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RAY TRACING EQUATIONS FOR A PARABOLIC DISH ANTENNA WITH A PHASED-ARRAY SECONDARY

by

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ABSTRACT. This report gives the derivation of equations for the ray trace analysis of an antenna consisting of a parabolic dish with a phased array in front of it. The derivation gives the equations for ray tracing in a form suitable for digital computer use. A number of rays from a plane wavefront can be traced through the antenna system to a feed point on the antenna axis. The rays are straight lines, and specular reflection is assumed at the dish. The phased array returns the reflected rays to the feed point (which is assumed to lie between the dish and phased array).

The major features provided by the analysis and the computer program are the relative phases and path lengths of the rays from the wavefront to the feed point (tabulation and computer plots), the total number of rays, the number obscured by the phased array, the number that arrive at the phased-array plane and lie within the phased array itself, a computer-generated plot of the starting points on the wavefront of the rays used in tracing, a plot of the locations of those rays that arrive at the phased-array plane, and an outline of the phased array.

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FOREWORD

Existing C-band instrumentation radar sets acquire and track targets by mechanically positioning the antenna but, because of the large mechanical mass that must be moved, it is impossible for one radar set to track more than one target simultaneously. However, electronic movement of the beam would permit instantaneous beam positioning without having to move the large mechanical mass of the reflector and feed assembly. Phased-array antennas, which provide electronic beam steering, have been demonstrated to be practical in other applications and could be used to advantage in instrumentation radar sets. The Naval Weapons Center is investigating a system consisting of a small phased array placed in front of a parabolic reflector. This system is expected to provide an electronically steered beam at relatively low cost and with little modification to existing sets. Before radar modifications are undertaken, however, NWC is investigating the advantages and possible difficulties inherent in the proposed system.

This report derives the equations for the ray-trace analysis of such an antenna with a phased array. An outline of a digital computer program that performs the ray-trace analysis is included.

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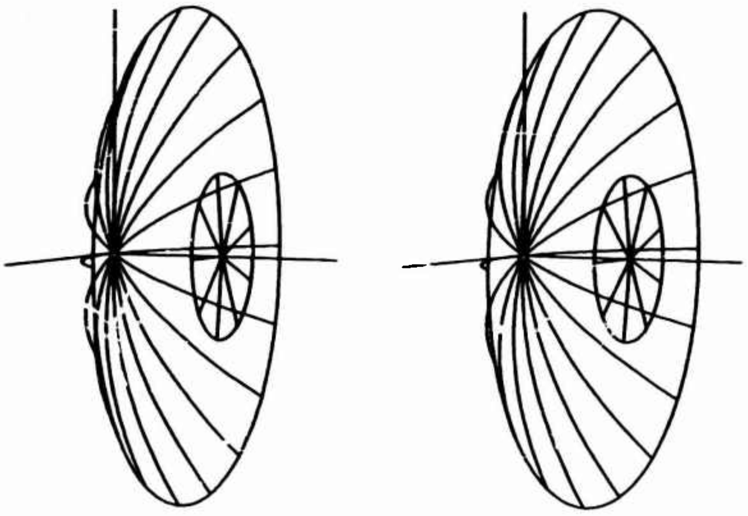


Stereo Views of the Phased-Array Antenna

The two drawings of the antenna system can be seen in stereo by viewing the right figure with the right eye and the left with the left eye so that the two images fuse. Without any aid there are three images. The central one is stereo. A sheet of paper held perpendicular to the drawing so that each eye can see only one drawing helps in learning to adjust the eyes for stereo viewing.

INTRODUCTION

The analysis given here provides the equations for tracing rays from a plane wavefront through an antenna system to a specified feed point. The antenna system consists of a parabolic dish (paraboloid of revolution about the axis of the parabola), a circular phased array perpendicular to the paraboloid axis and centered on this axis, and a feed point on the axis. The rim of the dish lies in a plane perpendicular to the axis and is, therefore, a circle centered on the axis. The feed point lies between the dish and the phased array. The phased array is assumed to lie outside the dish.

A ray normal to the wavefront is traced from the wavefront to the dish. It is specularly reflected from the dish to the phased array, which directs it to the feed point. Those rays which are not reflected by the dish are omitted and those which are intercepted by the back of the phased array can be omitted. Further, those rays which are not reflected to the face of the phased array can be rejected or noted. Conditions for multiple reflection are given so that such rays can be rejected also. The total geometric path length from the wavefront to the feed point is found and can be related to the relative phase.

In the process of ray tracing the projections of the dish rim and the phased array rim on the wavefront plane and the image of the dish rim on the plane of the phased array are found. Plots of these curves and the intersections of the rays on the phased array plane are useful in the evaluation of the ray tracing.

Since the ray tracing equations are complicated functions of the constants defining the antenna geometry, it is a practical necessity to use a large-scale digital computer for this ray tracing. The development of the equations in the body of the report is, therefore, oriented to numerical analysis. A summary of the equations suitable for the digital computer is given in Appendix A, and examples of computer output are given in Appendix B.

A computer program for ray tracing has been written in FORTRAN V for the UNIVAC 1108 with a Stromberg DatagraphiX, Inc., 4060 plotter. This program (WAVE) is available from Commander, Naval Weapons Center (Code 3037), China Lake. California 93555.

Thanks are due John Anderson and Lorene Paschal, Code 3037, and Edlin Patterson, Code 3045, for useful discussions during the course of this work.

DERIVATION OF THE RAY TRACING EQUATIONS

The geometry of the antenna system is shown in Fig. 1 and 2. The origin of the xyz coordinate system is at the vertex of the parabolic dish and the x axis is along the dish axis. The following notation is used:

$\underline{i}, \underline{j}, \underline{k}$	unit vectors along x, y, and z, respectively
$\underline{Q}$	position vector of a point on the dish
$\underline{n}$	unit (inward) normal to the dish at point $\underline{Q}$
$\underline{N}$	unit normal to wavefront plane (from vertex to plane)
N	distance from origin to xyz to wavefront plane along $\underline{N}$
$\underline{P}$	position vector of point on wavefront
$\underline{R}$	unit vector along reflected ray
$\underline{J}, \underline{K}$	unit vectors along Y and Z coordinate axes lying in the wavefront plane
$N\underline{N}$	the position vector of the origin of the YZ coordinate system
Y, Z	coordinates of point on wavefront in YZ coordinate system
A	radius of rim of dish
F	focal distance of dish
D	distance (along x) from origin to phased array
R_0	radius of phased array
f	distance (along x) from origin to feed point
y_p, z_p	position coordinates of intersection of reflected ray with phased array
L_1	path length from wavefront to dish
L_2	path length of reflected ray from dish to phased array
L_3	path length from phased array to feed point

Other notation is defined as it is used.

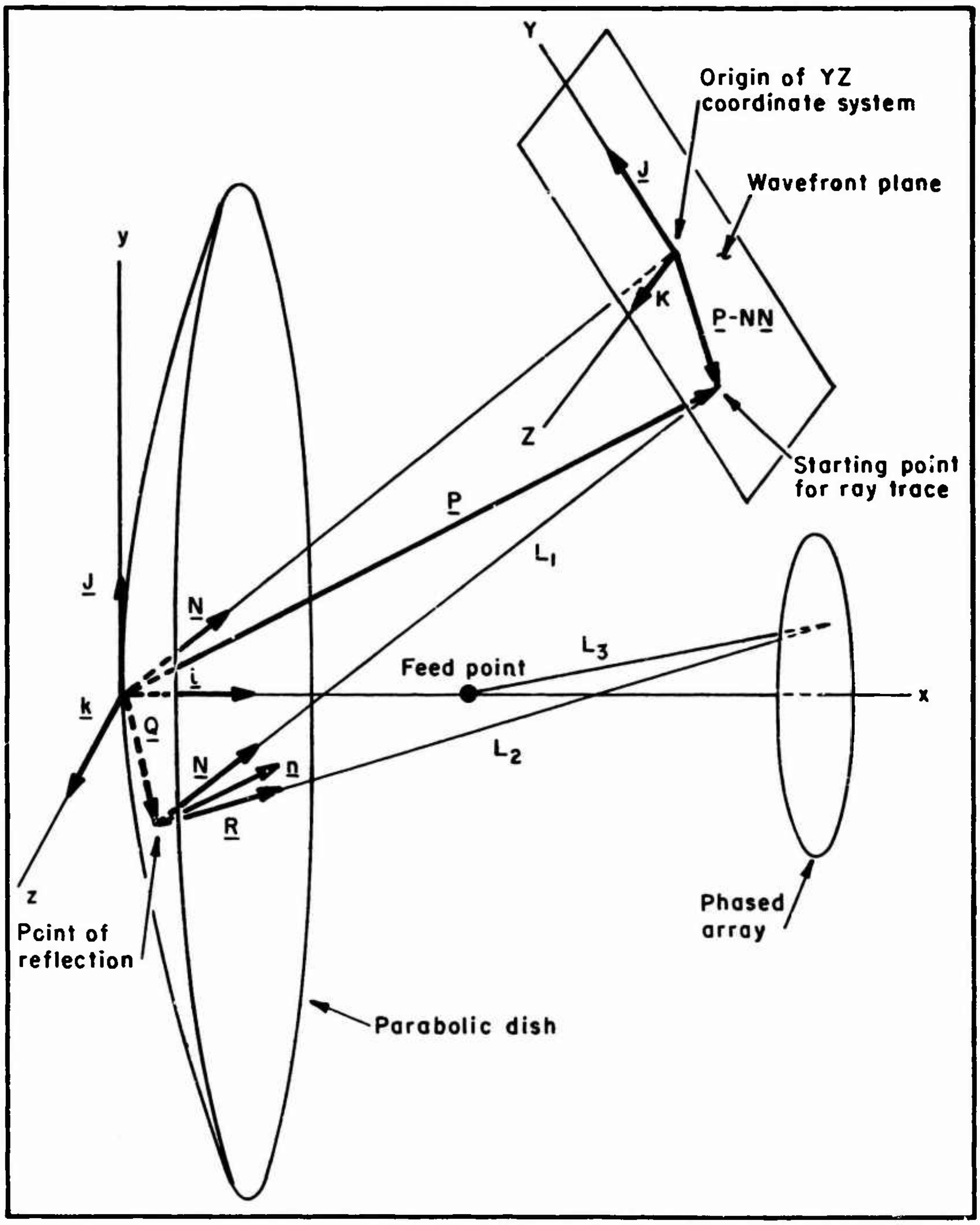


FIG. 1. Antenna System Showing the Three Segments, L_1, L_2, L_3, of the Path From the Wavefront to the Feedpoint.

