Finite Element Modelling of the Acoustic Properties of Rubber Containing Inclusions

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Part 1 – Theoretical Description of PCFEM Code

The theory behind the implemented Periodic Unit Cell based code is described. The work is based on references [1-3].

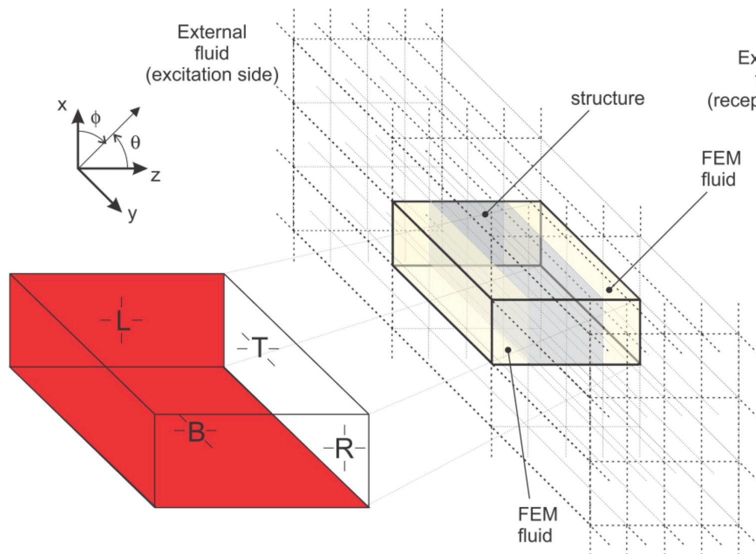

Figure 1

The material is periodic in the directions x and y and is excited by an oblique incidence plane wave given by:

$$\hat{p}_i\left(x,y,z\right)=\hat{A}e^{-jk\left(\sin\theta\cos\phi x+\sin\theta\sin\phi y+\cos\theta z\right)} \quad (1)$$

where we have assumed a $e^{j\omega t}$ temporal dependency.

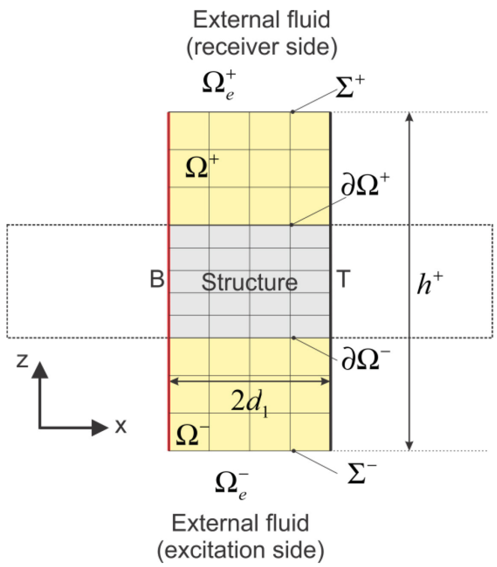

Figure 2

We start with the Unit Cell Finite Element system (see Figure 2 for a cut along plane xz and for notations):

$$\begin{pmatrix} [K]-\omega^2[M] & -[C] \\ -[C]^T & \dfrac{[H]}{\omega^2}-[Q] \end{pmatrix} \begin{Bmatrix} \{u\} \\ \{p\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \dfrac{1}{\rho_f \omega^2}\{\Phi\} \end{Bmatrix} \tag{2}$$

$\{u\}$ and $\{p\}$ contain the structural displacement (displacements along x,y,z) and pressure dof respectively before imposition of boundary conditions. $\{F\}$ and $\{\Phi\}$ represents the external force nodal vector acting on the structure and the external normal pressure gradient nodal vector acting on the fluid domain boundary respectively.

Note that for an acoustical excitation $\{F\}=0$ and $\{\Phi\}$ is only non-zero at degrees of freedom on boundaries Σ^+ and Σ^- . $\{\Phi\}$ is given by

$$\{\Phi\} = \int_{\Sigma^+\cup\Sigma^-\cup B\cup T\cup L\cup R} \{N(\underline{x})\} \frac{\partial \hat{p}}{\partial n_{ext}}(\underline{x})\, dS(\underline{x}) \tag{3}$$

where $\hat{p}$ is the acoustic pressure inside the finite element domain $\Omega^+\cup\Omega^-$. Actually, it will be shown after that only the contribution of Σ^+ and Σ^- needs to be calculated (see Eq(67)).

Since the material is periodic in direction x and y, any function solution of the problem must satisfy:

$$\begin{aligned} \Gamma(x+2d_1, y+2d_2, z) &= \Gamma(x,y,z)\,\mathrm{e}^{-j2d_1q_x}\,\mathrm{e}^{-j2d_2q_y} \\ &= \Gamma(x,y,z)\,\mathrm{e}^{-j\varphi_x}\,\mathrm{e}^{-j\varphi_y} \end{aligned} \tag{4}$$

with

$$\begin{cases} q_x = k^- \sin\theta\cos\phi \\ q_y = k^- \sin\theta\sin\phi \end{cases} \tag{5}$$

Note that a function $\Gamma(x,y,z)$ satisfying Eq(4) can be written as :

$$\Gamma(x,y,z) = \Theta(x,y,z)\,\mathrm{e}^{-jq_x x}\,\mathrm{e}^{-jq_y y} \tag{6}$$

where $\Theta(x+2d_1, y+2d_2, z) = \Theta(x,y,z)$. $\Theta(x,y,z)$ is a periodic function of x and y of period $2d_1$ and $2d_2$ in each direction.

To calculate this vector, the pressure in Ω_e^- is expanded in Bloch series. The reflected pressure in Ω_e^- can be written using Eq(6) as:

$$\hat{p}_r^-(x,y,z) = \hat{\Theta}_r^-(x,y,z)\,e^{-jk(\sin\theta\cos\phi x + \sin\theta\sin\phi y)} \tag{7}$$

And since $\hat{\Theta}_r^-(x,y,z)$ is a periodic function in x and y, it can be expanded in term of spatial Fourier series along x and y as:

$$\begin{aligned} \hat{\Theta}_r^-(x,y,z) &= \sum_{m,n=-\infty}^{+\infty} \hat{\Theta}_{r,mn}^-(z)\,e^{-j\gamma_x mx} e^{-j\gamma_y ny} \\ &= \sum_{m,n=-\infty}^{+\infty} \hat{\Theta}_{r,mn}^-\,e^{jk_{mn}z} e^{-j\gamma_x mx} e^{-j\gamma_y ny} \end{aligned} \tag{8}$$

With

$$\hat{\Theta}_{r,mn}^- = \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} \hat{\Theta}_r^-(x,y,z)\,e^{j\gamma_x mx} e^{j\gamma_y ny}\,dxdy \tag{9}$$

and

$$\begin{cases} \gamma_x = \dfrac{\pi}{d_1} \\ \gamma_y = \dfrac{\pi}{d_2} \end{cases} \tag{10}$$

For the z-dependence, the reflected wave is supposed to propagate in the –z direction. Therefore the total pressure in Ω_e^- can be written as:

$$\begin{aligned} \hat{p}^-(x,y,z) &= \hat{p}_i(x,y,z) + \sum_{m,n=-\infty}^{+\infty} \hat{p}_{mn}^-\,e^{jk_{mn}^- z} e^{-jk(\sin\theta\cos\phi x + \sin\theta\sin\phi y)} e^{-j\gamma_x mx} e^{-j\gamma_y ny} \\ &= \hat{p}_i(x,y,z) + \sum_{m,n=-\infty}^{+\infty} \hat{p}_{mn}^-\,e^{jk_{mn}^- z} e^{-j\alpha_m x} e^{-j\beta_n y} \end{aligned} \tag{11}$$

with

$$\begin{cases} \alpha_m = m\gamma_x + k^- \sin\theta\cos\phi \\ \beta_n = n\gamma_y + k^- \sin\theta\sin\phi \end{cases} \tag{12}$$

Inserting Eq(11) in Helmholtz equation leads to

$$\left(k_{mn}^-\right)^2 = \left(k^-\right)^2 - \alpha_m^2 - \beta_n^2 \tag{13}$$

Similarly, the pressure in Ω_e^+ can be written as

$$\hat{p}^+\left(x,y,z\right) = \sum_{m,n=-\infty}^{+\infty} \hat{p}_{mn}^+ e^{-jk_{mn}^+\left(z-h^+\right)} e^{-j\alpha_m x} e^{-j\beta_n y} \tag{14}$$

with

$$\left(k_{mn}^+\right)^2 = \left(k^+\right)^2 - \alpha_m^2 - \beta_n^2 \tag{15}$$

To calculate the right-hand side $\left\{\Phi_z^c\right\}$ (see Eq(67)), the idea is to express $\frac{\partial \hat{p}}{\partial z}$ in terms of the nodal values of the acoustic pressure in the finite element domain.

The normal pressure gradient on $\Sigma^+\left(z=h^+\right)$ can be calculated from Eq(14) as

$$\begin{aligned} \left.\frac{\partial \hat{p}^+}{\partial z}\right|_{z=h^+} &= \sum_{m=-M}^{M}\sum_{n=-N}^{N} -jk_{mn}^+ \hat{p}_{mn}^+ e^{-j\alpha_m x} e^{-j\beta_n y} \\ &= \left\langle -jk_{mn}^+ e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left\{ p_{mn}^+ \right\} \end{aligned} \tag{16}$$

where $\left(2M+1\right)$ terms along x and $\left(2N+1\right)$ terms along y have been kept in the series expansion. The vector $\left\langle -jk_{mn}^+ e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle$ is of size $\left(2M+1\right)\times\left(2N+1\right)$.

The acoustic pressure on $\Sigma^+\left(z=h^+\right)$ can be approximated as

$$\hat{p}^+\left(x,y,h^+\right) = \sum_{m=-M}^{M}\sum_{n=-N}^{N} \hat{p}_{mn}^+ e^{-j\alpha_m x} e^{-j\beta_n y} \tag{17}$$

Where the coefficient $\hat{p}_{mn}^+$ writes:

$$\hat{p}_{mn}^+ = \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} \hat{p}^+\left(x,y,h^+\right) e^{j\alpha_m x} e^{j\beta_n y} dxdy \tag{18}$$

$\hat{p}^+\left(x,y,h^+\right)$ can be expressed in term of pressure nodal values as:

$$\hat{p}^+\left(x,y,h^+\right) = \left\langle N^p\left(x,y,h^+\right)\right\rangle\left\{p\right\} \tag{19}$$

The vector $\left\langle N^p\left(x,y,h^+\right)\right\rangle$ is of size N^p where N^p is the total number of pressure dofs.

Therefore substituting Eq(19) in Eq(18) and putting the $\left(2M+1\right)\times\left(2N+1\right)$ terms in a vector leads to:

$$\begin{aligned}\left\{p_{mn}^+\right\} &= \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2}\left\{e^{j\alpha_m x}e^{j\beta_n y}\right\}\left\langle N^p\left(x,y,h^+\right)\right\rangle dxdy\left\{p\right\} \\ &= \left[A^+\right]\left\{p\right\}\end{aligned} \tag{20}$$

Matrix $\left[A^+\right]$ is a rectangular matrix of size $\left(\left(2M+1\right)\left(2N+1\right),N^p\right)$

The normal pressure gradient on $\Sigma^-\left(z=0\right)$ can be calculated from Eq(11) as

$$\begin{aligned}-\left.\frac{\partial\hat{p}^-}{\partial z}\right|_{z=0} &= -\left.\frac{\partial\hat{p}_i}{\partial z}\right|_{z=0} - \sum_{m=-M}^{M}\sum_{n=-N}^{N} jk_{mn}^-\hat{p}_{mn}^- e^{-j\alpha_m x}e^{-j\beta_n y} \\ &= -\left.\frac{\partial\hat{p}_i}{\partial z}\right|_{z=0} + \left\langle -jk_{mn}^- e^{-j\alpha_m x}e^{-j\beta_n y}\right\rangle\left\{p_{mn}^-\right\}\end{aligned} \tag{21}$$

where the same number of terms as before has been kept in the expansion.

