 facility, and the DOW rate of this path is strictly lower than that of all paths through the second Role 3 facility, otherwise the average DOW rate would be even higher than that with two cheaper Role 3 facilities. Without loss of generality, we assume that the first Role 2 facility lies on the path with lowest DOW rate. Then the first Role 2 facility must send all patients to that path because the Role 3 facility is assumed to have large enough capacity. Since we assume that the cheaper Role 3 facility is also receiving patients, then those patients must be from the second Role 2 facility. Since we assume the Role 3 facilities' capacities are large enough, then the reason why the second Role 3 facility sends patients to the cheaper Role 3 facility must be because the path with the cheaper Role 3 facility is the best choice for the second Role 2 facility. Therefore, the second Role 2 facility will send all patients to the cheaper Role 3 facility. In such case, each Role 2 facility will use only one Role 3 facility. So the the first Role 3 facility will be placed such that it can minimize the DOW rate of the path connecting the first Role 2 facility and the air base, and the second Role 3 facility will be placed such that it can minimize the DOW rate of the path connecting the second Role 2 facility and the air base. The only locations satisfying the requirement are those indicated by the two blue dots in Fig. 6.

□

Proposition 1. Consider the problem of optimal mixture of types of MTFs under the assumption that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$ in eq. (1). The optimal deployment of the three types of Role 3 facilities is then a mixture of one 100-bed unit and one 30-bed unit. If $\lambda_3 \gg 1$, then the optimal mixture of MTF type will involve only 10-bed units.

Proof. It is easy to prove the latter part that only 10-bed units will be used when \$ is a primary consideration. If $\lambda_3 \gg 1$, then the optimal mixture of MTF types will involve only 10-bed units or even no Role 3 facilities in the extreme case. The conclusion is supported by the fact the the decrease of DOW rate is not proportional to the investment, and the details are shown in Table 1. It is, in short, that the marginal cost of reducing one unit of DOW increases significantly when the DOW rate decreases. So here we only need to consider the case that that only 100-bed unit and 30-bed units will be used when the DOW rate is the

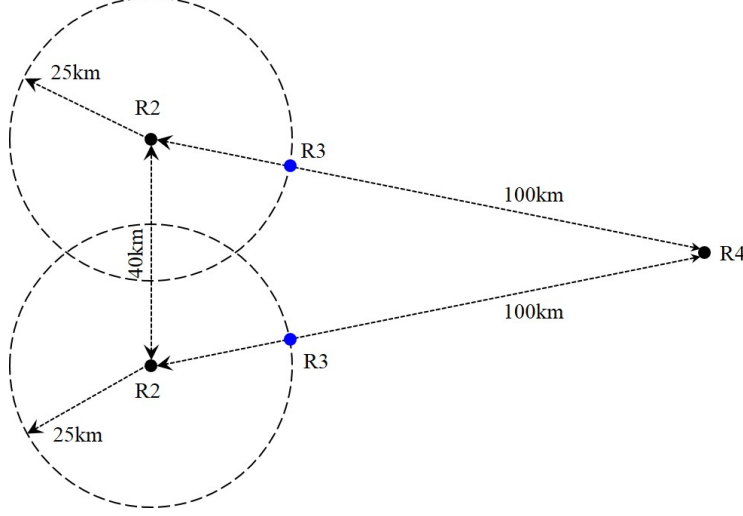


Figure 6: The potential locations for the two *Role 3* facilities that have enough capacities.

prime consideration.

Since one 30-bed unit has enough capacity to serve all patients from one Role 2 facility when it is a minimum of 25km from that Role 2 facility, it is easy to prove that the performance of two 30-bed units together must dominate the performance of all possible combinations of 30-bed units and 100-bed units. Also it is easy to prove that the performance of one 30-bed unit and one 100-bed unit together dominates the performance of two 30-bed units together. Then the only question left here is whether the the performance of one 30-bed unit and one 100-bed unit together is better than just one 100-bed unit.

By using Lemma 1, we know that the optimal location for one 100-bed unit is the blue dot in Fig. 4. The total of the expected casualties entering the DOW state in this configuration is

$$\eta_1 = \sum_{i=1}^2 15 \times (1 - .985^{25} \times 10\% - .985^{25} \times 80\% \times .995^{83}) = 17.0939.$$

By using Lemma 2, we know that if the 30-bed unit is helpful when we already have one 100-bed unit, then the optimal locations for them must be the two blue dots in Fig. 6. The total of the expected casualties entering the DOW state in this configuration is

$$\eta_2 = 15 \times (1 - .985^{25} \times 10\% - .985^{25} \times 80\% \times .995^{75}) + 15 \times (1 - .985^{25} \times 10\% - .985^{25} \times 75\% \times .995^{75}) = 17.0029.$$

Thus we will use 100-bed unit and 30-bed unit together when DOW rate is the prime consideration. $\square$

Remark: The difference in cost between the optimal and suboptimal deployment costs that conclude the above proof is very small. Clearly the solution might easily change if any of the constraints are changed. It's also likely that the difference in costs would not be detected by the approximate methods discussed in Section 6 below.

5.1.2 The investment frontier when trading off deployment costs against quality of patient care

The analysis of this subsection considers tradeoffs in cost under different facility deployment strategies. We now relax the assumption that all Role 3 treatments will conclude within 24 hours.

Definition 1. One Role 3 facility is *saturated* when its average length of stay is greater than 24 hours.

Patients can be transported directly from Role 2 to the Role 4. This transportation is unlimited and free with a DOW rate of $1 - (0.985^{100}) = 0.779$.

The minimum DOW rate of a path with one 10-bed unit is achieved when the path is a straight path connecting Role 2 and Role 4 with the 10-bed unit being 25km from Role 2 and 75km from Role 4. When the length of stay in the 10-bed unit is 24 hours and the 10-bed unit is not saturated, the minimum DOW rate of the path is achieved and it is $0.636 = 1 - 0.985^{25} \times 0.05 - 0.985^{25} \times 0.7 \times 0.995^{75}$. If the 10-bed unit is saturated, then the minimum DOW rate of those excess patients is $0.886 = 1 - 0.985^{25}/6$. Since 0.886 is even higher than the DOW rate of the free path (0.779), we know *the 10-bed unit will never be saturated no matter how many beds it has*.

The minimum DOW rate of a path with one 30-bed unit is achieved when the path is a straight path connecting Role 2 and Role 4 with the 30-bed unit being 25km from Role 2 and 75km from Role 4. When the length of stay in the 30-bed unit is 24 hours and the 30-bed unit is not saturated, the minimum DOW rate of the path is achieved and it is $0.579 = 1 - 0.985^{25} \times 0.1 - 0.985^{25} \times 0.75 \times 0.995^{75}$. If the 30-bed unit is saturated, then the minimum DOW rate of those excess patients is $0.726 = 1 - 0.985^{25} \times 0.4$.

The minimum DOW rate of a path with one 100-bed unit is achieved when the path is a straight path connecting Role 2 and Role 4 with the 100-bed unit being 25km from Role 2 and 75km from Role 4. When the length of stay in the 100-bed unit is 24 hours and the 100-bed unit is not saturated, the minimum DOW rate of the path is achieved and it is $0.555 = 1 - 0.985^{25} \times 0.1 - 0.985^{25} \times 0.8 \times 0.995^{75}$. If the 100-bed unit is saturated, then the minimum DOW rate of those excess patients is $0.657 = 1 - 0.985^{25} \times 0.5$.

We organize the above results as follows:

Path Type	Minimum Unsaturated DOW	Minimum Saturated DOW
Type 1 Role 3	0.555	0.657
Type 2 Role 3	0.579	0.726
Type 3 Role 3	0.636	0.886
Free Path	0.779	NA

Table 1. DOW.

Now let's gradually increase our budget from \$100k to \$2M.

Case 1: Budget $\leq$ \$400K. Clearly, we can have at most four 10-bed units. Since 10-bed units can never be saturated, we then only need to determine the locations of the 10-bed units by solving the constrained optimization problem. Since the minimum capacity of the path with one 10-bed unit is $14.591 = 10/0.985^{25}$ which is smaller than the casualty rate of one Role 2 (15/day), the optimal DOW rate will decrease when the budget is increased from \$100K to \$400k, but it must be strictly higher than 0.636.

Case 2: Budget = \$500K. Now we can have at most five 10-bed units or one 30-bed unit. It is easy to prove that the purchase of the fifth 10-bed unit is not useful, and we only need to consider the 30-bed unit. With only one 30-bed unit, saturation becomes possible because the DOW rate of the excess patients in the 30-bed unit may be lower than the DOW rate along the direct path between the Role 2 and Role 4 facilities. But the number of patients arriving at the 30-bed unit has an upper limit which is $30 \times 0.25 + 10 = 17.5$ patients everyday. Thus we need to determine one more variable, the length of stay in 30-bed unit or the number of patients arriving at 30-bed unit everyday, when solving the constrained optimization problem. It can be proved that one 30-bed unit can't beat four 10-bed units. Thus the increase of budget from \$400K to \$500K is useless.

Case 3: Budget = \$600K. As always, using the budget to only purchase multiple 10-bed units is useless. We now only consider the case that we have one 30-bed unit and one 10-bed unit. Here, we would have to solve the constrained optimization problem. It can be proved that the DOW rate here will be further decreased.

Case 4: Budget = \$700K. As always, using the budget to only purchase multiple 10-bed units is useless. We now only consider the case that we have one 30-bed unit and two 10-bed units. Again, we would have to solve the constrained optimization problem here. It is expected that the DOW rate here will be further

decreased.