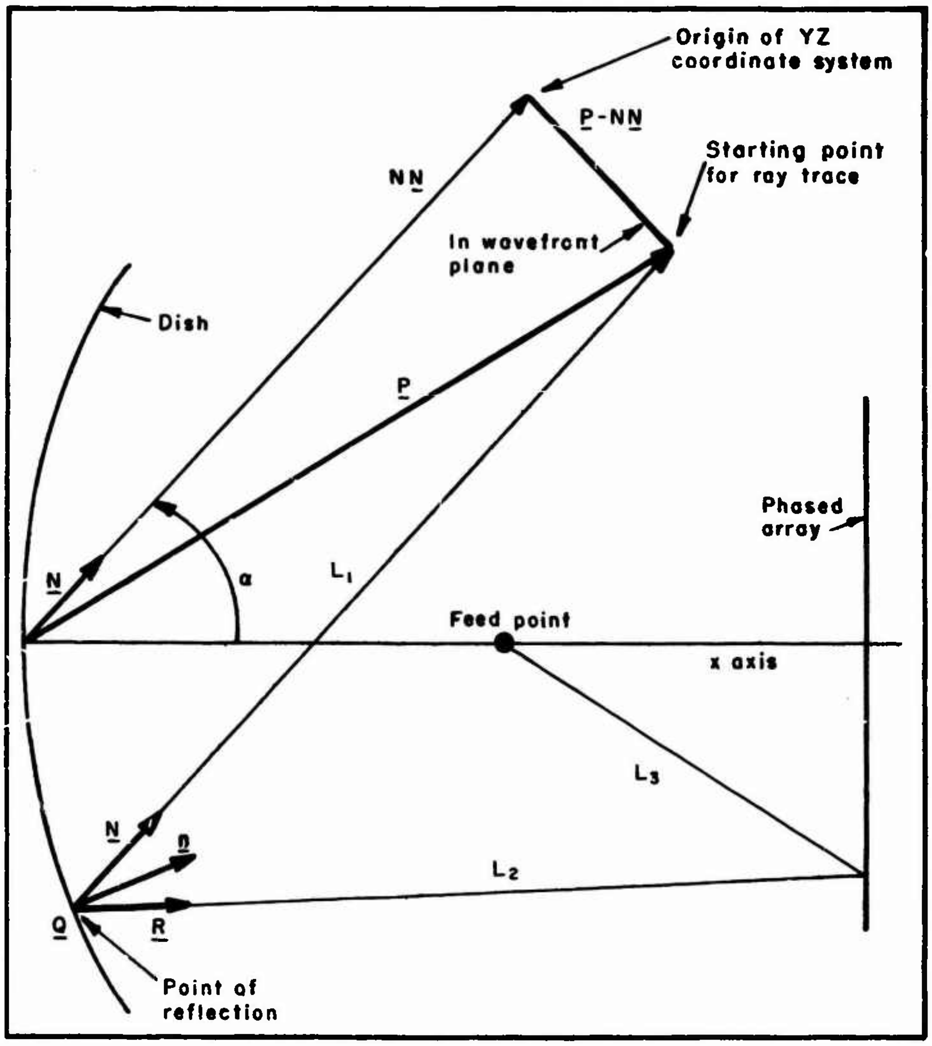


FIG. 2. Projection of the Incident Ray Onto the Dish and Specular Reflection. Vectors N, P-NN, Q, P lie in a common plane.

The equation of the parabolic dish is

$$Q_y^2 + Q_z^2 = 4FQ_x \tag{1}$$

where Q_x, Q_y, and Q_z are the coordinates of a point on the dish. Therefore, the position vector

$$\underline{Q} = \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \underline{i}\,Q_x + \underline{j}\,Q_y + \underline{k}\,Q_z \tag{2}$$

The unit vector $\underline{N}$, which defines the orientation of the wavefront plane has components that are just the direction cosines of the normal to the plane. Since the antenna system has rotational symmetry about the x axis, it is sufficient to keep $\underline{N}$ in the xy plane and define its direction by a single angle α. Then

$$\underline{N} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \tag{3}$$

where α is the angle from the x axis to $\underline{N}$. For the present purpose $0 \le \alpha < 90°$ is the only range of interest. Reflection from the convex side of the dish is not required. Negative α is not required since the system is mirror symmetric about the xz plane.

Projection of Dish and Phased Array Rims on Wavefront

With a plane wavefront, the rays from it to the dish are all parallel to $\underline{N}$. The ray tracing from the wavefront to the dish consists of finding the position $\underline{Q}$ of the ray which starts with coordinates Y and Z on the wavefront. Only the region on the wavefront within the projection of the rim of the dish need be considered. The projection of the rim is made parallel to $\underline{N}$. The projection of the rim of the phased array can obscure part of this area; rays within the intersection of the two projections can either be omitted or noted. It is of interest to trace such rays to show the extent of the "shadow".

The computational procedure is to trace rays from a rectangular grid of points on the wavefront. Only the points which lie in or on the projection of the dish rim are used. It is now useful to find the bounds of this region on the wavefront.

The unit vectors $\underline{J}$ and $\underline{K}$ in the wavefront plane are defined as

$$\underline{J} = \frac{\underline{k} \times \underline{N}}{|\underline{k} \times \underline{N}|} \quad \text{and} \quad \underline{K} = \underline{N} \times \underline{J} \tag{4}$$

Using Eq. (3) and the fact that

$$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

then,

$$\underline{k} \times \underline{N} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}$$

and

$$|\underline{k} \times \underline{N}| = \sqrt{\cos^2\alpha + \sin^2\alpha} = \pm\ 1$$

the positive value is taken, so that $\underline{J} = \underline{j}$ for $\alpha = 0$. Then

$$\underline{J} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} \tag{5}$$

It is clear from the definition of $\underline{K}$, Eq. (4), that

$$\underline{K} = \underline{k} \tag{6}$$

The components of the projection of $\underline{Q}$ onto the wavefront plane are

$$Y = \underline{Q} \cdot \underline{J} = -\ Q_x \sin\alpha + Q_y \cos\alpha \tag{7}$$

and

$$Z = \underline{Q} \cdot \underline{K} = Q_z \tag{8}$$

and the equation of the rim of the dish is

$$Q_y^2 + Q_z^2 = 4FQ_x = A^2 \tag{9}$$

Combining Eq. (7), (8), and (9) gives

$$Z^2 + \left(Y + \frac{A^2 \sin\alpha}{4F}\right)^2 \Big/ \cos^2\alpha = A^2 \tag{10}$$

This equation is the projection of the rim on the wavefront plane and is an ellipse with center at $Z=0$, $Y=-A^2 \sin\alpha/4F$. The semimajor axis is A, and the semiminor axis is $A\cos\alpha$. This is illustrated in Fig. 3.

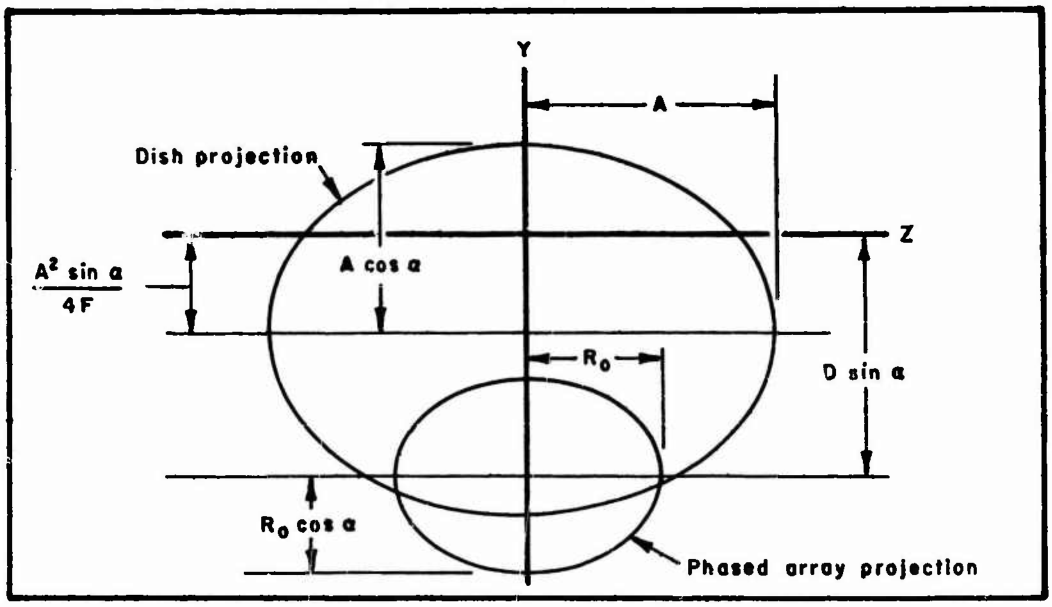


FIG. 3. Projections of the Dish and Phased Array on the Wavefront Plane.

For making computer plots, it is convenient to put Eq. (10) in parametric form. At the rim,

$$Q_y = A \sin \theta$$

$$Q_z = A \cos \theta$$

Then Eq. (7) and (8) become

$$Y = A \sin \theta \cos \alpha - \frac{A^2}{4F} \sin \alpha$$

$$Z = a \cos \alpha \tag{10a}$$

for the projection of the dish rim on the wavefront plane. It is sufficient to use $-\pi/2 \leq \theta \leq \pi/2$ and use symmetry about the Y axis to plot for $Z < 0$.

Next, the projection of the rim of the phased array is found. Since the phased array is circular, the equations of the rim are

$$x_p = D \quad \text{and} \quad y_p^2 + z_p^2 = R_0^2 \tag{11}$$

and the projection of the rim position vector $D\underline{i} + y_p\underline{j} + z_p\underline{k} = \underline{r}$ onto the wavefront plane gives

$$Y = \underline{r} \cdot \underline{J} = -D \sin\alpha + y_p \cos\alpha$$

$$Z = \underline{r} \cdot \underline{K} = z_p \tag{12}$$

Combining Eq. (11) and (12) gives the equation of the projection of the phased array rim,

$$Z^2 + (Y + D \sin\alpha)^2/\cos^2\alpha = R_0^2 \tag{13}$$

This is an ellipse with the center at $Z = 0$, $Y = -D \sin\alpha$, and the semimajor and semiminor axes are R_0 and $R_0 \cos\alpha$, respectively. See Fig. 3.

Again, for making computer plots it is convenient to have Eq. (13) in parametric form. Following the scheme to get Eq. (10a)

$$
\begin{aligned}
Y &= R_0 \sin\theta \cos\alpha - D \sin\alpha \\
Z &= R_0 \cos\theta \qquad\qquad (13a) \\
-\pi &< \theta \leq \pi
\end{aligned}
$$

for the coordinates of the projection of the phased array rim.

Points on Wavefront to Start Ray Tracing

For computational purposes, a grid of points on the YZ plane is used to start the ray tracing. Only those points on or within the ellipse of Eq. (10) are used.

If rays that start on points along a line parallel to the Y axis are required, the limits on Y imposed by Eq. (10) are

$$-(A^2 - Z^2)^{1/2} \cos\alpha - \frac{A^2 \sin\alpha}{4F} \leq Y \leq (A^2 - Z^2)^{1/2} \cos\alpha - \frac{A^2 \sin\alpha}{4F} \qquad (14)$$

where

$$Z^2 \leq A^2 \qquad (15)$$

Considering Eq. (13), if

$$R_0^2 \leq Z^2 \leq A^2 \qquad (16)$$

Then the full range of Y given in (14) is used. On the other hand, for

$$Z^2 < R_0^2 \qquad (17)$$

all Y in the range

$$-(R_j^2 - Z^2)^{1/2} \cos\alpha - D \sin\alpha < Y < (R_0^2 - Z^2)^{1/2} \cos\alpha - D \sin\alpha \tag{18}$$

are within the projection of the phased array.

Since the antenna system is mirror symmetric about the xy plane and $\underline{N}$ is in the xy plane, it is sufficient to use $Z \geq 0$ only.

For programming the computer it is convenient to have the coordinates of points on the wavefront plane indexed by integers and to have the limits in terms of these indices. For the Z coordinate in the range

$$0 \leq Z \leq A \tag{19}$$

$$Z(J) = \Delta Z \cdot (J-1), \quad J = 1, 2, \ldots, NZ \tag{20}$$

where ΔZ is the increment on Z, and

$$NZ = INT\left(\frac{A}{\Delta Z}\right) + 1 \tag{21}$$

is the number of points on Z on or within the limits (19) and INT means "the integer part of".

