ON THE SUPPORT OF DIFFUSION PROCESSES WITH APPLICATIONS TO THE STRONG MAXIMUM PRINCIPLE

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1. Introduction

Let $a\colon [0, \infty) \times R^d \twoheadrightarrow S_d$ and $b\colon [0, \infty) \times R^d \twoheadrightarrow R^d$ be bounded continuous functions, where S_d denotes the class of symmetric, nonnegative definite $d \times d$ matrices. From a and b form the operator

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i}. \tag{1.1}$$

A strong maximal principle for the operator $(\partial/\partial t) + L_t$ is a statement of the form: "for each open $\mathscr{G} \subseteqq [0, \infty) \times R^d$ and each $(t_0, x_0) \in \mathscr{G}$ there is a set $\mathscr{G}(t_0, x_0) \subseteqq \mathscr{G}$ with the property that $(\partial f/\partial t) + L_t f \geqq 0$ on $\mathscr{G}(t_0, x_0)$ and $f(t_0, x_0) = \sup_{\mathscr{G}(t_0,x_0)} f(t, x)$ imply $f \equiv f(t_0, x_0)$ on $\mathscr{G}(t_0, x_0)$." Of course, in order for a strong maximum principle to be very interesting it must describe the set $\mathscr{G}(t_0, x_0)$. Further, it should be possible to show that $\mathscr{G}(t_0, x_0)$ is maximal. That is, one wants to know that if $(t_1, x_1) \in \mathscr{G} - \mathscr{G}(t_0, x_0)$, then there is an f satisfying $(\partial f/\partial t) + L_t f \geqq 0$ on $\mathscr{G}$ (perhaps in a generalized sense) such that $f(t_0, x_0) = \sup f(t, x)$, and $f(t_1, x_1) < f(t_0, x_0)$.

In the case when $a(t, x)$ is positive definite for all (t, x), L. Nirenberg [6] has shown that $\mathscr{G}(t_0, x_0)$ can be taken as the closure in $\mathscr{G}$ of the set of $(t_1, x_1) \in \mathscr{G} \cap ([t_0, \infty) \times R^d)$ such that there exists a continuous map $\phi\colon [t_0, t_1] \twoheadrightarrow R^d$ with the properties that $\phi(t_0) = x_0$, $\phi(t_1) = x_1$, and $(t, \phi(t)) \in \mathscr{G}$ for all $t \in (t_0, t_1)$. We will give a probabilistic proof of the Nirenberg maximum principle in Section 3. Moreover, we will also prove there that Nirenberg's $\mathscr{G}(t_0, x_0)$ is maximal in the desired sense.

If a is only nonnegative definite, the problem of finding a suitable maximum principle is more difficult. Results in this direction have been proved by J.-M. Bony [1] and C. D. Hill [3]. Both of these authors employ a modification of the technique originally introduced by E. Hopf for elliptic operators and later adapted by Nirenberg for parabolic ones. The major drawback to Bony's

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work is his restrictive assumptions on the coefficients of L_t. As we will point out later, our own approach has not removed all his restrictions.

Before going into the details, we will conclude this section with an outline of our method. Suppose P_{t_0,x_0} is a probability measure on $C([t_0, \infty), R^d)$ with the properties that $P_{t_0,x_0}(x(t_0) = x_0) = 1$ and

$$(1.2)\qquad \int_{t_0}^{t} \left(\frac{\partial}{\partial u} + L_u\right) f(u, x(u))\, du$$

is a martingale for all $f \in C_0^\infty([t_0, \infty) \times R^d)$ (here and in what follows $C_0^\infty(S)$ denotes the space of infinitely differentiable functions having compact support in S). Given an open $\mathscr{G} \subseteqq [0, \infty) \times R^d$, let $\tau = \inf\{t \geqq t_0 : (t_0, x(t)) \notin \mathscr{G}\}$. Then it is easy to see that $f(t \wedge \tau, x(t \wedge \tau))$ is a submartingale if $f \in C_b^{1,2}(\mathscr{G})$ and $(\partial f/\partial t) + L_t \geqq 0$. (We use $C_b^{1,2}(S)$ to denote the class of bounded f having one bounded continuous t derivative and two bounded continuous x derivatives on S.) Hence,

$$(1.3)\qquad f(t_0, x_0) \leqq E^{P_{t_0,x_0}}[f(t \wedge \tau, x(t \wedge \tau))].$$

In particular, if $f(t_0, x_0) = \sup_{\mathscr{G}} f(t, x)$, then $f(t_1, x_1) = f(t_0, x_0)$ at all (t_1, x_1) for which there exists a path $\phi \in \operatorname{supp}(P_{t_0,x_0})$ such that $\phi(t_1) = x_1$ and $(t, \phi(t)) \in \mathscr{G}$ for $t \in [t_0, t_1]$. Thus, for example, if

$$(1.4)\qquad \operatorname{supp}(P_{t_0,x_0}) = \{\phi \in C([0, \infty), R^d) : \phi(t_0) = x_0\},$$

then $(\partial/\partial t) + L_t$ satisfies the Nirenberg maximum principle. What we are going to do is study the measure P_{t_0,x_0} and try to describe its support.

2. Background material

In this section, we discuss diffusions from the point of view introduced in [9]. Our notation throughout is the same as it was in that paper. Namely,

$$(2.1)\qquad \begin{aligned} &\Omega = C([0, \infty), R^d),\\ &x(t, \omega) = x_t(\omega) \text{ is the position of } \omega \text{ at time } t,\\ &\mathscr{M}_t^s = \mathscr{B}[x_u : s \leqq u \leqq t], \qquad \mathscr{M}^s = \mathscr{B}[x_u : u \geqq s]. \end{aligned}$$

In order to discuss the weak convergence of measures on $\langle \Omega, \mathscr{M}^s \rangle$, we will sometimes think of a measure on $\langle \Omega, \mathscr{M}^s \rangle$ as defined on $C([s, \infty), R^d)$. A useful criterion for the relative compactness of a set Γ of probability measures P on $\langle \Omega, \mathscr{M}^s \rangle$ is the following:

$$(2.2)\qquad \lim_{R \to \infty} \sup_{P \in \Gamma} P(|x(s)| \geqq R) = 0,$$

$$(2.3)\qquad \sup_{P \in \Gamma} E^P[|x(t_2) - x(t_1)|^4] \leqq C_T (t_2 - t_1)^2, \quad s \leqq t_1 \leqq t_2 \leqq T, \qquad T > 0.$$

A proof of this fact may be found in [7].

A function η on $[s, \infty) \times \Omega$ into a measurable space is said to be s *nonanticipating* if η is $\mathscr{B}_{[s,\infty]} \times \mathscr{M}^s$ measurable and $\eta(t)$ is $\mathscr{M}_t^s$ measurable for each $t \geqq s$. If P is a probability measure on $\langle \Omega, \mathscr{M}^s \rangle$, then η is a *P martingale* if η is complex valued, s nonanticipating, and

$$E^P[\eta(t_2) \mid \mathscr{M}_{t_1}^s] = \eta(t_1) \qquad \text{a.s. } P \tag{2.4}$$

for $s \leqq t_1 < t_2$.

Let $a\colon [0, \infty) \times R^d \twoheadrightarrow S_d$ and $b\colon [0, \infty) \times R^d \twoheadrightarrow R^d$ be bounded and measurable. Define

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) + \sum_{i=1}^{d} b_i(t, x). \tag{2.5}$$

A probability measure P on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ is said to solve the *martingale problem for L_t starting at (t_0, x_0) if $P(x(t_0) = x_0) = 1$ and $f(x(t)) - \int_{t_0}^t L_s f(x(u))\, du$* is a P martingale for all $f \in C_0^\infty(R^d)$. In [9] it was shown that if a is continuous and $a(t, x)$ is positive definite for all (t, x), then there is exactly one solution $P_{t,x}$ to the martingale problem for L_t starting at (t, x). Moreover, we proved there that the family $\{P_{t,x}\colon (t, x) \in [0, \infty) \times R^d\}$ forms a strong Feller, strong Markov process. The purpose of the present section is to extend this result to the case when a and b are smooth in x and $a(t, x)$ is only nonnegative definite. The idea is to reduce the martingale problem to a stochastic differential equation which can be solved by the techniques introduced by K. Itô in [4]. For purposes of easy reference, we state here the following theorem, whose proof may be found in [10].

THEOREM 2.1. *Let $a\colon [t_0, \infty) \times \Omega \twoheadrightarrow S_d$ and $b\colon [t_0, \infty) \times \Omega \twoheadrightarrow R^d$ be bounded t_0 nonanticipating functions and define*

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t) \frac{\partial}{\partial x_i}. \tag{2.6}$$

Suppose $\alpha\colon [t_0, \infty) \times \Omega \twoheadrightarrow R^d$ is a continuous (in time) t_0 nonanticipating function and that P is a probability measure on $\langle \Omega, \mathscr{M}^{t_0} \rangle$. Then the following are equivalent:

(i) *$f(\alpha(t)) - \int_{t_0}^t L_u f(\alpha(u))\, du$ is a P martingale for all $f \in C_0^\infty(R^d)$;*

(ii) *$f(t, \alpha(t)) - \int_{t_0}^t [(\partial/\partial u) + L_u] f(u, \alpha(u))\, du$ is a P martingale for all $f \in C_b^{1,2}([t_0, \infty) \times R^d)$;*

(iii) *$\exp\{\langle \theta, \alpha(t) - \alpha(t_0) - \int_{t_0}^t b(u)\, du \rangle - \frac{1}{2} \int_{t_0}^t \langle \theta, a(u)\, \theta \rangle\, du\}$ is a P martingale for all $\theta \in R^d$.*

Moreover, if P satisfies one of these and if $\bar{\alpha}(t) - \int_0^t b(u)\, du$, then $d\bar{\alpha}(t)$ stochastic integrals $\int_{t_0}^t \langle \theta(u), d\bar{\alpha}(u) \rangle$ can be defined when the integrand $\theta\colon [t_0, \infty) \times \Omega \twoheadrightarrow R^d$ is t_0 nonanticipating and satisfies

$$E^P\left[\int_{t_0}^t \langle \theta(u), a(u)\theta(u) \rangle\, du\right] < \infty, \qquad t \geqq t_0. \tag{2.7}$$

The process $\int_{t_0}^t \langle\theta(u), d\bar{\alpha}(u)\rangle$ *is a continuous P martingale; and if* $\langle\theta(u), a(u)\theta(u)\rangle$ *is bounded, then*

$$\exp\left\{\int_{t_0}^t \langle\theta(u), d\bar{\alpha}(u)\rangle - \frac{1}{2}\int_{t_0}^t \langle\theta(u), a(u)\theta(u)\rangle\, du\right\} \tag{2.8}$$

is a P martingale. Finally, for $f \in C_b^{1,2}([t_0, \infty) \times R^d)$, *one has*

$$f(t, \alpha(t)) - f(t_0, \alpha(t_0)) = \int_{t_0}^t \langle\nabla_x f(u, \alpha(u)), d\bar{\alpha}(u)\rangle + \int_{t_0}^t \left(\frac{\partial}{\partial u} + L_u\right) f(u, \alpha(u))\, du, \tag{2.9}$$

where $\nabla_x f(t, x) = [(\partial f/\partial x_1)(t, x), \cdots, (\partial f/\partial x_d)(t, x)]$.

In [9], we showed that if the a in Theorem 2.1 is uniformly positive definite, then $\beta(t) = \int_{t_0}^t a^{-1/2}(u)\, d\bar{\alpha}(u)$ is a P Brownian motion (that is, $P(\beta(t_0) = 0) = 1$ and $\exp\{\langle\theta, \beta(t)\rangle - \frac{1}{2}|\theta|^2(t - t_0)\}$ is a P martingale for all $\theta \in R^d$), where $a^{1/2}$ is the positive definite, symmetric square root of a. Hence,

$$\alpha(t) - \alpha(t_0) = \int_{t_0}^t a^{1/2}(u)\, d\beta(u) + \int_{t_0}^t b(u)\, du \quad \text{a.s. } P \tag{2.10}$$

for some P Brownian motion β. We will now extend this result to nonnegative definite a. As we will see, this entails enlarging the sample space.

