PROCEEDINGS *of the* SECOND BERKELEY SYMPOSIUM ON MATHEMATICAL STATISTICS AND PROBABILITY

Held at the Statistical Laboratory
Department of Mathematics
University of California
July 31–August 12, 1950

EDITED BY JERZY NEYMAN

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY AND LOS ANGELES
1951

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY AND LOS ANGELES
CALIFORNIA

CAMBRIDGE UNIVERSITY PRESS
LONDON, ENGLAND

PRINTED IN THE UNITED STATES OF AMERICA

A REMARK ON CHARACTERISTIC FUNCTIONS

A. ZYGMUND
UNIVERSITY OF CHICAGO

Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a sequence of distribution functions, and let

$$\varphi_n(x) = \int_{-\infty}^{+\infty} e^{ixt} dF_n(t)$$

be the corresponding characteristic functions. If the sequence $\{\varphi_n(x)\}$ converges over every finite interval, and if the limit is continuous at the point $x = 0$, then, as is very well known, the sequence $\{F_n(t)\}$ converges to a distribution function $F(t)$ at every point of continuity of the latter (see, for example, [1, p. 96]. It is also very well known that in this theorem convergence over every finite interval cannot be replaced by convergence over a fixed interval containing the point $x = 0$.

The situation is different if the random variables whose distribution functions are the F_n are uniformly bounded below (or above). Without loss of generality we may assume that the random variables in question are positive, so that all $F_n(t)$ are zero for t negative. The purpose of this note is to prove the following theorem.

THEOREM. *Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a sequence of distribution functions all vanishing for $t \leqq 0$, and let*

$$\varphi_n(x) = \int_0^{+\infty} e^{ixt} dF_n(t), \qquad -\infty < x < +\infty .$$

If the functions $\varphi_n(x)$ tend to a limit in an interval around $x = 0$, and if the limiting function is continuous at $x = 0$, then there is a distribution function $F(t)$ such that $F_n(t)$ tends to $F(t)$ at every point of continuity of F.

PROOF. Let $z = x + iy$, and let us consider the functions

$$\varphi_n(z) = \int_0^{+\infty} e^{izt} dF_n(t) = \int_0^{+\infty} e^{ixt} e^{-yt} dF_n(t) .$$

Each $\varphi_n(z)$ is regular for $y > 0$, continuous for $y \geqq 0$, and is of modulus $\leqq 1$ there. For z real, $\varphi_n(z)$ coincides with the characteristic function $\varphi_n(x)$. It is easy to see that the sequence $\{\varphi_n(z)\}$ converges in the half plane $y > 0$, and that the convergence is uniform over any closed and bounded set of this half plane. For let $z = \lambda(\zeta)$ be a conformal mapping of the half plane $y > 0$ onto the unit circle $|\zeta| < 1$, and let us consider the functions

$$(1) \qquad \varphi_n^*(\zeta) = \varphi_n[\lambda(\zeta)] .$$

These functions are regular for $|\zeta| < 1$, are numerically $\leqq 1$ there and their boundary values converge to a limit on a set of positive measure situated on the circumference $|\zeta| = 1$ (this set is actually an arc). By the theorem of Khintchine [2] and Ostrowski [3], the sequence $\{\varphi_n^*(\zeta)\}$ converges for $|\zeta| < 1$, and the convergence is uniform in every circle $|\zeta| \leqq \rho$, $\rho < 1$. Going back to the half plane $y > 0$, we see that the functions $\varphi_n(z)$ converge there to a regular function $\varphi(z)$, and that the convergence is uniform over any closed and bounded set in that half plane. In particular, the convergence is uniform over any finite segment of any line

$$y = y_0 , \qquad y_0 > 0 .$$

We shall now show that

$$\varphi(iy) \to 1 \quad \text{as} \quad y \to +0 . \tag{2}$$

It will again be slightly easier to consider the functions $\varphi_n^*(\zeta)$ defined by (1). They tend to a function $\varphi^*(\zeta)$ regular in $|\zeta| < 1$ and numerically $\leqq 1$ there. This function has nontangential boundary values $\varphi^*(e^{i\theta})$ for almost every θ and (as a bounded harmonic function) is the Poisson integral of $\varphi^*(e^{i\theta})$. Let us assume for simplicity that the mapping function $z = \lambda(\zeta)$ makes correspond $z = 0$ and $\zeta = 1$. If we can prove that in the neighborhood of $\theta = 0$ the function $\varphi^*(e^{i\theta})$ coincides almost everywhere with a function continuous at $\theta = 0$ and taking the value 1 at that point, then [since the values of $\varphi^*(e^{i\theta})$ in a set of measure zero are immaterial for the Poisson integral] the function $\varphi^*(\zeta)$ will tend to 1 as ζ approaches 1 along any nontangential path. This will immediately lead to relation (2).

Let us revert to the Khintchine-Ostrowski theorem used above. It can be completed as follows. *If the sequence of functions $\varphi_n^*(\zeta)$ regular and of modulus $\leqq 1$ for $|\zeta| < 1$, converges in a set E of positive measure on the circumference $|\zeta| = 1$, then on almost every radius $\zeta = \rho e^{i\theta}$, $0 \leqq \rho < 1$, terminating in the set E the sequence converges uniformly* (for the proof, see [4, p. 213]). Since the function $\varphi^*(\zeta) = \lim \varphi_n^*(\zeta)$ has nontangential limit $\varphi^*(e^{i\theta})$ for almost every θ, it immediately follows that $\varphi^*(e^{i\theta}) = \lim \varphi_n^*(e^{i\theta})$ *almost everywhere in E*. In our particular case, the functions $\varphi_n^*(\zeta)$ are continuous on $|\zeta| = 1$ except at the point ζ corresponding to $z = \infty$, and converge on an arc $-\delta \leqq \theta \leqq +\delta$ to a function $\gamma(\theta)$ continuous at $\theta = 0$ and taking the value 1 there [since $\varphi_n^*(1) = 1$ for all n]. Hence at almost every point θ in $(-\delta, \delta)$ the function $\varphi^*(e^{i\theta})$ coincides with $\gamma(\theta)$. Thus the proof of (2) is complete.

