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# RANDOM ERGODIC THEOREMS AND MARKOFF PROCESSES WITH A STABLE DISTRIBUTION

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## 1. Introduction

The purpose of this paper is to discuss the relations between random ergodic theorems and Markoff processes with a stable distribution. Random ergodic theorem concerning a finite number of measure preserving transformations was obtained by S. M. Ulam and J. von Neumann. This result was announced in abstract form [6] but the proof has never been published. In the present paper we shall first give a proof of random ergodic theorem concerning a family of (infinitely many) measure preserving transformations with a probability distribution on it. We shall then discuss the condition of ergodicity for a family of measure preserving transformations and its consequence in random ergodic theorems. It turns out that the theory of Markoff processes with a stable distribution which was previously discussed by J. L. Doob [2], [3], K. Yosida [8], and the author [4] has a very close connection with our problem. It will be shown that to any family $\Phi$ of measure preserving transformations with a probability distribution there corresponds a Markoff process $P(s, B)$ with a stable distribution in such a way that the ergodic theorems concerning the Markoff process $P(s, B)$ which were obtained in [8] and [4] are nothing but the "integrated form" of random ergodic theorems concerning the family $\Phi$ of measure preserving transformations. Further, the conditions of ergodicity for $P$ correspond exactly to those for $\Phi$. It is, indeed, by making use of this fact that we prove the equivalence of various conditions of ergodicity for the family $\Phi$ of measure preserving transformations.

In case $\Phi$ consists of a finite number of measure preserving transformations, our ergodic theorem is reduced to that of S. M. Ulam and J. von Neumann [6]. If, in particular, the space on which the measure preserving transformations act is finite (and hence the measure preserving transformations are reduced to a permutation) our theory is reduced to that of H. Anzai [1] on the relationship between the random ergodic theorem concerning a finite number of permutations and the theory of Markoff process with a finite number of possible states.

We do not discuss it in our present paper, but it is an interesting problem to investigate the conditions of (weak or strong) mixing for a family $\Phi$ of measure preserving transformations and for the corresponding Markoff process $P$ with a stable distribution.

## 2. Random ergodic theorems

Let $(S, \mathfrak{B}, m)$ be a *measure space:* $S = \{s\}$ is a set of elements $s$, $\mathfrak{B} = \{B\}$ is a $\sigma$-field of subsets $B$ of $S$, and $m(B)$ is a countably additive measure defined on $\mathfrak{B}$. We assume that $m(S) = 1$. A subset $B$ of $S$ which belongs to $\mathfrak{B}$ is called $\mathfrak{B}$*-measurable* and $m(B)$ is called the *m-measure* of $B$. A $\mathfrak{B}$-measurable subset $N$ of $S$ with $m$-measure zero is called a $\mathfrak{B}$*-m-null set.* A property $P(s)$ of an element $s$ of $S$ is said to hold $\mathfrak{B}$*-m-almost everywhere* on $S$ if there exists a $\mathfrak{B}$-$m$-null set $N$ such that $P(s)$ holds for all $s \in S - N$. A real valued function $f(s)$ defined on $S$ is called $\mathfrak{B}$*-measurable* if the set $\{s | \alpha < f(s) < \beta\}$ is $\mathfrak{B}$-measurable for any real numbers $\alpha$, $\beta$ with $\alpha < \beta$. The *m-integrability* of a $\mathfrak{B}$-measurable function $f(s)$ is defined as usual, and the *m-integral* of $f(s)$ on $S$ is denoted by

$$\int_S f(s)\, m(ds)\,. \tag{2.1}$$

Let $(X, \mathfrak{E}, \mu)$ be another measure space: $X = \{x\}$ is a set of elements $x$, $\mathfrak{E} = \{E\}$ is a $\sigma$-field of subsets $E$ of $X$, and $\mu(E)$ is a countably additive measure defined on $\mathfrak{E}$. We also assume that $\mu(X) = 1$. Let $(S \times X, \mathfrak{B} \otimes \mathfrak{E}, m \times \mu)$ be the *direct product measure* space of $(S, \mathfrak{B}, m)$ and $(X, \mathfrak{E}, \mu)$: $S \times X$ is the set of all pairs of elements $(s, x)$, $s \in S$, $x \in X$; $\mathfrak{B} \otimes \mathfrak{E}$ is the $\sigma$-field of subsets of $S \times X$ generated by sets of the form: $B \times E = \{(s, x) | s \in B, x \in E\}$, $B \in \mathfrak{B}$, $E \in \mathfrak{E}$; and $m \times \mu$ is a countably additive measure defined on $\mathfrak{B} \otimes \mathfrak{E}$ such that $(m \times \mu)(B \times E) = m(B)\mu(E)$ for any $B \in \mathfrak{B}$, $E \in \mathfrak{E}$.

A one to one mapping $\varphi$ of $S$ onto itself is called a $\mathfrak{B}$*-m-measure preserving* transformation if $B \in \mathfrak{B}$ implies $\varphi(B) \in \mathfrak{B}$, $\varphi^{-1}(B) \in \mathfrak{B}$ and $m[\varphi(B)] = m[\varphi^{-1}(B)] = m(B)$. Let $\Phi = \{\varphi_x | x \in X\}$ be a family of $\mathfrak{B}$-$m$-measure preserving transformations $\varphi_x$ defined on $S$ with a parameter $x \in X$. $\Phi$ is called $(\mathfrak{B}, \mathfrak{E})$*-measurable* if $B \in \mathfrak{B}$ implies $\{(s, x) | \varphi_x(s) \in B\} \in \mathfrak{B} \otimes \mathfrak{E}$. We also say that $\Phi$ is a *family of* $\mathfrak{B}$*-m-measure preserving transformations with a probability distribution* $(X, \mathfrak{E}, \mu)$. This condition is equivalent to saying that, for any $\mathfrak{B}$-measurable function $f(s)$ defined on $S$, the composite function $f(s, x) = f[\varphi_x(s)]$ is a $(\mathfrak{B} \otimes \mathfrak{E})$-measurable function defined on $S \times X$. Further, it is easy to see that the one to one mapping $\bar{\varphi}$ of $S \times X$ onto itself defined by

$$\bar{\varphi}(s, x) = [\varphi_x(s), x] \tag{2.2}$$

is a $(\mathfrak{B} \otimes \mathfrak{E})$-$(m \times \mu)$-measure preserving transformation defined on $S \times X$.

