THE MATHEMATICS OF SEX AND MARRIAGE

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1. Introduction

The models of this paper attempt to account for the age, sex, and marital status distributions of human populations. A marriage market develops around preferences for mates of different ages, and we study this market as changes in age distributions change the availability of mates. Unless we know how to relate marriages to the exposed population, we cannot even calculate rates that will tell us whether marriage is increasing or decreasing. Sections 11 to 16 below attempt an empirically based solution of the two-sex problem.

2. Separate treatment of the sexes

To suggest what constitutes a "solution" from a demographic viewpoint, think of the sense in which the one-sex problem is solved. A given and fixed set of birth and death rates, specific by age, say for females, determines the entire trajectory of a closed population. Theory permits a calculation of exactly how many individuals would be present at each future time if those rates applied; the ultimate stable age distribution, the ultimate stable rates of birth, death, and natural increase, are similarly calculable. For the shorter term, a spectral analysis specifies the waves through which the population at each age would move on its way to the stable exponentially increasing condition; we can in particular trace the echo effect by which an initial hollow in the age distribution tends to be reflected in later generations with gradually diminishing relative amplitude until it disappears.

Aside from this, the one-sex theory enables us to say just what a given degree of emigration will do to the level of the ultimate population; how birth control applied by women aged 40 will affect the rate of increase of the population, as compared with birth control applied by women aged 20; when we find that the United States has a much higher mean age than Mexico, the theory enables us to trace this to our low birth rates rather than to any advantage that we may have in lower mortality. Within its own assumptions, often a close approximation to reality, the model gives complete and consistent results.

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This is not true of the two sexes considered simultaneously, where models typically produce selfcontradictions. Consider two separate, uncoupled equations, one for the trajectory of each sex. Between the number F_t of females at time t and its derivative F'_t we might have the relation

$$F'_t = r_F F_t, \tag{2.1}$$

where r_F is the observed rate of increase, to which the solution is $F_t = F_0 \exp\{r_F t\}$. For males M_t at time t, we would correspondingly have

$$M'_t = r_M M_t, \tag{2.2}$$

with solution $M_t = M_0 \exp\{r_M t\}$.

If the parameters r_F and r_M are taken from a given period of observation of a real population, the two equations will provide solutions in which the male and female populations increase unequally, so that the sex ratio ultimately becomes zero or infinity. We know that in real populations the male and female populations do adjust in numbers so as to maintain equality, and the mechanism by which they do this eludes the representation (2.1) and (2.2).

3. Female dominance without recognition of age

The simplest way of coupling the equations is by supposing male or female *dominance* in the generation of births. In female dominance, both boy and girl babies are generated in fixed proportion to the number of females at time t. The pair of equations for the population trajectory becomes, if the birth rate is λ and the death rate μ,

$$\begin{aligned} M'_t &= -\mu_M M_t + \lambda_M F_t, \\ F'_t &= -\mu_F F_t + \lambda_F F_t. \end{aligned} \tag{3.1}$$

The form of these equations, with $\mu_M = \mu_F$ and $\lambda_M = \lambda_F$, is due to Kendall [15]; Goodman [7] permitted the male and female parameters to differ from one another. The ratio λ_M/λ_F is not arbitrary, but holds close to 1.05 for human populations.

The equations (3.1) are coupled in one direction only—the first depends on the second, but the second does not depend on the first. The second member of (3.1) is in fact the same as (2.1) if we identify $\lambda_F - \mu_F$ with r_F, and hence must be satisfied by $F_t = F_0 \exp\{(\lambda_F - \mu_F)t\}$. From the solution for females in the second member of (3.1), that for males follows by substitution in the first member, and it turns out that the male population ultimately increases according to the same exponential as the female.

To find the asymptotic ratio of males to females is to find the value at which M_t/F_t has a zero derivative. Where the derivative of M_t/F_t is zero, M_t/F_t will be equal to the ratio of derivatives M'_t/F'_t, and hence from (3.1),

$$\frac{M_\infty}{F_\infty} = \frac{M'_\infty}{F'_\infty} = \frac{-\mu_M M_\infty + \lambda_M F_\infty}{(\lambda_F - \mu_F)F_\infty}, \tag{3.2}$$

or if R_∞ is the ratio M_∞/F_∞,

$$R_\infty = \frac{-\mu_M R_\infty + \lambda_M}{\lambda_F - \mu_F}. \tag{3.3}$$

Solving this for R_∞, gives easily

$$R_\infty = \frac{\lambda_M}{\lambda_F - \mu_F + \mu_M}. \tag{3.4}$$

The one sided coupling of the equations (3.1) is a decided improvement on the one sex model. It leads to (3.4) which accords with common sense in telling us that insofar as male mortality is heavier than female the ultimate sex ratio of the population of all ages will be less than the sex ratio at birth (Goodman [7], Keyfitz [16], p. 297). But it has the drawback of supposing that births continue unchanged in the total absence of males.

A similar argument applies to male dominance and to an average of male and female dominance taken with fixed weights. A degree of male dominance D would mean that each male produces at the rate of $D\lambda_M$ male births, and each female produces at the rate of $(1 - D)\lambda_M$ male births, per unit time, so that male births would be $\lambda_M(DM_t + (1 - D)F_t)$. Similarly, female births would be $\lambda_F(DM_t + (1 - D)F_t)$. If D is unity, then the rate of increase of the system is $\lambda_M - \mu_M$. If D is zero, the rate of increase is $\lambda_F - \mu_F$. If D is fixed at some intermediate value, then the ultimate rate of increase of the system is a weighted average of the male rate of increase $\lambda_M - \mu_M$ and of the female rate $\lambda_F - \mu_F$. This is better than female dominance, where births depend not at all on males. Nonetheless, births would still continue, though at a lower rate, in the mixed dominance case if either sex was entirely absent, as long as D is fixed.

