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Further results on constructions of generalized bent Boolean functions

Fengrong ZHANG[1], Shixiong XIA[1*], Pantelimon STĂNICĂ[2] & Yu ZHOU[3]

[1]*School of Computer Science and Technology, China University of Mining and Technology, Xuzhou* 221116*, China;*
[2]*Naval Postgraduate School, Applied Mathematics Department, Monterey, CA* 93943*, USA;*
[3]*Science and Technology on Communication Security Laboratory, Chengdu* 610041*, PR China*

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Dear editor,

Boolean bent functions were introduced by Rothaus in 1976 as an interesting combinatorial object with the important property of having optimal nonlinearity [1]. Since bent functions have many applications in sequence design, cryptography and algebraic coding, they have been extensively studied during the last thirty years [2, 3]. Over the past decades, based on bent functions, several constructions of highly nonlinear balanced functions were presented [4, 5].

In recent years several researchers have proposed generalizations of Boolean functions [6–9] and studied the effect of the Walsh-Hadamard transform on these classes. In [6], Schmidt presented the connection between words in multicode code-division multiple access (MC-CDMA) systems and generalized bent functions from $\mathbb{Z}_2^m$ to $\mathbb{Z}_4$, and considered functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ from the viewpoint of cyclic codes over rings. Later, Solé and Tokareva [7] called these functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ generalized Boolean functions and presented the direct links between Boolean bent functions and generalized bent functions. More recently, Stănică et al. [9] investigated the properties of generalized bent functions and presented several constructions of such generalized bent functions for both n even and n odd. They characterized a class of generalized bent functions symmetric with respect to two variables and generalized bent functions defined on $\mathbb{Z}_2^n$ in $\mathbb{Z}_8$. However, is there a technique that provides generalized bent functions symmetric with respect to m variables, where m is even? Additionally, in [9, Example 20, 21] the authors provided an explicit construction only for the even case. These give us a motivation to identify those generalized bent functions.

Let us denote the set of integers, real numbers and complex numbers by $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, respectively and let the ring of integers modulo r be denoted by $\mathbb{Z}_r$. We denote the addition over $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ by '+'. Moreover, addition modulo q ($\neq 2$) is also denoted by '+' and it is understood from the context. Let $\mathbb{Z}_2^n$ be the n-dimensional vector space over $\mathbb{Z}_2$. We denote the addition over $\mathbb{Z}_2^n$ and $\mathbb{Z}_2$ by '$\oplus$'. Letting $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$, we define the inner (or scalar) product by $\boldsymbol{\omega} \cdot \boldsymbol{x} = \omega_1 x_1 \oplus \ldots \oplus \omega_n x_n$. If $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$, then $|z| = \sqrt{a^2 + b^2}$ denotes the absolute value of z, where $i^2 = -1$. We denote the vectors $(0, 0, \ldots, 0) \in \mathbb{Z}_2^n$ by $\mathbf{0}_n$.

A function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ ($q \geqslant 2$ a positive integer) is called a *generalized Boolean function* in n variables [7]. Let $\mathcal{GB}_n^q$ be the set of all n-variable

* Corresponding author (email: xiasx@cumt.edu.cn)
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generalized Boolean functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$. If $q=2$, we obtain the classical Boolean functions in n variables, whose set will be denoted by $\mathcal{B}_n$. The *Hamming weight* $\mathrm{wt}(\boldsymbol{u})$ of a vector $\boldsymbol{u}\in\mathbb{Z}_2^n$ is the weight (number of 1's) of the binary string.

The *(generalized) Walsh-Hadamard transform* of $f\in\mathcal{GB}_n^q$ is the complex valued function over $\mathbb{Z}_2^n$ which is defined by $\mathcal{H}_f(\boldsymbol{\omega})=\sum_{\boldsymbol{x}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{x})}(-1)^{\boldsymbol{\omega}\cdot\boldsymbol{x}}$ where $\zeta(=e^{2\pi i/q})$ is the complex q-primitive root of unity. When $q=2$, we obtain the Walsh transform of $f\in\mathcal{B}_n$, which will be denoted by $\mathcal{W}_f$.

