Discontinuous Galerkin for Navier-Stokes

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14. ABSTRACT

Recovery-Based Discntinuous Galerkin Methods for modeling the viscous/conducting terms in the Navier-Stokes equations with a polynomial basis of order p have been shown (under a previous AFOSR grant) to achieve on Cartesian grids an unmatched order of accuracy of $3p+1$ (p odd) or $3p+2$ (p even). On triangular grids all DG methods not based on recovery only achieve the order $p+1$, except Hybrid DG (Peraire and Persson), which achieves order $p+2$. We demonstrate that recovery-based DG achieves at least the order $2p$ on a triangular grid; for $p>2$ this exceeds the accurary of any existing DG method. The result has been found first for linear diffusion, but is expected to remain valid for nonlinear diffusion and for shear terms, i.e., cross-derivative terms, such as appear in nthe Navier-Stokes equations. This is still being investigated in a six-month continuation of the current project, supported by AFRL.

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Discontinuous Galerkin for Navier-Stokes

Bram van Leer
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Foreword

This Final Performance report covers the research accomplishments in the project "Discontinuous Galerkin for Navier-Stokes," which ran with funding from AFOSR/NM under Grant Nr. FA9550-11-1-0257, during the period of August 15, 2011 through February 14, 2013. Additional support by AFRL-WPAFB will allow the project to continue through August 15, 2013. The Discontinuous Galerkin methods studied here are of a special kind, *viz.*, based on *recovery* (RDG), and inclluing additional *solution-enhancement* techniques.

1   Initial status of RDG

At the start of the project, RDG was well-established on Cartesian grids up to 2 dimenions; the state of the art is summed up in the doctoral thesis of Marcus Lo[1] and the conference paper by Lo and Van Leer[2]. These results were soon extended to 3-D Cartesian grids by Varadan[3].

The most eye-catching findings were that RDG on Cartesian grids, using a tensor basis in each element and face-sharing neighbors only, achieves an order of accuracy of $3p+1$ or $3p+2$, depending on whether the degree p of the polybonial basis is odd or even. This result was demonstrated for the linear diffusion equation combined with the formulation RDG-2x, which uses partial integration twice. For it to remain valid with the RDG-1x formulation and/or nonlinear diffusion, an additional step of "solution enhancement" has to be taken. In this technique the interface data produced by recovery are used to enhance the element's polynomial basis by weak interpolation. The improved internal representation increases the accuracy of the volume integral appearing in the RDG-1x formulas, which suffices to restore the above superconvergence.

The Navier-Stokes equations, though, also contain shear terms, repre-

sented by mixed derivatives, which are not sufficiently well treated by RDG; the solution-enhancement step by itself does not improve this situation. In order to restore the superconvergence for a diffusion-shear equation, a second enhancement step is needed, which simply is a repeat of the recovery step but using the already enhanced interior solutions. Thus, the improved accuracy of the internal representation gets moved to the cell boundary and improves the accuracy of the boundary integral of the shear-term. Note that this second enhancement step, which by itself only uses face-sharing neighbors, does increase the method's stencil beyond that of the original RDG method; in the most frugal formulation on a Carteisan grid the domain of dependence only includes those elements that touch the element being updated.

If a minimal basis of order p is used, instead of a tensor basis, the order of accuracy drops to $2p + 2$, for both odd and even p. Since using a tensor basis is not feasible on triangles or tetrahedra, the accuracy of RDG extended to simplex grids is expected not to exceed the order $2p + 2$.

At the start of the project, no superconvergence results were yet available for triangular grids. All DG methods, including standard RDG, show degradation of the order of accuracy to $p + 1$ on a triangular grid, except Hybrid DG due to Peraire et al.[4], which achieves the order $p + 2$. In the presently terminating project we have attempted to improve on this result by using solution enhancement.

2   From Cartesian to triangular

When attempting solution enhancement after switching from tensor bases on Cartesian elements to minimal bases on triangular elements, we face the non-trivial problem of extending the polynomial basis by a set of basis functions that increase the order of the solution representation at least by one, while leading to a solvable system of equations when constrained by the recovered data on the element boundary.

