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STREAMLINE MOTION OF A VISCOUS, INCOMPRESSIBLE FLUID IN A CURVED PIPE WITH AN ELLIPTICAL CROSS SECTION

by

Bobby F. Mullinix

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U.S. ARMY MISSILE COMMAND

Redstone Arsenal, Alabama

28 May 1969

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STREAMLINE MOTION OF A VISCOUS, INCOMPRESSIBLE FLUID IN A CURVED PIPE WITH AN ELLIPTICAL CROSS SECTION

by

Bobby F. Mullinix

D. A. Project No. 1M222901A206

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ABSTRACT

This study concerns the laminar flow of an incompressible fluid through a curved pipe with an elliptical cross section. The governing equations are derived by applying the Navier-Stokes and continuity equations in cylindrical coordinates and the method of successive approximations to get the five partial differential equations. These equations are solved by the perturbation method, and twenty numerical examples are presented. For each example, arbitrary numerical parameters are assumed for input into an IBM 7094 Computer to obtain solutions for the simultaneous algebraic equations. For an eccentricity of one, the ellipse degenerates to a circle and Dean's solution for the streamline flow of an incompressible fluid through a curved pipe with a circular cross section is obtained.

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CHAPTER I

INTRODUCTION

The design of present and future high performance missiles, rockets, and aircraft necessitates the optimum use of all available space. To help accomplish this requirement, bent tubing and curved pipes are frequently used. In the design of these piping systems, it is generally assumed that the circular cross-sectional area remains constant to simplify the analysis. This is a true assumption only if the pipe is cast or forged, or if a fitting with a circular cross section is used to obtain the required curvature. However, a fitting is used only for small curvatures. Large curvatures are usually desired to reduce the pressure drop, decrease the size of pipe or increase the flow rate. This is accomplished by bending a piece of straight pipe to the desired curvature, thus distorting the circular cross section and making it elliptical.

The purpose of this report is to study the effect of the eccentricity of an elliptical cross section in a pipe with a small curvature and to get a second-order approximation of the flow rate through the pipe.

The term "curved pipe" as used in this report is a pipe with an elliptical cross section bent so that the center of the ellipse forms a circular arc. It is assumed that the flow is fully developed throughout the region and that the straight pipes are connected to the bend so that they always lie in the plane of the bend with their center lines tangent to the center line of the bend.

Dean [1927] showed that the motion of a viscous, incompressible fluid in a curved pipe with a circular cross section consists of a primary motion along and parallel to the center line of the pipe, and the secondary motion which is in the plane of the cross section. His solution is the most detailed and perhaps the best theoretical study performed to this date although it is severely limited to Dean's numbers less than about 400. This results from neglect of the unsymmetrical terms in the second-order approximation of the flow rate and the insufficient number of terms used with the method of successive approximations. He also assumed small pipe curvature and that the circular cross section remained circular after the pipe was bent. Within its limitations, Dean's solution agrees very well with experimental data, and his theory has been widely used and extended.

Thomas and Walters [1963] used Dean's approach and analyzed his problem using an elastico-viscous liquid. Their work showed that the elastico-viscous property of the fluid reduced the curvature of the streamlines in the central plane and increased the rate of flow through the pipe.

Baura [1963] took Dean's [1928] basic equations and integrated across the boundary layer to get the momentum integral form. The equations were then solved by Polyhausen's method. The results agreed very well with experimental data but are probably no more accurate than Dean's solution.

Clegg and Power [1963] analyzed the flow of a Bingham fluid* in a slightly curved pipe, but their solution was accurate only to the first-order approximation and did not include the effect of the curvature. The effects of a plug being inserted in the center of the pipe were also studied.

Thomas and Walters [1965] studied the flow of a fluid through a curved pipe with an elliptical cross section, but their results were accurate only to the first-order approximation. Their solution indicated that the rate of flow through the pipe was independent of the curvature. It was then concluded that the second-order terms were required to determine the effect of curvature on the rate of flow through the pipe.

The experimental approach to special cases of this problem was taken by White [1929] and Keulegan and Beij [1937]. The results obtained conformed very well with Dean's solution. White performed experiments with water and oil flowing through curved pipes with oval and circular cross sections to determine the law of resistance for streamline flow. Keulegan and Beij conducted experiments with water flowing through a curved pipe, with a circular cross section prior to bending, to determine the pressure losses due to the large curvature of the pipe. The first experiments resulted in an equation for the prediction of head loss, and the second experiments resulted in an equation for the increase in resistance in a bend as compared to that of a straight pipe.

This study considers the streamline motion of a viscous, incompressible

* A Bingham fluid is a material that can support a finite stress elastically without flow and flows with constant plastic fluidity when the stresses are sufficiently great.

fluid in a curved pipe with an elliptical cross section. The angle of bend in the pipe is unrestricted provided that the curvature is small and the flow in the bend is fully developed.

The governing partial differential equations are derived in Chapter II by applying the equations of motion (Navier-Stokes equations) and the continuity equation in cylindrical coordinates. The method of successive approximations is then used to obtain equations which are accurate to the second-order approximation.

In Chapter III these nonlinear partial differential equations are solved. Simultaneous equations involving the constants are then obtained by matching coefficients. The unsymmetrical terms, although negligible for small Dean's numbers, are not small for large Dean's numbers and are not neglected in the solutions. Therefore, these solutions are more accurate than the corresponding second-order approximation solution presented by Dean for a pipe of circular cross section. By letting the cross-sectional ellipse degenerate to a circle, the equations of the present work reduce to Dean's [1928] equations except for the last equation, in which Dean neglected the unsymmetrical terms.

The four sets of simultaneous equations in matrix form are presented in Chapter IV, and their computer solutions are in Appendix B. The equations for the rate of flow through the curved pipe are integrated, and equations for 20 different eccentricities of the cross-sectional ellipse are presented in Chapter IV.

The discussion, summary and conclusions, and recommendations for future research are given in Chapters V, VI and VII, respectively.

CHAPTER II

GOVERNING DIFFERENTIAL EQUATIONS

A. Assumptions

This work is a study of the streamline motion of a Newtonian fluid in a curved pipe with an elliptical cross section. The equations of motion in cylindrical coordinates (r', Θ, y') are applied to the arrangement shown in Figure 1.

The following assumptions are made in the derivation of the governing partial differential equations:

1. Streamline motion theory applies.
2. Body forces are negligible.
3. The flow is steady, uniform and incompressible.
4. The curvature of the pipe is small.
5. The flow is axisymmetric and all flow variables except pressure are independent of Θ.
6. The flow is fully developed throughout the region of study.

For laminar, incompressible flow the unknown variables consist of the pressure and three components of velocity. To solve for these four unknowns the three equations of motion (Navier-Stokes equations) and the continuity equation are used. These equations appear in Schlichting [1968] as well as

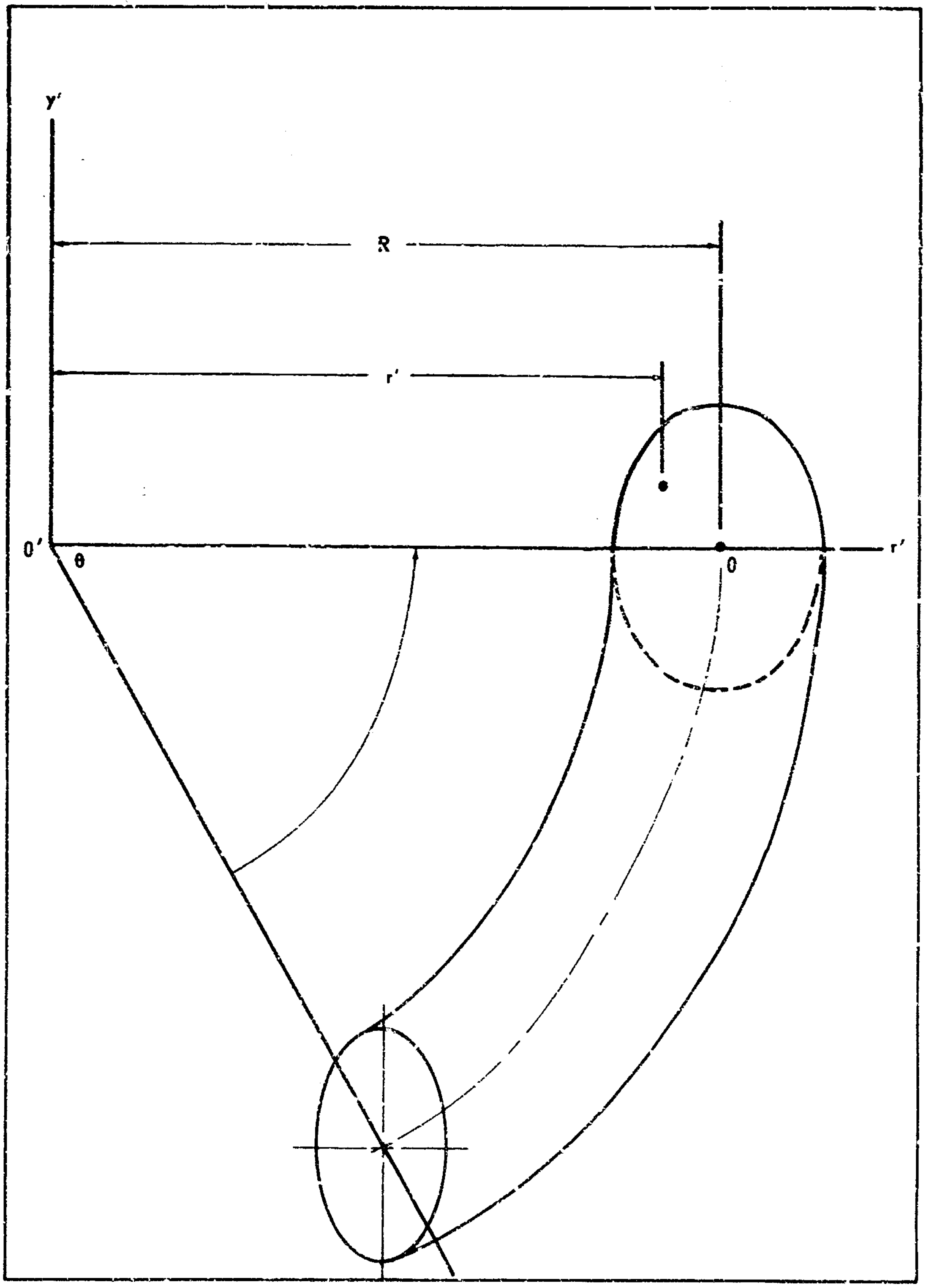

Figure 1. Cylindrical Coordinate System

most text books on fluid mechanics, and are given in the following sections.

B. Navier-Stokes Equations in Cylindrical Coordinates

For the r'-direction:

$$\rho\left(\frac{\partial q_r}{\partial t} + q_r\frac{\partial q_r}{\partial r'} + \frac{q_\Theta}{r'}\frac{\partial q_r}{\partial \Theta} + q_y\frac{\partial q_r}{\partial y'} - \frac{q_\Theta^2}{r'}\right)$$

$$= \rho f_r - \frac{\partial P}{\partial r'} + \mu\left(\nabla'^2 q_r - \frac{q_r}{r'^2} - \frac{2}{r'^2}\frac{\partial q_\Theta}{\partial \Theta}\right) \tag{2.1}$$

For the Θ-direction:

$$\rho\left(\frac{\partial q_\Theta}{\partial t} + q_r\frac{\partial q_\Theta}{\partial r'} + \frac{q_\Theta}{r'}\frac{\partial q_\Theta}{\partial \Theta} + q_y\frac{\partial q_\Theta}{\partial y'} + \frac{q_r q_\Theta}{r'}\right)$$

$$= \rho f_\Theta - \frac{1}{r'}\frac{\partial P}{\partial \Theta} + \mu\left(\nabla'^2 q_\Theta + \frac{2}{r'^2}\frac{\partial q_r}{\partial \Theta} - \frac{q_\Theta}{r'^2}\right) \tag{2.2}$$

For the y'-direction:

$$\rho\left(\frac{\partial q_y}{\partial t} + q_r\frac{\partial q_y}{\partial r'} + \frac{q_\Theta}{r'}\frac{\partial q_y}{\partial \Theta} + q_y\frac{\partial q_y}{\partial y'}\right)$$

$$= \rho f_y - \frac{\partial P}{\partial y'} + \mu\nabla'^2 q_y \tag{2.3}$$

C. Continuity Equation in Cylindrical Coordinates

$$\frac{\partial q_r}{\partial r'} + \frac{q_r}{r'} + \frac{1}{r'}\frac{\partial q_\Theta}{\partial \Theta} + \frac{\partial q_y}{\partial y'} = 0 \tag{2.4}$$

where f_r, f_Θ, and f_y are the components of the body force; q_r, q_Θ, and q_y are the components of velocity; P is the pressure; ρ is the fluid density; μ is the absolute viscosity of the fluid and $\nabla'^2 = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2}{\partial \Theta^2} + \frac{\partial^2}{\partial y'^2}$.