The acoustic pressure on $\Sigma^-\left(z=0\right)$ can be approximated as

$$\hat{p}^-\left(x,y,0\right) = \hat{p}_i\left(x,y,0\right) + \sum_{m=-M}^{M}\sum_{n=-N}^{N}\hat{p}_{mn}^- e^{-j\alpha_m x}e^{-j\beta_n y} \tag{22}$$

Where the Fourier coefficient $\hat{p}_{mn}^-$ writes:

$$\hat{p}_{mn}^- = \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2}\left(\hat{p}^-\left(x,y,0\right)-\hat{p}_i\left(x,y,0\right)\right)e^{j\alpha_m x}e^{j\beta_n y}dxdy \tag{23}$$

$\hat{p}^-\left(x,y,0\right)$ can be expressed in term of pressure nodal values as:

$$\hat{p}^-\left(x,y,0\right) = \left\langle N^p\left(x,y,0\right)\right\rangle\left\{p\right\} \tag{24}$$

$\hat{p}_i\left(x,y,0\right)$ can also be expressed in term of incident pressure evaluated on the FEM nodes:

$$\hat{p}_i\left(x,y,0\right) = \left\langle N^p\left(x,y,0\right)\right\rangle\left\{p_i\right\} \tag{25}$$

Therefore substituting Eq(24) in Eq(23) and putting the $\left(2M+1\right)\times\left(2N+1\right)$ terms in a vector leads to:

$$\begin{aligned}\left\{p_{mn}^-\right\} &= \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2}\left\{e^{j\alpha_m x}e^{j\beta_n y}\right\}\left\langle N^p\left(x,y,0\right)\right\rangle dxdy\left\{p-p_i\right\} \\ &= \left[A^-\right]\left\{p-p_i\right\}\end{aligned} \tag{26}$$

Finally, Eq(3) rewrites :

$$
\begin{aligned}
\left\{\Phi_z^c\right\} = & \int_{\Sigma^+} \left\{N\left(x,y,h^+\right)\right\} \left\langle -jk_{mn}^+ e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left[A^+\right] dS \left\{p\right\} \\
& + \int_{\Sigma^-} \left\{N\left(x,y,0\right)\right\} \left\langle -jk_{mn}^- e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left[A^-\right] dS \left\{p\right\} \\
& - \int_{\Sigma^-} \left\{N\left(x,y,0\right)\right\} \left\langle -jk_{mn}^- e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left[A^-\right] dS \left\{p_i\right\} \\
& - \int_{\Sigma^-} \left\{N\left(x,y,0\right)\right\} \left.\frac{\partial \hat{p}_i}{\partial z}\right|_{z=0} dS
\end{aligned} \tag{27}
$$

Besides, it can be noted that

$$
\left\langle -jk_{mn}^- e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left[A^-\right] \left\{p_i\right\} = \left.\frac{\partial \hat{p}_i}{\partial z}\right|_{z=0} \tag{28}
$$

Indeed, $\hat{p}_i\left(x,y,z\right)$ can be expanded as

$$
\hat{p}_i\left(x,y,z\right) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{p}_{i,mn} e^{-jk_{mn}^- z} e^{-j\alpha_m x} e^{-j\beta_n y} \tag{29}
$$

So that

$$
\begin{aligned}
\left.\frac{\partial \hat{p}_i}{\partial z}\right|_{z=0} & = -\sum_{m=-M}^{M} \sum_{n=-N}^{N} jk_{mn} \hat{p}_{i,mn} e^{-j\alpha_m x} e^{-j\beta_n y} \\
& = \left\langle -jk_{mn}^- e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left\{p_{i,mn}\right\}
\end{aligned} \tag{30}
$$

with

$$
\begin{aligned}
\left\{p_{i,mn}\right\} & = \frac{1}{2d_1} \frac{1}{2d_2} \int_0^{2d_1} \int_0^{2d_2} \left\{e^{j\alpha_m x} e^{j\beta_n y}\right\} \left\langle N^p\left(x,y,0\right)\right\rangle dxdy \left\{p_i\right\} \\
& = \left[A^-\right] \left\{p_i\right\}
\end{aligned} \tag{31}
$$

Therefore Eq(27) becomes :

$$
\begin{aligned}
\left\{\Phi_z^c\right\} = & \int_{\Sigma^+} \left\{N\left(x,y,h^+\right)\right\} \left\langle -jk_{mn}^+ e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left[A^+\right] dS \left\{p\right\} \\
& + \int_{\Sigma^-} \left\{N\left(x,y,0\right)\right\} \left\langle -jk_{mn}^- e^{-j\alpha_m x} e^{-j\beta_n y} \right\rangle \left[A^-\right] dS \left\{p\right\} \\
& - 2\int_{\Sigma^-} \left\{N\left(x,y,0\right)\right\} \frac{\partial \hat{p}_i}{\partial z} dS
\end{aligned} \tag{32}
$$

Or

$$
\left\{\Phi_z^c\right\} = \left(\left[\Delta^+\right] + \left[\Delta^-\right]\right) \left\{p\right\} - 2\int_{\Sigma^-} \left\{N\left(x,y,0\right)\right\} \frac{\partial \hat{p}_i}{\partial z} dS \tag{33}
$$

System (2) becomes :

$$\begin{pmatrix} [K]-\omega^2[M] & -[C] \\ -[C]^T & \dfrac{[H]}{\omega^2}-[Q]-\left(\dfrac{[\Delta^+]}{\rho_f^+\omega^2}+\dfrac{[\Delta^-]}{\rho_f^-\omega^2}\right) \end{pmatrix} \begin{Bmatrix} \{u\} \\ \{p\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \dfrac{2}{\rho_f^+\omega^2}\{\Phi_i\} \end{Bmatrix} \tag{34}$$

With $\{\Phi_i\} = \int_{\Sigma^-} \{N(x,y,0)\} \dfrac{\partial \hat{p}_i}{\partial n_{ext}} dS$

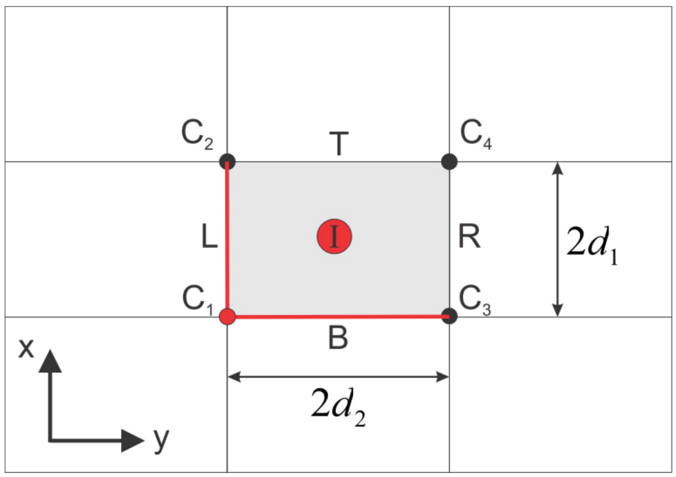

Figure 3

The nodal vector $\{u\}$ and $\{p\}$ can be partitioned in nine parts: surfaces B, L, T, R, corner lines C_1, C_2, C_3 and C_4, inner domain I (see Figure 3).

$$\{u^{ren}\} = \langle \langle u_L \rangle \langle u_R \rangle \langle u_B \rangle \langle u_T \rangle \langle u_{C_1} \rangle \langle u_{C_2} \rangle \langle u_{C_3} \rangle \langle u_{C_4} \rangle \langle u_I \rangle \rangle^T \tag{35}$$

$$\{p^{ren}\} = \langle \langle p_L \rangle \langle p_R \rangle \langle p_B \rangle \langle p_T \rangle \langle p_{C_1} \rangle \langle p_{C_2} \rangle \langle p_{C_3} \rangle \langle p_{C_4} \rangle \langle p_I \rangle \rangle^T \tag{36}$$

Where $\langle . \rangle$ denotes the transpose of $\{.\}$. $\langle u_L \rangle$, $\langle u_R \rangle$, $\langle u_B \rangle$, $\langle u_T \rangle$ contain the displacement degrees of freedom on surfaces L, R, B, T except those on their edges. $\langle u_{C_1} \rangle$, $\langle u_{C_2} \rangle$, $\langle u_{C_3} \rangle$, $\langle u_{C_4} \rangle$ contain the displacement degrees of freedom on the unit cell edges along direction z. $\langle u_I \rangle$ contains the internal displacement degrees of freedom (all those which are not on the boundaries of the unit cell). The same notations are used for the pressure degrees of freedom.

Similarly, the force and normal pressure gradient nodal vectors can be partitioned as:

$$\{F^{ren}\} = \langle \langle F_L \rangle \langle F_R \rangle \langle F_B \rangle \langle F_T \rangle \langle F_{C_1} \rangle \langle F_{C_2} \rangle \langle F_{C_3} \rangle \langle F_{C_4} \rangle \langle F_I \rangle \rangle^T \tag{37}$$

$$\left\{\Phi^{ren}\right\} = \left\langle \left\langle \Phi_L \right\rangle \left\langle \Phi_R \right\rangle \left\langle \Phi_B \right\rangle \left\langle \Phi_T \right\rangle \left\langle \Phi_{C_1} \right\rangle \left\langle \Phi_{C_2} \right\rangle \left\langle \Phi_{C_3} \right\rangle \left\langle \Phi_{C_4} \right\rangle \left\langle 0 \right\rangle \right\rangle^T + \left\langle \left\langle \Phi_L^z \right\rangle \left\langle \Phi_R^z \right\rangle \left\langle \Phi_B^z \right\rangle \left\langle \Phi_T^z \right\rangle \left\langle \Phi_{C_1}^z \right\rangle \left\langle \Phi_{C_2}^z \right\rangle \left\langle \Phi_{C_3}^z \right\rangle \left\langle \Phi_{C_4}^z \right\rangle \left\langle \Phi_I^z \right\rangle \left\langle 0 \right\rangle \right\rangle^T \quad (38)$$

where $\left\langle \Phi_L \right\rangle, \left\langle \Phi_R \right\rangle, \left\langle \Phi_B \right\rangle, \left\langle \Phi_T \right\rangle$ are nodal vectors containing the normal pressure gradient to surfaces L, R, B, T[1] except those on their edges. $\left\langle \Phi_{C_1} \right\rangle, \left\langle \Phi_{C_2} \right\rangle, \left\langle \Phi_{C_3} \right\rangle, \left\langle \Phi_{C_4} \right\rangle$ are corner-line nodal vectors containing the sum of the normal pressure gradient to surfaces L and B, L and T, B and R, L and T respectively[2]. $\left\langle 0 \right\rangle$ is the nodal vector which refers to the PUC internal nodes not on its lateral boundaries. $\left\langle \Phi_L^z \right\rangle, \left\langle \Phi_R^z \right\rangle, \left\langle \Phi_B^z \right\rangle, \left\langle \Phi_T^z \right\rangle, \left\langle \Phi_{C_1}^z \right\rangle, \left\langle \Phi_{C_2}^z \right\rangle, \left\langle \Phi_{C_3}^z \right\rangle, \left\langle \Phi_{C_4}^z \right\rangle, \left\langle \Phi_I^z \right\rangle$ are nodal vectors containing the normal pressure gradient in the z direction to faces Σ^- and Σ^+. The last vector $\left\langle 0 \right\rangle$ is the nodal vector which refers to the PUC internal nodes not on Σ^- and Σ^+ (i.e internal fluid nodes comprised between Σ^- and Σ^+).

$$\begin{aligned}
\left\{\Phi_L^z\right\} &= \left\langle \left\langle \Phi_L^- \right\rangle \left\langle \Phi_L^+ \right\rangle \right\rangle^T \\
\left\{\Phi_R^z\right\} &= \left\langle \left\langle \Phi_R^- \right\rangle \left\langle \Phi_R^+ \right\rangle \right\rangle^T \\
\left\{\Phi_B^z\right\} &= \left\langle \left\langle \Phi_B^- \right\rangle \left\langle \Phi_B^+ \right\rangle \right\rangle^T \\
\left\{\Phi_T^z\right\} &= \left\langle \left\langle \Phi_T^- \right\rangle \left\langle \Phi_T^+ \right\rangle \right\rangle^T \\
\left\{\Phi_{C_1}^z\right\} &= \left\langle \left\langle \Phi_{C_1}^- \right\rangle \left\langle \Phi_{C_1}^+ \right\rangle \right\rangle^T \\
\left\{\Phi_{C_2}^z\right\} &= \left\langle \left\langle \Phi_{C_2}^- \right\rangle \left\langle \Phi_{C_2}^+ \right\rangle \right\rangle^T \\
\left\{\Phi_{C_3}^z\right\} &= \left\langle \left\langle \Phi_{C_3}^- \right\rangle \left\langle \Phi_{C_3}^+ \right\rangle \right\rangle^T \\
\left\{\Phi_{C_4}^z\right\} &= \left\langle \left\langle \Phi_{C_4}^- \right\rangle \left\langle \Phi_{C_4}^+ \right\rangle \right\rangle^T \\
\left\{\Phi_I^z\right\} &= \left\langle \left\langle \Phi_I^- \right\rangle \left\langle \Phi_I^+ \right\rangle \right\rangle^T
\end{aligned} \quad (39)$$

where the vectors have been partitioned in two domains Σ^-, Σ^+.
$\left\langle \Phi_J^- \right\rangle, \left\langle \Phi_J^+ \right\rangle, J = L, R, B, T, C_1, C_2, C_3, C_4, I$ refer to the vectors containing the normal pressure gradient in the z-direction on faces Σ^- and Σ^+.

There remains to apply the boundary conditions on the unit cell which read :

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[1] Terms in $\dfrac{\partial \hat{p}}{\partial n_y}$ for $\left\langle \Phi_L \right\rangle, \left\langle \Phi_R \right\rangle$ and in $\dfrac{\partial \hat{p}}{\partial n_x}$ for $\left\langle \Phi_B \right\rangle, \left\langle \Phi_T \right\rangle$.