The points on the Y axis have the limits (14) (see Fig. 3). For a given $Z(J)$, the number of Y points from the major axis to the ellipse with the increment ΔY is

$$NY(J) = INT\left(\frac{[A^2 - Z^2(J)]^{1/2}}{\Delta Y} \cos\alpha\right) \tag{22}$$

This equation does not include the point on the major axis. The number of points along the Y axis is

$$NY(1) = INT\left(\frac{A \cos\alpha}{\Delta Y}\right) \tag{23}$$

For $Z = 0$ the $Y(I)$ are

$$Y(I) = \Delta Y \cdot (I-NY(1)-1) - \frac{A^2 \sin\alpha}{4F} \tag{24}$$

where

$$I = 1, 2, \ldots, 2NY(1) + 1 \tag{25}$$

For other Z(J), I has the range

$$I = 1 + NY(1) - NY(J), \ldots, 1 + NY(1) + NY(J) \tag{26}$$

Equations (20) and (21) provide the Z coordinates, and Eq. (22) through (26) provide the Y coordinates and range of the index I for the Z that cover the right half of the projection of the dish rim.

Those points Z(J), Y(I) lying within the projection of the phased array are now readily found from conditions (16) through (18). Although the rays which fall within the phased array projection cannot reach the dish, it is still useful to trace them to find the number in this "shadow area" and to find the image of this shadow on the phased array plane.

Ray From Wavefront to Dish

Next, the point $\underline{Q}$ on the dish is found given the coordinates X and Y of a point on the wavefront. Solving Eq. (1), (7), and (8) for Q_x, Q_y, and Q_z gives

$$Q_x = \frac{Q_y^2 + Z^2}{4F}$$

$$Q_y = 2F \frac{\cos\alpha}{\sin\alpha} \left\{ 1 \pm \left[1 - \frac{Z^2 \sin\alpha + 4FY}{4F^2 \cos^2\alpha} \sin\alpha \right]^{1/2} \right\} \tag{27}$$

$$Q_z = Z$$

for $\alpha \neq 0$. The choice of sign in the Q_y equation is made as follows. For $Y=Z=0$, $Q_y=0$, since the origin of the YZ coordinate system is taken as the position vector $\underline{NN}$ starting at the origin of xyz. At the origin $Q_x=Q_y=Q_z=0$. The negative sign in the Q_y equation is therefore required. Since the antenna system has mirror symmetry about the xz plane, only positive α need be considered. This can be seen from the Q_y equation. If both X and Y change sign, Q_y changes sign. Q_x and Q_z remain unchanged. These changes are just the expected mirror symmetry.

For $\alpha=0$ Eq. (7) becomes $Y=Q_y$, then the components of $\underline{Q}$ are

$$Q_x = \frac{Z^2+Y^2}{4F}$$

$$Q_y = Y \qquad (28)$$

$$Q_z = Z$$

For small but nonzero α equation (27) are valid, but the Q_y equations may lead to computational difficulties. This equation can be expanded in power series in $\sin\alpha$, and to second order

$$Q_y = Y/\cos\alpha + \frac{1}{4F\cos\alpha}\,[Z^2 + (Y/\cos\alpha)^2]\left\{\sin\alpha + \frac{Y}{2F\cos^2\alpha}\sin^2\alpha\right\} \qquad (29)$$

Using 8 figure computations and typical numbers for Y, Z, and F, the Q_y equation of (27) gives about 5-figure accuracy for $\alpha=0.01$ radian. For angles smaller than 0.01 radian Eq. (29) will, in general, give equal or better accuracy.

Considering Fig. 2, and recalling Eq. (3), the path length from the wavefront to point $\underline{Q}$ on the dish is

$$L_1 = N - \underline{Q}\cdot\underline{N} = N - Q_x\cos\alpha - Q_y\sin\alpha \qquad (30)$$

Later, these path lengths will be converted into relative phases, and $N=0$ can be used, since path difference is all that is necessary. Therefore,

$$L_1 = -(Q_x \cos\alpha + Q_y \sin\alpha) \qquad (31)$$

In this case the sign of L_1 must be retained.

Reflection From Dish to Phased Array

The next path required is that of the reflected ray from the dish to the phased array. For this purpose, the unit normal $\underline{n}$ to the dish surface is required. The gradient of a surface $V = V(x, y, z)$ is normal to that surface so, in the case of the parabolic dish $V = Q_y^2 + Q_z^2 - 4FQ_x$ and the gradient

$$\underline{\nabla} V = -4F\underline{i} + 2Q_y\underline{j} + 2Q_z k$$

The two unit normals are then

$$\underline{n} = \frac{\underline{\nabla} V}{\pm [(4F)^2 + (2Q_y)^2 + (2Q_z)^2]^{1/2}}$$

For the normal on the concave side of the dish the negative sign on the square root is taken. In terms of components

$$\underline{n} = \frac{1}{[4F^2 + Q_y^2 + Q_z^2]^{1/2}} \begin{pmatrix} 2F \\ -Q_y \\ -Q_z \end{pmatrix} = \frac{1}{2[F(F+Q_x)]^{1/2}} \begin{pmatrix} 2F \\ -Q_y \\ -Q_z \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \qquad (32)$$

The conditions for specular reflection are that the angle of incidence and reflection are equal and that the incident and reflected rays and the normal to the reflecting surface element are coplanar. In vector form (Fig. 2) these conditions are

$$\begin{aligned} \underline{N} \cdot \underline{n} &= \underline{n} \cdot \underline{R} \\ \underline{N} \times \underline{n} &= \underline{n} \times \underline{R} \end{aligned} \qquad (33)$$

where $\underline{R}$ is a unit vector in the direction of the reflected ray. Recall that all the rays projected onto the dish are parallel to $\underline{N}$. To solve Eq. (25) for $\underline{R}$, take the dot product equation and the y and z component equations of the cross product. In matrix form, these are

$$\begin{pmatrix} n_x & n_y & n_z \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}\begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}=\begin{pmatrix} \cos\alpha\cdot n_x + \sin\alpha\cdot n_y \\ -\cos\alpha\cdot n_z \\ \cos\alpha\cdot n_y - \sin\alpha\cdot n_x \end{pmatrix} \qquad (34)$$

the determinate of the left square matrix is n_x; therefore, Eq. (34) are not singular unless $n_x=0$. This certainly will not occur for finite sized dishes, Eq. (32). The solutions of (34) are

$$\underline{R} = \begin{pmatrix} (2n_x^2-1)\cos\alpha + 2n_xn_y\sin\alpha \\ (2n_y^2-1)\sin\alpha + 2n_xn_y\cos\alpha \\ 2n_z(n_x\cos\alpha + n_y\sin\alpha) \end{pmatrix}=\begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} \qquad (35)$$

It can be verified that $\underline{R}$ is, indeed, the solution to the four equations (33) and that it is a unit vector.

Under some conditions, the reflected ray will not leave the dish without a second reflection. A necessary, but not sufficient, condition is that

$$R_x > 0 \qquad (36)$$

for the ray to be reflected only once. Reflected rays not meeting the condition (36) can be rejected at this point. In the following, $R_x>0$ is assumed.

The equation of the line with a direction specified by $\underline{R}$ and passing through the point $\underline{Q}$ on the dish is

$$\frac{x-Q_x}{R_x} = \frac{y-Q_y}{R_y} = \frac{z-Q_z}{R_z} \qquad (37)$$

the equation of the plane of the rim of the dish is

$$A^2 = 4Fx \tag{38}$$

The coordinates of the intersection of the line (37) and the plane (38) are, therefore,

$$\begin{aligned} y_d &= \frac{R_y}{R_x}(A^2/4F - Q_x) + Q_y \\ z_d &= \frac{R_z}{R_x}(A^2/4F - Q_x) + Q_z \end{aligned} \tag{39}$$

The condition that the reflected ray "escape" from the dish without a second reflection is

$$y_d^2 + z_d^2 < A^2 \tag{40}$$

Rays not meeting condition (40) are rejected. The conditions, (36) and (40), are for the phased array outside of the dish.

The equation of the plane of the phased array is

$$x = D$$

and the intersection of the phased-array plane and the reflected ray, Eq. (37) has coordinates

$$\begin{aligned} y_p &= \frac{R_y}{R_x}(D-Q_x) + Q_y \\ z_p &= \frac{R_z}{R_x}(D-Q_x) + Q_z \end{aligned} \tag{41}$$

The condition for this point to lie on or within the rim of the phased array is

$$y_p^2 + z_p^2 \leq R_0^2 \tag{42}$$

Rays not meeting condition (42) are outside the phased array circle, but they will intersect the plane of the phased array.

Image of Dish Rim on Plane of Phased Array

The image of the rim of the dish on the phased array plane can be useful in evaluating the ray tracing. This image is the projection of the rim along the rays reflected from the rim. The parametric form of the image curve is found as follows: At the rim,

$$\begin{aligned} Q_x &= A^2/4F \\ Q_y &= y_{rim} = A \sin\theta \\ Q_z &= z_{rim} = A \cos\theta \end{aligned} \tag{43}$$

Where the parameter θ has the range $-\pi < \theta \leq \pi$ to cover the entire rim. Substitution of Eq. (43) into Eq. (32) and (35) gives

$$\begin{aligned} \left(\frac{R_y}{R_x}\right)_{rim} &= \frac{(2y_{rim}^2 - 4F^2 - A^2)\sin\alpha - 4Fy_{rim}\cos\alpha}{(4F^2 - A^2)\cos\alpha - 4Fy_{rim}\sin\alpha} \\ \left(\frac{R_z}{R_x}\right)_{rim} &= \frac{2(y_{rim}\sin\alpha - 2F\cos\alpha)\, z_{rim}}{(4F^2 - A^2)\cos\alpha - 4Fy_{rim}\sin\alpha} \end{aligned} \tag{44}$$

Using Eq. (41) the coordinates of an image point on the phased array plane are

$$y_p = \left(\frac{R_y}{R_x}\right)_{rim} \left(D - \frac{A^2}{4F}\right) + y_{rim}$$

$$z_p = \left(\frac{R_z}{R_x}\right)_{rim} \left(D - \frac{A^2}{4F}\right) + z_{rim} \tag{45}$$

$R_x > 0$ is the necessary and sufficient condition for a ray reflected at the rim to avoid multiple reflections, and, if the phased array is outside the dish, $R_x > 0$ insures the ray intersects the plane of the phased array. The denominator in Eq. (44) comes from R_x and must be positive for $R_x > 0$. Thus,

$$y_{rim} < \frac{4F^2 - A^2}{4F \tan\alpha} \tag{46}$$

for $0 \leq \alpha \leq 90°$. For the entire rim to have an image in a finite region of the plane the greatest y_{rim} (i.e., A) must meet condition (46) or

$$\tan\alpha < \frac{4F^2 - A^2}{4FA} \tag{47}$$

It is interesting to examine the case of $\alpha = 0$. For this case, the image of the dish is a circle having a radius of

$$(y_p^2 + z_p^2)^{1/2} = 4A \frac{F^2 - FD}{4F^2 - A^2}$$

As is expected with the phased array at the focal point, that is $F = D$, the radius is zero. With $A = 2F$, all reflections at the rim are perpendicular to the dish axis, and the image is infinite.

The path length from the dish to the phased array is

$$L_2 = [(D-Q_x)^2 + (y_p-Q_y)^2 + (z_p-Q_z)^2]^{1/2} \tag{48}$$

The positive square root is taken.