The assumption of steady flow and negligible body forces reduces equations (2.1) through (2.3) to:

$$\rho\left(q_r \frac{\partial q_r}{\partial r'} + \frac{q_\Theta}{r'}\frac{\partial q_r}{\partial \Theta} + q_y \frac{\partial q_r}{\partial y'} - \frac{q_\Theta^2}{r'}\right) = -\frac{\partial P}{\partial r'} + \mu\left(\nabla'^2 q_r - \frac{q_r}{r'^2} - \frac{2}{r'^2}\frac{\partial q_\Theta}{\partial \Theta}\right) \quad (2.5)$$

$$\rho\left(q_r \frac{\partial q_\Theta}{\partial r'} + \frac{q_\Theta}{r'}\frac{\partial q_\Theta}{\partial \Theta} + q_y \frac{\partial q_\Theta}{\partial y'} + \frac{q_r q_\Theta}{r'}\right) = -\frac{1}{r'}\frac{\partial P}{\partial \Theta} + \mu\, \nabla'^2 q_\Theta + \frac{2}{r'}\frac{\partial q_r}{\partial \Theta} - \frac{q_\Theta}{r'^2} \quad (2.6)$$

$$\rho\left(q_r \frac{\partial q_y}{\partial r'} + \frac{q_\Theta}{r'}\frac{\partial q_y}{\partial \Theta} + q_y \frac{\partial q_y}{\partial y'}\right) = -\frac{\partial P}{\partial y'} + \mu \nabla'^2 q_y \quad (2.7)$$

Since the flow is uniform and q_r, q_Θ, and q_y are independent of Θ, $\frac{\partial q_r}{\partial \Theta} = \frac{\partial q_y}{\partial \Theta} = \frac{\partial q_\Theta}{\partial \Theta} = 0$. The kinematic viscosity, $\nu = \frac{\mu}{\rho}$ and the equations (2.4) through (2.7) become:

$$\frac{\partial q_r}{\partial r'} + \frac{q_r}{r'} + \frac{\partial q_y}{\partial y'} = 0 \quad (2.8)$$

$$q_r \frac{\partial q_r}{\partial r'} + q_y \frac{\partial q_r}{\partial y'} - \frac{q_\Theta^2}{r'} = -\frac{1}{\rho}\frac{\partial P}{\partial r'} + \nu\left(\nabla'^2 q_r - \frac{q_r}{r'^2}\right) \quad (2.9)$$

$$q_r \frac{\partial q_\Theta}{\partial r'} + q_y \frac{\partial q_\Theta}{\partial y'} + \frac{q_r q_\Theta}{r'} = - \frac{1}{\rho r'} \frac{\partial P}{\partial \Theta} + \nu \left(\nabla'^2 q_\Theta - \frac{q_\Theta}{r'^2} \right) \tag{2.10}$$

$$q_r \frac{\partial q_y}{\partial r'} + q_y \frac{\partial q_y}{\partial y'} = - \frac{1}{\rho} \frac{\partial P}{\partial y'} + \nu \nabla'^2 q_y \tag{2.11}$$

where ∇'^2 is redefined by

$$\nabla'^2 = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'} \frac{\partial}{\partial r'} + \frac{\partial^2}{\partial y'^2} .$$

For incompressible flow the density is constant. Also, from Figure 1, $r' = R + x'$. For small curvatures the radius of curvature of the pipe, R, is large when compared to the distance x' shown in Figure 2. Therefore, r' approximately equals R and equations (2.9) through (2.11) can be expressed as:

$$q_r \frac{\partial q_r}{\partial r'} + q_y \frac{\partial q_r}{\partial y'} - \frac{q_\Theta^2}{r'} = - \frac{\partial \left(\frac{P}{\rho} \right)}{\partial r'} + \nu \left(\nabla'^2 q_r - \frac{q_r}{r'^2} \right) \tag{2.12}$$

$$q_r \frac{\partial q_\Theta}{\partial r'} + q_y \frac{\partial q_\Theta}{\partial y'} + \frac{q_r q_\Theta}{r} = - \frac{1}{R} \frac{\partial \left(\frac{P}{\rho} \right)}{\partial \Theta} + \nu \left(\nabla'^2 q_\Theta - \frac{q_\Theta}{r'^2} \right) \tag{2.13}$$

$$q_r \frac{\partial q_y}{\partial r'} + q_y \frac{\partial q_y}{\partial y'} = - \frac{\partial \left(\frac{P}{\rho} \right)}{\partial y'} = \nu \nabla'^2 q_y \quad , \tag{2.14}$$

where R is a constant.

Since q_r, q_Θ, and q_y are independent of Θ, all terms in equations

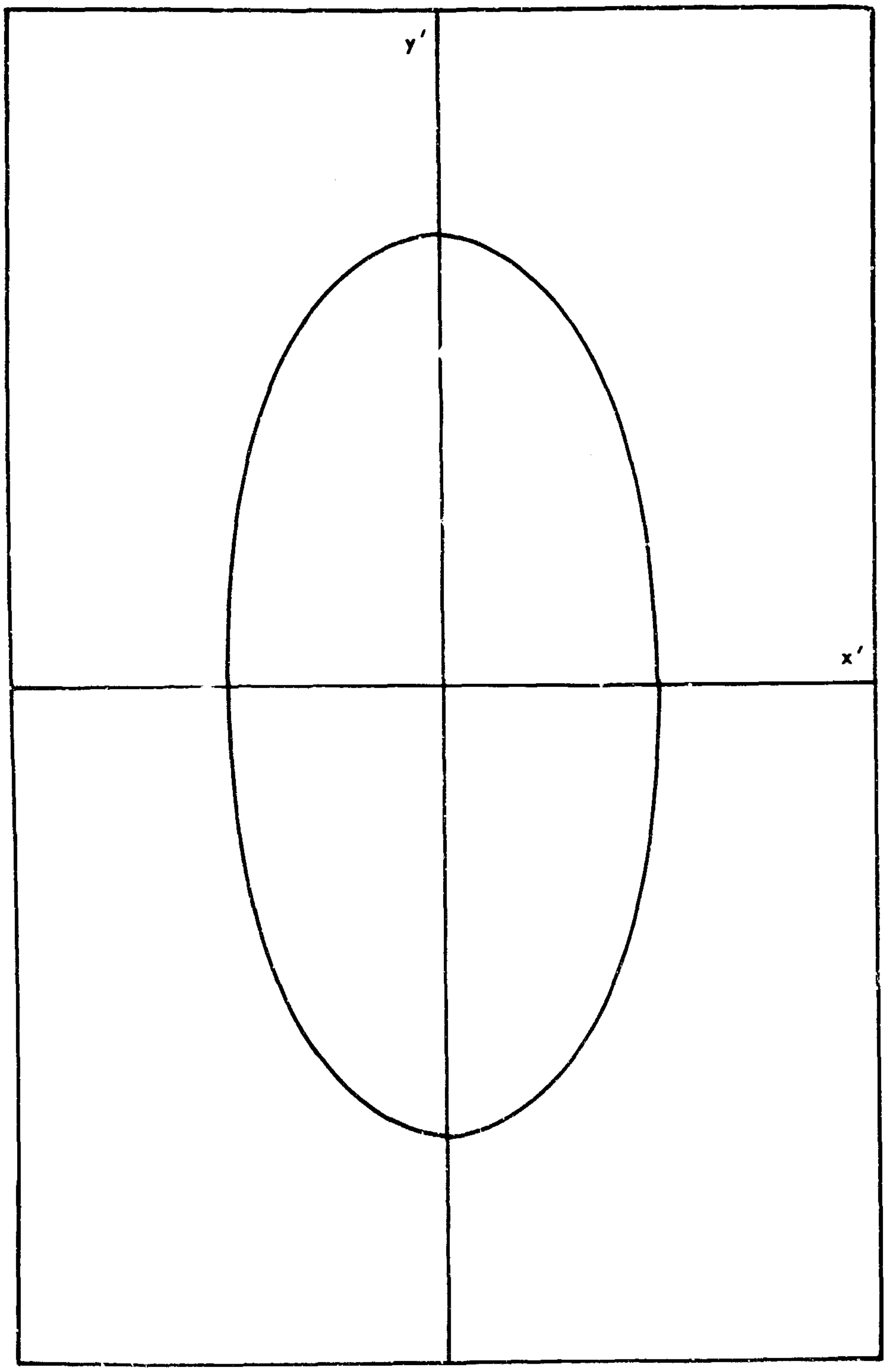

Figure 2. Cartesian Coordinate System for Any Θ

(2.12), (2.13), and (2.14) are functions of r' and y' only except the pressure terms $-\frac{1}{R}\frac{\partial(P/\rho)}{\partial\Theta}$, $-\frac{\partial(P/\rho)}{\partial r'}$, and $-\frac{\partial(P/\rho)}{\partial y'}$. Solving each of these equations for its pressure term shows that the three pressure terms are also functions of

r' r' and y' only. Therefore, $\frac{P}{\rho} = e_1\Theta + f(r',y')$, where e_1 is a constant. Then

$$-\frac{1}{R}\frac{\partial\left(\frac{P}{\rho}\right)}{\partial\Theta} = -\frac{1}{R}\frac{\partial[e_1\Theta + f(r',y')]}{\partial\Theta} = -\frac{e_1}{R} = \text{constant} \quad . \qquad (2.15)$$

Redefining the constant $-\frac{e_1}{R} = \frac{G}{\rho}$ makes G a constant which can be called the mean pressure gradient. It is the space-rate of decrease in the pressure along the central line traced out by the center of the pipe.

Differentiating equations (2.12) and (2.14) with respect to y' and r', respectively, and subtracting (2.14) from (2.12) eliminates the pressure terms. The results are

$$\frac{\partial}{\partial r'}\left(q_r\frac{\partial q_y}{\partial r'} - q_y\frac{\partial q_y}{\partial y'}\right) - \frac{\partial}{\partial y'}\left(q_r\frac{\partial q_r}{\partial r'} + q_y\frac{\partial q_r}{\partial y'} - \frac{q_\Theta^2}{r'}\right) \qquad (2.16)$$

$$= \nu\frac{\partial}{\partial r'}\left(\nabla'^2 q_y\right) - \nu\frac{\partial}{\partial y'}\left(\nabla'^2 q_r - \frac{q_r}{r'^2}\right) \quad .$$

Using equation (2.15) and $-\frac{e_1}{R} = \frac{G}{\rho}$, equation (2.13) can be expressed as

$$q_r\frac{\partial q_\Theta}{\partial r'} + q_y\frac{\partial q_\Theta}{\partial y'} + \frac{q_r q_\Theta}{r'} = \frac{G}{\rho} + \nu\left(\nabla'^2 q_\Theta - \frac{q_\Theta}{r'^2}\right) \quad . \qquad (2.17)$$

Because of the highly nonlinear character of these equations, a solution cannot be obtained in this form. It is, therefore, necessary to simplify them in such a way that the nonlinear and curvature effects are retained. To accomplish this and transfer the origin from 0' to 0 (Figure 1) it is assumed that ∇'^2 in cylindrical coordinates equals ∇'^2 in Cartesian coordinates, $\frac{\partial}{\partial x'} = \frac{\partial}{\partial r'} + \frac{1}{r'}$ and r' is approximately R. The small curvature of the pipe provides the basis for these assumptions. It is evident that these assumptions are equivalent to the ones made by Dean [1928] for a curved pipe with a circular cross section because the equations obtained degenerate to those presented by Dean [1928]. for the circular case.* Since the r'- and x'-directions are the same, $q_r = q_x$, where q_x is the velocity in the x'-direction. Use of these assumptions reduces equations (2.8), (2.16), and (2.17) to:

$$\left(\frac{\partial}{\partial x'} - \frac{1}{r'}\right) q_x + \frac{q_x}{r'} + \frac{\partial q_y}{\partial y'} = 0 \tag{2.18}$$

$$\left(\frac{\partial}{\partial x'} - \frac{1}{r'}\right)\left[q_x \frac{\partial q_y}{\partial x'} - \frac{q_x q_y}{r'} + q_y \frac{\partial q_y}{\partial y'}\right] - \frac{\partial}{\partial y'}\left[q_x \frac{\partial q_x}{\partial x'} - \frac{q_x^2}{r'} + q_y \frac{\partial q_x}{\partial y'} - \frac{q_\Theta^2}{R}\right]$$
$$= \nu\left(\frac{\partial}{\partial x'} - \frac{1}{r'}\right)\nabla'^2 q_y - \nu \frac{\partial}{\partial y'}\left(\nabla'^2 q_x - \frac{q_x}{r'^2}\right) \tag{2.19}$$

$$\left(\frac{\partial}{\partial x'} - \frac{1}{r'}\right)\left[q_x \frac{\partial q_\Theta}{\partial x'}\right] + q_y \frac{\partial q_y}{\partial y'} + \frac{q_x q_\Theta}{r'} = \frac{G}{\rho} + \left(\nu\ \nabla'^2 q_\Theta - \frac{q_\Theta}{r'^2}\right), \tag{2.20}$$

where, by definition, $\nabla'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$ in the Cartesian coordinate system.