[2] Terms in $\dfrac{\partial \hat{p}}{\partial n_x} + \dfrac{\partial \hat{p}}{\partial n_y}$

$$\left\{u_R\right\} = \left\{u_L\right\} e^{-j\varphi_y} \tag{40}$$

$$\left\{u_T\right\} = \left\{u_B\right\} e^{-j\varphi_x} \tag{41}$$

$$\left\{u_{C_2}\right\} = \left\{u_{C_1}\right\} e^{-j\varphi_x} \tag{42}$$

$$\left\{u_{C_3}\right\} = \left\{u_{C_1}\right\} e^{-j\varphi_y} \tag{43}$$

$$\left\{u_{C_4}\right\} = \left\{u_{C_1}\right\} e^{-j\left(\varphi_x+\varphi_y\right)} \tag{44}$$

The same holds for the pressure dofs. These equations allows for expressing the nodal vectors in terms of reduced nodal vectors $\left\{u^c\right\} = \left\langle \left\langle u_L \right\rangle \left\langle u_B \right\rangle \left\langle u_{C_1} \right\rangle \left\langle u_I \right\rangle \right\rangle^T$ and $\left\{p^c\right\} = \left\langle \left\langle p_L \right\rangle \left\langle p_B \right\rangle \left\langle p_{C_1} \right\rangle \left\langle p_I \right\rangle \right\rangle^T$:

$$\left\{u^{ren}\right\} = \left[Q^U\right]\left\{u^c\right\} \tag{45}$$

$$\left\{p^{ren}\right\} = \left[Q^p\right]\left\{p^c\right\} \tag{46}$$

With

$$\left[Q^U\right] = \begin{bmatrix} \left[Q^{u_L}\right] & [0] & [0] & [0] \\ [0] & \left[Q^{u_B}\right] & [0] & [0] \\ [0] & [0] & \left[Q^{u_{C_1}}\right] & [0] \\ [0] & [0] & [0] & \left[I_I^u\right] \end{bmatrix} \tag{47}$$

and

$$\left[Q^{u_L}\right] = \begin{bmatrix} \left[I_L^u\right] \\ \left[I_L^u\right] e^{-j\varphi_y} \end{bmatrix} \tag{48}$$

$$\left[Q^{u_B}\right] = \begin{bmatrix} \left[I_B^u\right] \\ \left[I_B^u\right] e^{-j\varphi_x} \end{bmatrix} \tag{49}$$

$$\left[Q^{u_{C_1}}\right] = \begin{bmatrix} \left[I_{C_1}^u\right] \\ \left[I_{C_1}^u\right] e^{-j\varphi_x} \\ \left[I_{C_1}^u\right] e^{-j\varphi_y} \\ \left[I_{C_1}^u\right] e^{-j\left(\varphi_x+\varphi_y\right)} \end{bmatrix} \tag{50}$$

In the previous equations $\left[I_A^u\right]$ denotes the identity matrix of size equal to the number of structural degrees of freedom belonging to entity A (face or edge). Similarly

$$\left[Q^p\right]=\begin{bmatrix}\left[Q^{p_L}\right] & [0] & [0] & [0]\\ [0] & \left[Q^{p_B}\right] & [0] & [0]\\ [0] & [0] & \left[Q^{p_{C_1}}\right] & [0]\\ [0] & [0] & [0] & \left[I_I^p\right]\end{bmatrix} \tag{51}$$

and

$$\left[Q^{p_L}\right]=\begin{bmatrix}\left[I_L^p\right]\\ \left[I_L^p\right]e^{-j\varphi_y}\end{bmatrix} \tag{52}$$

$$\left[Q^{p_B}\right]=\begin{bmatrix}\left[I_B^p\right]\\ \left[I_B^p\right]e^{-j\varphi_x}\end{bmatrix} \tag{53}$$

$$\left[Q^{p_{C_1}}\right]=\begin{bmatrix}\left[I_{C_1}^p\right]\\ \left[I_{C_1}^p\right]e^{-j\varphi_x}\\ \left[I_{C_1}^p\right]e^{-j\varphi_y}\\ \left[I_{C_1}^p\right]e^{-j\left(\varphi_x+\varphi_y\right)}\end{bmatrix} \tag{54}$$

For the external forces, we have the following relationships:

$$\left\{F_R\right\}+\left\{F_L\right\}e^{-j\varphi_y}=\left\{0\right\} \tag{55}$$

$$\left\{F_T\right\}+\left\{F_B\right\}e^{-j\varphi_x}=\left\{0\right\} \tag{56}$$

$$\left\{F_{C_1}\right\}+\left\{F_{C_2}\right\}e^{-j\varphi_x}+\left\{F_{C_3}\right\}e^{-j\varphi_y}+\left\{F_{C_4}\right\}e^{-j\left(\varphi_x+\varphi_y\right)}=\left\{0\right\} \tag{57}$$

Let assume that each component of corner line nodal vectors $\left\{F_{C_2}\right\}$, $\left\{F_{C_3}\right\}$ and $\left\{F_{C_4}\right\}$ is proportional to the corresponding component of $\left\{F_{C_1}\right\}$, that is:

$$F_{C_2,i}=\varpi_i F_{C_1,i},\ F_{C_3,i}=\vartheta_i F_{C_1,i},\ F_{C_4,i}=\xi_i F_{C_1,i} \tag{58}$$

Then using Eq(57), we get the following relationship between the proportionality constants ϖ_i, ϑ_i and ξ_i:

$$1+\varpi_i^F e^{-j\varphi_x}+\vartheta_i^F e^{-j\varphi_y}+\xi_i^F e^{-j(\varphi_x+\varphi_y)}=\{0\} \tag{59}$$

The same relations Eq(55) to Eq (59) holds for the normal pressure gradients but with different constants.

We then have

$$\{F^{ren}\}=[Q^F]\{F^c\} \tag{60}$$

$$[Q^F]=\begin{bmatrix} [Q^{F_L}] & [0] & [0] & [0] \\ [0] & [Q^{F_B}] & [0] & [0] \\ [0] & [0] & [Q^{F_{C_1}}] & [0] \\ [0] & [0] & [0] & [I_I^F] \end{bmatrix} \tag{61}$$

and

$$[Q^{F_L}]=\begin{bmatrix} [I_L^F] \\ -[I_L^F]e^{-j\varphi_y} \end{bmatrix} \tag{62}$$

$$[Q^{F_B}]=\begin{bmatrix} [I_B^F] \\ -[I_B^F]e^{-j\varphi_x} \end{bmatrix} \tag{63}$$

$$[Q^{F_{C_1}}]=\begin{bmatrix} [I_{C_1}^F] \\ [\varpi_{C_1}^F]e^{-j\varphi_x} \\ [\vartheta_{C_1}^F]e^{-j\varphi_y} \\ [\xi_{C_1}^F]e^{-j(\varphi_x+\varphi_y)} \end{bmatrix} \tag{64}$$

Matrices $[\varpi_{C_1}^F]$, $[\vartheta_{C_1}^F]$ and $[\xi_{C_1}^F]$ are diagonal matrices of size the number of structural dofs on corner line C_1 whose generic term are $\varpi_{C_1,ii}^F=\varpi_i^F$, $\vartheta_{C_1,ii}^F=\vartheta_i^F$ $\xi_{C_1,ii}^F=\xi_i^F$ respectively.

We also have

$$\{\Phi^{ren}\}=[Q^\Phi]\{\Phi^c\}+\{\Phi_z\} \tag{65}$$

with

$$\{\Phi^c\}=\langle\langle\Phi_L\rangle\langle\Phi_B\rangle\langle\Phi_{C_1}\rangle\langle\Phi_I\rangle\rangle^T \tag{66}$$

and

$$\begin{aligned}\{\Phi_z\} &= \left\langle \langle\Phi_L^z\rangle\langle\Phi_R^z\rangle\langle\Phi_B^z\rangle\langle\Phi_T^z\rangle\langle\Phi_{C_1}^z\rangle\langle\Phi_{C_2}^z\rangle\langle\Phi_{C_3}^z\rangle\langle\Phi_{C_4}^z\rangle\langle\Phi_I^z\rangle\langle 0\rangle \right\rangle^T \\ &= \left\langle \langle\Phi_z^c\rangle\langle 0\rangle \right\rangle^T\end{aligned} \tag{67}$$

For the normal gradients in the plane (x,y) we have,

$$\left[Q^{\Phi}\right] = \begin{bmatrix} \left[Q^{\Phi_L}\right] & [0] & [0] & [0] \\ [0] & \left[Q^{\Phi_B}\right] & [0] & [0] \\ [0] & [0] & \left[Q^{\Phi_{C_1}}\right] & [0] \\ [0] & [0] & [0] & \left[I_I^{\Phi}\right] \end{bmatrix} \tag{68}$$

and

$$\left[Q^{\Phi_L}\right] = \begin{bmatrix} \left[I_L^{\Phi}\right] \\ -\left[I_L^{\Phi}\right]e^{-j\varphi_y} \end{bmatrix} \tag{69}$$

$$\left[Q^{\Phi_B}\right] = \begin{bmatrix} \left[I_B^{\Phi}\right] \\ -\left[I_B^{\Phi}\right]e^{-j\varphi_x} \end{bmatrix} \tag{70}$$

$$\left[Q^{\Phi_{C_1}}\right] = \begin{bmatrix} \left[I_{C_1}^{\Phi}\right] \\ \left[\varpi_{C_1}^{\Phi}\right]e^{-j\varphi_x} \\ \left[\vartheta_{C_1}^{\Phi}\right]e^{-j\varphi_y} \\ \left[\xi_{C_1}^{\Phi}\right]e^{-j(\varphi_x+\varphi_y)} \end{bmatrix} \tag{71}$$

Matrices $\left[\varpi_{C_1}^{\Phi}\right]$, $\left[\vartheta_{C_1}^{\Phi}\right]$ and $\left[\xi_{C_1}^{\Phi}\right]$ are diagonal matrices of size the number of pressure dofs on cornerline C_1 whose generic term are $\varpi_{C_1,ii}^{\Phi} = \varpi_i^{\Phi}$, $\vartheta_{C_1,ii}^{\Phi} = \vartheta_i^{\Phi}$ $\xi_{C_1,ii}^{\Phi} = \xi_i^{\Phi}$ respectively.

Matrices $\left[I_L^{\Phi}\right], \left[I_B^{\Phi}\right], \left[I_{C_1}^{\Phi}\right]$ are diagonal matrices of size the number of pressure dofs on the respective partition.

Substituting Eq(45), Eq(46), Eq(60) and Eq(65) in Eq(2), multiplying the first and the second line of the resulting system by $\left[Q^u\right]^{*T}$ and $\left[Q^p\right]^{*T}$ respectively, the following reduced system is obtained:

$$\begin{pmatrix} [Z^{uu}] & [Z^{up}] \\ [Z^{up}]^{*T} & [Z^{pp}] \end{pmatrix} \begin{Bmatrix} \{u^c\} \\ \{p^c\} \end{Bmatrix} = \begin{Bmatrix} [Q^u]^{*T} [Q^F] \{F^c\} \\ \dfrac{2}{\rho_f \omega^2} \left([Q^p]^{*T} [Q^\Phi] \{\Phi^c\} + [Q^p]^{*T} \{\Phi^z\} \right) \end{Bmatrix} \tag{72}$$

$$[Z^{uu}] = [Q^u]^{*T} \left[[K] - \omega^2 [M] \right] [Q^u]$$

$$[Z^{pp}] = [Q^p]^{*T} \left[\frac{[H]}{\omega^2} - [Q] - \left(\frac{[\Delta^+]}{\rho_f^+ \omega^2} + \frac{[\Delta^-]}{\rho_f^- \omega^2} \right) \right] [Q^p]$$

$$[Z^{up}] = -[Q^u]^{*T} [C][Q^p]$$

Using Eq(47) and Eq(61) on the one hand and Eq (51) and Eq(68) on the other hand, it can be shown that:

$$[Q^u]^{*T} [Q^F] \{F^c\} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{F_I\} \end{Bmatrix} \underbrace{\equiv}_{\text{no internal force acting on the structure}} \{0\} \tag{73}$$

and

$$[Q^p]^{*T} [Q^\Phi] \{\Phi^c\} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} \tag{74}$$

Eq(72) becomes

$$\begin{pmatrix} [Z^{uu}] & [Z^{up}] \\ [Z^{up}]^{*T} & [Z^{pp}] \end{pmatrix} \begin{Bmatrix} \{u^c\} \\ \{p^c\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \dfrac{2}{\rho_f \omega^2} [Q_p]^{*T} \{\Phi^z\} \end{Bmatrix} \tag{75}$$

System in Eq(75) is Hermitian and can be solved using appropriate resolution algorithm. The matrices $\left[[K] - \omega^2 [M] \right]$, $[C]$ and $\left[\dfrac{[H]}{\omega^2} - [Q] \right]$ are obtained directly from Novafem without imposing any boundary conditions on the fluid and the structure. System (75) requires to build the condensation matrices $[Q^u]$ and $[Q^p]$. Note that in $\{\Phi^z\}$ only the normal incident pressure gradient degrees of freedom (on $\Sigma^-(z=0)$) are non-zero.