Since, as seen from the formula for $\varphi_n(z)$, all the quantities $\varphi_n(iy)$ are positive for $y > 0$, the quantity $\varphi(iy) = \lim \varphi_n(iy)$ is nonnegative. On account of (2), we have $\varphi(iy_0) > 0$ for all y_0 small enough. Let us fix such a y_0 and let us consider the nonnegative and nondecreasing functions

$$G_n(t) = \frac{1}{\varphi_n(iy_0)} \int_{-\infty}^{t} e^{-uy_0} dF_n(u) \tag{3}$$

[thus $G_n(t) = 0$ for $t \leqq 0$]. As seen from the formula defining $\varphi_n(z)$, the characteristic function $\psi_n(x)$ of $G_n(t)$ is

$$\int_0^\infty e^{ixt} dG_n(t) = \frac{1}{\varphi_n(iy_0)} \int_0^\infty e^{ixt} e^{-ty_0} dF_n(t) = \frac{\varphi_n(x + iy_0)}{\varphi_n(iy_0)} .$$

Since

$$1 = \psi_n(0) = \int_0^\infty dG_n(t),$$

it follows that the G_n are distribution functions. We know that the functions $\psi_n(x) = \varphi_n(x + iy_0)/\varphi_n(iy_0)$ converge uniformly over any finite interval of the variable x. Hence the functions $G_n(t)$ converge to a distribution function $G(t)$ at the points of continuity of G.

From (3) we see that

$$F_n(t) = \varphi_n(iy_0) \int_{-\infty}^t e^{uy_0} dG_n(u).$$

The right side here can be written

$$\varphi_n(iy_0) \left\{ e^{ty_0} G_n(t) - y_0 \int_{-\infty}^t e^{uy_0} G_n(u)\, du \right\}.$$

Hence the functions $F_n(t)$ tend to a nondecreasing function $F(t)$ at every point t at which G is continuous, and

$$(4)\quad F(t) = \varphi(iy_0) \left\{ e^{ty_0} G(t) - y_0 \int_{-\infty}^t e^{uy_0} G(u)\, du \right\} = \varphi(iy_0) \int_{-\infty}^t e^{uy_0} dG(u).$$

From this formula we see that the points of discontinuity of F are the same as those of G. It remains to show that F is a distribution function, that is that

$$(5)\qquad F(+\infty) - F(-\infty) = 1.$$

That the left side here is $\leqq 1$ is obvious since $0 \leqq F_n(t) \leqq 1$ for all n. Observing that both F and G vanish for $t < 0$, we deduce from (4) that

$$F(a) - F(-0) \geqq \varphi(iy_0) \{G(a) - G(-0)\} \quad \text{for} \quad a > 0.$$

Taking first a large, and then y_0 small, and using (2), we find that $F(+\infty) - F(-0) \geqq 1$, which gives (5). This completes the proof of the theorem.

Remark 1. The theorem can be extended to nonnegative random variables in the k-dimensional space R_k. The requirement is that the characteristic functions $\varphi_n(x_1, \ldots, x_k)$ converge in the neighborhood of $(0, \ldots, 0)$ to a function continuous at that point. The proof follows the same line as for $k = 1$, and the proofs of the corresponding lemmas for functions $\varphi_n(z_1, \ldots, z_k)$ of several complex variables offer no serious difficulties. The details are omitted here.

Remark 2. It is easy to see that the condition of the theorem, namely that all of the $F_n(t)$ vanish for $t \leqq 0$ (or for $t \leqq t_0$), can be replaced by a less stringent one:

$$F_n(t) \leqq Ae^{-\epsilon|t|}, \qquad t \leqq t_0,$$

where the positive constants A, ϵ and the constant t_0 are all independent of n.

The proof of this generalization remains essentially the same as before. For, applying integration by parts in the formula defining the function $\varphi_n(z)$, we see that the $\varphi_n(z)$ are regular in the strip

$$0 < y < \epsilon,$$

and are continuous and uniformly bounded in every closed strip

$$0 \leqq y \leqq \epsilon', \qquad \epsilon' < \epsilon.$$

In the proof given above it is therefore enough to take for $\lambda(\zeta)$ the function mapping the latter strip onto the unit circle $|\zeta| \leqq 1$ and consider only the values of y_0 sufficiently small ($y_0 < \epsilon$).

REFERENCES

[1] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1946.
[2] J. Khintchine, "Sur les suites de fonctions analytiques bornées dans leur ensemble," *Fund. Math.*, Vol. 4 (1923), pp. 72–75.
[3] A. Ostrowski, "Ueber die Bedeutung der Jensenschen Formel für einige Fragen der komplexen Funktionentheorie," *Acta Litt. ac Sci., Sect. Sci. Math., Szeged*, Vol. 1 (1922), pp. 80–87.
[4] A. Zygmund, "On the boundary values of functions of several complex variables, I," *Fund. Math.*, Vol. 36 (1949), pp. 207–235.