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*The only exception is Dean's solution for the second-order approximation of the velocity in the Θ-direction, in which he neglected the unsymmetrical terms of the solution.

Performing some of the operations indicated in equations (2.18) through (2.20) and neglecting the terms $\frac{1}{r'} \nabla'^2 q_y$, $\frac{q_x q_y}{r'}$, and $\frac{q_x^2}{r'}$, which are of the same order of magnitude as the small curvature squared, reduces these equations to:

$$\frac{\partial q_x}{\partial x'} + \frac{\partial q_y}{\partial y'} = 0 \tag{2.21}$$

$$\left(\frac{\partial}{\partial x'} - \frac{1}{r'}\right)\left[q_x \frac{\partial q_y}{\partial x'} + q_y \frac{\partial q_y}{\partial y'}\right] - \frac{\partial}{\partial y'}\left[q_x \frac{\partial q_x}{\partial x'} + q_y \frac{\partial q_x}{\partial y'}\right] - \frac{2q_\Theta}{R} \frac{\partial q_\Theta}{\partial y'}$$
$$= \nu \left[\frac{\partial\left(\nabla'^2 q_y\right)}{\partial x'} - \frac{\partial\left(\nabla'^2 q_x\right)}{\partial y'} - \frac{1}{r'^2} \frac{\partial q_x}{\partial y'}\right] \tag{2.22}$$

$$q_x \frac{\partial q_\Theta}{\partial x'} + q_y \frac{\partial q_\Theta}{\partial y'} = \frac{G}{\rho} + \nu\left(\nabla'^2 q_\Theta - \frac{q_\Theta}{r'^2}\right) . \tag{2.23}$$

Equations (2.21) through (2.23) are put in nondimensional form by the substitutions:

$$x' = ax \qquad y' = ay \qquad r' = ar \tag{2.24}$$
$$q_x = \frac{\nu}{a} U \qquad q_y = \frac{\nu}{a} V \qquad q_\Theta = W_0 W \ ,$$

where W_0 has the dimensions of velocity and a has the dimensions of length. Equations (2.21) through (2.23) become:

$$\frac{\nu}{a^2}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0 \tag{2.25}$$

$$\frac{\nu^2}{a^4}\left(\frac{\partial}{\partial x} - \frac{1}{r}\right)\left(U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial y}\left(U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y}\right) + \frac{2W_0^2}{aR} W \frac{\partial W}{\partial y}$$

$$= \frac{\nu^2}{a^4}\left[\frac{\partial(\nabla^2 V)}{\partial x} - \frac{\partial(\nabla^2 U)}{\partial y} - \frac{1}{r^2}\frac{\partial U}{\partial y}\right] \tag{2.26}$$

$$\frac{\nu W_0}{a^2}\left(U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y}\right) = \frac{G}{\rho} + \frac{\nu W_0}{a^2}\left(\nabla^2 W - \frac{W}{r^2}\right), \tag{2.27}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Dividing equation (2.25) by $\frac{\nu}{a^2}$, equation (2.26) by $\frac{\nu^2}{a^4}$, equation (2.27) by $\frac{\nu W_0}{a^2}$ and neglecting the term $\frac{W}{r^2}$, which does not affect the second-order approximation, yields:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{2.28}$$

$$\left(\frac{\partial}{\partial x} - \frac{1}{r}\right)\left(U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial y}\left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) + KW\frac{\partial W}{\partial y} \tag{2.29}$$

$$= \frac{\partial(\nabla^2 V)}{\partial x} - \frac{\partial(\nabla^2 U)}{\partial y} - \frac{1}{r^2}\frac{\partial U}{\partial y}$$

$$U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} = C + \nabla^2 W \quad , \tag{2.30}$$

where

$$K = \frac{2W_0^2 a^3}{R\nu^2} \tag{2.31}$$

and

$$C = \frac{Ga^2}{\mu W_0} \quad . \tag{2.32}$$

The stream function, $\psi(x,y)$, is defined such that equation (2.28) is satisfied and in Cartesian coordinates is:

$$U = \frac{\partial \psi}{\partial y} \qquad V = -\frac{\partial \psi}{\partial x} \quad . \tag{2.33}$$

Substituting equations (2.33) into equation (2.29) yields:

$$\left(\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\nabla^2\psi + \frac{1}{r}\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2\psi}{\partial x \partial y}\right] + KW\frac{\partial W}{\partial y} \tag{2.34}$$

$$= -\nabla^2(\nabla^2\psi) + \frac{1}{r}\frac{\partial^2\psi}{\partial y^2}$$

The terms $\frac{1}{r}\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2\psi}{\partial x \partial y}\right]$ and $\frac{1}{r}\frac{\partial^2\psi}{\partial y^2}$ are third-order terms and are negligible for a solution accurate to the second approximation. Therefore, equation (2.34) is expressed as:

$$\left(\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\nabla^2\psi + KW\frac{\partial W}{\partial y} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\nabla^2\psi \quad . \tag{2.35}$$

Substituting equations (2.33) into equation (2.30) yields:

$$\frac{\partial \psi}{\partial y}\frac{\partial W}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial W}{\partial y} = C + \nabla^2 W \quad . \tag{2.36}$$

The asymptotic expansions used to apply the method of successive approximations are as follows:

$$\psi = K\psi_1 + K^2\psi_2 + \ldots \tag{2.37}$$

$$W = W_0 + KW_1 + K^2W_2 + \ldots$$

Differentiating equations (2.37) and substituting into equation (2.35) yields:

$$\left[\left(K\frac{\partial\psi_1}{\partial x}+K^2\frac{\partial\psi_2}{\partial x}\right)\frac{\partial}{\partial y}-\left(K\frac{\partial\psi_1}{\partial y}+K^2\frac{\partial\psi_2}{\partial y}\right)\frac{\partial}{\partial x}\right]\left(K\frac{\partial^2\psi_1}{\partial x^2}+K^2\frac{\partial^2\psi_2}{\partial x^2}\right. \tag{2.37}$$

$$\left.+K\frac{\partial^2\psi_1}{\partial y^2}+K^2\frac{\partial^2\psi_2}{\partial y^2}\right)+K(W_0+KW_1+K^2W_2)\left(\frac{\partial W_0}{\partial y}\right.$$

$$\left.+K\frac{\partial W_1}{\partial y}+K^2\frac{\partial W_2}{\partial y}\right)=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(K\frac{\partial^2\psi_1}{\partial x^2}+K^2\frac{\partial^2\psi_2}{\partial x^2}\right.$$

$$\left.+K\frac{\partial^2\psi_1}{\partial y^2}+K^2\frac{\partial^2\psi_2}{\partial y^2}\right).$$

Equating the zero-order terms (involving K^0):

$$0 = 0 \quad .$$

Equating the first-order terms (involving K^1):

$$\nabla^4\psi_1 = -W_0\frac{\partial W_0}{\partial y}\quad, \tag{2.38}$$

where $\nabla^4\psi_1 = \nabla^2(\nabla^2\psi_1)$ by definition and

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\quad .$$

Equating the second-order terms (involving K^2):

$$\nabla^4\psi_2 = \frac{\partial\psi_1}{\partial y}\frac{\partial^3\psi_1}{\partial x^3}+\frac{\partial\psi_1}{\partial y}\frac{\partial^3\psi_1}{\partial x\partial y^2}-\frac{\partial\psi_1}{\partial x}\frac{\partial^3\psi_1}{\partial x^2\partial y}-\frac{\partial\psi_1}{\partial x}\frac{\partial^3\psi_1}{\partial y^3} \tag{2.39}$$

$$-W_0\frac{\partial W_1}{\partial y}-W_1\frac{\partial W_0}{\partial y}$$

Differentiating equations (2.37) and substituting into equation (2.36) yields:

$$\left(K\frac{\partial\psi_1}{\partial y} + K^2\frac{\partial\psi_2}{\partial y}\right)\left(\frac{\partial W_0}{\partial x} + K\frac{\partial W_1}{\partial x} + K^2\frac{\partial W_2}{\partial x}\right)$$

$$- \left(K\frac{\partial\psi_1}{\partial x} + K^2\frac{\partial\psi_2}{\partial x}\right)\left(\frac{\partial W_0}{\partial y} + K\frac{\partial W_1}{\partial y} + K^2\frac{\partial W_2}{\partial y}\right)$$

$$= C + \left(\frac{\partial^2 W_0}{\partial x^2} + K\frac{\partial^2 W_1}{\partial x^2} + K^2\frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_0}{\partial y^2}\right.$$

$$\left. + K\frac{\partial^2 W_1}{\partial y^2} + K^2\frac{\partial^2 W_2}{\partial y^2}\right) .$$

Equating the zero-order terms (involving K^0):

$$\nabla^2 W_0 = -C \quad . \tag{2.40}$$

Equating the first-order terms (involving K^1):

$$\nabla^2 W_1 = \frac{\partial\psi_1}{\partial y}\frac{\partial W_0}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial W_0}{\partial y} \quad . \tag{2.41}$$

Equating the second-order terms (involving K^2):

$$\nabla^2 W_2 = \frac{\partial\psi_2}{\partial y}\frac{\partial W_0}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial W_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial W_1}{\partial y} - \frac{\partial\psi_2}{\partial x}\frac{\partial W_0}{\partial y} \quad . \tag{2.42}$$

Terms involving K to powers higher than two are negligible because the equations are only accurate to the second-order approximation.

Equations (2.38) through (2.42) are in a solvable form and are the governing partial differential equations. The boundary conditions for equations (2.38) and (2.39) are:

$$\psi_1 = \psi_2 = 0 \quad \text{when } 1 - x^2 - m^2y^2 = 0 \tag{2.43}$$

and

$$\frac{\partial\psi_1}{\partial x} = \frac{\partial\psi_2}{\partial x} = \frac{\partial\psi_1}{\partial y} = \frac{\partial\psi_2}{\partial y} = 0 \quad , \quad \text{when } 1 - x^2 - m^2y^2 = 0 \quad , \qquad (2.43)$$

where m is a positive constant related to the eccentricity of the ellipse, e, by the equations

$$e = \frac{1}{m}\sqrt{m^2 - 1} \quad , \qquad m \geq 1$$

and

$$e = \frac{1}{m}\sqrt{1 - m^2} \quad , \qquad 0 < m \leq 1 \quad .$$

The boundary conditions for equations (2.40) through (2.42) are:

$$W_0 = W_1 = W_2 = 0 \quad , \qquad \text{when } 1 - x^2 - m^2y^2 = 0 \quad . \qquad (2.44)$$

Equation (2.40), $\nabla^2 W_0 = -C$, is the governing partial differential equation for flow through a straight pipe. This is the same equation presented by Dean [1928]. Therefore, the boundary value problem for flow through a straight pipe with a circular cross section differs with that for a straight pipe with an elliptical cross section only in the boundary conditions.