Case 5: $\$800K \leq \text{Budget} \leq \$900K$. It is easy to prove that we need at most two 10-bed units when we have one 30-bed unit. The idea is described as follows. First, the location of the 30-bed unit must fall into the triangle region in Fig. 4, and thus DOW rate of the paths with 10-bed units must be strictly lower than $0.645 = 1 - 0.985^{25} \times 0.05 - 0.985^{25} \times 0.7 \times 0.995^{80}$. So no patients will be transported on the Role 2-to-Role 4 direct path. Second, there must exist at least one path with 30-bed unit whose DOW rate is lower than that of the paths with 10-bed units because otherwise the optimal DOW rate here would be even higher than that of Case 3, and this is impossible. Here, the following cases may happen. (1) Only the patients from one Role 2 are transported to the 30-bed unit. Then all patients from that Role 2 must be transported to 30-bed unit, and all patients from another Role 2 must be transported to 10-bed units. If this case is true, we only need two 10-bed units. (2) Both the first Role 2 and the second Role 2 transport the patients to the 30-bed unit. If the DOW rate from the first Role 2 to the 30-bed unit is lower than the DOW rate from the second Role 2 to the 30-bed unit, then all patients from the first Role 2 will be transported to the 30-bed unit. So only one 10-bed unit will be used. If such case is true, then the purchase of the second 10-bed unit is useless. However, if the DOW rate from the first Role 2 to the 30-bed unit is equal to the DOW rate from the second Role 2 to the 30-bed unit, then each Role 2 will use one 10-bed unit, and both 10-bed units are not saturated. In such case the purchase of additional 10-bed unit is clearly useless. Thus the increase of budget from $\$700K$ to $\$900K$ is useless.

Case 6: Budget = $\$1M$. Now we can have 2 30-bed units. Since the minimum capacity of the path with one 30-bed unit is $19.455 = 10/0.75/0.985^{25}$ which is larger than the casualty rate of one Role 2 (15/day), we can easily get the optimal DOW rate 0.579 with two 30-bed units.

Case 7: $\$110K \leq \text{Budget} \leq \$140K$. There is no basis for purchasing more 10-bed units because the paths in Case 6 are not saturated. Thus the increase of budget from $\$1M$ to $\$140K$ is useless.

Case 8: Budget = $\$150K$. Now we can have one 100-bed unit. The minimum capacity of path with one 100-bed unit is $20/0.8/0.985^{25} = 36.478$, so saturation can't happen here. It must be true that one 100-bed unit can beat two 30-bed units as the DOW rate with one 100-bed unit is $1 - 0.985^{25} \times 0.1 - 0.985^{25} \times 0.8 \times 0.995^{83} = 0.570$ which is obtained when the 100-bed unit is placed at the location shown in Fig. 4.

Case 9: $\$160K \leq \text{Budget} \leq \$190K$. There is no basis for purchasing any 10-bed units if we have Type 1 MTF3. The reason is given below. First there must exist at least one path with 100-bed unit whose DOW rate is lower than that of the path with 10-bed unit otherwise the average DOW rate would be even higher than Case 3. Second, since the 100-bed unit can't be saturated, then at least one Role 2 station will send all patients to that 100-bed unit. Another Role 2 will either send its patients to 100-bed unit or 10-bed unit. If another Role 2 does send patients to 10-bed unit, then the DOW rate of its patients can't be lower than 0.636 because 100-bed unit is not saturated. In such case the average DOW rate can't be better than $(0.636 + 0.555)/2 = 0.596$ which is even higher than that of Case 8.

Case 10: Budget = $\$2M$. By Proposition 1, we know the optimal configuration would be placing one 100-bed unit and one 30-bed unit at the locations indicated by the two blue dots in Fig. 6.

The cost and return relationship is graphically represented in Fig. 7 where we use one minus DOW rate as the measure of return.

5.2 Approaches to Relaxation of the Placement and Routing Problems

Finding a solution to the non-linear mixed integer optimization problem in the above is in general very difficult as the scale of the problem increases. Thus we resort to approximation and relaxation techniques.

5.2.1 From nonlinear mixed integer optimization to mixed integer hybrid linear programs

In the parameter ranges of interest, the cost function (5) is nonconvex and thus not easily solved with standard optimization problems. Despite this difficulty, we note that for the ranges of parameters that

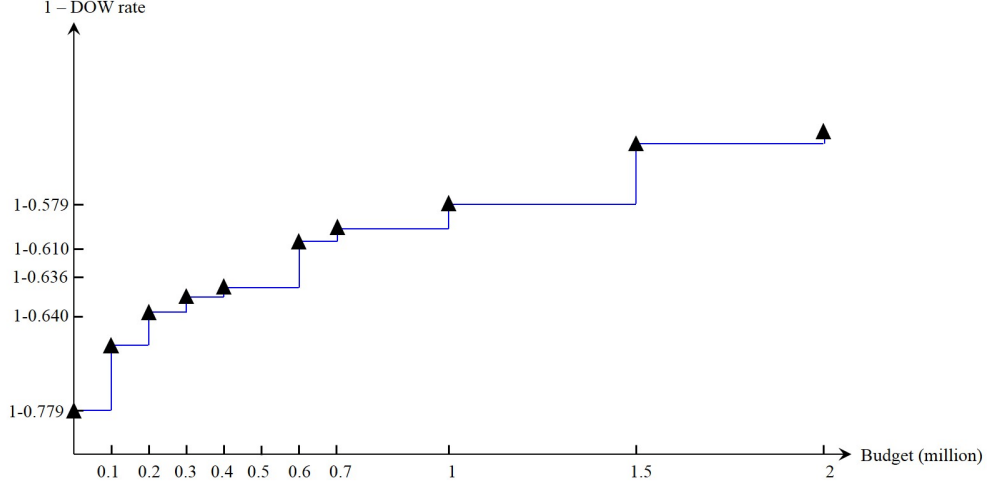


Figure 7: Investment frontier.

specify casualty flows, T_{ij} , the cost of each Type 1 or Type 2 facility

$$\eta_j = \sum_{i=1}^2 T_{ij} (1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 80\% \times .995^{e_1})$$

is not too far from being linear in the location variables, d_{1j}, d_{2j} , as illustrated in Fig. 8. These functional components are thus well approximated by linear surrogates, and the cost function η in eq. (5) can be relaxed to

$$\begin{aligned} \eta_{lin} = & \sum_{j=1}^{M_1} a_{1j}(T_{1j}, T_{2j})d_{1j} + a_{2j}(T_{1j}, T_{2j})d_{2j} + b_j(T_{1j}, T_{2j}) + \\ & + \sum_{j=M_1+1}^{M_1+M_2} a_{1j}(T_{1j}, T_{2j})d_{1j} + a_{2j}(T_{1j}, T_{2j})d_{2j} + b_j(T_{1j}, T_{2j}) \\ & + \sum_{j=M_1+M_2+1}^M a_{1j}(T_{1j}, T_{2j})d_{1j} + a_{2j}(T_{1j}, T_{2j})d_{2j} + b_j(T_{1j}, T_{2j}) + \sum_{i=1}^2 (1 - .985^{100})D_i. \end{aligned} \quad (8)$$

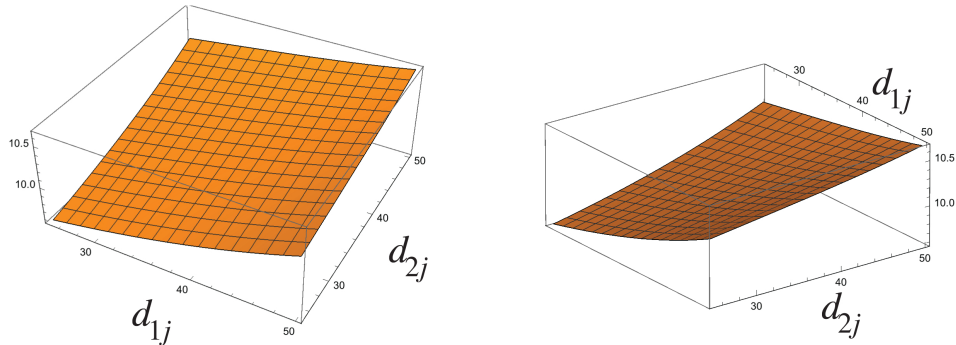


Figure 8: The cost components of the individual *Role 3* facilities are in general nonconvex but nearly linear in the parameter ranges of interest. Two views of a typical facility cost component are plotted for the parameter ranges $25 \leq d_{ij} \leq 50$.

Because the applicable constraints (6) are also linear, this relaxed version may be solved for each set of flow parameters T_{ij} using readily available software such as *scipy.optimize.linprog*, [7]. By cycling through

T_{ij} -values that are projected to be of interest, selections of MTF types together with their placements may be determined. At the same time, these approximate linear models may be profitably used in conjunction with the FIncke and Posht relaxation of the next subsection.

Results: Linear components of the cost function have been found for the cases that were discussed in Section 3. The sensitivities to changes in the patient flow parameters T_{ij} are under investigation.

5.2.2 Fincke and Posht concepts for relaxation of nonlinear mixed integer

One of our approaches to relaxation involves mathematical techniques inspired by the work of Fincke and Pohst [6]. We illustrate the technique using the example given above. The generalization to other scenarios is obvious. In our case, the underlying optimization variables are real values d_{ij} (as the e_j 's are functions of d_{1j}, d_{2j}), and integer values D_j, T_{ij} . In general not all d_{ij} etc. are independent (This depends on the geometry). In general, it follows from the implicit function theorem that there exists a set of free variables X_i, Y_i such that every variable d_{ij}, D_j and T_{ij} is expressible in terms of $X_l, 1 \leq l \leq p$ and $Y_m, 1 \leq m \leq q$ with X_l real valued and Y_m discrete valued. This means that the objective of design is to find values of X_l and Y_m that optimize an objective function $O(X_1, \dots, X_p, Y_1, \dots, Y_q)$ subject to various constraints of X_l and Y_m . In our example, X_l are the d_{ij} and Y_m are T_i and D_j .

We will first solve this optimization problem subject by assuming that both X_l and Y_m are real valued. This turns our nonlinear mixed integer programming problem into a real valued nonlinear programming problem. For such problems, there exists numerical solvers (In some cases, these may output sub-optimal solutions). Let the output of the numerical solver be $x_1, \dots, x_p$ and $y_1, \dots, y_q$. Clearly, the output of numerical solver may not be a solution to the problem at hand, as the value y_m may not be integers. Consider the integer vector $([y_1], [y_2], \dots, [y_q])$ where $[A]$ denotes the nearest integer to A . Consider the interval

$$I_m = [y_m - j_m, y_m + j_m], m = 1, 2, \dots, q,$$

where j_m is a pre-specified positive integer. Then the following algorithm produces a near-optimal solution to the mixed integer optimization problem of interest.