A generalized Boolean function $f\in\mathcal{GB}_n^q$ is called *generalized bent* (or *gbent*, for short) if and only if $|\mathcal{H}_f(\boldsymbol{\omega})|=2^{n/2}$ for all $\boldsymbol{\omega}\in\mathbb{Z}_2^n$. Note that when $q=2$, Boolean bent functions exists only if the number n of variables is even. For $q>2$, if f is a gbent function in n variables, it does not follow that n must be even. Such functions for $q=4$ were investigated by Schmidt [6], Solé and Tokareva [7], Stănică, Martinsen, Gangopadhyay, and Singh [9], etc.

The sum $\mathcal{C}_{f,g}(\boldsymbol{u})=\sum_{\boldsymbol{x}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{x})-g(\boldsymbol{x}\oplus\boldsymbol{u})}$ is the *crosscorrelation* of f and g at $\boldsymbol{u}\in\mathbb{Z}_2^n$. The *autocorrelation* of $f\in\mathcal{GB}_n^q$ at $\boldsymbol{u}\in\mathbb{Z}_2^n$ is $\mathcal{C}_{f,g}(\boldsymbol{u})$ above, which is denoted by $\mathcal{C}_f(\boldsymbol{u})$.

Lemma 1. Let $f\in\mathcal{GB}_n^q$. Then f is a gbent function if and only if

$$\mathcal{C}_f(\boldsymbol{u})=\begin{cases}2^n, & \text{if } \boldsymbol{u}=\boldsymbol{0}_n,\\ 0, & \text{if } \boldsymbol{u}\neq\boldsymbol{0}_n.\end{cases}$$

By using Lemma 1, we can prove the following theorem.

Theorem 1. Let n be a positive integer and m,q be even positive integers. Let $f\in\mathcal{GB}_n^q$ be gbent. Let $f+\frac{q}{2}g_i\in\mathcal{GB}_n^q$ be gbent, where $i=0,1$. Let $\boldsymbol{y}=(\boldsymbol{y}',\boldsymbol{y}'')$, $\boldsymbol{y}'=(y_1,y_2,\ldots,y_{m/2})$, $\boldsymbol{y}''=(y_{m/2+1},y_{m/2+2}\ldots y_m)$ and $\vartheta(\boldsymbol{y})=\boldsymbol{y}'\cdot\boldsymbol{y}''$. Let $\boldsymbol{c}\in\mathbb{Z}_2^m$ and $\mathrm{wt}(\boldsymbol{c})$ be even. Then the function $h\in\mathcal{GB}_n^q$, defined by

$$h(\boldsymbol{x},\boldsymbol{y})=f(\boldsymbol{x})+\frac{q}{2}(\boldsymbol{c}\cdot\boldsymbol{y})g_{\boldsymbol{c}\cdot\boldsymbol{y}}(\boldsymbol{x})+\frac{q}{2}\vartheta(\boldsymbol{y}) \quad (1)$$

is a gbent function in $n+m$ variables.

In Table 1, we compare our approach to other methods [9, 10] in terms of the form of gbent functions.

In what follows, we first provide some notations.

If $f\in\mathcal{B}_n$ is bent, then the dual function $\widetilde{f}$ of f, defined on $\mathbb{Z}_2^n$ by $\mathcal{W}_f(\boldsymbol{\omega})=2^{n/2}(-1)^{\widetilde{f}(\boldsymbol{\omega})}$ is also bent and it is known that $\widetilde{\widetilde{f}}=f$.

Lemma 2. For every $\boldsymbol{a},\boldsymbol{b}\in\mathbb{Z}_2^n$ and for every bent function f, the dual of the function $f(\boldsymbol{x}\oplus\boldsymbol{b})\oplus\boldsymbol{a}\cdot\boldsymbol{x}$ equals $\widetilde{f}(\boldsymbol{x}\oplus\boldsymbol{a})\oplus\boldsymbol{b}\cdot(\boldsymbol{x}\oplus\boldsymbol{a})$.