CHAPTER III

SOLUTIONS TO EQUATIONS

The nature of differential equations (2.38) through (2.42) and the geometry of the cross section of the pipe suggest that the solutions for W_0, W_1, W_2, ψ_1, and ψ_2 may be assumed to be polynomials in x and y. For convenience, the solutions are assumed in a form so that the symmetry requirements and boundary conditions are satisfied in advance. The boundary conditions on ψ_1 and ψ_2 are given by equation (2.43), and the condition on W_0, W_1, and W_2 is that each vanishes on the boundary as required by equation (2.44). Inspection of the governing differential equations shows that W_0 and W_2 are symmetric in both x and y; W_1 is symmetric in y but antisymmetric with respect to x; ψ_1 is symmetric in x but antisymmetric in y; and ψ_2 is antisymmetric with respect to both x and y. Assumed solutions which satisfy these requirements are:

$$W_0 = A(1 - x^2 - m^2y^2) \tag{3.1}$$

$$\psi_1 = (1 - x^2 - m^2y^2)^2 y (A_0 + A_1x^2 + A_2y^2) \tag{3.2}$$

$$\begin{aligned} W_1 = (1 - x^2 - m^2y^2)\, x\, (b_0 &+ b_1x^2 + b_2y^2 + b_3x^4 + b_4x^2y^2 + b_5y^4 \\ &+ b_6x^6 + b_7x^4y^2 + b_8x^2y^4 + b_9y^6) \end{aligned} \tag{3.3}$$

$$\begin{aligned} \psi_2 = (1 - x^2 - m^2y^2)^2\, x\, y\, (C_0 &+ C_1x^2 + C_2y^2 + C_3x^4 + C_4x^2y^2 \\ &+ C_5y^4 + C_6x^6 + C_7x^4y^2 + C_8x^2y^4 + C_9y^6 + C_{10}x^8 \\ &+ C_{11}x^6y^2 + C_{12}x^4y^4 + C_{13}x^2y^6 + C_{14}y^8) \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
W_2 = (1 - x^2 - m^2y^2)\,(d_0 &+ d_1x^2 + d_2y^2 + d_3x^4 + d_4x^2y^2 + d_5y^4 \\
&+ d_6x^6 + d_7x^4y^2 + d_8x^2y^4 + d_9y^6 + d_{10}x^8 + d_{11}x^6y^2 + d_{12}x^4y^4 \\
&+ d_{13}x^2y^6 + d_{14}y^8 + d_{15}x^{10} + d_{16}x^8y^2 + d_{17}x^6y^4 + d_{18}x^4y^6 \\
&+ d_{19}x^2y^8 + d_{20}y^{10} + d_{21}x^{12} + d_{22}x^{10}y^2 + d_{23}x^8y^4 + d_{24}x^6y^6 \\
&+ d_{25}x^4y^8 + d_{26}x^2y^{10} + d_{27}y^{12} + d_{28}x^{14} + d_{29}x^{12}y^2 + d_{30}x^{10}y^4 \\
&+ d_{31}x^8y^6 + d_{32}x^6y^8 + d_{33}x^4y^{10} + d_{34}x^2y^{12} + d_{35}y^{14})
\end{aligned}
\tag{3.5}
$$

The coefficients of the polynomial terms in these solutions will be found by substituting them into the governing differential equations and by matching the coefficients.

Partially differentiating equation (3.1), substituting into equation (2.40) and solving for A gives:

$$A = \frac{C}{2(m^2 - 1)} .$$

Partial differentiation of equations (3.1) through (3.5), substituting the expressions into each of equations (2.38) through (2.42) and equating the coefficients of like terms reduces the problem to the solution of four sets of simultaneous equations, each involving the constants in the corresponding assumed solution.

Partially differentiating equations (3.1) and (3.2), substituting into equation (2.38) and equating the coefficients of like terms yields:

$$(5m^4 + 2m^2 + 1)A_0 + (-2m^2 - 2)A_1 + (-10m^2 - 2)A_2 = \frac{A^2m^2}{12} \quad (3.7)$$

$$(10m^4 + 6m^2)A_1 + (105m^4 + 20m^2 + 3)A_2 = -\frac{A^2m^4}{4} \quad (3.8)$$

$$(5m^4 + 12m^2 + 15)A_1 + (10m^2 + 6)A_2 = -\frac{A^2m^2}{12} \quad (3.9)$$

Partially differentiating equations (3.1) through (3.3), substituting into equation (2.41) and equating the coefficients of like terms yields:

$$(m^2 + 3)b_0 - 3b_1 - b_2 = AA_0 \quad (3.10)$$

$$(m^2 + 10)b_1 - b_2 - 10b_3 - b_4 = AA_1 - 2AA_0 \quad (3.11)$$

$$3m^2b_1 + (6m^2 + 3)b_2 - 3b_4 - 6b_5 = 3AA_2 + 4AA_0m^2 - 6AA_0m^2 - 2AA_1m^2 \quad (3.12)$$

$$(m^2 + 21)b_3 + b_4 - 21b_6 - b_7 = AA_0 - 2AA_1 \quad (3.13)$$

$$5m^2b_3 - (3m^2 + 5)b_4 + 3b_5 - 5b_7 - 3b_8 = AA_0m^2 + AA_1m^2 - 3AA_2 \quad (3.14)$$

$$3m^2b_4 + (15m^2 + 3)b_5 - 3b_8 - 15b_9 = AA_0m^4 - 6AA_2m^2 - 4AA_1m^4 \quad (3.15)$$

$$(m^2 + 36)b_6 + b_7 = AA_1 \quad (3.16)$$

$$21m^2b_6 + (6m^2 + 21)b_7 + 6b_8 = 3AA_2 \quad (3.17)$$

$$10m^2b_7 + (15m^2 + 10)b_8 + 15b_9 = 6AA_2m^2 - 3AA_1m^4 \quad (3.18)$$

$$3m^2b_8 + (28m^2 + 3)b_9 = 3AA_2m^4 - 2AA_1m^6 \quad (3.19)$$

Partially differentiating equations (3.1) through (3.4), substituting into equation (2.39) and equating the coefficients of like terms yields:

$$(120m^4 + 144m^2 + 120)C_0 + (-144m^2 - 240)C_1 + (-240m^2 - 144)C_2 \tag{3.20}$$
$$+ 120C_3 + 72C_4 + 120C_5 = 8A_0^2 - 24A_0^2m^2 - 32A_0A_1$$
$$-4A_1^2 - 12A_1A_2 + 4Ab_0m^2 - 2Ab_2$$

$$(120m^4 + 480m^2 + 840)C_1 + (240m^2 + 240)C_2 + (-480m^2 - 1680)C_3 \tag{3.21}$$
$$+ (-240m^2 - 480)C_4 - 240C_5 + 840C_6 + 240C_7 + 120C_8$$
$$= 80A_0A_1 - 48A_0A_1m^2 + 16A_0^2 + 48A_0^2m^2 + 16A_1^2 + 48A_1A_2$$
$$- 2Ab_4 + 4Ab_2 + 4Ab_1m^2 - 4Ab_0m^2$$

$$(240m^4 + 240m^2)C_1 + (840m^4 + 480m^2 + 120)C_2 - 240m^2C_3 \tag{3.22}$$
$$+ (-480m^2 - 240)C_4 + (-1680m^2 - 480)C_5 + 120C_7$$
$$+ 240C_8 + 840C_9 = 240A_0A_1m^2 + 16A_0A_1m^4 + 32A_0A_2$$
$$- 256A_0A_2m^2 - 80A_0^2m^2 + 144A_0^2m^4 - 112A_1A_2$$
$$+ 192A_1A_2m^2 - 48A_2^2 + 32A_1^2m^2 - 4Ab_5 + 8Ab_2m^2 - 4Ab_0m^4$$

$$(120m^4 + 1008m^2 + 3024)C_3 + (240m^2 + 504)C_4 + 120C_5 \tag{3.23}$$
$$+ (-1008m^2 - 6048)C_6 + (-240m^2 - 1008)C_7 - 240C_8$$
$$+ 3024C_{10} + 504C_{11} + 120C_{12} = -24A_0^2$$
$$- 24A_0^2m^2 + 96A_0A_1m^2 - 48A_1^2 - 24A_1^2m^2 - 72A_1A_2$$
$$- 2Ab_7 + 4Ab_4 - 2Ab_2 + 4Ab_3m^2 - 4Ab_1m^2$$

$$(800m^4 + 1680)C_3 + (840m^4 + 1600m^2 + 840)C_4 + (-1680m^2 + 800)C_5 \tag{3.24}$$
$$- 1680m^2C_6 + (-1600m^2 - 1680)C_7 + (-1680m^2 - 1600)C_8$$
$$-1680C_9 + 840C_{11} + 800C_{12} + 840C_{13} = -640A_0A_1m^2$$
$$+ 256A_0A_1m^4 + 64A_0A_2 + 512A_0A_2m^2 + 400A_1A_2 - 832A_1A_2m^2$$
$$+ 48A_0^2m^2 - 144A_0^2m^4 - 16A_1^2m^2 + 16A_1^2m^4 - 4Ab_8 + 8Ab_5$$
$$+ 8Ab_4m^2 - 8Ab_2m^2 - 4Ab_1m^4$$

$$120m^4C_3 + (504m^4 + 240m^2)C_4 + (3024m^4 + 1008m^2 + 120)C_5 \tag{3.25}$$
$$- 240m^2C_7 + (-1008m^2 - 240)C_8 + (-6048m^2 - 1008)C_9$$
$$+ 120C_{12} + 504C_{13} + 3024C_{14} = 72A_0^2m^4 - 120A_0^2m^6$$
$$- 384A_0A_1m^4 - 96A_0A_1m^6 - 192A_0A_2m^2 + 768A_0A_2m^4$$
$$+ 480A_1A_2m^2 - 552A_1A_2m^4 - 72A_1^2m^4 + 24A_2^2 - 24A_2^2m^2$$
$$- 6Ab_9 \quad 12Ab_5m^2 - 6Ab_2m^4$$

$$(120m^4 + 1728m^2 + 7920)C_6 + (240m^2 + 864)C_7 + 120C_8 \tag{3.26}$$
$$+ (-1728m^2 - 15,840)C_{10} + (-240m^2 - 1728)C_{11}$$
$$- 240C_{12} = - 48A_0A_1 - 48A_0A_1m^2 + 96A_1^2 + 48A_1^2m^2$$
$$+ 48A_1A_2 + 4Ab_7 - 2Ab_4 + 4Ab_6m^2 - 4Ab_3m^3$$

$$(1680m^4 + 6048m^2)C_6 + (840m^4 + 3360m^2 + 3024)C_7 \tag{3.27}$$
$$+ (1680m^2 + 1680)C_8 + 840C_9 - 6048m^2C_{10}$$
$$+ (-3360m^2 - 6048)C_{11} + (-1680m^2 - 3360)C_{12}$$
$$- 1680C_{13} = 336A_0A_1m^2 - 272A_0A_1m^4 - 96A_0A_2$$
$$- 256A_0A_2m^2 - 144A_1^2m^2 + 112A_1^2m^4 - 336A_1A_2$$
$$+ 1088A_1A_2m^2 - 144A_2^2 + 8Ab_7m^2 + 8Ab_8 - 4Ab_5$$
$$- 8Ab_4m^2 - 4Ab_3m^4$$

$$840m^4C_6 + (1680m^4 + 1680m^2)C_7 + (3024m^4 + 3360m^2 + 840)C_8$$
$$+ (6048m^2 + 1680)C_9 - 1680m^2C_{11} + (-3360m^2 - 1680)C_{12} \tag{3.28}$$
$$+ (-6048m^2 - 3360)C_{13} - 6048C_{14} = 560A_0A_1m^4$$
$$- 144A_0A_1m^6 + 64A_0A_2m^2 - 768A_0A_2m^4 - 1280A_1A_2m^2$$
$$+ 1872A_1A_2m^4 - 16A_1^2m^4 - 96A_1^2m^6 + 48A_2^2 + 48A_2^2m^2$$
$$+ 12Ab_9 + 12Ab_8m^2 - 12Ab_5m^2 - 6Ab_4m^4$$

$$120m^4C_7 + (864m^4 + 240m^2)C_8 + (7920m^4 + 1728m^2 + 120)C_9 \tag{3.29}$$

$$- 240m^2C_{12} + (-1728m^2 - 240)C_{13} + (-15,840m^2 - 1728)C_{14}$$

$$= 176A_0A_1m^6 + 80A_0A_1m^8 + 160A_0A_2m^4 - 512A_0A_2m^6$$

$$- 624A_1A_2m^4 + 512A_1A_2m^6 + 64A_1^2m^6 - 112A_2^2m^2 + 352A_2^2m^4$$

$$+ 16Ab_9m^2 - 8Ab_5m^4$$

$$(120m^4 + 2640m^2 + 17,160)C_{10} + (240m^2 + 1320)C_{11} \tag{3.30}$$

$$+ 120C_{12} = -60A_1^2 - 24A_1^2m^2 - 12A_1A_2 - 2Ab_7$$

$$- 4Ab_6m^2$$

$$(2880m^4 + 15,840m^2)C_{10} + (840m^4 + 5760m^2 + 7920)C_{11} \tag{3.31}$$

$$+ (1680m^2 + 2880)C_{12} + 840C_{13} = 96A_1^2m^2 - 128A_1^2m^4$$

$$+ 48A_1A_2 - 448A_1A_2m^2 + 48A_2^2 - 4Ab_8 - 8Ab_7m^2$$

$$- 4Ab_6m^4$$

$$3024m^4C_{10} + (3528m^4 + 6048m^2)C_{11} + (3024m^4 + 7056m^2 + 3024)C_{12}$$

$$+ (6048m^2 + 3528)C_{13} + 3024C_{14} = 192A_1^2m^4 - 24A_1^2m^6 \tag{3.32}$$

$$+ 672A_1A_2m^2 - 1320A_1A_2m^4 - 72A_2^2 - 24A_2^2m^2 - 6Ab_9$$

$$- 12Ab_8m^2 - 6Ab_7m^4$$

$$840m^4C_{11} + (2880m^4 + 1680m^2)C_{12} + (7920m^4 + 5760m^2 + 840)C_{13}$$

$$+ (15,840m^2 + 2880)C_{14} = 16A_1^2m^6 + 80A_1^2m^8 \tag{3.33}$$

$$+ 880A_1A_2m^4 - 1024A_1A_2m^6 + 16A_2^2m^2 - 352A_2^2m^4$$

$$- 16Ab_9m^2 - 8Ab_8m^4$$

$$120m^4C_{12} + (1320m^4 + 240m^2)C_{13} + (17,160m^4 + 2640m^2 + 120)C_{14}$$

$$= 256A_1A_2m^6 - 140A_1A_2m^8 + 88A_2^2m^4 - 280A_2^2m^6 \tag{3.34}$$

$$- 20A_1^2m^8 - 10Ab_9m^4$$

Partially differentiating equations (3.1) through (3.5), substituting into equation (2.42) and equating the coefficients of like terms yields:

$$(-2m^2 - 2)d_0 + 2d_1 + 2d_2 = A_0b_0 \tag{3.35}$$

$$(-2m^2 - 12)d_1 - 2d_2 + 12d_3 + 2d_4 = -2AC_0 + 3A_0b_1 - 5A_0b_0 + A_1b_0 \tag{3.36}$$

$$\begin{aligned}(-2m^2)d_1 + (-12m^2 - 2)d_2 + 2d_4 + 12d_5 &= 2AC_0m^2 - 7A_0b_0m^2 \\ &+ A_0b_2 + 3A_2b_0\end{aligned} \tag{3.37}$$

$$\begin{aligned}(-2m^2 - 30)d_3 - 2d_4 + 30d_6 + 2d_7 &= -2AC_1 + 4AC_0 + 5A_0b_3 \\ &- 11A_0b_1 + 7A_0b_0 + 3A_1b_1 - 5A_1b_0\end{aligned} \tag{3.38}$$

$$\begin{aligned}(-12m^2)d_3 + (-12m^2 - 12)d_4 - 12d_5 + 12d_7 + 12d_8 &= -6AC_2 \\ &+ 6AC_1m^2 + 3A_0b_4 - 5A_0b_2 - 21A_0b_1m^2 + 18A_0b_0m^2 \\ &- 3A_1b_2 - 3A_1b_0m^2 + 9A_2b_1 - 15A_2b_0 + 8A_0b_2\end{aligned} \tag{3.39}$$

$$\begin{aligned}(-2m^2)d_4 + (-30m^2 - 2)d_5 + 2d_8 + 30d_9 &= 2AC_2m^2 - 4AC_0m^4 \\ &+ A_0b_5 - 7A_0b_2m^2 + 11A_0b_0m^4 + 3A_2b_2 - 13A_2b_0m^2\end{aligned} \tag{3.40}$$

$$\begin{aligned}(-2m^2 - 56)d_6 - 2d_7 + 56d_{10} + 2d_{11} &= 4AC_1 - 2AC_3 - 2AC_0 \\ &+ 7A_0b_6 - 17A_0b_3 + 13A_0b_1 - 3A_0b_0 + 5A_1b_3 - 11A_1b_1 \\ &+ 7A_1b_0\end{aligned} \tag{3.41}$$

$$\begin{aligned}(-30m^2)d_6 + (-12m^2 - 30)d_7 - 12d_8 + 30d_{11} + 12d_{12} &= 12AC_2 - 6AC_4 \\ &- 8AC_1m^2 - 2AC_0m^2 + 10AC_3m^2 + 5A_0b_7 - 3A_0b_4 \\ &- 35A_0b_3m^2 - 9A_0b_2 - 11A_0b_0m^2 + 46A_0b_1m^2 - A_1b_4 \\ &+ 15A_1b_2 - 17A_1b_1m^2 + 10A_1b_0m^2 + 15A_2b_3 - 33A_2b_1 + 21A_2b_0\end{aligned} \tag{3.42}$$

$$(-12m^2)d_7 + (-30m^2 - 12)d_8 - 30d_9 + 12d_{12} + 30d_{13} = 8AC_2m^2 \tag{3.43}$$
$$- 10AC_5 + 6AC_4m^2 - 12AC_1m^4 + 2AC_0m^4 + 3A_0b_8$$
$$+ 11A_0b_5 - 21A_0b_4m^2 + 33A_0b_1m^4 - 13A_0b_0m^4 + 2A_0b_2m^2$$
$$- 7A_1b_5 + 9A_1b_2m^2 + 3A_1b_0m^4 + 9A_2b_4 - 7A_2b_2 - 39A_2b_1m^2$$
$$+ 38A_2b_0m^2$$

$$(-2m^2)d_8 + (-56m^2 - 2)d_9 + 2d_{13} + 56d_{14} = 2AC_5m^2 - 4AC_2m^4 \tag{3.44}$$
$$+ 2AC_0m^6 + A_0b_9 + 11A_0b_2m^4 - 5A_0b_0m^6 - 7A_0b_5m^2$$
$$+ 3A_2b_5 - 13A_2b_2m^2 + 17A_2b_0m^4$$

$$(-2m^2 - 90)d_{10} - 2d_{11} + 90d_{15} + 2d_{16} = 4AC_3 - 2AC_6 - 2AC_1 \tag{3.45}$$
$$- 23A_0b_6 + 19A_0b_3 - 5A_0b_1 + 7A_1b_6 - 17A_1b_3 + 13A_1b_1$$
$$- 3A_1b_0$$

$$(-56m^2)d_{10} + (-12m^2 - 56)d_{11} - 12d_{12} + 56d_{16} + 12d_{17} = 12AC_4 \tag{3.46}$$
$$- 6AC_7 - 16AC_3m^2 + 2AC_1m^2 - 6AC_2 + 14AC_6m^2$$
$$- 25A_0b_1m^2 - 9A_0b_7 - 49A_0b_6m^2 - 3A_0b_4 + 5A_0b_2$$
$$+ 74A_0b_3m^2 + A_1b_7 + 9A_1b_4 - 31A_1b_3m^2 - 21A_1b_2$$
$$- 7A_1b_0m^2 + 38A_1b_1m^2 + 21A_2b_6 - 51A_2b_3 + 39A_2b_1 - 9A_2b_0$$

$$(-30m^2)d_{11} + (-30m^2 - 30)d_{12} - 30d_{13} + 30d_{17} + 30d_{18} = 20AC_5 \tag{3.47}$$
$$- 10AC_8 - 10AC_2m^2 + 10AC_1m^4 + 10AC_7m^2$$
$$- 20AC_0m^4 + 5A_0b_8 - 35A_0b_7m^2 - 25A_0b_5 + 5A_0b_2m^2$$
$$+ 55A_0b_3m^4 - 35A_0b_1m^4 + 30A_0b_4m^2 - 5A_1b_8 + 35A_1b_5$$
$$- 5A_1b_4m^2 + 25A_1b_1m^4 - 5A_1b_0m^4 - 30A_1b_2m^2 + 15A_2b_7$$
$$- 25A_2b_4 - 65A_2b_3m^2 + 5A_2b_2 - 25A_2b_0m^2 + 90A_2b_1m^2$$

$$(-12m^2)d_{12} + (-56m^2 - 12)d_{13} - 56d_{14} + 12d_{18} + 56d_{19} = 16AC_5m^2$$
$$- 14AC_9 - 2AC_2m^4 + 6AC_8m^2 - 12AC_4m^4 + 6AC_1m^6 \qquad (3.48)$$
$$+ 19A_0b_9 - 21A_0b_8m^2 + 33A_0b_4m^4 - 5A_0b_2m^4 - 15A_0b_1m^6$$
$$- 14A_0b_5m^2 - 11A_1b_9 + 21A_1b_5m^2 - 9A_1b_2m^4 - A_1b_0m^6$$
$$+ 9A_2b_8 + A_2b_5 - 39A_2b_4m^2 + 51A_2b_1m^4 - 25A_2b_0m^4$$
$$+ 22A_2b_2m^2$$

$$(-2m^2)d_{13} + (-90m^2 - 2)d_{14} + 2d_{19} + 90d_{20} = 2AC_9m^2 - 4AC_5m^1$$
$$+ 2AC_2m^6 - 7A_0b_9m^2 + 11A_0b_5m^4 - 5A_0b_2m^6 + 3A_2b_9 \qquad (3.49)$$
$$- 13A_2b_5m^2 + 17A_2b_2m^4 - 7A_2b_0m^6$$

$$(-2m^2 - 132)d_{15} - 2d_{16} + 132d_{21} + 2d_{22} = 4AC_6 - 2AC_{10} \qquad (3.50)$$
$$- 2AC_3 + 25A_0b_6 - 7A_0b_3 - 23A_1b_6 + 19A_1b_3 - 5A_1b_1$$

$$(-90m^2)d_{15} + (-12m^2 - 90)d_{16} - 12d_{17} + 90d_{22} + 12d_{23} = 12AC_7$$
$$- 6AC_{11} - 24AC_6m^2 - 6AC_1 + 6AC_3m^2 + 18AC_{10}m^2 \qquad (3.51)$$
$$+ 3A_0b_7 - 3A_0b_1 - 39A_0b_3m^2 + 102A_0b_6m^2 + 3A_1b_7$$
$$- 45A_1b_6m^2 - 15A_1b_1 + 9A_1b_2 - 21A_1b_1m^2 + 66A_1b_3m^2$$
$$- 69A_2b_6 + 57A_2b_3 - 15A_2b_1$$

$$(-56m^2)d_{16} + (-30m^2 - 56)d_{17} - 30d_{18} + 56d_{23} + 30d_{21} \qquad (3.52)$$
$$= -10AC_{12} + 20AC_8 - 8AC_7m^2 - 10AC_5 - 6AC_4m^2$$
$$+ 18AC_3m^1 + 14AC_{11}m^2 - 28AC_6m^1 - 19A_0b_8$$
$$- 13A_0b_5 - 9A_0b_1m^2 + 77A_0b_6m^1 - 57A_0b_3m^1$$
$$+ 58A_0b_7m^2 + 29A_1b_8 - 19A_1b_7m^2 - 49A_1b_5 + 21A_1b_2m^2$$
$$+ 47A_1b_3m^1 - 27A_1b_1m^1 - 2A_1b_1m^2 - 43A_2b_7$$
$$- 91A_2b_6m^2 + 23A_2b_1 - A_2b_2 - 51A_2b_1m^2 + 142A_2b_3m^2$$

$$(-30m^2)\,d_{17} + (-56m^2 - 30)\,d_{18} - 56d_{23} + 30d_{24} + 56d_{25} \tag{3.53}$$

$$= -14AC_{13} + 28AC_9 + 8AC_8m^2 - 18AC_5m^2$$

$$+ 6AC_4m^4 + 10AC_{12}m^2 - 20AC_7m^4 + 10AC_3m^6$$

$$- 41A_0b_9 + 21A_0b_5m^2 + 55A_0b_7m^4 - 27A_0b_4m^4$$

$$- 25A_0b_3m^6 - 14A_0b_8m^2 + 55A_1b_9 - 7A_1b_8m^2$$

$$+ 13A_1b_4m^4 + 15A_1b_2m^4 - 11A_1b_1m^6 - 70A_1b_5m^2$$

$$- 17A_2b_8 - 65A_2b_7m^2 - 11A_2b_5 - 9A_2b_2m^2$$

$$+ 85A_2b_3m^4 - 57A_2b_1m^4 + 74A_2b_4m^2$$

$$(-12m^2)\,d_{18} + (-90m^2 - 12)\,d_{19} - 90d_{20} + 12d_{25} + 90d_{26} = -18AC_{14} \tag{3.54}$$

$$+ 24AC_9m^2 - 6AC_5m^4 + 6AC_{13}m^2 - 12AC_8m^4$$

$$+ 6AC_4m^6 + 33A_0b_8m^4 + 3A_0b_5m^4 - 15A_0b_4m^6$$

$$- 30A_0b_9m^2 + 33A_1b_9m^2 - 28A_1b_5m^4 - 3A_1b_2m^6$$

$$+ 9A_2b_9 - 39A_2b_8m^2 - 51A_2b_1m^4 - 15A_2b_2m^4$$

$$- 21A_2b_1m^6 + 6A_2b_5m^2$$

$$(-2m^2)\,d_{19} + (-132m^2 - 2)\,d_{20} + 2d_{26} + 132d_{27} = 2AC_{11}m^2 \tag{3.55}$$

