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U. S. NAVAL ORDNANCE LABORATORY

WHITE OAK, MARYLAND

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INEQUALITIES FOR COMPLEX LINEAR DIFFERENTIAL SYSTEMS OF THE SECOND ORDER

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ABSTRACT: Inequalities have been obtained for solutions of systems of linear differential equations of the form $\begin{cases} \dot{w} = \alpha w + \beta u \\ \dot{u} = \gamma w + \delta u \end{cases}$ where α, β, γ, δ are complex functions of the independent variable.

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This Report contains results of an investigation of certain systems of linear differential equations with complex coefficients. Equations of this type are fundamental in the linearized theory of exterior ballistics of symmetric projectiles, and results of this type are a necessary preliminary for further investigation of the nonlinear equations. The work was carried out at the Naval Ordnance Laboratory under the sponsorship of the Office of Naval Research, project number NR-044,003, entitled "Numerical Analysis and Theoretical Mechanics". This information is forwarded for use of outside research laboratories engaged in the development of missiles.

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By direction

INEQUALITIES FOR COMPLEX LINEAR DIFFERENTIAL SYSTEMS OF THE SECOND ORDER

1. The purpose of this paper is to obtain inequalities to be satisfied by solutions of systems of the form

$$dw/dt = \alpha w + \beta u$$

(1)

$$du/dt = \gamma w + \delta u,$$

where α, β, γ, δ are complex valued functions of the real variable t. These functions are assumed to be of class C' on some interval $t_0 \leq t < T$, where T may be ∞. We also restrict attention to the "general case" in which $\Delta = (\delta - \alpha)^2 + 4\beta\gamma \neq 0$ and $\gamma \neq 0$ on this interval. The inequalities obtained are the best possible ones of their type and are especially adapted to the case in which α, β, γ, δ have numerically small derivatives.

2. In terms of the notation,

$$\Delta = r(t)e^{i\theta(t)} \doteq (\delta - \alpha)^2 - 4\beta\gamma.$$

$R = R(t) \doteq |\Delta|/4\gamma\bar{\gamma}$

$$D(t) = \left[|w(t)|^2 + (2\bar{\gamma})^{-1}(\bar{\delta}-\bar{\alpha})w\bar{u} + (2\gamma)^{-1}(\delta-\alpha)\bar{w}u + (|2\gamma|^{-2}|\delta-\alpha|^2 + R)|u|^2\right]^{\frac{1}{2}}$$

$$\sigma(t) = \alpha + \bar{\alpha} + \delta + \bar{\delta}$$

$$\Omega(t) = (1+e^{i\theta})\bar{\gamma}\frac{d}{dt}\left(\frac{\bar{\delta}-\bar{\alpha}}{2\bar{\gamma}}\right) + (1+e^{-i\theta})\gamma\frac{d}{dt}\left(\frac{\delta-\alpha}{2\gamma}\right) + R^{-1}\left|\frac{d}{dt}\left(\frac{\delta-\alpha}{2\gamma}\right)\right|^2$$

where $\bar{w}$, $\bar{u}$, $\bar{\alpha}$, $\bar{\gamma}$, etc. are used to indicate the conjugate imaginaries of the corresponding undashed quantities, namely w, u, α, γ, etc., our fundamental inequalities can be written in the following reasonably compact form,

(2)
$$D_0(R/R_0)^{1/4}\exp\int_{t_0}^{t}\left[\sigma/4 - \left\{(r/4)\cos^2(\theta/2) + (\dot{R}/4R)^2 + \Omega/4\right\}^{1/2}\right]dt \leq D(t) \leq$$
$$D_0(R/R_0)^{1/4}\exp\int_{t_0}^{t}\left[\sigma/4 + \left\{(r/4)\cos^2(\theta/2) + (\dot{R}/4R)^2 + \Omega/4\right\}^{1/2}\right]dt$$

where, of course, the functions w(t) and u(t) are assumed to constitute an arbitrary solution of (1). Here, as in the sequel, a dot over a letter indicates differentiation with respect to t, and $D_0 = D(t_0)$ and $R_0 = R(t_0)$.