The original Maiorana-McFarland's (M-M) class of bent functions is the set of all the (bent) Boolean functions on $\mathbb{Z}_2^{2n}=\{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}_2^n\}$ of the form

$$f(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{x}\cdot\phi(\boldsymbol{y})\oplus g(\boldsymbol{y}), \quad (2)$$

where ϕ is any permutation of $\mathbb{Z}_2^n$ and $g\in\mathcal{B}_n$.

Let $f\in\mathcal{B}_n$. If there exists an even integer $0\leqslant r\leqslant n$, such that $\|\{\boldsymbol{\omega}|\mathcal{W}_f(\boldsymbol{\omega})\neq 0,\boldsymbol{\omega}\in\mathbb{F}_2^n\}\|=2^r$, where $\|\cdot\|$ denotes the size (cardinality) of a set, and $(\mathcal{W}_f(\boldsymbol{\omega}))^2$ equals 2^{2n-r} or 0, for every $\boldsymbol{\omega}\in\mathbb{F}_2^n$, then f is called an *r-order plateaued* function in n variables. If f is a $2\lceil(n-2)/2\rceil$-order plateaued function in n variables, then f is also called a *semibent* function.

Let $f\in\mathcal{GB}_n^8$ be as

$$f(\boldsymbol{x})=\upsilon_0(\boldsymbol{x})+\upsilon_1(\boldsymbol{x})\cdot 2+\upsilon_2(\boldsymbol{x})\cdot 2^2, \quad (3)$$

where $\upsilon_i(\boldsymbol{x})\in\mathcal{B}_n, i=0,1,2$.

In [9, Theorem 19], Stănică et al. presented a sufficient and necessary condition for a function f as in (3) to be gbent.

Theorem 2 ([9])**.** Let $f\in\mathcal{GB}_n^8$ be as in (3). The following are true:

(i) If n is even, then f is gbent if and only if $\upsilon_2,\upsilon_0\oplus\upsilon_2,\upsilon_1\oplus\upsilon_2,\upsilon_0\oplus\upsilon_1\oplus\upsilon_2$ are all bent, and $\mathcal{W}_{\upsilon_0\oplus\upsilon_2}(\boldsymbol{u})\mathcal{W}_{\upsilon_1\oplus\upsilon_2}(\boldsymbol{u})=\mathcal{W}_{\upsilon_2}(\boldsymbol{u})\mathcal{W}_{\upsilon_0\oplus\upsilon_1\oplus\upsilon_2}(\boldsymbol{u})$ for all $\boldsymbol{u}\in\mathbb{Z}_2^n$;

(ii) If n is odd, then f is gbent if and only if $\upsilon_2,\upsilon_0\oplus\upsilon_2,\upsilon_1\oplus\upsilon_2,\upsilon_0\oplus\upsilon_1\oplus\upsilon_2$ are all semibent, and $\mathcal{W}_{\upsilon_0\oplus\upsilon_2}(\boldsymbol{u})=\mathcal{W}_{\upsilon_2}(\boldsymbol{u})=0$ and $|\mathcal{W}_{\upsilon_1\oplus\upsilon_2}(\boldsymbol{u})|=|\mathcal{W}_{\upsilon_0\oplus\upsilon_1\oplus\upsilon_2}(\boldsymbol{u})|=2^{\frac{n+1}{2}}$; or, $|\mathcal{W}_{\upsilon_0\oplus\upsilon_2}(\boldsymbol{u})|=|\mathcal{W}_{\upsilon_2}(\boldsymbol{u})|=2^{\frac{n+1}{2}}$ and $\mathcal{W}_{\upsilon_1\oplus\upsilon_2}(\boldsymbol{u})=\mathcal{W}_{\upsilon_0\oplus\upsilon_1\oplus\upsilon_2}(\boldsymbol{u})=0$, for all $\boldsymbol{u}\in\mathbb{Z}_2^n$.