Let $(\Omega, \mathfrak{E}^*, \mu^*)$ be the *two sided infinite direct product measure space* of $(X_n, \mathfrak{E}_n, \mu_n) = (X, \mathfrak{E}, \mu)$, $n = 0, \pm 1, \pm 2, \ldots$ . This means that $\Omega = \mathop{\mathbf{P}}_{n=-\infty}^{\infty} X_n$ is the set of all two sided infinite sequences

$$\omega = \{x_n | n = 0, \pm 1, \pm 2, \ldots\}\,, \tag{2.3}$$

where $x_n = x_n(\omega)$ (= the $n$-th coordinate of $\omega$) $\in X_n = X$, $n = 0, \pm 1, \pm 2, \ldots$ ; $\mathfrak{E}^* = \mathop{\mathbf{P}}_{n=-\infty}^{\infty} \otimes \mathfrak{E}_n$ is the $\sigma$-field of subsets of $\Omega$ generated by sets of the form: $\mathop{\mathbf{P}}_{n=-\infty}^{\infty} E_n = \{\omega | x_n(\omega) \in E_n,\ n = 0, \pm 1, \pm 2, \ldots\}$, where $E_n \in \mathfrak{E}_n = \mathfrak{E}$,

$n = 0, \pm 1, \pm 2, \ldots$ ; and $\mu^* = \prod_{n=-\infty}^{\infty} \mu_n$ is a countably additive measure defined on $\mathfrak{E}^*$ such that $\mu^*\left(\mathop{\mathrm{P}}_{n=-\infty}^{\infty} E_n\right) = \prod_{n=-\infty}^{\infty} \mu_n(E_n) = \prod_{n=-\infty}^{\infty} \mu(E_n)$ for any $E_n \in \mathfrak{E}_n = \mathfrak{E}$, $n = 0, \pm 1, \pm 2, \ldots$ .

A one to one mapping $\psi$ defined on $\Omega$ by

$$x_n[\psi(\omega)] = x_{n+1}(\omega) , \tag{2.4}$$

[this means that the $n$-th coordinate of $\psi(\omega)$ is equal to the $(n + 1)$-st coordinate of $\omega$], $n = 0, \pm 1, \pm 2, \ldots$ is called the *shift transformation*. It is clear that $\psi$ is an $\mathfrak{E}^*$-$\mu^*$-measure preserving transformation defined on $\Omega$.

For any $p \geqq 1$, let $L^p(S) = L^p(S, \mathfrak{B}, m)$ be the $L^p$-space of all real valued $\mathfrak{B}$-measurable functions $f(s)$ defined on $S$ with

$$\|f\|_{L^p(S)} = \left(\int_S |f(s)|^p m(ds)\right)^{1/p} < \infty \tag{2.5}$$

as its norm. Two functions from $L^p(S)$ which coincide with each other $\mathfrak{B}$-$m$-almost everywhere on $S$ are identified.

The *random ergodic theorem* of S. M. Ulam and J. von Neumann [6] may be stated as follows:

THEOREM 1. *Let $(S, \mathfrak{B}, m)$ and $(X, \mathfrak{E}, \mu)$ be two measure spaces with $m(S) = \mu(X) = 1$. Let $\Phi = \{\varphi_x | x \in X\}$ be a $(\mathfrak{B}, \mathfrak{E})$-measurable family of $\mathfrak{B}$-$m$-measure preserving transformation $\varphi_x$ defined on $S$. Then, for any function $f(s) \in L^p(S)$ $(p \geqq 1)$, there exists an $\mathfrak{E}^*$-$\mu^*$-null set $N^*$ of $\Omega$ such that, for any $\omega \in \Omega - N^*$, there exists a function $\bar{f}_\omega(s) \in L^p(S)$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f[\varphi_{x_{k-1}(\omega)} \cdots \varphi_{x_0(\omega)}(s)] - \bar{f}_\omega(s) \right\|_{L^p(S)} = 0 \tag{2.6}$$

*and*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f[\varphi_{x_{k-1}(\omega)} \cdots \varphi_{x_0(\omega)}(s)] = \bar{f}_\omega(s) \tag{2.7}$$

*$\mathfrak{B}$-$m$-almost everywhere on $S$.*

*Remark.* The $\mathfrak{B}$-$m$-null set $N_\omega$ on which the convergence (2.7) does not hold may depend on $\omega$, and it might happen that the union $\cup_{\omega\in\Omega-N^*} N_\omega$ of all $N_\omega$ is no more a $\mathfrak{B}$-$m$-null set.

PROOF. Let us consider the direct product measure space $(S \times \Omega, \mathfrak{B} \otimes \mathfrak{E}^*, m \times \mu^*)$ of $(S, \mathfrak{B}, m)$ and $(\Omega, \mathfrak{E}^*, \mu^*)$: $S \times \Omega$ is the set of all pairs $(s, \omega)$, $s \in S$, $\omega \in \Omega$; $\mathfrak{B} \otimes \mathfrak{E}^*$ is the $\sigma$-field of subsets of $S \times \Omega$ generated by sets of the form: $B \times E^* = \{(s, \omega) | s \in B, \omega \in E^*\}$, $B \in \mathfrak{B}$, $E^* \in \mathfrak{E}^*$; and $m \times \mu^*$ is a countably additive measure defined on $\mathfrak{B} \otimes \mathfrak{E}^*$ such that $(m \times \mu^*)(B \times E^*) = m(B)\mu^*(E^*)$ for any $B \in \mathfrak{B}$, $E^* \in \mathfrak{E}^*$.

Let us put

$$\varphi^*(s, \omega) = [\varphi_{x_0(\omega)}(s), \psi(\omega)] , \tag{2.8}$$

where $x_0(\omega)$ is the 0-th coordinate of $\omega$ and $\psi$ is the shift transformation defined by (2.4). It is easy to see that $\varphi^*$ is a $(\mathfrak{B} \otimes \mathfrak{C}^*)$-$(m \times \mu^*)$-measure preserving transformation defined on $S \times \Omega$, and it is clear that

$$\varphi^{*n}(s, \omega) = [\varphi_{x_{n-1}(\omega)} \cdots \varphi_{x_0(\omega)}(s), \psi^n(\omega)], \tag{2.9}$$

for $n = 1, 2, \ldots$.