For a given p the number of functions in a minimal basis spanning all polynomials up to the degree p is $(p + 1)(p + 2)/2$. The number of basis functions in the space containing the recovered solution f at any interface is twice that, or $(p + 1)(p + 2)$. Among these, only $p + 1$ define the solution values on the interface; the remaining $(p + 1)^2$ describe the variation of the solution normal to the face.

When increasing the order of the elemental basis from $p = P$ to $p = P+1$,

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the number of basis functions to be added is $P + 1$. On the other hand, the number of basis functions resulting from recovery at each interface is $(P + 1)(P + 2)$, so in total there are $3(P + 1)(P + 2)$ face data available to find only $P + 1$ new interior coefficients, truly a lavish choice. If we restrict the face data to properties of the distribution along the faces, we still have $3(P+1)$ data to work with. It is clear that not all of the boundary information need to be used; the question is: which should be ignored? The choice must be robust, i. e., the rule for choosing the boundary data must remain valid for all P, and the resulting system of equations must always be solvable.

Inversely, we could also choose to raise the order of the interior basis by more than 1, for instance by 2, or even 3. Choosing 3 is particularly attractive, because then the number of additional interior basis functions asymptotically scales with $3P$, while the number of independent data defining the solution values on the faces also scales with $3P$, because the element has 3 faces

3 Accomplishment: a new convergence result

During most of the grant peiod we conducted strings of experiments in which the order of the interior basis was raised by 1 or 2. These are reported in detail in an accepted AIAA conference abstract[5], which we have attached to this report as an Appendix. The convergence results for these experiments were not robust and the findings inconclusive. In the abstract, submitted in November 2012, the possibility of enhancing the basis from P to $P + 3$ is discussed but not yet implemented.

When we finally implemented this choice, success came after some experimentation. Robust convergence results became available around the closing date of the project. The main conclusion is that the order of triangular RDG-1x with solution enhancement is $2p$, rather than $p + 1$, for $p \geq 1$. The computational expense of the method, though, is high, and our current research effort, supported by AFRL, is aimed at achieving the same accuracy at much reduced cost. We expect to be able to show results of this kind in the final version of the AIAA conference paper, to be presented in June in San Diego, CA.

4   Method of analysis

The accuracy assessment of the RDG methods on a triangular grid is done with Fourier analysis. To make Fourier analysis feasible, we keep the grid structured; it is obtained from a uniform Cartesian grid by inserting cell diagonals with the same orientation. In spite of the regularity of this grid the order of accuracy of standard RDG-1x drops to $p + 1$; the mere presence of one diagonal face ruins the superconvergence. We therefore feel certain that, inversely, if superconvergence can be restored on this grid by solution enhancement, this will carry over to any irregular triangular grid. Figure 1 of the Appendix shows the stencil on which the Fourier analysis for our class of RDG methods (including two enhancement steps) is based.

Acknowledgement/Disclaimer

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[1] M. Lo, *A Space-Time Discontinuous Galerkin Method for Navier-Stokes witth Recovery*, PhD thesis, University of Michigan, 2011.

[2] M. Lo and B. van Leer, "Recovery-based discontinuous Galerkin for Navier-Stokes viscous terms," AIAA Paper 2011-3406, 2011.

[3] E. Johnsen, S. Varadan, and B. van Leer, "A three-dimensional recovery-based Discontinuous Galerkin method for turbulence simulations," AIAA Paper 2013-0515, 2013.

[4] N. C. Nguyen, J. Peraire, and B. Cockburn, "An implicit high-order hybridizable discontinuous Galerkin method for the incompressible

Navier.Stokes equations," *J. Computational Phys.* **230** (2011), pp. 1147–1170.

[5] H. L. Khieu, B. van Leer, and M. Lo, "Optimal Accuracy of Discontinuous Galerkin for Diffusion." To be presented at the 21st AIAA Computational Fluid Dynamics Conference
24-27 June 2013, San Diego, CA.