$$- 4AC_9m^4 + 2AC_5m^6 + 11A_0b_9m^4 - 5A_0b_5m^6$$

$$- 13A_2b_9m^2 + 17A_2b_5m^4 - 7A_2b_2m^6$$

$$(-2m^2 - 182)\,d_{21} - 2d_{22} + 182d_{28} + 2d_{29} = 4AC_{10} - 2AC_6 \tag{3.56}$$

$$- 9A_0b_6 + 25A_1b_3 - 7A_1b_4$$

$$(-132m^2)\,d_{21} + (-12m^2 - 132)\,d_{22} - 12d_{23} + 132d_{29} + 1[illegible]d_{30} \tag{3.57}$$

$$= 12AC_{11} - 32AC_{10}m^2 [illegible] 6AC [illegible] 10AC [illegible] m^2 [illegible] A_0b_7$$

$$- 52A_0b_6m^2 - 9A_1b_7 [illegible] b_4m^2 + 91A_1b_6m^2$$

$$+ 75A_2b_6 - 21A_2b_7$$

$$(-90m^2)d_{22} + (-30m^2 - 90)d_{23} - 30d_{24} + 90d_{30} + 30d_{31} \tag{3.58}$$
$$= 20AC_{12} - 16AC_{11}m^2 - 10AC_8 - 2AC_7m^2$$
$$+ 26AC_6m^4 - 36AC_{10}m^4 + 11A_0b_8 - 23A_0b_7m^2$$
$$- 79A_0b_6m^4 - 43A_1b_8 + 21A_1b_5 + 7A_1b_4m^2$$
$$+ 69A_1b_6m^4 - 49A_1b_3m^4 + 26A_1b_7m^2 + 41A_2b_7$$
$$- 7A_2b_4 - 77A_2b_3m^2 + 194A_2b_6m^2$$

$$(-56m^2)d_{23} + (-56m^2 - 56)d_{24} - 56d_{25} + 56d_{31} + 56d_{32} \tag{3.59}$$
$$= 28AC_{15} - 14AC_9 - 14AC_8m^2 + 14AC_7m^4$$
$$- 28AC_{11}m^4 + 14AC_6m^6 + 21A_0b_9 + 7A_0b_8m^2$$
$$- 49A_0b_7m^4 - 35A_0b_6m^6 - 77A_1b_9 + 49A_1b_5m^2$$
$$+ 35A_1b_7m^4 - 7A_1b_4m^4 - 21A_1b_3m^6 - 42A_1b_8m^2$$
$$+ 7A_2b_8 + 7A_2b_5 - 35A_2b_4m^2 + 119A_2b_6m^4$$
$$- 91A_2b_3m^4 + 126A_2b_7m^2$$

$$(-30m^2)d_{24} + (-90m^2 - 30)d_{25} - 90d_{26} + 30d_{32} + 90d_{33} = 36AC_{14} \tag{3.60}$$
$$+ 16AC_{13}m^2 - 26AC_9m^2 + 2AC_8m^4 - 20AC_{12}m^4$$
$$+ 10AC_7m^6 + 37A_0b_9m^2 - 19A_0b_8m^4 - 25A_0b_7m^6 + A_1b_8m^4$$
$$+ 35A_1b_5m^4 - 7A_1b_4m^6 - 110A_1b_9m^2 - 27A_2b_9 + 7A_2b_5m^2$$
$$+ 85A_2b_7m^4 - 49A_2b_4m^4 - 35A_2b_3m^6 + 58A_2b_8m^2$$

$$(-12m^2)d_{25} + (-132m^2 - 12)d_{26} - 132d_{27} + 12d_{33} + 132d_{34} \tag{3.61}$$
$$= 32AC_{14}m^2 - 10AC_9m^4 - 12AC_{13}m^4 + 6AC_8m^6$$
$$+ 11A_0b_9m^4 - 15A_0b_8m^6 - 33A_1b_9m^4 + 7A_1b_5m^6$$
$$+ 51A_2b_8m^4 - 7A_2b_5m^4 - 21A_2b_4m^6 - 10A_2b_9m^2$$

$$(-2m^2)d_{26} + (-182m^2 - 2)d_{27} + 2d_{34} + 182d_{35} = 17A_2b_9m^4 - 5A_0b_9m^6 - 7A_2b_5m^6 + 2AC_9m^6 - 4AC_{14}m^4 \tag{3.62}$$

$$(-2m^2 - 240)d_{28} - 2d_{29} = -9A_1b_6 - 2AC_{10} \tag{3.63}$$

$$(-182m^2)d_{28} + (-12m^2 - 182)d_{29} - 12d_{30} = 14AC_{10}m^2 - 6AC_{11} + 5A_1b_7 - 49A_1b_6m^2 - 27A_2b_6 \tag{3.64}$$

$$(-132m^2)d_{29} + (-30m^2 - 132)d_{30} - 30d_{31} = 2AC_{11}m^2 - 10AC_{12} + 34AC_{10}m^4 + 19A_1b_8 - 7A_1b_7m^2 - 71A_1b_6m^4 - 13A_2b_7 - 103A_2b_6m^2 \tag{3.65}$$

$$(-90m^2)d_{30} + (-56m^2 - 90)d_{31} - 56d_{32} = -14AC_{13} - 10AC_{12}m^2 + 22AC_{11}m^4 + 18AC_{10}m^6 + 33A_1b_9 + 35A_1b_8m^2 + A_2b_8 - 29A_1b_7m^4 - 31A_1b_6m^6 - 61A_2b_7m^2 - 125A_2b_6m^4 \tag{3.66}$$

$$(-56m^2)d_{31} + (-90m^2 - 56)d_{32} - 90d_{33} = -18AC_{14} - 22AC_{13}m^2 + 10AC_{12}m^4 + 14AC_{11}m^6 + 77A_1b_9m^2 + 13A_1b_8m^4 - 17A_1b_7m^6 + 15A_2b_9 - 19A_2b_8m^2 - 83A_2b_7m^4 - 49A_2b_6m^6 \tag{3.67}$$

$$(-30m^2)d_{32} + (-132m^2 - 30)d_{33} - 132d_{34} = -34AC_{14}m^2 - 2AC_{13}m^4 + 10AC_{12}m^6 + 55A_1b_9m^4 - 3A_1b_8m^6 + 23A_2b_9m^2 - 41A_2b_8m^4 - 35A_2b_7m^6 \tag{3.68}$$

$$(-12m^2)d_{33} + (-182m^2 - 12)d_{34} - 182d_{35} = -14AC_{14}m^4 + 6AC_{13}m^6 + 11A_1b_9m^6 + A_2b_9m^4 - 21A_2b_8m^6 \tag{3.69}$$

$$(-2m^2)d_{34} + (-240m^2 - 2)d_{35} = 2AC_{14}m^6 - 7A_2b_9m^6 \tag{3.70}$$

CHAPTER IV

COMPUTER ANALYSIS AND NUMERICAL EXAMPLES

Equations (3.7) through (3.9) for the first-order approximation of ψ are written in matrix form and designated as the W-matrix with the following nonzero elements:

$$W_{1\,1} = 5m^4 + 2m^2 + 1 \qquad W_{1\,2} = -2m^2 - 2$$

$$W_{1\,3} = -10m^2 - 2 \qquad W_{1\,4} = \frac{A^2m^2}{12}$$

$$W_{2\,2} = 10m^4 + 6m^2 \qquad W_{2\,3} = 105m^4 + 20m^2 + 3$$

$$W_{2\,4} = -\frac{A^2m^4}{4}$$

$$W_{3\,2} = 5m^4 + 12m^2 + 15 \qquad W_{3\,3} = 10m^2 + 6$$

$$W_{3\,4} = -\frac{A^2m^2}{12}$$

$$\begin{bmatrix} W_{1,1} & W_{1,2} & W_{1,3} \\ W_{2,1} & W_{2,2} & W_{2,3} \\ W_{3,1} & W_{3,1} & W_{3,3} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} W_{1,4} \\ W_{2,4} \\ W_{3,4} \end{bmatrix}$$

W-Matrix

Equations (3.10) through (3.19) for the first-order approximation of W are written in matrix form and designated as the X-matrix with the following nonzero elements:

$X_{1\,1} = m^2 + 3$ $\qquad$ $X_{1\,2} = -3$

$X_{1\,3} = -1$ $\qquad$ $X_{1\,11} = AA_0$

$X_{2\,2} = m^2 + 10$ $\qquad$ $X_{2\,3} = 1$

$X_{2\,4} = -10$ $\qquad$ $X_{2\,5} = -1$

$X_{2\,11} = AA_1 - 2AA_0$

$X_{3\,2} = 3m^2$ $\qquad$ $X_{3\,3} = 6m^2 + 3$

$X_{3\,5} = -3$ $\qquad$ $X_{3\,6} = -6$

$X_{3\,11} = 3AA_2 - 2AA_0m^2 - 2AA_1m^2$

$X_{4\,4} = m^2 + 21$ $\qquad$ $X_{4\,5} = 1$

$X_{4\,7} = -21$ $\qquad$ $X_{4\,8} = -1$

$X_{4\,11} = AA_0 - 2AA_1$

$X_{5\,4} = 5m^2$ $\qquad$ $X_{5\,5} = 3m^2 + 5$

$X_{5\,6} = 3$ $\qquad$ $X_{5\,8} = -5$

$X_{5\,9} = -3$ $\qquad$ $X_{5\,11} = AA_0m^2 + AA_1m^2 - 3AA_2$

$X_{6\,5} = 3m^2$ $\qquad$ $X_{6\,6} = 15m^2 + 3$

$X_{6\,9} = -3$ $\qquad$ $X_{6\,10} = -15$

$X_{6\,11} = AA_0m^4 + 4AA_1m^4 - 6AA_2m^2$

$X_{7\,7} = m^2 + 36$ $\qquad$ $X_{7\,8} = 1$

$X_{7\,11} = AA_1$

$X_{8\,7} = 21m^2$ $\qquad$ $X_{8\,8} = 6m^2 + 21$

$X_{8\,9} = 6$ $\qquad$ $X_{8\,11} = 3AA_2$

$X_{9\ 8} = 10m^2$ $\qquad$ $X_{9\ 9} = 15m^2 + 10$

$X_{9\ 10} = 15$ $\qquad$ $X_{9\ 11} = 6AA_2m^2 - 3AA_1m^4$

$X_{10\ 9} = 3m^2$ $\qquad$ $X_{10\ 10} = 28m^2 + 3$

$X_{10\ 11} = 3AA_2m^4 - 2AA_1m^6$

$$
\begin{bmatrix}
X_{1,1} & X_{1,2} & X_{1,3} & \cdots & X_{1,10} \\
X_{2,1} & \cdot & \cdot & \cdots & \cdot \\
X_{3,1} & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
X_{10,1} & \cdot & \cdot & \cdots & X_{10,10}
\end{bmatrix}
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \end{bmatrix}
=
\begin{bmatrix} X_{1,11} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_{10,11} \end{bmatrix}
$$

X-Matrix

Equations (3.20) through (3.34) for the second-order approximation of ψ are written in matrix form and designated as the Y-matrix with the following nonzero elements:

$Y_{1\ 1} = 120m^4 + 144m^2 + 120$ $\qquad$ $Y_{1\ 2} = -144m^2 - 240$

$Y_{1\ 3} = -240m^2 - 144$ $\qquad$ $Y_{1\ 4} = 120$

$Y_{1\ 5} = 72$ $\qquad$ $Y_{1\ 6} = 120$

$$Y_{1\,16} = 8A_0^2 - 24A_0^2m^2 - 32A_0A_1 - 4A_1^2 - 12A_1A_2 + 4Ab_0m^2 - 2Ab_2$$

$$Y_{2\,2} = 120m^4 + 480m^2 + 840 \qquad Y_{2\,3} = 240m^2 + 240$$

$$Y_{2\,4} = -480m^2 - 1680 \qquad Y_{2\,5} = -240m^2 - 480$$

$$Y_{2\,6} = -240 \qquad Y_{2\,7} = 840$$

$$Y_{2\,8} = 240 \qquad Y_{2\,9} = 120$$

$$Y_{2\,16} = 80A_0A_1 - 48A_0A_1m^2 + 16A_0^2 + 48A_0^2m^2 + 16A_1^2 + 48A_1A_2$$

$$- 2Ab_4 + 4Ab_2 + 4Ab_1m^2 - 4Ab_0m^2$$

$$Y_{3\,2} = 240m^4 + 240m^2 \qquad Y_{3\,3} = 840m^4 + 480m^2 + 120$$

$$Y_{3\,4} = -240m^2 \qquad Y_{3\,5} = -480m^2 - 240$$

$$Y_{3\,6} = -1680m^2 - 480 \qquad Y_{3\,8} = 120$$

$$Y_{3\,9} = 240 \qquad Y_{3\,10} = 840$$

$$Y_{3\,16} = 240A_0A_1m^2 + 16A_0A_1m^4 + 32A_0A_2 - 256A_0A_2m^2$$

$$- 80A_0^2m^2 + 144A_0^2m^4 - 112A_1A_2 + 192A_1A_2m^2 - 48A_2^2$$

$$+ 32A_1^2m^2 - 4Ab_5 + 8Ab_2m^2 - 4Ab_0m^4$$

$$Y_{4\,4} = 120m^4 + 1008m^2 + 3024 \qquad Y_{4\,5} = 240m^2 + 504$$

$$Y_{4\,6} = 120 \qquad Y_{4\,7} = -1008m^2 - 6048$$

$$Y_{4\,8} = -240m^2 - 1008 \qquad Y_{4\,9} = -240$$

$$Y_{4\,11} = 3024 \qquad Y_{4\,12} = 504$$

$$Y_{4\,13} = 120$$

$$Y_{4\,16} = -24A_0^2 - 24A_0^2m^2 + 96A_0A_1m^2 - 48A_1^2 - 24A_1^2m^2 - 72A_1A_2$$

$$- 2Ab_7 + 4Ab_1 - 2Ab_2 + 4Ab_3m^2 - 4Ab_1m^2$$

$Y_{5\,4} = 800m^4 + 1680$

$Y_{5\,5} = 840m^4 + 1600m^2 + 840$

$Y_{5\,6} = 1680m^2 + 800$

$Y_{5\,7} = -1680m^2$

$Y_{5\,8} = -1600m^2 - 1680$

$Y_{5\,9} = -1680m^2 - 1600$

$Y_{5\,10} = -1680$

$Y_{5\,12} = 840$

$Y_{5\,13} = 800$

$Y_{5\,14} = 840$

$$Y_{5\,16} = 256A_0A_1m^4 - 640A_0A_1m^2 + 64A_0A_2 + 512A_0A_2m^2 + 400A_1A_2 - 832A_1A_2m^2 + 48A_0^2m^2 - 144A_0^2m^4 - 16A_1^2m^2 + 16A_1^2m^4 - 4Ab_8 + 8Ab_5 + 8Ab_4m^2 - 8Ab_2m^2 - 4Ab_1m^4$$

$Y_{6\,4} = 120m^4$

$Y_{6\,5} = 504m^4 + 240m^2$

$Y_{6\,6} = 3024m^4 + 1008m^2 + 120$

$Y_{6\,8} = -240m^2$

$Y_{6\,9} = -1008m^2 - 240$

$Y_{6\,10} = -6048m^2 - 1008$

$Y_{6\,13} = 120$

$Y_{6\,14} = 504$

$Y_{6\,15} = 3024$

$$Y_{6\,16} = 72A_0^2m^4 - 120A_0^2m^6 - 384A_0A_1m^4 - 96A_0A_1m^6 - 192A_0A_2m^2 + 768A_0A_2m^4 + 480A_1A_2m^2 - 552A_1A_2m^4 - 72A_1^2m^4 + 24A_2^2 - 24A_2^2m^2 - 6Ab_9 + 12Ab_5m^2 - 6Ab_2m^4$$

$Y_{7\,7} = 120m^4 + 1728m^2 + 7920$

$Y_{7\,8} = 240m^2 + 864$

$Y_{7\,9} = 120$

$Y_{7\,11} = -1728m^2 - 15,840$

$Y_{7\,12} = -240m^2 - 1728$

$Y_{7\,13} = -240$

$$Y_{7\,16} = -48A_0A_1 - 48A_0A_1m^2 + 96A_1^2 + 48A_1^2m^2 + 48A_1A_2 + 4Ab_7 - 2Ab_4 + 4Ab_6m^2 - 4Ab_3m^2$$

$Y_{8\ 7} = 1680m^4 + 6048m^2$

$Y_{8\ 8} = 840m^4 + 3360m^2 + 3024$

$Y_{8\ 9} = 1680m^2 + 1680$

$Y_{8\ 10} = 840$

$Y_{8\ 11} = -6048m^2$

$Y_{8\ 12} = -3360m^2 - 6048$

$Y_{8\ 13} = -1680m^2 - 3360$

$Y_{8\ 14} = -1680$

$$Y_{8\ 16} = 336A_0A_1m^2 - 272A_0A_1m^4 - 96A_0A_2 - 256A_0A_2m^2 - 144A_1^2m^2$$
$$+ 112A_1^2m^4 - 336A_1A_2 + 1088A_1A_2m^2 - 144A_2^2 + 8Ab_7m^2$$
$$+ 8Ab_8 - 4Ab_5 - 8Ab_4m^2 - 4Ab_3m^4$$

$Y_{9\ 7} = 840m^4$

$Y_{9\ 8} = 1680m^4 + 1680m^2$

$Y_{9\ 9} = 3024m^4 + 3360m^2 + 840$

$Y_{9\ 10} = 6048m^2 + 1680$

$Y_{9\ 12} = -1680m^2$

$Y_{9\ 13} = -3360m^2 - 1680$

$Y_{9\ 14} = -6048m^2 - 3360$

$Y_{9\ 15} = -6048$

$$Y_{9\ 16} = 560A_0A_1m^4 - 144A_0A_1m^6 + 64A_0A_2m^2 - 768A_0A_2m^4$$
$$- 1280A_1A_2m^2 + 1872A_1A_2m^4 - 16A_1^2m^4 - 96A_1^2m^6$$
$$+ 48A_2^2 + 48A_2^2m^2 + 12Ab_9 + 12Ab_8m^2 - 12Ab_5m^2$$
$$- 6Ab_4m^4$$

$Y_{10\ 8} = 120m^4$

$Y_{10\ 9} = 864m^4 + 240m^2$

$Y_{10\ 10} = 7920m^4 + 1728m^2 + 120$

$Y_{10\ 13} = -240m^2$

$Y_{10\ 11} = -1728m^2 - 240$

$Y_{10\ 15} = -15,840m^2 - 1728$

$$Y_{10\ 16} = 176A_0A_1m^6 + 80A_0A_1m^8 + 160A_0A_2m^4 - 512A_0A_2m^6$$
$$- 624A_1A_2m^4 + 512A_1A_2m^6 + 64A_1^2m^6 - 112A_2^2m^2$$
$$+ 352A_2^2m^4 + 16Ab_9m^2 - 8Ab_5m^4$$

$Y_{11\ 11} = 120m^4 + 2640m^2 + 17,160 \qquad Y_{11\ 12} = 240m^2 + 1320$

$Y_{11\ 13} = 120$

$Y_{11\ 16} = -60A_1^2 - 24A_1^2m^2 - 12A_1A_2 - 2Ab_7 - 4Ab_6m^2$

$Y_{12\ 11} = 2880m^4 + 15,840m^2 \qquad Y_{12\ 12} = 840m^4 + 5760m^2 + 7920$

$Y_{12\ 13} = 1680m^2 + 2880 \qquad Y_{12\ 14} = 840$

$Y_{12\ 16} = 96A_1^2m^2 - 128A_1^2m^4 + 48A_1A_2 - 448A_1A_2m^2 + 48A_2^2 - 4Ab_8$

$\quad - 8Ab_7 - 4Ab_6m^4$

$Y_{13\ 11} = 3024m^4 \qquad Y_{13\ 12} = 3528m^4 + 6048m^2$

$Y_{13\ 13} = 3024m^4 + 7056m^2 + 3024 \qquad Y_{13\ 14} = 6048m^2 + 3528$

$Y_{13\ 15} = 3024$

$Y_{13\ 16} = 192A_1^2m^4 - 24A_1^2m^6 + 672A_1A_2m^2 - 1320A_1A_2m^4 - 72A_2^2$

$\quad - 24A_2^2m^2 - 6Ab_9 - 12Ab_8m^2 - 6Ab_7m^4$

$Y_{14\ 12} = 840m^4 \qquad Y_{14\ 13} = 2880m^4 + 1680m^2$

$Y_{14\ 14} = 7920m^4 + 5760m^2 + 840 \qquad Y_{14\ 15} = 15,840m^2 + 2880$

$Y_{14\ 16} = 16A_1^2m^6 + 80A_1^2m^8 + 880A_1A_2m^4 - 1024A_1A_2m^6 + 16A_2^2m^2$

$\quad - 352A_2^2m^4 - 16Ab_9m^2 - 8Ab_8m^4$

$Y_{15\ 13} = 120m^4 \qquad Y_{15\ 14} = 1320m^4 + 240m^2$

$Y_{15\ 15} = 17,160m^4 + 2640m^2 + 120$

$Y_{15\ 16} = 256A_1A_2m^6 - 140A_1A_2m^8 + 88A_2^2m^4 - 280A_2^2m^6 - 20A_1^2m^8$

$\quad - 10Ab_9m^4$

$$
\begin{bmatrix}
Y_{1,1} & \cdots & \cdots & Y_{1,15} \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdot \\
Y_{15,1} & \cdots & \cdots & Y_{15,15}
\end{bmatrix}
\begin{bmatrix}
C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \\ C_{13} \\ C_{14}
\end{bmatrix}
=
\begin{bmatrix}
Y_{1,16} \\ Y_{2,16} \\ Y_{3,16} \\ Y_{4,16} \\ Y_{5,16} \\ Y_{6,16} \\ Y_{7,16} \\ Y_{8,16} \\ Y_{9,16} \\ Y_{10,16} \\ Y_{11,16} \\ Y_{12,16} \\ Y_{13,16} \\ Y_{14,16} \\ Y_{15,16}
\end{bmatrix}
$$

Y-Matrix

Equations (3.35) through (3.70) for the second-order approximation of W are written in matrix form and designated as the Z-matrix with the following nonzero elements:

$Z_{1\,1} = -2m^2 - 2$ $\qquad$ $Z_{1\,2} = 2$

$Z_{1\,3} = 2$ $\qquad$ $Z_{1\,37} = A_0 b_0$

$Z_{2\,2} = -2m^2 - 12$ $\qquad Z_{2\,3} = -2$

$Z_{2\,4} = 12$ $\qquad Z_{2\,5} = 2$

$Z_{2\,37} = -2AC_0 + 3A_0b_1 - 5A_0b_0 + A_1b_0$

$Z_{3\,2} = -2m^2$ $\qquad Z_{3\,3} = -12m^2 - 2$

$Z_{3\,5} = 2$ $\qquad Z_{3\,6} = 12$

$Z_{3\,37} = 2AC_0m^2 - 7A_0b_0m^2 + A_0b_2 + 3A_2b_0$

$Z_{4\,4} = -2m^2 - 30$ $\qquad Z_{4\,5} = -2$

$Z_{4\,7} = 30$ $\qquad Z_{4\,8} = 2$

$Z_{4\,37} = 4AC_0 - 2AC_1 + 5A_0b_3 - 11A_0b_1 + 7A_0b_0 + 3A_1b_1 - 5A_1b_0$

$Z_{5\,4} = -12m^2$ $\qquad Z_{5\,5} = -12m^2 - 12$

$Z_{5\,6} = -12$ $\qquad Z_{5\,8} = 12$

$Z_{5\,9} = 12$

$$Z_{5\,37} = 6AC_1m^2 + 3A_0b_4 - 6AC_2 - 5A_0b_2 - 21A_0b_1m^2 + 18A_0b_0m^2$$
$$- 3A_1b_2 - 3A_1b_0m^2 + 9A_2b_1 - 15A_2b_0 + 8A_0b_2$$

$Z_{6\,5} = -2m^2$ $\qquad Z_{6\,6} = -30m^2 - 2$

$Z_{6\,9} = 2$ $\qquad Z_{6\,10} = 30$

$$Z_{6\,37} = 2AC_2m^2 - 4AC_0m^4 + A_0b_5 - 7A_0b_2m^2 + 11A_0b_0m^4 + 3A_2b_2$$
$$- 13A_2b_0m^2$$

$Z_{7\,7} = -2m^2 - 56$ $\qquad Z_{7\,8} = -2$

$Z_{7\,11} = 56$ $\qquad Z_{7\,12} = 2$

$$Z_{7\,37} = 4AC_1 - 2AC_3 - 2AC_0 + 7A_0b_6 - 17A_0b_3 + 13A_0b_1 - 3A_0b_0$$
$$+ 5A_1B_3 - 11A_1b_1 + 7A_1b_0$$