- Algorithm

- For each integer value $(z_1, z_2, \dots, z_q)$ in $I_1 \times I_2 \times \dots \times I_q$ do
 - * Let $(Y_1, Y_2, \dots, Y_q) = (z_1, z_2, \dots, z_q)$
 - * Replace these values for $(Y_1, Y_2, \dots, Y_q)$ in both the constraints and the objective function $O(X_1, \dots, X_p, Y_1, \dots, Y_q)$
 - * Solve the new optimization problem (non-integer valued nonlinear program) for $X_1, \dots, X_p$. Let the solution be $(u_1, u_2, \dots, u_p)$.
 - * Compute the value

$$\tilde{O}(z_1, z_2, \dots, z_q) = O(u_1, u_2, \dots, u_p, z_1, z_2, \dots, z_q).$$

- If no solutions are found, then output that the mixed-integer nonlinear problem is infeasible.
- Find the value $(z_1, z_2, \dots, z_q)$ that optimizes the value $\tilde{O}(z_1, z_2, \dots, z_q)$ and output the corresponding values $(X_1, \dots, X_p) = (u_1, u_2, \dots, u_p)$ and $(Y_1, \dots, Y_q) = (z_1, z_2, \dots, z_q)$ as the optimizing values.

- END

Remarks: Clearly the choice of values $j_1, j_2, \dots, j_q$ give various trade-offs between the complexity of search and optimality of the solution produced by the above algorithm.

5.3 Results

We have coded the Fincke and Posht algorithm in Matlab and the software has successfully returned solutions that agree with the provably correct results of Section 5.1. The code is several orders of magnitude faster than exhaustive search. Development is continuing with the aim of finding approaches to optimal algorithm seeding with feasible initial conditions.

5.4 Dictionary Building

Although for presentation purposes we have assumed that the problem and its constraints to be static, in general we do not expect this to be the case. In fact, we expect that in most tactical scenarios, these values are given by outcomes of some random variables.

Clearly, even in the reduced complexity algorithms outlined above may be too complex to solve if the constraints are dynamically changing. This means that will need adaptation techniques if the changes in conditions are small from time to time, or simplified methods of recalculation of the optimizing values in more general cases. This motivates the use of *dictionaries* proposed below.

In our setting a dictionary $\mathcal{D}$ will be created that lists the constraints and solutions for various cases of interest. This dictionary may be created off-line, or it may consist of some known scenarios. In one of **the simplest cases**, the dictionary may only have one element consisting of the constraints of the last time update (24 hours earlier stylized problem of Section 2) and the corresponding computed solution.

The dictionary aided optimization algorithm is as follows.

- Dictionary Aided Optimization Algorithm
 - Find the "Closest" scenario in the dictionary $\mathcal{D}$ to the present scenario (The closeness metric must be specified).
 - Let the solution of the corresponding dictionary element be $(X_1, \dots, X_p) = (u_1, u_2, \dots, u_p)$ and $(Y_1, \dots, Y_q) = (z_1, z_2, \dots, z_q)$.
 - Perform an search analogous to Algorithm I with y_m replaced by z_m (with possibly different pre-selected values of j_m)
 - Output the corresponding solution .
- END

Remark: In the event that $\mathcal{D}$ has only one element (e.g. with the only element consisting of the constraints of last time interval, and the corresponding computed solution), then the search build on the intuition that the underlying solutions may change slowly with time. If there are more dictionary elements, the algorithm learns more from the "past experience" and use this learned knowledge to reduce the search complexity.

5.5 Extensions and future work

The stylized problem of Section 3 presents a caricature of an actual theater of operations. Decision support tools that will be of value in the field will require relaxation of a number of the assumptions in the toy problem — the most notable of which is that the dynamics of the interacting system components are stationary. Among the goals for extending the approaches discussed in the preceding sections, there is the need to validate the computational methods for conducting rapid scenario planning in which the numbers of casualty streams may change and the daily flows on each stream may vary widely. Software support for such scenario planning will need to provide worst case bounds on facility capacities in order to identify failure modes. Work is also needed to be able to provide robustness guarantees for the solutions that are found.

Other possible extensions include considering a finer granularity of MTF types — with certain facilities having specializations such as for treating burns and hazardous chemical exposures. Further development of decision support systems should take a variety of potentially disruptive externalities into account. These include:

1. Compromised operation of a Role 3 MTF due to shortage of medicine, water, or fuel,
2. Changes in capability of moving patients due to unavailability of MEDEVAC vehicles,
3. Consideration of constraints on moving patients related to capabilities of vehicle types — e.g. helicopters vs. land vehicles,
4. Disruption of routes of transit due to weather, military operations, etc.

Some indication of the complexity of extending the analysis of this report to broader and more realistic settings is given in the following subsection, where facility saturation and the effects of having treatment regimens of varying duration in the MTFs are considered. Clearly, significant investment in further research is needed to realize the objectives of a system-of-systems decision support system for this class of problems.

5.5.1 The complete nonlinear model

The baseline optimization problem in Section 5.1 is based on the assumption that the length of stay of each patient in the Role 3 facilities is always 24 hours. Now let's consider the more general case that allows the patients to stay for more than 24 hours if such a decision is beneficial.

In order to simplify the calculation, we assume that every day the j -th Role 3 facility only selects from those patients that arrived 24 hours ago for transportation which means it operates on a last in, first out (LIFO) protocol. This means that those patients that can't be transferred to the air base in 24 hours will stay in the MTF forever until they return to duty or die. (This assumption is made without loss of generality because the computation of average DOW rate in this case is simpler yet equivalent to other, possibly fairer, protocols.) By using this idea, we can classify the patients arriving at the j -th facility everyday into two subgroups with the first subgroup having a 24 hour length of stay and the second subgroup remaining until they enter one of the two absorbing states DOW or RTD. Although this arrangement is unfair to patients assigned to the second group, for the purpose of analyzing capacity, it provides good insight. It can be easily shown that this classification dichotomy has no influence on the *average* DOW rate.

Proposition 2. *In the steady state of the systems, the average DOW rate that is obtained on a last in, first out (LIFO) basis is equal to the DOW rate that is obtained on a first in, first out (FIFO) basis.*

Proof. It is easy to prove that the average DOW rate of the system in the steady state is completely determined by the values of T_{ij} , S_j , and D_i . The above classification clearly can't change the values of T_{ij} and D_i . Also it is easy to prove that a FIFO basis and a LIFO basis have same S_j since assigning the priority to the first subgroup won't change the number of patients sent from the j -th Role 3 facility to the Air Base. $\square$

It is easy to see that the DOW rate of a patient that was sent from the i -th Role 2 facility to the j -th Role 3 and then fell into the first subgroup is equal to the the DOW rate of the first case in the baseline model of Section 3. And the DOW rate of a patient that was sent from the i -th Role 2 facility to the j -th Role 3 and then fell into the second subgroup is thus given by

$$1 - .985^{d_{ij}} \times \frac{RTD \text{ rate in MTF}}{DOW \text{ rate in MTF} + RTD \text{ rate in MTF}}.$$

To be more specific, the DOW rate of a patient in the second subgroup is

$$\begin{aligned} 1 - .985^{d_{ij}} \times \frac{10\%}{10\%+10\%} &= 1 - 0.5 \times .985^{d_{ij}} \quad \text{for 100 bed unit,} \\ 1 - .985^{d_{ij}} \times \frac{10\%}{15\%+10\%} &= 1 - 0.4 \times .985^{d_{ij}} \quad \text{for 30 bed unit,} \\ 1 - .985^{d_{ij}} \times \frac{5\%}{25\%+5\%} &= 1 - \frac{1}{6} \times .985^{d_{ij}} \quad \text{for 10 bed unit.} \end{aligned}$$

Then the average DOW rate of all patients in the general case of MTF stay possibly longer than 24 hours would be the weighed average of the DOW rates of the two subgroups. (As always the patients in 10-bed unit can't be classified into the second subgroup.)

Thus the DOW rate of the patients that are sent from the i -th Role 2s facility to the j -th Role 3 facility in the second case can be given by

$$\begin{aligned} DOW_{ij} &= \frac{T_{ij,1}}{T_{ij}} \times DOW \text{ rate of the first subgroup} + \\ &\quad \frac{T_{ij,2}}{T_{ij}} \times DOW \text{ rate of the second subgroup of the general case} \\ &= \frac{T_{ij,1}}{T_{ij}} \times DOW \text{ rate of the baseline case} + \\ &\quad \frac{T_{ij,2}}{T_{ij}} \times DOW \text{ rate of the second subgroup of the general case,} \end{aligned} \tag{9}$$

where $.985^{d_{ij}} \times T_{ij,1}$ is the number of patients that were sent from the i -th Role 2 facility to the j -th Role 3 and then fell into the first subgroup, and $.985^{d_{ij}} \times T_{ij,2}$ is the number of patients that were sent from the

i -th Role 2 facility to the j -th Role 3 and then fell into the second subgroup. These variables satisfy the following constraints

$$\begin{aligned}
\sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} &\leq 25 \quad \text{for 100 bed unit,} \\
\sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} &\leq 13.3 \quad \text{for 30 bed unit,} \\
\sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} &\leq 10 \quad \text{for 10 bed unit.} \\
\sum_{i=1}^2 T_{ij,2} \times .985^{d_{ij}} &\leq 20 - 0.2 \times \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \quad \text{for 100 bed unit,} \\
\sum_{i=1}^2 T_{ij,2} \times .985^{d_{ij}} &\leq 7.5 - 0.25 \times \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \quad \text{for 30 bed unit,} \\
\sum_{i=1}^2 T_{ij,2} \times .985^{d_{ij}} &\leq 3 - 0.3 \times \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \quad \text{for 10 bed unit.}
\end{aligned} \tag{10}$$

The general model DOW rate is thus given by

$$\begin{aligned}
DOW_{ij} &= \frac{T_{ij,1}(1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 80\% \times .995^{e_j})}{T_{ij}} + \\
&\quad \frac{T_{ij,2}(1 - 0.5 \times .985^{d_{ij}})}{T_{ij}} \quad \text{for 100 bed unit,}
\end{aligned} \tag{11}$$

$$\begin{aligned}
DOW_{ij} &= \frac{T_{ij,1}(1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 75\% \times .995^{e_j})}{T_{ij}} + \\
&\quad \frac{T_{ij,2}(1 - 0.4 \times .985^{d_{ij}})}{T_{ij}} \quad \text{for 30 bed unit,}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
DOW_{ij} &= \frac{T_{ij,1}(1 - .985^{d_{ij}} \times 5\% - .985^{d_{ij}} \times 70\% \times .995^{e_j})}{T_{ij}} + \\
&\quad \frac{T_{ij,2}(1 - \frac{1}{6} \times .985^{d_{ij}})}{T_{ij}} \quad \text{for 10 bed unit.}
\end{aligned} \tag{13}$$