THEOREM 2.2. *Let a, b, and* α *be as in Theorem* 2.1 *and suppose P is a probability measure of* $\langle\Omega, \mathscr{M}^{t_0}\rangle$ *satisfying one of the conditions* (i), (ii), *or* (iii) *of that theorem. Assume* $\sigma\colon [t_0, \infty) \times \Omega \twoheadrightarrow R^d \times R^d$ *is* t_0 *nonanticipating and satisfies* $a(u) = \sigma(u)\sigma^*(u)$. *Then there is an extension* $\hat{P}$ *of P to* $\langle\Omega \times \Omega, \mathscr{M}^{t_0} \times \mathscr{M}^{t_0}\rangle$ *and a* $\hat{P}$ *Brownian motion* $\hat{\beta}$ *such that*

$$\alpha(t) - \alpha(t_0) = \int_{t_0}^t \sigma(u)\, d\hat{\beta}(u) + \int_{t_0}^t b(u)\, du \quad \text{a.s. } \hat{P}. \tag{2.11}$$

PROOF. It suffices to treat the case when $\sigma = a^{1/2}$. Indeed, if $a = \sigma\sigma^*$ and $U = (a + \varepsilon I)^{-1/2}\sigma$, then $U_\varepsilon \to U_0$ as $\varepsilon \downarrow 0$, where U_0 is an orthogonal transformation such that $a^{1/2} = U_0\sigma$. Hence, if

$$\alpha(t) - \alpha(t_0) = \int_{t_0}^t a^{1/2}(u)\, d\hat{\beta}(u) + \int_{t_0}^t b(u)\, du \quad \text{a.s. } \hat{P}, \tag{2.12}$$

then

$$\alpha(t) - \alpha(t_0) = \int_{t_0}^t \sigma(u)\, d\hat{\hat{\beta}}(u) + \int_{t_0}^t b(u)\, du \quad \text{a.s. } \hat{P}, \tag{2.13}$$

where $\hat{\hat{\beta}}(t) = \int_{t_0}^t U_0^*(u)\, d\hat{\beta}(u)$ is again a $\hat{P}$ Brownian motion.

To prove the assertion when $\sigma = a^{1/2}$, define $\tilde{a}(u) = \lim_{\varepsilon\downarrow 0} a^{1/2}(u)(a(u) + \varepsilon I)^{-1}$. Then $\tilde{a}(u)a^{1/2}(u) = a^{1/2}(u)\tilde{a}(u) = E_R(u)$, where $E_R(u)$ is the orthogonal projection onto the range of $a(u)$. Let $\beta(t)$ be a W Brownian motion on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ and define $\hat{P} = P \times W$. Then, by Theorem 2.1,

$$\hat{\beta}(t) = \int_{t_0}^{t} \tilde{a}(u)\, d\bar{\alpha}(u) + \int_{t_0}^{t} E_N(u)\, d\beta(u) \tag{2.14}$$

is a $\hat{P}$ Brownian motion, where $E_N(u) = I - E_R(u)$. Moreover,

$$\begin{aligned} \int_{t_0}^{t} a^{1/2}(u)\, d\hat{\beta}(u) &= \int_{t_0}^{t} E_R(u)\, d\bar{\alpha}(u) + \int_{t_0}^{t} a^{1/2}(u) E_N(u)\, d\beta(u) \\ &= \int_{t_0}^{t} E_R(u)\, d\bar{\alpha}(u), \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &E^{\hat{P}}\left[\left|\bar{\alpha}(t) - \bar{\alpha}(t_0) - \int_{t_0}^{t} E_R(u)\, d\bar{\alpha}(u)\right|^2\right] \\ &\quad = E^{\hat{P}}\left[\left|\int_{t_0}^{t} E_N(u)\, d\bar{\alpha}(u)\right|^2\right] \\ &\quad = E^{\hat{P}}\left[\left|\int_{t_0}^{t} \operatorname{tr}\left(E_N(u) a(u) E_N(u)\right) du\right]\right. = 0, \end{aligned} \tag{2.16}$$

where tr means trace. *Q.E.D.*

Theorem 2.2 is the multidimensional analogue of Theorem 5.3 in J. L. Doob's book [2].

THEOREM 2.3. *Let $a: [0, \infty) \times R^d \to S_d$ and $b: [0, \infty) \times R^d \to R^d$ be bounded measurable functions. Assume that there is a bounded measurable $\sigma: [0, \infty) \times R^d \to S_d$ such that $a = \sigma\sigma^*$ and*

$$\sup_{0 \leqq t \leqq T, |x| + |y| \leqq R} \left(\|\sigma(t, x) - \sigma(t, y)\| + |b(t, x) - b(t, y)|\right) \leqq C(T, R)|x - y| \tag{2.17}$$

for all $T, R > 0$. Then for each (t_0, x_0) there is exactly one solution P_{t_0, x_0} to the martingale problem for

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i} \tag{2.18}$$

starting at (t_0, x_0). Moreover, the family $\{P_{t_0, x_0}: (t_0, x_0) \in [0, \infty) \times R^d\}$ forms a Feller continuous, strong Markov process.

PROOF. We will only prove the first assertion, since the second one follows by standard methods used in [9]. Moreover, we will restrict ourselves to the case when $C(T, R)$ is independent of T and R, because the general case can then be handled by the techniques employed in Theorem 5.6 of [9].

To prove existence, let β be a W Brownian motion on $\langle \Omega, \mathscr{M}^{t_0} \rangle$. Define $\xi_0(t) \equiv x_0$ and

$$\xi_{n+1}(t) = x_0 + \int_{t_0}^{t} \sigma(u, \xi_n(u))\, d\beta(u) + \int_{t_0}^{t} b(u, \xi_n(u))\, du. \tag{2.19}$$

Following H. P. McKean [5], it is easy to show that

$$W\big(\sup_{t_0 \leqq t \leqq T} |\xi_n(t) - \xi(t)| > \varepsilon\big) \to 0 \tag{2.20}$$

for all $T > t_0$, where

$$\xi(t) = x_0 + \int_{t_0}^{t} \sigma(u, \xi(u))\, d\beta(u) + \int_{t_0}^{t} b(u, \xi(u))\, du. \tag{2.21}$$

Letting P be the distribution of $\xi(t)$ (that is, P is defined on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ by

$$P\big(x(t_1) \in \Gamma_1, \cdots, x(t_n) \in \Gamma_n\big) = W\big(\xi(t_1) \in \Gamma_1, \cdots, \xi(t_n) \in \Gamma_n\big), \tag{2.22}$$

and using Theorem 2.1, one sees that P solves the martingale problem for L_t starting at (t_0, x_0).

Turning to uniqueness, suppose that P is a solution. Choose $\hat{P}$ and $\hat{\beta}(t)$ as in Theorem 2.2. Using the technique of the preceding paragraph, we can find $\xi_n(t)$ such that

$$\xi_{n+1}(t) = x_0 + \int_{t_0}^{t} a^{1/2}(u, \xi_n(u))\, d\hat{\beta}(u) + \int_{t_0}^{t} b(u, \xi_n(u))\, du, \tag{2.23}$$

and $\xi_n(t) \to x(t)$ in probability uniformly on finite intervals. Because the distribution of each $\xi_n(t)$ is uniquely determined, P is unique. *Q.E.D.*

We next state another theorem for reference purposes. Its proof can be found in [9].

THEOREM 2.4 (Cameron-Martin). *Let $\beta(t)$ be a W Brownian motion on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ and suppose $c\colon [t_0, \infty) \times \Omega \twoheadrightarrow R^d$ is bounded and t_0 nonanticipating. Then*

$$R(t) = \exp\left\{\int_{t_0}^{t} \langle c(u), d\beta(u) \rangle - \frac{1}{2}\int_{t_0}^{t} |c(u)|^2\, du\right\} \tag{2.24}$$

is a W martingale. In particular, there is a unique probability measure Q on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ such that

$$\frac{dQ}{dW} = R(t) \text{ on } \mathscr{M}_t^{t_0}, \qquad t \geqq t_0. \tag{2.25}$$

Moreover, $\bar{\beta}(t) = \beta(t) - \int_{t_0}^{t} c(u)\, du$ is a Brownian motion.

COROLLARY 2.1. *Let a, b, L_t, and σ be as in Theorem 2.3, and let $c\colon [0, \infty) \times R^d \twoheadrightarrow R^d$ be bounded and measurable. Suppose P_{t_0, x_0} is the solution to the martingale problem for L_t starting at (t_0, x_0), and choose $\hat{P}_{t_0, x_0}$ and $\hat{\beta}(t)$ as in Theorem 2.2 so that*

$$x(t) = x_0 + \int_{t_0}^{t} \sigma(u, x(u))\, d\hat{\beta}(u) + \int_{t_0}^{t} b(u, x(u))\, du \qquad a.s.\ \hat{\Gamma}_{t_0, x_0}. \tag{2.26}$$

Define

$$R(t) = E^{P_{t_0,x_0}}\left[\exp\left\{\int_{t_0}^{t} \langle c(u, x(u)), d\hat{\beta}(u)\rangle - \frac{1}{2}\int_{t_0}^{t} |c(u)|^2\, du\right\}\middle| \mathscr{M}_t^{t_0}\right], \tag{2.27}$$

and determine Q_{t_0,x_0} by the relations

$$\frac{dQ_{t_0,x_0}}{dP_{t_0,x_0}} = R(t) \text{ on } \mathscr{M}_t^{t_0}, \qquad t \geqq t_0. \tag{2.28}$$

Then Q_{t_0,x_0} is the only solution to the martingale problem for

$$L_t^c = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}(b + \sigma c)_i(t, x)\frac{\partial}{\partial x_i} \tag{2.29}$$

starting at (t_0, x_0). Moreover, the family $\{Q_{t,x}: (t, x) \in [0, \infty) \times R^d\}$ forms a strong Markov process which is Feller continuous when c is continuous in x.

REMARK 2.1. If $a: [0, \infty) \times R^d \to S_d$ is bounded and measurable, then

$$\sup_{t \leqq T, |x| + |y| \leqq R} \|a^{1/2}(t, x) - a^{1/2}(t, y)\| \leqq C(T, R)|x - y|, \tag{2.30}$$

if $a(t, x)$ is twice continuous differentiable in x and

$$\max_{1 \leqq i,j \leqq d}\ \sup_{t \leqq T, |x|+|y| \leqq R} \left\|\frac{\partial^2 a}{\partial x_i \partial x_j}(t, x)\right\| \leqq C(T, R). \tag{2.31}$$

This fact is proved by R. S. Philips and L. Sarason [8].

REMARK 2.2. Unfortunately there is no nice criterion on $a: [0, \infty) \times R^d \to S_d$ which guarantees the existence of a smooth $\sigma: [0, \infty) \times R^d \to R^d \otimes R^d$ such that $a = \sigma\sigma^*$. Nonetheless, we will often assume in what follows that L_t can be written in the form

$$L_t = \tfrac{1}{2}\sigma^*\nabla_x \cdot \sigma^*\nabla_x + b \cdot \nabla_x, \tag{2.32}$$

where

$$\sigma^*\nabla_x \cdot \sigma^*\nabla_x = \sum_{i,j=1}^{d} \sigma^{i\ell}\frac{\partial}{\partial x_i}\left(\sigma^{j\ell}\frac{\partial}{\partial x_j}\right). \tag{2.33}$$

(Here, and in what follows, repeated indices are summed.) Notice that (2.32) can be written as

$$L_t = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\left(b + \frac{1}{2}\sigma'\sigma\right)_i \frac{\partial}{\partial x_i}, \tag{2.34}$$

where $a = \sigma\sigma^*$ and the vector $\sigma'\sigma$ is defined by

$$(\sigma'\sigma)_i = \sigma^{ij}_{,\ell}\sigma^{\ell j}. \tag{2.35}$$

(We have used here, and will continue to use, the notation $f_{,\ell}$ to stand for $\partial f/\partial x_\ell$.) What specific assumptions are made about the smoothness σ will depend on our immediate needs. But in any case, it will be necessary to assume that σ is once differentiable in x in order to even define $\sigma^*\nabla_x \cdot \sigma^*\nabla_x$.