From the above theorem, we know that the sufficient conditions that a function f as in (3) is gbent are abstract. Hence, we provide some sufficient conditions for a function f as in (3) to be gbent.

Theorem 3. Let n be an even integer, $\upsilon_0,\upsilon_1,\upsilon_2\in\mathcal{B}_n$ and $f\in\mathcal{GB}_n^8$ be as in (3). The following $\upsilon_0,\upsilon_1,\upsilon_2$ satisfy the sufficient conditions of Theorem 2 for the even case.

(i) Let $\upsilon_0,\upsilon_1,\upsilon_2$ be bent functions and $\upsilon_2,\upsilon_0\oplus\upsilon_2,\upsilon_1\oplus\upsilon_2,\upsilon_0\oplus\upsilon_1\oplus\upsilon_2$ be all bent, and $(\upsilon_0\oplus\upsilon_2)(\boldsymbol{x})=\widetilde{\upsilon_0}(\boldsymbol{x})\oplus\widetilde{\upsilon_2}(\boldsymbol{x})$, $(\upsilon_1\oplus\upsilon_2)(\boldsymbol{x})=\widetilde{\upsilon_1}(\boldsymbol{x})\oplus\widetilde{\upsilon_2}(\boldsymbol{x})$, $(\upsilon_0\oplus\upsilon_1\oplus\upsilon_2)(\boldsymbol{x})=\widetilde{\upsilon_0}(\boldsymbol{x})\oplus\widetilde{\upsilon_1}(\boldsymbol{x})\oplus\widetilde{\upsilon_2}(\boldsymbol{x})$.

(ii) Let $\upsilon_2\in\mathcal{B}_n$ be a bent function, $\upsilon_0=\upsilon_1$ and $\upsilon_0\oplus\upsilon_2$ be bent.

(iii) Let $\upsilon_0(\boldsymbol{x})=\boldsymbol{a}_0\cdot\boldsymbol{x}$ and $\upsilon_1(\boldsymbol{x})=\boldsymbol{a}_1\cdot\boldsymbol{x}$ respectively, be two linear functions. Let $\upsilon_2\in\mathcal{B}_n$ be a bent function, and $\widetilde{\upsilon_2}(\boldsymbol{x})\oplus\widetilde{\upsilon_2}(\boldsymbol{x}\oplus\boldsymbol{a}_0)\oplus\widetilde{\upsilon_2}(\boldsymbol{x}\oplus\boldsymbol{a}_1)\oplus\widetilde{\upsilon_2}(\boldsymbol{x}\oplus\boldsymbol{a}_0\oplus\boldsymbol{a}_1)=0$.

Table 1 Form of gbent functions comparison

Number of variables	q	From	Resource
$n+2$	2	$h(\boldsymbol{x},\boldsymbol{y})=f(\boldsymbol{x})\oplus(y_1\oplus y_2)g(\boldsymbol{x})\oplus y_1y_2$	Ref. [10]
$n+2$	Even integer	$h(\boldsymbol{x},\boldsymbol{y})=f(\boldsymbol{x})+(y_1\oplus y_2)g(\boldsymbol{x})+\frac{q}{2}y_1y_2$	Ref. [9]
$n+m$	Even integer	$h(\boldsymbol{x},\boldsymbol{y})=f(\boldsymbol{x})+\frac{q}{2}(\boldsymbol{c}\cdot\boldsymbol{y})g_{\boldsymbol{c}\cdot\boldsymbol{y}}(\boldsymbol{x})+\frac{q}{2}\vartheta(\boldsymbol{y})$	New

(iv) Let $\upsilon_0(\boldsymbol{x})=\boldsymbol{a}_0\cdot\boldsymbol{x}$, be a linear function. Let $\upsilon_2\in\mathcal{B}_n$ be a bent function, $\upsilon_1\oplus\upsilon_2$ be bent and $\widetilde{\upsilon_2}(\boldsymbol{x})\oplus\widetilde{\upsilon_2}(\boldsymbol{x}\oplus\boldsymbol{a}_0)\oplus(\widetilde{\upsilon_1\oplus\upsilon_2})(\boldsymbol{x})\oplus(\widetilde{\upsilon_1\oplus\upsilon_2})(\boldsymbol{x}\oplus\boldsymbol{a}_0)=0$.

