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**Extending the Riemann-Solver-Free High-Order Space-Time discontinuous galerkin Cell Vertex Scheme (DG-CVS) to Solve Compressible Magnetohydrodynamics Equations**

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# Extending the Riemann-Solver-Free High-Order Space-Time Discontinuous Galerkin Cell Vertex Scheme (DG-CVS) to Solve Compressible Magnetohydrodynamics Equations

Final Report to AFOSR  
Grant Number: FA9550-13-1-0092  
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This final report summarizes our major accomplishments on the research under AFOSR Grant FA9550-13-1-0092 during the project period between March, 2013 - March, 2016.

In this project, we continue our development of our Riemann-solver-free spacetime discontinuous Galerkin method for general conservation laws to solve compressible magnetohydrodynamics (MHD) equations. The method is first applied to solve the  $3 \times 3$  MHD model system in the phase space which exactly preserves the MHD hyperbolic singularities. Numerical results show that the method is able to solve the model system correctly, which makes the method very promising in solving the complete ideal MHD equations. The method is then extended to solve the 1-D  $7 \times 7$  and 2-D  $8 \times 8$  MHD equations. The Powell's approach by adding appropriate source terms is adopted to handle the  $\nabla \cdot \mathbf{B} = 0$  condition. Again, the numerical results show that the present method is able to resolve the complex MHD waves without the need of any type of Riemann solvers or other flux functions. The success of solving MHD equations further strengthens our belief that the DG-CVS is an effective approach in solving systems where accurate and reliable Riemann solvers are difficult to design.

## 1 Background

This project continues our previous AFOSR projects (Grant No. FA9550-08-1-0122 and Grant No. FA9550-10-1-0045) to extend and verify our high order spacetime discontinuous Galerkin cell vertex scheme (DG-CVS) toward solving the magnetohydrodynamics (MHD) equations.

The magnetohydrodynamics (MHD) equations are often used to model electrically conducting fluid flows where the electromagnetic forces can be of the same order as, or even greater than, the hydrodynamic forces. Plasmas in hypersonic and astrophysical flows are one of the most typical examples of such conductive fluids. Though MHD models are a low order approximation of actual plasma processes, they have been successfully applied to simulate many important plasma processes. Today, the MHD models remain powerful tools in helping researchers to understand the complex physical processes in the geospace environment.

For example, the ideal MHD equations combines the Euler equations of gas dynamics with Maxwell equations of electromagnetics for problems in which viscous, resistive and relativistic effects can be neglected. The ideal MHD equations contain a set of nonlinear hyperbolic equations as shown in the following conservative

form:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1a)$$

$$\partial_t(\rho \mathbf{v}) + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + \left( p + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) \mathbf{I} - \frac{1}{\mu_0} \mathbf{B} \mathbf{B} \right] = \mathbf{0} \quad (1b)$$

$$\partial_t(\rho E) + \nabla \cdot \left[ \left( \rho E + p + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) \mathbf{v} - \frac{1}{\mu_0} \mathbf{B} (\mathbf{v} \cdot \mathbf{B}) \right] = 0 \quad (1c)$$

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) = \mathbf{0} \quad (1d)$$

Here,  $\rho$ ,  $\mathbf{v}$ ,  $E$ ,  $P$ ,  $\mu_0$  and  $\mathbf{B}$  represent the mass density, the velocity vector, the total energy, the hydrodynamic pressure, permeability of vacuum and the magnetic field. In addition to satisfying the induction equation (Eq. (1d) above), the magnetic field is also divergence free, i.e.  $\nabla \cdot \mathbf{B} = 0$  which can be considered as an extra constraint.

As can be seen in Eqs. (1a-1d), the MHD equations are complicated nonlinear partial differential equations (PDEs) that require some numerical method to solve for general problems. Since MHD flow fields may develop shock waves, contact discontinuities and shear layers, the numerical schemes are required to be of high resolution which are able to resolve all kinds of discontinuities accurately and robustly.

## 1.1 Traditional Numerical Methods for MHD Equations

Due to its close mathematical similarity to the Euler equations of gas dynamics, the MHD equation system is often solved by various methods extended from those originally designed for Euler equations. The Godunov-typed methods use exact or approximate Riemann solvers to provide numerical inter-cell spatial fluxes. Since the MHD equations contain much more complicated wave structures than the compressible Euler equations, the (approximate) Riemann solvers for Euler equations must be sophisticatedly modified or even redesigned to handle the degeneracies and instabilities of the MHD equations.

Two of the most widely used approximate Riemann solver families, the Roe scheme [1] and the HLL (Harten, Lax and van Leer) scheme [2] and its two-state variant, HLLC scheme [3, 4], have gained increasing popularity in solving the MHD equations. For example, see [5, 6, 7] for MHD solvers based on the Roe's scheme and [8, 9, 10, 11] for MHD solvers based on the HLL or HLLC scheme. Roe's scheme requires eigen-decomposition and becomes very complicated in MHD equations. For example, the Roe averages of quantities are not clearly defined in MHD system. Moreover, due to the complexity and non-strict hyperbolicity of the MHD system, the validity of the eigensystems of the MHD system is not unanimously agreed upon among researchers. The HLL or HLLC based Riemann solvers exhibits several advantages over Roe's scheme. For example, the HLLC scheme preserves positivity, satisfies the entropy condition and does not need eigen-decomposition of the system. The accuracy of MHD solvers is dependent on the underlying Riemann solvers [12]. Therefore, the use of Riemann solvers adds uncertainties to the accuracy of MHD solvers. The kinetic MHD scheme [13, 14, 15] is a flux vector splitting method with the consideration of particle transport. It is claimed to be more efficient than other Riemann-solver based MHD schemes due to the simplicity of the kinetic flux functions [14].

In addition to upwind schemes, the fully discrete [16] or semi-discrete [17] central schemes can also be used for MHD equations. Fully discrete central schemes use staggered grids to evolve cell averages and eliminate the need for the information of the eigenstructure of the system. Moreover, central schemes eliminate the need for dimensional splitting [16]. Central schemes are considered as Riemann-solver-free methods. Central schemes are very attractive in that the notorious pathological phenomena associated with some Riemann solvers are absent in Riemann-solver-free methods.

Another issue when solving the MHD equations using numerical methods is how to preserve  $\nabla \cdot \mathbf{B} = 0$  for the magnetic field. Failure to enforcing this divergence-free constraint may lead to a nonlinear numerical instability. Several approaches are available to preserve this constraint. One approach is to use the projection method [18] by solving a Poisson equation for the magnetic potential which can be used to correct the non-divergence-free magnetic field. In Ref. [19], a DG method using locally divergence-free piecewise polynomials as the solution space is presented to solve the MHD equations. Another so-called Powell approach [7] is to add source terms proportional to  $\nabla \cdot \mathbf{B}$  to the ideal MHD equations. Despite some success, this approach leads to a non-conservative formulation which may result in incorrect weak solution. It is also possible to

use a generalized Lagrange multiplier method [20] to “clean” the divergence error. Instead of solving an elliptic Poisson equation, this approach solves a damped hyperbolic equation for the divergence error with appropriately chosen parameters. In Ref. [21], the magnetic vector potential  $\mathbf{A}$  is solved in place of the magnetic field  $\mathbf{B}$  itself. Since  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$  is guaranteed mathematically, solving  $\mathbf{A}$  automatically leads to  $\nabla \cdot \mathbf{B} = 0$ . However, setting the boundary conditions for the vector potential is not straightforward. Another very popular method is the constrained transport (CT) method of Evans and Hawley [22]. The CT method can be considered as a predictor-corrector approach for the magnetic field [23]. It uses the computed conservative variables to approximate the electric field which is used to update the magnetic vector potential. The magnetic vector potential is then used to compute the divergence-free magnetic field. The CT method does not need to solve a global elliptic equation and has no free parameters. The CT method and its variants have been investigated by many researchers [24, 25, 23].