4. The harmonic mean birth function

The defect may be rectified by making dominance vary with time, say letting the degree of male dominance be given by

$$D_t = \frac{F_t}{M_t + F_t}. \tag{4.1}$$

In support of such a value of D_t we note that when there are few females D_t would be low, which is to say that the model would make births depend largely on the number of females, and on males when males are few. The equations are now

$$\begin{aligned} M'_t &= -\mu_M M_t + \lambda_M(D_t M_t + (1 - D_t)F_t), \\ F'_t &= -\mu_F F_t + \lambda_F(D_t M_t + (1 - D_t)F_t), \\ D_t &= \frac{F_t}{M_t + F_t}, \end{aligned} \tag{4.2}$$

where as usual λ_M/λ_F equals the sex ratio at birth. By entering $D_t = F_t/(M_t + F_t)$ in the first two equations, we obtain

$$(4.3)\qquad \begin{aligned} M'_t &= -\mu_M M_t + 2\lambda_M \frac{M_t F_t}{M_t + F_t}, \\ F'_t &= -\mu_F F_t + 2\lambda_F \frac{M_t F_t}{M_t + F_t}. \end{aligned}$$

These equations are homogeneous and of the first degree in M_t and F_t, though not linear.

We can study the asymptotic properties of the system (4.3) by entering me^{rt} for M_t and fe^{rt} for F_t. Making this substitution and eliminating m and f, we obtain the condition for consistency, and it may be arranged to provide the intrinsic rate of natural increase as

$$(4.4)\qquad r = \frac{2}{\dfrac{1}{\lambda_M} + \dfrac{1}{\lambda_F}} - \frac{\dfrac{\mu_M}{\lambda_M} + \dfrac{\mu_F}{\lambda_F}}{\dfrac{1}{\lambda_M} + \dfrac{1}{\lambda_F}}.$$

Of the two terms the first is the harmonic mean of the given birth rates λ_M and λ_F, and the second a weighted arithmetic mean of the given death rates μ_M and μ_F, the weights being the reciprocals of the birth rates.

The corresponding ultimate sex ratio is m/f or

$$(4.5)\qquad \frac{m}{f} = \frac{\lambda_M - \frac{1}{2}(\mu_M - \mu_F)}{\lambda_F + \frac{1}{2}(\mu_M - \mu_F)},$$

whose numerator and denominator, respectively, are the averages of the numerator and denominator of the male and female dominant models.

Robert Traxler has gone on to solve (4.3) more completely. He divides the first equation by M_t and the second by F_t, subtracts the first equation from the second, substitutes $\log z$ for M_t/F_t, and so reduces the problem to a standard quadrature in z.

Our set (4.3) was reached by making dominance a function D_t of time. Fredrickson ([6], p. 121) and Pollard [26] reach the same equations by a different and instructive argument. They suggest the conditions that: (1) if linear equations are not possible, then let us at worst have equations that are homogeneous of degree one; (2) when either males or females are lacking, the births must be zero; and (3) when males are relatively plentiful the number of births must be proportional to the number of females, and *vice versa*. The quantity $M_tF_t/(M_t + F_t)$ tends simply to F_t when M_t is large:

$$(4.6)\qquad \lim_{M_t\to\infty} \frac{M_tF_t}{M_t + F_t} = \lim_{M_t\to\infty} \frac{F_t}{1 + \dfrac{F_t}{M_t}} = F_t.$$

A similar result holds when F_t becomes large and M_t remains finite.

The theoretical argument in favor of the harmonic mean birth function $2M_tF_t/(M_t + F_t)$ applies to marriage as well as to birth. But, as we will see in an empirical test below, the fluctuations in available males and females existing in real populations are not great enough to enable the harmonic mean to stand out over the arithmetic and geometric means.

5. An asymmetric birth function

A desirable degree of asymmetry is introduced by replacing $M_tF_t/(M_t + F_t)$ in (4.3) by $M_t^{1-\varepsilon}F_t^{1+\varepsilon}/(M_t + F_t)$. The homogeneity is retained, and setting ε greater than zero would make births depend more on fluctuations in the number of females than of males.

The pair of equations corresponding to (4.3), but with M_tF_t on the right replaced by $M_t^{1-\varepsilon}F_t^{1+\varepsilon}$, may again be studied by entering $M_t = me^{rt}$ and $F_t = fe^{rt}$. We have two homogeneous equations in m and f:

$$(5.1)\qquad \begin{aligned} mr &= -\mu_M m + 2\lambda_M \frac{m^{1-\varepsilon}f^{1+\varepsilon}}{m+f}, \\ fr &= -\mu_F f + 2\lambda_F \frac{m^{1-\varepsilon}f^{1+\varepsilon}}{m+f}, \end{aligned}$$

equivalent to two nonhomogeneous equations in r and the ultimate sex ratio $m/f = x$:

$$(5.2)\qquad \begin{aligned} r &= -\mu_M + 2\lambda_M \frac{x^{-\varepsilon}}{1+x}, \\ r &= -\mu_F + 2\lambda_F \frac{x^{1-\varepsilon}}{1+x}. \end{aligned}$$

First seeking x by equating the two right sides, we have

$$(5.3)\qquad x \equiv \frac{m}{f} = \frac{\lambda_M - \frac{1}{2}(\mu_M - \mu_F)x^{\varepsilon}}{\lambda_F + \frac{1}{2}(\mu_M - \mu_F)x^{\varepsilon}},$$

which expresses x in terms of the other quantities including x^{ε}. This can be used for iteration, starting with an arbitrary x, say 1, entered on the right side to start the process. With ε zero, this is the same as (4.5), and with ε small, convergence will be rapid.

Once x is known, r may be obtained from either member of the pair (5.2):

$$(5.4)\qquad r = 2\lambda_M \left(\frac{x^{-\varepsilon}}{1+x}\right) - \mu_M.$$

If x were unity, then r would be $\lambda_M - \mu_M$. Having x greater than unity and ε positive lowers the value of r. In words, if there are fewer women than men, then making births depend more on women than on men reduces the birth rate. This is hardly surprising, but it shows that the model makes sense in one important respect.