As in Section 3, the Role 3 MTF's are labeled so that $1 \dots M_1$ are the 100-bed facilities, $M_1 + 1, \dots, M_1 + M_2$ are the 30-bed facilities, and $M_1 + M_2 + 1, \dots, M_1 + M_2 + M_3 = M$ are the 10-bed facilities. The cost function of the total expected casualties entering the DOW state is thus given as

$$\begin{aligned}
\eta &= \sum_{j=1}^{M_1} \sum_{i=1}^2 [T_{ij,1}(1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 80\% \times .995^{e_j}) + T_{ij,2}(1 - 0.5 \times .985^{d_{ij}})] \\
&\quad + \sum_{j=M_1+1}^{M_1+M_2} \sum_{i=1}^2 [T_{ij,1}(1 - .985^{d_{ij}} \times 10\% - .985^{d_{ij}} \times 75\% \times .995^{e_j}) + T_{ij,2}(1 - 0.4 \times .985^{d_{ij}})] \\
&\quad + \sum_{j=M_1+M_2+1}^M \sum_{i=1}^2 [T_{ij,1}(1 - .985^{d_{ij}} \times 5\% - .985^{d_{ij}} \times 70\% \times .995^{e_j}) + T_{ij,2}(1 - \frac{1}{6} \times .985^{d_{ij}})] \\
&\quad + \sum_{i=1}^2 (1 - .985^{100}) D_i,
\end{aligned} \tag{14}$$

subject to

$$\begin{aligned}
& \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \leq 25 \quad \forall j \in \{1, \dots, M_1\}, \\
& \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \leq 13.3 \quad \forall j \in \{M_1, \dots, M_1 + M_2\}. \\
& \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \leq 10 \quad \forall j \in \{M_1 + M_2, \dots, M_1 + M_2 + M_3\} \\
& \sum_{i=1}^2 T_{ij,2} \times .985^{d_{ij}} \leq 20 - 0.2 \times \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}}, \quad \forall j \in \{1, \dots, M_1\}, \\
& \sum_{i=1}^2 T_{ij,2} \times .985^{d_{ij}} \leq 7.5 - 0.25 \times \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \quad \forall j \in \{M_1, \dots, M_1 + M_2\}, \\
& \sum_{i=1}^2 T_{ij,2} \times .985^{d_{ij}} \leq 3 - 0.3 \times \sum_{i=1}^2 T_{ij,1} \times .985^{d_{ij}} \quad \forall j \in \{M_1 + M_2, \dots, M_1 + M_2 + M_3\}, \\
& D_i + \sum_{j=1}^M \sum_{k=1}^1 T_{ij,k} = 15 \quad \forall i, \\
& T_{ij,1} + T_{ij,2} = T_{ij} \quad \forall i, j, \\
& T_{ij,1}, T_{ij,2}, T_{ij}, D_i \geq 0 \quad \forall i, j, \\
& T_{ij}, D_i \text{ are integers } \forall i, j, \text{ and} \\
& \text{All geometry constraints.}
\end{aligned} \tag{15}$$

Remark 1. In the toy problem, as we can see from the discussion about the investment frontier, there is no loss of generality in restricting to assuming all patients leave Role 3 facilities at the end of each 24 hour period as all routings on the investment frontier involves no saturated paths.

6 Enumerative techniques for system-of-systems decision support

As demonstrated by the detailed analysis of Section 5.1, counting arguments and enumeration of feasible cases can be a useful means of narrowing the range of solutions to complex resource allocation problems. In the case of the problem considered in this report, even if the budget for deployment of MTFs is increased, the numbers of possible combinations of the three Role 3 MTF types are fairly easy to describe. For investment thresholds of \$2M (the case of Section 4), \$4M, \$8M, and \$16M, for instance, the possible combinations of Role 3 facilities at exactly these costs are 7, 18, 57, and 198, respectively. Of course, as in Section 5.1.1, the actual mixtures of facility types that are worth considering are expected to be much fewer in number.

Future work on enumerative approaches to composition of system components may be informed by the team's prior work on semigroups of partial functions. By way of a brief summary, following [1], we recall Green's equivalence relations on semigroups as follows.

Definition 2. Let $(S, \circ)$ be a semigroup. The following four equivalence relations $\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{H}$ are called *Green's relations*: For $f, g \in S$,

1. $f\mathcal{R}g \Leftrightarrow f \circ S = g \circ S$;
2. $f\mathcal{L}g \Leftrightarrow S \circ f = S \circ g$;
3. $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$; and
4. $\mathcal{H} = \mathcal{R} \wedge \mathcal{L}$.

These definitions can be specialized in the case that $(S, \circ)$ is the semigroup of partial functions on a set X . (Recall that f is a partial function on X if f maps a subset called $Dom f \subseteq X$ onto another subset $Ran f \subseteq X$. The set of all partial functions on a set is a semigroup under the operation of normal function composition under the convention that in the case that $Ran f \cap Dom g = \emptyset$, $g \circ f$ is just the *empty function*.)

Proposition 3. Let $(S, \circ)$ be the semigroup of partial functions on a finite set X . The equivalence relations of Definition 2 can be rendered as

1. $f\mathcal{R}g \Leftrightarrow Ran f = Ran g$;
2. $f\mathcal{L}g \Leftrightarrow \begin{cases} Dom f = Dom g \text{ and} \\ f(a) = f(b) \Leftrightarrow g(a) = g(b) \end{cases}$;
3. $f\mathcal{D}g \Leftrightarrow$ the cardinality of $Ran f$ equals the cardinality of $Ran g$; and
4. $f\mathcal{H}g \Leftrightarrow f\mathcal{R}g$ and $f\mathcal{L}g$.

From this proposition, it follows immediately that if $(S_n, \circ)$ denotes the semigroup of partial functions on a set with n elements that there are $n + 1$ $\mathcal{D}$ classes in S_n . Counting the numbers of $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{H}$ classes is a bit more complex. We recall the following:

Proposition 4. ([1]) Let $\mathcal{R}_{D_k}, \mathcal{L}_{D_k}, \mathcal{H}_{D_k}$ denote, respectively, the numbers of $\mathcal{R}, \mathcal{L}$, and $\mathcal{H}$ subclasses of the k -th $\mathcal{D}$ class. These are given explicitly by

1. $\mathcal{R}_{D_k} = \binom{n}{k}$;
2. $\mathcal{L}_{D_k} = S(k + 1, n + 1)$, where $S(m, n)$ is the Stirling number of the second kind, denoting the number of ways of partitioning n elements into m cells;
3. $\mathcal{H}_{D_k} = \binom{n}{k} \cdot S(k + 1, n + 1)$.

	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$
$\{\{x, y\}, z\}$	$f_{11} = \{x, y\}$ $g_{11} = \{y, x\}$	$f_{12} = \{x, z\}$ $g_{12} = \{z, x\}$	$f_{13} = \{y, z\}$ $g_{13} = \{z, y\}$
$\{\{x, z\}, y\}$	$f_{21} = \{x, y\}$ $g_{21} = \{y, x\}$	$f_{22} = \{x, z\}$ $g_{22} = \{z, x\}$	$f_{23} = \{y, z\}$ $g_{23} = \{z, y\}$
$\{x, \{y, z\}\}$	$f_{31} = \{x, y\}$ $g_{31} = \{y, x\}$	$f_{32} = \{x, z\}$ $g_{32} = \{z, x\}$	$f_{33} = \{y, z\}$ $g_{33} = \{z, y\}$
$\{x, y\}$	$f_{41} = \{x, y\}$ $g_{41} = \{y, x\}$	$f_{42} = \{x, z\}$ $g_{42} = \{z, x\}$	$f_{43} = \{y, z\}$ $g_{43} = \{z, y\}$
$\{x, z\}$	$f_{51} = \{x, y\}$ $g_{51} = \{y, x\}$	$f_{52} = \{x, z\}$ $g_{52} = \{z, x\}$	$f_{53} = \{y, z\}$ $g_{53} = \{z, y\}$
$\{y, z\}$	$f_{61} = \{x, y\}$ $g_{61} = \{y, x\}$	$f_{62} = \{x, z\}$ $g_{62} = \{z, x\}$	$f_{63} = \{y, z\}$ $g_{63} = \{z, y\}$

Figure 9: The $\mathcal{D}$ -class D_2 in S_3 . The rectangles bounded by horizontal lines define the $\mathcal{L}$ -classes, the rectangles bounded by vertical lines define the $\mathcal{R}$ -classes, and the small rectangles defined by the intersections define the $\mathcal{H}$ -classes.

6.1 An enumerative approach to “compositionality”

To study “compositionality” at a finer level of granularity, we consider $\mathcal{L}$ classes associated with each subdomain and each subdomain size— $m, m+1, \dots, n$. The numbers of subdomains of size m are

$$\binom{n}{m}, \binom{n}{m+1}, \dots, 1.$$

Consider the functions from the “full domain”

$$f : X_n \rightarrow Y_j, \quad j = 1, \dots, \binom{n}{m},$$

where Y_j is j -th m -element range. For each Y_j and each partition of X_n into m cells, there are $m!$ functions mapping X_n to Y_j .

Function composition can be used as an abstraction for modeling assignment of resources to address situational needs. The construct of $\mathcal{L}$ classes is of interest in connection with models in which there are classes of situational needs that are mitigated by common sets of resources. The concept of *right inverses* in S_n can be employed to map *resources* in a 1-1 fashion back to situational needs. Right inverses assign representative elements of X_n to each cell in the domain partition associated with the $\mathcal{L}$ -class.

6.2 Complexity and generality of semigroups of partial functions on finite sets

The number of partial functions on a set with n elements is $(n+1)^n$. There is no significant additional generality in studying sets of functions mapping one finite set to another, $f : X_n \rightarrow Y_m$, say, because such function will be members of the semigroup of partial functions on the $n+m$ -element union $X_n \cup Y_m$.