3. The nondegenerate case

Throughout this section $a: [0, \infty) \times R^d \rightsquigarrow S_d$ will be bounded, continuous, positive definite valued function and $b: [0, \infty) \times R^d \rightsquigarrow R^d$ will be bounded and measurable. We will use P_{t_0, x_0} to denote the unique solution to the martingale problem for

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i} \tag{3.1}$$

starting at (t_0, x_0), and $P^0_{t_0, x_0}$ will denote the unique solution for

$$L^0_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} \tag{3.2}$$

starting at (t_0, x_0). Our aim is to prove that

$$\operatorname{supp}(P_{t_0, x_0}) = \Omega(t_0, x_0), \tag{3.3}$$

where $\Omega(t_0, x_0)$ is the set of $\omega \in \Omega$ such that $x(t_0, \omega) = x_0$.

Before proceeding, we make two simplifying observations. First, since P_{t_0, x_0} and $P^0_{t_0, x_0}$ are equivalent (that is, mutually absolutely continuous) on $\mathscr{M}_t^{t_0}$ for all $t \geqq t_0$, it suffices to work with $P^0_{t_0, x_0}$. Second, by an obvious transformation, we can always assume that $t_0 = 0$ and $x_0 = 0$. Thus, what we need to show is that

$$\operatorname{supp}(P_0) = \Omega_0, \tag{3.4}$$

where $P_0 = P^0_{0,0}$ and $\Omega_0 = \Omega(0, 0)$.

LEMMA 3.1. *Let $\phi: [0, \infty) \rightsquigarrow R^d$ be once continuously differentiable such that $\phi(0) = 0$. Then for all $T > 0$ and $\varepsilon > 0$, $P_0(\|x(t) - \phi(t)\|^0_T < \varepsilon) > 0$, where $\|\cdot\|^s_T$ denotes the sup norm on the interval $[s, T]$.*

PROOF. Let $\psi(t) = \chi_{[0,T]}(t)\dot{\phi}(t)$ and define Q_0 by

$$\frac{dQ_0}{dP_0} = R(t) = \exp\left\{\int_0^t \langle a^{-1}(u, x(u)), \psi(u)\, dx(u)\rangle - \frac{1}{2}\int_0^t \langle \psi(u), a^{-1}(u, x(u)), \psi(u)\rangle\, du\right\} \tag{3.5}$$

on $\mathscr{M}_t^0$, $t \geqq 0$. Then Q_0 is the unique solution to the martingale problem for

$$L^\psi_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \psi_i(t) \frac{\partial}{\partial x_i} \tag{3.6}$$

starting at (0, 0). In particular, there is a Q_0 Brownian motion $\beta(t)$ such that

$$\bar{x}(t) = \int_0^t a^{1/2}(u, x(u))\, d\beta(u), \tag{3.7}$$

where $\bar{x}(t) = x(t) - \int_0^t \psi(u)\, du = x(t) - \phi(t \wedge T)$. Hence, by Theorem 1 in [11], $Q_0(\|\bar{x}(t)\|_T^0 < \varepsilon) > 0$ for all $\varepsilon > 0$. Since Q_0 and P_0 are equivalent on $\mathscr{M}_T^0$, this implies $P_0(\|x(t) - \phi(t)\|_T^0 < \varepsilon) > 0$ for all $\varepsilon > 0$. *Q.E.D.*

Using Lemma 3.1, we see that $\operatorname{supp}(P_0)$ contains all differentiable paths which start at 0. Because these are dense in Ω_0, equation (3.4) is now proved. We have therefore proved the following theorem.

THEOREM 3.1. *If P_{t_0,x_0} is the solution to the martingale problem for L_t starting at (t_0, x_0), then equation* (3.3) *holds.*

As we saw in Section 1, equation (3.3) implies Nirenberg's strong maximum principle. In fact, it implies more. Let $\mathscr{G}$ be an open set in $[0, \infty) \times R^d$ and let $(t_0, x_0) \in \mathscr{G}$. Define $\mathscr{H}_{L_t}^-(t_0, x_0, \mathscr{G})$ to be the set of $f: \mathscr{G} \to R \cup \{-\infty\}$ which are upper semicontinuous, bounded above, and have the property that $f(t \wedge \tau, x(t \wedge \tau))$ is a P_{t_0,x_0} submartingale, that is,

$$f(t_1 \wedge \tau, x(t_1 \wedge \tau)) \leqq E^{P_{t_0,x_0}}[f(t_2 \wedge \tau, x(t_2 \wedge \tau)) | \mathscr{M}_{t_1 \wedge \tau}^{t_0}], \qquad t_0 \leqq t_1 \leqq t_2, \tag{3.8}$$

where $\tau = \inf\{t \geqq t_0 : (t, x(t)) \notin \mathscr{G}\}$. Note that $\mathscr{H}_{L_t}^-(t_0, x_0, \mathscr{G})$ is closed under nonincreasing limits, multiplication by nonnegative constants, and maximums (that is, if $f_1, f_2 \in \mathscr{H}_{L_t}^-(t_0, x_0, \mathscr{G})$, then so is $f_1 \vee f_2$). Next define (t_0, x_0) to be the closure in $\mathscr{G}$ of the set of $(t_1, x_1) \in \mathscr{G} \cap ([t_0, \infty) \times R^d)$ such that $x_0 = \phi(t_0)$ and $x_1 = \phi(t_1)$ for some $\phi \in C([t_0, t_1], R^d)$ satisfying $(t, \phi(t)) \in \mathscr{G}$, $t \in [t_0, t_1]$. Observe that if $f \in C_b^{1,2}(\mathscr{G})$ and $((\partial/\partial t) + L_t)f \geqq 0$ on $\mathscr{G}(t_0, x_0)$, then $f \in \mathscr{H}_{L_t}^-(t_0, x_0)$. Finally, define

$$\mathscr{H}_{L_t}^-(\mathscr{G}) = \bigcap_{\mathscr{G}} \mathscr{H}_{L_t}^-(t_0, x_0, \mathscr{G}). \tag{3.9}$$

THEOREM 3.2. *If $f \in \mathscr{H}_{L_t}^-(t_0, x_0, \mathscr{G})$ and $f(t_0, x_0) = \sup f(t, x)$, then $f \equiv f(t_0, x_0)$ on $\mathscr{G}(t_0, x_0)$. Moreover, if $(t_1, x_1) \in \mathscr{G} - \mathscr{G}(t_0, x_0)$, then there is an $f \in \mathscr{H}_{L_t}^-(\mathscr{G})$ such that $f(t_0, x_0) = \sup f(t, x)$ and $f(t_1, x_1) < f(t_0, x_0)$.*

PROOF. The first assertion follows from the argument given in Section 1. To prove the second assertion, let $(t_1, x_1) \in \mathscr{G} - \mathscr{G}(t_0, x_0)$. If $t_1 < t_0$, take $f(t, x) = t \wedge t_0$. If $t_1 \geqq t_0$, choose an open neighborhood N of (t_1, x_1) such that $N \subseteqq \mathscr{G}$ and $N \cap \mathscr{G}(t_0, x_0) = \phi$. Let $h \in C_0^\infty(N)$ such that $-1 \leqq h \leqq 0$ and $h(t_1, x_1) = -1$. Define

$$f(t, x) = E^{P_{t,x}}\left[\int_t^{\tau_t} e^{-u} h(u, x(u))\, du\right], \tag{3.10}$$

where $\tau_t = \inf\{u \geqq t : (u, x(u)) \notin \mathscr{G}\}$. Since $\{P_{t,x} : (t, x) \in [0, \infty) \times R^d\}$ is strongly Feller continuous, f is continuous. Moreover,

$$E^{P_{s,x}}[f(t \wedge \tau_s, x(t \wedge \tau_s))] = E^{P_{s,x}}\left[\chi_{\tau_s > t} \int_t^{\tau_t} e^{-u} h(u, x(u))\, du\right] \geqq f(s, x), \tag{3.11}$$

for $(s, x) \in \mathscr{G}$ and $t \geqq s$; and therefore

$$\begin{aligned} E^{P_{s,x}}&[f(t_2 \wedge \tau_s, x(t_2 \wedge \tau_s)) | \mathscr{M}^s_{t_1 \wedge \tau_s}] \\ &= E^{P_{t_1 \wedge \tau_s, x(t_1 \wedge \tau_s)}}[f(t_2 \wedge \tau_{t_1}, x(t_2 \wedge \tau_{t_1}))] \\ &\geqq f(t_1 \wedge \tau_3, x(t_1 \wedge \tau_s)) \end{aligned} \tag{3.12}$$

for $s \leqq t_1 \leqq t_2$. This proves that $f \in \mathscr{H}^-_{L_t}(\mathscr{G})$. Clearly, $f \leqq 0, f(t_0, x_0) = 0$, and $f(t_1, x_1) < 0$. *Q.E.D.*

REMARK 3.1. It is important to know in what sense $\mathscr{H}^-_{L_t}(\mathscr{G})$ is an extension of the class of $f \in C_b^{1,2}(\mathscr{G})$ satisfying $(\partial f/\partial t) + L_t f \geqq 0$ on $\mathscr{G}$. Using the estimates obtained in [9], one can show that $\mathscr{H}^-_{L_t}(\mathscr{G})$ contains the class of $f \in W_p^{1,2}(\mathscr{G})$ (see [9] for the definition of $W^{1,2}$) satisfying $(\partial f/\partial t) + L_t f \geqq 0$ when $p > (d + 2)/2$. To give a complete analytic description of $\mathscr{H}^-_{L_t}(\mathscr{G})$, consider the transition function $\hat{P}(t, (s, x), \cdot)$ defined by

$$\hat{P}(t, (s, x), \Delta \times \Gamma) = \chi_\Delta(s + t) P_{s,t}(x(s + t) \wedge \tau_s) \in \Gamma) \tag{3.13}$$

for $\Delta \in \mathscr{B}_{[0,\infty)}$ and $\Gamma \in \mathscr{B}[\bar{\mathscr{G}}]$, where $\tau_s = \inf\{t \geqq s : (t, x(t)) \notin \mathscr{G}\}$. It is easy to see that

$$\hat{P}(t_1 + t_2, (s, x), \cdot) = \int_0^\infty \int_{\mathscr{G}} \hat{P}(t_1, (s, x), du \times dy) \hat{P}(t_2, (u, y), \cdot). \tag{3.14}$$

Thus, we can define a Markov semigroup $\{\hat{T}_t\}_{t \geqq 0}$ on $\mathscr{B}(\mathscr{G})$ by setting

$$\hat{T}_t f(s, x) = \int_0^\infty \int_{\mathscr{G}} \hat{P}(t, (s, x), du \times dy) f(u, y). \tag{3.15}$$

Furthermore, $\{\hat{T}_t\}_{t \geqq 0}$ is the only Markov semigroup having the property that

$$\hat{T}_t f(s, x) - f(s, x) = \int_0^t \left(\hat{T}_u\left[\chi_{\mathscr{G}} \cdot \left(\frac{\partial f}{\partial u} + L_u f\right)\right]\right)(u, x)\, du \tag{3.16}$$

for all $f \in C_b^{1,2}(\bar{\mathscr{G}})$. Finally, $\mathscr{H}^-_{L_t}(\mathscr{G})$ coincides with the class of f on $\bar{\mathscr{G}}$ such that f is bounded above and $\hat{T}_t f(s, x) \downarrow f(s, x)$ for $(s, x) \in \mathscr{G}$. When $L_t = \frac{1}{2}\Delta$, it is easy to see that $\mathscr{H}^-_{L_t}(\mathscr{G})$ is just the class of subparabolic functions on $\mathscr{G}$.