Let $L^p(S \times \Omega) = L^p(S \times \Omega, \mathfrak{B} \otimes \mathfrak{C}^*, m \times \mu^*)$ be the $L^p$-space of all $\mathfrak{B} \otimes \mathfrak{C}^*$-measurable real valued functions $f^*(s, \omega)$ defined on $S \times \Omega$ with

$$\|f^*\|_{L^p(S\times\Omega)} = \left(\int\int_{S\times\Omega} |f^*(s, \omega)|^p m(ds)\, \mu^*(d\omega)\right)^{1/p} < \infty \tag{2.10}$$

as its norm. For any function $f(s) \in L^p(S)$, let us put $f^*(s, \omega) = f(s)$. Then it is easy to see that $f^*(s, \omega) \in L^p(S \times \Omega)$. If we apply the ordinary mean and individual ergodic theorems to $f^*(s, \omega)$ and $\varphi^*$, then we see that there exists a function $\bar{f}^*(s, \omega) \in L^p(S \times \Omega)$ such that

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f[\varphi_{x_{k-1}(\omega)} \cdots \varphi_{x_0(\omega)}(s)] - \bar{f}^*(s, \omega)\right\|_{L^p(S\times\Omega)} = 0 \tag{2.11}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f[\varphi_{x_{k-1}(\omega)} \cdots \varphi_{x_0(\omega)}(s)] = \bar{f}^*(s, \omega) \tag{2.12}$$

$(\mathfrak{B} \otimes \mathfrak{C}^*)$-$(m \times \mu^*)$-almost everywhere on $S \times \Omega$. It is easy to see that theorem 1 follows from (2.11) and (2.12) by Fubini's theorem.

It is interesting to note that the limit function $\bar{f}_\omega(s) = \bar{f}^*(s, \omega)$ depends ordinarily on $\omega$ and is in general not equal to a constant $\mathfrak{B}$-$m$-almost everywhere on $S$. $\bar{f}_\omega(s) = \bar{f}^*(s, \omega)$ will be equal to a constant $(\mathfrak{B} \otimes \mathfrak{C}^*)$-$(m \times \mu^*)$-almost everywhere on $S \times \Omega$ if $\varphi^*$ is ergodic on $(S \times \Omega, \mathfrak{B} \otimes \mathfrak{C}^*, m \times \mu^*)$. In order to discuss this problem we introduce the notion of ergodicity for the family $\Phi = \{\varphi_x | x \in X\}$ of $\mathfrak{B}$-$m$-measure preserving transformations $\varphi_x$ defined on $S$.

Given a $(\mathfrak{B}, \mathfrak{C})$-measurable family $\Phi = \{\varphi_x | x \in X\}$ of $\mathfrak{B}$-$m$-measure preserving transformations $\varphi_x$ defined on $S$, a $\mathfrak{B}$-measurable subset $B$ of $S$ is called *$\Phi$-invariant* if $m[\varphi_x(B)\Delta(B)] = 0$ for $\mathfrak{C}$-$\mu$-almost all $x \in X$, where $A\Delta B$ denotes the symmetric difference of $A$ and $B$. $\Phi$ is called *ergodic* if every $\Phi$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ satisfies either $m(B) = 0$ or $m(S - B) = 0$. This means that, if a subset of $S \times X$ of the form: $B \times X$, $B \in \mathfrak{B}$, is invariant under the $(\mathfrak{B} \otimes \mathfrak{C})$-$(m \times \mu)$-measure preserving transformation $\bar{\varphi}$ defined by (2.2), then either $m(B) = 0$ or $m(S - B) = 0$. It is easy to see that, in case $X$ consists of a single element, that is, in case $\Phi$ consists of a single $B$-$m$-measure preserving transformation $\varphi$, our definition of ergodicity for $\Phi$ coincides with the usual definition of ergodicity for $\varphi$.

Further, it is to be noticed that, in case $X$ contains an $\mathfrak{C}$-measurable subset $E$ with $\mu(E) > 0$ and $\mu(X - E) > 0$, $\varphi$ cannot be ergodic on $(S \times X, \mathfrak{B} \otimes \mathfrak{C}, m \times \mu)$. In fact, the set $S \times E$ is invariant under $\bar{\varphi}$ and clearly satisfies $(m \times \mu)(S \times E) > 0$ and $(m \times \mu)(S \times X - S \times E) > 0$.

It is, however, possible to prove that the ergodicity of $\Phi = \{\varphi_x | x \in X\}$ is equivalent to the ergodicity of a $(\mathfrak{B} \otimes \mathfrak{C}^*)$-$(m \times \mu^*)$-measure preserving transformation $\varphi^*$ defined on $S \times \Omega$ by (2.8). It is easy to see that the ergodicity of $\varphi^*$ on $(S \times \Omega, \mathfrak{B} \otimes \mathfrak{C}^*, m \times \mu^*)$ implies that of $\Phi = \{\varphi_x | x \in X\}$ on $(S, \mathfrak{B}, m)$ with respect to $(X, \mathfrak{C}, \mu)$, but the converse is not obvious. This will be proved in theorem 3.

## 3. Markoff process with a stable distribution

Let $(S, \mathfrak{B}, m)$ be a measure space with $m(S) = 1$. A real valued nonnegative function $P(s, B)$ of two variables $s$, $B$ defined for $s \in S$, $B \in \mathfrak{B}$, is called a *Markoff process with a stable distribution* $m(B)$ if the following conditions are satisfied: (i) for any fixed $s \in S$, $P(s, B)$ is a countably additive set function of $B$ defined on $\mathfrak{B}$ and satisfies $P(s, S) = 1$, (ii) for any fixed $B \in \mathfrak{B}$, $P(s, B)$ is a $\mathfrak{B}$-measurable function of $s$ defined on $S$ and satisfies

$$\int_S P(s, B)\, m(ds) = m(B). \tag{3.1}$$

Similarly, a real valued nonnegative function $R(B, s)$ defined for $B \in \mathfrak{B}$, $s \in S$ is called an *inverse Markoff process with a stable distribution* $m(B)$ if (i) for any $s \in S$, $R(B, s)$ is a countably additive set function of $B$ defined on $\mathfrak{B}$ and satisfies $R(S, s) = 1$, (ii) for any fixed $B \in \mathfrak{B}$, $R(B, s)$ is a $\mathfrak{B}$-measurable function of $s$ on $S$ and satisfies

$$\int_S R(B, s)\, m(ds) = m(B). \tag{3.2}$$

$P(s, B)$ and $R(B, s)$ are said to be *associated with* each other if

$$\int_A P(s, B)\, m(ds) = \int_B R(A, s)\, m(ds) \tag{3.3}$$

for any $A \in \mathfrak{B}$, $B \in \mathfrak{B}$, and this common value is denoted by $Q(A, B)$.