In this project, we solve the following system [26, 7].

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2a)$$

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + \left( p + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) \mathbf{I} - \frac{1}{\mu_0} \mathbf{B} \mathbf{B} \right] = -\frac{1}{\mu_0} \mathbf{B} \nabla \cdot \mathbf{B} \quad (2b)$$

$$\partial_t (\rho E) + \nabla \cdot \left[ \left( \rho E + p + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) \mathbf{v} - \frac{1}{\mu_0} \mathbf{B} (\mathbf{v} \cdot \mathbf{B}) \right] = -\frac{1}{\mu_0} (\mathbf{v} \cdot \mathbf{B}) \nabla \cdot \mathbf{B} \quad (2c)$$

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) = -\mathbf{v} \nabla \cdot \mathbf{B} \quad (2d)$$

## 1.2 Riemann-Solver-Free DG-CVS for General Conservation Laws

A novel high order discontinuous Galerkin cell-vertex scheme (DG-CVS) [27, 28, 29, 30, 31, 32, 33] has been developed for hyperbolic conservation laws under the support from AFOSR. DG-CVS was inspired by the spacetime Conservation Element/Solution Element (CE/SE) [34] method and the discontinuous Galerkin (DG) [35] method. The main features of DG-CVS are summarized as follows:

- *based on space-time formulation.* Therefore, DG-CVS automatically satisfies the Geometric Conservation Law (GCL) and solves moving mesh problems [36] without special care as needed by ALE-based methods.
- *arbitrary high-order accuracy in both space and time.* Space and time are handled in a unified way based on space-time flux conservation and high-order space-time discontinuous basis functions. This is in contrast to semi-discrete methods where the temporal order of accuracy is limited by the Backward Difference Formula (BDF) or the multi-step Runge-Kutta method.
- *Riemann-solver free.* DG-CVS does not need a (approximate) Riemann solver to provide numerical fluxes as needed in finite volume or traditional DG methods. The Riemann-solver-free feature offers two-fold advantages. First, this Riemann-solver-free approach eliminates some pathological behaviors (e.g., carbuncle phenomenon, expansion shocks, etc.) associated with some Riemann solvers. Second, it is suitable for any hyperbolic PDE systems whose eigenstructures are not explicitly known. The DG-CVS based solvers have been successfully applied to solve scalar advection-diffusion equations [31], compressible Euler equations [27, 33], shallow water equations [37] and the level set equation [38] (see Fig. ?? for simulation examples using the DG-CVS method).
- *reconstruction free.* DG-CVS solves for the solution and its all spatial and temporal derivatives simultaneously at each space-time node, thus eliminating the need of reconstruction.
- *suitable for arbitrary spatial meshes.* The CE and SE definitions in DG-CVS are independent of the underlying spatial mesh and the same definitions can be easily extended from 1-D (cf. Fig. 1) to higher-dimensions (cf. Fig. 2) without any ambiguity. Though not shown, the same CE and SE definitions also apply to meshes with hanging nodes [30].
- *highly compact regardless of order of accuracy.* Only information at the immediate neighboring nodes will be needed to update the solution at the new time level. Compactness eases the parallelization of the flow solver.
- *also accurately solves time-dependent diffusion equations.* Note that the treatment of diffusion terms in traditional semi-discrete DG methods are non-trivial because of the so-called “variational crime”. In DG-CVS, the inclusion of diffusion terms is straightforward and simple in exactly the same way

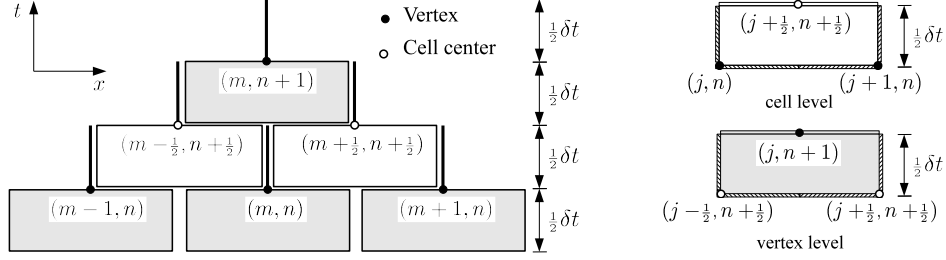


Figure 1: Solution elements (SEs) and conservation elements (CEs) in the  $x-t$  domain. Left: solution elements and right: conservation elements.

how advection terms are handled by simply incorporating the diffusive flux into the space-time flux [31]. No extra reconstruction or recovery or ad hoc penalty and coupling terms are needed to ensure the consistency of the variational form for diffusive terms. For this reason, DG-CVS is conceptually simpler than other existing DG methods for diffusion equations.

The DG-CVS method integrates the best features of the space-time Conservation Element/Solution Element (CE/SE) method [34] and the discontinuous Galerkin (DG) method [35]. The core idea is to construct a staggered space-time mesh through alternate cell-centered CEs and vertex-centered CEs (cf. Fig. 1 (right)) within each time step. Inside each SE (cf. Fig. 1 (left)), the solution is approximated using high-order space-time DG basis polynomials. The space-time flux conservation is enforced inside each CE using the DG discretization. The solution is updated successively at the cell level and at the vertex level within each physical time step. For this reason and the method's DG ingredient, the method was named as the space-time discontinuous Galerkin cell-vertex scheme (DG-CVS).

DG-CVS equally works on higher dimensions on arbitrary grids. Figure 2 shows the conservation elements and solution elements on quadrilateral meshes and triangular meshes. Obviously, the definitions of CEs and SEs on higher dimensions are analogous to that for 1-D meshes (cf. Fig. 1). Figure 3 demonstrates the resulting dual mesh at the cell level and the vertex level for both rectangular meshes and triangular meshes, respectively.

As a Riemann-solver free method, DG-CVS differs from other Riemann-solver free central schemes [16, 17] in that DG-CVS employs spacetime discontinuous Galerkin polynomials inside the solution element and therefore is arbitrarily high order in both space and time. Some space-time DG methods like the STE-DG (space-time expansion discontinuous Galerkin) scheme [39] also approximates solution using high-order space-time basis polynomials but it requires some numerical flux function and thus is subject to the shortcomings associated with Riemann solvers.

## 2 Summary of Major Accomplishments

During the period of this project, we have accomplished the following:

- It has been verified that the DG-CVS method solves  $3 \times 3$  MHD model equations in the phase space.
- The method has been successfully extended to solve the ideal 1D 7-eq. MHD equations.
- The method has been successfully extended to solve the ideal 2D 8-eq. MHD equations.
- It has been verified that Powell's approach by adding appropriate source terms in the governing equations is effective in handling the  $\nabla \cdot \mathbf{B} = 0$  condition in the current solution framework.