6. A simple marriage model

Permanent monogamous marriage modifies the two-sex problem by fixing the difference between ages of father and mother through any sequence of births to a particular couple. We can here, in preliminary fashion because age is omitted, follow Kendall ([15], p. 248) and Goodman ([7], p. 216) in setting down the conditions for marriage and reproduction with the two sexes. This section is confined to female marriage dominance, meaning that the number of marriages is proportional to the number of females. The equations for single females F_t and for married couples N_t are

$$\begin{aligned} F'_t &= -(\mu_F + \nu)F_t + (\lambda_F + \mu_M)N_t, \\ N'_t &= \nu F_t - (\mu_F + \mu_M)N_t, \end{aligned} \tag{6.1}$$

where ν is the fraction of females marrying per unit time, applied continuously to F_t. We suppose that all births occur to married couples, at rate λ_F for girl babies, and that death rates are the same for the married and single population, conditions that could be relaxed at the cost of a slight complication in the equations. The first of the equations (6.1) says that the number of single females (a) declines with female mortality and with marriage, and (b) increases with births and with the death of married males. (We are counting the widows as "single.") The equation for single males can be easily written down, but is omitted here because it cannot affect the trajectory of females in our female dominant model.

Unlike equations that we will meet later, the pair (6.1) is easily solved. The trajectory from any starting point F_0 and N_0 and any set of the Greek letter fixed parameters is a sum of two exponentials each multiplied by a constant. But to try to answer our questions without explicit solution will be more suggestive for the unsolvable models that follow. We may be interested in the ratio of married to single females N_t/F_t and especially its rate of change. This is given by

$$\frac{d\left(\frac{N_t}{F_t}\right)}{dt} = \frac{N'_t}{F_t} - \frac{N_t}{F_t}\frac{F'_t}{F_t}, \tag{6.2}$$

in which we may enter the derivatives from (6.1):

$$\frac{d\left(\frac{N_t}{F_t}\right)}{dt} = \nu + \frac{N_t}{F_t}(\nu - \mu_M) - \left(\frac{N_t}{F_t}\right)^2 (\lambda_F + \mu_M). \tag{6.3}$$

Equation (6.3) tells us that the ratio of married to single women goes up with ν, the marriage rate, and it goes down with increased births of females or deaths of males. These entirely reasonable results are evidence that we have not put into our equations (6.1) conditions that contradict common sense. It also follows from (6.3) that the proportion married is not affected by the female death rate, provided, as in this model, mortality is the same for single and married females.

7. Introduction of age in a continuous model

However, we cannot be satisfied with any model that fails to take account of age, since we are hoping to answer questions concerning the marriage market, which is clearly sorted out by age as well as sex. Sets of equations such as (3.1) or (6.1) can distinguish ages if we interpret F_t and other variables as column vectors of age distributions—say in five year age intervals, with the first item the number of persons aged zero to four at last birthday, the second element the numbers five to nine, and so forth. The Greek letter constants in (3.1) or (6.1) would now be interpreted as square matrices, with as many rows and columns as the number of ages recognized.

In fact there is good reason to object to matrix differential equations in this application. To take the same interval of time as of age offers a real advantage in population analysis, because then the whole group of one age moves into the next age interval in the time interval. If the time interval is shorter than the age interval only a part of the group moves on, and we face the complication of calculating what part. Thus, staying with discrete ages while making the time interval infinitesimal is to be avoided. We could easily write an analogue to any of our equations in discrete time; that for the first member of (6.1) would be given by replacing F'_t on the left by $F_{t+1} - F_t$. The Leslie [17] theory then applies in full detail.

Here we will proceed to infinitesimal intervals for both age and time, a device developed by von Foerster [5] in his work on cellular proliferation. In application to equations (2.1) and (2.2), F_t becomes a function of age and time, say $F_{a,t}$, and the female population of age $a + \Delta a$ at time $t + \Delta t$ is $F_{a+\Delta a, t+\Delta t}$. These latter include the same individuals as were counted in $F_{a,t}$, only a little older and subject to deductions for mortality (as well as for emigration if one wishes, but we shall confine ourselves here to populations closed to migration). The equation corresponding to (2.1) and (2.2) becomes

$$F_{a+\Delta a, t+\Delta t} = F_{a,t} - \mu_a F_{a,t} \Delta t. \tag{7.1}$$

Expanding on the left by Taylor's theorem for two independent variables and cancelling $F_{a,t}$ from both sides, we have

$$\frac{\partial F_{a,t}}{\partial t} \Delta t + \frac{\partial F_{a,t}}{\partial a} \Delta a = -\mu_a F_{a,t} \Delta t. \tag{7.2}$$

Dividing by Δa, which is supposed to be the same as Δt, and allowing $\Delta a = \Delta t$ to tend to zero, we have

$$\frac{\partial F_{a,t}}{\partial t} + \frac{\partial F_{a,t}}{\partial a} = -\mu_a F_{a,t}. \tag{7.3}$$

This is von Foerster's equation for one sex, for all values of $0 < a < \omega$, where ω is the oldest age to which anyone lives. The corresponding equation for males at age a' is

$$\frac{\partial M_{a',t}}{\partial t} + \frac{\partial M_{a',t}}{\partial a'} = -\mu_{a'} M_{a',t} \tag{7.4}$$

(Fredrickson [6]). Note that (7.3) and (7.4) are concerned with mortality only; $\mu_{a'}$ is our way of writing the male force of mortality. Deaths are uncoupled as usual; the numbers of males and their deaths are not taken as having any effect on the deaths of females.

Births enter as a boundary condition at age zero:

$$F_{0,t} = \int_{\alpha}^{\beta} F_{a,t}\lambda_{a,t}\, da, \tag{7.5}$$

where α is the youngest age of childbearing and β the oldest, and $\lambda_{a,t}$ is the age specific birth rate at time t. We will consider birth rates fixed in time and accordingly write $\lambda_{a,t}$ as λ_a. But this is unsatisfactory once again, because births really depend on males as well as on females, and our model must somehow take account of this.