5 Personnel supported

1. Bram van Leer, Professor (PI); Professor Emeritus starting September1, 2012

2. Huu Loc Khieu, doctoral student; postdoctoral fellow starting May 1, 2012

3. Kentaro Hara, doctoral student

4. Sreenivas Varadan, doctoral student

6 Publications

Reference Nr. 5 above

7 Honors and awards

1. B. van Leer was Arthur B. Modine Professor of Aerospace Engineering from September 1, 2007 through August 31, 2012. He was awarded Emeritus status per September 1, 2012. His lifetime achievements include an Honorary Doctorate from the Free University of Brussels (1990) and the AIAA 2010 Fluid Dynamics Award.

2. Loc Khieu was awarded the Ph. D. degree in May 2012.

8 AFRL point of contact

John Benek, AFRL/WPAFB

9   Interactions/transitions

(a) *Conference presentations:*

    (1) First International Workshop on High-Order CFD Methods, Nashville, TN, January 2012 (co-organizer: B. van Leer, speakers: B. van Leer, K. Hara, S. Varadan)

    (2) AFOSR Computational Mathematics Program Review, August 2012, Arlington, VA (speaker: Loc Khieu)

(b) *Consulting/Advising:* None

(c) *Transition:* Subcontract with HyperComp (Westlake Village, CA) in STTR Phase 2 Project titled "High-order modeling of applied multiphysics phenomena;" through July 2012.

10   New discoveries

None patentable

6

Apeendix

Optimal Accuracy of
Discontinuous Galerkin for Diffusion

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I.   Introduction

The Discontinuous Galerkin (DG) method was originally developed for advection-type operators, for which it is pre-eminently suited,[1] but soon got applied to diffusion operators because of the need to model advection-diffusion processes with one numerical strategy. DG, however, is not naturally suited for diffusion operators, precisely because of the discontinuous solution representation, and requires a special step to overcome this handicap. Most of the newer methods require rewriting the second-order differential operator as a system of first-order operators, in order to arrive at a stable and accurate approximation.[2] In recovery-based DG (RDG) a smooth solution basis, weakly identical to the discontinuous basis, is introduced for computing diffusive fluxes.[3]

For a given order p of the elemental polynomial basis, what is the maximum order of accuracy DG can reach for diffusion, if we allow only the direct neighbors of an element to participate in the discretization? The answer to this question is only known in part. On a Cartesian grid, which brings out the best in all DG methods, RDG has been demonstrated

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to achieve the order $3p+2$ or $3p+1$ for p even or odd, respectively;[4] this is the highest order found so far among DG schemes for diffusion. The result is robust: it holds in any number of dimensions, for nonlinear equations and equations with mixed derivatives, provided that (1) a tensor-product basis is used for the solution, (2) solution enhancement,[5] a technique of weak interpolation from element boundary to interior, is used to improve the volume integral in the DG equation, and (3) an extra recovery step is used to improve the element-boundary integral. Lo[6,5] was the first to show the optimal accuracy for 2D Navier-Stokes terms; Varadan et al.[7] showed it for 3D turbulence calculations.

When the tensor-product basis is abandoned for a lean basis of order p, the resulting order of RDG on a Cartesian grid immediately reduces to 2p+2, which is still attractive. On simplex elements one can not maintain a tensor basis anyway, so we expect that on unstructured triangular or tetrahedral grids the order barrier is at best 2p+2. Most DG methods for diffusion, including basic RDG, reduce to the order of accuracy $p+1$; only HDG[8] has been shown to yield $p+2$, owing to solution enhancement.