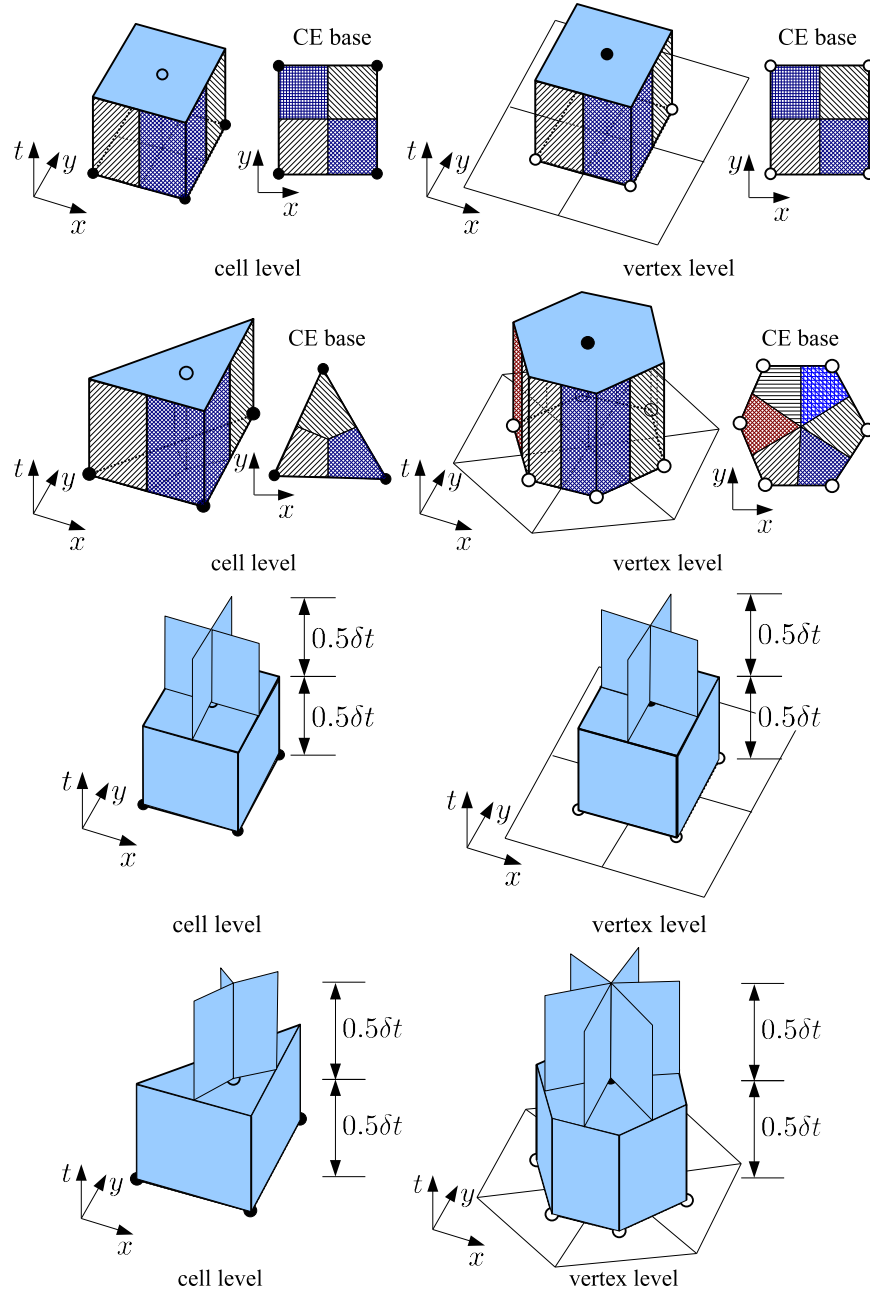


Figure 2: Conservation elements (CEs) and solution elements (SEs) in the  $x-y-t$  domain. First row: CEs for rectangular meshes, second row: CEs for triangular meshes, third row: SEs for rectangular meshes, and fourth row: SEs for triangular meshes.

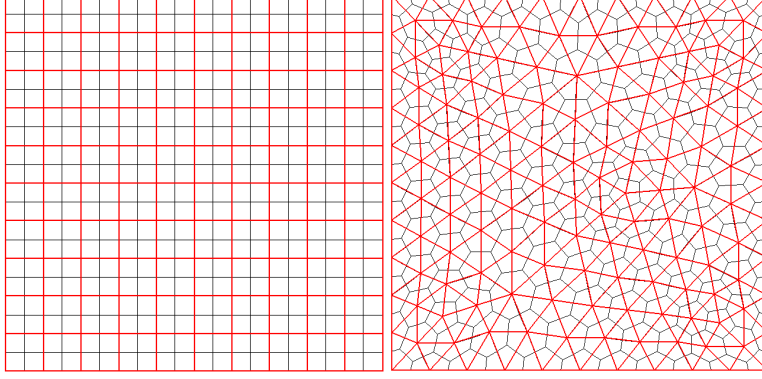


Figure 3: Dual meshes for the solution updating at the cell level (in red) and the vertex level (in black). Left: rectangular mesh and right: triangular mesh.

### 3 Description of the Spacetime Discontinuous Galerkin Cell-Vertex Scheme

To illustrate the DG-CVS formulation, without loss of generality, we consider the following general one-dimensional time dependent partial differential equation system

$$\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{r} \quad (3)$$

that arises from some physics. Here,  $\mathbf{u}$  is the solution vector containing the physical quantities to be determined.  $\mathbf{f}$  is the spatial flux vector. Without loss of generality, a source term  $\mathbf{r}$  is also included in Eq. (3). Note that since DG-CVS solves both advection and diffusion problems in an identical way,  $\mathbf{f}$  may contain both advective and diffusive fluxes. Therefore,  $\mathbf{f}$  can be functions of  $\mathbf{u}$  or  $\nabla \mathbf{u}$  or both. Appropriate initial and boundary conditions must also be provided to solve Eq. (3).

#### 3.1 Space-Time Discontinuous Galerkin Formulation

Following the idea of the discontinuous Galerkin (DG) method, an approximate solution  $\mathbf{u}^h$  is sought within each space-time solution element (SE), denoted as  $K$ . When restricted to the SE,  $\mathbf{u}^h$  belongs to the finite dimensional space  $\mathcal{U}(K)$ . Assume all the components of the conservative variables inside each SE are approximated by polynomials of the same degree, i.e

$$\mathbf{u}^h(\mathbf{x}, t) = \sum_{j=1}^N \mathbf{s}_j \phi_j(\mathbf{x}, t) \quad (4)$$

where  $\{\phi_j(\mathbf{x}, t)\}_{j=1}^N$  are some type of space-time polynomial basis functions defined within the solution element,  $\{\mathbf{s}_j\}_{j=1}^N$  are the unknowns to be determined and  $N$  is the number of basis functions depending on the degree of the polynomial function. *Using such space-time DG solution space, the solution approximation can be arbitrarily high order accurate in both space and time.*

Following the Galerkin orthogonality principle, multiply (3) with each of the basis functions  $\phi_i$  ( $i = 1, 2, \dots, N$ ) and integrate over a space-time CE to obtain the weak form

$$\int_{\Omega_K} \phi_i \left( \frac{\partial \mathbf{u}^h}{\partial t} + \frac{\partial \mathbf{f}^h}{\partial x} - \mathbf{r}^h \right) d\Omega = 0, \quad i = 1, 2, \dots, N \quad (5)$$

where  $\Omega_K$  is the space-time conservation element (CE) associated with the solution element  $K$ . Note that the conservation element is identical to the solution element except for the volumeless vertical spike in the solution element as shown in Fig. ???. The space-time flux conservation in weak form as in (5) is for each



individual space-time conservation element. Therefore, the current method can be considered as a space-time discontinuous Galerkin method. As can be seen, the space-time flux conservation is enforced in the variational form (5).