4. The degenerate case, part I

Let $\sigma: [0, \infty) \times R^d \rightsquigarrow R^d \otimes R^d$ and $b: [0, \infty) \times R^d \rightsquigarrow R^d$ be bounded measurable functions. In this section, we will assume that b and the first spatial derivatives of σ are uniformly Lipschitz continuous in x. From σ and b, we form the operator

$$L_t = \tfrac{1}{2}\sigma^*\nabla_x \cdot \sigma^*\nabla_x + b \cdot \nabla_x, \tag{4.1}$$

by the prescription given in Remark 2.2. Under the above assumptions, we know from Theorem 2.3 that there is exactly one solution P_{t_0,x_0} to the martingale problem for L_t starting at (t_0, x_0). The purpose of this section is to prove that

$$\operatorname{supp}(P_{t_0,x_0}) \subseteq \overline{\mathscr{S}_{\sigma,b}(t_0, x_0)}, \tag{4.2}$$

where $\overline{\mathscr{S}_{\sigma,b}(t_0, x_0)}$ is the class of $\phi \in C([0, \infty), R^d)$ for which there exists a piecewise constant $\psi: [t_0, \infty) \to R^d$ such that

$$\phi(t) = x_0 + \int_{t_0}^t \sigma(u, \phi(u))\psi(u)\,du + \int_{t_0}^t b(u, \phi(u))\,du, \qquad t \geqq t_0. \tag{4.3}$$

Clearly, it suffices to treat the case when $t_0 = 0$ and $x_0 = 0$.

Let $\beta(t)$ be a W Brownian motion on $\langle \Omega, \mathscr{M}^0 \rangle$. Given $n \geqq 0$, define $t_n = [2^n t]/(2^n)$, $t_n^+ = ([2^n t] + 1)/2^n$, and

$$\dot\beta^{(n)}(t) = 2^n(\beta(t_n^+) - \beta(t_n)). \tag{4.4}$$

Next, let $\xi^{(n)}(t)$ be the stochastic process determined by the ordinary integral equation

$$\xi^{(n)}(t) = \int_0^t \sigma(u, \xi^{(n)}(u))\dot\beta^{(n)}(u)\,du + \int_0^t b(u, \xi^{(n)}(u))\,du; \tag{4.5}$$

and denote by P_n the distribution of $\xi^{(n)}(t)$. Clearly, $\operatorname{supp}(P_n) \subseteq \overline{\mathscr{S}_{\sigma,b}(0, 0)}$ for all $n \geqq 0$. Hence, if we show that P_n tends weakly to $P_{0,0}$ as $n \to \infty$, then it will follow that $\operatorname{supp}(P_{0,0}) \subseteq \overline{\mathscr{S}_{\sigma,b}(0, 0)}$. Thus, we must prove that $P_n \Rightarrow P_{0,0}$. Results of this sort are familiar in various branches of applied mathematics when $d = 1$ (see E. Wong and M. Zakai [12]). However, to the best of our knowledge, the proof which follows is the first complete one for $d > 1$.

The procedure which we will use consists of two steps. The first of these is to prove that $\{P_n\}_{n\geqq 1}$ is relatively weakly compact. Once this is done, we will then show that every convergent subsequence of $\{P_n\}_{n\geqq 1}$ converges to a solution of the martingale problem for L_t starting at $(0, 0)$. For convenience in writing, we will use the following notation:

$$\eta^{(n)}(t) = \int_0^t \sigma(u, \xi^{(n)}(u))\dot\beta^{(n)}(u)\,du, \tag{4.6}$$

$$\alpha^{(n)}(t) = \sigma\left(t, \xi^{(n)}\left(\frac{[2^n t]}{2^n}\right)\right)\dot\beta^{(n)}(t), \tag{4.7}$$

$$(\sigma'\sigma)_i^{\ell,\ell'}(t, x) = \left(\frac{\partial}{\partial x_j}\sigma^{i\ell}(t, x)\right)\sigma^{j\ell'}(t, x), \tag{4.8}$$

$$\Delta_k^{(n)} = \beta\left(\frac{k+1}{2^n}\right) - \beta\left(\frac{k}{2^n}\right). \tag{4.9}$$

LEMMA 4.1. *The set $\{P_n\}_{n \geqq 1}$ is relatively weakly compact.*

PROOF. It suffices to prove that

$$\sup_n E^{P_n}[|x(t) - x(s)|^4] \leqq C_T|t_2 - s|^2, \quad 0 \leqq s \leqq t \leqq T, \quad T > 0. \tag{4.10}$$

To do this, first observe that

$$\begin{aligned} &E^{P_n}|x(t) - x(s)|^4 \\ &\quad \leqq 8\left(E^W[|\eta^{(n)}(t) - \eta^{(n)}(s)|^4] + E^W\left[\left|\int_s^t b(u, \xi^{(n)}(u))\,du\right|^4\right]\right). \end{aligned} \tag{4.11}$$

Hence, it suffices to examine

$$E^W[|\eta^{(n)}(t) - \eta^{(n)}(s)|^4]. \tag{4.12}$$

But

$$\begin{aligned} &\eta^{(n)}(t) - \eta^{(n)}(s) \\ &\quad = \int_s^t \alpha^{(n)}(u)\,du + \int_s^t du \int_{u_n}^u dv(\sigma'\sigma)^{\ell,\ell'}(u, \xi^{(n)}(v))\dot\beta_\ell^{(n)}(v)\dot\beta_{\ell'}^{(n)}(v), \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} E^W\left[\left|\int_s^t \alpha^{(n)}(u)\,du\right|^4\right] &= E^W\left[\left|\int_s^t \sigma(u, \xi^{(n)}(u_n))\dot\beta^{(n)}(u)\,du\right|^4\right] \\ &= E^W\left[\left|\int_{s_n}^{t_n} \sigma^{(n)}(u)\,d\beta(u)\right|^4\right] \leqq C_1(t - s)^2, \end{aligned} \tag{4.14}$$

where

$$\sigma^{(n)}(u) = 2^n \int_{u_n \vee s}^{u_n^+ \wedge t} \sigma(v, \xi^{(n)}(u_n))\,dv. \tag{4.15}$$

Finally,

$$\begin{aligned} &E^W\left[\left|\int_s^t du \int_{u_n}^u dv(\sigma'\sigma)^{\ell,\ell'}(u, \xi^{(n)}(v))\dot\beta_\ell^{(n)}(v)\dot\beta_{\ell'}^{(n)}(v)\right|^4\right] \\ &\quad \leqq (t - s)^3 E^W\left[\int_s^t du \left|\int_{u_n}^u dv(\sigma'\sigma)^{\ell,\ell'}(u, \xi^{(n)}(v))\dot\beta_\ell^{(n)}(v)\dot\beta_{\ell'}^{(n)}(v)\right|^4\right] \\ &\quad \leqq C_2(t - s)^3 \sum_{k=[2^n s]}^{[2^n t]} 2^{8n} \int_{k/2^n}^{k+1/2^n} du\left(u - \frac{k}{2^n}\right)^4 E^W[|\Delta_k^{(n)}|^8] \\ &\quad \leqq C_3(t - s)^3. \end{aligned} \tag{4.16}$$

Q.E.D.

We now have to show that if P is the limit of a convergent subsequence of $\{P_n\}$, then

$$E^P[F\cdot(f(x(t)) - f(x(s)))] = E^P\left[F\cdot\int_s^t L_u f(x(u))\,du\right] \tag{4.17}$$

for all $f \in C_0^\infty(R^d)$, $0 \leqq s < t$, and bounded $\mathscr{M}_s^0$ measurable $F\colon \Omega \rightsquigarrow R$. Clearly, it will suffice to do this when s and t have the form $k/2^N$ and F is continuous as well as bounded and $\mathscr{M}_s^0$ measurable. For the sake of convenience, we will use $\{P_n\}_{n\geqq 1}$ to denote the subsequence which converges to P. Observe that

$$\begin{aligned} E^{P_n}&[F\cdot(f(x(t)) - f(x(s)))] \\ &= E^{P_n}\left[F\cdot\int_s^t \langle\nabla_x f(x(u)), b(u, x(u))\rangle\,du\right] \\ &\quad + E^W\left[F\cdot\int_s^t \langle\nabla_x f(\xi^{(n)}(u)), \alpha^{(n)}(u)\rangle\,du\right] \\ &\quad + E^W\left[F\cdot\int_s^t \langle\nabla_x f(\xi^{(n)}(u)), \dot\eta^{(n)}(u) - \alpha^{(n)}(u)\rangle\,du\right] \\ &= I_1^{(n)} + I_2^{(n)} + I_3^{(n)}. \end{aligned} \tag{4.18}$$

Obviously, $I_1^{(n)} \to E^P[F\cdot\int_s^t \langle\nabla_x f(x(u)), b(u, x(u))\rangle\,du]$. Before examining the limiting behavior of the other two terms, we need the following simple lemma.

LEMMA 4.2. *If ϕ and ψ are bounded measurable functions on $[s, t]$, then*

$$2^n\int_s^t du\phi(u)\int_{u_n}^u dv\psi(v) \to \frac{1}{2}\int_s^t \phi(u)\psi(u)\,du. \tag{4.19}$$

PROOF. Let $\psi_n(u) = 2^n\int_{u_n}^u \psi(v)\,dv$. Then $\{\psi_n\}_{n\geqq 1}$ is obviously relatively compact in the weak topology on $\mathsf{L}^1([s, t])$ induced by $\mathsf{L}^\infty([s, t])$. Hence, it suffices to prove the result when ϕ is continuous. Next observe that

$$2^n\int_s^t du\phi(u)\int_{u_n}^u dv\psi(v) = 2^n\int_s^t du\psi(u)\int_u^{u_n^+} dv\phi(v). \tag{4.20}$$

Thus, by the argument just used, we may assume that both ϕ and ψ are continuous. But the result is obvious for continuous ϕ and ψ, and so we are done. *Q.E.D.*

LEMMA 4.3. *Let $L_u^0 = \frac{1}{2}(\sigma\sigma^*)^{ij}(t, x)(\partial^2/\partial x_i\partial x_j)$. Then*

$$I_2^{(n)} \to E^P\left[F\cdot\int_s^t L_u^0 f(x(u))\,du\right]. \tag{4.21}$$

PROOF. We will use $H(x)$ to denote the Hessian matrix of f. Note that

$$E^W[\alpha^{(n)}(u)\,|\,\mathscr{M}_{u_n}] = 0, \tag{4.22}$$

and therefore

$$(4.23)\qquad \begin{aligned} I_2^{(n)} &= E^W\left[F\cdot\int_s^t \langle \nabla_x f(\xi^{(n)}(u)) - \nabla_x f(\xi^{(n)}(u_n)), \alpha^{(n)}(u)\rangle\, du\right] \\ &= E^W\left[F\cdot\int_u^t du \int_{u_n}^u dv\langle \dot{\eta}^{(n)}(v), H(\xi^{(n)}(v))\alpha^{(n)}(u)\rangle\right] \\ &\qquad + E^W\left[F\cdot\int_s^t du\int_{u_n}^u dv\langle b(v, \xi^{(n)}(v)), H(\xi^{(n)}(v))\alpha^{(n)}(u)\rangle\right] \\ &= J_1^{(n)} + J_2^{(n)}. \end{aligned}$$