To improve on the female model with fixed (female, male, or mixed) dominance we need a simultaneous birth function of the females aged a to $a + da$ *and* males aged a' to $a' + da'$. By an extension of the varying D_t incorporated in (4.3), we could use here an analogue to the harmonic mean for births

$$F_{0,t} = \frac{1}{F_t + M_t} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \lambda_{a,a'} F_{a,t} M_{a',t}\, da\, da', \tag{7.6}$$

where $F_t + M_t$ is the total population of both sexes and all ages. This substantially meets the requirements mentioned earlier; for example, if the male population of all ages becomes very large then the birth rate $\lambda_{a,a'}$ is multiplied by a number proportional to $F_{a,t}$. Only if some ages and not others became very large would (7.6) be unsatisfactory.

Since in any large population the ratio of boy to girl babies is nearly constant, we can take for the boundary condition on males,

$$M_{0,t} = sF_{0,t}, \tag{7.7}$$

s being the sex ratio at birth.

Finally, to start the system on its way we need initial age distributions for males and females:

$$\begin{aligned} F_{a,0} &= F_a, & 0 < a < \omega \\ M_{a',0} &= M_{a'}, & 0 < a' < \omega. \end{aligned} \tag{7.8}$$

To convert (7.3) to a homogeneous form, apply the substitution

$$F_{a,t} = \exp\left\{-\int_0^a \mu_b\, db\right\} G_{a,t}, \tag{7.9}$$

which results in

$$\frac{\partial G_{a,t}}{\partial a} + \frac{\partial G_{a,t}}{\partial t} = 0. \tag{7.10}$$

But any function of $t - a$, say $G_{a,t} = f(t - a)$, obviously satisfies this homogeneous equation. Hence, a general solution of (7.3) is

$$F_{a,t} = f_0 \exp\left\{-\int_0^a \mu_b\, db\right\} f(t - a), \tag{7.11}$$

where f_0 is an arbitrary constant, and $f(t - a)$ an arbitrary function.

To deal more easily with the boundary conditions, we will specialize $f(t - a)$ to an exponential, writing $f(t - a) = \exp\{(t - a)r\}$. Entering in (7.6) the value

$$F_{a,t} = f_0 \exp\{(t - a)r\} \exp\left\{-\int_0^a \mu_b \, db\right\} \tag{7.12}$$

and the corresponding function for $M_{a',t}$, we can solve for the asymptotic sex ratio and rate of increase as t becomes large.

We will make the usual assumption that the sex ratio at birth is the same for all ages of mothers and fathers, that is, in the present notation that $s = \lambda'_{a,a'}/\lambda_{a,a'}$, where $\lambda'_{a,a'}$ is defined as the rate of birth of boy babies to couples of which the mother is aged a and the father a'. For ease in writing let $\exp\left\{-\int_0^a \mu_b \, db\right\}$ be called ℓ_a as is usual in demographic work.

Then we have from (7.6) and the corresponding equation for $M_{0,t}$

$$\frac{M_{0,t}}{F_{0,t}} = \frac{\int_\alpha^\beta \int_\alpha^\beta \lambda'_{a,a'} \exp\{-r(a + a')\} \ell_a \ell_{a'} \, da \, da'}{\int_\alpha^\beta \int_\alpha^\beta \lambda_{a,a'} \exp\{-r(a + a')\} \ell_a \ell_{a'} \, da \, da'}, \tag{7.13}$$

and with the supposition $\lambda'_{a,a'} = s\lambda_{a,a'}$, the ratio of integrals reduces to s. The assumption of a fixed ratio of boy to girl babies at each age of parents evidently leads to a fixed sex ratio for total births at all times. The sex ratio in the population as a whole is obtained from (7.12) and the corresponding equation for males as

$$\frac{M_t}{F_t} = \frac{\int_0^\omega M_{a,t} \, da}{\int_0^\omega F_{a,t} \, da} = \frac{s \int_0^\omega e^{-ra} \ell'_a \, da}{\int_0^\omega e^{-ra} \ell_a \, da}. \tag{7.14}$$

The boundary condition (7.6) provides the intrinsic rate of increase of the system r. Entering the general solution for $F_{a,t}$ of (7.12) and the corresponding solution for $M_{a,t}$, (7.6) becomes

$$1 = \frac{\int_\alpha^\beta \int_\alpha^\beta \lambda_{a,a'} e^{-ra} \ell_a e^{-ra'} \ell'_{a'} \, da \, da'}{\int_0^\omega (e^{-ra} \ell_a + s e^{-ra} \ell'_a) \, da}, \tag{7.15}$$

on cancelling e^{rt} from both sides. (See Fredrickson [6], Equation (38).) This equation for r in the two sex model corresponds to Lotka's characteristic equation for one sex. It is solvable by iterative methods—for example, multiplying both sides by $e^{27.5r}$ and taking 1/27.5 of the logarithms gives an improved r on the left when an arbitrary r is inserted on the right. (The method is exhibited in detail for the characteristic equation of the one-sex model in Keyfitz [16], p. 108.) P. Das Gupta [2] has developed and applied in some detail a two-sex model similar to that of the present section.

The model consisting of equations (7.3), (7.4), (7.6), (7.7), and (7.8) can be shown to reduce to the Lotka integral equation, where one sex only, say females, is under consideration and the analysis is concentrated on female births $F_{0,t}$.

The discrete approach using matrices (Leslie [17]) and that of Goodman [7] reduce to the above partial differential equations when the intervals of time and age, always remaining equal to one another, tend to zero. The partial differential equations may be extended to an explicit incorporation of marriage, and to this we proceed.