Integrating (5) by parts results in

$$\int_{\Omega_K} \left[ \frac{\partial \phi_i}{\partial t} \mathbf{u}^h + \frac{\partial \phi_i}{\partial x} \mathbf{f}^h + \phi_i \mathbf{r}^h \right] d\Omega = \int_{\Gamma} \phi_i \mathbf{H}^h \cdot \mathbf{n} d\Gamma \quad (6)$$

where

$$\mathbf{H}^h = (\mathbf{f}^h, \mathbf{u}^h) \quad (7)$$

is the space-time flux tensor and  $\mathbf{n} = (n_x, n_t)$  is the outward unit normal of the CE boundary, i.e.  $\Gamma = \partial\Omega_K$ , of the space-time conservation element (CE) under consideration. Note that the partial integration is also performed on the time-dependent term, which is a salient difference between space-time DG methods and semi-discrete DG methods. As can be seen, the formulation in (6) contains both the volume integral and the surface integral.

### 3.2 Cell-Vertex Solution Updating Strategy

The DG-CVS inherits the core idea of the CE/SE method using a staggered space-time mesh to enforce the space-time flux conservation. However, the construction of the staggered space-time slabs in DG-CVS deviates from the CE/SE method. In DG-CVS, unknowns are stored at both vertices and cell centroids of the spatial mesh, and the solutions at vertices and cell centroids are updated at different time levels within each time step. At the beginning of each physical time step, the solution is assumed known at the vertices of the mesh, either given as the initial condition or obtained from the previous time step. Inside each new time step, the solution is updated in two successive steps. The first step updates the solution at cell centroids at the half-time level ( $t^{n+1/2}$ ) based on the known vertex solutions at the previous time level ( $t^n$ ). The second step updates the solution at vertices at the new time level ( $t^{n+1}$ ) based on the known cell solutions at the previous half-time level ( $t^{n+1/2}$ ). The same process is repeated for new time steps.

The solution updating at the cell level or the vertex level is based on the key equation (6). First divide the conservation element into the following portions:

- the interior volume  $\Omega_K$  where the solution is associated with the space-time node at the new time level.
- the top surface  $\Gamma_{\text{top}}$  where the solution is also associated with the space-time node at the new time level.
- the side surfaces  $\Gamma_{\text{side}}$  where the solution is associated with the space-time node at the previous time level.
- the bottom surfaces  $\Gamma_{\text{bott}}$  where the solution is also associated with the space-time node at the previous time level.

Correspondingly, Eq. (6) can be rearranged to yield

$$\begin{aligned} \int_{\Omega_K} \left[ \frac{\partial \phi_i}{\partial t} \mathbf{u}^h + \frac{\partial \phi_i}{\partial x} \mathbf{f}^h + \phi_i \mathbf{r}^h \right] d\Omega - \int_{\Gamma_{\text{top}}} \phi_i \mathbf{H}^h \cdot \mathbf{n} d\Gamma & \quad i = 1, 2, \dots, N \\ = \int_{\Gamma_{\text{side}}} \phi_i \mathbf{H}^h \cdot \mathbf{n} d\Gamma + \int_{\Gamma_{\text{bott}}} \phi_i \mathbf{H}^h \cdot \mathbf{n} d\Gamma & \end{aligned} \quad (8)$$

where the left hand side contains the unknowns and the right hand side contains the known information.

Since the top and the bottom surface of the CE are horizontal, which leads to  $\mathbf{H}^h \cdot \mathbf{n} = \mathbf{u}^h$  on the top face and  $\mathbf{H}^h \cdot \mathbf{n} = -\mathbf{u}^h$  on the bottom face, Eq. (8) can be simplified to

$$\begin{aligned} \int_{\Omega_K} \left[ \frac{\partial \phi_i}{\partial t} \mathbf{u}^h + \frac{\partial \phi_i}{\partial x} \mathbf{f}^h + \phi_i \mathbf{r}^h \right] d\Omega - \int_{\Gamma_{\text{top}}} \phi_i \mathbf{u}^h d\Gamma & \quad i = 1, 2, \dots, N \\ = \int_{\Gamma_{\text{side}}} \phi_i \mathbf{H}^h \cdot \mathbf{n} d\Gamma - \int_{\Gamma_{\text{bott}}} \phi_i \mathbf{u}^h d\Gamma & \end{aligned} \quad (9)$$

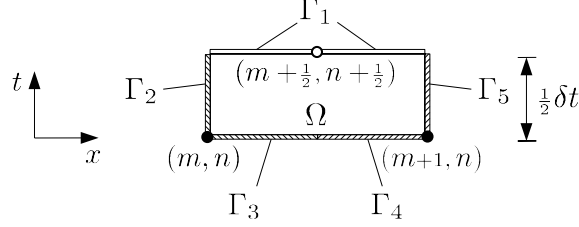


Figure 4: Illustration of space-time flux conservation on a 1-D cell-level CE.

To further illustrate the idea of enforcing the space-time flux conservation, consider the conservation element shown in Fig. 4. Suppose the solution at the spacetime node  $(m + \frac{1}{2}, n + \frac{1}{2})$  is to be determined. Here,  $m$  and  $n$  represents the spatial and the temporal locations of the space-time node, respectively. The boundaries of the CE associated with the spacetime node  $(m + \frac{1}{2}, n + \frac{1}{2})$  is divided into five sections  $\Gamma_1$ ,  $\Gamma_2$ ,  $\Gamma_3$ ,  $\Gamma_4$  and  $\Gamma_5$ , as shown in Figure 4. Among these sections,  $\Gamma_1$  belongs to the SE associated with node  $(m + \frac{1}{2}, n + \frac{1}{2})$  whose solutions are to be determined,  $\Gamma_2$  and  $\Gamma_3$  the SE associated with node  $(m, n)$  and  $\Gamma_4$  and  $\Gamma_5$  the SE associated with node  $(m + 1, n)$ . The interior volume of the conservation element is also associated with node  $(m + \frac{1}{2}, n + \frac{1}{2})$ . Eq. (9) is a nonlinear equation system for each space-time node which can be solved by the Newton-Raphson iterative method.

As can be seen, *with this staggered space-time cell-vertex solution updating strategy, no Riemann-solver-typed flux functions are needed for the interface flux.* There is no “left state” and “right state” when evaluating inter-cell fluxes as Riemann solvers do. Our Riemann-solver free DG-CVS method is able to capture all flow features (shock waves, contact discontinuities, etc.) without pathological phenomena such as the expansion shock. DG-CVS equally works for the shallow water equations, scalar advection-diffusion equations and the level set equation in addition to compressible Euler equations.

### 3.3 Quadrature-Free Implementation

To improve efficiency, the integrals on the left hand side of Eq. (9) are evaluated using a quadrature-free approach.