Ray From Phased Array to Feed Point

Finally the phased array is assumed to direct the reflected ray to a feed point on the x axis. The distance to the feed point $x=f$ from the point $x=D$, $y=y_p$, $z=z_p$ is

$$L_3 = [(D-f)^2 + y_p^2 + z_p^2]^{1/2} \tag{49}$$

and the total path length is Eq. (31), (48), (49)

$$L = L_1 + L_2 + L_3 \tag{50}$$

Path length L (or for that matter any of its components) can be converted to phase angle by

$$\varphi = \frac{360°}{\lambda} L \quad \text{in degrees} \tag{51}$$

Where λ is the wave length of the incident radiation. (λ and L must be in the same units.) Equation (51) assumes that the medium in the antenna system is homogeneous and that the phase changes on reflection at the dish and at the phased array are constants. It is convenient to adjust the path lengths by subtracting one L from the others; since only relative phase is required.

This treatment of the antenna system does not consider such things as phase changes at reflection, which may depend on the angle of incidence; polarization changes at various places in the antenna system; or the wave nature of the radiation that leads to diffraction effects at each boundary. These phenomena have more effect on either the power radiated by the antenna or the power received at the feed point. However, they deserve some consideration in the overall understanding of the operation of the antenna described here.

Appendix A

SUMMARY OF EQUATIONS FOR COMPUTATION

The numerical computations run as follows, with the necessary equations repeated here for convenience:

A. Projections of Dish and Phased-Array Rims on Wavefront

The coordinates of the dish rim in the xyz coordinate system are

$$z_{rim} = A \cos \theta$$

$$y_{rim} = A \sin \theta$$

The coordinates of the projection of the dish rim on the wavefront plane are

$$\begin{aligned} Z &= z_{rim} \\ Y &= y_{rim} \cos\alpha - \frac{A^2}{4F} \sin\alpha \end{aligned} \qquad (10a)$$

The coordinates of the phased-array rim are

$$z'_{rim} = R_0 \cos \theta$$

$$y'_{rim} = R_0 \sin \theta$$

The coordinates of the projection of the phased array are

$$Z' = z'_{rim}$$
$$Y' = y'_{rim} \cos\alpha - D \sin\alpha \qquad (13a)$$

It is sufficient to let the parameter θ have the range

$$-\pi/2 \le \theta \le \pi/2$$

for the right half of either projection. The Y axis is taken as the vertical axis. Mirror symmetry about the Y axis gives the left half of each projection. With a θ increment of 1° or 2°, these equations give sufficient points to draw smooth curves with the automatic plotters used with digital computers.

B. Image of the Dish Rim on the Phased-Array Plane

Using z_{rim} and y_{rim}, the ratios of the components of the unit reflected vector R at the dish rim are

$$\left(\frac{R_y}{R_x}\right)_{rim} = \frac{(2y^2_{rim} - 4F^2 - A^2)\sin\alpha - 4Fy_{rim}\cos\alpha}{(4F^2 - A^2)\cos\alpha - 4Fy\sin\alpha}$$
$$\left(\frac{R_z}{R_x}\right)_{rim} = \frac{2(y_{rim}\sin\alpha - 2F\cos\alpha)\, z_{rim}}{(4F^2 - A^2)\cos\alpha - 4F\, y_{rim}\sin\alpha} \qquad (44)$$

and the coordinates of the intersection of the reflected rays and the phased-array plane are

$$y_p = \left(\frac{R_y}{R_x}\right)_{rim}\left(D - \frac{A^2}{4F}\right) + y_{rim}$$
$$z_p = \left(\frac{R_z}{R_x}\right)_{rim}\left(D - \frac{A^2}{4F}\right) + z_{rim} \qquad (45)$$

For a finite rim image and no multiple reflections, the following conditions must be met

$$R_x > 0 \tag{46}$$

and

$$\tan\sigma < \frac{4F^2 - A^2}{4FA} \tag{47}$$

Condition (47) is for a finite image, and $R_x > 0$ is for no multiple reflection. The ray tracing can be discontinued at this point if either of these conditions is not met. Again, an increment of 1° or 2° for θ is sufficient to draw smooth curves for the dish image on the phased-array plane. The circle which is the phased array rim can also be drawn by plotting y'_{rim} and z'_{rim} on the phased-array plane.

C. Grid of Points on the Phased Array Plane

The coordinates of the grid of points on the wavefront plane (which are the starting points of ray tracing) are computed from the following equations. These points all lie on or within the projection of the dish rim. The number of Z coordinates is

$$NZ = INT\left(\frac{A}{\Delta Z}\right) + 1 \tag{21}$$

This is the number of points for one half of the Z axis including the origin The Z(J) are

$$Z(J) = \Delta Z \quad (J-1); \quad J=1,2,\ldots,NZ \tag{20}$$

The number of Y(J) coordinates along the semiminor axis is

$$NY(1) = INT\left(\frac{A\cos\alpha}{\Delta Y}\right) \tag{23}$$

not including the point at the origin. The Y(I) coordinates are

$$Y(I) = \Delta Y \cdot (I - NY(1) - 1) - \frac{A^2 \sin\alpha}{4F} \tag{24}$$

with

$$I = 1, 2, \ldots, 2NY(1) + 1 \tag{25}$$

For Z(J) (other than J = 1), the range of the index I is found from

$$NY(J) = \mathrm{INT}\left(\frac{[A^2 - Z^2(J)]^{1/2}}{\Delta Y} \cos\alpha\right) \tag{22}$$

and

$$I = 1 + NY(1) - NY(J), \ldots, 1 + NY(1) + NY(J) \tag{26}$$

The scheme is to trace a set of rays for a given Z(J) for all Y(I) in the range (26). Path lengths and phases for this set of Y(I) can then be listed and plotted as functions of Y(I) for this Z(J).

D. Tests for Rays in "Shadow Area"

Certain of the Y(I) will fall within the projection of the phased array. If

$$R_0 \le Z(J) \le A \tag{16}$$

then all Y(I) with indices in range (26) are on or within the dish rim projection and on or outside of the phased-array projection ("shadow area"). However, if

$$0 \le Z(J) < R_0 \tag{17}$$

then the Y(I) in the range

$$-[R_0^2 - Z^2(J)]^{1/2} \cos\alpha - D \sin\alpha < Y(I) < [R_0^2 - Z^2(J)]^{1/2} - D \sin\alpha \tag{18}$$

lie within the shadow area. Rays with these Y (I) can still be traced and identified so that the shadow area can be indicated on the various plots and listings. It is sufficient to note the smallest and greatest I meeting conditions (18) for each Z (J).

E. Coordinates of the Intersection of a Ray and the Dish

Three cases are considered

(a) $\alpha = 0$

(b) $0 < \alpha \leq 0.01$ (radian)

(c) $0.01 < \alpha < \pi/2$ (radian)

The equations for Q_y are (omitting the indices)

$$\text{(a)} \quad Q_y = Y, \quad \alpha = 0 \tag{28}$$

$$\text{(b)} \quad Q_y = Y/\cos\alpha + \frac{1}{4F\cos\alpha}\left[Z^2 + (Y/\cos\alpha)^2\right]\left\{\sin\alpha + \frac{Y}{2F\cos^2\alpha}\sin^2\alpha\right\} \tag{29}$$

$$0 < \alpha \leq 0.01$$

$$\text{(c)} \quad Q_y = 2F\frac{\cos\alpha}{\sin\alpha}\left\{1 - \left[1 - \frac{Z^2\sin\alpha + 4FY}{4F^2\cos^2\alpha}\sin\alpha\right]^{1/2}\right\}$$

$$0.01 < \alpha < \pi/2$$

For all α,

$$Q_z = Z \tag{27}$$

$$Q_x = \frac{Q_y^2 + Q_z^2}{4F}$$

F. Normal to Dish at Point (Q_x, Q_y, Q_z), $\underline{n}$

The components of the unit normal are

$$n_x = \frac{F}{[F(F+Q_x)]^{1/2}}$$

$$n_y = \frac{-Q_y}{2[F(F+Q_x)]^{1/2}} \qquad (32)$$

$$n_z = \frac{-Q_z}{2[F(F+Q_x)]^{1/2}}$$

G. Unit Vector Along Reflected Ray, $\underline{R}$

The components of $\underline{R}$ are

$$R_x = (2n_x^2 - 1)\cos\alpha + 2n_x n_y \sin\alpha$$

$$R_y = (2n_y^2 - 1)\sin\alpha + 2n_x n_y \cos\alpha$$

$$R_z = 2n_z(n_x \cos\alpha + n_y \sin\alpha)$$

H. Tests for Internal Reflections

(a) A necessary condition is that

$$R_x > 0 \qquad (36)$$

If this test is not met, stop ray tracing for current α, make note of this fact, and go to next α.

(b) With

$$y_d = \frac{R_y}{R_x}(A^2/4F - Q_x) + Q_y$$

$$z_d = \frac{R_z}{R_x}(A^2/4F - Q_x) + Q_z \qquad (39)$$

the condition

$$y_d^2 + z_d^2 < A^2 \qquad (40)$$

must be met for the reflected ray to "escape" from the dish without internal reflection. If it is not met, discontinue ray tracings as in (a) above.

I. Test for Ray to Intersect Phased Array

With

$$y_p = \frac{R_y}{R_x}(D - Q_x) + Q_y$$

$$z_p = \frac{R_z}{R_x}(D - Q_x) + Q_z \qquad (41)$$

the intersection of the ray and phased-array plane lies within the phased array if

$$y_p^2 + z_p^2 \leq R_0^2 \qquad (42)$$

If (42) is not met, ray tracing can continue but the ray should be identified as being outside the phased array. After all rays are traced for a given α, these points of intersection can be plotted on the phased-array plane along with the image of the dish and the rim of the phased-array plane (see Section B).

J. Path Lengths and Phase

The path lengths are

$$L_1 = -(Q_x \cos\alpha + Q_y \sin\alpha) \tag{31}$$

$$L_2 = [(D-Q_x)^2 + (y_p-Q_y)^2 + (z_p-Q_z)^2]^{1/2} \tag{48}$$

$$L_3 = [(D-f)^2 + y_p^2 + z_p^2]^{1/2} \tag{49}$$

and the total path length is

$$L = L_1 + L_2 + L_3 \tag{50}$$

This path can be adjusted by subtracting some constant from all path lengths. The phase is

$$\varphi = \frac{360°}{\lambda} L \tag{51}$$

provided λ and L are in the same units. If L is in feet and λ is in centimeters then

$$\varphi = 10972.8 \frac{L_{feet}}{\lambda_{cm}} \quad \text{(degrees)} \tag{51a}$$

Path length L and phase φ can be tabulated and plotted as functions of Y for each Z. Note which rays are in the shadow area (see Section D).