The steps in the methodology are the following:

1. Identify corner line, lateral face and internal nodes
2. Partition global solution vector into nine components
3. Build the periodic boundary condition condensation matrix for the structure $\left[Q^u\right]$
4. Build the periodic boundary condition condensation matrices for the fluid $\left[Q^p\right]$ and $\left[Q^\Phi\right]$
5. Calculate matrices $\left[\Delta^+\right]$ and $\left[\Delta^-\right]$
6. Calculate the normal incident pressure gradient nodal vector $\left[\Phi\right]$
7. Build system (72) (Hermitian projection of Eq(2) on $\left[Q^g\right]=\begin{bmatrix}\left[Q^u\right] & [0] \\ [0] & \left[Q^p\right]\end{bmatrix}$)
8. Solve the projected Hermitian system

Calculation of $\left[\Delta^+\right]$

$\left[\Delta^+\right]$ can be rewritten as

$$\begin{aligned}\left[\Delta^+\right] &= \int_{\Sigma^+}\left\{N\left(x,y,h^+\right)\right\}\left\langle -jk_{mn}^+ e^{-j\alpha_m x}e^{-j\beta_n y}\right\rangle dS\left[A^+\right] \\ &= \frac{1}{\Sigma^+}\left[A^{+1}\right]^{*T}\left[D^+\right]\left[A^{+1}\right]\end{aligned} \tag{76}$$

where $\left[D^+\right]$ is a diagonal matrix of dimension $\left(\left(2M+1\right)\times\left(2N+1\right),\left(2M+1\right)\times\left(2N+1\right)\right)$

$$\left[D^+\right]=\begin{bmatrix} -jk_{-M-N}^+ & 0 & 0 & 0 & 0 \\ 0 & -jk_{-M-N+1}^+ & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -jk_{MN-1}^+ & 0 \\ 0 & 0 & 0 & 0 & -jk_{MN}^+ \end{bmatrix} \tag{77}$$

where $\left[A^{+1}\right]$ is a matrix of dimension $\left(\left(2M+1\right)\times\left(2N+1\right),N^p\right)$ given by

$$
\begin{aligned}
\left[A^{+1}\right] &= \int_{\Sigma^+} \left\{ e^{j\alpha_m x} e^{j\beta_n y} \right\} \left\langle N^p\left(x, y, h^+\right) \right\rangle dS \\
&= \left[e^{j\alpha_m x_i} e^{j\beta_n y_i} \right] \int_{\Sigma^+} \left\{ N^p\left(x, y, h^+\right) \right\} \left\langle N^p\left(x, y, h^+\right) \right\rangle dS \\
&= \left[e^{j\alpha_m x_i} e^{j\beta_n y_i} \right] \left[C^p \right] \\
&= \left[\Upsilon \right] \left[C^p \right]
\end{aligned}
\tag{78}
$$

$\left[\Upsilon\right] = \left[e^{j\alpha_m x_i} e^{j\beta_n y_i} \right]$ is a matrix of dimension $\left(\left(2M+1\right)\times\left(2N+1\right), N^p\right)$ which contains on each line the nodal values of term $e^{j\alpha_m x} e^{j\beta_n y}$. Actually, only the dof on face Σ^+ are considered in the calculation.

Calculation of the transmitted power (power exchanged between the structure and Ω^+)

$$
\begin{aligned}
W^+ &= \frac{1}{2} \Re\left[\int_{\partial\Omega^+} \hat{p}^+ \hat{\vec{v}}^* . \vec{n} dS \right] \\
&= \frac{1}{2} \Re\left[-j\omega \left\langle p^+ \right\rangle \left[C\right] \left\{u^*\right\} \right]
\end{aligned}
\tag{79}
$$

Calculation of the power exchanged between the structure and Ω^-

$$
\begin{aligned}
W^- &= \frac{1}{2} \Re\left[\int_{\partial\Omega^-} \hat{p}^- \hat{\vec{v}}^* . \vec{n} dS \right] \\
&= \frac{1}{2} \Re\left[-j\omega \left\langle p^- \right\rangle \left[C\right] \left\{u^*\right\} \right]
\end{aligned}
\tag{80}
$$

Calculation of the Reflection and transmission coefficients

In the case where only the normal modes taken care of, the reflection coefficient is given by:

$$
\begin{aligned}
R = \hat{p}_{00}^- &= \frac{1}{2d_1} \frac{1}{2d_2} \int_0^{2d_1} \int_0^{2d_2} \hat{p}^-\left(x, y, 0\right) e^{jk^- \sin\theta\cos\phi x} e^{jk^- \sin\theta\sin\phi y} dxdy \\
&= \frac{1}{2d_1} \frac{1}{2d_2} \left\langle e^{jk^- \sin\theta\cos\phi x_i} e^{jk^- \sin\theta\sin\phi y_i} \right\rangle \left[C^p \right] \left\{ \hat{p}^- \right\}
\end{aligned}
\tag{81}
$$

Similarly, the transmission coefficient is given by $\hat{p}_{00}^+$ with:

$$T = \hat{p}_{00}^{+} = \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} \hat{p}^{+}\left(x, y, h^{+}\right) e^{jk^{-}\sin\theta\cos\phi x} e^{jk^{-}\sin\theta\sin\phi y} dxdy$$
$$= \frac{1}{2d_1}\frac{1}{2d_2}\left\langle e^{jk^{-}\sin\theta\cos\phi x_i} e^{jk^{-}\sin\theta\sin\phi y_i}\right\rangle \left[C^{p}\right]\left\{\hat{p}^{+}\right\} \quad (82)$$

To validate these statements, note that the incident pressure $\hat{p}_i\left(x, y, z\right)$ can be expanded as

$$\hat{p}_i\left(x, y, z\right) = \sum_{m=-M}^{M}\sum_{n=-N}^{N} \hat{p}_{i,mn} e^{-jk_{mn}^{-}z} e^{-j\alpha_m x} e^{-j\beta_n y} \quad (83)$$

with

$$\left\{p_{i,mn}\right\} = \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} \left\{e^{j\alpha_m x} e^{j\beta_n y}\right\}\left\langle N^{p}\left(x, y, 0\right)\right\rangle dxdy \left\{p_i\right\}$$
$$= \left[A^{-}\right]\left\{p_i\right\} \quad (84)$$

Compare to the reflected pressure:

$$\hat{p}^{-}\left(x, y, z\right) = \sum_{m,n=-\infty}^{+\infty} \hat{p}_{mn}^{-} e^{jk_{mn}^{-}z} e^{-j\alpha_m x} e^{-j\beta_n y}. \quad (85)$$

At z=0:

$$\hat{p}_i\left(x, y, 0\right) = \sum_{m=-M}^{M}\sum_{n=-N}^{N} \hat{p}_{i,mn} e^{-j\alpha_m x} e^{-j\beta_n y}, \quad (86)$$

and thus,

$$\hat{p}^{-}\left(x, y, 0\right) = \sum_{m,n=-\infty}^{+\infty} \hat{p}_{mn}^{-} e^{-j\alpha_m x} e^{-j\beta_n y} = R\hat{p}_i\left(x, y, 0\right) \quad (87)$$

Explicitly,

$$\frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} \sum_{m,n=-\infty}^{+\infty} \hat{p}_{mn}^{-} e^{-j\alpha_m x} e^{-j\beta_n y}\left(e^{j\alpha_p x} e^{j\beta_q y}\right) dxdy =$$
$$\frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} R\sum_{m=-M}^{M}\sum_{n=-N}^{N} \hat{p}_{i,mn} e^{-j\alpha_m x} e^{-j\beta_n y}\left(e^{j\alpha_p x} e^{j\beta_q y}\right) dxdy, \quad (88)$$

So that,

$$\hat{p}_{mn}^{-} = R\hat{p}_{i,mn} \Rightarrow R = \frac{\hat{p}_{mn}^{-}}{\hat{p}_{i,mn}} \neq f(m,n) = \frac{\hat{p}_{00}^{-}}{\hat{p}_{i,00}} = \hat{p}_{00}^{-}. \qquad (89)$$

Note that we can verify :

$$\frac{1}{2d_1}\frac{1}{2d_2}\hat{p}_{mn}^{-} = \frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2} R\hat{p}_i\left(x,y,0\right)\left(e^{j\alpha_p x}e^{j\beta_q y}\right)dxdy =$$

$$R\frac{1}{2d_1}\frac{1}{2d_2}\int_0^{2d_1}\int_0^{2d_2}\hat{A}\left(e^{j\frac{p\pi}{d_1}x}e^{j\frac{q\pi}{d_2}y}\right)dxdy = R\hat{A} = R\hat{p}_{i,00}$$

The same proof can be given for the transmission coefficient.

REFERENCES

[1] Hladky-Hennion, A.C. and Decarpigny, J.N. (1991), Analysis of the scattering of a plane acoustic wave by a doubly periodic structure using the finite element method: Application to Alberich anechoic coatings, The Journal of the Acoustical Society of America, 90, 3356

[2] V. Easwaran and M. L. Munjal, Analysis of reflection characteristics of a normal incidence plane wave on resonant sound absorbers: A finite element approach, JASA, 93(3) 1993, 1308-1318

[3] P.Langlet (1993). Analyse de la propagation d’ondes acoustiques dans les matériaux périodiques à l’aide de la méthode des éléments finis, PhD thesis, 200p. Université de Valenciennes, Valenciennes, France.

Part 2 – Validation of PCFEM Code: Materials with no Inclusions

Progress report

Preliminary TL Validation tests of embedded Helmholtz resonators in a porous frame

N. Atalla / F. Sgard / R. Panneton

January 31, 2013

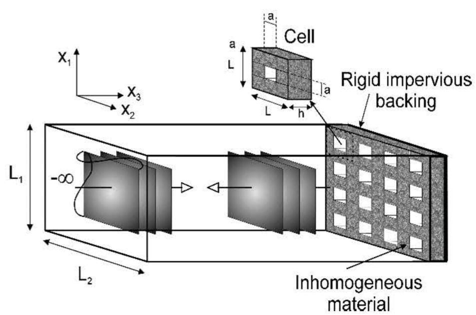
Methods

Transfer matrix Methods

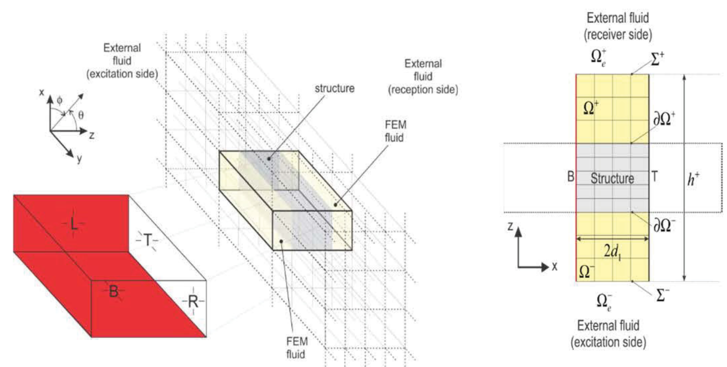
- DRDC data (reference)
- NOVA (Mecanum's commercial TMM code for multilayered sound packages)

FE

- Solid in a waveguide using NOVAFEM software (limited to normal incidence)
- Periodic cell model : specifically coded in NOVAFEM for this project (Not fully validated)

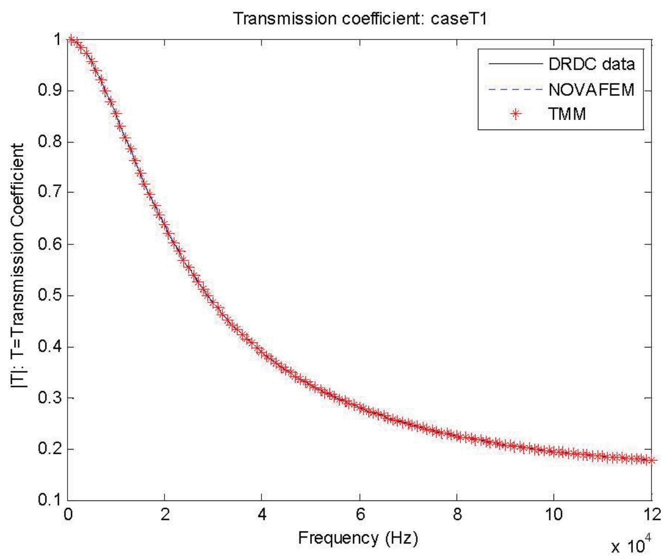


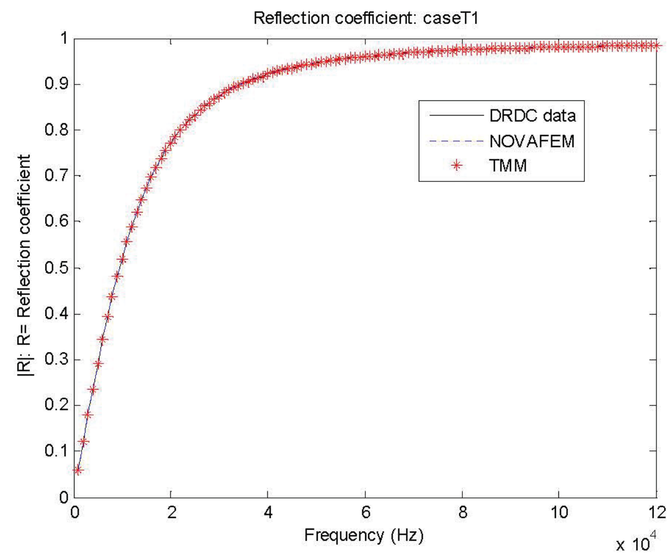
Waveguide model in NOVAFEM (inlcusion were not tested yet)



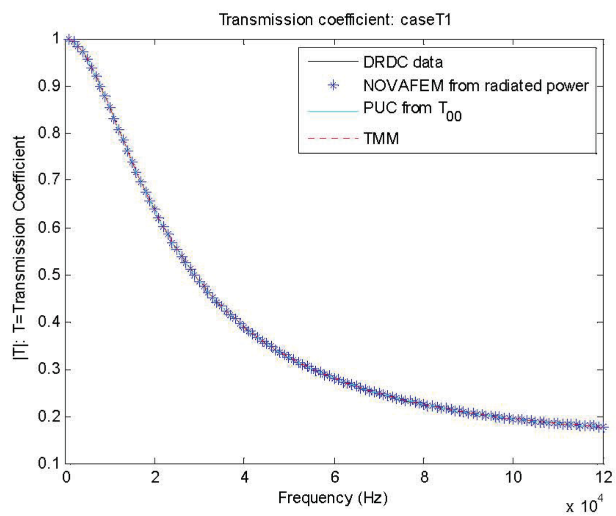
Periodic Unit Cell model (PUC) in NOVA (under development; (so far only homogeneous solids meshed with HEXA 8 elements were tested)