To allow the quadrature-free implementation, the flux and source vectors must be expanded in terms of the basis polynomials similar to those used to expand the conservative variables, namely,

$$\mathbf{f}^h = \sum_{j=1}^{\tilde{N}} \mathbf{s}_j^f \phi_j, \mathbf{g}^h = \sum_{j=1}^{\tilde{N}} \mathbf{s}_j^g \phi_j, \text{ and } \mathbf{r}^h = \sum_{j=1}^{\tilde{N}} \mathbf{s}_j^r \phi_j. \quad (10)$$

where  $\tilde{N}$  is the number of basis polynomials of one degree higher than those to expand the conservative variables as in Eq. (4). The requirement of using basis polynomials of degree  $p + 1$  is necessary to ensure the accuracy of the scheme.

The method based on the Cauchy-Kovalewski (CK) procedure[40, 41] is especially suitable for our purpose. In DG-CVS, the conservative variables are expanded using the Taylor polynomials. With the Taylor polynomials, the spatial and temporal derivatives of the solution are explicitly available. Using the CK procedure, the space-time derivatives of the flux vectors can be obtained using the space-time derivatives of the conservative variables. The detailed procedure has been described for Euler equation in our earlier paper[33] and will not be repeated here.

## 4 Major Progress #1: Solving the MHD Model Equations

The present Riemann-solver-free method has been first applied to solve the following  $3 \times 3$  MHD model system in the phase space which exactly preserves the MHD singularities[42]:

$$\mathbf{u}_t + \mathbf{f}_x = \mathbf{d}\mathbf{u}_{xx} \quad (11)$$

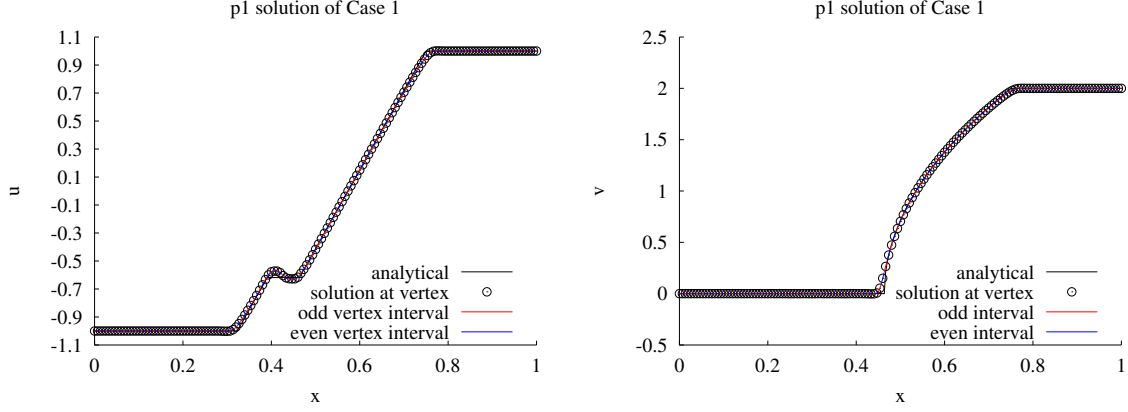


Figure 5: Solutions of Case 1 by solving the model equations.  $\mathbf{u}_L = (-u_0, 0, 0)$  and  $\mathbf{u}_R = (u_0, 2u_0, 0)$  with  $u_0 = 1$  at  $t = 0.0295085$ .

with

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \mathbf{f} = \begin{pmatrix} cu^2 + v^2 + w^2 \\ 2uv \\ 2uw \end{pmatrix}, \text{ and } \mathbf{d} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \eta & -\chi \\ 0 & \chi & \eta \end{pmatrix}$$

where the longitudinal viscosity  $\mu$  and the magnetic resistivity  $\eta$  are the dissipative coefficients, and the Hall coefficient  $\chi$  is a dispersive coefficient.

The above quantities are related to the MHD system via

$$u = \left(\frac{a}{c_a}\right)^2 - 1, v = \frac{B_y}{B_x}, w = \frac{B_z}{B_x}, \text{ and } c = \gamma + 1$$

where  $a = \sqrt{\gamma p / \rho}$  is the acoustic wave speed,  $c_a = \sqrt{\rho} |B_x|$  is the Alfvén wave speed,  $p$  is the pressure,  $\rho$  is the density and  $\gamma$  is the ratio of specific heats.

The model system (11) generates three wave families which are the Alfvén wave, the slow and the fast MHD waves. This model system has been investigated by several authors [42, 43] to provide insights to improve Godunov-typed numerical schemes for MHD equations.

The two test cases presented in this section are taken from Ref. [43] where conventional and modified Roe’s scheme were used to solve the system. In all the cases,  $c = 3$  and the computational domain  $[0, 1]$  is evenly divided into 150 cells. In both test cases, the dissipative term on the right hand side of (11) is assumed to be zero.

#### 4.1 Case 1

The initial discontinuity is a fast switch-off expansive shock defined as

$$\mathbf{u}_L = (-u_0, 0, 0) \text{ and } \mathbf{u}_R = (u_0, 2u_0, 0)$$

with  $u_0 = 1$  in this case. The final time is  $t = 0.0295085$ . 150 time steps are run to reach the time step.

Figure 5 shows  $u$  vs.  $x$  and  $v$  vs.  $x$  solutions from the  $p1$  (second order) simulation. As can be seen, without any entropy fix, the present Riemann solver free method does not produce expansion shocks. The comparison between the analytical solutions and the present solutions demonstrate the accuracy of the present method.

#### 4.2 Case 2

The initial discontinuity is an entropy-violating shock defined as

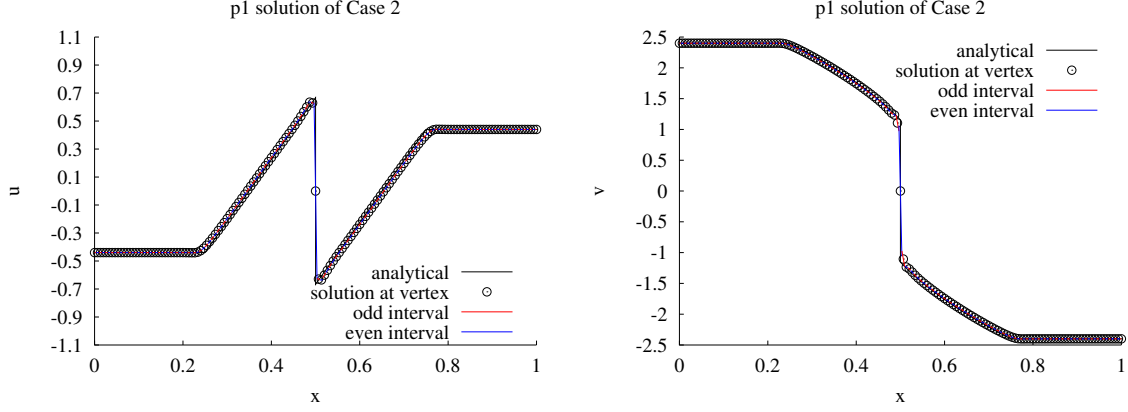


Figure 6: Solutions of Case 2 by solving the model equations.  $\mathbf{u}_L = -\mathbf{u}_R = (-0.44, 2.4, 0)$  at  $t = 0.0376506024$ .

$$\mathbf{u}_L = -\mathbf{u}_R = (-0.44, 2.4, 0)$$

The final time is  $t = 0.0376506024$ . 150 time steps have been run to reach the final time.