We now discuss the case when n is odd. Let n be a positive odd integer and $g_1,g_2\in\mathcal{B}_n$. We say that g_1 and g_2 are *complementary semibent functions* in n variables if they are semibent (that is, $(n-1)$-order plateaued) functions and satisfy the property that $\mathcal{W}_{g_1}(\boldsymbol{\omega})=0$ if and only if $\mathcal{W}_{g_2}(\boldsymbol{\omega})\neq 0$.

Lemma 3. Let n be an even integer and $f\in\mathcal{B}_n$. Then f is bent if and only if the two functions on $\mathbb{Z}_2^{n-1}$, $f(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)$ and $f(x_1,\ldots,x_{j-1},1,x_{j+1},\ldots,x_n)$, are complementary semibent functions on $\mathbb{Z}_2^{n-1}$, where $j=1,\ldots,n$.

Theorem 4. Let k,n be two integers and $n=2k-1$. Let $\varphi=(\varphi_1,\ldots,\varphi_k),\phi=(\phi_1,\ldots,\phi_k)$ be Boolean maps from $\mathbb{Z}_2^k$ to $\mathbb{Z}_2^k$ such that both ϕ and $\phi\oplus\varphi=(\phi_1\oplus\varphi_1,\ldots,\phi_k\oplus\varphi_k)$ are permutations on $\mathbb{Z}_2^k$. Set $\Delta_j=\{\phi(\boldsymbol{y})|\boldsymbol{y}\in\mathbb{Z}_2^{j-1}\times\{0\}\times\mathbb{Z}_2^{k-j}\}$, $\boldsymbol{y}_\epsilon^{(j)}=(y_1,\ldots,y_{j-1},\epsilon,y_{j+1}\ldots,y_k)$, where $\epsilon\in\mathbb{Z}_2,j=1,2,\cdots,k$. Let $f\in\mathcal{GB}_n^8$ be as in (3), and let $\upsilon_0(\boldsymbol{x},\boldsymbol{y}_0^{(j)})=\boldsymbol{a}_0\cdot\boldsymbol{x}\oplus\varphi(\boldsymbol{y}_0^{(j)})\cdot\boldsymbol{x}$, $\upsilon_1(\boldsymbol{x})=\Big(\phi(\boldsymbol{y}_0^{(j)})\oplus\phi(\boldsymbol{y}_1^{(j)})\Big)\cdot\boldsymbol{x}\oplus g(\boldsymbol{y}_0^{(j)})\oplus g(\boldsymbol{y}_1^{(j)})$ and $\upsilon_2(\boldsymbol{x})=\phi(\boldsymbol{y}_0^{(j)})\cdot\boldsymbol{x}\oplus g(\boldsymbol{y}_0^{(j)})$, where $\boldsymbol{a}_0\in\mathbb{Z}_2^k$. If there exists one positive integer $\varrho(\leqslant k)$ such that

$$\{(\phi\oplus\varphi)(\boldsymbol{y})|\boldsymbol{y}\in\mathbb{Z}_2^{\varrho-1}\times\{0\}\times\mathbb{Z}_2^{k-\varrho}\}=\Delta_\varrho \quad (4)$$

(if $\boldsymbol{a}_0\neq\boldsymbol{0}_k$ we further require Δ_ϱ to be a linear subspace of $\mathbb{Z}_2^k$ and $\boldsymbol{a}_0\in\Delta_\varrho$), then $\upsilon_0,\upsilon_1,\upsilon_2$ satisfy the conditions of Theorem 2 for the odd case, that is, f is gbent.

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