Clearly, $|J_2^{(n)}| \to 0$ and

$$(4.24)\qquad \begin{aligned} J_1^{(n)} &= E^W\left[F\cdot\int_s^t du\int_{u_n}^u dv\langle \alpha^{(n)}(v), H(\xi^{(n)}(v)\alpha^{(n)}(u)\rangle\right] \\ &\qquad + E^W\Big[F\cdot\int_s^t du\int_{u_n}^u dv\int_{u_n}^v dw\langle(\sigma'\sigma)^{\ell,\ell'}(v, \xi^{(n)}(w))\mathring{\beta}_\ell^{(n)}\mathring{\beta}_{\ell'}^{(n)}, \\ &\qquad\qquad H[\xi^{(n)}(v)]\alpha^{(n)}(u)\rangle\Big] \\ &= J_3^{(n)} + J_4^{(n)}. \end{aligned}$$

Again, it is obvious that $|J_4^{(n)} \to 0$ and that

$$(4.25)\qquad \begin{aligned} J_3^{(n)} &= E^W\left[F\cdot\int_s^t du\int_{u_n}^u dv\langle \alpha^{(n)}(v), H(\xi^{(n)}(u_n))\alpha^{(n)}(u)\rangle\right] \\ &\qquad + E^W\left[F\cdot\int_s^t du\int_{u_n}^u dv\langle\alpha^{(n)}(v), (H(\xi^{(n)}(v)) - H(\xi^{(n)}(u_n)))\alpha^{(n)}(u)\rangle\right] \\ &= J_5^{(n)} + J_6^{(n)}. \end{aligned}$$

Since $|J_6^{(n)}| \to 0$, it remains to examine $J_5^{(n)}$. Clearly,

$$(4.26)\qquad \begin{aligned} J_5^{(n)} &= E^W\left[2^nF\cdot\int_s^t du\int_{u_n}^u dv\ \mathrm{tr}\,[\sigma^*(v, \xi^{(n)}(v_n))H(\xi^{(n)}(u_n))\sigma(u, \xi^{(n)}(u_n))]\right] \\ &= E^{P_n}\left[2^nF\cdot\int_s^t du\int_{u_n}^u dv\ \mathrm{tr}\,[\sigma^*(v, x(v_n))H(x(u_n))]\sigma(u, x(u_n))\right]. \end{aligned}$$

Hence, by Lemma 4.2 and the fact that $P_n \Rightarrow P$, we have that

$$(4.27)\qquad J_5^{(n)} \to \frac{1}{2}E^P\left[F\cdot\int_s^t \mathrm{tr}\,[\sigma^*(u, v(u))H(x(u))\sigma(u, x(u))]\, du\right].$$

Q.E.D.

LEMMA 4.4. *Let* $(\sigma'\sigma)_i(t, x) = (\sigma'\sigma)_i^{\ell\ell}(t, x)$. *Then*

$$(4.28)\qquad I_3^{(n)} \to \frac{1}{2}E^P\left[F\cdot\int_s^t \nabla_x f(x(u)), (\sigma'\sigma)(u, x(u))\, du\right].$$

PROOF. Note that

$$(4.29)\quad \begin{aligned} I_3^{(n)} &= E^W\left[F\cdot\int_s^t du\int_{u_n}^u dv\, \nabla_x f(\xi^{(u)}(u)), (\sigma\sigma')^{\ell\ell'}(u, \xi^{(n)}(v))\mathring{\beta}_\ell(v)\mathring{\beta}_{\ell'}(v)\right] \\ &= E^W\left[F\cdot\int_s^t \nabla_x f(\xi^{(n)}(u_n)), (\sigma\sigma')(u, \xi^{(n)}(u_n))(u-u_n)\,du\right] \\ &\quad + E^W\Big[F\cdot\int_s^t du\int_{u_n}^u dv\, \nabla_x f(\xi^{(n)}(u_n)), \\ &\qquad [(\sigma\sigma')^{\ell\ell'}(u, \xi^{(n)}(v)) - (\sigma\sigma')^{\ell\ell'}(u, \xi^{(n)}(u_n))]\mathring{\beta}_\ell(v)\mathring{\beta}_{\ell'}(v)\Big] \\ &\quad + E^W\Big[F\cdot\int_s^t du\int_{u_n}^u dv\langle\nabla_x f(\xi^{(n)}(u)) - \nabla_x f(\xi^{(n)}(u_n)), \\ &\qquad (\sigma'\sigma)^{\ell\ell'}(u, \xi^{(n)}(v))\mathring{\beta}_\ell(v)\mathring{\beta}_{\ell'}(v)\rangle\Big] \\ &= J_1^{(n)} + J_2^{(n)} + J_3^{(n)}. \end{aligned}$$

Clearly, $|J_2^{(n)}|$ and $|J_3^{(n)}|$ tend to zero. Moreover,

$$(4.30)\quad \begin{aligned} J_1^{(n)} &= E^{P_n}\left[F\cdot\int_s^t \langle\nabla_x f(x(u_n)), (\sigma\sigma')(u, x(u_n))\rangle\,(u-u_n)\,du\right] \\ &\to \frac{1}{2}E^P\left[F\cdot\int_s^t \nabla_x f(x(u)), (\sigma\sigma')(u, x(u))\,du\right]. \end{aligned}$$

Q.E.D.

THEOREM 4.1. *Let P_n be the distribution of the process $\xi^{(n)}(t)$ defined in* (4.5). *Then P_n converges weakly to $P_{0,0}$ as $n\to\infty$. In particular, equation* (4.2) *is valid for all (t_0, x_0).*

COROLLARY 4.1. *If $c\colon [0,\infty)\times R^d \twoheadrightarrow R^d$ is bounded and measurable, then for each (t_0, x_0) the unique solution $P^c_{t_0,x_0}$ to the martingale problem for*

$$(4.31)\qquad L_t^c = \tfrac{1}{2}\sigma^*\nabla_x\cdot\sigma^*\nabla_x + (b+\sigma c)\cdot\nabla_x$$

starting at (t_0, x_0) satisfies

$$(4.32)\qquad \operatorname{supp}(P^c_{t_0,x_0}) \subseteq \overline{\mathscr{S}_{\sigma,b}(t_0, x_0)}.$$

PROOF. According to Corollary 2.1, $P^c_{t_0,x_0}$ and P_{t_0,x_0} are equivalent on $\mathscr{M}_t^{t_0}$ for all $t\geqq t_0$. Hence, $\operatorname{supp}(P^c_{t_0,x_0}) = \operatorname{supp}(P_{t_0,x_0})$. *Q.E.D.*

REMARK 4.1. Let $a_i\colon [0,\infty)\times R^d \twoheadrightarrow S_d$, $i = 1, 2$, and $b\colon [0,\infty)\times R^d \twoheadrightarrow R^d$ be bounded measurable functions which are uniformly Lipschitz continuous in x. Define $\mathscr{S}_{a_i,b}(t_0, x_0)$ to be the class of $\phi\in C([0,\infty), R^d)$ for which there exists a piecewise constant $\psi\colon [0,\infty)\times R^d \twoheadrightarrow R^d$ such that

$$(4.33)\qquad \phi(t) = x_0 + \int_{t_0}^t a_i(u, \phi(u))\psi(u)\,du + \int_{t_0}^t b(u, \phi(u))\,du, \qquad t\geqq t_0.$$

Then Range $(a_1(t, x)) \subseteqq$ Range $(a_2(t, x))$, $(t, x) \in [t_0, \infty) \times R^d$, implies

$$\overline{\mathscr{S}_{a_1,b}(t_0, x_0)} \subseteqq \overline{\mathscr{S}_{a_2,b}(t_0, x_0)}. \tag{4.34}$$

REMARK 4.2. Suppose $a: [0, \infty) \times R^d \twoheadrightarrow S_d$ satisfies the conditions stated in Remark 2.1. Then for each (t_0, x_0) there is exactly one solution P_{t_0,x_0} to the martingale problem for $L_t = \frac{1}{2}\nabla_x \cdot (a\nabla_x) + b \cdot \nabla_x$, where

$$\nabla_x \cdot (a\nabla_x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a^{ij}(t, x)\frac{\partial}{\partial x_j}\right) \tag{4.35}$$

and $b: [0, \infty) \times R^d \twoheadrightarrow R^d$ is bounded, measurable, and uniformly Lipschitz continuous in s. Moreover, if $a^{1/2}(t, x)$ possesses first spatial derivatives which are uniformly Lipschitz continuous in x, then L_t can be written in the form

$$L_t = \tfrac{1}{2}a^{1/2}\nabla_x \cdot a^{1/2}\nabla_x + (b + a^{1/2}c) \cdot \nabla_x; \tag{4.36}$$

and therefore

$$\text{supp}\,(P_{t_0,x_0}) \subseteqq \overline{\mathscr{S}_{a^{1/2},b}(t_0, x_0)}. \tag{4.37}$$

By Remark 4.1, $\overline{\mathscr{S}_{a^{1/2},b}(t_0, x_0)} = \overline{\mathscr{S}_{a,b}(t_0, x_0)}$, and so we have

$$\text{supp}\,(P_{t_0,x_0}) \subseteqq \overline{\mathscr{S}_{a,b}(t_0, x_0)}. \tag{4.38}$$

Using a localization procedure, it is possible to prove (4.38) under the assumptions on a stated in Remark 2.1, without any further conditions on $a^{1/2}$.

REMARK 4.3. Suppose $\sigma: [0, \infty) \times R^d \twoheadrightarrow R^d \times R^d$ and $b: [0, \infty) \times R^d \twoheadrightarrow R^d$ are bounded infinitely differentiable functions. Define vector fields

$$X_\ell = \sum_{i=1}^{d} \sigma^{i\ell}(t, x)\frac{\partial}{\partial x_i}, \qquad 1 \leqq \ell \leqq d, \tag{4.39}$$

and

$$Y = \sum_{i=1}^{d} b_i(t, x)\frac{\partial}{\partial x_i}. \tag{4.40}$$

Then

$$\tfrac{1}{2}\sigma^*\nabla_x \cdot \sigma^*\nabla_x + b \cdot \nabla_x = \tfrac{1}{2}\sum_{\ell=1}^{d} X_\ell^2 + Y. \tag{4.41}$$

Using the techniques of Bony [1], one can show that $\overline{\mathscr{S}_{\sigma,b}(t_0, x_0)}$ contains all $\phi \in C([0, \infty), R^d)$ such that

$$\phi(t) = x_0 + \int_{t_0}^{t} Z(u, \phi(u))\,du + \int_{t_0}^{t} Y(u, \phi(u))\,du, \qquad t \geqq t_0, \tag{4.42}$$

where Z is an element of the Lie algebra $\mathscr{L}(X_1, \cdots, X_d)$ generated by $X_1, \cdots, X_d$. In particular, if $\mathscr{L}(X_1, \cdots, X_d)$ has rank d at every point, then $\overline{\mathscr{S}_{\sigma,b}(t_0, x_0)}$ coincides with the set of $\phi \in C([0, \infty), R^d)$ such that $\phi(t_0) = x_0$.

5. The degenerate case, part II

Throughout this section $\sigma\colon [0, \infty) \times R^d \rightsquigarrow R^d \otimes R^d$ will satisfy $\sigma^{ij} \in C_b^{1,2}([0, \infty) \times R^d)$, $1 \leqq i, j \leqq d$, and $b\colon [0, \infty) \times R^d \rightsquigarrow R^d$ will be a bounded measurable function which is uniformly Lipschitz continuous in x. For each (t_0, x_0), P_{t_0,x_0} will denote the unique solution to the martingale problem for $L_t = \frac{1}{2}\sigma^*\nabla_x \cdot \sigma^*\nabla_x + b \cdot \nabla_x$, starting at (t_0, x_0). Our aim is to prove that

$$\operatorname{supp}(P_{t_0,x_0}) = \overline{\mathscr{S}_{\sigma,b}(t_0, x_0)}, \tag{5.1}$$

where $\mathscr{S}_{\sigma,b}(t_0, x_0)$ is defined as in Section 4. In view of Theorem 4.1, equation (5.1) will be proved once we have shown that for all $T > t_0$, $\varepsilon > 0$, and ϕ in a dense subset of $\mathscr{S}_{\sigma,b}(t_0, x_0)$,

$$P_{t_0,x_0}\left(\|x(t) - \phi(t)\|_T^{t_0} < \varepsilon\right) > 0 \tag{5.2}$$

and $x_0 = 0$.