Appendix B

COMPUTER PROGRAM OUTPUT

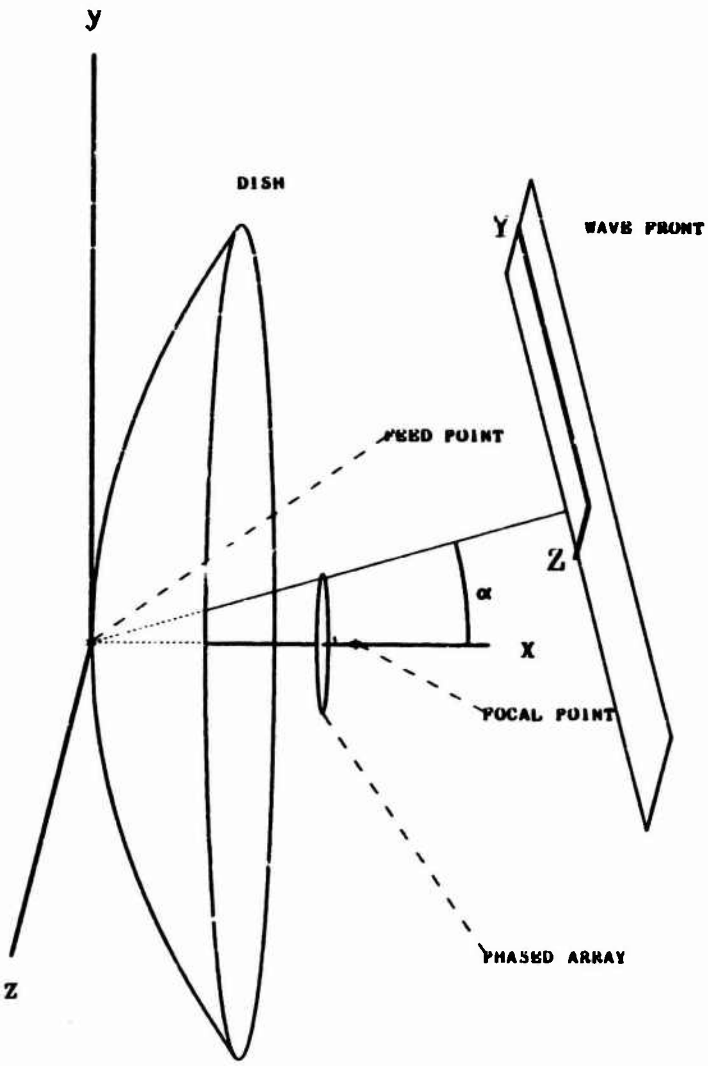
The ray tracing equations developed in this report have been programmed in FORTRAN V for the UNIVAC 1108. Much of the output of this program (named WAVE) is graphic and requires the Stromberg DatagraphiX, Inc., 4060 plotter and the associated INTEGRATED GRAPHICS SYSTEM to be available on the UNIVAC 1108. The graphic outputs are as follows:

First, a scale drawing of the antenna system showing the major coordinate systems is made with the plotter. A table of input data, such as the focal distance of the dish and the radius of the phased array, follows this drawing.

Next, a plot of the grid of points on the wavefront plane is made along with the projection of the rims of the dish and the phased array. Curves of phase change as functions of the Y coordinate on the wavefront plane are made for each $Z \geq 0$. Those points, which lie on or within the phased-array rim, are marked with the symbol *; those outside are plotted with the symbol •. The "shadow area"; that is, those rays which are blocked by the phased array, are indicated by the shaded area on the plots. On another plot, the points of intersection of the rays with the phased-array plane are drawn on this plane. The circle, which is the phased-array rim, and the image of the dish rim are also drawn. In this plot, those rays in the shadow area are not shown.

Printer listings of the path lengths and phases and the coordinates of the intersections of the rays with the phased-array plane can be made. The total number of rays (those in the shadow and those on or within the phased array) can also be tabulated.

Program control permits selecting the several outputs individually so that only those which are required can be produced. Examples of graphic outputs are presented on the following pages. The charts, which are exact replicas of the computer output, have been reduced to 85% of actual size to conform to the page size of this report.



RADIUS OF DISH RIM,FT	A =	6.00	LOCATION OF PHASED ARRAY,FT	D =	3.50
FOCAL DISTANCE,FT	F =	4.00	FEED POINT,FT	FEED =	0.00
RADIUS OF PHASED ARRAY,FT	RO =	1.00	WAVE LENGTH,CM	LAMBDA =	5.31

Computer-Generated Scale Drawing of Antenna System. The dish and phased array dimensions and locations are to scale. A typical wavefront angle α is shown. The constants in the table hold for all computer plots which follow.

RADAR RAY TRACING

INPUT DATA

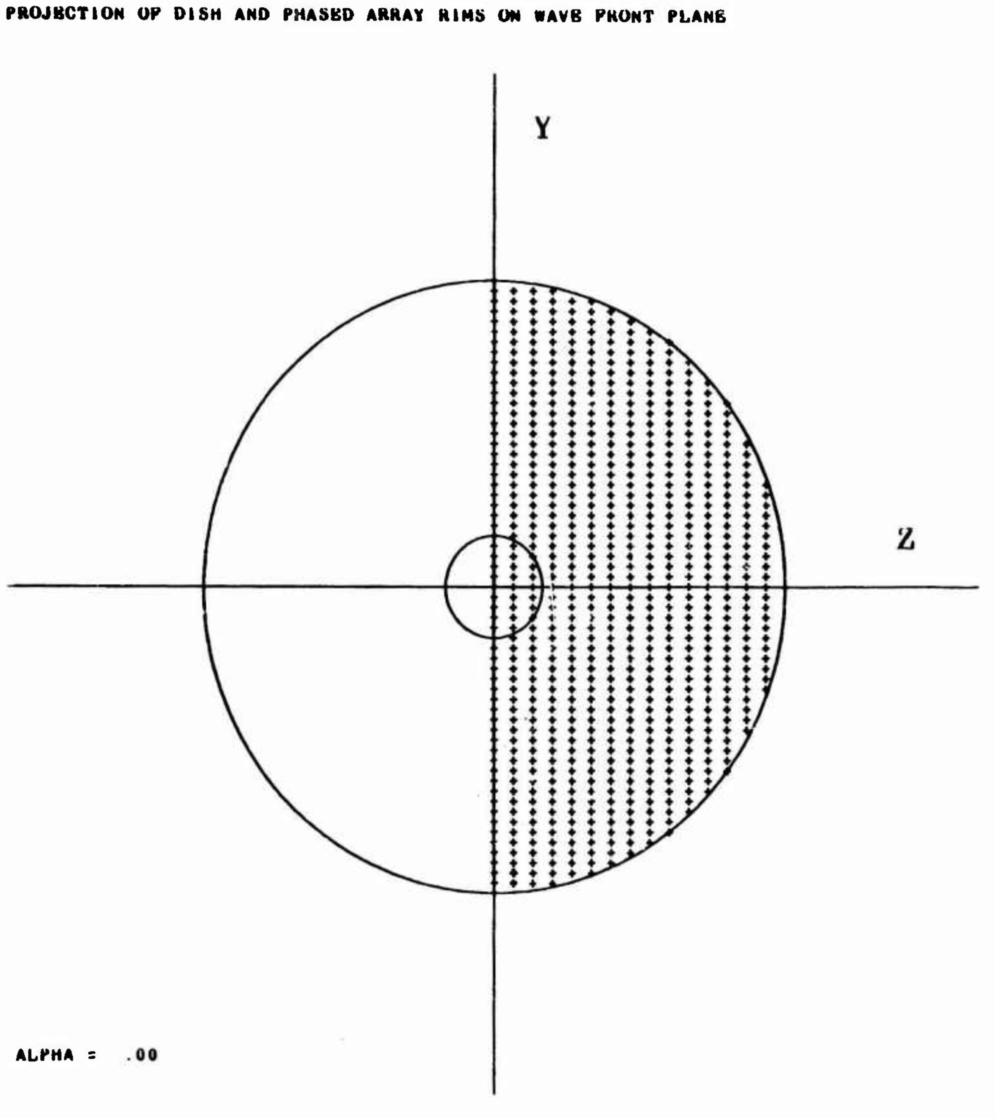
ANTENNA CONSTANTS (FT)

RADIUS OF DISH RIM	=	6.00
FOCAL DISTANCE OF DISH	=	4.00
FEED POINT DISTANCE	=	.00
DISTANCE OF PHASED AREA FROM VERTEX OF DISH	=	3.50
RADIUS OF PHASED AREA	=	1.00
LAMBDA (WAVE LENGTH) (CM)	=	5.31

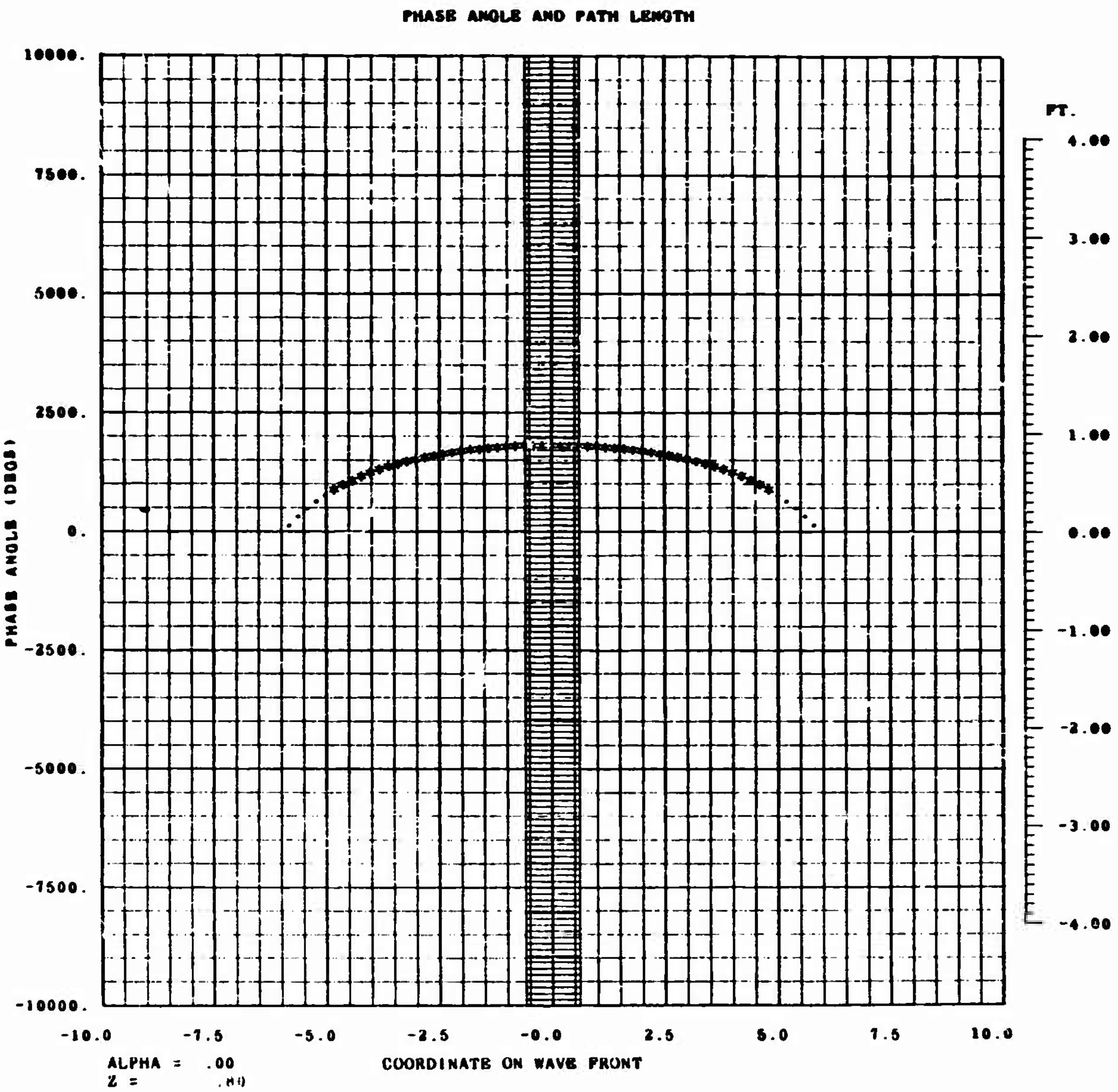
WAVE FRONT PARAMETERS

ALPHA (INITIAL ANGLE OF WAVE FRONT PLANE	=	.00
INCREMENT OF ALPHA (DEGREES)	=	2.50
NUMBER OF ALPHAS	=	3
INCREMENT OF Z COORDINATE ON WAVE FRONT GRID(FT)	=	.40
INCREMENT OF Y COORDINATE ON WAVE FRONT GRID(FT)	=	.20
PROJECTION PARAMETER INCREMENT	=	1.00