Figure 6 shows  $u$  vs.  $x$  and  $v$  vs.  $x$  solutions from the second order simulation along with the analytical solutions. Again, the agreement is very well and the overshoots/undershoots around the discontinuity region have been suppressed.

## 5 Major Progress #2: Solving the Ideal 1D 7-Eq MHD Equations

Assuming  $B_x = \text{const.}$ , the ideal MHD equations reduce to a  $7 \times 7$  system where the state vector and the flux vector are

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ \rho E \\ B_y \\ B_z \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} \rho u_x \\ \rho u_x^2 + p + \frac{1}{2}|\mathbf{B}|^2 - B_x^2 \\ \rho u_x u_y - B_x B_y \\ \rho u_x u_z - B_x B_z \\ u_x (\rho E + p + \frac{1}{2}|\mathbf{B}|^2) - B_x (u_x B_x + u_y B_y + u_z B_z) \\ u_x B_y - u_y B_x \\ u_x B_z - u_z B_x \end{bmatrix}$$

Since  $B_x$  is constant, the divergence-free condition of the magnetic field is automatically satisfied.

We are testing the test cases proposed by [5]. The problems are simulated by solving the  $7 \times 7$  1-D Ideal MHD Equations. The problem to be solved is a one-dimensional Riemann problem on the computational domain  $[-1, 1]$ . For both cases below, the primitive variables are defined as  $\mathbf{q} = [\rho, u, v, w, p, B_y, B_z]^T$ .

### 5.1 Brio-Wu Case 1:

In this case, the initial states are given as

$$\mathbf{q}_L = (1.000, 0, 0, 0, 1000, 1.0, 0.0)^T \quad \text{and} \quad \mathbf{q}_R = (0.125, 0, 0, 0, 0.1, -1.0, 0.0)^T$$

with  $B_x = 0$  and  $\gamma = 2$ .

If one define

$$p^* = p + \frac{1}{2}|\mathbf{B}|^2$$

this case is then exactly the same as the gas dynamics shock tube problem except the equation for  $B_y$ . Figure 7 shows the numerical solution at  $T = 0.012$  together with the available analytical solution.

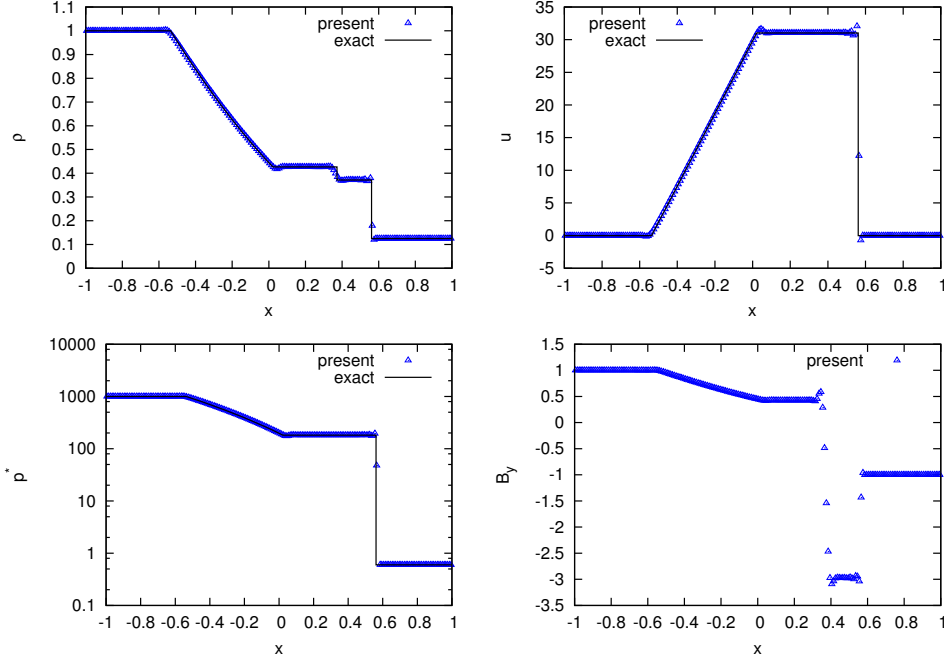


Figure 7: Solutions of Brio-Wu Case 1.  $B_x = 0$ .  $T = 0.012$ .

It can be seen that the numerical results agrees well with the analytical solution except for the slight overshoots/undershoots around discontinuities. Note that no any type of limiting procedure is employed to obtain the current solution shown in Fig. 7.

## 5.2 Brio-Wu Case 2:

In the second case, the initial states, separated at  $x = 0$ , are given as

$$\mathbf{q}_L = (1.000, 0, 0, 0, 1.0, 1.0, 0.0)^T \text{ and } \mathbf{q}_R = (0.125, 0, 0, 0, 0.1, -1.0, 0.0)^T$$

with  $B_x = 0.75$  and  $\gamma = 2$ . Figure 8 shows the unlimited (left) and limited (right) solutions at  $T = 0.2$ .

As can be seen, the complex MHD waves including the compound wave, the shock wave, contact discontinuity and rarefaction waves can be captured by the present Riemann-solver free method. However, the unlimited solutions exhibit some non-physical overshoots and undershoots near discontinuities (left column in Fig. 8). These overshoots/undershoots can be suppressed (right column in Fig. 8) using a minmod-typed limiting procedure.

## 6 Major Progress #3: Solving the Ideal 2D 8-Eq MHD Equations

The  $8 \times 8$  ideal MHD equations in 2-D is then solved. Here the state vector  $\mathbf{u} = [\rho, \rho u_x, \rho u_y, \rho u_z, \rho E, B_x, B_y, B_z]^T$  and the flux vectors are

$$\mathbf{f} = \begin{bmatrix} \rho u_x \\ \rho u_x^2 + p + \frac{1}{2} |\mathbf{B}|^2 - B_x^2 \\ \rho u_x u_y - B_x B_y \\ \rho u_x u_z - B_x B_z \\ u_x (\rho E + p + \frac{1}{2} |\mathbf{B}|^2) - B_x (\mathbf{v} \cdot \mathbf{B}) \\ 0 \\ u_x B_y - u_y B_x \\ u_x B_z - u_z B_x \end{bmatrix} \text{ and } \mathbf{g} = \begin{bmatrix} \rho u_y \\ \rho u_x u_y - B_x B_y \\ \rho u_y^2 + p + \frac{1}{2} |\mathbf{B}|^2 - B_y^2 \\ \rho u_y u_z - B_y B_z \\ u_y (\rho E + p + \frac{1}{2} |\mathbf{B}|^2) - B_y (\mathbf{v} \cdot \mathbf{B}) \\ 0 \\ u_y B_x - u_x B_y \\ u_y B_z - u_z B_y \end{bmatrix}$$