Using Theorem 2.2, one sees that the desired result is equivalent to proving that, for a dense set of $\phi \in \mathscr{S}_{\sigma,b}(0, 0)$,

$$W\left(\|\eta(t) - \phi(t)\|_T^0 < \varepsilon\right) > 0, \qquad T > 0, \varepsilon > 0, \tag{5.3}$$

where

$$\eta(t) = \int_0^t \sigma(u, \eta(u))\, d\beta(u) + \int_0^t \tilde{b}(u, \eta(u))\, du \qquad \text{a.s. } W, \tag{5.4}$$

$\beta(t)$ being a W Brownian motion and $\tilde{b}$ standing for $b + \frac{1}{2}\sigma'\sigma$ (see equation (2.35) for the definition of $\sigma'\sigma$). Actually, what we will show is more; namely, we will prove that if $\psi \in C^2([0, \infty) \times R^d)$, $\psi(0) = 0$, and

$$\phi(t) = x_0 + \int_0^t \sigma(u, \phi(u))\dot{\psi}(u)\, du + \int_0^t b(u, \phi(u))\, du, \tag{5.5}$$

then, for all $\varepsilon > 0$ and $T > 0$,

$$W\left(\|\eta(t) - \phi(t)\|_T^0 < \varepsilon \,\middle|\, \||\beta(t) - \psi(t)\||_T^0 < \delta\right) \to 1 \tag{5.6}$$

as $\delta \downarrow 0$, where $\||\alpha(t)\||_T^0 = \max_{1 \leqq i \leqq d} \|\alpha_i(t)\|_T^0$. We will first prove (5.6) when $\psi \equiv 0$.

After some easy manipulations involving Itô's formula (see McKean [5]), one can show that (5.4) is equivalent to

$$\begin{aligned}\eta(t) = \int_0^t b(u, \eta(u))\, du &+ \sigma(t, \eta(t))\beta(t) \\ &- \int_0^t \left[\left(\frac{\partial}{\partial u} + L_u\right)\sigma\right](u, \eta(u))\beta(u)\, du - \Delta(t),\end{aligned} \tag{5.7}$$

where

$$\Delta_i(t) = \sum_{j \neq k} \int_t^t (\sigma_\ell^{ij} \sigma^{\ell k})(u, \eta(u))\beta_j(u)\, d\beta_k(u) + \sum_j \int_0^t (\sigma_\ell^{ij} \sigma^{\ell j})(u, \eta(u))\, d\beta_j^2(u). \tag{5.8}$$

Thus, in order to prove that

$$W(\|\eta(t) - \phi(t)\|_T^0 < \varepsilon \mid |||\beta(t)|||_T^0 < \delta) \to 1 \tag{5.9}$$

as $\delta \downarrow 0$, where $\phi(t) = \int_0^t b(u, \phi(u))\, du$, it suffices to show that

$$W(\|\Delta(t)\|_T^0 < \varepsilon \mid |||\beta(t)|||_T^0 < \delta) \to 1 \tag{5.10}$$

as $\delta \downarrow 0$.

LEMMA 5.1. *Let $\eta(t)$ be given by* (5.4) *and suppose f: $[0, \infty) \times R^d \twoheadrightarrow R$ is bounded and uniformly continuous. Then for all $\varepsilon > 0$*

$$W\left(\left\| \int_0^t f(u, \eta(u))\, d\beta_i^2(u) \right\|_T^0 < \varepsilon \mid |||\beta(t)|||_T^0 < \delta\right) \to 1 \tag{5.11}$$

as $\delta \downarrow 0$, and

$$W\left(\left\| \int_0^t f(u, \eta(u))\beta_i(u)\, d\beta_j(u) \right\|_T^0 < \varepsilon \mid |||\beta(t)|||_T^0 < \delta\right) \to 1 \tag{5.12}$$

as $\delta \downarrow 0$, where $1 \leqq i \leqq d$ and $j \neq i$.

PROOF. We will first prove (5.11) under the assumption that $f \in C_b^\infty([0, \infty) \times R^d)$. Applying Itô's formula, we have

$$\begin{aligned}\int_0^t f(u, \eta(u))\, d\beta_i^2(u) &= f(t, \eta(t))\beta_i^2(t) - \int_0^t \beta_i^2(u)(f_u + L_u f)(u, \eta(u))\, du \\ &\quad - 2\int_0^t \beta_i(u)(\sigma^* \nabla_x f)_i(u, \eta(u))\, du \\ &\quad - \int_0^t \beta_i^2(u)\langle(\sigma^* \nabla_x f)(u, \eta(u)), d\beta(u)\rangle.\end{aligned} \tag{5.13}$$

Clearly, the first three terms on the right tend to 0 as $|||\beta(t)|||_T^0 \to 0$. Moreover, by standard estimates,

$$\begin{aligned}&W\left(\left\| \int_0^t \beta_i^2(u)\langle(\sigma^* \nabla_x f)(u, \eta(u)), d\beta(u)\rangle \right\|_T^0 > \varepsilon, |||\beta(t)|||_T^0 < \delta\right) \\ &\leqq A_1 \exp\left\{\frac{-B_1\varepsilon^2}{\delta^4 T}\right\}\end{aligned} \tag{5.14}$$

and

$$W(\|\beta(t)\|_T^0 < \delta) \geqq A_2 \exp\left\{\frac{-B_2T}{\delta^2}\right\}. \tag{5.15}$$

Hence, (5.11) is proved in the case when $f \in C_b^\infty([0, \infty) \times R^d)$. Now suppose f is uniformly continuous and choose $\{f_n\}_0^\infty \subseteq C_b^\infty([0, \infty) \times R^d)$ so that $f_n \to f$ uniformly. Then

$$\begin{aligned}\int_0^t f(u, \eta(u))\, d\beta_i^2(u) - \int_0^t f_n(u, \eta(u))\, d\beta_i^2(u) \\ = 2\int_0^t (f - f_n)(y, \eta(u))\beta_i(u)\, d\beta_i(u) + \int_0^t (f - f_n)(u, \eta(u))\, du.\end{aligned} \tag{5.16}$$

Given $\varepsilon > 0$, and $T > 0$, it is clear that for large n,

$$\begin{aligned}&W\left(\left\|\int_0^t f(u, \eta(u))\, d\beta_i^2(u) - \int_0^t f_0(u, \eta(u))\, d\beta_i^2(u)\right\|_T^0 > \varepsilon, \||\beta(t)\||_T^0 < \delta\right) \\ &\leqq W\left(\left\|\int_0^t (f - f_n)(u, \eta(u))\beta_i(u)\, d\beta_i(u)\right\|_T^0 > \tfrac{1}{2}\varepsilon, \||\beta(t)\||_T^0 < \delta\right) \\ &\leqq A \exp\left\{\frac{-B_1\varepsilon^2}{\|f - f_n\|\delta^2T}\right\}.\end{aligned} \tag{5.17}$$

Hence, for all $\varepsilon > 0$ and $\alpha > 0$, there is an $n(\varepsilon, \alpha)$ such that

$$\begin{aligned}W\Bigg(\Bigg\|\int_0^t f(u, \eta(u))\, d\beta_i^2(u) \\ - \int_0^t f_n(u, \eta(u))\, d\beta_i^2(u)\Bigg\|_T^0 > \varepsilon \,\Big|\, \||\beta(t)\||_T^0 < \delta\Bigg) < \alpha\end{aligned} \tag{5.18}$$

independent of $0 < \delta \leqq 1$. Combining this with our original result, (5.11) follows.

Next we will prove (5.12) under the assumption that $f \in C_b^\infty([0, \infty) \times R^d)$. Again the general result follows from this by the technique used above. For convenience, we will let $\xi(t) = \int_0^t \beta_i(u)\, d\beta_j(u)$. Using Itô's formula, we have

$$\begin{aligned}&\int_0^t f(u, \eta(u))\beta_i(u)\, d\beta_j(u) \\ &= f(t, \eta(t))\zeta(t) - \int_0^t \xi(u)(f_n + L_n f)(u, \eta(u))\, du \\ &\quad - \int_0^t (\sigma^* \nabla_x f)_j(u, \eta(u))\beta_i(u)\, du \\ &\quad - \int_0^t \xi(u)\langle(\sigma^* \nabla_x f)(u, \eta(u)), d\beta(u)\rangle.\end{aligned} \tag{5.19}$$

Applying Theorem A.1 of the Appendix, we see that only the last term on the right need be examined. Let $\phi = (\sigma^* \nabla_x f)_\ell$. Then

$$\begin{aligned}(5.20)\qquad &\int_0^t \xi(u)(\sigma^*\nabla_x f)_\ell(u, \eta(u))\, d\beta_\ell(u)\\ &\quad = \int_0^t \phi(u, \eta(u))\, d(\xi(u)\beta_\ell(u))\\ &\quad\quad - \int_0^t \phi(u, \eta(u))\beta_i(u)\beta_\ell(u)\, d\beta_j(u) - \delta_{j,\ell}\int_0^t \phi(u, \eta(u))\beta_i(u)\, du.\end{aligned}$$

The third term on the right gives no trouble. Moreover, the second term can be handled by the estimates used to prove (5.11). Thus, we need only worry about the first term. But

$$\begin{aligned}(5.21)\qquad &\int_0^t \phi(u, \eta(u))\, d(\xi(u)\beta_\ell(u))\\ &\quad = \phi(t, \eta(t))\xi(t)\beta_\ell(t) - \int_0^t \xi(u)\beta_\ell(u)(\phi_u + L_u\phi)(u, \eta(u))\, du\\ &\quad\quad - \int_0^t \xi(u)(\sigma^*\nabla_x)_\ell(u, \eta(u))\, du\\ &\quad\quad - \int_0^t \beta_\ell(u)\beta_i(u)(\sigma^*\nabla_x\phi)_j(u, \eta(u))\, du\\ &\quad\quad - \int_0^t \langle \xi(u)\beta_\ell(u)(\sigma^*\nabla_x\phi)(u, \eta(u)), d\beta(u)\rangle.\end{aligned}$$

Again, only the final term need be examined. Using $\gamma(t)$ to denote this term, we have

$$\begin{aligned}(5.22)\qquad &W(\|\gamma(t)\|_T^0 \geqq \varepsilon, |||\beta(t)|||_T^0 < \delta)\\ &\quad = W(\|\gamma(t)\|_T^0 \geqq \varepsilon, |||\beta(t)|||_T^0 < \delta, \|\xi(t)\|_T^0 < M\delta)\\ &\quad\quad + W(\|\gamma(t)\|_T^0 \geqq \varepsilon, |||\beta(t)|||_T^0 < \delta, \|\xi(t)\|_T^0 \geqq M\delta).\end{aligned}$$

By the standard estimates,

$$(5.23)\qquad W(\|\gamma(t)\|_T^0 \geqq \varepsilon, \|\beta(t)\|_T^0 \leqq \delta, \|\xi(t)\|_T^0 < M\delta) \leqq A \exp\left\{-\frac{B\varepsilon^2}{M^2\delta^4}\right\}$$

By Theorem A.1,

$$(5.24)\qquad \lim_{M\to\infty}\ \sup_{0<\delta\leqq 1} \frac{W(\|\gamma(t)\|_T^0 \geqq \varepsilon, |||\beta(t)|||_T^0 < \delta, \|\xi(t)\|_T^0 \geqq M\delta)}{W(|||\beta(t)|||_T^0 < \delta)} = 0.$$

Combining these, we see that

$$(5.25)\qquad W(\|\gamma(t)\|_T^0 \geqq \varepsilon \mid |||\beta(t)|||_T^0 < \delta) \to 0$$

as $\delta \downarrow 0$. *Q.E.D.*

Clearly, (5.10) is an immediate consequence of Lemma 5.1. Hence, we have proved (5.6) when $\psi \equiv 0$.