In the present paper we describe our efforts to extend to a triangular grid the RDG techniques that yield the optimal accuracy on a 2-D Cartesian grid. Our grid of orthogonal triangles, shown in Figure 1 allows a Fourier analysis to find the order of accuracy of the DG discretization. As is always the case with RDG schemes, the key issues to be addressed are what basis to choose in an element before and after solution enhancement, and how to balance the number of conditions available for enhancement with the number of functions enriching the basis in the enhancement process.
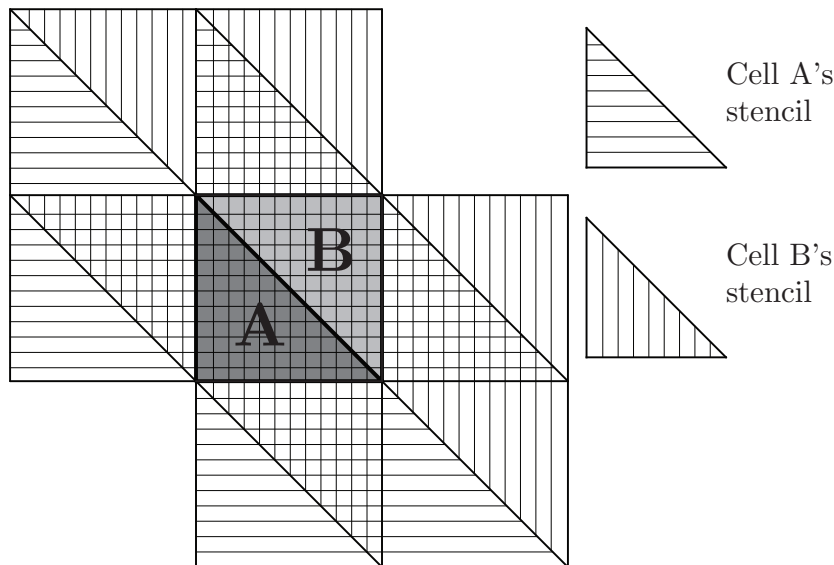

Figure 1. The stencil for Fourier analysis is the union of the stencils used for updating the triangles A and B which together form s square element of the underlying Cartesian grid.

II.   Counting basis functions and equations

We start discussing the balance of basis functions and enhancement equations for RDG-1x++CO,[5] where CO stands for "Cartesian optimization". This discretization, designed

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exclusively for a Cartesian grid, achieves the solution accuracy of order $3p + 1(3p + 2)$ for p odd(even), while using only the nearest neighbors; Figure 2 shows the scheme's stencil.
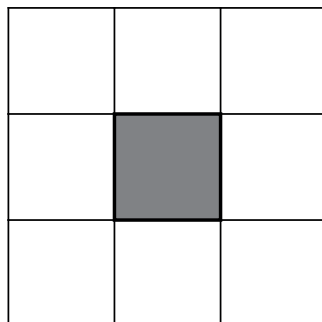
RDG-1x++CO

Figure 2. Stencil for RDG-1x++CO.

The scheme achieves its high order of accuracy in three steps: recovery is followed by enhancement which is followed by another recovery. Since recovery spreads the stencil in the direction normal to the cell face, recovering twice would spread the stencil beyond the first ring of neighbors. To avoid this, the standard enhancement is replaced by two independent enhancements biased along the cell faces. An enhanced solution $\hat{u}_x$ is obtained by combining the solution u with the solution on the left and right faces, while an enhanced solution $\hat{u}_y$ is obtained using the solution on the bottom and top faces. These enhanced solutions figure in the second recovery step as follows: $\hat{u}_x$ is used to obtain the solution on the bottom and top faces, while $\hat{u}_y$ is used to obtain the solution at the left and right faces. Thus the solution along each faces has been enhanced to the order $p + 2$ without further raising of the order normal to the faces and further spreading of the stencil. To compute the volume integral in the DG equation we use the unbiased enhanced solution

$$\hat{u} = \hat{u}_x + \hat{u}_y - u. \tag{1}$$

The basis of this solution is almost a tensor-product basis of the order $p+2$, except that the highest-order 2×2 block is missing. But the basis does contain a full basis of order $p + 2$.

On a triangular grid, using the cell's face-neighbors and their face-neighbors leads to the stencil shown in Figure 3. In this case the issue of the stencil spreading with the second recovery does not exist, but directionally biased enhancement may still be helpful in better balancing the number of face conditions with the number of interior basis functions. (At the time this Extended Abstract was submitted, such bias had not yet been explored.)

