



ARL-TN-0775 • SEP 2016



Expressions for the Total Yaw Angle

by Benjamin A Breech

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REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188		
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1. REPORT DATE (DD-MM-YYYY) September 2016		2. REPORT TYPE Technical Note		3. DATES COVERED (From - To) June-July 2016	
4. TITLE AND SUBTITLE Expressions for the Total Yaw Angle			5a. CONTRACT NUMBER		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Benjamin A Breech			5d. PROJECT NUMBER AH80		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) U.S. Army Research Laboratory ATTN: RDRL-WML-A Aberdeen Proving Ground, MD 21005-5066			8. PERFORMING ORGANIZATION REPORT NUMBER ARL-TN-0775		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)		
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.					
13. SUPPLEMENTARY NOTES Author email: <benjamin.a.breech.civ@mail.mil>					
14. ABSTRACT The total yaw angle is the angle between the velocity and symmetry axis of a projectile. The total yaw angle can be related to the pitch and yaw angles through a widely available expression. Recently, we needed the total yaw angle and attempted to derive that expression as an exercise. We obtained a different, and seemingly contradictory, expression instead. The 2 expressions are actually equivalent, which we demonstrate.					
15. SUBJECT TERMS total yaw angle, pitch angle, yaw angle, ballistic projectile					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT UU	18. NUMBER OF PAGES 14	19a. NAME OF RESPONSIBLE PERSON Benjamin A Breech
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (Include area code) 410-278-3976

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1. Introduction

The total yaw angle, γ , of a ballistic projectile is the angle between the velocity, \mathbf{V} , of the projectile, and the axis of symmetry, \mathbf{S} , in the plane defined by those 2 vectors.* The symmetry axis can be aligned with the velocity by moving down through the pitch angle, α , and across through the yaw angle, β . These angles are related to the total yaw angle through the equation

$$\sin^2 \gamma = \sin^2 \beta + \cos^2 \beta \sin^2 \alpha. \quad (1)$$

Recently, we had reason to use the total yaw angle in some system effectiveness studies. As a first step, we attempted to derive Eq. 1 but obtained a different expression (see Section 3.2) that, at first, appears to be incorrect. In the process of investigating the discrepancy, we discovered a minor mistake in a reference text and that the expression we derived was actually equivalent to Eq. 1. We document these in the following sections.

2. Mathematical Notation

We first provide a few remarks on the mathematical notation we have adopted. Points in space are denoted by italicized text, such as A , and B . Any 2 points can be joined by a line segment, which we specify by its endpoints, as in AB . The length of the segment AB is written as $|AB|$.

Vectors are written in boldface, as in \mathbf{V} . The length of vector \mathbf{V} is simply V , which is not to be confused with a point. We only use 2 vectors, \mathbf{V} and \mathbf{S} , so this should not cause too much confusion. Technically, \mathbf{S} is not a vector; it is simply a line in space with no set magnitude. For convenience, we treat it as a vector whose magnitude can be adjusted as needed.

Triangles are specified using the symbol \triangle followed by the vertices of the triangle, as in $\triangle ABC$. The vertices may be listed in any order. Specific angles are specified by Greek letters, such as α for the pitch angle and β for the yaw angle. Other angles between line segments are specified by using \angle followed by the endpoints. For example, $\angle AOC$ would be the angle between the segments AO and OC .

*Various references we have seen list the total yaw angle as α_t . We chose to use γ to avoid confusion with the pitch angle.

3. Total Yaw Expression Derivations

The derivations for the different total yaw angle expressions generally follow the same pattern. Given the vectors \mathbf{V} and \mathbf{S} , and the angles α and β , we can set up a series of triangles and compute the lengths of each side. Through some elementary trigonometry, the lengths can be related to the angles and eventually an expression for γ emerges. Certain triangles will, by construction, be right triangles for which the trigonometry is particularly simple. For other triangles, though, care must be exercised to determine if the triangle is a right triangle or not.