6.3 Elementary aspects of function composition

The *data-processing theorem* of information theory states that if X_n represents data from an experiment, and there is a function f mapping X_n to a (possibly proper) subset of X_n , there can never be more information about the data in $f(X_n)$ than there is in X_n itself. The essential idea is captured by the following:

Theorem 1. The function that maximizes information preservation, $H(f(\tilde{X}))$ for a finite valued random variable X minimizes the conditional entropy $H(\tilde{X}|f(\tilde{X}))$.

PROOF: Using notation that can be found in any standard text on information theory, we have

$$\begin{aligned} I(\tilde{X}; f(\tilde{X})) &= H(\tilde{X}) - H(\tilde{X}|f(\tilde{X})) \\ &= H(f(\tilde{X})). \end{aligned}$$

□

Despite the unavoidable compression of information that is frequently encountered in mapping one set to another, the approaches to applications that will be developed show that the most relevant information is frequently preserved. To set the stage for applications, a few basic results on the composition of partial function will be presented.

Proposition 5. For any $f, g \in S_n$, it is generally not the case that $f \circ g \mathcal{D} g \circ f$.

The proposition is most easily understood in terms of how elements in a domain partition associated with an $\mathcal{L}$ -class are mapped. Looking at the functions in the $\mathcal{D}$ -class D_2 in Fig. 9, we see that f_{11} and g_{11} map $X_3 = \{x, y, z\}$ to $\{x, y\}$. The common range of these two functions is mapped to one of the cells in the domain partition that is defined by their $\mathcal{L}$ -class—namely $\{x, y\}$. For any function h in the same $\mathcal{L}$ -class, we find that $h \circ f_{11}(\zeta)$ is constant for $\zeta \in X_3$. Similarly for g_{11} . Hence, for all h such that $h\mathcal{L}f_{11}$, we have $h \circ f_{11} \in D_1$. Considering similar possibilities, it is easy to find pairs of functions f and g in the table of Fig. 9 such that $f \circ g \in D_2$ but $g \circ f \in D_1$.

This very simple observation implies that the order in which function composition is carried out matters in terms of data compression. It can also have consequences in terms of algorithmic complexity of optimal composing functions.

6.4 Right and left inverses, partitions, partial identities, and connections with axiomatic category theory

The notions of composition of partial functions on a given set are similar in certain respects to aspects of axiomatic category theory as developed in early work of Eilenberg and MacLane, [2]. In particular, for any partial function f on a finite set X , we define *partial identities*

$$Rf = \{(x, x) : x \in \text{Dom } f\}, \text{ and} \\ Lf = \{(f(x), f(x)) : x \in \text{Dom } f\}.$$

Interpreted as functions, these clearly have the property that $Lf \circ f = f$ and $f \circ Rf = f$. These are clearly similar to the left and right *identity objects* defined in [2]. In [2], compositions of functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are defined *only* when $\text{Dom } g = \text{Ran } f$. By relaxing this requirement to $\text{Ran } f \subseteq \text{Dom } g$, we are able to prove the existence of right inverses.

Definition 3. Let S be a semigroup of partial functions. Given $f \in S$, f_ℓ is said to be a *left inverse* of f if $f_\ell \circ f(x) = x$ for all $x \in \text{Dom } f$. f_r is said to be a *right inverse* of f if $f \circ f_r(x) = x$ for all $x \in \text{Ran } f$.

Because functions need not be 1-1 onto their ranges, left inverses need not exist. Right inverses always exist, but they are generally not unique. This is the content of the following:

Proposition 6. Let S be a finite semigroup of partial functions, and let $f \in S$. There exists at least one function $f_r \in S$ such that $f \circ f_r(y) = y$ for all $y \in \text{Ran } f$.

PROOF: For each $y \in \text{Ran } f$ let $U_y = \{x : f(x) = y\}$. Clearly $\cup_{y \in \text{Ran } f} U_y = \text{Dom } f$. We define f_r to be a function that assigns to each $y \in \text{Ran } f$ a single element in U_y . Because there are finitely many elements in $\text{Ran } f$ and because each U_y is a finite set, this construction can be carried out, and it yields the desired right inverse. $\square$

6.5 Partial function representations of resource allocations in SAROPS

Problems of optimal resource allocation in search and rescue optimal planning systems (SAROPS) are well represented by the function composition abstraction described above. Search and rescue systems are viewed as *small* categories whose objects are sets made up of subsets whose elements are system resources, mission requirements, operational constraints, behaviors, operating scenarios, and possibly other system elements. Each of the subsets is referred to as a block, and the morphisms between blocks are finite functions mapping mapping one block to another. Within this abstraction, *composition* is just the usual composition of category theory—or more concretely, the usual notion of composition of functions or mapping on finite sets. In addition to blocks and mappings, a very important part of this abstraction is *composition protocols* that specify the order in which functions are evaluated.

Equivalence relations are frequently represented by simplices. D. Spivak, for instance, has introduced the notion of an *olog* or *ontology log*, [3]. Also introducing the notion of *aspects* functional relationships among objects may be established. Indeed, Spivak views *aspects* informally as ways of viewing objects and more formally as functions in the sense in which we have been using the term in the above.

In another direction, we wish to study the use finite sets of partial functions to allocate resources so as to meet situational requirements while satisfying operating constraints. As a simple example, suppose the are four classes of resources for *search and rescue*:

- R1** Land vehicle with large personnel capacity.
- R2** Land vehicles with large cargo capacity,
- R3** Air vehicles with capability of landing in wilderness terrain,
- R4** High speed air vehicles with reconnaissance capability.

In addition, we assume that there are four classes of mission requirements:

- S1** Multiple stranded disaster victims in a single location.
- S2** Stranded disaster victims in multiple locations,
- S3** Stranded victims in wilderness locations,
- S4** Stranded victims in unknown locations.

An important distinction between classes **S1** and **S2**— is that **S1** envisions requiring a single vehicle picking up individuals from a single location and returning to base, whereas **S2** requires a single vehicle to visit multiple locations to complete its mission. This notional example set thus illustrates the possibility of multiple levels of granularity in requirement classes. Continuing in this fashion we also specify classes of operating constraints, that affect the optimal selection of resource allocation functions as discussed below. To keep this stylized example maximally simple, we consider two classes of constraints:

- C1** Weather related constraints that affect the allocation of resources to each mission type as well as mission execution,
- C2** Roadway and flight corridor disruptions.

6.6 SAROPS scenario planning

Situational requirements are specified by lists of requirements —e.g. $(s_{11}, s_{12}, \dots, s_{1m_1}, s_{21}, \dots, s_{2m_2}, s_{31}, s_{41})$ where, for each of the four classes of search and rescue missions designated by the first subscript i , $s_{i,j}$ is a vector specifying essential details of the mission requirements with some components giving coordinates of the rescue site and other coordinates giving other mission parameters, such as the number of people needing to be rescued from the site(s). To keep the discussion within the chosen mathematical domain, it is necessary to employ a quantizer for continuous data such as geographic coordinates. The *situation* represented in this example has m_1 occurrences of disaster victims in locations that each require deployment of one or more vehicles with adequate capacity that will return those that are rescued to the base station. There are also m_2 groups victims in scattered locations for which a single vehicle can sequentially traverse the locations to effect the rescue of all victims in the groups. There is also one individual needing rescue from a wilderness location and one individual whose location is not known. The numbers of people in the m_1 groups associated with class **S1** requirements are elements of the vectors $s_{11}, s_{12}, \dots, s_{1m_1}$; the numbers of people in groups associated with **S2** requirements are $s_{21}, \dots, s_{2m_2}$; and we shall assume that $m_3 = 1, m_4 = 1$ (there is only single group in each rescue class **S3, S4**), and the vector components that specify the number of people needing rescue in each of the type 3 and 4 missions is 1 for both s_{31} and s_{41} (there is only one person requiring rescue from difficult-to-reach wilderness, and only one person whose location is not known.).

Weather related constraints are given in a living dictionary (wiki) of historical weather conditions. To utilize this dictionary, there is also a continuously updated set of *availability/usability* functions that map weather conditions to the resource set. These dictionaries are constructed over time, as the system gains experience. The road and flight corridor network constraints can be more model based and serve to model the possible inability of some of our vehicle resources may be unable to operate in certain edges in our road or flight corridor networks.

We also define resource sets associated with each resource class. Specifically, we label each resource in each class, and let $R_k = \{r_{k1}, \dots, r_{kn_k}\}$ denote the set of the n_k resources in class **Rk** ($\mathbf{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$). The entire resource set is $\cup_{k=1}^4 R_k$. Pursuing the example in the case that $m_1 = m_2 = 1$, but $s_{11} > 1, s_{12} > 1$, we consider deployment of resources to meet requirements. Consider a resource set

$$\{r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{31}, r_{32}, r_{41}, r_{42}\},$$

where the meaning of the notation is that there are 3 personnel carrying vehicles, (labeled $r_{1,j}, j = 1, 2, 3$) two cargo carrying vehicles, two wilderness capable air vehicles, and two reconnaissance vehicles. At this point, we encounter a constraint on allocation of resources—namely that the number of people to be rescued may exceed the capacity of any of the resource vehicles, in which case it is required to either deploy multiple vehicles or else carry out sequential trips between the operations base and the rescue site. The magnitudes of the numbers s_{ij} will determine the numbers of vehicle trips that will be needed to complete the rescue mission. The most desirable circumstance is that in which the available resources are adequate to complete

the mission in a single deployment form base. That is to say, the preferred case is when no ground or air vehicle needs to make multiple trips. An example of this situation is the case of resource and requirement sets

$$R = \{r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{31}, r_{32}, r_{41}, r_{42}\}, \quad S = \{s_{11}, s_{21}, s_{31}, s_{41}\},$$

under the following assumptions:

1. The number s_{11} is too large for the entire group to fit into a single personnel carrier, but it is small enough to fit into two such vehicles;
2. s_{21} is also too large to fit into a single personnel carrier, but small enough to fit into two; and
3. Cargo carriers may be used to transport people if necessary.