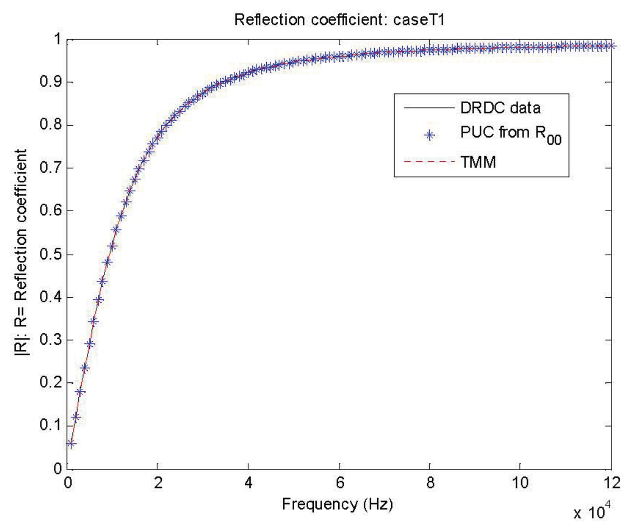
Case T1 (Waveguide option)





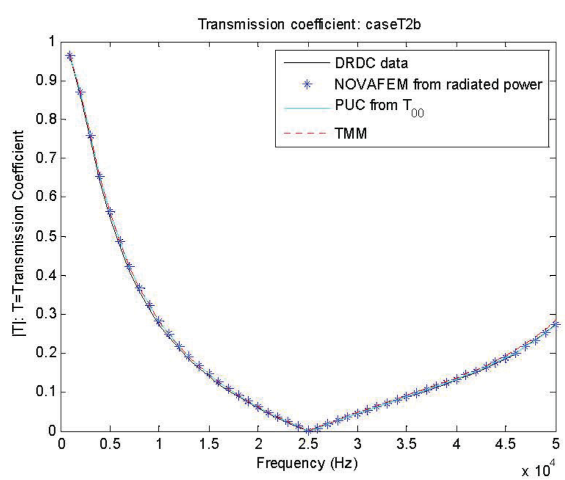
Case T1 (PUC option)

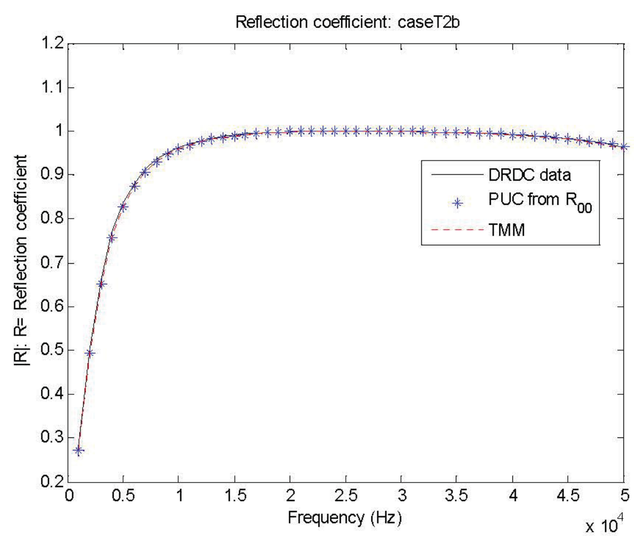




Case T2b (PUC option)

Test Case 2b: Water/ 50 mm aluminum / water, 15 degrees, no absorption

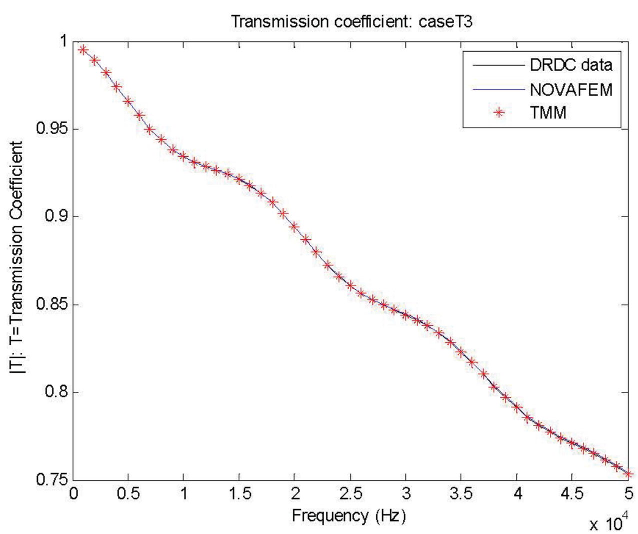


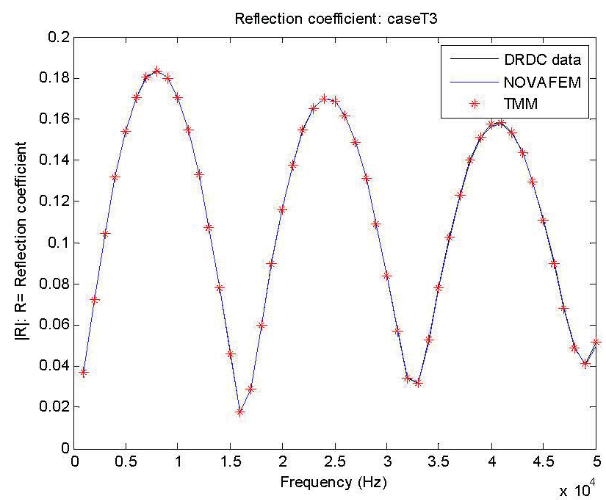


Case2b.mat uses 5cm thick al contrary to 10.7mm in the Words document.

Case T3 (Waveguide option)

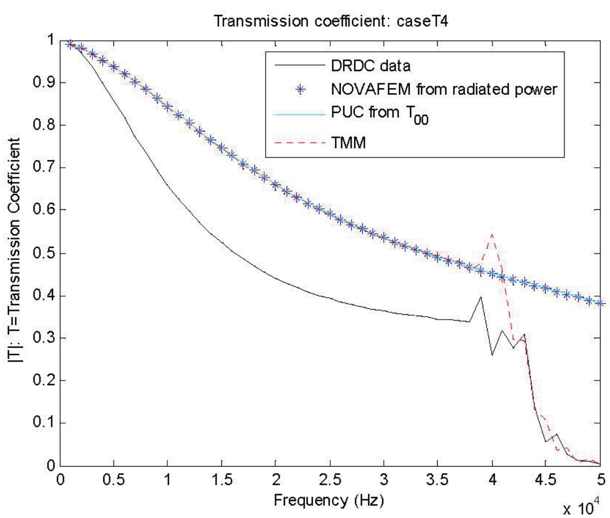
Case 3: Water/ 50 mm Rubber #1/ water, 0 degrees with absorption

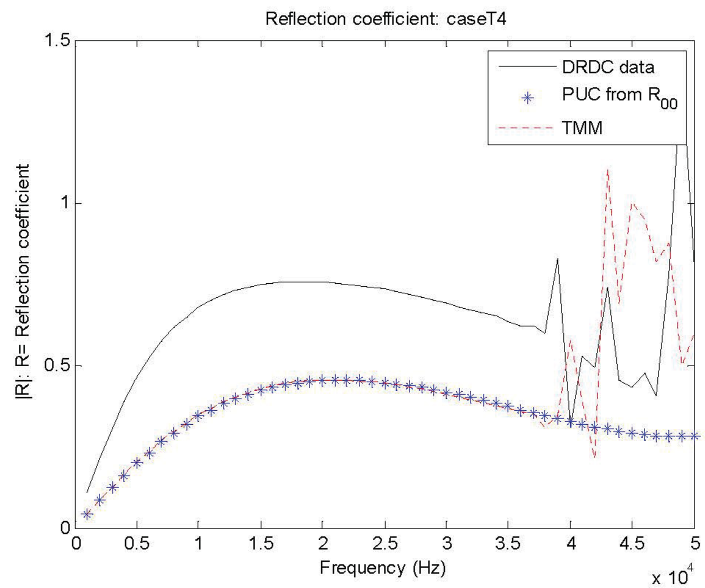




Case T4 (PUC option)

Test Case 4: Water/ 50 mm Rubber #1/ water, 60 degrees absorption

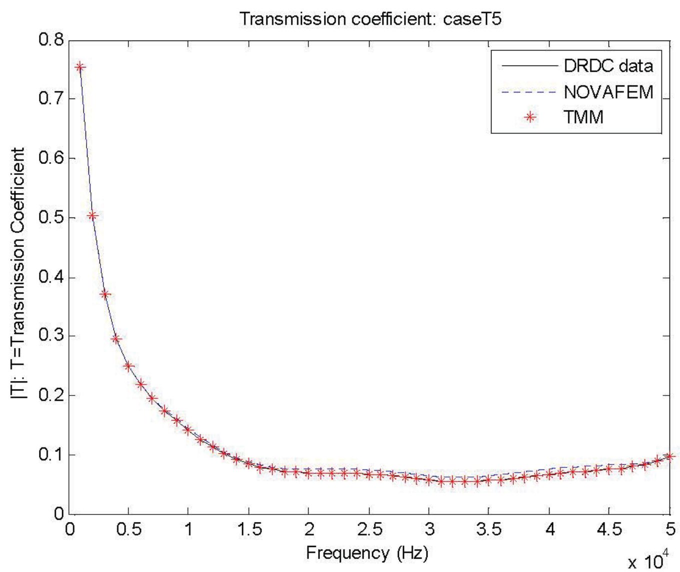


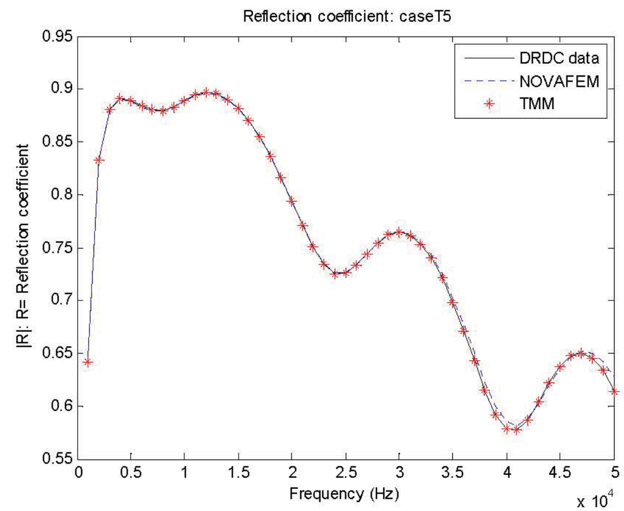


It seems that the reference solution given in caseT4.mat is wrong?

Case T5 (Waveguide option)

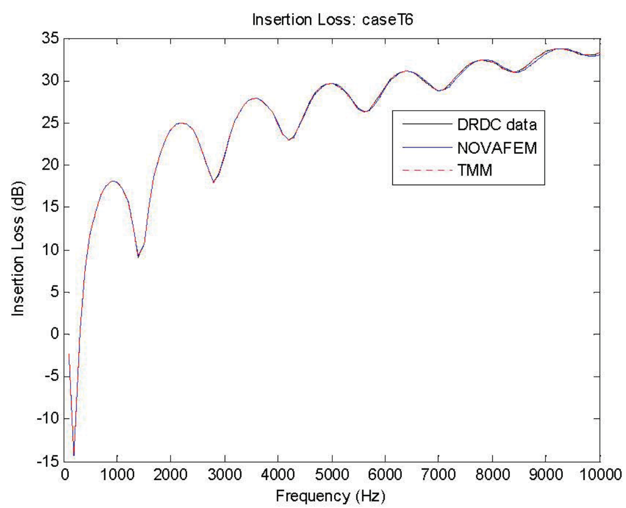
Test Case 5: Water/ 50 mm Rubber #1/ 50 mm Steel/ water, 0 degrees

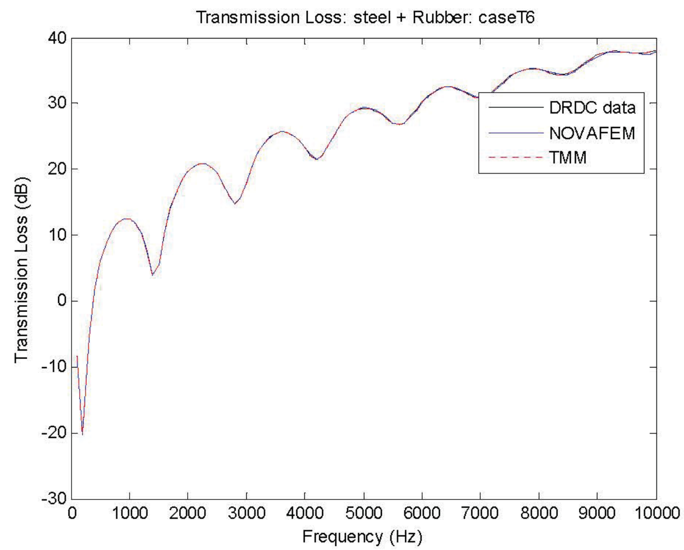




Case T6 (Waveguide option)

Test Case 6: air/ 10 mm steel/ 30 mm Rubber #2/ water, 0 degrees





Conclusion

1. Excellent agreement for all homogeneous test cases

Part 3 – Validation of PCFEM Code: Materials with Inclusions

PCFEM - Validation cases with inclusions

CASE 1 : DOUBLY PERIODIC CICURLAR CYLINDRICAL AIR INCLUSIONS

REFERENCES

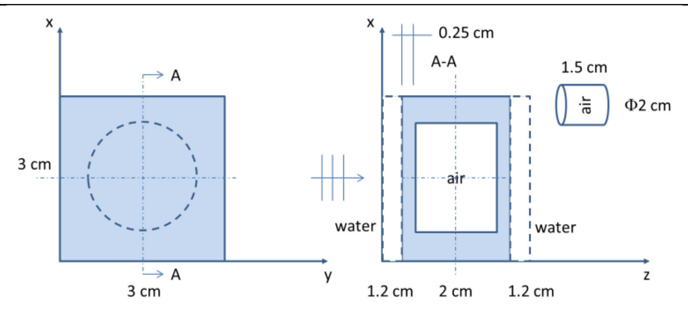
- Easwaran, V. and Munjal, M. (1993), Analysis of reflection characteristics of a normal incidence plane wave on resonant sound absorbers: A finite element approach, The Journal of the Acoustical Society of America, 93, 1308.
- Hladky-Hennion, A.C. and Decarpigny, J.N. (1991), Analysis of the scattering of a plane acoustic wave by a doubly periodic structure using the finite element method: Application to Alberich anechoic coatings, The Journal of the Acoustical Society of America, 90, 3356.