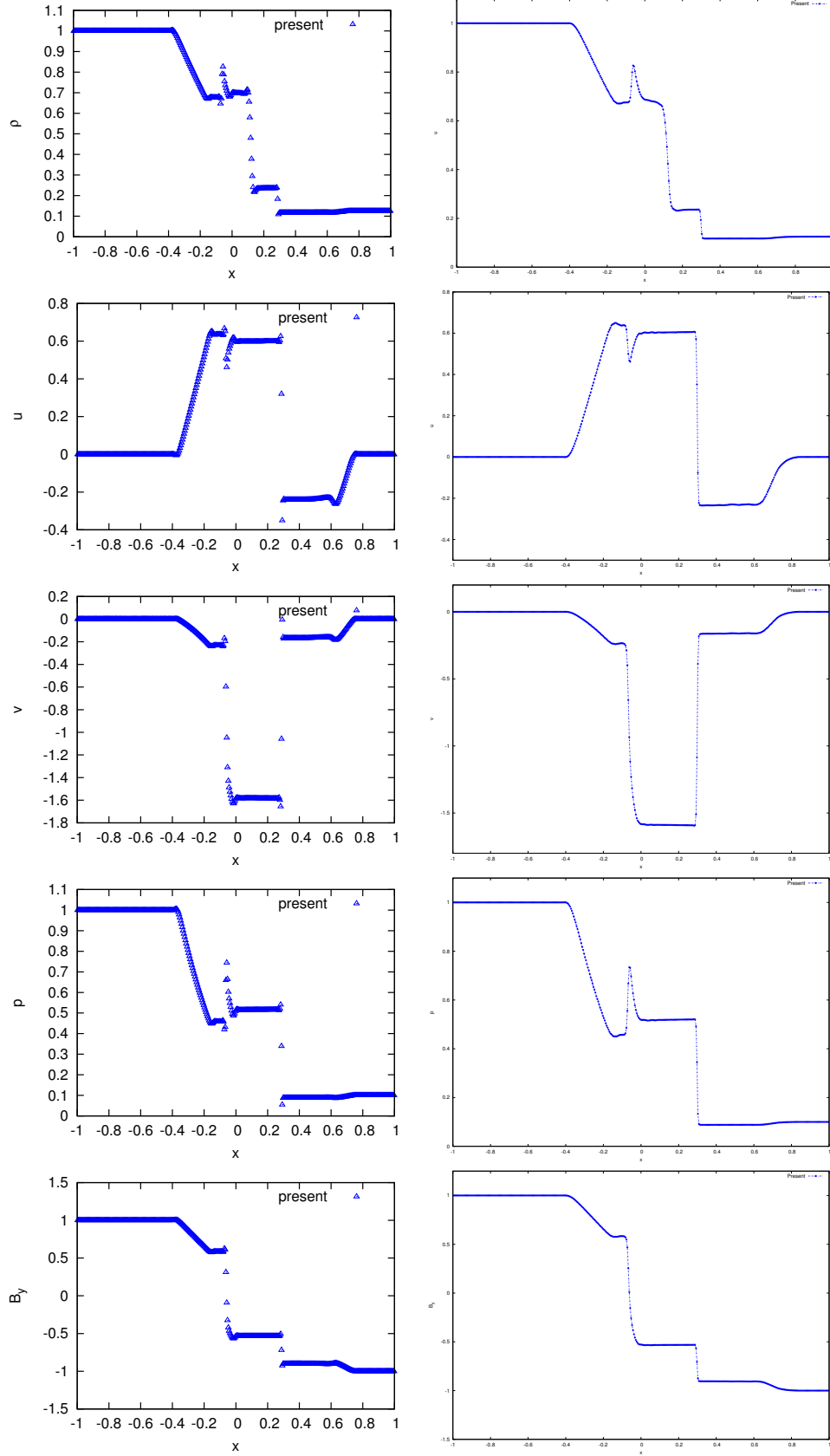


Figure 8: Solutions of Brio-Wu Case 2.  $B_x = 0.75$ .  $T = 0.2$ . Left column: unlimited solutions. Right column: limited solutions.

The 2-D case solved here is a MHD vortex advection problem. The solution is a vortex advected with the magnetic field diagonally in a square  $[-5, 5] \times [-5, 5]$  domain. Periodic boundary conditions are assumed. The computational domain is discretized by a  $32 \times 32$  rectangular mesh. The solution is initialized as follows [19]:

$$\begin{aligned}\rho &= 1.0 \\ u &= 1.0 + \frac{1}{2\pi} e^{0.5(1-r^2)}(-y) \\ v &= 1.0 + \frac{1}{2\pi} e^{0.5(1-r^2)}x \\ w &= 0.0 \\ p &= 1.0 + \frac{1}{8\pi^2} e^{1-r^2}(-r^2) \\ B_x &= \frac{1}{2\pi} e^{0.5(1-r^2)}(-y) \\ B_y &= \frac{1}{2\pi} e^{0.5(1-r^2)}x \\ B_z &= 0.0\end{aligned}$$

where  $r^2 = x^2 + y^2$ .

Figure 9 shows the magnetic field and velocity magnitude field at  $T = 0.0$  and  $T = 10.0$ . It can be seen that the solution fields have been preserved well after one period's advection.

## 7 Acknowledgment/Disclaimer

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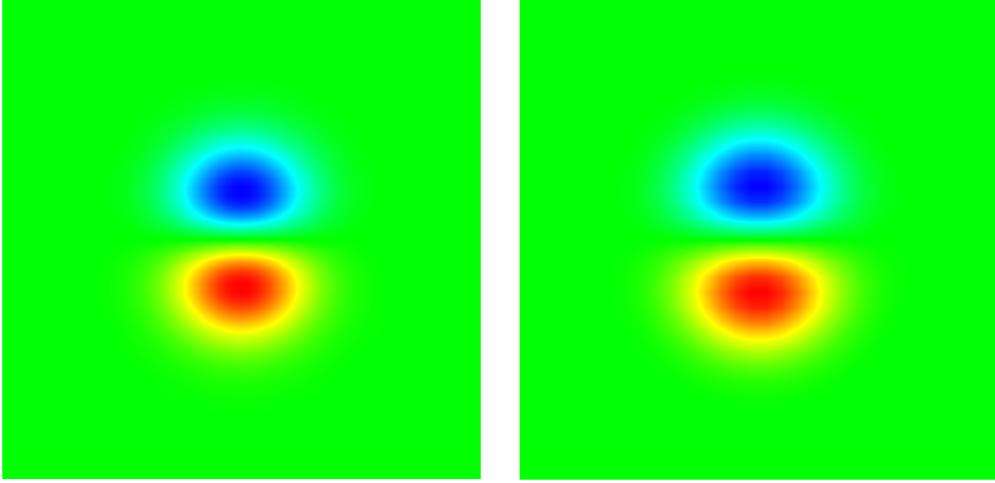
## 8 Publications

This project has produced the following full-length journal and conference articles.