*Remark.* In case $(S, \mathfrak{B}, m)$ is the *Lebesgue measure space* [that is, a measure space in which $S = \{s\}$ is the set of real numbers $s$, $0 \leqq s < 1$; $\mathfrak{B} = \{B\}$ is the $\sigma$-field of all Lebesgue measurable subsets $B$ of $S$, and $m(B)$ is the ordinary Lesbesgue measure of $B$ with the normalization $m(S) = 1$], it is easy to see that, for any Markoff process $P(s, B)$ with the stable distribution $m(B)$, there exists an inverse Markoff process $R(B, s)$ with the same stable distribution $m(B)$ which is associated with $P(s, B)$. In fact, if we put

$$Q(A, B) = \int_A P(s, B)\, m(ds) \tag{3.4}$$

for any $A, B \in \mathfrak{B}$, then, for any fixed $A$, $Q(A, B)$ is a countably additive set function of $B$ defined on $\mathfrak{B}$ which satisfies $Q(A, B) \leqq Q(S, B) = m(B)$ for all $B \in \mathfrak{B}$. Hence, by Radon-Nikodym's theorem, for any $A \in \mathfrak{B}$, there exists a $\mathfrak{B}$-measurable function $R(A, s)$ of $s$ such that $0 \leqq R(A, s) \leqq 1$ for all $s \in S$ and

$$Q(A, B) = \int_B R(A, s)\, m(ds) \tag{3.5}$$

for all $B \in \mathfrak{B}$. By a well known argument due to J. von Neumann [5] and J. L. Doob [2], in case $(S, \mathfrak{B}, m)$ is a Lebesgue measure space, we can find a solution

$R(A, s)$ of (3.5) which is countably additive on $\mathfrak{B}$ as a set function of $A$ for almost all fixed $s \in S$. This $R(A, s)$ clearly is an inverse Markoff process with a stable distribution $m(B)$ which is associated with $P(s, B)$. Conversely, starting from $R(A, s)$ we can easily obtain $P(s, B)$ by similar arguments.

Let $L^p(S) = L^p(S, \mathfrak{B}, m)$ $(p \geqq 1)$ be the $L^p$-space defined on the measure space $(S, \mathfrak{B}, m)$ as in section 2. It is easy to see that (compare [9])

$$\Delta f(s) = \int_S P(s, dt) f(t) \tag{3.6}$$

is a bounded linear transformation of $L^p(S)$ into itself such that

$$f \geqq 0 \text{ implies } \Delta f \geqq 0 , \tag{3.7}$$

where $f \geqq 0$ means that $f(s) \geqq 0$ $\mathfrak{B}$-$m$-almost everywhere on $S$. It is also easy to see, because $P(s, S) = 1$ for all $s \in S$, that the constant function $f(s) \equiv 1$ is invariant under $\Delta$: $\Delta(1) \equiv 1$.

Similarly, we see that

$$\Gamma f(s) = \int_S f(t) R(dt, s) \tag{3.8}$$

is a bounded linear transformation of $L^p(S)$ into itself such that

$$f \geqq 0 \text{ implies } \Gamma f \geqq 0 \quad \text{and} \quad \Gamma(1) \equiv 1 . \tag{3.9}$$

Let further $\mathfrak{M}(\mathfrak{B})$ be the Banach space of all real valued countably additive set functions $F(B)$ defined on $\mathfrak{B}$ with

$$\|F\|_{\mathfrak{M}(\mathfrak{B})} = \underset{B \in \mathfrak{B}}{\text{total variation}} |F(B)| \tag{3.10}$$

as its norm. It is then easy to see that

$$\Delta^* F(B) = \int_S F(ds) P(s, B) \tag{3.11}$$

is a bounded linear transformation of $\mathfrak{M}(\mathfrak{B})$ into itself such that

$$F \geqq 0 \text{ implies } \Delta^* F \geqq 0 , \tag{3.12}$$

where $F \geqq 0$ means that $F(B) \geqq 0$ for all $B \in \mathfrak{B}$. We observe that (3.1) means that $m(B)$ is invariant under $\Delta^*$. Let now $\mathfrak{M}_0(\mathfrak{B})$ be the closed linear subspace of $\mathfrak{M}(\mathfrak{B})$ consisting of all $F(B)$ which are absolutely continuous with respect to $m(B)$. Then it is easy to see that $\Delta^*$ maps $\mathfrak{M}_0(\mathfrak{B})$ into itself [follows easily from (3.11)]. On the other hand, it is well known (by Radon-Nikodym's theorem) that $\mathfrak{M}_0(\mathfrak{B})$ is isometrically isomorphic with the $L^1$-space $L^1(S) = L^1(S, B, m)$ by means of the correspondence $F(B) \leftrightarrow f(s)$, where $f(s) \in L^1(S)$ is the derivative of $F(B) \in \mathfrak{M}_0(\mathfrak{B})$ with respect to $m(B)$, or, in other words, $F(B) \in \mathfrak{M}_0(\mathfrak{B})$ is the indefinite integral of $f(s) \in L^1(S)$. Thus $\Delta^*$ may be considered as a bounded linear transformation of $L^1(S)$ into itself:

$$\begin{aligned} \Delta^* f(s) = g(s) : \int_B g(s)\, m(ds) &= \int_S f(s) P(s, B)\, m(ds) \\ &= \int_S f(s) Q(ds, B) \end{aligned} \tag{3.13}$$

for all $B \in \mathfrak{B}$ or

$$\Delta^* f(s) = \frac{d}{m(ds)} \int_S f(s) Q(ds, B). \tag{3.14}$$

It is now easy to see that this bounded linear transformation $\Delta^*$ defined on $L^1(S)$ is exactly the same as the bounded linear transformation $\Gamma$ defined on $L^1(S)$ by (3.8).

Similarly, if we consider a bounded linear transformation $\Gamma^*$ defined on $\mathfrak{M}(\mathfrak{B})$ by

$$\Gamma^* F(B) = \int_S R(B, s) F(ds), \tag{3.15}$$

then $m(B)$ is invariant under $\Gamma^*$, and $\Gamma^*$ may be considered as a bounded linear transformation of $\mathfrak{M}_0(\mathfrak{B})$ into itself. Further, the bounded linear transformation obtained on $L^1(S)$ from $\Gamma^*$ by means of the isometric isomorphism $F(B) \leftrightarrow f(s)$ between $\mathfrak{M}_0(\mathfrak{B})$ and $L^1(S)$, is exactly the same as the bounded linear transformation $\Delta$ defined on $L^1(S)$ by (3.6).

We now notice that the iterations $\Delta^n$ and $\Gamma^n$ of $\Delta$ and $\Gamma$ are given by

$$\Delta^n f(s) = \int_S P^{(n)}(s, dt) f(t), \tag{3.16}$$

$$\Gamma^n f(s) = \int_S f(t) R^{(n)}(dt, s), \tag{3.17}$$

respectively, where $P^{(n)}(s, B)$ and $R^{(n)}(B, s)$ are defined recurrently by

$$\begin{aligned} P^{(n)}(s, B) &= \int_S P^{(n-1)}(s, dt) P(t, B) \\ &= \int_S P(s, dt) P^{(n-1)}(t, B), \end{aligned} \tag{3.18}$$

$$\begin{aligned} R^{(n)}(B, s) &= \int_S R^{(n-1)}(B, t) R(dt, s) \\ &= \int_S R(B, t) R^{(n-1)}(dt, s), \end{aligned} \tag{3.19}$$

$n = 2, 3, \ldots$ . It is easy to see that

$$\int_A P^{(n)}(s, B)\, m(ds) = \int_B R^{(n)}(A, s)\, m(ds) \tag{3.20}$$

for all $A, B \in \mathfrak{B}$. This common value is denoted by $Q^{(n)}(A, B)$.