DESCRIPTION

- Alberich anechoic coating with circular cylindrical **air** inclusion immersed in **water** on both sides.
- Normal incidence acoustic wave.
- Air is not modelled in References – see as void.
- Only transmission results are available.

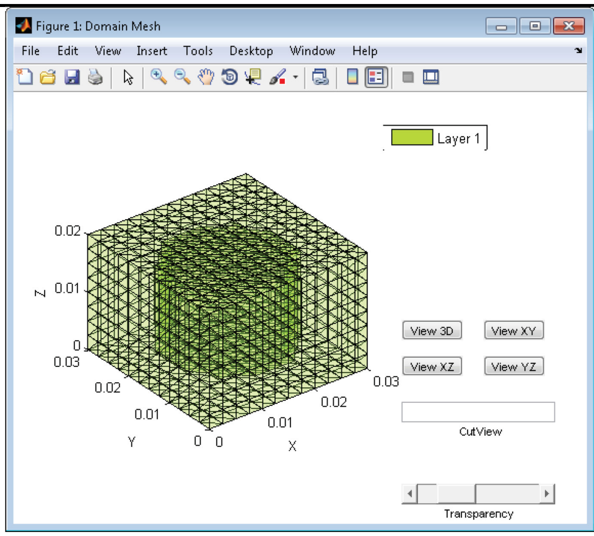
PCF MODEL

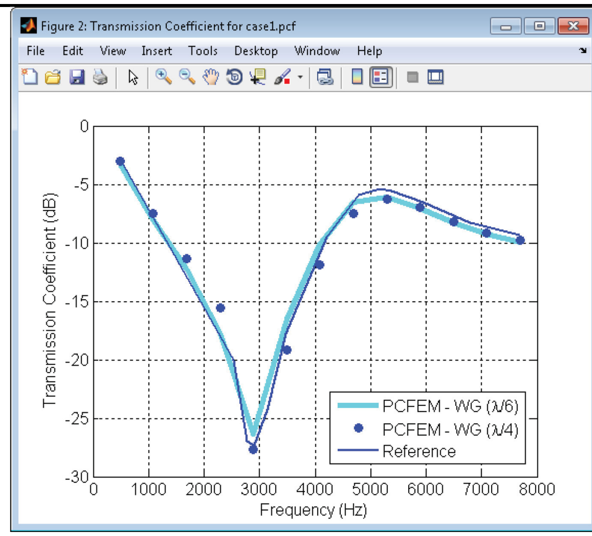
- \PCFEM\work\Test cases with inclusion\case1\case1.pcf
- Waveguide solver with TETRA 10 elements
- λ/4 criterion CPU time is 19 s; Number of elements: 1 268 Number of nodes: 2 474
- λ/6 criterion CPU time is 98 s; Number of elements: 5 080 Number of nodes: 8 784



Elastic properties of coating material	
E (Pa)	1.4×10^{8}
ν (-)	0.49
η_L (%)	23
ρ (kg/m³)	1100

PCFEM - Validation cases with inclusions





Good correlation obtained compared with reference.

CASE 2 : DOUBLY PERIODIC CICURLAR CYLINDRICAL AIR INCLUSIONS

REFERENCES

- Easwaran, V. and Munjal, M. (1993), Analysis of reflection characteristics of a normal incidence plane wave on resonant sound absorbers: A finite element approach, The Journal of the Acoustical Society of America, 93, 1308.
- Hladky-Hennion, A.C. and Decarpigny, J.N. (1991), Analysis of the scattering of a plane acoustic wave by a doubly periodic structure using the finite element method: Application to Alberich anechoic coatings, The Journal of the Acoustical Society of America, 90, 3356.

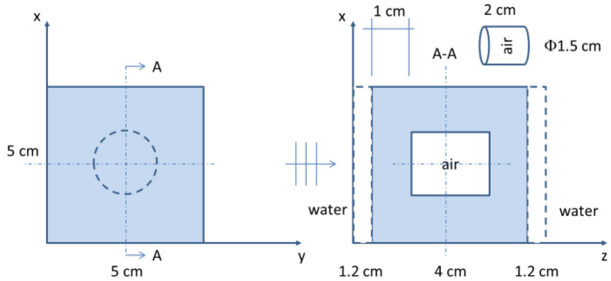
DESCRIPTION

- Alberich anechoic silicon coating with circular cylindrical **air** inclusion immersed in **water** on both sides.
- Normal incidence acoustic wave.
- Air is not modelled in References – see as void.

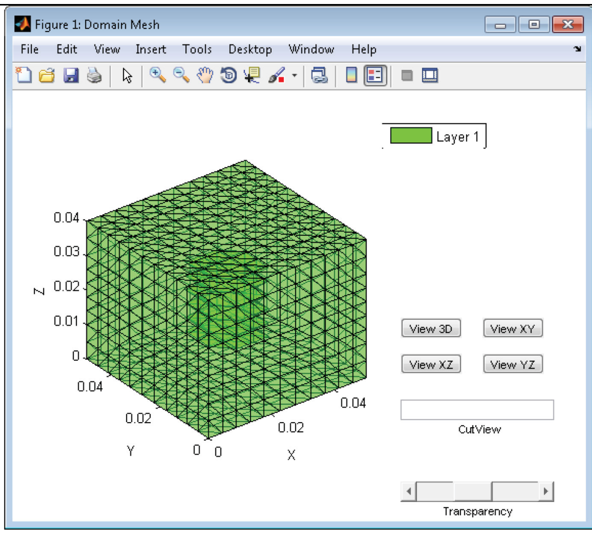
PCF MODEL

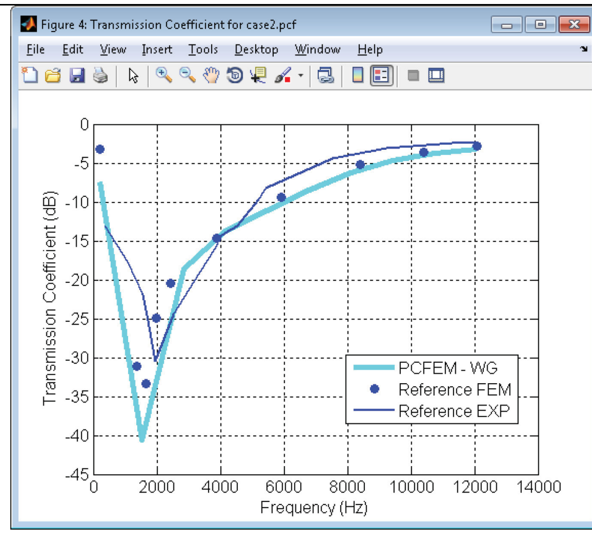
- \PCFEM\work\Test cases with inclusion\case2\case2.pcf
- Waveguide solver with TETRA 10 elements
- λ/2 criterion CPU time is 95 s   Number of elements: 6 314   Number of nodes: 9 888
- λ/3 criterion CPU time is 581 s   Number of elements: 21 674   Number of nodes: 32 240
- λ/3 yields results very very close to λ/2 for a fraction of time.

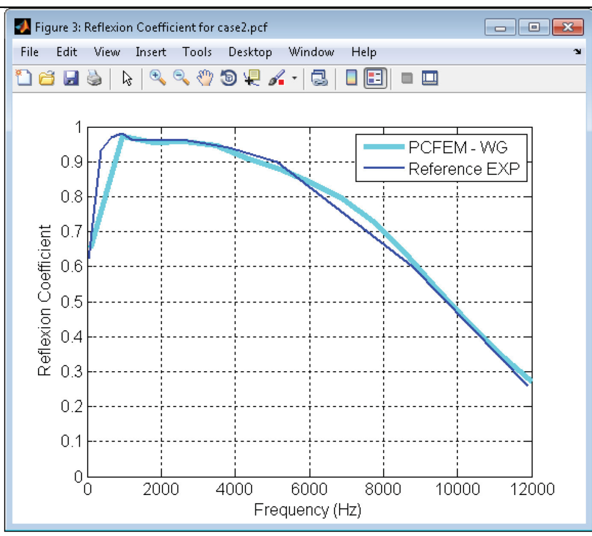
PCFEM - Validation cases with inclusions



Elastic properties of silicon coating	
E (Pa)	$1.8x10^6$
ν (-)	0.49976
η_L (%)	15
ρ (kg/m³)	1000







Good correlation obtained compared with FEM and EXP results from reference.

Validation cases with inclusions

CASE 3 : SINGLE PERIODIC INFINITE RECTANGULAR CYLINDRICAL AIR INCLUSIONS

REFERENCES

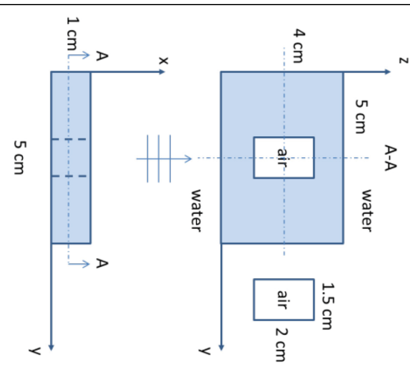
- Hladky-Hennion, A.C. and Decarpigny, J.N. (1991), Analysis of the scattering of a plane acoustic wave by a doubly periodic structure using the finite element method: Application to Alberich anechoic coatings, The Journal of the Acoustical Society of America, 90, 3356.

DESCRIPTION

- Alberich anechoic polyurethane coating with an infinite rectangular **air** inclusion immersed in water on both sides.
- It is similar to a 2D case (**arbitrary 1 cm height is imposed**).
- Normal incidence acoustic wave.
- Air is not modelled in References – see as void.

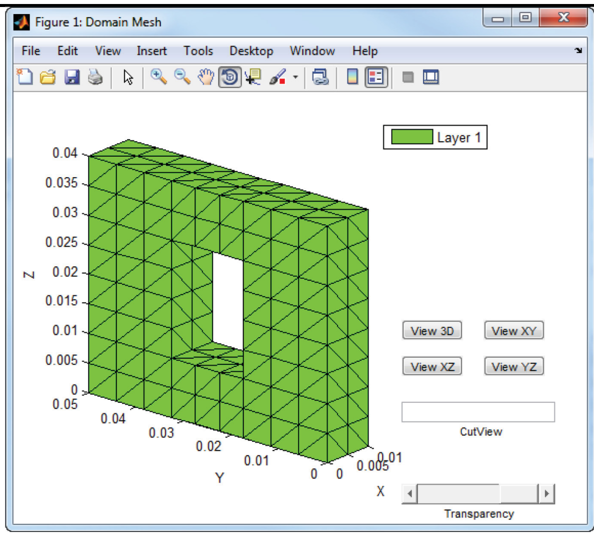
PCF MODEL

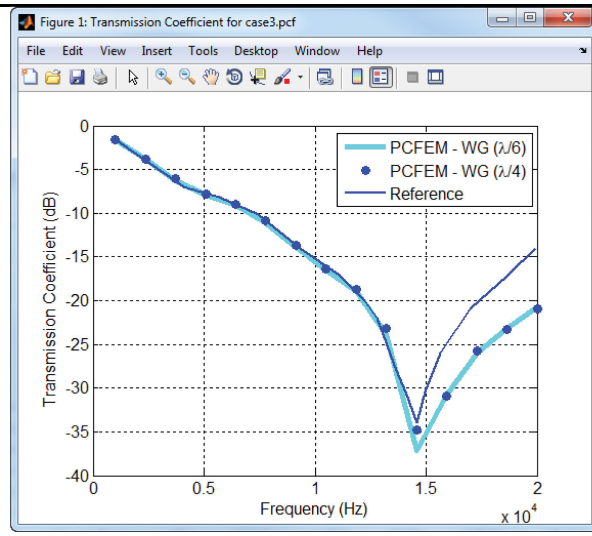
- \PCFEM\work\Test cases with inclusion\case3\case3.pcf
- Waveguide solver with TETRA 10 elements
- $\lambda/4$ criterion CPU time is 12 s   Number of elements: 720   Number of nodes: 1 440
- $\lambda/6$ criterion CPU time is 38 s   Number of elements: 2 412   Number of nodes: 4 228



Elastic properties of polyurethane material	
E (Pa)	2.81×10^{8}
ν (-)	0.479
η_L (%)	45
η_S (%)	1.78
ρ (kg/m³)	1100 (estimation)

PCFEM - Validation cases with inclusions





Good correlation obtained compared with reference except above 15 000 Hz. We have checked several meshes, and the curve doesn't change. However the results are very sensitive to damping. Here we apply, separate damping to E and Poisson and deduce G… So we are wondering if there is an error in the data especially the damping…It is surprising that only case 1 was reproduced in various papers.