3. These inequalities are greatly simplified in case $\alpha = \delta$, as one sees at once from the definition of Ω. It is also easier to establish the above inequalities in this special case; moreover it is possible to obtain certain inequalities

quite similar, but not identical, to (2) in the case $\alpha \neq \delta$ by means of a preliminary transformation on the dependent variables, such as

$$w^* = w + (\delta - \alpha)u/2\gamma, \quad u^* = u$$

which reduces (1) to a similar system,

$$\dot{w}^* = \alpha^* \; w^* + \beta^* \; u^*$$

$$\dot{u}^* = \gamma^* \; w^* + \delta^* \; u^*,$$

in which $\delta^* = \alpha^* = 2^{-1}(\alpha + \delta)$, $\beta^* = \Delta/4\gamma + (d/dt)[(\delta - \alpha)/2\gamma]$

and $\gamma^* = \gamma$. But there are certain disadvantages in setting up inequalities in this manner. For instance, if the transformation cited above is used, we must later assume that α^*, β^*, γ^*, are of class C', and hence we would have to assume that the original coefficients are of class C" or at least that $(\delta - \alpha)/2\gamma$ is of class C". We shall therefore establish (2) without any assumption as to a relationship between α and δ. Our proof runs as follows:

Consider the positive definite Hermitian form,

$$Q = w\bar{w} + \bar{A}w\bar{u} + A\bar{w}u + Bu\bar{u},$$

where $A = (\delta - \alpha)/2\gamma$, $B = (|\delta - \alpha|^2 + |\Delta|)/4\gamma\bar{\gamma} = A\bar{A} + R$

Assuming that w and u satisfy (1), we differentiate Q with respect to t and replace $\dot{w}$ by $\alpha w + \beta u$, $\dot{\bar{w}}$ by $\bar{\alpha}\bar{w} + \bar{\beta}\bar{u}$, $\dot{u}$ by $\gamma w + \delta u$, and $\dot{\bar{u}}$ by $\bar{\gamma}\bar{w} + \bar{\delta}\bar{u}$. We thus find that

$$\dot{Q} = (\alpha + \bar{\alpha} + A\gamma + \bar{A}\bar{\gamma})w\bar{w} + (\bar{A}\alpha + \bar{\beta} + \bar{A}\bar{\delta} + B\gamma + \dot{\bar{A}})w\bar{u}$$
$$+ (A\bar{\alpha} + \beta + A\delta + B\bar{\gamma} + \dot{A})\bar{w}u + (\bar{A}\beta + A\bar{\beta} + B\bar{\delta} + B\delta + \dot{B})u\bar{u}.$$

If we denote by Q^* the Hermitian form on the right, we have

(3) $\qquad \min(Q^*/Q) \leq \dot{Q}/Q \leq \max(Q^*/Q).$

Evidently the problem of minimizing or maximizing Q^*/Q is equivalent to minimizing or maximizing Q^* for values of w and u which make Q = 1. This problem can be handled by the method of Lagrange, which yields the result that the minimum and maximum values of Q^*/Q are precisely the two roots of the following determinantal equation in λ:

$$\begin{vmatrix} \alpha + \bar{\alpha} + A\gamma + \bar{A}\bar{\gamma} - \lambda & \bar{A}\alpha + \bar{\beta} + \bar{A}\bar{\delta} + B\gamma + \dot{\bar{A}} - \bar{A}\lambda \\ A\bar{\alpha} + \beta + A\delta + B\bar{\gamma} + \dot{A} - A\lambda & \bar{A}\beta + A\bar{\beta} + B\bar{\delta} + B\delta + \dot{B} - B\lambda \end{vmatrix} = 0$$

When we replace A and B by their expressions in terms of α, β, γ, δ we find that

$$\alpha+\bar{\alpha}+A\gamma+\bar{A}\bar{\gamma} = (\alpha+\bar{\alpha}+\delta+\bar{\delta})/2 = \sigma/2$$
$$\bar{A}\alpha+\bar{\beta}+\bar{A}\bar{\delta}+B\gamma = \big(|\Delta|+\bar{\Delta}+\sigma(\bar{\delta}-\bar{\alpha})\big)/4\bar{\gamma}$$
$$A\bar{\alpha}+\beta+A\delta+B\bar{\gamma} = \big(|\Delta|+\Delta+\sigma(\delta-\alpha)\big)/4\gamma$$
$$\bar{A}\beta+A\bar{\beta}+B\bar{\delta}+B\delta = \big[\sigma(|\Delta|+|\delta-\alpha|^2)+(\delta-\alpha)(|\Delta|+\bar{\Delta})+(\bar{\delta}-\bar{\alpha})(|\Delta|+\Delta)\big]/8\gamma\bar{\gamma}$$
$$B-A\bar{A} = |\Delta|/4\gamma\bar{\gamma} = R.$$