It is further assumed, for the sake of simplicity that $s_{31} = s_{41} = 1$, and under these assumptions, a coordinate SAR mission is feasible because by using any mixture of personnel and cargo carriers (four vehicles in total), all rescue and reconnaissance requirements can be met. Recalling that the s_{ij} represent both the numbers of individuals requiring each type of rescue and labels for the type of rescue, a typical allocation function is

$$\begin{aligned} f(r_{11}) &= s_{11} & f(r_{21}) &= s_{21} \\ f(r_{12}) &= s_{11} & f(r_{31}) &= s_{31} = 1 \\ f(r_{13}) &= s_{21} & f(r_{41}) &= s_{41} = 1, \end{aligned}$$

meaning that vehicles r_{11} and r_{12} are both employed to meet requirement s_{11} and vehicles r_{13} and r_{21} are both assigned to requirement s_{21} . Obviously, this is only one of a number of ways to assign resources to this particular mission requirement. In the language of Section 1 above, all resource assignment functions that meet the requirements are in the $\mathcal{R}$ -class associated with the range value $\{s_{11}, s_{21}, 1, 1\}$. Because of the assumed interchangeability of rescue vehicle assets, it is easy to see that there are 20 subsets of the resource set R that can satisfy the mission requirements. For each such subset, there are three ways to group the resources to be assigned, and hence we have 60 $\mathcal{H}$ classes of allocation functions. Within each $\mathcal{H}$ class, there are two ways to assign vehicle sets to rescuing groups s_{11} and s_{21} . There are thus a total of 120 possible ways to assign rescue assets in this example problem. $\square$

Remark 2. The example serves to illustrate intrinsic complexity in managing deployment of assets in SAR applications, and such complexity will undoubtedly be a part of other application (e.g. urban infrastructure system emergencies) as well. A few observations are in order:

- There is a need for additional notions of equivalence of asset assignment functions that account for features specific to the application. In the example at hand, it is more useful to consider ranges of group sizes s_{11} and s_{21} , because as long as the rescue vehicles can be assigned in the same way, difference in sizes of the groups being transported are unimportant. More generally, care must be taken in defining abstractions so as to avoid combinatorial explosions to the greatest possible extent.
- On the flip side of the combinatorial explosion problem, research is needed to identify what parts of proposed models of functional composition lead to NP-hard problems and to what extent can *greedy* and other sub-optimal approaches suffice.

7 *Compositionality* and Service Protocols for Deployed Medical Treatment Facilities

A scenario in which it will be important to consider protocol alternatives for deployment of resources is depicted in the figure. Medevac transportation will be needed in this setting of deployed treatment facilities in which casualties arrive at field treatment facilities and must be transported to primary treatment facilities that are varying distances away. The medevac vehicles that do this have limited capacities, and once one of them picks up casualties, it will be unavailable for further service until it discharges those it is carrying at the destination primary treatment facility. The primary treatment facilities are a various distance from the

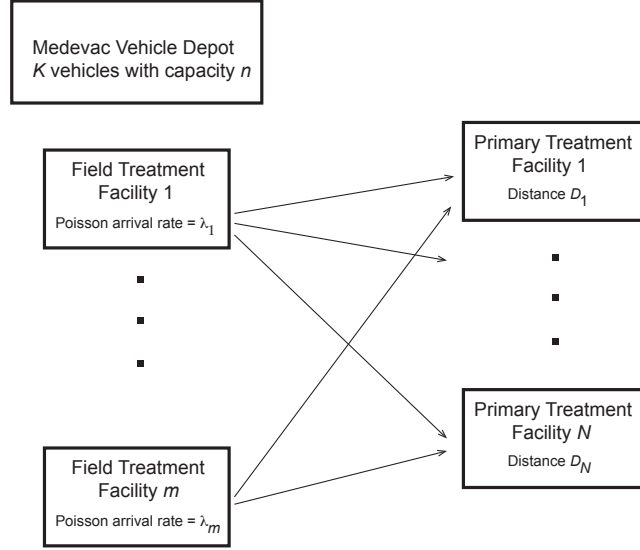


Figure 10: Medevac vehicles must be dispatched to transport casualties from field treatment facilities (FTF's) to primary treatment facilities. The FTFs serve as the earliest staging areas for casualties from the theater of operations. Arrivals at the i -th FTF are Poisson with rate λ_i .

field treatment facilities, and some of these distances are great enough that the travel time is concerning in terms of having the medevac vehicle effectively out of service for the duration of the trip.

We assume that casualties arrive at the field treatment facilities (FTFs) at Poisson rates λ_i , and these rates remain stationary for the time intervals being modeled. For a given facility, the expected number of casualties arriving in a time interval $[0, T]$ will be λT . There is urgency in transshipping arriving casualties to primary treatment facilities, and for this reason the model would require $n > \lambda T$ if there is only one medevac vehicle of capacity n and one FTF from which travel time to the nearest primary treatment facility is T .

7.1 Medevac vehicle routing protocols.

We consider three different routing protocols. We offer some analysis of the baseline case in which there is a single field treatment facility (FTF) being served by a single medevac vehicle. The additional complexity of multiple facilities and multiple vehicles is left to future work.

Deployment protocol 1: Continuous circulation of the medevac vehicle making the round trip from FTF to the primary treatment facility in T units of time.. Clearly we must have $\lambda T < n$. If $\lambda T \ll n$, then this is a wasteful use of the vehicle resource. On the other hand, if $\lambda T \sim n$, there will frequently be larger numbers of casualties than can be transported, and our modeling assumes that those left behind will be at grave risk.

Figure 11(a) indicates that if the round trip time for the vehicle is $< \frac{1}{4} \cdot \frac{n}{\lambda}$, then there is little likelihood that the number of casualties that arrive during the time that the vehicle is deployed on its round trip will exceed the vehicle capacity n . If the round trip time is $\sim n/\lambda$, however, there is a significant chance that the number of newly arrived casualties will exceed the vehicle's capacity. It is also apparent from the same figure that if the round trip time is a little longer (by, say, 20%), then there will be a greater than 50% chance of new casualties exceeding the vehicle's capacity. Figures 11(b), 11(c) show that small ((b)) to moderate ((c)) increases in casualty arrival rates can easily overwhelm the capacity of the medevac vehicle.

Deployment protocol 2: The vehicle leaves its depot every T_s units of time. This is essentially the same as Deployment Protocol 1, except that return-to-depot is part of the circuit. In the case of a single vehicle and single FTF, the analysis is the same. In case we wish to monitor capability-to-respond to emergency field situations, vehicle inventory may be an important variable in the case of multiple vehicles.

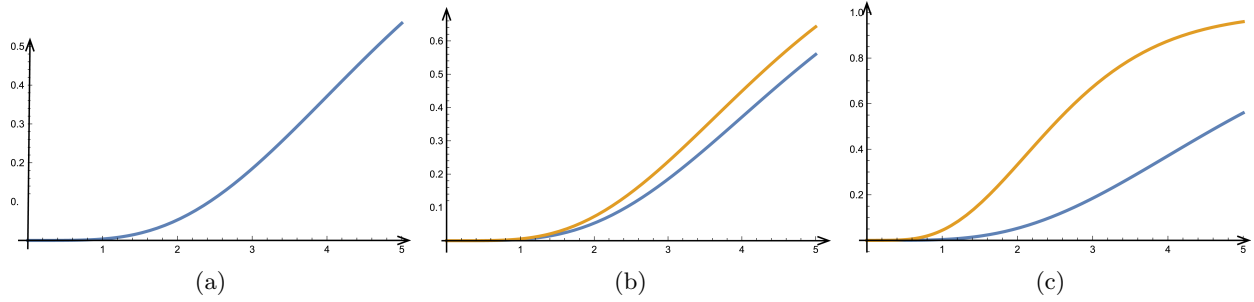


Figure 11: The probabilities that more than n casualties will arrive at an FTF during the time interval $T = n/\lambda$. (a) depicts $\lambda = 1$; (b) compares $\lambda = 1$ (blue) with $\lambda = 1.1$ (tan); and (c) depicts $\lambda = 1$ (blue) with $\lambda = 1.9$ (tan).

This situation will be left as future work.

Deployment protocol 3: The final protocol is termed *call-for-service*. Here, the deployment of the medevac vehicle is initiated by either an explicit call from the FTF requesting transport of the casualties that have arrived, or it could also be initiated by a probabilistic model of expected numbers of casualties. (I.e., after a time $T < n/\lambda$, the medevac vehicle goes to the FTF under the assumption that an average of λT casualties will need to be picked up.)

While a detailed analysis of these protocols will be of greatest interest in the case of multiple medevac vehicles, multiple FTFs, and different distances to primary treatment facilities (which may have different capacities), the baseline case of a single vehicle (with capacity n) and single FTF and PTF is illuminating. Consider the case in which $n \sim \lambda T$. In the case that the vehicle is busy for T units of time, a certain number of new casualties will arrive at the FTF. On average, the number will be λT , but in general, the probability that it will exceed this average is fairly significant. Figure 11 shows this for $n=4$ and $\lambda = 1$. The Figure 11(b) shows how much the probability of exceeding the capacity (n) changes if there is a small change in the casualty arrival rate, and Figure 11(c) shows a very dramatic change - with probability approaching 1 - when there is a significant change in the arrival rate. It is also of interest to note that any extension (e.g. due to weather or road conditions) of the deployment interval T can also have a dramatically adverse effect on the vehicle's ability to meet the need for service.

Open Problems: The simple analysis of the single medevac vehicle serving a single FTF is instructive in its demonstrating the sensitivity of operations to unforeseen delays and random fluctuations in casualty arrival rates. Research is needed to understand the possible compounding of effects that can occur if there are more facilities and more vehicles whose routes through the facility network must be planned. For these problems as well as for the closely related problems described in Section 5, there will be important tradeoffs between cost of operation and level of resources deployed on the one hand and the possibility of ensuring that all casualties get the highest level of service that can be provided. Excess capacity will inevitably be needed to keep mortality rates uniformly low, but of course this capacity comes at a cost. The exact amount of that cost and its dependence on system parameters can only be determined by further research that is focused on larger treatment facility networks.