Validation cases with inclusions

CASE 4 : DOUBLY PERIODIC ALBERICH ANECHOIC COATING ON STEEL PLATE – WATER/SLAB/AIR

REFERENCES

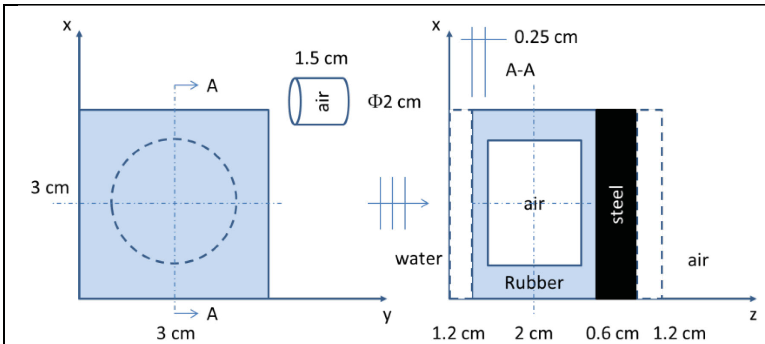
- Meng, H., Wen, J., Zhao, H., Lv, L. and Wen, X. (2012), Analysis of absorption performances of anechoic layers with steel plate backing, The Journal of the Acoustical Society of America, 132, 69.

DESCRIPTION

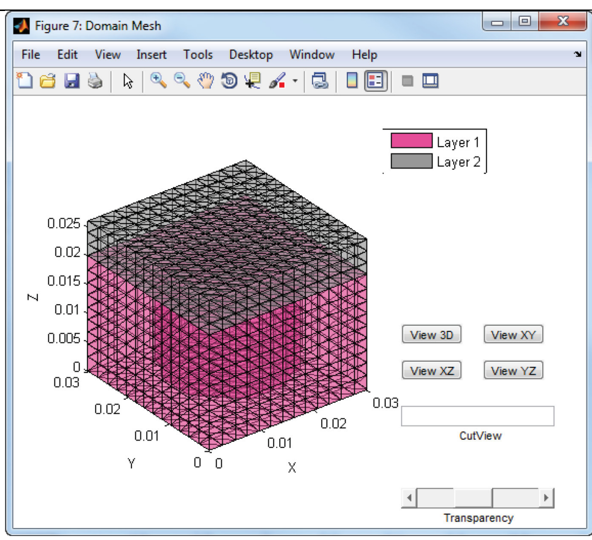
- Alberich anechoic coating with a cylindrical **air** inclusion as in CASE 1 backed by a STEEL plate.
- Water on incident side and air behind the steel plate.
- Normal incidence acoustic wave.

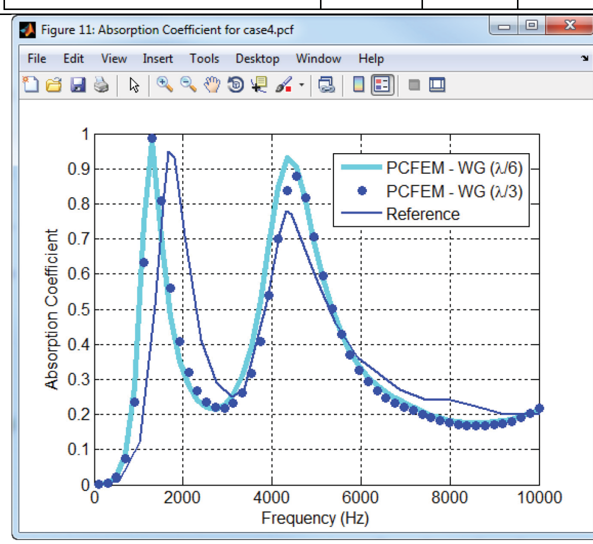
PCF MODEL

- \PCFEM\work\Test cases with inclusion\case4\case4.pcf
- Waveguide solver with TETRA 10 elements
- $\lambda/3$ criterion CPU time is 47 s Number of elements: 1 050 Number of nodes: 1 996
- $\lambda/6$ criterion CPU time is 539 s Number of elements: 9 452 Number of nodes: 15 310



Elastic properties of material	Rubber	Steel	Water
E (Pa)	1.4e8	2.1e11	n/a
ν (-)	0.49	0.3	n/a
η_E (%)	0.23	n/a	n/a
ρ (kg/m³)	1100	7890	1000
c (m/s)			1489



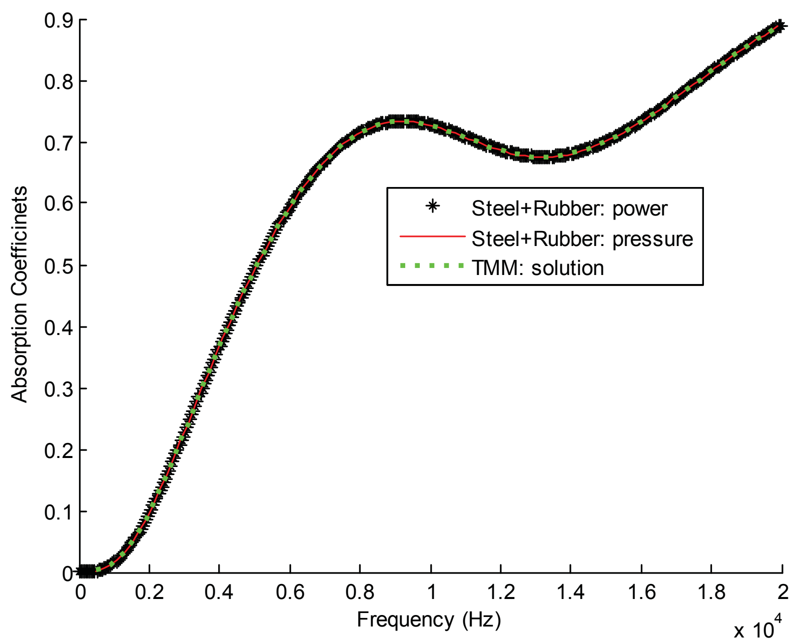


PCFEM - Validation cases with inclusions

Good correlation with reference. However we have tried to explain differences. We have checked several meshes and also meshed various air-layers upstream and downstream slab… the curves do not change. We believe that our results are more representative compared the reference since (1) the anechoic coating alone is exactly case 1 that was validated; and (2) the homogeneous case (=without the inclusion) agrees with the Transfer Matrix Method (TMM) – **see next page**.
Still, we are trying to find out what's wrong with the input data!

Homogeneous case (without inclusion)

*Comparison between PCFEM (waveguide solver *) and TMM (green curve)*



PCFEM - Validation cases with inclusions

CASE 4B : DOUBLY PERIODIC ALBERICH ANECHOIC COATING ON WATER – WATER/SLAB/WATER

REFERENCES

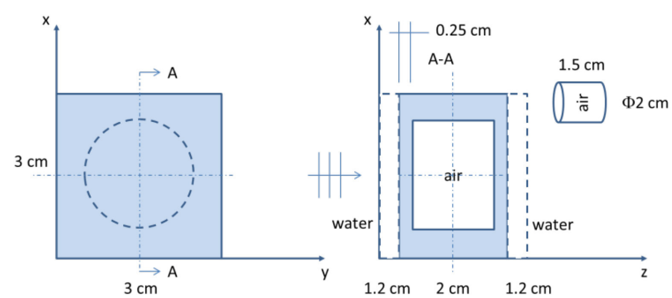
- Meng, H., Wen, J., Zhao, H., Lv, L. and Wen, X. (2012), Analysis of absorption performances of anechoic layers with steel plate backing, The Journal of the Acoustical Society of America, 132, 69.

DESCRIPTION

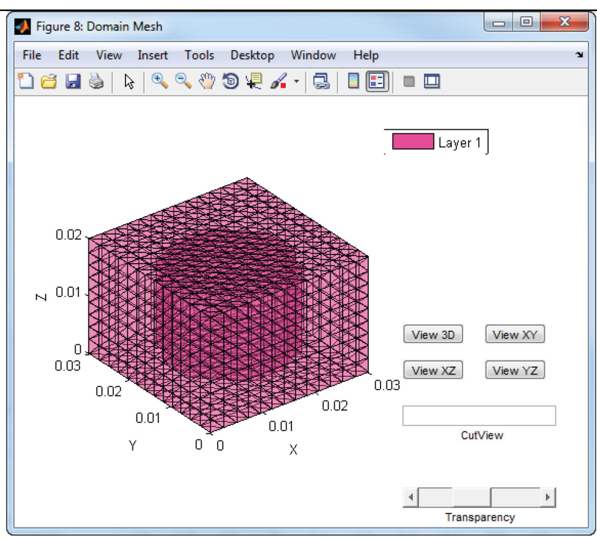
- Alberich anechoic coating with a cylindrical **air** inclusion with water of both sides **= CASE 1**.
- As in case 1, however this time in absorption.
- Normal incidence acoustic wave.

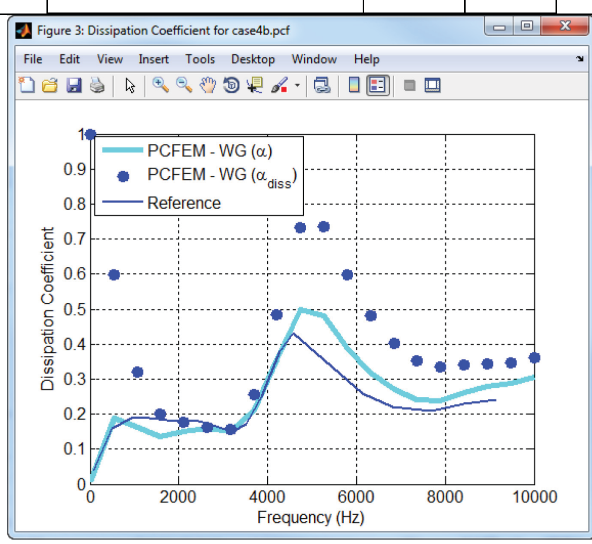
PCF MODEL

- \PCFEM\work\Test cases with inclusion\case4b\case4b.pcf
- Waveguide solver with TETRA 10 elements
- λ/6 criterion CPU time is 176 s Number of elements: 6 410 Number of nodes: 10 936



Elastic properties of material	Rubber	Water
E (Pa)	1.4e8	n/a
ν (-)	0.49	n/a
η_E (%)	0.23	n/a
ρ (kg/m³)	1100	1000
c (m/s)		1489





PCFEM - Validation cases with inclusions

Good correlation with reference with same conclusions as in case 4 relatively to the differences.

NOTE that usually the absorption coefficient is defined as $\alpha = 1-|R|^2$. Here the absorbance they defined seems to be equal to $\alpha_{diss} = D = 1-|R|^2 - |T|^2$ which is in fact the energy dissipated in the coating. While for the previous case both absorptions are very close due to the steel plate causing very low value of *T*, in this case this is no longer true.

CASE 5 : DOUBLY PERIODIC COATED RIGID SPHERE COATING ON STEEL PLATE OR ON WATER

REFERENCES

- Meng, H., Wen, J., Zhao, H., Lv, L. and Wen, X. (2012), Analysis of absorption performances of anechoic layers with steel plate backing, The Journal of the Acoustical Society of America, 132, 69. **(FIGURE 6)**

DESCRIPTION

- Doubly periodic ALUMINUM coated sphere in a host rubber backed by a STEEL plate or WATER. Water on incident side and air behind the steel plate if not water backing. The core of the coated sphere is in Aluminum, while is coating is a soft silicon rubber.
- Normal incidence acoustic wave.