8. Age and marriage in a continuous model

If husbands and wives were always of exactly the same age, the contradictions between the male and female uncoupled models could be easily overcome, and much of the problem with which we are here concerned would disappear. In fact husbands are of different ages from their wives, but not independently for the several children: for any given marriage the difference of ages remains always the same. We take advantage of this fact by distinguishing the married population from the single, and suppose that all births are to married couples (Fredrickson, [6]).

Now let $M_{a',t}\, da'$ be the number of single males of age a' to $a' + da'$, let $F_{a,t}\, da$ be single females of age a to $a + da$, and let $N_{a,a',t}\, da\, da'$ be the number of couples in which the age of the wife is a to $a + da$ and of the husband a' to $a' + da'$. The partial differential equation for unmarried females that corresponds to (7.3) is

$$\text{(8.1)}\quad \frac{\partial F_{a,t}}{\partial t} + \frac{\partial F_{a,t}}{\partial a} = -\mu_a F_{a,t} - \int_0^\omega \nu_{a,a'} f(M_{a',t}, F_{a,t})\, da' + \int_0^\omega \mu_{a'} N_{a,a',t}\, da' + \frac{1}{1+s} \int_0^\omega \lambda_{a,a'} N_{a,a',t}\, da',$$

where μ_a is the death rate of females of age a; $\nu_{a,a'}$ is the marriage rate between girls aged a and men aged a'; $\mu_{a'}$ is the death rate of males of age a'; $\lambda_{a,a'}$ is the rate of childbearing at time t of couples of which the wife is aged a and the husband aged a'; and f is a function whose nature will be discussed below. Of the four terms shown on the right side of (8.1) the first allows for the deaths of unmarried females, the second for marriages of women to men of all ages, the third for the deaths of married males, each of which releases one female into the unmarried group, and the fourth for childbearing among married couples.

The last term of (8.1) makes births perfectly straightforward in the two sex model. Admittedly its $\lambda_{a,a'}$ demands a kind of data that is not commonly produced, namely births by age of father and of mother, along with the number of married couples in the population by age of husband and of wife. These could easily be tabulated from existing censuses and vital statistics, and such tabulations are available for a few countries. The two items of data, on births and population, provide a discrete version of $\lambda_{a,a'}$; birth rates specific for age of father and of mother, say in five year age intervals, would be obtained by dividing births by population in each class. If seven age intervals are recognized for women, and nine for males, this means 63 rates altogether, not an impossible number to handle.

9. Varying marriage dominance

The second term on the right of (8.1), to allow for the women who leave the single state for marriage, now inherits the difficulties that appeared for births in models not recognizing marriage. All the possibilities in the function $f(M_{a',t}F_{a,t})$ that were previously open for births are now possibilities for marriage, and each brings with it the earlier disadvantages. To use female dominance would cause marriages to take place in the model even in the absence of males, and with male dominance in the absence of females. Mixed dominance in any fixed ratio would entail at least part of the same drawback.

The only escape is to allow the degree of dominance to vary according to the availabilities of individuals of the two sexes at the ages in question. The analogue for marriage and age to (4.3) would give for the second term on the right side of (8.1)

$$(9.1) \qquad -\int_0^\omega \nu_{a,a'} \frac{2M_{a',t}F_{a,t}}{M_{a',t}+F_{a,t}}\, da'.$$

Yet (9.1) does not embrace the full complexity of the problem. For marriages between women of age a and men of age a' evidently depend on much more than $F_{a,t}$ and $M_{a',t}$. They depend also on the numbers of individuals at other ages. If for example the number of women at ages $a-1$ and $a+1$ is increased, all other circumstances remaining the same, then the number of marriages between women aged a and men aged a' will be reduced. To take this into account we would have to make the second term on the right of (8.1) depend on other ages of women, say represented by x, so it would become

$$(9.2) \qquad -\int_0^\omega \int_0^\omega \nu_{a,x,a'} f(M_{a',t}, F_{a,t}, F_{x,t})\, dx\, da',$$

where the function f would have to be specified. Equally, it could be argued that other ages of men should be taken into account.

10. Measuring the marriage rate

Until we know the form of the marriage function, we have no denominator constituting exposures that will permit calculation of a marriage rate. With a suitable marriage function, we can calculate marriage rates just as we calculate death rates. An example will suffice to present the problem and the proposed solution.

Let us try to decide whether marriage among single persons 20 to 24 years of age went up or down between 1963 and 1967 in the registration area of the United States. The facts are that 207,211 first marriages at those ages were reported for 1963, and 315,650 first marriages for 1967, in official vital statistics. In 1963, the estimated number of single males 20 to 24 was 2,355,000, and in 1967 it was 3,379,000, so in relation to single males the marriage rate was 0.0880 in 1963 and 0.0934 in 1967, a six per cent increase. But relating the marriages to the number of single women in the two years (1,339,000 and 2,183,000, respectively) produces the ratios 0.1547 for 1963 and 0.1446 for 1967, a seven per cent decline. Did the marriage rate go up or down?

Clearly, the best denominator for the rate is some kind of average of the single men and women. If we take the geometric mean the rate declines from 0.1167 to 0.1162 or 0.4 per cent. We will see, however, that such a symmetric treatment of the men and women exposed is not quite appropriate.

11. The marriage coefficients—response of preferences to availabilities

Running through all work on this subject is an implicit juxtaposition between the ages of mates preferred by young people and the demographic availabilities of persons of those ages. (Griffith Feeney [3], Henry [10], Hirschman and Matras [12], Hoem [13], and others have elaborated this point.) It is this juxtaposition that must somehow be incorporated in the marriage function. We want a function that will apply in the face of considerable departures from the availabilities that are "normal," that is, from those pertaining to a population of fixed marriage, birth and death rates and no migration.