III.   Experiments on triangles

In a triangle we will use a lean basis of order p, which has only $(p + 1)(p + 2)/2$ basis functions, as compared to $(p + 1)^2$ for a tensor basis. Again using just the $p + 1$ solution values at the faces (no normal-derivative values) we may thus enhance the interior solution u with contributions from $3(p + 1)$ higher-order basis functions. To increase the order of the basis by 1 from $p = P$ to $p = P + 1$, only $P + 2$ extra basis functions are required; there are more than enough face equations to determine these. To increase the order by
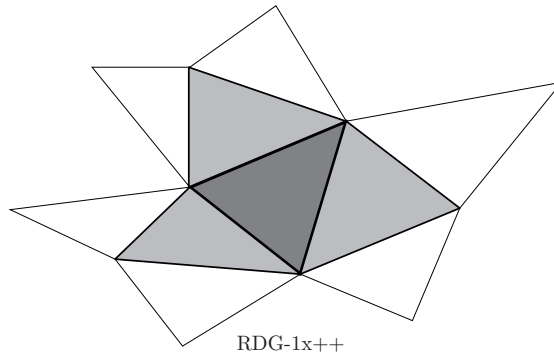
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RDG-1x++

Figure 3. Stencil of RDG-1x++ on an unstructured triangular mesh.

2, to $p = P + 2$, would require a total of $2P + 5$ extra basis functions; the number of face equations will suffice for $P \geq 2$. This is illustrated in Figure 4.
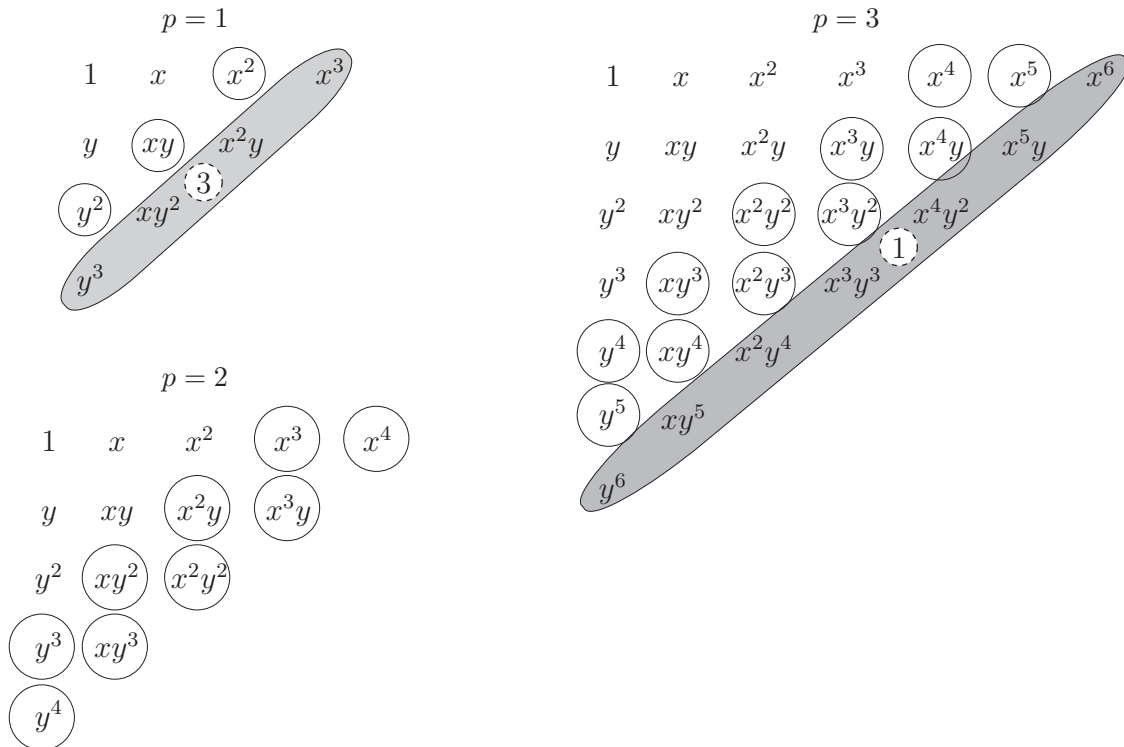