The following result concerning Markoff processes with a stable distribution is known:

THEOREM 2. *Let $P(s, B)$ be a Markoff process with a stable distribution $m(B)$. Let $R(B, s)$ be an inverse Markoff process with the same stable distribution $m(B)$ which is associated with $P(s, B)$. Let $\Delta$ and $\Gamma$ be the bounded linear transformations defined on $L^p(S)$ ($p \geqq 1$) by (3.6) and (3.8). Then for any function $f(s) \in L^p(S)$ ($p \geqq 1$) there exist two functions $\bar{f}(s)$, $\bar{\bar{f}}(s) \in L^p(S)$ such that*

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \Delta^k f(s) - \bar{f}(s) \right\|_{L^p(S)} = 0, \tag{3.21}$$

$$(3.22)\qquad \lim_{n\to\infty}\left\| \frac{1}{n}\sum_{k=0}^{n-1}\Gamma^k f(s) - \bar{\bar{f}}(s)\right\|_{L^p(S)} = 0 .$$

*If further, $f(s)$ is bounded, then*

$$(3.23)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\Delta^k f(s) = \bar{f}(s),$$

$$(3.24)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\Gamma^k f(s) = \bar{\bar{f}}(s)$$

*$\mathfrak{B}$-$m$-almost everywhere on $S$.*

These results were obtained by K. Yosida [8] and the author [4], and are called the mean and individual ergodic theorems concerning Markoff process with a stable distribution. It is interesting to observe that the individual ergodic theorems (3.23), (3.24) were proved only when $f(s)$ is bounded. It was proved lately by J. L. Doob [3] that we have (3.23) and (3.24) if $f(s) \in L^p(S)$ $(p > 1)$ or if $|f(s)|\log^+|f(s)| \in L^1(S)$. This follows from the fact that individual ergodic theorems in theorem 3 are the "integrated form" of individual ergodic theorems in an infinite product space and that the "integration" of individual ergodic theorems is permitted only when we have Wiener's dominated ergodic theorem [7].

Let $P(s, B)$ be a Markoff process with a stable distribution $m(B)$. A $\mathfrak{B}$-measurable subset $B$ of $S$ is called *P-invariant* if (i) $P(s, B) = 1$ for $\mathfrak{B}$-$m$-almost all $s \in B$ and (ii) $P(s, B) = 0$ for $\mathfrak{B}$-$m$-almost all $s \in S - B$. It is easy to see that the conditions (i), (ii) are equivalent. Further, $B$ is $P$-invariant if and only if the characteristic function $\chi_B(s)$ of $B$ is invariant under the bounded linear transformation $\Delta$ defined by (3.6).

Similarly, we can introduce the notion of $R$-invariance for an inverse Markoff process $R(B, s)$ with a stable distribution. *A* $\mathfrak{B}$-measurable subset $B$ of $S$ is called *R-invariant* if (i) $R(B, s) = 1$ for $\mathfrak{B}$-$m$-almost all $s \in B$, and (ii) $R(B, s) = 0$ for $\mathfrak{B}$-$m$-almost all $s \in S - B$. Again these two conditions are equivalent. Further, $B$ is $R$-invariant if and only if the characteristic function $\chi_B(s)$ of $B$ is invariant under the bounded linear transformation $\Gamma$ defined by (3.8).

We observe that, in case $P(s, B)$ and $R(B, s)$ are associated with each other, the notion of $P$-invariance and that of $R$-invariance are equivalent. This follows from the observation that each of these invariance conditions is equivalent to

$$(3.25)\quad Q(B,B) = \int_B P(s,B)\, m(ds) = \int_B R(B, s)\, m(ds) = m(B).$$

A Markoff process $P(s, B)$ with a stable distribution $m(B)$ is called *ergodic* if $P$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ satisfies either $m(B) = 0$ or $m(S - B) = 0$. Similarly, an inverse Markoff process $R(B, s)$ with a stable distribution $m(B)$ is *ergodic* if every $R$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ satisfies either $m(B) = 0$ or $m(S - B) = 0$. It is clear that in case $P(s, B)$ and $R(B, s)$ are associated with each other, these two conditions of ergodicity are equivalent.

Further, we can introduce the notion of ergodicity for bounded linear transformations: A bounded linear transformation $\Delta$ (or $\Gamma$) of $L^p(S)$ into itself is *ergodic* if constant functions are the only functions which are invariant under $\Delta$ (or $\Gamma$). In case the mean ergodic theorem holds for $\Delta$ (or $\Gamma$), that is, in case $\Delta^n$ (or $\Gamma^n$) are uniformly bounded and $\frac{1}{n}\sum_{k=0}^{n-1}\Delta^k$ $\left(\text{or } \frac{1}{n}\sum_{k=0}^{n-1}\Gamma^k\right)$ converges strongly to $\bar{\Delta}$ (or $\bar{\Gamma}$), this condition means that $\bar{\Delta}$ (or $\bar{\Gamma}$) is a projection to a one dimensional subspace of $L^p(S)$ consisting only of constant functions. Thus, in case our $\Delta$ [or $\Gamma$] is obtained from $P(s, B)$ [or $R(B, s)$] by (3.6) [or (3.8)], theorem 2 implies the following result: $\Delta$ (or $\Gamma$) is ergodic if and only if, for any $f(s) \in L^p(S)$, the corresponding limit function $\bar{f}(s)$ [or $\bar{\bar{f}}(s)$] in theorem 2 is a constant $\mathfrak{B}$-$m$-almost everywhere on $S$. Further it is clear that the ergodicity of $\Delta$ (or $\Gamma$) implies that of $P(s, B)$ [or $R(B, s)$], but the converse is not so obvious. This will be proved in theorem 3.