A substantial departure from normal availabilities occurs in the wake of the postwar baby boom. United States births in 1947 numbered 3,817,000 (*Historical Statistics*, p. 22). About twenty years later the girls of this cohort reach marrying age, and they would ordinarily marry men about two years older than themselves, which is to say men born about 1945. But the births of 1945 numbered only 2,858,000, about one quarter fewer than those of 1947. This is what Paul Glick [24] has termed the marriage squeeze, and it is occurring in most western countries. When the births decline, on the other side of the baby boom, the squeeze will be in the opposite direction—the shortage will be of girls. This has already come about in Japan, and will gradually appear in the United States towards the end of the 1970's. Insofar as the fall of births from 1957 to 1968 was more gradual than the rise from 1945 to 1947 the squeeze of the mid-1980's will be less spectacular than that of the early 1970's.

Having now brought the presentation of the problem to its maximum of difficulty, I shall show how it can be simplified again, and how data may be made to bear on at least some aspects. What is needed is a way of using the observation of marriages in successive years, along with the known availabilities by age, to tell what function of the ages of available men and women determines the unions that take place. If the availabilities were always the same, we could not make the inference I propose, and under this condition of stability the problem would not arise. It is the changing availabilities that both give rise to the problem of this paper and provide the data for its solution.

12. Forcing the data to decide among marriage functions

The first confrontation with data in effect asks the observed marriages to discriminate among the five marriage functions:

$$N_{i,j,t} = \nu_{i,j}F_{i,t} \quad \text{(female dominance)}, \tag{12.1}$$

$$N_{i,j,t} = \nu_{i,j}M_{j,t} \quad \text{(male dominance)}, \tag{12.2}$$

$$N_{i,j,t} = \nu_{i,j}(0.5F_{i,t} + 0.5M_{j,t}) \quad \text{(arithmetic mean)}, \tag{12.3}$$

$$N_{i,j,t} = \nu_{i,j}(F_{i,t}^{0.5}\, M_{j,t}^{0.5}) \quad \text{(geometric mean)}, \tag{12.4}$$

$$N_{i,j,t} = 2\nu_{i,j}\frac{F_{i,t}M_{j,t}}{F_{i,t} + M_{j,t}} \quad \text{(harmonic mean)}, \tag{12.5}$$

and to tell us which fits the observed marriages best.

We should be able to decide which of the functions (12.1) to (12.5) above is appropriate once we know marriages by age of bride and groom for two dates t and t', and the numbers of males and females exposed to the risk of marriage at the two dates. If the marriage function to be used is $f(F_{i,t}, M_{j,t})$, where f stands for any of (12.1) to (12.5) or some other function altogether, and if f contains one constant $\nu_{i,j}$ for each combination of ages as do (12.1) to (12.5), then the constant can be evaluated for time t and the f so completely specified can be applied to time t'. Suppose that the resulting estimate of marriages between brides aged i and grooms aged j at time t' is

$$\hat{N}_{i,j,t'} = f(F_{i,t'}, M_{j,t'}). \tag{12.6}$$

Then the difference $d_{i,j,t'}$ between this estimate $\hat{N}_{i,j,t'}$ and the observation $N_{i,j,t'}$,

$$d_{i,j,t'} = N_{i,j,t'} - \hat{N}_{i,j,t'}, \tag{12.7}$$

averaged somehow over the ages of women i and of men j, is a measure of the appropriateness of the marriage function f.

Three kinds of average of (12.7) will be used—its root mean square, mean $3/2$ power, and mean absolute value, shown as the columns of Table I. The mean square gives relatively more weight to the large deviations, and the mean absolute value more weight to the smaller ones. We will see that the three measures do not always report in quite the same way on a given set of data.

The work will be confined to first marriages, partly because the exposure to risk is more clearcut for these. If divorce is easy the whole of the married population is at risk of further marriage, but the degree of risk varies greatly among individuals and we have no way of establishing a cutoff. For first marriages the single population is the obvious measure of exposure, even though some members may be immune and others highly susceptible.

TABLE I

MEASURE OF DEPARTURE FROM OBSERVATIONS OF ESTIMATES OF MARRIAGES BASED ON FIVE MARRIAGE FUNCTIONS, DATA FOR UNITED STATES 1963 AND 1967, AND SWEDEN 1959 AND 1963

n is 9 for the U.S. and 36 for Sweden.

	$[\sum (d^2/n)]^{1/2}$	$[\sum \lvert d\rvert^{3/2}/n]^{2/3}$	$\sum \lvert d\rvert/n$
United States 1963 and 1967			
(12.1) Female dominance	16,256	14,465	12,079
(12.2) Male dominance	18,860	16,498	13,642
(12.3) Arithmetic mean	14,316	11,896	9,069
(12.4) Geometric mean	14,678	11,874	8,608
(12.5) Harmonic mean	15,320	12,421	9,195
Sweden 1959 and 1963			
(12.1) Female dominance	309.0	227.9	149.1
(12.2) Male dominance	375.9	266.4	160.9
(12.3) Arithmetic mean	291.2	201.0	120.7
(12.4) Geometric mean	297.6	202.6	115.1
(12.5) Harmonic mean	309.8	214.4	125.9

13. Measuring departures of observed from expected marriages

The first set of data on which the several marriage functions are to be tested is for the registration area of the United States, 1963 and 1967, using age groups 15–19, 20–24, and 25–44 at last birthday. (The official publication does not show any finer breakdown of ages of the single under 45.) The second set of data for Sweden in 1959 and 1963, in which six age groups (15–19, 20–24, 25–29, 30–34, 35–39, and 40–44) are recognized, was provided to me by David McFarland [23].

The measures of departure of Table I correspond to the two sets of data and three criteria of fit ranging from mean square to mean absolute value. They show the one-sex models (12.1) and (12.2) to be generally inferior to the various averages of the two sexes (12.3), (12.4), and (12.5). The sequence is essentially the same for the three measures of departure, and for the United States and Sweden, with the arithmetic and geometric means tied for first place.

A surprising feature of the outcome is that the theoretical merits of the harmonic mean over the arithmetic and geometric do not assert themselves. Our theoretical argument revolved around ensuring that when one sex was altogether lacking there would be no marriages, and when one sex was plentiful marriages would be proportional to the other. The data apparently vary too little from the stable case to discriminate on the basis of what happens when the number of one sex goes towards zero or infinity.