THEOREM 5.1. *Let $\eta(t)$ be given by*

$$(5.26)\qquad \eta(t) = x_0 + \int_{t_0}^{t} \sigma(u, \eta(u))\, d\beta(u) + \int_{t_0}^{t} \tilde{b}(u, \eta(u))\, du \quad a.s.\ W, \qquad t \geqq t_0,$$

where $\beta(t)$ is a W Brownian motion. Given $\psi \in C^2([0, \infty), R^d)$ satisfying $\psi(t_0) = 0$, define $\phi(t)$ by

$$(5.27)\qquad \phi(t) = x_0 + \int_{t_0}^{t} \sigma(u, \phi(u))\dot{\psi}(u)\, du + \int_{t_0}^{t} b(u, \phi(u))\, du, \qquad t \geqq t_0.$$

Then for all $\varepsilon > 0$

$$(5.28)\qquad W(\|\eta(t) - \phi(t)\|_T^{t_0} < \varepsilon \mid |||\beta(t) - \psi(t)|||_T^{t_0} < \delta) \to 1$$

as $\delta \downarrow 0$.

PROOF. We may assume that $t_0 = 0$ and $x_0 = 0$. The case when $\psi \equiv 0$ has just been proved. To handle the general case, define $\bar{W}$ so that

$$(5.29)\qquad \frac{d\bar{W}}{dW} = R(t) = \exp\left\{\int_0^t \langle \dot{\psi}(u), d\beta(u)\rangle - \frac{1}{2}\int_0^t |\dot{\psi}(u)|^2\, du\right\},$$

on $\mathscr{M}_t^0$, $t \geqq 0$. Then, by Theorem 2.4, $\bar{\beta}(t) = \beta(t) - \psi(t)$ is a $\bar{W}$ Brownian motion and

$$(5.30)\qquad \eta(t) = \int_0^t \sigma(u, \eta(u))\, d\bar{\beta}(u) + \int_0^t \tilde{c}(u, \eta(u))\, du \qquad \text{a.s. } \bar{W}, \qquad t \geqq 0,$$

where $c = b + \sigma\dot{\psi}$ and $\tilde{c} = c + \frac{1}{2}\sigma'\sigma$. Hence,

$$(5.31)\qquad \bar{W}(\|\eta(t) - \phi(t)\|_T^0 < \varepsilon \mid |||\bar{\beta}(t)|||_T^0 < \delta) \to 1$$

as $\delta \downarrow 0$, where $\phi(t) = \int_0^t c(u, \phi(u))\, du$. But this means that

$$\begin{aligned}(5.32)\qquad &\lim_{\delta\downarrow 0} W(\|\eta(t) - \phi(t)\|_T^0 < \varepsilon \mid |||\beta(t) - \psi(t)|||_T^0 < \delta)\\ &= \lim_{\delta\downarrow 0} \frac{W(\|\eta(t) - \phi(t)\|_T^0 < \varepsilon,\ |||\beta(t) - \psi(t)|||_T^0 < \delta)}{E^{W}[R(T)\chi_{[0,\varepsilon)}(\|\eta(t) - \phi(t)\|_T^0)\chi_{[0,\delta)}(|||\beta(t) - \psi(t)|||_T^0)]}\\ &\qquad\qquad \cdot \frac{E^{W}[R(T)\chi_{[0,\delta)}(|||\beta(t) - \psi(t)|||_T^0)]}{W(|||\beta(t) - \psi(t)|||_T^0 < \delta)}\\ &= 1,\end{aligned}$$

since

$$(5.33)\qquad R(t) = \exp\left\{\dot{\psi}(t)\beta(t) - \int_0^t \langle \beta(u), \ddot{\psi}(u)\rangle\, du - \frac{1}{2}\int_0^t |\dot{\psi}(u)|^2\, du\right\},$$

is a continuous functional of β. *Q.E.D.*

THEOREM 5.2. *Let* $c\colon [0, \infty) \times R^d \to R^d$ *be bounded and measurable, and set*

$$L_t = \tfrac{1}{2}\sigma^* \nabla_x \cdot \sigma^* \nabla_x + (b + \sigma c) \cdot \nabla_x \tag{5.34}$$

(*as before*). *If* P_{t_0,x_0} *is the solution to the martingale problem for* (5.34) *starting at* (t_0, x_0), *then equation* (5.1) *obtains.*

PROOF. By Corollary 2.1, we may take $c \equiv 0$. Choose $\hat{P}_{t_0,x_0}$ and $\hat{\beta}(t)$ as in Theorem 2.2. Then

$$x(t) + \int_{t_0}^t \sigma(u, x(u))\, d\hat{\beta}(u) + \int_{t_0}^t \tilde{b}(u, x(u))\, du \quad \text{a.s. } \hat{P}_{t_0,x_0}, \qquad t \geqq t_0. \tag{5.35}$$

Hence, by Theorem 5.1, for all $T \geqq t_0$ and $\varepsilon > 0$,

$$\hat{P}_{t_0,x_0}(\|x(t) - (t)\|_T^{t_0} < | \,\|\hat{\beta}(t) - \psi(t)\|\|_T^{t_0} < \delta) \to 1 \tag{5.36}$$

as $\delta \downarrow 0$, where $\psi \in C^2([t_0, \infty), R^d)$ satisfies $\psi(t_0) = 0$ and (5.27). By Theorem 3.1, $\hat{P}_{t_0,x_0}(\|\hat{\beta}(t) - \psi(t)\|_T^{t_0} < \delta) > 0$ for all $T > t_0$ and $\delta > 0$. In particular, supp (P_{t_0,x_0}) contains a dense subset of $\mathscr{S}_{\sigma,b}(t_0, x_0)$.

REMARK 5.1. Let $\langle X, \rho \rangle$ be a metric space and suppose that μ is a probability measure on X. Given a μ measurable transformation $T\colon X \to X$, we say that $x \in X$ is a *continuity point* of the transformation T if for all $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \frac{\mu(T^{-1}(B(T(x), \varepsilon)) \cap B(x, \delta))}{\mu(B(x, \delta))} = 1, \tag{5.37}$$

where $B(y, \alpha) = \{z \in X\colon \rho(y, z) < \alpha\}$. In the terminology of continuity points, Theorem 5.1 says the following. Define $T\colon C^2([0, \infty), R^d) \to \Omega$ so that $T(\psi) = \phi$, where

$$\phi(t) = \int_0^t \sigma(u, \phi(u))\dot{\psi}(u)\, du + \int_0^t b(u, \phi(u))\, du, \qquad t \geqq 0, \tag{5.38}$$

and let W be Wiener measure on Ω (that is, $x(t)$ is a W Brownian motion). Then the transformation $\hat{T}$ such that $\hat{T}(\omega)(t) = \eta(t, \omega)$, where

$$\eta(t) = \int_0^t \sigma(u, \eta(u))\, dx(u) + \int_0^t \tilde{b}(u, \eta(u))\, du \quad \text{a.s. } W, \tag{5.39}$$

is a W measurable extension of T of Ω with the property that all elements in $C^2([0, \infty), R^d)$ are continuity points of $\hat{T}$.

REMARK 5.2. It seems likely that the hypotheses under which Theorem 5.1 was proved are close to the best possible. However, the authors believe that Theorem 5.2 is valid under weaker assumptions. In particular, it seems likely that if $a\colon [0, \infty) \times R^d \to S_d$ satisfies $a^{ij} \in C_b^{1,2}([0, \infty) \times R^d)$, $1 \leqq i, j \leqq d$ and $b\colon [0, \infty) \times R^d \to R^d$ is a bounded, measurable function which is uniformly Lipschitz continuous in x, then

$$\text{supp}\,(P_{t_0,x_0}) = \overline{\mathscr{S}_{a,b}(t_0, x_0)}, \tag{5.40}$$

where P_{t_0,x_0} is the solution to the martingale problem for $L_t = \frac{1}{2}\nabla_x \cdot (a\nabla_x) + b \cdot \nabla_x$ starting at (t_0, x_0).

6. Applications

Let σ and b be as in Section 5 and define $\mathscr{S}_{\sigma,b}(t_0, x_0)$, $(t_0, x_0) \in [0, \infty) \times R^d$, accordingly. Given an open $\mathscr{G} \subseteq [0, \infty) \times R^d$, take $\mathscr{G}_{\sigma,b}(t_0, x_0)$ to be the closure in $\mathscr{G}$ of $\{(t, \phi(t)) : t \geqq t_0, \phi \in \mathscr{S}_{\sigma,b}(t_0, x_0)$, and $(s, \phi(s)) \in \mathscr{G}$ for $t_0 \leqq s \leqq t\}$. Let $c: [0, \infty) \times R^d \twoheadrightarrow R^d$ be bounded, measurable, and continuous in x. Denote by P_{t_0,x_0} the solution to the martingale problem for $L_t = \frac{1}{2}\sigma^* \nabla_x \cdot \sigma^* \nabla_x + (b + \sigma c) \cdot \nabla_x$, starting at (t_0, x_0). We will use $\mathscr{H}^-_{L_t}(t_0, x_0, \mathscr{G})$ to denote the class of upper semicontinuous functions f on $\mathscr{G}$ into $R \cup \{-\infty\}$ which are bounded above and have the property that $f(t \wedge \tau, x(t \wedge \tau))$, $t \geqq t_0$, is a P_{t_0,x_0} submartingale, where $\tau = \inf\{t \geqq t_0 : (t, x(t)) \notin \mathscr{G}\}$. We use $\mathscr{H}^-_{L_t}(\mathscr{G})$ to denote the class $\bigcap_{\mathscr{G}} \mathscr{H}^-_{L_t}(t_0, x_0, \mathscr{G})$. Observe that $\mathscr{H}^-_{L_t}(t_0, x_0, \mathscr{G})$ and $\mathscr{H}^-_{L_t}(\mathscr{G})$ have the same closure properties as the analogously defined classes in Section 3. Further, note that if $f \in C_b^{1,2}(\mathscr{G})$, then $f \in \mathscr{H}^-_{L_t}(t_0, x_0, \mathscr{G})$ if and only if $(\partial f/\partial t) + L_t f \geqq 0$ on $\mathscr{G}(t_0, x_0)$.

THEOREM 6.1. *If $f \in \mathscr{H}^-_{L_t}(t_0, x_0, \mathscr{G})$ and $f(t_0, x_0) = \sup_{\mathscr{G}} f(t, x)$, then $f \equiv f(t_0, x_0)$ on $\mathscr{G}(t_0, x_0)$. Moreover, if $(t_1, x_1) \in \mathscr{G} - \mathscr{G}(t_0, x_0)$, then there is an $f \in \mathscr{H}^-_{L_t}(\mathscr{G})$ such that $f(t_0, x_0) = \sup_{\mathscr{G}} f(t, x)$ and $f(t_1, x_1) < f(t_0, x_0)$.*

PROOF. The proof is exactly the same as that of Theorem 3.2. The only difference lies in the fact that the $P_{t,x}$ are no longer strongly Feller continuous and therefore $E^{P_{t,x}}[\int_t^{\tau_t} e^{-u} h(u, x(u))\, du]$ will not, in general, be continuous. Nonetheless, if $h \leqq 0$, then it will still be upper semicontinuous, by virtue of the Feller continuity of the $P_{t,x}$. *Q.E.D.*

REMARK 6.1. The class $\mathscr{H}^-_{L_t}(\mathscr{G})$ admits the same semigroup interpretation as we gave in Remark 3.1, only it is no longer true that every measurable function f which is bounded above and satisfies $\hat{T}_t f \downarrow f$ is upper semicontinuous in the ordinary sense. However, it will be upper semicontinuous in the "intrinsic topology" of the time-space process; and one can still show that such an f will be constant on the intrinsic closure of $\{(t, \phi(t)) : t \geqq t_0, \phi \in \mathscr{S}_{\sigma,b}(t_0, x_0)$, and $(s, \phi(s)) \in \mathscr{G}$ for $t_0 \leqq s \leqq t\}$ if $f(t_0, x_0) = \sup_{\mathscr{G}} f(t, x)$.