## 4. Random ergodic theorems and Markoff process with a stable distribution

Let $(S, \mathfrak{B}, m)$, $(X, \mathfrak{C}, \mu)$ be two measure spaces with $m(S) = \mu(X) = 1$. Let $\Phi = \{\varphi_x | x \in X\}$ be a $(\mathfrak{B}, \mathfrak{C})$-measurable family of $\mathfrak{B}$-$m$-measure preserving transformations $\varphi_x$ defined on $S$. Let us put, for any $s \in S$ and for any $B \in \mathfrak{B}$,

$$(4.1) \qquad P(s, B) = \mu\{x \mid \varphi_x(s) \in B\} = \int_X \chi_B[\varphi_x(s)]\,\mu(dx),$$

where $\chi_B(s)$ is the characteristic function of $B$. It is then easy to see that $P(s, B)$ is a Markoff process with a stable distribution $m(B)$. In fact,

$$(4.2) \qquad \begin{aligned} \int P(s, B)\, m(ds) &= \int_S m(ds) \int_X \chi_B[\varphi_x(s)]\,\mu(dx) \\ &= \int_X \mu(dx) \int_S \chi_B[\varphi_x(s)]\, m(ds) \\ &= \int_X \mu(dx) \int_S \chi_B(s)\, m(ds) \\ &= \mu(X)\, m(B) = m(B) \end{aligned}$$

for all $s \in S$ and for all $B \in \mathfrak{B}$. $P(s, B)$ is called the *Markoff process with a stable distribution* $m(B)$ *induced* by $\Phi = \{\varphi_x | x \in X\}$.

Similarly, if we put

$$(4.3) \qquad R(B, s) = \mu\{x \mid \varphi_x^{-1}(s) \in B\} = \int_X \chi_B[\varphi_x^{-1}(s)]\,\mu(dx),$$

then $R(B, s)$ is an inverse Markoff process with a stable distribution $m(B)$ which is associated with $P(s, B)$. In fact,

$$(4.4)\qquad \int_S R(B, s)\, m(ds) = \int_S m(ds) \int_X \chi_B[\varphi_x^{-1}(s)]\, \mu(dx)$$
$$= \int_X \mu(dx) \int_S \chi_B[\varphi_x^{-1}(s)]\, m(ds)$$
$$= \int_X \mu(dx) \int_S \chi_B(s)\, m(ds)$$
$$= \mu(X)\, m(B) = m(B)$$

for all $s \in S$ and for all $B \in \mathfrak{B}$; and further

$$(4.5)\qquad \int_A P(s, B)\, m(ds) = \int_A m(ds) \int_X \chi_B[\varphi_x(s)]\, \mu(dx)$$
$$= \int_X \mu(dx) \int_A \chi_B[\varphi_x(s)]\, m(ds)$$
$$= \int_X \mu(dx) \int_A \chi_{\varphi_x^{-1}(B)}(s)\, m(ds)$$
$$= \int_X m[A \cap \varphi_x^{-1}(B)]\, \mu(dx)$$
$$= \int_X m[\varphi_x(A) \cap B]\, \mu(dx)$$
$$= \int_X \mu(dx) \int_B \chi_{\varphi_x(A)}(s)\, m(ds)$$
$$= \int_X \mu(dx) \int_B \chi_A[\varphi_x^{-1}(s)]\, m(ds)$$
$$= \int_B m(ds) \int_X \chi_A[\varphi_x^{-1}(s)]\, \mu(dx)$$
$$= \int_B R(A, s)\, m(ds)$$

for all $A, B \in \mathfrak{B}$. The relation:

$$(4.6)\qquad Q(A, B) = \int_A P(s, B)\, m(ds)$$
$$= \int_B R(A, s)\, m(ds) = \int_X m[\varphi_x(A) \cap B]\, \mu(dx)$$

is useful in the later arguments.

For each $x \in X$, let $V_x$ be a bounded linear transformation defined on $L^p(S)$ ($p \geqq 1$) by

$$(4.7)\qquad V_x f(s) = f[\varphi_x(s)]\,.$$

Then it is easy to see that the bounded linear transformation $\Delta$ defined on $L^p(S)$

($p \geqq 1$) obtained from $P(s, B)$ by (3.6) is given by

$$\Delta f(s) = \int_X f[\varphi_x(s)]\,\mu(dx) = \int_X V_x f(s)\,\mu(dx) \tag{4.8}$$

or, symbolically,

$$\Delta = \int_X V_x \mu(dx). \tag{4.9}$$

Similarly, the bounded linear transformation $\Gamma$ defined on $L^p(S)$ ($p \geqq 1$) obtained from $R(B, s)$ by (3.8) is given

$$\Gamma f(s) = \int_X f[\varphi_x^{-1}(s)]\,\mu(dx) = \int_X V_x^{-1} f(s)\,\mu(dx) \tag{4.10}$$

or, symbolically,

$$\Gamma = \int_X V_x^{-1}\mu(dx), \tag{4.11}$$

where $V_x^{-1}$ denotes the inverse transformation of $V_x$ for each $x \in X$.

We also notice that the iterations $\Delta^n$ and $\Gamma^n$ of $\Delta$ and $\Gamma$ are given by

$$\Delta^n f(s) = \int_X \cdots \int_X f[\varphi_{x_n} \cdots \varphi_{x_1}(s)]\,\mu(dx_1) \cdots \mu(dx_n) = \int_S P^{(n)}(s, dt)\, f(t), \tag{4.12}$$

$$\Gamma^n f(s) = \int_X \cdots \int_X f[\varphi_{x_1}^{-1} \cdots \varphi_{x_n}^{-1}(s)]\,\mu(dx_1) \cdots \mu(dx_n) = \int_S f(t) R^{(n)}(dt, s), \tag{4.13}$$

where $P^{(n)}(s, B)$ and $R^{(n)}(B, s)$ are obtained recurrently from $P(s, B)$ and $R(B, s)$ by (3.18) and (3.19), respectively. It is easy to see that

$$P^{(n)}(s, B) = \int_X \cdots \int_X \chi_B[\varphi_{x_n} \cdots \varphi_{x_1}(s)]\,\mu(dx_1) \cdots \mu(dx_n), \tag{4.14}$$

$$R^{(n)}(B, s) = \int_X \cdots \int_X \chi_B[\varphi_{x_1}^{-1} \cdots \varphi_{x_n}^{-1}(s)]\,\mu(dx_1) \cdots \mu(dx_n) \tag{4.15}$$

and

$$Q^{(n)}(A, B) = \int_A P^{(n)}(s, B)\, m(ds) = \int_B R^{(n)}(A, s)\, m(ds) = \int_X \cdots \int_X m[\varphi_{x_n} \cdots \varphi_{x_1}(A) \cap B]\,\mu(dx_1) \cdots \mu(dx_n) \tag{4.16}$$

for any $A, B \in \mathfrak{B}$.