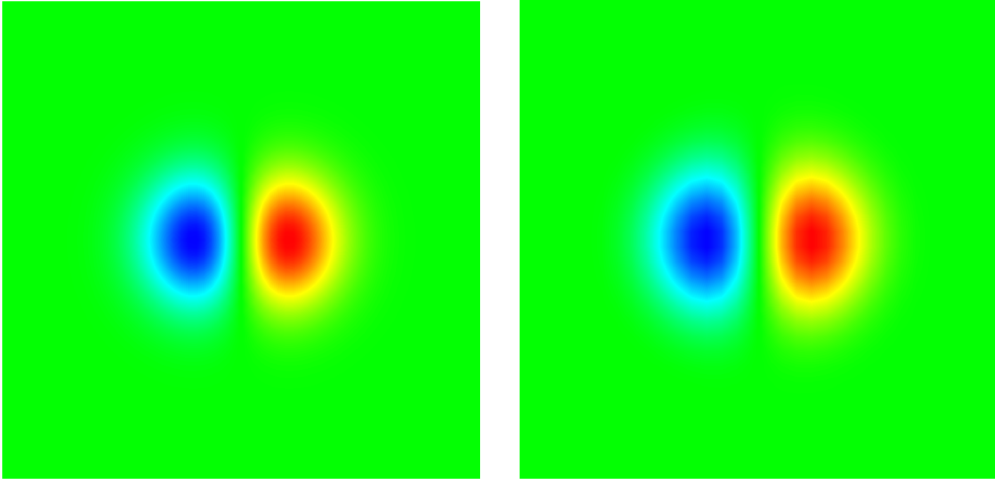
1. **S. Tu**, "A Riemann-solver Free Spacetime Discontinuous Galerkin Method for General Conservation Laws," *American Journal of Computational Mathematics*, Vol. 5, No. 2, (2015), pp. 55-74.
2. **S. Tu** and H. Song and L. Ji and Q. Pang, "Further Development of a Riemann-solver Free Space-time Discontinuous Galerkin Method for Compressible Magnetohydrodynamics (MHD) Equations", presented at AIAA Orlando SciTech Conference 2015, AIAA Paper 2015-0567.
3. **S. Tu**, Q. Pang and R.S. Myong, "Riemann-solver Free Space-time Discontinuous Galerkin Method for Magnetohydrodynamics," presented at AIAA San Diego Summer Conference 2013, AIAA Paper 2013-2755.

## 9 Personnel Supported

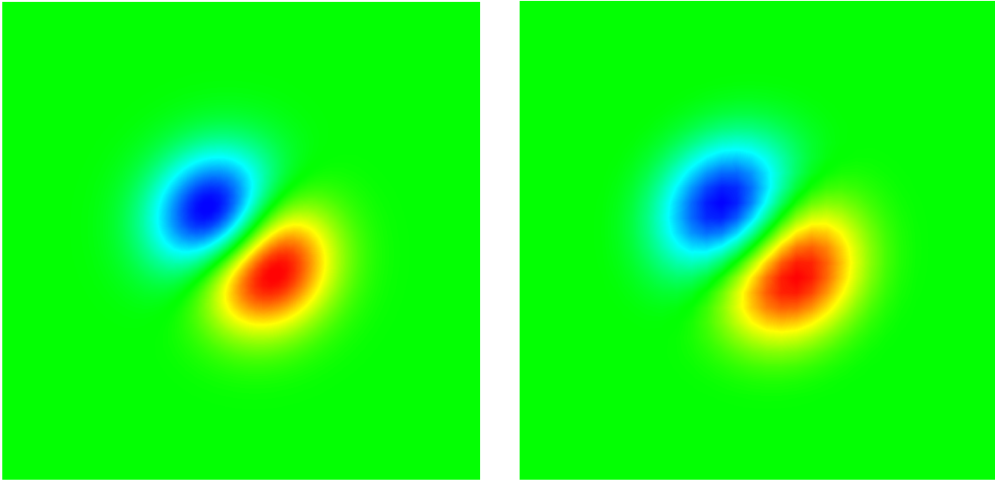
- Shuang Z. Tu, faculty member (PI).
- Yongze Liu, graduate student.
- Fariba Samiel, graduate student.
- Lei Zhang, Master's graduate student.
- LeSamuel Jimel Gardner, undergraduate student.
- Chao Jiang, undergraduate student.



(a)  $B_x$ -field. Left  $t = 0.0$ , right  $t = 10.0$ .



(b)  $B_y$ -field. Left  $t = 0.0$ , right  $t = 10.0$ .



(c) Velocity magnitude field. Left  $t = 0.0$ , right  $t = 10.0$ .

Figure 9: MHD vortex advection problem.



## 10 AFOSR Point of Contact

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The program manager has been changed to Dr. Jean-Luc Cambier in 2015. His contact email is [jean\\_luc.cambier@us.af.mil](mailto:jean_luc.cambier@us.af.mil).

## 11 Interactions/Transitions

*Conference presentations since 2013:*

1. One talk at the SIAM Conferences on Computational Science and Engineering, Salt Lake City, Utah, March 2015. (speaker: S. Tu).
2. One talk at the SIAM Central States Conferences. Rolla, Missouri, April 2015. (speaker: S. Tu).
3. One talk at the AIAA Orlando SciTech Conference 2015, January 2015. (speaker: S. Tu).
4. One talk at the AIAA San Diego Summer Conference 2013, June 2013. (speaker: S. Tu).
5. One talk at the SIAM CSE Conference, Feb., 2013, Boston, MA. (speaker: S. Tu).

## 12 Changes in Research Objectives

None.

## 13 Change in AFOSR Program Manager

The program manager has been changed to Dr. Jean-Luc Cambier in 2015.

## 14 Extensions Granted or Milestones Slipped

None.

## 15 New Discoveries

None patentable.

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1.

**1. Report Type**

Final Report

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shuang.z.tu@jsums.edu

**Primary Contact Phone Number****Contact phone number if there is a problem with the report**

601-979-1275

**Organization / Institution name**

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**Grant/Contract Title****The full title of the funded effort.**

Extending the Riemann-Solver-Free High-Order Spacetime Discontinuous Galerkin Cell Vertex Scheme (DG-CVS) to Solve Compressible Magnetohydrodynamics Equations

**Grant/Contract Number****AFOSR assigned control number. It must begin with "FA9550" or "F49620" or "FA2386".**

FA9550-13-1-0092

**Principal Investigator Name****The full name of the principal investigator on the grant or contract.**

Shuang Zhang Tu

**Program Manager****The AFOSR Program Manager currently assigned to the award**

Dr. Jean-Luc Cambier

**Reporting Period Start Date**

03/15/2013

**Reporting Period End Date**

03/14/2016

**Abstract**

In this project, we continue our development of our Riemann-solver-free spacetime discontinuous Galerkin method for general conservation laws to solve compressible magnetohydrodynamics (MHD) equations. The method is first applied to solve the 3 by 3 MHD model system in the phase space which exactly preserves the MHD hyperbolic singularities. Numerical results show that the method is able to solve the model system correctly, which makes the method very promising in solving the complete ideal MHD equations. The method is then extended to solve the 1-D 7 by 7 and 2-D 8 by 8 MHD equations. The Powell's approach by adding appropriate source terms is adopted to handle the divergence-free magnetic field condition. Again, the numerical results show that the present method is able to resolve the complex MHD waves without the need of any type of Riemann solvers or other flux functions. The success of solving MHD equations further strengthens our belief that the DG-CVS is an effective approach in solving systems where accurate and reliable Riemann solvers are difficult to design.

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**Changes in research objectives (if any):**

**Change in AFOSR Program Manager, if any:**

The program manager was changed from Dr. Fariba Fahroo to Dr. Jean-Luc Cambier in 2015.

**Extensions granted or milestones slipped, if any:**

**AFOSR LRIR Number**

**LRIR Title**

**Reporting Period**

**Laboratory Task Manager**

**Program Officer**

**Research Objectives**

**Technical Summary**

**Funding Summary by Cost Category (by FY, \$K)**

	Starting FY	FY+1	FY+2
Salary			
Equipment/Facilities			
Supplies			
Total			

**Report Document**

**Report Document - Text Analysis**

**Report Document - Text Analysis**

**Appendix Documents**

**2. Thank You**

**E-mail user**

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