3.1 First Derivation

The basics of the first derivation of $\sin \gamma$ can be found in various reference texts, such as Carlucci and Jacobson.¹ We are given \mathbf{V} , \mathbf{S} , α , and β from which we immediately set up a series of triangles, as shown in Fig. 1. We label the endpoint of \mathbf{V} as the point A . We draw another line through O such that the line is both an angle α down from \mathbf{S} and an angle β from \mathbf{V} . There is only one such line. Along this line, we choose the point C such that AC is perpendicular to OC . By construction, we must have

$$|OC| = V \cos \beta \quad \text{and} \quad |AC| = V \sin \beta. \quad (2)$$

Next, choose the point B along \mathbf{S} such that the segment CB is perpendicular to OB . This makes the triangle $\triangle OBC$ a right triangle, and so we must have

$$|BC| = |OC| \sin \alpha = V \cos \beta \sin \alpha. \quad (3)$$

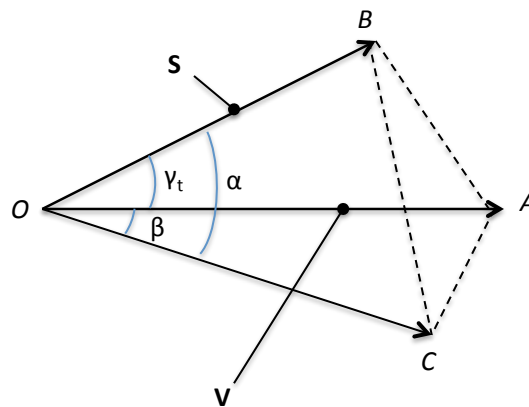


Fig. 1 Layout for the first derivation

We must also have

$$|OB|^2 = |OC|^2 - |BC|^2 = V^2 \cos^2 \beta - V^2 \cos^2 \beta \sin^2 \alpha, \quad (4)$$

which will be useful later.

The next triangle to examine is $\triangle ABC$. We chose the points B and C to make certain segments perpendicular. As such, there is no particular reason to assume that $\triangle ABC$ is a right triangle. As it turns out, $\triangle ABC$ is a right triangle with the angle $\angle ACB$ being the right angle. This provides a very minor correction to Carlucci and Jacobson's book,¹ which stated that $\triangle ABC$ was not a right triangle.* Recall from earlier that the segment AC is perpendicular to OC . Further, OC lies along the projection of OB , so AC must also be perpendicular to OB , and thus AC is perpendicular to the plane defined by OC and OB . Since the segment BC is in that plane, we conclude that AC is perpendicular to BC , and thus $\angle ACB$ is a right angle. We already know the lengths of AC and BC from earlier, so we must have

$$|AB|^2 = |AC|^2 + |BC|^2 = V^2 \sin^2 \beta + V^2 \cos^2 \beta \sin^2 \alpha.$$

Finally, we examine $\triangle OBA$, which includes the γ_f angle we are looking for. This is a right triangle since $|OB|^2 + |BA|^2 = V^2 = |OA|^2$. As such, we have

$$\begin{aligned} V \sin \gamma_f &= |AB| \\ &= \sqrt{V^2 \sin^2 \beta + V^2 \cos^2 \beta \sin^2 \alpha} \\ \sin \gamma_f &= \sqrt{\sin^2 \beta + \cos^2 \beta \sin^2 \alpha}, \end{aligned} \quad (5)$$

which is our answer.

*The correction has been noted and will appear in errata for the text.

3.2 Second Derivation

Another, perhaps simpler, derivation is to create a coordinate system where \mathbf{V} lies along the x -axis, with \mathbf{S} arbitrary, as shown in Fig. 2. In such a system, β and α become the familiar azimuthal and elevation angles from spherical coordinates.* We again place point A at the end point of \mathbf{V} . Now imagine a plane parallel to the y - z plane that includes the point A . Let the point B lie at the intersection of this plane and \mathbf{S} . The result is shown in Fig. 2. Using the spherical coordinate transformations, the coordinates of A are $(S \cos \alpha \cos \beta, 0, 0)$ and the coordinates of B are $(S \cos \alpha \cos \beta, S \cos \alpha \sin \beta, S \sin \alpha)$. In the plane, we have

$$|AB|^2 = |OB|^2 \cos^2 \alpha \sin^2 \beta + |OB|^2 \sin^2 \alpha.$$

Since the plane was chosen to be perpendicular to the x axis, we know that $\triangle OAB$ is a right triangle with $\angle OAB$ being the right angle. This means we must have $|OB| \sin \gamma = |AB|$ and thus