THEOREM 3. *Let* $(S, \mathfrak{B}, m)$ *and* $(X, \mathfrak{C}, \mu)$ *be two measure spaces with* $m(S) = \mu(X) = 1$. *Let* $\Phi = \{\varphi_x | x \in X\}$ *be a* $(\mathfrak{B}, \mathfrak{C})$*-measurable family of* $\mathfrak{B}$*-m-measure preserving transformations* $\varphi_x$ *defined on* $S$. *Then the following conditions are mutually equivalent:*

(a) $\Phi = \{\varphi_x | x \in X\}$ is ergodic, that is, every $\Phi$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ satisfies either $m(B) = 0$ or $m(S - B) = 0$.

(b) The induced Markoff process $P(s, B)$ with a stable distribution $m(B)$ defined from $\Phi = \{\varphi_x | x \in X\}$ by (4.1) is ergodic, that is, every $P$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ satisfies either $m(B) = 0$ or $m(S - B) = 0$.

(c) The induced inverse Markoff process $R(B, s)$ with a stable distribution $m(B)$ defined from $\Phi = \{\varphi_x | x \in X\}$ by (4.3) is ergodic, that is, every $R$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ satisfies either $m(B) = 0$ or $m(S - B) = 0$.

(d) The bounded linear transformation $\Delta$ defined on $L^p(S)$ $(p \geqq 1)$ by (4.8) is ergodic; that is, if a function $f(s) \in L^p(S)$ is invariant under $\Delta$, then $f(s)$ is equal to a constant $\mathfrak{B}$-$m$-almost everywhere on $S$.

(e) The bounded linear transformation $\Gamma$ defined on $L^p(S)$ $(p \geqq 1)$ by (4.10) is ergodic; that is, if a function $f(s) \in L^p(S)$ is invariant under $\Gamma$, then $f(s)$ is constant $\mathfrak{B}$-$m$-almost everywhere on $S$.

(f) The $(\mathfrak{B} \otimes \mathfrak{C}^*)$-$(m \times \mu^*)$-measure preserving transformation $\varphi^*$ defined on the product space $S \times \Omega$ by (2.8) is ergodic on $(S \times \Omega, \mathfrak{B} \otimes \mathfrak{C}^*, m \times \mu^*)$, that is, every $(\mathfrak{B} \otimes \mathfrak{C}^*)$-measurable subset $B^*$ of $S \times \Omega$ which is invariant under $\varphi^*$ satisfies either $(m \times \mu^*)(B^*) = 0$ or $(m \times \mu^*)(S \times \Omega - B^*) = 0$.

PROOF. It suffices to show that (a) → (b) → (d) → (f) → (a). In fact the implications (a) → (c) → (e) → (f) → (a) can be proved similarly.

*Proof of* (a) → (b). Assume that $\Phi = \{\varphi_x | x \in X\}$ is ergodic. If $P(s, B)$ is not ergodic, then there exists a $\mathfrak{B}$-measurable subset $B$ of $S$ with $m(B) > 0$, $m(S - B) > 0$ which is $P$-invariant. From (3.25) and (4.6) follows

$$m(B) = Q(B, B) = \int_X m[\varphi_x(B) \cap B]\, \mu(dx). \tag{4.17}$$

Since the integrand is $\leqq m(B)$ for all $x \in X$, this shows that $m[\varphi_x(B) \cap B] = m(B)$ for $\mathfrak{C}$-$\mu$-almost all $x$, that is, $m[\varphi_x(B) \Delta B] = 0$ for $\mathfrak{C}$-$\mu$-almost all $x$. This is a contradiction to the assumption that $\Phi = \{\varphi_x | x \in X\}$ is ergodic.

*Proof of* (b) → (d). Assume that $P(s, B)$ is ergodic. If $\Delta$ is not ergodic then there exists a function $f(s) \in L^p(S)$ $(p \geqq 1)$, not equal to constant $\mathfrak{B}$-$m$-almost everywhere on $S$, such that

$$f(s) = \int_S P(s, dt)\, f(t) \tag{4.18}$$

$\mathfrak{B}$-$m$-almost everywhere on $S$. If we put $f^+(s) = \max[f(s), 0]$, then, by (3.7), it is easy to see that

$$f^+(s) \leqq \int_S P(s, dt)\, f^+(t) \tag{4.19}$$

$\mathfrak{B}$-$m$-almost everywhere on $S$. If the inequality in (4.19) holds on a $\mathfrak{B}$-measurable set of positive $m$-measure, then we would have the inequality:

$$(4.20)\qquad \int_S f^+(s)\, m(ds) < \int_S m(ds) \int_S P(s, dt)\, f^+(t) = \int_S f^+(t) \int_S m(ds) P(s, dt) = \int_S f^+(t)\, m(dt)$$

which is a contradiction. Thus we must have

$$(4.21)\qquad f^+(s) = \int_S P(s, dt)\, f^+(t)$$

$\mathfrak{B}$-$m$-almost everywhere on $S$. Similarly, if we put $f^-(s) = \max[-f(s), 0]$, then

$$(4.22)\qquad f^-(s) = \int_S P(s, dt)\, f^-(t)$$

$\mathfrak{B}$-$m$-almost everywhere on $S$. By similar argument we have

$$(4.23)\qquad f_{\alpha,\beta}(s) = \int_S P(s, dt)\, f_{\alpha,\beta}(t)$$

$\mathfrak{B}$-$m$-almost everywhere on $S$, for any real numbers $\alpha$, $\beta$ with $\alpha < \beta$, where

$$(4.24)\qquad f_{\alpha,\beta}(s) = \begin{cases} 1 & \text{if} \quad f(s) \geqq \beta \\ \dfrac{f(s) - \alpha}{\beta - \alpha} & \text{if} \quad \alpha < f(s) < \beta \\ 0 & \text{if} \quad f(s) \leqq \alpha . \end{cases}$$

Since $f(s)$ is not equal to a constant $\mathfrak{B}$-$m$-almost everywhere on $S$, there exists a real number $\beta$ such that the set $B = \{s \mid f(s) \geqq \beta\}$ satisfies $m(B) > 0$ and $m(S - B) > 0$. Let us put $\alpha_n = \beta - \frac{1}{n}$ in (4.23) and let $n \to \infty$. Then, since $f_{\alpha_n,\beta}(s) \to \chi_B(s)$ for all $s \in S$ and since all functions $f_{\alpha_n,\beta}(s)$ are uniformly bounded,

$$(4.25)\qquad \chi_B(s) = \int_S P(s, dt)\, \chi_B(t) = P(s, B)$$

$\mathfrak{B}$-$m$-almost everywhere on $S$. This shows that $B$ is $P$-invariant. This is a contradiction to the assumption that $P(s, B)$ is ergodic. Thus $\Delta$ must be ergodic.