The above determinantal equation can be expanded with the help of these relationships and we thus arrive at the quadratic equation in λ,

$$(4) \qquad \lambda^2-(\sigma+R^{-1}\dot{R})\lambda+4^{-1}\big(\sigma^2-(\Delta^{1/2}+\bar{\Delta}^{1/2})^2+2\sigma R^{-1}\dot{R}\big)-\Omega=0,$$

where $\Delta^{1/2}$ and $\bar{\Delta}^{1/2}$ are chosen so as to be conjugate imaginaries of each other. The roots of (4), both real, of course, are

$$\lambda = 2^{-1}(\sigma+R^{-1}\dot{R}) \pm 2^{-1}\big[(\Delta^{1/2}+\bar{\Delta}^{1/2})^2+R^{-2}\dot{R}^2+4\Omega\big]^{1/2}$$
$$= 2^{-1}(\sigma+R^{-1}\dot{R}) \pm \big[h\cos^2(\theta/2)+4^{-1}R^{-2}\dot{R}^2+\Omega\big]^{1/2}.$$

The $\pm$ sign is to be taken $+$ to yield the maximum for Q^*/Q and $-$ for the minimum. Both, of course, are functions of t. Our fundamental inequalities (2) follow at once upon intergration of (3) and upon observing that $D = Q^{1/2}$.

4. We add the following two remarks:

a. In case α, β, γ, and δ are constants, the roots of (4) reduce to $2^{-1}(\alpha+\delta+\bar{\alpha}+\bar{\delta}) \pm 2^{-1}(\Delta^{1/2}+\bar{\Delta}^{1/2})$, which are precisely twice the real parts of the roots of the usual characteristic equation,

$$\begin{vmatrix} \alpha-\mu & \beta \\ \gamma & \delta-\mu \end{vmatrix} = 0$$

in μ. It follows that in this case of constant coefficients there will always exist a solution for which either one of the $\leq$ signs in (2) may be replaced by an equal sign. This remark explains the precise meaning of our claim that our inequalities are the best of their type, and it was in order to reach this result that the Hermitian form Q was chosen in the peculiar form stated. Certain other forms would lead to a similar result, but the form chosen seems to be the simplest. The algebraic background behind our choice of Q was given in a previous paper[1].

b. As simple corollaries of our inequalities, we may read off criteria for boundedness of the functions $[w(t),\ u(t)]$ constituting a solution of (1). For instance, *if the larger of the two roots of* (4), *namely*

$$\lambda = 2^{-1}(\sigma + R^{-1}\dot{R}) + [n \cos^2(\theta/2) + 4^{-1}\dot{R}^2/R^2 + \Omega]^{1/2}$$

does not exceed $\dot{R}/R$ (*or even if it has only the property that*

$$\int_{t_0}^{t} (\lambda - \dot{R}/R)\,dt$$

is bounded from above), *then the function* $u(t)$ *must be bounded*. In fact $D(t) = (|w + (2\gamma)^{-1}(\delta - \alpha)u|^2 + R|u|^2)^{1/2} \leqslant D_0 \exp \int_{t_0}^{t} (\lambda/2)\,dt$.

Hence $|u| \leqslant R^{-1/2} D_0 \exp \int_{t_0}^{t} (\lambda/2)\,dt = D_0 R_0^{-1/2} \exp \int_{t_0}^{t} (\lambda/2 - \dot{R}/2R)\,dt,$

from which the italicized statement is obvious.

5. A slight generalization of a theorem of Leighton is to the effect that every solution of the equation $(d/dt)(Ge^{1\varphi}\,du/dt) + Fe^{1\varphi}u = 0$ is bounded, if $F > 0$, $G > 0$, and are real and of class C' in t, and if FG is monotonically increasing on $(t_0 \leqslant t < \infty)$. This is an extremely special case of the above corollary. For, if we take $w = Ge^{1\varphi}u$, the differential equation may be written in the form (1) with $\alpha = \delta = 0$, $\beta = -Fe^{1\varphi}$, $\gamma = G^{-1}e^{-1\varphi}$. Hence $\sigma = \Omega = 0$, $\Delta = -4FG^{-1}$, $\theta \equiv \pi \pmod{2\pi}$, $R = FG$. Remembering that this R is monotonically increasing, we can write $(\dot{R}^2)^{1/2} = +\dot{R}$. A simple calculation, now shows that $\lambda = \dot{R}/R$, so that the aoove criterion is satisfied.

References

1. D. C. Lewis, "Differential Equations Referred to a Variable Metric", American Journal of Mathematics, vol. 73 (1951), pp. 48-58. Especially Theorem 3, pp. 56 and 57.

2. W. Leighton, "Bound for the Solutions of a Second order Linear Differential Equation", Proceedings of the National Academy of Sciences, vol. 35 (1949), pp. 190-191.