$$\sin \gamma = \sqrt{\cos^2 \alpha \sin^2 \beta + \sin^2 \alpha}. \quad (6)$$

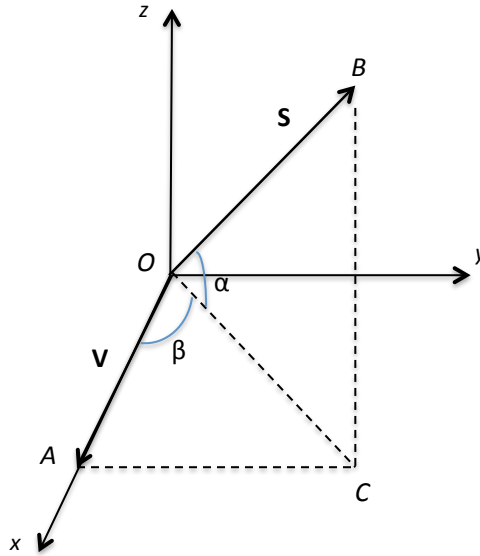


Fig. 2 Layout for the second derivation

* α is the elevation angle. Spherical coordinates typically uses the inclination angle, θ , which is related to α by $\alpha + \theta = \pi/2$.

The result in Eq. 6 is functionally identical to the result in Eq. 1, but α and β are interchanged. At first look, this may suggest that either Eq. 1 or Eq. 6 is incorrect. However, the 2 expressions are equivalent. Starting from Eq. 6,

$$\begin{aligned}
 \sin^2 \gamma_t &= \cos^2 \alpha \sin^2 \beta + \sin^2 \alpha \\
 &= (1 - \sin^2 \alpha) \sin^2 \beta + \sin^2 \alpha \\
 &= \sin^2 \beta + (1 - \sin^2 \beta) \sin^2 \alpha \\
 &= \sin^2 \beta + \cos^2 \beta \sin^2 \alpha.
 \end{aligned} \tag{7}$$

Thus, Eq. 6 is simply an alternative expression for finding γ_t .

3.3 Other Expressions

We can obtain other expressions for γ_t by modifying the derivation or manipulating the known expressions. For example, we can use the coordinate system from Section 3.2 to derive another expression by drawing the plane perpendicular to the line segment that makes an angle β from \mathbf{V} . Working out the side lengths of the various triangles will lead to an expression for $\cos \gamma_t$. Alternatively, we could simply manipulate the earlier $\sin \gamma_t$ expressions to find

$$\begin{aligned}
 \sin^2 \gamma_t &= \sin^2 \beta + \cos^2 \beta \sin^2 \alpha \\
 1 - \cos^2 \gamma_t &= 1 - \cos^2 \beta + \cos^2 \beta - \cos^2 \beta \cos^2 \alpha \\
 \cos^2 \gamma_t &= \cos^2 \beta \cos^2 \alpha.
 \end{aligned} \tag{8}$$

Since the $\sin \gamma_t$ and $\cos \gamma_t$ are now known, it would be a trivial matter to construct expressions for $\tan \gamma_t$. We again find multiple expressions:

$$\tan^2 \gamma_t = \frac{\tan^2 \beta + \sin^2 \alpha}{\cos^2 \beta} = \frac{\tan^2 \alpha + \sin^2 \beta}{\cos^2 \alpha}. \tag{9}$$

Note that, once again, multiple expressions are found that have the same functional form but have α and β interchanged. The same was true for $\cos \gamma_t$.

4. Conclusion

We have presented multiple expressions relating the total yaw angle, γ , to the pitch angle, α , and the yaw angle, β , of a ballistic projectile with velocity \mathbf{V} and symmetry axis \mathbf{S} . The derivations all follow roughly the same procedure of drawing triangles and working out the lengths of their sides. None of the derivations are particularly difficult; however, care must be taken to determine which triangles are right triangles and which are not, as it is seldom obvious.

It is well worth noting that the multiple expressions for $\sin \gamma$ (and also $\cos \gamma$ and $\tan \gamma$) have the same functional form but allow α and β to be interchanged. This type of symmetry between α and β does not appear to be noted in various reference texts. This can easily cause confusion as one may derive the alternative expression and not realize that it is equivalent to the often quoted expression shown in Eq. 1.

5. References

1. Carlucci DE, Jacobson SS. Ballistics: theory and design of guns and ammunition. 2nd ed. Boca Raton (FL): CRC Press; 2014.

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