14. Asymmetry

The second question we will put to the data concerns the degree of dominance: whether males or females are more important for marriage. Merely to illustrate the logic that will be employed in this argument, suppose that we have statistics for available men and women, along with the year's marriages, in three situations that are in all other respects identical, as in Table II.

TABLE II

HYPOTHETICAL MARRIAGES WITH DIFFERING NUMBERS OF SINGLE MEN AND SINGLE WOMEN

	Single men	Single women	Year's marriages
(a)	100,000	100,000	10,000
(b)	120,000	100,000	11,000
(c)	100,000	120,000	11,500

Apparently the extra 20,000 males in (b) increase marriages by 1000, while the extra 20,000 females in (c) increase them by 1500. Various mechanisms are imaginable; we will not attempt to discriminate among them.

Instead, we empirically investigate whether fluctuations in the number of females affect marriages more or less than do equal fluctuations in the number of males. To test for such asymmetry requires some kind of weighting of the numbers of males and females, and the easiest way to weight is by modifying the arithmetic or geometric mean. Weighting the females with $0.5 + \varepsilon$ and the males with $0.5 - \varepsilon$ alters the arithmetic mean to

$$\hat{N}_{i,j,t} = \nu_{i,j}[(0.5 + \varepsilon)F_{i,t} + (0.5 - \varepsilon)M_{j,t}], \tag{14.1}$$

and the geometric mean to

$$\hat{N}_{i,j,t} = \nu_{i,j}F_{i,t}^{0.5+\varepsilon}M_{j,t}^{0.5-\varepsilon}. \tag{14.2}$$

If it turns out that $\varepsilon > 0$, we will consider that females are more determining, if $\varepsilon < 0$, that males are more determining.

From another point of view, the estimates of nuptiality for the more recent date can be regarded as a projection of the nuptiality at the earlier date, using as an index the marriage function based on the available males and females at the two dates:

$$\hat{N}_{i,j,t'} = N_{i,j,t}\frac{f(F_{i,t'}, M_{j,t'})}{f(F_{i,t}, M_{j,t})}. \tag{14.3}$$

With the weighted arithmetic mean, for example, the departure for the United States in 1967 would become

$$\begin{aligned} d_{i,j}^{1967} &= N_{i,j}^{1967} - \hat{N}_{i,j}^{1967} \\ &= N_{i,j}^{1967} - N_{i,j}^{1963}\left[\frac{(0.5+\varepsilon)F_i^{1967} + (0.5-\varepsilon)M_j^{1967}}{(0.5+\varepsilon)F_i^{1963} + (0.5-\varepsilon)M_j^{1963}}\right]. \end{aligned} \tag{14.4}$$

Table III shows with the United States data the three kinds of average of the $d_{i,j}$, for values of ε from -0.5 to 0.5 at intervals of 0.1, and an enlargement of the arithmetic mean formula for the interval 0.1 to 0.2. On the whole, ε is clearly

TABLE III

DEPARTURE OF OBSERVED FROM CALCULATED FIRST MARRIAGES
The calculated first marriages are based on arithmetic and geometric weighted means, with weights ε from -0.5 to $+0.5$ and three measures of departure ranging from mean square to mean absolute value, United States 1963 and 1967 in three age groups.
Asterisks indicate minimum points.

Values of ε	$[\sum d^2/9]^{1/2}$	$[\sum \|d\|^{3/2}/9]^{2/3}$	$\sum \|d\|/9$
Arithmetic mean			
−0.5	18860	16498	13642
−0.4	17680	15414	12673
−0.3	16638	14416	11744
−0.2	15719	13489	10899
−0.1	14933	12631	10012
0	14316	11896	9069
0.1	13923	11506*	8805*
0.2	13830*	11584	9087
0.3	14127	12187	9997
0.4	14907	13149	10999
0.5	16256	14465	12079
Geometric mean			
−0.5	18860	16498	13642
−0.4	17745	15381	12607
−0.3	16736	14314	11554
−0.2	15865	13323	10484
−0.1	15166	12458	9397
0	14678	11874	8608*
0.1	14436*	11873*	9009
0.2	14468	12242	9749
0.3	14785	12822	10507
0.4	15387	13572	11283
0.5	16256	14465	12079
Enlargement for arithmetic mean			
0.1	13923	11506	8805
0.11	13899	11484	8778
0.12	13878	11466	8750
0.13	13860	11452	8722
0.14	13845	11444	8694
0.15	13834	11442*	8666*
0.16	13826	11458	8748
0.17	13821*	11482	8832
0.18	13821*	11511	8916
0.19	13823	11545	9001
0.2	13830	11584	9087

positive, showing a tendency to female dominance. The data are very skimpy—we can hardly expect to learn much from only three age groups.

The somewhat more detailed ages for Sweden resulted in Table IV, showing ε at 0.13 for the mean square and at 0.10 and 0.11 for the other two averages of the $d_{i,j}$, all based on the arithmetic mean. The geometric mean is less consistent, with ε ranging from 0.21 down to 0.10, depending on which average of the $d_{i,j}$ is taken. In both parts of Table IV, however, the ε is positive for best fit, and

TABLE IV

DEPARTURES OF ACTUAL FROM CALCULATED FIRST MARRIAGES
The calculated first marriages are based on arithmetic and geometric weighted means, Sweden 1959 and 1963 in six age groups.
Asterisks indicate minimum points.