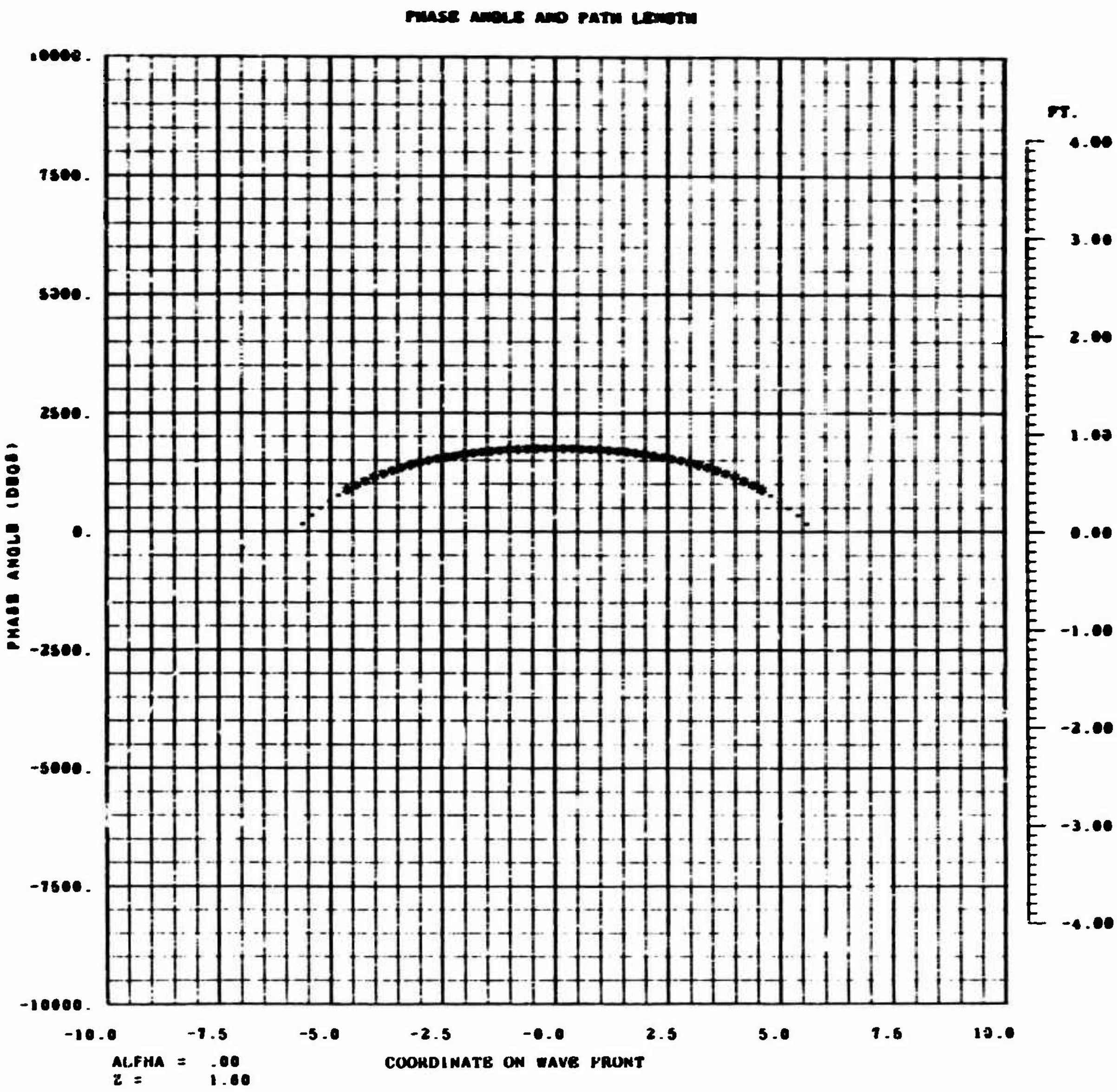
Table of Constants for Antenna Geometry and Computer-Control Parameters. (A plotter output)



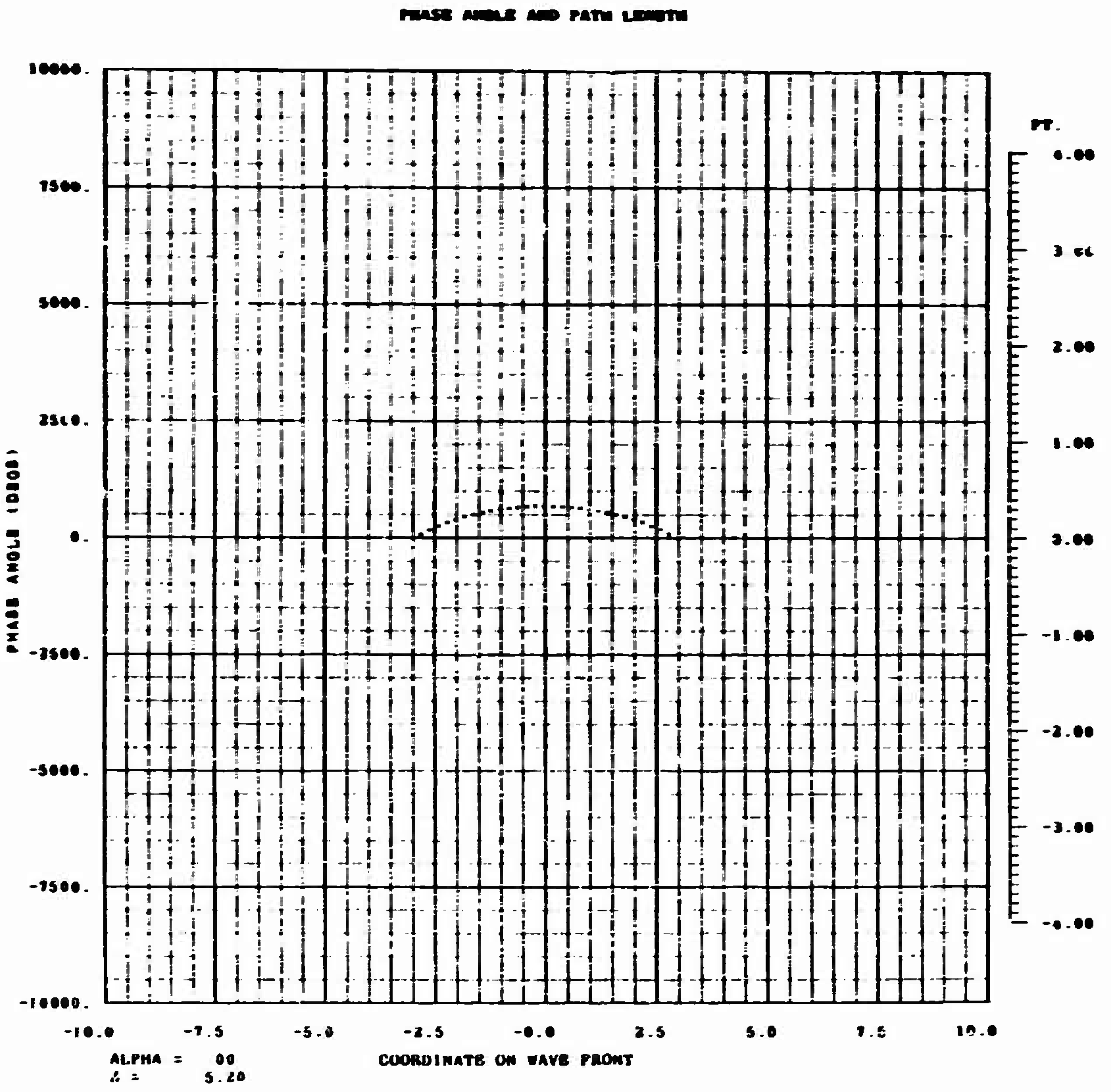
The Wavefront Plane, the Dish and Phased-Array Rim Projections, and the Grid of Points (+) Used to Start Ray Tracing. The smaller circle is the phased-array rim projection. Rays starting on or within this circle are blocked by the phased array and constitute the "shadow area" in the phase-angle and path-length plots. $\alpha = 0$.



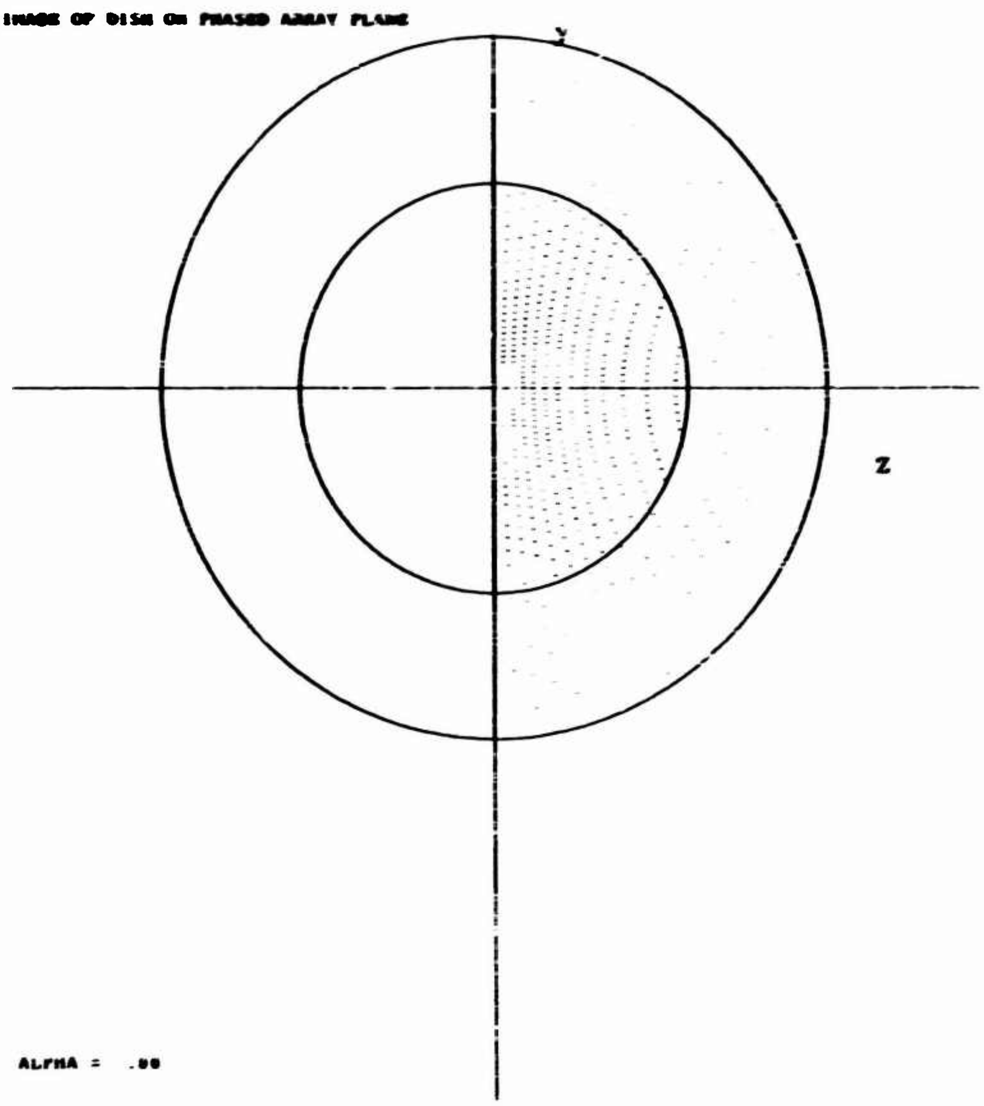
Phase Angles and Path Lengths of Rays From the Wavefront to the Feed Point as Functions of the Y Coordinate on the Wavefront. This plot is for all Y which lie along the line Z = 0.80 and $\alpha = 0$ and which are on or within the dish rim projection. The shading indicates the "shadow area"; that is, the region in which the rays are blocked by the phased array. The symbol * indicates rays which are reflected so that they intersect the phased-array plane on or within the phased-array rim. The symbol · indicates rays which arrive at the phased-array plane but lie outside its rim.