PCF MODEL

- \PCFEM\work\Test cases with inclusion\case5\case5.pcf (for steel backing)
- \PCFEM\work\Test cases with inclusion\case5b\case5b.pcf (for steel backing)
- Waveguide solver with TETRA 10 elements
- λ/10 criterion CPU time is 2202 s Number of elements: 29 184 (24 576) Number of nodes: 42 471 (35 937)
- λ/6 criterion CPU time is 115 s Number of elements: 6 528 (3 456) Number of nodes: 10 115 (5 491)

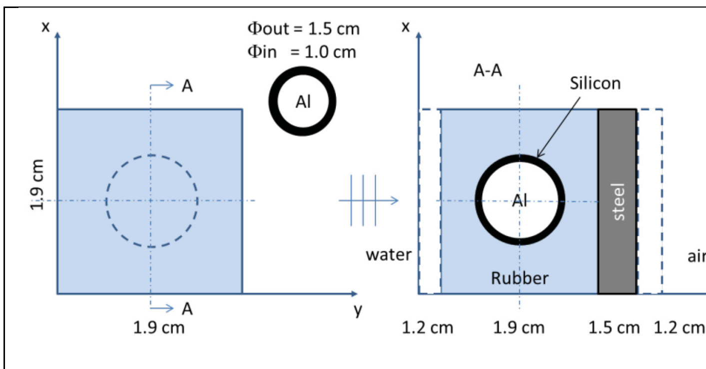
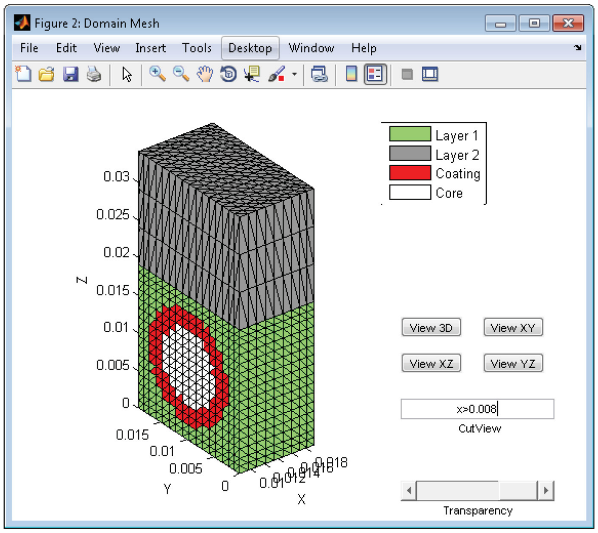


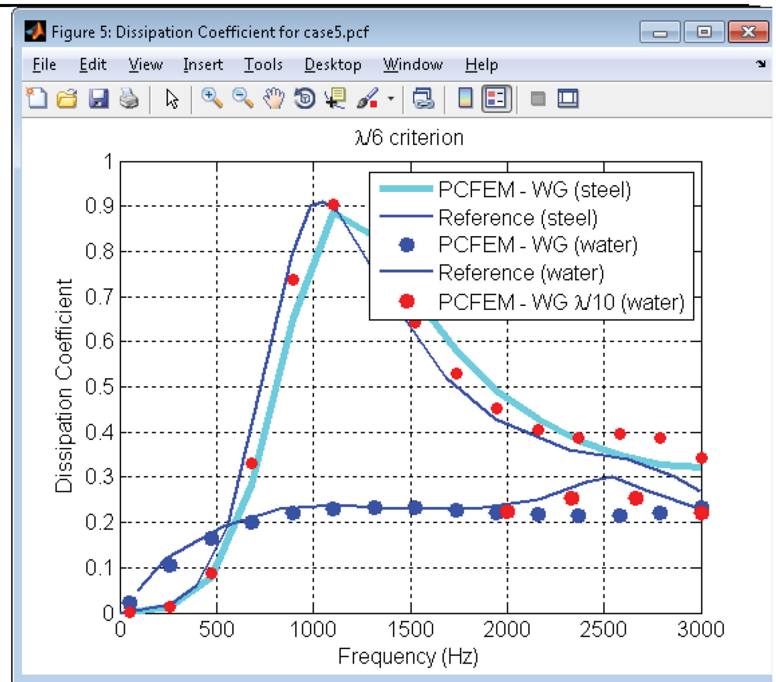
TABLE I. Physical parameters of materials for the local resonance slab.

Material	ρ (kg m^{-3})	Young's modulus (Pa)	Poisson's ratio (ν)	Loss factor (μ)
Steel	7890	2.09×10^{11}	0.275	0
Al	2690	7.44×10^{10}	0.334	0
Host rubber	1100	2.97×10^{7}	0.498	0.3
Silicon rubber	1300	2.0×10^{6}	0.462	0.3

PCFEM - Validation cases with inclusions



λ/10 criterion for layer 1. Manual for layer 2.



Good correlation with reference. Finer mesh yields better correlation around peaks.

PCFEM - Validation cases with inclusions

CASE 6 : DOUBLY PERIODIC COATED SPHERE COATING – WATER/SLAB/AIR

REFERENCES

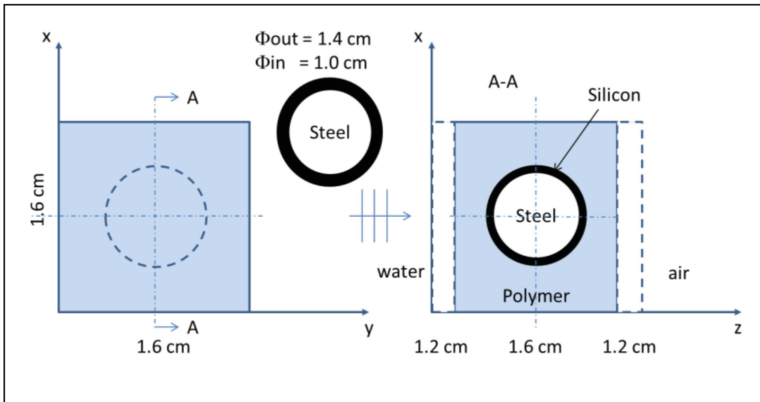
- Wen, J., Zhao, H., Lv, L., Yuan, B., Wang, G. and Wen, X. (2011), Effects of locally resonant modes on underwater sound absorption in viscoelastic materials, The Journal of the Acoustical Society of America, 130, 1201. **(FIGURE 3)**

DESCRIPTION

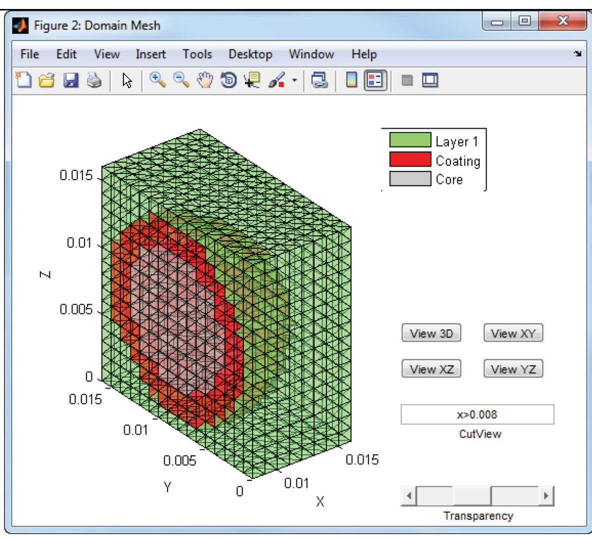
- Doubly periodic STEEL coated sphere in a host rubber backed by a STEEL plate. Water on incident side and air behind the steel plate. The core of the coated sphere is in Aluminum, while is coating is a soft silicon rubber.
- Normal incidence acoustic wave.

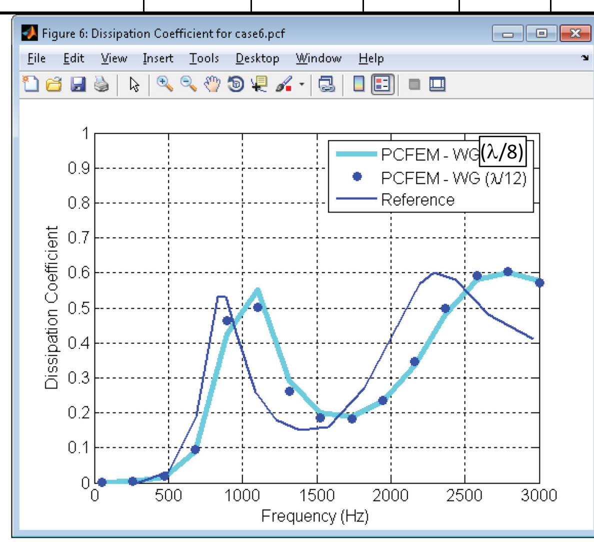
PCF MODEL

- \PCFEM\work\Test cases with inclusion\case6\case6.pcf (for steel backing)
- Waveguide solver with TETRA 10 elements
- λ/8 criterion CPU time is 901 s Number of elements: 18 816 Number of nodes: 27 753
- λ/10 criterion CPU time is s Number of elements: 34 992 Number of nodes: 50 653



Elastic properties	Steel	Polymer	Silicon	Water	Air
E (Pa)	2.09e11	2.66e7	2.01e5		
ν (-)	0.275	0.4952	0.4895		
η_E (%)		0.4	0.3		
ρ (kg/m³)	7890	1100	1300	1000	1.29
c (m/s)				1480	340





PCFEM - Validation cases with inclusions

Good correlation with reference; however some discrepancies are not explained. We thought that coarser mesh was not enough to discretize geometry; however coarse and very fine meshes yield same results. In the paper, only sound speeds in media were given. So we deduced from equations the properties given in the table. We expect the discrepancies result from errors in the input data given in the reference paper.

PCFEM - Validation cases with inclusions

CASE 7 : RUBBER/STEEL PLATE WITH DOUBLY PERIODIC PRISMATIC VOIDS AT JUNCTION (PUC CASE)

REFERENCES

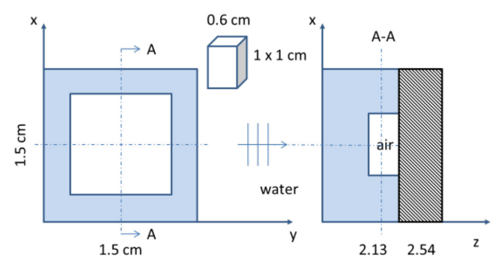
- Email by Jeff Szabo entitled "Test Case" and sent on February 20th , 2014.

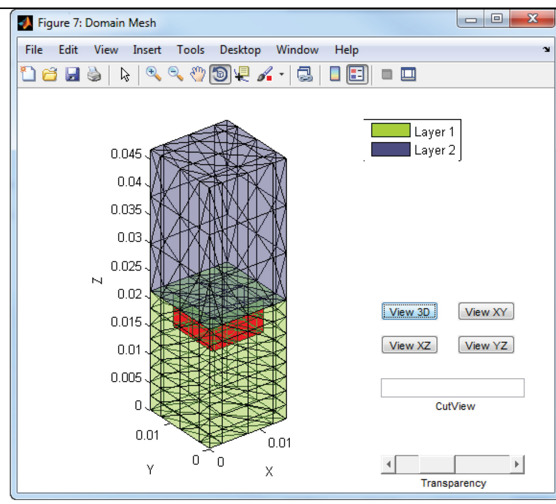
DESCRIPTION

- Rubber/Steel plate with doubly periodic prismatic voids at junction.
- Normal incidence acoustic wave.
- Properties of material are frequency dependent and given in MAT file "rubber3.mat" and "steel.mat" of reference.

PCF MODEL

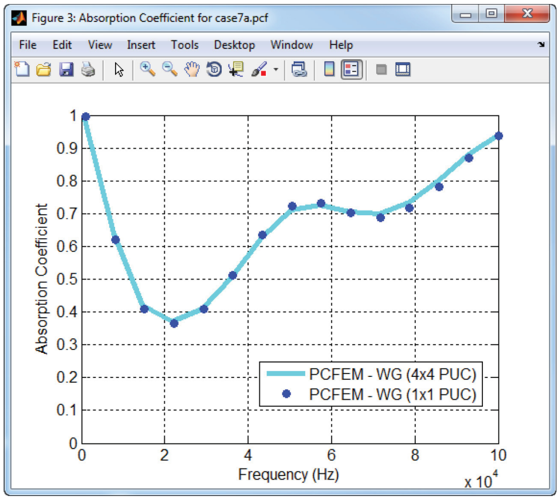
- \PCFEM\work\Test cases with inclusion\case7\case7.pcf (for 1x1 PUC)
- \PCFEM\work\Test cases with inclusion\case7a\case7a.pcf (for 4x4 PUC)
- Waveguide solver with TETRA10 elements
- ~λ/2 (USER DEFINED) criterion CPU time is 60 s (347 s) Num. of elements: 1 008 (16 128) Num. of nodes: 1 836 (24 615)
- Dynamic complex material properties. Their spectra are stored in MAT files: **Rubber3_table.mat** and **Steel_table.mat** located in folder: **\PCFEM\work\Material Library**

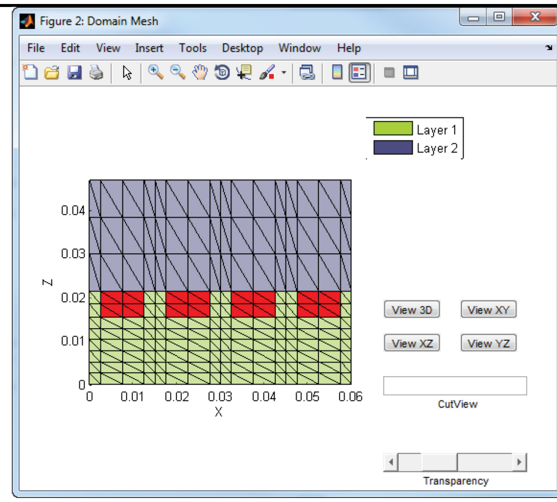




hx=hy=[2.5 5 2.5] and hz=[2.5 2.5 8]

PCFEM - Validation cases with inclusions





hx=hy= [2.5 5 2.5 5 2.5 5 2.5 5 2.5] and hz=[2.5 2.5 8]

For the same mesh definition, one 1x1 PUC cell yields exactly the same results as one 4x4 PUC cell