Figure 4. Monomial bases suitable for RDG-1x+ and RDG-1x++ on a triangular mesh, for various values of p; in practice the bases are orthogonalized. Monomials added through solution enhancement are encircled; for the enhancement $p + 1$ data per face are used. Monomials that do not add a full order to the solution representation are found in the shaded are; the number of constraints on this group is indicated in a dashed circle. None of these monomials are used in the second recovery step.

An increase to the order $P + 3$ is not possible, as it requires the addition of $3P + 9$ basis functions, more than the number of face equations available. However, adding two properties of the normal derivative to the face data will make the number of equations and unknowns exactly match for an enhancement of three orders, for any value of $P \geq 1$.
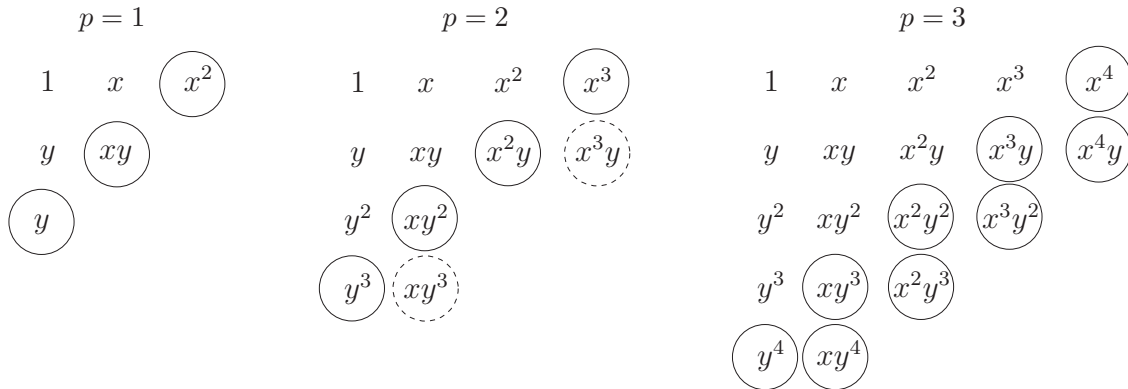
IV.   Some numerical experiments

Figure 5. Same as in Figure 4, except that only p data per face are used for solution enhancement. Here the monomials dropped before the second recovery are in dashed circles.

When experimenting with solution enhancement and subsequent additional recovery, we started out with limiting the number of face data to p per face. The possibilities of basis enhancement are shown in Figure 5. Both Figures 4 and 5 show the basis functions as monomials; in reality we orthogonalize these bases via a Gram-Schmidt procedure in order to bring down the condition number for the recovery- and enhancement-equation systems. A further benefit of an orthogonal basis is that we may drop higher-order polynomials we do not wish to carry along, without affecting the coefficients of the other basis functions used to build up the solution. Especially if the second recovery step is made, enhancing the face values, it is important to trim the basis in each triangle down to a complete basis; this makes the interior bases rotationally invariant and facilitates the choice of basis for the enhanced recovered solution.

Tables 1 and 2 contain some results of steady-state calculations with a time-dependent Poisson equation on the unit square $[1, 0] \times [-1, 0]$ with periodic boundary conditions; the source term is such that the steady solution becomes

$$u(x, y) = \sin(2\pi x) \sin(2\pi y). \tag{2}$$

The grid was that of Figure 1: Cartesian elements cut into orthogonal triangles by parallel-running diagonals. The solution was advanced to the steady state with the 3-stage Runge-Kutta method. These grid-refinement studies are equivalent to Fourier analysis with equal x- and y-frequencies.

The first table shows results for from experiments in which we used p data on each face for the enhancement of the solution in the element; this enhanced solution was used to evaluate the volume integral in the DG update equations. No second recovery step was performed. Without solution enhancement the accuracy of the solution average in the element is only $p + 1$ (results not shown here). The enhancement step appears to improve this order by 1 for $p = 2$. It is too early to present a rule-of-thumb for the order of this RDG-1x+ scheme.