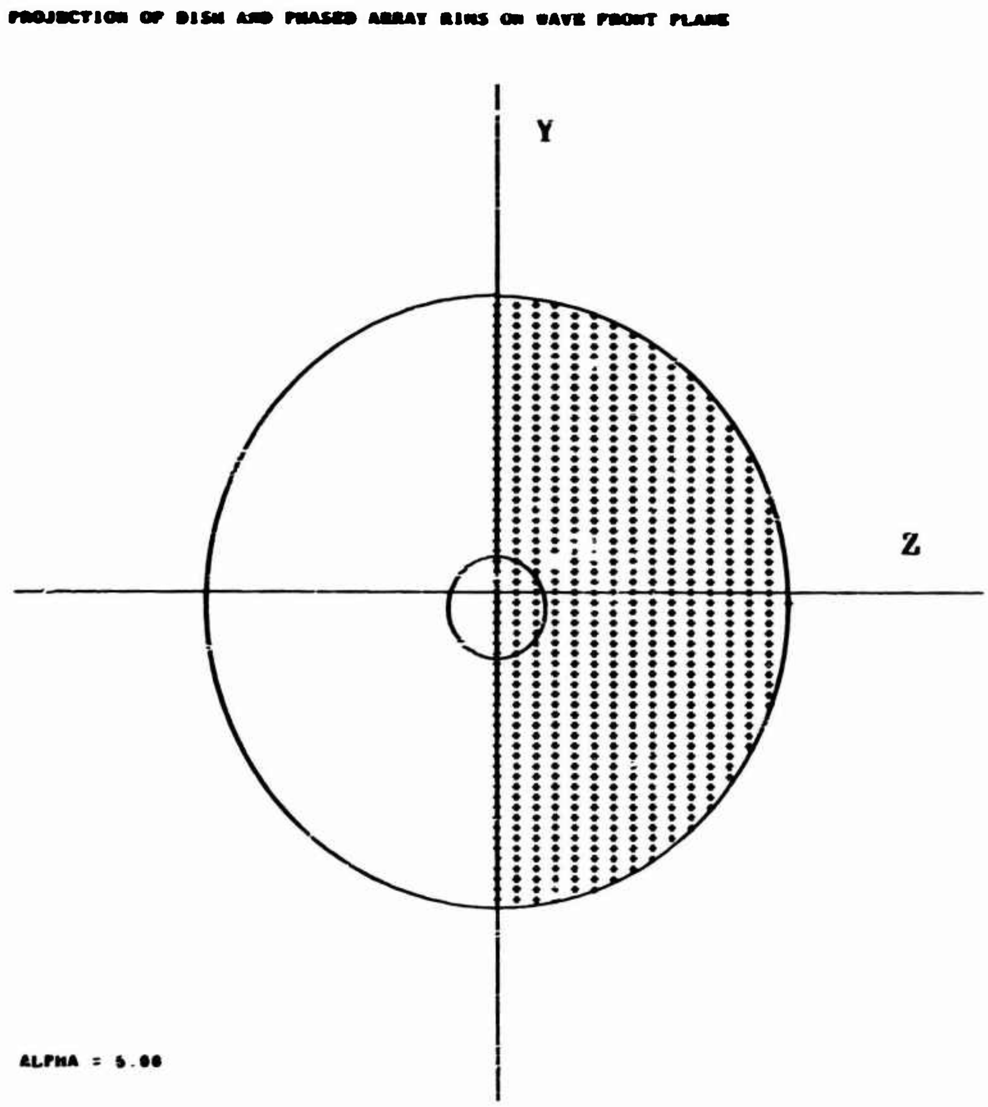
Phase Angles and Path Lengths From the Wavefront to the Feed Point as Functions of the Y Coordinate on the Wavefront. This plot is for Z = 1.60 and $\alpha = 0$. Note that, in this case, no rays are in the "shadow area". The symbols * and • have the same meanings as in the previous plot.



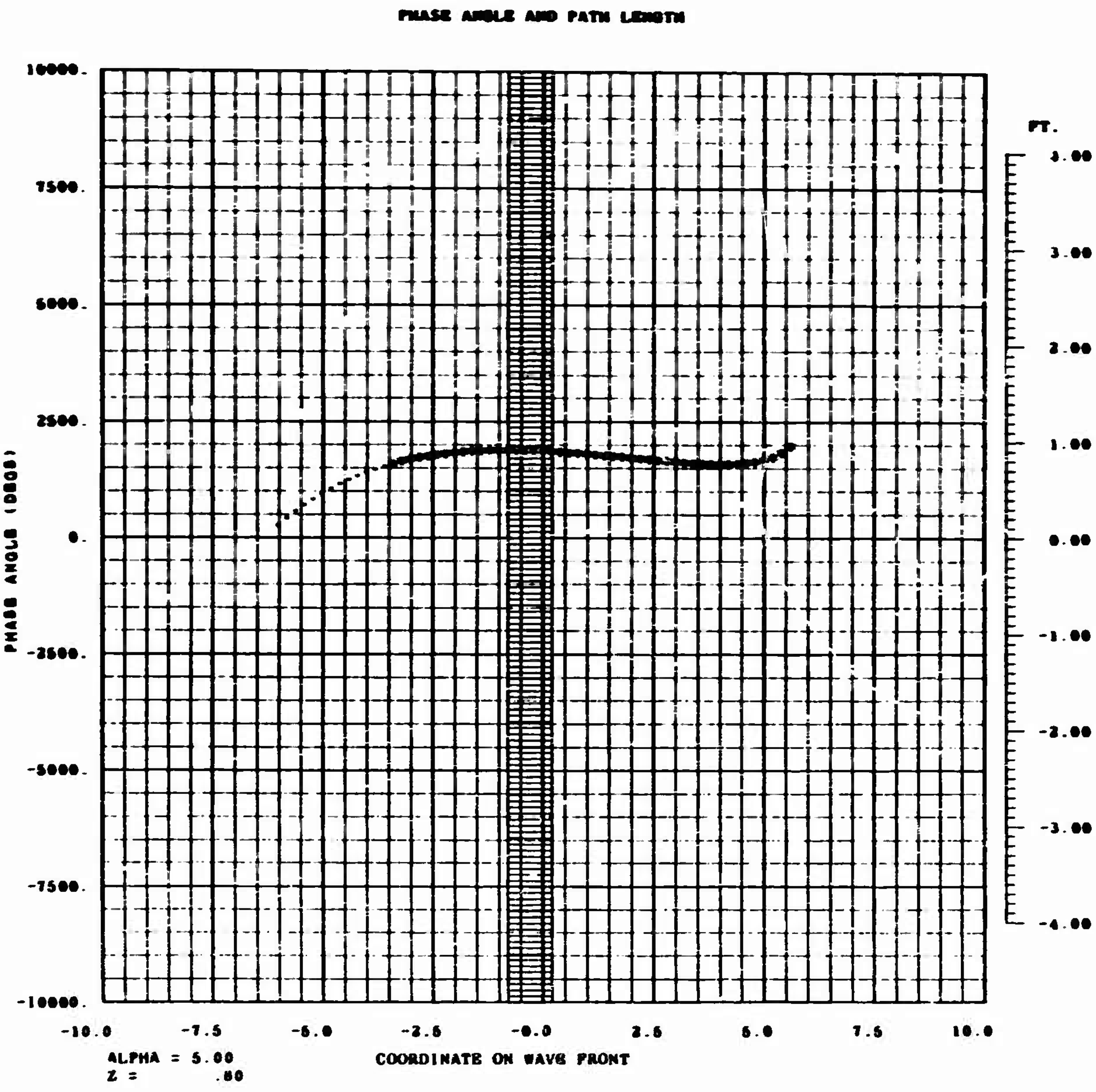
Phase Angles and Path Lengths From the Wavefront to the Feed Point as Functions of the Y Coordinate on the Wavefront. This plot is for Z = 5.20 and $\alpha = 0$. Note that all rays lie outside the phased-array rim (plotted with the symbol •).



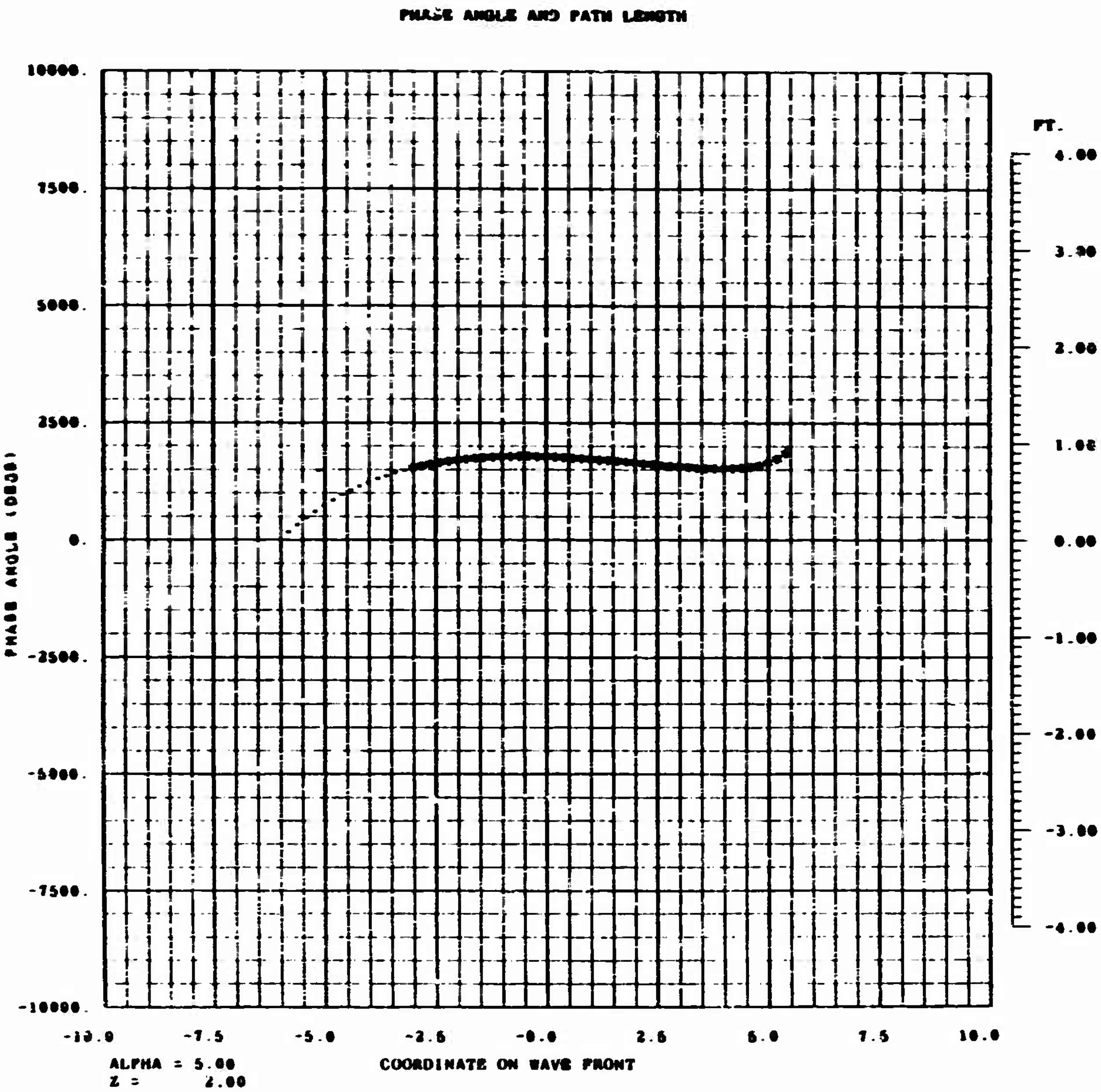
The Phased-Array Plane. The phased-array rim is the large circle. The smaller one is the dish rim image. The points are the intersections of the rays with the phased-array plane. The ray intersections in the "shadow area" are not plotted. In this case, the shadow area lies in a circular area in the center of the phased-array plane. $\alpha = 0$.