◇ ◇ ◇ ◇ ◇

APPENDIX

Let $\beta(t)$ be a W Brownian motion and set $\xi(t) = \int_0^t \beta_i(u)\, d\beta_j(u)$, where $i \neq j$. The purpose of this section is to prove that for all $T > 0$,

$$\lim_{M \to \infty} \inf_{0 < \delta \leqq 1} W(\|\xi(t)\|_T^0 < M\delta \mid \|\beta(t)\|_T^0 < \delta) = 1. \tag{A.1}$$

It is clear that we may assume that $d = 2$, $i = 1$, and $j = 2$.

For each $(s, x) \in [0, \infty) \times R^2$, let $W_{s,x}$ be Wiener measure starting at (s, x) (that is, $W_{s,x}$ is the measure on $\langle \Omega, \mathscr{M}^s \rangle$ such that $x(t) - x$ is a $W_{s,x}$ Brownian motion). Define

$$\tau_s = \inf \{t \geqq s : |x_1(t)| \vee |x_2(t)| \geqq \delta\}, \tag{A.2}$$

and let

$$Q_{s,x}(A) = W_{s,x}(A \mid \{\tau_s > T\}), \qquad A \in \mathscr{M}^s. \tag{A.3}$$

LEMMA A.1. *Let* $g(s, x) = W_{s,x}(\tau_s > T)$ *and* $h(s, a) = W_{s,(a,0)}(\|x_1(t)\|_T^s < \delta)$. *Then* $g(s, x) = h(s, x_1) \cdot h(s, x_2)$, *and* $h \in C^\infty[(0, T) \times (-\delta, \delta)]$. *Next, let* $b(s, a) = h_a(s, a)/h(s, a)$ *in* $(0, T) \times (-\delta, \delta)$. *Then* $b(s, \cdot)$ *is nonincreasing and* $b(s, -a) = -b(s, a)$. *Finally, define* $B(s, x) = (b(s, x_1), b(s, x_2))$. *Then for all* $s \in [0, \tau)$ *and* x *such that* $|x_1| \vee |x_2| < \delta$,

$$\beta(t) = x(t) - x - \int_s^{t \wedge \tau} B(u, x(u))\, du \tag{A.4}$$

is a $Q_{s,x}$ *Brownian motion.*

PROOF. The first assertion is trivial. To prove the second assertion, note that $h_s + \frac{1}{2} h_{aa} = 0$ and that $h_s \geqq 0$. Hence,

$$b_a(s, a) = \frac{h h_{aa} - h_a^2}{h^2} \leqq 0, \tag{A.5}$$

and so $b(s, \cdot)$ is nonincreasing. Also, note that $h(s, -a) = h(s, a)$, and therefore $h_a(s, -a) = -h_a(s, a)$. Hence, $b(s, -a) = -b(s, a)$.

Finally, to prove the last assertion, let $\{\delta_n\}_1^\infty \subseteq (0, \delta)$ such that $\delta_n \uparrow \delta$ and define

$$\tau_s^{(n)} = (\inf \{t \geqq s : |x_1(t)| \vee |x_2(t)| \geqq \delta_n\}) \wedge T. \tag{A.6}$$

Then, by Itô's formula,

$$\begin{aligned} &g(t \wedge \tau_s^{(n)}, x(t \wedge \tau_s^{(n)})) - g(s, x) \\ &\qquad = \int_s^{t \wedge \tau_s^{(n)}} g(u, x(u)) \langle B(u, x(u)), dx(u) \rangle \quad \text{a.s. } W_{s,x}. \end{aligned} \tag{A.7}$$

Hence,

$$\begin{aligned} &\frac{g(t \wedge \tau_s^{(n)}, x(t \wedge \tau_s^{(n)}))}{g(s, x)} \\ &\quad = R^{(n)}(t) \\ &\quad = \exp \left\{ \int_s^{t \wedge \tau_s^{(n)}} \langle B(u, x(u)), dx(u) \rangle - \frac{1}{2} \int_s^{t \wedge \tau_s^{(n)}} |B(u, x(u))|^2\, du \right\} \\ &\qquad\qquad \text{a.s. } W_{s,x}. \end{aligned} \tag{A.8}$$

Given $A \in \mathscr{M}_{t \wedge \tau_s^{(n)}}^s$, we now have

$$\text{(A.9)} \qquad Q_{s,x}(A) = \frac{W_{s,x}(A \cap \{\tau_s > T\})}{g(s,x)} = \frac{E^{W_{s,x}}[\chi_A g(t \wedge \tau_s^{(n)}, x(t \wedge \tau_s^{(n)})]}{g(s,x)}$$
$$= E^{W_{s,x}}[R^{(n)}(t)\chi_A].$$

Thus, by Theorem 2.1,

$$\text{(A.10)} \quad X_\theta^{(n)}(t) = \exp\left\{\langle \theta, x(t \wedge \tau_s^{(n)}) - x - \int_s^{t \wedge \tau_s^{(n)}} B(u, x(u))\,du\rangle - \tfrac{1}{2}|\theta|^2(t \wedge \tau_s^{(n)} - s)\right\},$$

is a $Q_{s,x}$ martingale for all $\theta \in R^2$. Since $\tau_s^{(n)} \uparrow T$ as $n \to \infty$ and

$$\text{(A.11)} \qquad E^{Q_{s,x}}[|X_\theta^{(n)}(t)|^2] = E^{Q_{s,x}}[X_{2\theta}^{(n)}(t) \exp\{|\theta|^2(t \wedge \tau_s^{(n)} - s)\}]$$
$$\leqq \exp\{|\theta^{2(T-s)}\},$$

it follows that

$$\text{(A.12)} \quad \exp\left\{\langle \theta, x(t \wedge T) - x - \int_s^{t \wedge T} B(u, x(u))\,du\rangle - \tfrac{1}{2}|\theta|^2(t \wedge T - s)\right\},$$

is a $Q_{s,x}$ martingale for all $\theta \in R^2$. Combining these, we have that

$$\text{(A.13)} \qquad x(t) - x - \int_s^{t \wedge T} B(u, x(u))\,du$$

is a $Q_{s,x}$ Brownian motion. *Q.E.D.*

LEMMA A.2. *Let* $\xi(t) = \int_0^t x_1(u)\,dx_2(u)$. *Then*

$$\text{(A.14)} \qquad \sup_{0 \leqq s \leqq T, |x_1| \vee |x_2| < \delta} E^{Q_{s,x}}[|\xi(T) - \xi(s)|^2] \leqq C\delta^2.$$

PROOF. Note that by Lemma A.1

$$\text{(A.15)} \qquad \xi(t) = \int_s^t x_1(u)\,d\beta_1(u) + \int_s^{t \wedge T} x_1(u) b(u, x_2(u))\,du,$$

where $\beta_1(t) = x_1(t) - x_1 - \int_s^{t \wedge T} b(u, x_1(u))\,du$ is a one dimensional $Q_{s,x}$ Brownian motion. Hence,

$$\text{(A.16)} \quad E^{Q_{s,x}}[|\xi(T) - \xi(s)|^2]$$
$$\leqq 2E^{Q_{s,x}}\left[\left|\int_s^T x_1(u)\,d\beta_1(u)\right|^2\right] + 2E^{Q_{s,x}}\left[\left|\int_s^T x_1(u) b(u, x_2(u))\,du\right|^2\right].$$

Since $x_1(u) < \delta$ a.s. $Q_{s,x}$ for $s \leqq u \leqq T$,

$$\text{(A.17)} \qquad E^{Q_{s,x}}\left[\left|\int_s^T x_1(u)\,d\beta_1(u)\right|^2\right] \leqq \delta^2(T - s).$$

Using the independence of $x_1(\cdot)$ and $x_2(\cdot)$ under $Q_{s,x}$, we have

$$(A.18)\quad E^{Q_{s,x}}\left[\left|\int_s^T x_1(u)b(u, x_2(u))\,du\right|^2\right] = 2\iint_{s\leqq u_1\leqq u_2\leqq T} E^{Q_{s,x}}[x_1(u_1)x_1(u_2)]\; E^{Q_{s,x}}\cdot[b(u_1, x_2(u_1))b(u_2, x_2(u_2))]\,du_1du_2.$$

Observe that

$$(A.19)\quad E^{Q_{s,x}}[b(u_1, x_2(u_1))b(u_1, x_2(u_2))] = E^{Q_{s,x}}[b(u_1, x_2(u_1))E^{Q_{u_1,(0,x_2(u_1))}}[b(u_2, x_2(u_2))]].$$

Since $(b(u,\cdot)$ is nonincreasing and antisymmetric, it is easy to see that $v(u_1,\cdot) = E^{Q_{u_1,(0,\cdot)}}[b(u_2, x_2(u_2))]$ has the same properties. In particular,

$$(A.20)\quad E^{Q_{s,x}}[b(u_1, x_2(u_1))b(u, x_2(u_2))] \geqq 0.$$

Hence,

$$(A.21)\quad E^{Q_{s,x}}\left[\left|\int_s^T x_1(u)b(u, x_2(u))\,du\right|^2\right] \leqq \delta^2 E^{Q_{s,x}}\left[\left|\int_s^T b(u, x_2(u))\,du\right|^2\right] = \delta^2 E^{Q_{s,x}}[|\beta_2(T) - x_2(T) - x_2|^2] \leqq C\delta^2.$$

Q.E.D.

THEOREM A.1. *Define $\xi(t)$ as in Lemma* A.2 *and let $W = W_{0,0}$. Then*

$$(A.22)\quad \lim_{M\to\infty}\ \inf_{0<\delta\leqq 1}\ W(\|\xi(t)\|_T^0 < M\delta \,|\, \||x(t)\||_T^0 < \delta) = 1.$$

PROOF. Let σ be a stopping time which is dominated by T. Then for $A \in \mathscr{M}_\sigma^0$,

$$\begin{aligned}(A.23)\quad & E^W[\chi_{A\cap\{\tau_s>T\}}|\xi(T) - \xi(\sigma)|^2] \\ &= E^W[\chi_{A\cap\{\tau_s>\sigma\}}E^{W_{\sigma,x(\sigma)}}(\chi_{\{\tau_0>T\}}|\xi(T) - \xi(\sigma)|^2)] \\ &= E^W[\chi_{A\cap\{\tau_s>\sigma\}}W_{\sigma,x(\sigma)}(\tau_\sigma > T)E^{Q_{\sigma,x(\sigma)}}[|\xi(T) - \xi(\sigma)|^2]] \\ &\leqq C\delta^2 W(A\cap\{\tau_s > T\}).\end{aligned}$$

Thus if $Q = Q_{0,0}$, then

$$(A.24)\quad E^Q[|\xi(T) - \xi(\sigma)|^2 \,|\, \mathscr{M}_\sigma^0] \leqq C\delta^2.$$

Now let $\sigma_\ell = (\inf\{t \geqq 0: |\xi(t)| \geqq \ell\delta) \wedge T$. Then

$$\begin{aligned}(A.25)\quad & Q(\{\sigma_{2\ell} < T\}\cap\{|\xi(T)| \leqq \ell\delta\}) \\ &\leqq Q(\{\sigma_{2\ell} < T\}\cap\{|\xi(T) - \xi(\sigma_{2\ell})| \geqq \ell\delta\}) \\ &\leqq \frac{C}{\ell^2}Q(\sigma_{2\ell} < T).\end{aligned}$$

Thus,

$$Q(\sigma_{2\ell} < T) \leqq \frac{Q(|\xi(T)| > \ell\delta)}{1 - (C/\ell^2)} \leqq \frac{C}{\ell^2}\left(1 - \frac{C}{\ell^2}\right)^{-1}. \tag{A.26}$$

Q.E.D.

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