*Proof of* (d) $\to$ (f). It suffices to show that under the assumption that

$$(4.26)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (\Delta^k f, g) = (f, 1)(1, g)$$

for any functions $f(s), g(s) \in L^2(\mathfrak{S})$, where $(f, g)$ denotes the inner product of $f$, $g$ in $L^2(S)$, we have

$$(4.27)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (W^k f^*, g^*) = (f^*, 1)(1, g^*)$$

for any $f^*(s, \omega)$, $g^*(s, \omega) \in L^2(S \times \Omega)$, where $(f^*, g^*)$ denotes the inner product of $f^*$, $g^*$ in $L^2(S \times \Omega)$ and $W$ is the unitary transformation defined on $L^2(S \times \Omega)$ by

$$(4.28)\qquad W f^*(s, \omega) = f^*[\varphi^*(s, \omega)] = f^*[\varphi_{x_0(\omega)}(s), \psi(\omega)] .$$

Let $f^*(s, \omega)$, $g^*(s, \omega)$ be of the form:

$$f^*(s, \omega) = f(s) \prod_{i=a}^{b-1} f_i[x_i(\omega)] \tag{4.29}$$

$$g^*(s, \omega) = g(s) \prod_{j=c}^{d-1} g_j[x_j(\omega)], \tag{4.30}$$

where $f(s)$, $g(s) \in L^2(S)$; $a$, $b$, $c$, $d$ are integers, $a \leqq 0 < b$, $c \leqq 0 < d$; $f_i(x) \in L^2(X)$, $i = a, \ldots, b-1$; $g_j(x) \in L^2(X)$, $j = c, \ldots, d-1$. Since the linear combinations of such functions are everywhere dense in $L^2(S \times \Omega)$, it suffices to prove (4.27) only for the case when $f^*(s, \omega)$, $g^*(s, \omega)$ are of the forms (4.29), (4.30). In this case we have $\left(\text{note that we take the sum } \sum_{k=d-a}^{n-1} \text{ instead of } \sum_{k=0}^{n-1}\right)$:

$$\begin{aligned}\frac{1}{n} \sum_{k=d-a}^{n-1} (W^k f^*, g^*) &= \frac{1}{n} \sum_{k=d-a}^{n-1} \int_S m(ds) \prod_{i=a}^{b-1} f_i[x_{k+i}(\omega)] \\ &\quad \times \prod_{j=c}^{d-1} g_j[x_j(\omega)]\, f[\varphi_{x_{k-1}(\omega)} \cdots \varphi_{x_0(\omega)}(s)]\, g(s)\, \mu^*(d\omega) \\ &= \frac{1}{n} \sum_{k=d-a}^{n-1} \int_S m(ds) \int_X \cdots \int_X \prod_{i=a}^{b-1} f_i(x_{k+i}) \prod_{j=c}^{d-1} g_j(x_j) \\ &\quad \times f[\varphi_{x_{k-1}} \cdots \varphi_{x_0}(s)]\, g(s)\, \mu(dx_c) \ldots \mu(dx_{k+b-1}) \\ &= \frac{1}{n} \sum_{k=d-a}^{n-1} (\Delta_2 \Delta^{a+k-d} \Delta_1 f, g),\end{aligned} \tag{4.31}$$

where $\Delta$ is a bounded linear transformation defined on $L^2(S)$ by (4.8) and

$$\Delta_1 f(s) = \int_X \cdots \int_X \prod_{i=a}^{b-1} f_i(x_i)\, f[\varphi_{x_{b-1}} \cdots \varphi_{x_a}(s)] \times \mu(dx_{b-1}) \ldots \mu(dx_a), \tag{4.32}$$

$$\Delta_2 f(s) = \int_X \cdots \int_X \prod_{j=c}^{d-1} g_j(x_j)\, f[\varphi_{x_{d-1}} \cdots \varphi_{x_c}(s)] \times \mu(dx_{d-1}) \ldots \mu(dx_c). \tag{4.33}$$

Thus, denoting by $\Delta_2^*$ the adjoint operator of $\Delta_2$ in $L^2(S)$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=d-a}^{n-1} (W^k f^*, g^*) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=d-a}^{n-1} (\Delta_2 \Delta^{a+k-d} \Delta_1 f, g) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=d-a}^{n-1} (\Delta^{a+k-d} \Delta_1 f, \Delta_2^* g) \\ &= (\Delta_1 f, 1)(1, \Delta_2^* g) = (f^*, 1)(1, g^*),\end{aligned} \tag{4.34}$$

where we used the assumption (4.26) in the third equality, and the fourth equality follows from

$$(4.35)\quad (\Delta_1 f, 1) = \int_S m(ds) \int_X \cdots \int_X \prod_{i=a}^{b-1} f_i(x_i) \times f[\varphi_{x_{b-1}} \cdots \varphi_{x_a}(s)]\, \mu(dx_{b-1}) \cdots \mu(dx_a)$$

$$= \int \cdots \int_X \prod_{i=a}^{b-1} f_i(x_i)\, \mu(dx_{b-1}) \cdots \mu(dx_a) \times \int_S f[\varphi_{x_{b-1}} \cdots \varphi_{x_a}(s)]\, m(ds)$$

$$= \int_X \cdots \int \prod_{i=a}^{b-1} f_i(x_i)\, \mu(dx_{b-1}) \cdots \mu(dx_a) \int_S f(s)\, m(ds)$$

$$= (f^*, 1)$$

and

$$(4.36)\quad (1, \Delta_2^* g) = (\Delta_2 1, g)$$

$$= \int_S g(s)\, m(ds) \int_X \cdots \int_X \prod_{j=c}^{d-1} g_j(x_j)\, \mu(dx_{d-1}) \cdots \mu(dx_c)$$

$$= (1, \overset{*}{g}).$$

*Proof of* (f) $\rightarrow$ (a). Assume that $\varphi^*$ is ergodic on $(S \times \Omega, \mathfrak{B} \otimes \mathfrak{C}^*, m \times \mu^*)$. If $\Phi = \{\varphi_x | x \in X\}$ is not ergodic, then there exists a $\Phi$-invariant $\mathfrak{B}$-measurable subset $B$ of $S$ such that $m(B) > 0$ and $m(S - B) > 0$. Let us put $B^* = B \times \Omega$. Then $B^*$ is a $(\mathfrak{B} \otimes \mathfrak{C}^*)$-measurable subset of $S \times \Omega$ with $(m \times \mu^*)(B^*) > 0$, $(m \times \mu^*)(S \times \Omega - B^*) > 0$ and it is easy to see that $(m \times \mu^*)[\varphi^*(B^*)\Delta B^*] = 0$. This is a contradiction to the assumption that $\varphi^*$ is ergodic on $(S \times \Omega, \mathfrak{B} \otimes \mathfrak{C}^*, m \times \mu^*)$.

This completes the proof of theorem 3.

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