Table 2 shows the results obtained with RDG-1x++. This scheme builds on RDG-1x+ by down-projecting the solution in an element onto a complete orthogonal basis, then performing an additional recovery step with the enhanced interior solutions, orthogonalized the resulting interior basis then projected it down to the nearest complete basis of order

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p	Grid	L_1	L_2	L_∞	OOA_1	OOA_2	OOA_∞
1	3×3	0.0126	0.0151	0.0284			
	6×6	0.00419	0.00489	0.00942	1.59	1.63	1.59
	9×9	0.00209	0.00245	0.00464	1.72	1.70	1.75
	12×12	0.00124	0.00145	0.00272	1.81	1.82	1.86
2	3×3	0.00356	0.00383	0.00612			
	6×6	0.000398	0.000424	0.00066	3.16	3.18	3.21
	9×9	7.98E-05	9.00E-05	15.4E-05	3.96	3.82	3.59
	12×12	2.57E-05	2.93E-05	5.29E-05	3.94	3.90	3.71

Table 1. Accuracy of steady solution of time-dependent diffusion problem obtained with RDG-1x+ on an orthogonal triangular grid, for $p = 1$ and $p = 2$. Column 2: underlying Cartesian grid; columns 3-5: L_1, L_2 and L_∞ errors of solution average; columns 6-8: respective order of accuracy from finest pair of grids.

p	Grid	L_1	L_2	L_∞	OOA_1	OOA_2	OOA_∞
1	3×3	0.0626	0.0686	0.115			
	6×6	0.00692	0.00748	0.0116	3.18	3.20	3.31
	9×9	0.00189	0.00212	0.00344	3.20	3.11	3.00
	12×12	0.000785	0.000876	0.0014	3.05	3.07	3.12
2	3×3	0.0123	0.0143	0.03			
	6×6	0.00242	0.00276	0.00497	2.35	2.37	2.59
	9×9	0.00112	0.00128	0.00237	1.90	1.90	1.83
	12×12						

Table 2. Same as Table 1 but for RDG-1x++, which includes an extra recovery step after solution enhancement.

$> p$, then performed the second recovery. We observe that the order of accuracy for $p = 1$ is now increased to 3, but the order for $p = 2$ is, surprisingly, only 2. We conjecture that this is caused by the inconsistency between the enhanced interior basis used for the volume integral and the down-projected basis used to obtain the enhanced recovered solution. For $p = 1$ these two bases are identical; for $p = 2$ they are not.

V. Further development

We are currently exploring different numbers and choices of face data to be used for the solution enhancement, and apply these for a larger range of values of p. We do expect to find a strategy that will systematically increase the order of accuracy of RDG-1x++ to $2p$ plus a constant.

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[7]S. Varadan, E. Johnsen, and B. van Leer, "A three-dimensional recovery-based Discontinuous Galerkin method for turbulence simulations," AIAA Paper 2013-xxxx, 2013. To be presented at the 51st Aerospace Sciences Meeting, January 7-10, 2013, Dallas, TX.

[8]J. Peraire, N. Nguyen, and B. Cockburn, "A hybridizable Discontinuous Galerkin method for the compressible Euler and Navier-Stokes equations," AIAA Paper 2010-363, 2010.

American Institute of Aeronautics and Astronautics

1.

Report Submission Form

If you have any questions, please contact your Program Manager or Assistant Program Manager.

Air Force Office of Science and Research
875 Randolph Street
Suite 325 Room 3112
Arlington, VA 22203

1. Report Type

Final Report

Primary Contact E-mail
Contact email if there is a problem with the report.

carmenr@umich.edu

Primary Contact Phone Number
Contact phone number if there is a problem with the report

734-764-6927

Organization / Institution name

Regents at the University of Michigan

Award Information

Grant/Contract Title
The full title of the funded effort.