8 Simulation of Flows through the Medical Treatment Network

8.1 Dynamic models of patient flow

In addition to optimizing the system in the steady state it is useful to formulate a dynamic model for patient flow through the system built from role 2, role 3, and AFB/CONUS. Dynamic modeling can verify the validity steady flow assumptions and lead to tools for understanding and optimizing transient behavior. Moreover, though this problem has natural sink and source dynamics, extensions to systems of systems involving transport will certainly have oscillatory dynamics.

The current concept of operations (CONOPS) for expeditionary medical care emphasizes quickly moving patients to a series of successively more sophisticated medical facilities that provide the patients with the

care necessary to ultimately treat their injuries or conditions. A primary to obstruction to planning is thinking in terms of beds and aeromedical units versus operational practice (i.e. when patients can be moved). Measuring patient flow is more directly aligned operational practice and represents a common unit of measurement through the system of systems spectrum. We do not include treatment compartments (as seen as in RAND TR 1003) in this discussion and model description, but the framework is certainly capable of doing so.

In addition to the notation that was set in Section 3, we define the following variables.

- P_j^n is the population in the j th labelled, type n MTF.
- Let $DOW_{2 \rightarrow 3}$ be the DOW rate per kilometer of transport from role 2 to role 3 facilities. In the problem statement, this is 1.5% per kilometer.
- Let DOW_3^n be the DOW rate for a role 3 MTF. These numbers are given in Figure 1.
- Let $DOW_{3 \rightarrow AFB}$ be the DOW rate per kilometer of transport from role 3 to AFB. This is 0.5% is the previous sections of the report.
- Let AFB^n be the number that can be sent to an AFB by a type n MTF.
- Let RTD_3^n be the RTD rate for a role 3 MTF.
- C_i is the casualty stream count from role 2 facility i
- α_{ij} is the proportion of casualties from role 2 facility i to role 3 facility j .
- $cap(n)$ is the capacity in an MTF of type n .
- Let t be time in days.

The equations for MTF population are only weakly coupled through their casualty stream distribution (i.e. α_{ij}). In this case we have the following

$$\begin{aligned}
 \frac{dP_j^n}{dt} = & \underbrace{\chi_{\{P_j^n < cap(n)\}} \sum_{i=1,2} C_i \alpha_{ij} (1 - DOW_{2 \rightarrow 3})^{d^{ij}}}_{\text{Input from Role 2}} \\
 & - \underbrace{P_j^n (DOW_3^n + RTD_3^n)}_{\text{Leave MTF via DOW, RTD}} \\
 & - \underbrace{\min\{AFB^n, P_j^n - P_j^n \cdot DOW_3^n - P_j^n \cdot RTD_3^n\}}_{\text{To AFB}}
 \end{aligned} \tag{16}$$

We can also define an auxiliary variable Ω to track total DOW count at time t

$$\begin{aligned}
 \frac{d\Omega}{dt} = & \sum_{i,j} C_i \alpha_{ij} \left(1 - (1 - DOW_{2 \rightarrow 3})^{d^{ij}} \right) \\
 & + \sum_j P_j^n DOW_3^n \\
 & + \sum_j \min\{AFB^n, P_j^n - P_j^n \cdot DOW_3^n - P_j^n \cdot RTD_3^n\} DOW_{3 \rightarrow AFB}.
 \end{aligned} \tag{17}$$

8.2 Dynamic behaviors of interest to planning CONOPS

For the following discussion we explore the design space of the MTF. We provide time series data from numerically integrating the *hybrid dynamics* present in (16). We remark that obtaining more accurate and accelerated numerical integration and more complicated switching logic will require technical effort— there is little off-the-shelf software for hybrid dynamics. For the parameter values prescribed by the problem statement in Section 2, we verify that a patient does establish a steady state. We follow this discussion up

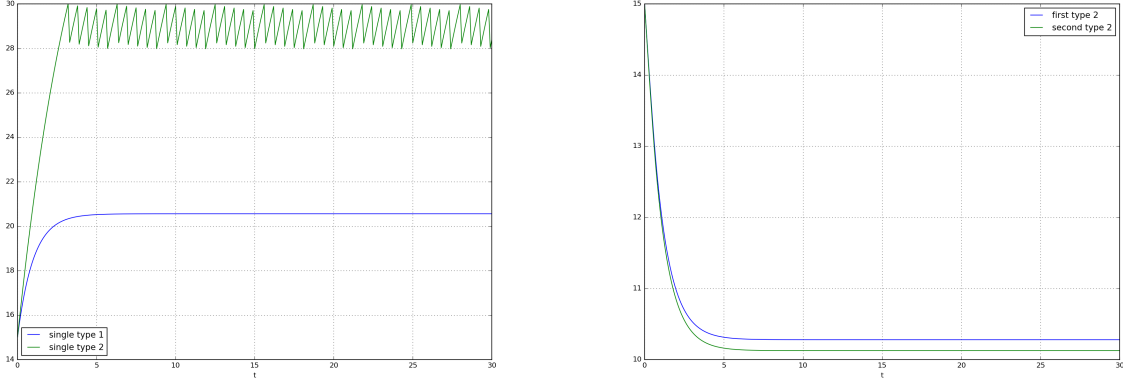


Figure 12: The left hand figure plots time dynamics for individual type 1 and type 2 MTF. The right hand figure gives the time dynamics for two type 2 MTF acting in tandem. Initial population in both was 15.

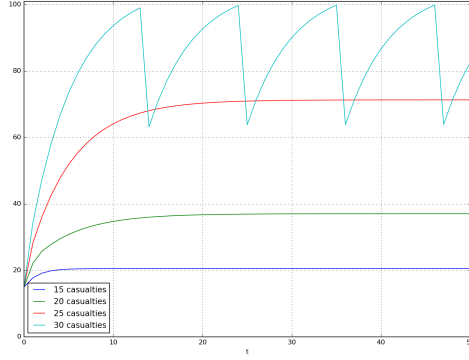


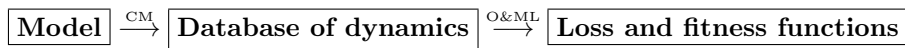
Figure 13: In this figure we have increased casualty rate to demonstrate appearance of oscillatory dynamics. The solution curves correspond to a single type 1 MTF

by showing other dynamic behavior in this design space if one allows casualty streams to grow. In particular, even in this simple scenario, we have oscillatory behavior when we approach MTF capacity.

Figure 12a shows time dynamics with either a single type 1 or single type 2 MTF. Observe that the patient population obtains a steady steady state which is near the 100 person capacity for a type 1 MTF whereas a single type 3 oscillates near capacity. The other types of MTF will have similar phenomena. Initial population in either MTF is initialized to be 15. Figure 12b shows time dynamics of two type 2 facilities acting in tandem. Figure 9 demonstrates the effects of increased casualty on behavior of a single type 1 MTF. These figures demonstrate that even in a simple problem there will be recurrent/oscillatory dynamics. Moreover, it is clear that a database of dynamics would allow one to know the shape regions in parameter space that have desirable or undesirable dynamics.

8.3 Pipeline to database of dynamics and decisions

Our proposed pipeline can be diagrammed as follows.



Here the **Model** must include parameters and constraints, examples of which are given above. In future iterations, it's noted that our pipeline requires not only model input and output, but control over parameters.

In what follows, we flesh out contents of the **Database of dynamics** and layer for calculating **Loss and fitness functions**. We also give some strategies for implementing $\xrightarrow{\text{CM}}$ and $\xrightarrow{\text{O\&ML}}$.

Constructing a database of dynamics. The work of Mischaikow and collaborators provides a road map for modern qualitative analysis dynamical system that employs computation in a rigorous way through the use of algebraic topology (where this can be discrete [10], continuous [11], or hybrid [14]). It has long been understood [13] that it is critical to understand qualitative behavior as parameters vary. However, classical methodology falls short when the number of parameters explodes, when dynamics of interest are only achieved by specific parameter choices, and when exact measurement of parameters is not possible or likely [17,18]. Course grained models of SoS involving human interaction must be phenomenological (i.e not from first principles) and therefore will eventually contain parameters for which measurement only makes sense as an interval of values. This complication and the existence of chaos have severe implications for computation and modeling: exact simulation for non-exact parameters may lead to dynamics that do not qualitatively match those of the observed system and in the presence of chaotic dynamics implies behavior of individual simulated orbits may incorrectly represent the true nature of orbits of the system. These obstacles led Mischaikow and others to explore coarser, but more robust descriptions of dynamics.

To implement $\xrightarrow{\text{CM}}$ our pipeline will construct a database of dynamics using one of the following depending on the nature of the problem and model.

1. Proceed as in [10,11,17] by discretizing phase and parameter space for the dynamic model at an a priori resolution that could be varied for precision or tractability. These discretizations are used to generate Morse graphs (summaries of global dynamics) and Conley–Morse graphs (a labeling by Conley index) as follows. The state space of the system is divided into regions at a chosen spatial scale, and the flow is discretized at a temporal scale. These scales are often given by the limits of measurement of the system, but can be chosen to be larger to reduce complexity. A directed graph Γ is built by taking a vertex for each region and a directed edge between two vertices when the flow from one region moves some of its points into the other. The directed graph Γ then induces the (also directed) Morse graph G by replacing all vertices in a strongly connected component (SCC) of Γ by a single vertex for G , and joining two vertices of G by a directed edge when Γ has at least one directed edge connecting vertices in the corresponding SCCs. The result is called a Morse graph because it is directed acyclic and describes a Morse flow between regions. Note that each vertex of G corresponds to a neighborhood of a separatrix of the system, with vertices at the "bottom" of G corresponding to stable attractors. Labeling each vertex with the Conley Index, i.e. the homology group of the neighborhood relative to the region where the flow leaves it (compare this idea to Morse theory, which inspired Conley), gives a combinatorial approximation/description of the dynamics called the Conley–Morse (CM) graph.

Note that different parameters can give rise to the same CM-graphs, so we obtain a decomposition of parameter space into regions which have similar dynamics. The keys of the database are the CM-graphs, and these store the regions of parameter space.

This method is the most general, but is often computationally expensive, we foresee the need to employ restrictions on parameter space, exploit the geometry and topology of the constraints and parameters, and carefully devise machine learning exploiting the structure of the database (see [16] and related).