Values of ε	$[\sum d^2/36]^{1/2}$	$[\sum \|d\|^{3/2}/36]^{2/3}$	$\sum \|d\|/36$
Arithmetic mean			
0.09	288.36	198.83	118.97
0.10	288.24	198.79*	118.76
0.11	288.16	198.82	118.54*
0.12	288.11	199.01	118.97
0.13	288.10*	199.31	119.69
0.14	288.12	199.68	120.44
Geometric mean			
0.09	291.30	197.64	113.16
0.10	290.81	197.28	113.05*
0.11	290.36	196.97	113.10
0.12	289.96	196.70	113.16
0.13	289.60	196.48	113.21
0.14	289.28	196.30	113.26
0.15	289.00	196.17	113.31
0.16	288.77	196.09	113.37
0.17	288.59	196.06*	113.43
0.18	288.45	196.10	113.48
0.19	288.36	196.22	113.54
0.20	288.31	196.54	114.27
0.21	288.31*	197.01	115.58
0.22	288.35	197.56	116.50

this is the most solid evidence so far attained of the tendency to female marriage dominance.

One would like to see clearer minima than are exhibited in Tables III and IV. Presumably the curves showing departure of observed from expected as functions of ε would rise more sharply on either side of the minimum if the disequilibrium of the numbers single of the two sexes was more extreme, a circumstance that may show itself for the years about 1970. A more sharply defined minimum would also appear if more age groups were used. One confirmation that an ε on the female dominant side at least has the right sign comes from English data of 1891 and 1961 (P. R. Cox [1]). Cox finds a higher correlation between marriages and the supply of nonmarried women than between marriages and the supply of nonmarried men, which confirms the tendency to female dominance.

This presentation cannot but end on the need for further testing. Until the same calculation is made on a variety of times and places, no one can be certain that it is the inequality of available males and females that results in the asymmetry of the marriage function found here. The same marriage function need not apply to the two years being compared if circumstances have changed. One is tempted to complicate the marriage function by introducing into it other variables than the single males and single females present, by analogy to proceeding from zero order correlation to partial correlation in the familiar linear model. Unfortunately, in our problem it is not obvious what variables ought to be partialled out to obtain the pure effect of availabilities of single men and single women. Failing some knowledge or at least suspicion of possible extraneous variables, we can only seek further data of the kind used here. Insofar as the disturbing variables are unspecifiable and all disturbances have to be regarded merely as noise, the recourse is to perform the same calculation on the most varied populations for which data are to be had.

15. A least square estimate for ε

Richard Cohen suggests a way of reducing our problem of estimating ε to classical least squares. He linearizes (14.2) by taking logarithms, and finds

$$\log \hat{N}_{i,j,t} = \log \nu_{i,j} + (0.5 + \varepsilon) \log F_{i,t} + (0.5 - \varepsilon) \log M_{j,t}. \tag{15.1}$$

The problem is now to find the ε that minimizes

$$\sum [-\log \hat{N}_{i,j,t} + \log \nu_{i,j} + (0.5 + \varepsilon) \log F_{i,t} + (0.5 - \varepsilon) \log M_{j,t}]^2. \tag{15.2}$$

With the United States data of 1963 and 1967, Cohen obtains $\varepsilon = 0.1035$.

16. Conclusion

While we would like to experiment with more countries and more years, and certainly wish that more ages could be recognized than three for the United States and six for Sweden, yet the present materials strongly suggest that the degree of female dominance is $\varepsilon = 0.1$ or more. This means that number of marriages is some constant times $F_i^{0.6}M_j^{0.4}$, where F_i is the number of females aged i and M_j the number of males aged j at any time.

The 0.6 and 0.4 will be recognized as elasticities of marriages for females and males, respectively. If single females increase by one per cent marriages go up by 0.6 per cent. If single males increase by one per cent marriages go up by 0.4 per cent. Pending further data, we conclude that at the margin marriages depend at least 60 per cent on the number of women and at most 40 per cent on the number of men. Should the nonlinearity of such a geometric function prove awkward, an almost equally good fit can be obtained from the arithmetic and linear $\nu_{i,j}\,(0.6F_i + 0.4M_j)$.

Once the marriage function $\nu_{i,j}(F_i^{0.6}M_j^{0.4})$ (or some other) is ascertained, then given childbearing rates by age of father and of mother, and given mortality specific by age for single and married persons of each sex, we are at last in a position to make statements on the two-sex model corresponding to any that are possible on the one-sex model. An authentic projection can be made into the future that generalizes the standard one-sex projection. Over each finite time period, we can calculate age by age and for each sex the number of marriages. Recognizing the single and the never married, we apply life tables to find the number of survivors, and fertility tables for the number of births. If this applies over any particular time interval, it applies over all together, and so provides the asymptotic ultimate values.

The projection in its turn permits examination of the effect of change in any specific rate on the ultimate rate of increase and age distribution. The effects would probably not be very different from those given by the simpler one-sex model for any question that can be asked both of the one-sex model and the age-sex-marriage model. However, the latter permits altogether new questions—effects of changes in age specific marriage rates, for example. It also could show the effect of constantly changing proportions of the two sexes, including, for example, how the growth of a population would be affected if its birth rate oscillated about a given mean, so that it experienced an endless succession of marriage squeezes.

Among other benefits of a marriage function, it tells how to calculate the marriage rate. To answer the question asked earlier, whether the marriage rate for persons 20–24 in the United States went up or down between 1963 and 1967, we would note that in relation to the weighted geometric mean $F_i^{0.6}M_j^{0.4}$, the rate was

$$\frac{207{,}211}{(1{,}339{,}000)^{0.6}(2{,}355{,}000)^{0.4}} = 0.1235 \tag{16.1}$$

in 1963 and

$$\frac{315{,}650}{(2{,}183{,}000)^{0.6}(3{,}379{,}000)^{0.4}} = 0.1214 \tag{16.2}$$

in 1967. The last word based on this argument is that the rate went down from 0.1235 to 0.1214, a decline of 1.7 per cent.

Where we cannot count on the relative permanence of marriage, or where a large fraction of births are illegitimate, we will have to drop marriage from the model and use birth rates to the whole population by age of father and of mother. In the resulting simpler and cruder two-sex model, a birth function would take the place of the marriage function, and one could still perform the projection and other inferences. Presumably the birth function would be asymmetric; in the extreme case of promiscuous mating it would be weighted heavily towards the female side.

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