Discontinuous Galerkin for Navier-Stokes

Grant/Contract Number
AFOSR assigned control number. It must begin with "FA9550" or "F49620".

FA9550-11-1-0257

Principal Investigator Name
The full name of the principal investigator on the grant or contract.

Bram van Leer

Program Manager
The AFOSR Program Manager currently assigned to the award

Dr. Fariba Fahroo

Report Information - Annual Report

Report Information - Final Report

DISTRIBUTION A: Distribution approved for public release.

Report Information - Conference/Workshop Report

Report Information - Equipment Report

Report Information - Patent/Invention Report, DD882

Report Information - Financial Report, SF425

Report Information - STTR Status Report

Report Information - STTR Annual Progress Report

For an annual report, the reporting period start date is either start date of the grant, if this is the first report, or 1 day after the due date of the previous report. The end date is due date of this report.

The reporting period start and end dates are the start and end dates of the award.

Reporting Period Start Date

08/15/2011

Reporting Period End Date

02/14/2013

Report Abstract:

In the Abstract section, please list any accomplishments that have been made since the last report submission (or since the beginning of the award if this is the first report).
Please do not type "see report" here, include at least an abstract, 250 words or more, of the accomplishments mentioned in your report.

Report Abstract:

In the Abstract section, enter the Final Conference Report. This is a summary of all scientific papers presented and a list of all attendees.

Report Abstract:

In the Abstract section, enter the Final Performance Report.

The Final Performance Report will identify the acquired equipment (although it may vary from that described in your proposal) by name and associated costs. The Final Performance Report shall summarize the research or educational project for which the equipment will be used.

The patent and inventions coverage contained in Article 36, Intangible Property, of the Research Terms and Conditions does not apply to this award.

Article 15, Intangible Property, in the AFOSR Agency Specific Requirements

does not apply to this award.

Abstract

Recovery-Based Discntinuous Galerkin Methods for modeling the viscous/conducting terms in the Navier-Stokes equations with a polynonial basis of order p have been shown (under a previous AFOSR grant) to achieve on Cartesian grids an unmatched order of accuracy of 3p+1 (p odd) or 3p+2 (p even). On triangular grids all DG methods not based on recovery only achieve the order p+1, except Hybrid DG (Peraire and Persson), which achieves order p+2. We demonstrate that recovery-based DG achieves at least the order 2p on a triangular grid; for p>2 this exceeds the accurary of any existing DG method. The result has been found first for linear diffusion, but is expected to remain valid for nonlinear diffusion and for shear terms, i.e., cross-derivative terms, such as appear in nthe Navier-Stokes equations. This is still being investigated in a six-month continuation of the current project, supported by AFRL.

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Additional Information

Archival Publications (published) during reporting period:

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Principal Investigator Name:	Bram van Leer
Primary Contact E-mail:	carmenr@umich.edu
Primary Contat Phone Number:	734-764-6927
Grant/Contract Title:	Discontinuous Galerkin for Navier-Stokes
Grant/Contract Number:	FA9550-11-1-0257

Program Manager:	Dr. Fariba Fahroo
Report Type:	Final Technical
Reporting Period Start Date:	08/15/2011
Reporting Period End Date:	02/14/2013
Abstract:	Recovery-Based Discntinuous Galerkin Methods for modeling the viscous/conducting terms in the Navier-Stokes equations with a polynonial basis of order p have been shown (under a previous AFOSR grant) to achieve on Cartesian grids an unmatched order of accuracy of 3p+1 (p odd) or 3p+2 (p even). On triangular grids all DG methods not based on recovery only achieve the order p+1, except Hybrid DG (Peraire and Persson), which achieves order p+2. We demonstrate that recovery-based DG achieves at least the order 2p on a triangular grid; for p>2 this exceeds the accurary of any existing DG method. The result has been found first for linear diffusion, but is expected to remain valid for nonlinear diffusion and for shear terms, i.e., cross-derivative terms, such as appear in nthe Navier-Stokes equations. This is still being investigated in a six-month continuation of the current project, supported by AFRL.
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Changes in Research objectives:	
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