2. Use switching structure to obtain a variation on the approach above [10,11,17] that applies to switching systems (see Cummins et al [14]). In the previous schema the structure of the database is contained in what the authors of [10] call the *continuation graph*. If the system can be given an explicit switching structure, a much more refined understanding of parameter space and the structure of the database can be made. This yields huge performance gains

We note that these approaches are not necessarily distinct, one may use switching systems to model dynamics we know to have transient behavior and use information to generate a *subset of interest* in discretized parameter space [15].

Evaluating and generating loss and fitness functions. With a database of dynamics in hand, one will still want to evaluate loss or fitness to achieve design goals. This could be done adaptively at the time of computing the database, or could use a stored database to give regions of interest in parameter space. To be explicit, functionality like those about DOW rates expressed in the Section 2 problem description can be searched for computing DOW quantities across regions in the database. Since the methods of database construction above allow for variable resolution of discretization, one can guarantee reasonable database size.

Evaluation of various loss or fitness functions may require evaluating models over parameter regions—this can be done efficiently even for large systems of ODEs.

Problem growth in parameter space is likely to be more computationally expensive—implementing and executing $\xrightarrow{\text{O\&ML}}$ will likely require the use of learning methods that exploit the structure of the database.

9 Combinatorial Topology for Parametric Sensitivity and Robustness in System-of-System Designs

Another set of tools that are being considered for applications like the agile field medicine problem considered in this report come from the theory of order sets, ([8]). The partial orders in system designs such as those considered in Section 4 are amenable to analysis using the language of lattice theory. As parameters such as casualty flow rates change, the order relationships of treatment schedules, planning for transfer to Role 4 facilities, and other key decisions will change, and we can use tools from partially ordered sets to describe the magnitude and impact of such changes. Some specific ideas are as follows.

Witness complexes and optimal resource allocation functions In the previous section, there was no consideration given to differences in cost of one allocation versus another. Cost and other metric considerations can be approached within our framework using the concept of a witness complex. We adopt the following from [4].

Definition 4. Given two finite sets of points A, B in some d -dimensional space, $\mathbb{R}^d$, such that all distances $\|x - y\|$ with $x \in A$ and $y \in B$ are different, the abstract simplicial complex $W(A, B)$, called the *witness complex* consists of all subsets $\sigma \subseteq B$ such that for every $\tau \subseteq \sigma$ there exists a point w in A with the property that every point in τ is closer to w than every point in $B \setminus \tau$.

In the standard terminology of witness complex constructions, points in A are called *data points*, and points in B are called *landmark points*. In the problems of allocating resources to requirements, we relax the assumption that both A and B lie in the same space, but we assume that there is a *pairing metric* $d(x, y)$ that gives a measure of how well suited each element $y \in B$ (i.e. each resource) is to each element $x \in A$ (requirement). Every requirement, x , then induces an ordering on the resource set where the resources are sorted in terms of their pairing metric with respect to x . Every such ordering can be associated with a path in the Hasse diagram of the Boolean lattice of subsets of B , starting from the point with the smallest pairing metric with respect to the given requirement and then proceeding to the union of the two most well paired points (well-paired means smallest value of the pairing metric), followed by the three best paired and so-on. The witness complex $W(A, B)$ is the maximal abstract simplicial complex whose face poset is contained in the union of paths over the entire requirement set.

To discuss this construction, we recall the following definition:

Definition 5. (See [4].) Let Δ be an abstract simplicial complex. The *face poset* associated with Δ , $\mathcal{F}(\Delta)$, is the partially ordered set of all non-empty simplices of Δ with the partial order relation being the inclusion relation on the set of simplices. $\square$

Fig. 14 shows an example with four landmark points (think of resources as in the previous section) and five data points (requirements). For the purpose of illustration in the figure, the landmark points and data points are depicted in the same space. The left hand component of the figure depicts the “distances” between resources and requirements. The right hand part of the figure is the Boolean lattice of subsets of resources, and the fat black paths correspond to the “witness” points representing requirements. The witness complex corresponding to the union of paths on the right of Fig. 14 is depicted in Fig. 15. $\square$

Finally, as we consider enumerative solutions (Sect. 4) and their extensions to relaxed search techniques (Sect. 6), we will be looking at possible use of the Kendall *tau* distance [9], to compare orderings of costs, η , as given in (5) or (14) as key parameters such as casualty flows T_{ij} are varied. The working assumption is that large values of such a metric indicate a noteworthy sensitivity.

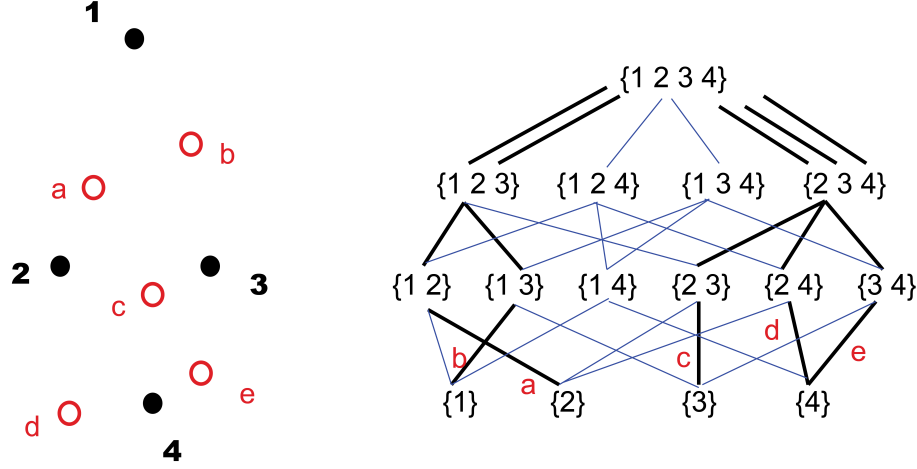


Figure 14: The landmark points appear in black, with the data points in red. In the Boolean lattice, the dark solid paths are the distance chains associated with each data point.

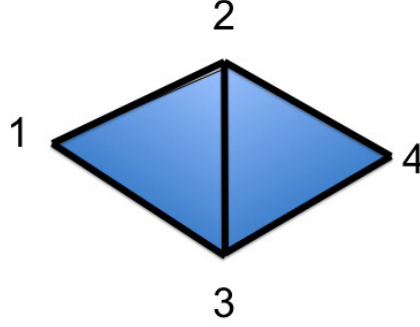


Figure 15: The witness complex corresponding to the construction depicted in Fig. 14.

10 Conclusion

A number of interrelated ideas have been explored with the goal of developing new insights on design and operation of complex systems using methods with roots in algebraic topology and category theory. Many of the questions that were discussed in the five month course of the project remain only partially answered, and the research has led to the conclusion that further work should be devoted to the quest for novel approaches based on heretofore under exploited concepts in abstract mathematics. It is the opinion of the author that further research on probabilistic topological and algebraic structures could pay large dividends for the kinds of applications treated in the report.

References

- [1] J. Baillieul, “Green’s Relations in Finite Function Semigroups,” *Aequationes Mathematicae*, 7, fasc. 1 (1972) 22-27.
- [2] S. MacLane, “Duality for Groups,” *Bull. Amer. Math. Soc.*, **56**:6, 485-516.
- [3] D.I. Spivak and R.E. Kent, “Ologs: A categorical framework for knowledge representations,” *PLoS ONE* 7(1), 2012, e24274.

- [4] D. Kozlov, *Combinatorial Algebraic Topology*, Springer, Berlin, 2008.
- [5] U. Fincke and M. Pohst, “On reduction algorithms in nonlinear integer mathematical programming,” *Operations Research Proceedings 1983* (Mannheim, 1983) Springer, Berlin, 1984, pp.289295]
- [6] U. Fincke and M. Pohst, “Improved Methods for Calculating Vectors of Short Length in a Lattice, Including a Complexity Analysis,” *Mathematics of Computation*, volume 44, number 170 April, 1985, p. 463-471.
- [7] Documentation: <http://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.linprog.html>
- [8] B.A. Davey and H.A. Priestly, *Introduction to Lattices and Order*, Second Ed., Cambridge University Press, ISBN 0 521 78451 4, 2002.
- [9] M. Kendall, (1938).“A New Measure of Rank Correlation”. *Biometrika*, 30 (12): 8189. doi:10.1093/biomet/30.1-2.81, 1938.
- [10] Zin Arai, William Kalies, Hiroshi Kokubu, Konstantin Mischaikow, Hiroe Oka, and Pawel Pilarczyk. A database schema for the analysis of global dynamics of multiparameter systems. *SIAM Journal on Applied Dynamical Systems*, 8(3):757–789, 2009.
- [11] Justin Bush, Marcio Gameiro, Shaun Harker, Hiroshi Kokubu, Konstantin Mischaikow, Ippei Obayashi, and Pawe\l Pilarczyk. Combinatorial-topological framework for the analysis of global dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4):047508, 2012.
- [12] Edward W. Chan, Heather Krull, Beth E. Lachman, Tom LaTourrette, Rachel Costello, Don Snyder, Mahyar A. Amouzegar, Hans V. Ritschard, and D. Scott Guernonprez. Improving Joint Expeditionary Medical Planning Tools Based on a Patient Flow Approach, 2012.
- [13] Shui-Nee Chow and Jack K. Hale. *Methods of bifurcation theory*. Springer-Verlag, 1982. Google-Books-ID: 5TXvAAAAMAAJ.
- [14] Bree Cummins, Tomas Gedeon, Shaun Harker, Konstantin Mischaikow, and Kafung Mok. Combinatorial Representation of Parameter Space for Switching Systems. *arXiv:1512.04131 [math]*, December 2015. arXiv: 1512.04131.
- [15] Tomas Gedeon, Shaun Harker, Hiroshi Kokubu, Konstantin Mischaikow, and Hiroe Oka. Global Dynamics for Steep Sigmoidal Nonlinearities in Two Dimensions. *arXiv:1508.02438 [math]*, August 2015. arXiv: 1508.02438.
- [16] Kelly Spendlove, Tomas Gedeon, and Jesse Berwald. Predicting Critical Transitions in Dynamical Systems Using Active Learning. *Preprint*.
- [17] Hiroshi Kokubu and Kelly Spendlove. A Users Guide to the Conley-Morse Database.
- [18] Jacob Palis and Floris Takens. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors in Dynamics*. Cambridge University Press, January 1995. Google-Books-ID: pwydPA23KVUC.