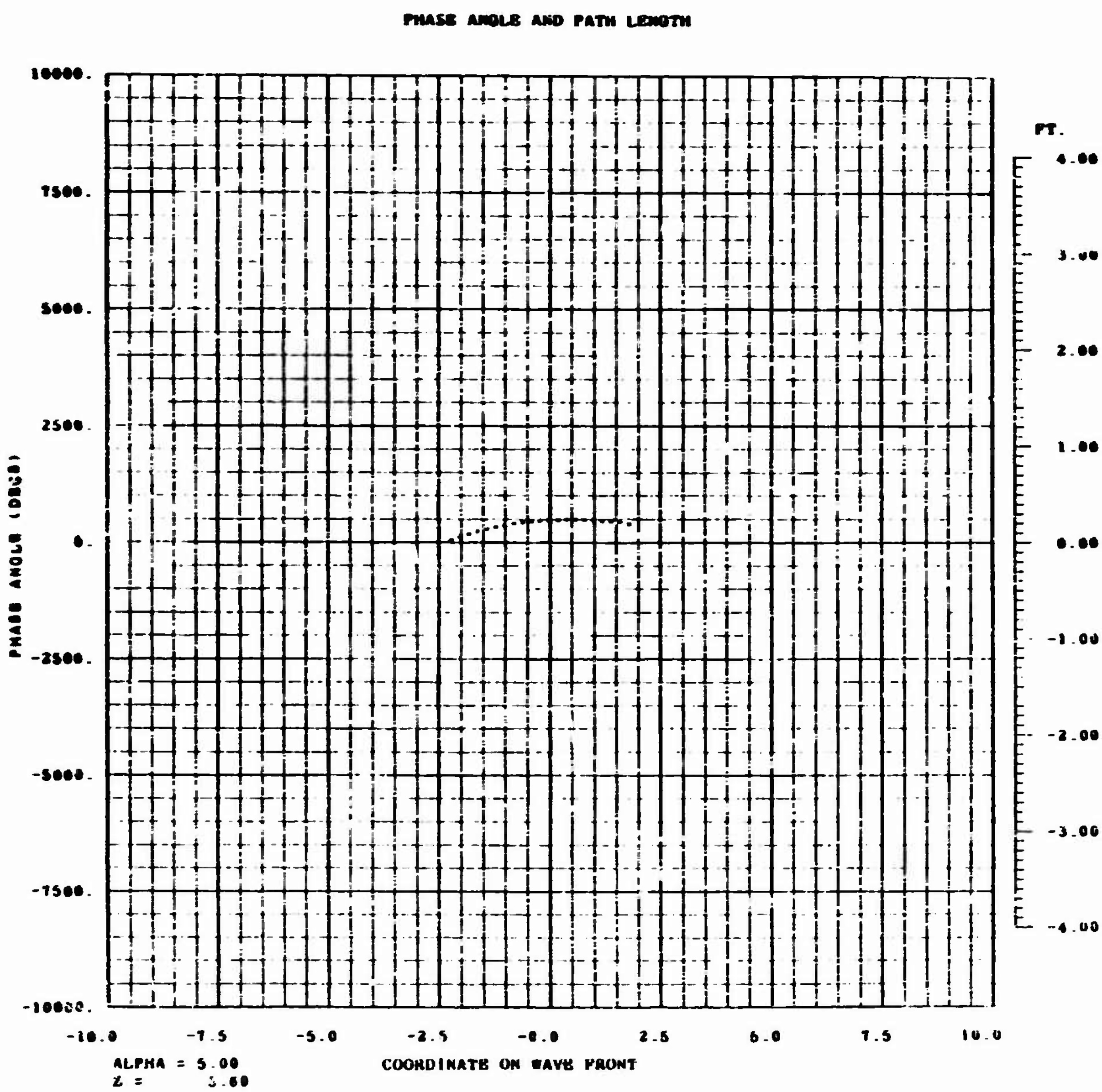
The Wavefront Plane, the Dish and Phased-Array Rim Projections, and the Grid of Points (+) Used to Start Ray Tracing. The smaller ellipse is the phased-array rim projection. Rays starting on or within this ellipse are blocked by the phased-array and constitute the "shadow area" in the phase-angle and path-length plots. $\alpha = 5.00$.



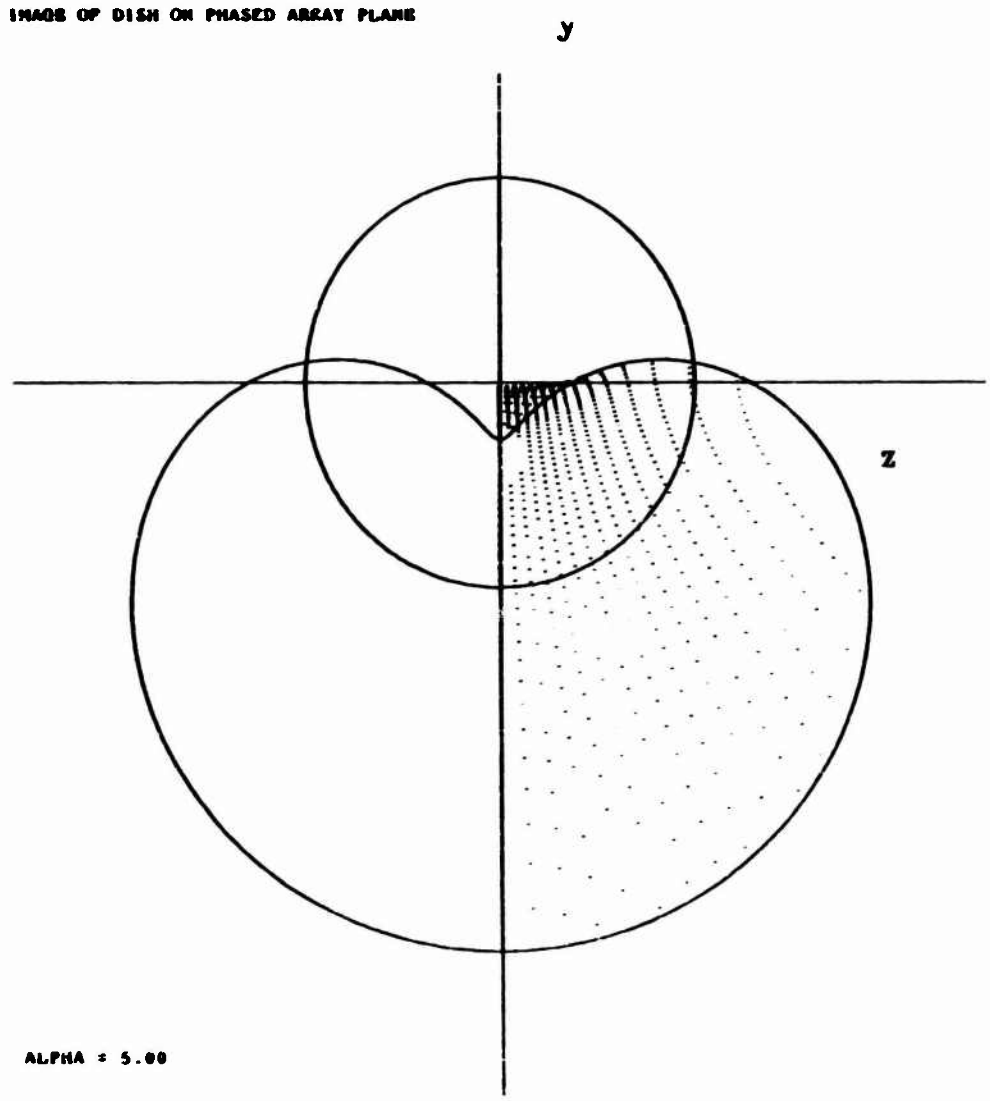
Phase Angles and Path Lengths of Rays From the Wavefront to the Feed Point as Functions of the Y Coordinate on the Wavefront. This plot is for all Y which lie along the line Z = 0.40, α = 5.00 and which are on or within the dish rim projection. The shading indicates the "shadow area"; that is, the region in which the rays are blocked by the phased array. The symbol * indicates rays which are reflected so that they intersect the phased-array plane on or within the phased-array rim. The symbol • indicates rays which arrive at the phased-array plane but lie outside the rim.



Phase Angles and Path Lengths From the Wavefront to the Feed Point as Functions of the Y Coordinate on the Wavefront. This plot is for Z = 2.00 and $\alpha = 5.00$. Note in this case that there are no rays in the shadow area.



Phase Angles and Path Lengths From the Wavefront to the Feed Point as Functions of the Y Coordinate on the Wavefront. This plot is for Z = 5.60 and $\alpha = 5.00$. Note that all rays lie outside the phased-array rim (plotted with the symbol •).



The Phased-Array Plane. The phased-array rim is the circle. The image of the dish rim is the cardioid curve. The points are intersections of the rays with the phased-array plane. The "shadow area" is seen below the center of the phased array. $\alpha = 5.00$. At this angle, the aberrations of the antenna system become apparent. For example, in the region near YZ origin, there is no longer a unique phase for a given point on the phased-array plane. In other words, more than one ray from the wavefront plane can intersect the phased-array plane at a given point in some regions of the plane.

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13. ABSTRACT

ABSTRACT. This report gives the derivation of equations for the ray trace analysis of an antenna consisting of a parabolic dish with a phased array in front of it. The derivation gives the equations for ray tracing in a form suitable for digital computer use. A number of rays from a plane wavefront can be traced through the antenna system to a feed point on the antenna axis. The rays are straight lines, and specular reflection is assumed at the dish. The phased array returns the reflected rays to the feed point (which is assumed to lie between the dish and phased array).

The major features provided by the analysis and the computer program are the relative phases and path lengths of the rays from the wavefront to the feed point (tabulation and computer plots), the total number of rays, the number obscured by the phased array, the number that arrive at the phased-array plane and lie within the phased array itself, a computer-generated plot of the starting points on the wavefront of the rays used in tracing, a plot of the locations of those rays that arrive at the phased-array plane, and an outline of the phased array.

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