$Z_{8\ 7} = -30m^2$ $\qquad$ $Z_{8\ 8} = -12m^2 - 30$

$Z_{8\ 9} = -12$ $\qquad$ $Z_{8\ 12} = 30$

$Z_{8\ 13} = 12$

$$Z_{8\ 37} = 12AC_2 - 6AC_4 - 8AC_1m^2 - 2AC_0m^2 + 10AC_3m^2 + 5A_0b_7$$
$$- 3A_0b_4 - 35A_0b_3m^2 - 9A_0b_2 - 11A_0b_0m^2 + 46A_0b_1m^2$$
$$- A_1b_4 + 15A_1b_2 - 17A_1b_1m^2 + 10A_1b_0m^2 + 15A_2b_3$$
$$- 33A_2b_1 + 21A_2b_0$$

$Z_{9\ 8} = -12m^2$ $\qquad$ $Z_{9\ 9} = -30m^2 - 12$

$Z_{9\ 10} = -30$ $\qquad$ $Z_{9\ 13} = 12$

$Z_{9\ 14} = 30$

$$Z_{9\ 37} = 8AC_2m^2 - 10AC_5 + 6AC_4m^2 - 12AC_1m^4 + 2AC_0m^4 + 3A_0b_8$$
$$+ 11A_0b_5 - 21A_0b_4m^2 + 33A_0b_1m^4 - 13A_0b_0m^4 + 2A_0b_2m^2$$
$$- 7\ A_1b_5 + 9A_1b_2m^2 + 3A_1b_0m^4 + 9A_2b_4 - 7A_2b_2 - 39A_2b_1m^2$$
$$+ 38A_2b_0m^2$$

$Z_{10\ 9} = -2m^2$ $\qquad$ $Z_{10\ 10} = -56m^2 - 2$

$Z_{10\ 14} = 2$ $\qquad$ $Z_{10\ 15} = 56$

$$Z_{10\ 37} = 2AC_5m^2 - 4AC_2m^4 + 2AC_0m^6 + A_0b_9 + 11A_0b_2m^4 - 5A_0b_0m^6$$
$$- 7A_0b_5m^2 + 3A_2b_5 - 13A_2b_2m^2 + 17A_2b_0m^4$$

$Z_{11\ 11} = -2m^2 - 90$ $\qquad$ $Z_{11\ 12} = -2$

$Z_{11\ 16} = 90$ $\qquad$ $Z_{11\ 17} = 2$

$$Z_{11\ 37} = 4AC_3 - 2AC_6 - 2AC_1 - 23A_0b_6 + 19A_0b_3 - 5A_0b_1 + 7A_1b_6$$
$$- 17A_1b_3 + 13A_1b_1 - 3A_1b_0$$

$Z_{12\ 11} = -56m^2$

$Z_{12\ 12} = -12m^2 - 56$

$Z_{12\ 13} = -12$

$Z_{12\ 17} = 56$

$Z_{12\ 18} = 12$

$$\begin{aligned} Z_{12\ 37} = {} & 12AC_4 - 6AC_7 - 16AC_3m^2 + 2AC_1m^2 - 6AC_2 + 14AC_6m^2 \\ & - 25A_0b_1m^2 - 9A_0b_7 - 49A_0b_6m^2 - 3A_0b_4 + 5A_0b_2 \\ & + 74A_0b_3m^2 + A_1b_7 + 9A_1b_4 - 31A_1b_3m^2 - 21A_1b_2 \\ & - 7A_1b_0m^2 + 38A_1b_1m^2 + 21A_2b_6 - 51A_2b_3 + 39A_2b_1 - 9A_2b_0 \end{aligned}$$

$Z_{13\ 12} = -30m^2$

$Z_{13\ 13} = -30m^2 - 30$

$Z_{13\ 14} = -30$

$Z_{13\ 18} = 30$

$Z_{13\ 19} = 30$

$$\begin{aligned} Z_{13\ 37} = {} & 20AC_5 - 10AC_8 - 10AC_2m^2 + 10AC_1m^4 + 10AC_7m^2 \\ & - 20AC_3m^4 + 5A_0b_8 - 35A_0b_7m^2 - 25A_0b_5 + 5A_0b_2m^2 \\ & + 55A_0b_3m^4 - 35A_0b_1m^4 + 30A_0b_4m^2 - 5A_1b_8 + 35A_1b_5 \\ & - 5A_1b_4m^2 + 25A_1b_1m^4 - 5A_1b_0m^4 - 30A_1b_2m^2 + 15A_2b_7 \\ & - 25A_2b_4 - 65A_2b_3m^2 + 5A_2b_2 - 25A_2b_0m^2 + 90A_2b_1m^2 \end{aligned}$$

$Z_{14\ 13} = -12m^2$

$Z_{14\ 14} = -56m^2 - 12$

$Z_{14\ 15} = -56$

$Z_{14\ 19} = 12$

$Z_{14\ 20} = 56$

$$\begin{aligned} Z_{14\ 37} = {} & 16AC_5m^2 - 14AC_9 - 2AC_2m^4 + 6AC_8m^2 - 12AC_4m^4 \\ & + 6AC_1m^6 + 19A_0b_9 - 21A_0b_8m^2 + 3A_0b_4m^4 - 5A_0b_2m^4 \\ & - 15A_0b_1m^6 - 14A_0b_5m^2 - 11A_1b_9 + 21A_1b_5m^2 - 9A_1b_2m^4 \\ & - A_1b_0m^6 + 9A_2b_8 + A_2b_5 - 39A_2b_4m^2 \quad 51A_2b_1m^4 \\ & - 23A_2b_0m^4 + 22A_2b_2m^2 \end{aligned}$$

$Z_{15\,14} = -2m^2$ $\quad$ $Z_{15\,15} = -90m^2 - 2$

$Z_{15\,20} = 2$ $\quad$ $Z_{15\,21} = 90$

$$Z_{15\,37} = 2AC_9m^2 - 4AC_5m^4 + 2AC_2m^6 - 7A_0b_9m^2 + 11A_0b_5m^4 - 5A_0b_2m^6 + 3A_2b_9 - 13A_2b_5m^2 + 17A_2b_2m^4 - 7A_2b_0m^6$$

$Z_{16\,16} = -2m^2 - 132$ $\quad$ $Z_{16\,17} = -2$

$Z_{16\,22} = 132$ $\quad$ $Z_{16\,23} = 2$

$$Z_{16\,37} = 4AC_6 - 2AC_{10} - 2AC_3 + 25A_0b_6 - 7A_0b_3 - 23A_1b_6 + 19A_1b_3 - 5A_1b_1$$

$Z_{17\,16} = -90m^2$ $\quad$ $Z_{17\,17} = -12m^2 - 90$

$Z_{17\,18} = -12$ $\quad$ $Z_{17\,23} = 90$

$Z_{17\,24} = 12$

$$Z_{17\,37} = 12AC_7 - 6AC_{11} - 24AC_6m^2 - 6AC_4 + 6AC_3m^2 + 18AC_{10}m^2 + 3A_0b_7 + 3A_0b_4 - 39A_0b_3m^2 + 102A_0b_6m^2 + 3A_1b_7 - 45A_1b_6m^2 - 15A_1b_4 + 9A_1b_2 - 21A_1b_1m^2 + 66A_1b_3m^2 - 69A_2b_6 + 57A_2b_3 - 15A_2b_1$$

$Z_{18\,17} = -56m^2$ $\quad$ $Z_{18\,18} = -30m^2 - 56$

$Z_{18\,19} = -30$ $\quad$ $Z_{18\,24} = 56$

$Z_{18\,25} = 30$

$$Z_{18\,37} = 20AC_8 - 10AC_{12} - 8AC_7m^2 - 10AC_5 - 6AC_4m^2 + 18AC_3m^4 + 14AC_{11}m^2 - 28AC_6m^4 - 19A_0b_8 + 13A_0b_5 - 9A_0b_4m^2 + 77A_0b_6m^4 - 57A_0b_3m^4 + 58A_0b_7m^2 + 29A_1b_8 - 19A_1b_7m^2 - 49A_1b_5 + 21A_1b_2m^2 + 47A_1b_3m^4 - 27A_1b_1m^4 - 2A_1b_4m^2$$

$$- 43A_2b_7 - 91A_2b_6m^2 + 23A_2b_4 - A_2b_2 - 51A_2b_1m^2 + 142A_2b_3m^2$$

$$Z_{19\ 18} = -30m^2 \qquad Z_{19\ 19} = -56m^2 - 30$$

$$Z_{19\ 20} = -56 \qquad Z_{19\ 25} = 30$$

$$Z_{19\ 26} = 56$$

$$\begin{aligned} Z_{19\ 37} = {} & 28AC_9 - 14AC_{13} + 8AC_8m^2 - 18AC_5m^2 + 6AC_4m^4 \\ & + 10AC_{12}m^2 - 20AC_7m^4 + 10AC_3m^6 - 41A_0b_9 + 21A_0b_5m^2 \\ & + 55A_0b_7m^4 - 27A_0b_4m^4 - 25A_0b_3m^6 + 14A_0b_8m^2 + 55A_1b_9 \\ & - 7A_1b_8m^2 + 13A_1b_4m^4 + 15A_1b_2m^4 - 11A_1b_1m^6 - 70A_1b_5m^2 \\ & - 17A_2b_8 - 65A_2b_7m^2 - 11A_2b_5 - 9A_2b_2m^2 + 85A_2b_3m^4 \\ & - 57A_2b_1m^4 - 74A_2b_4m^2 \end{aligned}$$

$$Z_{20\ 19} = -12m^2 \qquad Z_{20\ 20} = -90m^2 - 12$$

$$Z_{20\ 21} = -90 \qquad Z_{20\ 26} = 12$$

$$Z_{20\ 27} = 90$$

$$\begin{aligned} Z_{20\ 37} = {} & 24AC_9m^2 - 18AC_{14} - 6AC_5m^4 + 6AC_{13}m^2 - 12AC_8m^4 \\ & + 6AC_4m^6 + 33A_0b_8m^4 + 3A_0b_5m^4 - 15A_0b_4m^6 - 30A_0b_9m^2 \\ & + 33A_1b_9m^2 + 28A_1b_5m^4 + 3A_1b_2m^6 + 9A_2b_9 - 39A_2b_8m^2 \\ & + 51A_2b_4m^4 - 15A_2b_2m^4 - 21A_2b_1m^6 + 6A_2b_5m^2 \end{aligned}$$

$$Z_{21\ 20} = -2m^2 \qquad Z_{21\ 21} = -132m^2 - 2$$

$$Z_{21\ 27} = 2 \qquad Z_{21\ 28} = 132$$

$$\begin{aligned} Z_{21\ 37} = {} & 2AC_{14}m^2 - 4AC_9m^4 + 2AC_5m^6 + 11A_0b_9m^4 - 5A_0b_5m^6 \\ & - 13A_2b_9m^2 + 17A_2b_5m^4 - 7A_2b_2m^6 \end{aligned}$$

$$Z_{22\ 22} = -2m^2 - 182 \qquad Z_{22\ 23} = -2$$

$Z_{22\ 29} = 182$ $\qquad$ $Z_{22\ 30} = 2$

$Z_{22\ 37} = 4AC_{10} - 2AC_6 - 9A_0b_6 + 25A_1b_6 - 7A_1b_3$

$Z_{23\ 22} = -132m^2$ $\qquad$ $Z_{23\ 23} = -12m^2 - 132$

$Z_{23\ 24} = -12$ $\qquad$ $Z_{23\ 30} = 132$

$Z_{23\ 31} = 12$

$$Z_{23\ 37} = 12AC_{11} - 32AC_{10}m^2 - 6AC_7 + 10AC_6m^2 + A_0b_7 - 53A_0b_6m^2$$
$$- 9A_1b_7 + 7A_1b_4 - 35A_1b_3m^2 + 94A_1b_6m^2 + 75A_2b_6 - 21A_2b_3$$

$Z_{24\ 23} = -90m^2$ $\qquad$ $Z_{24\ 24} = -30m^2 - 90$

$Z_{24\ 25} = -30$ $\qquad$ $Z_{24\ 31} = 90$

$Z_{24\ 32} = 30$

$$Z_{24\ 37} = 20AC_{12} - 16AC_{11}m^2 - 10AC_8 - 2AC_7m^2 + 26AC_6m^4$$
$$- 36AC_{10}m^4 + 11A_0b_8 - 23A_0b_7m^2 - 79A_0b_6m^4 - 43A_1b_8$$
$$+ 21A_1b_5 + 7A_1b_4m^2 + 69A_1b_6m^4 - 49A_1b_3m^4 + 26A_1b_7m^2$$
$$+ 41A_2b_7 - 7A_2b_4 - 77A_2b_3m^2 + 194A_2b_6m^2$$

$Z_{25\ 24} = -56m^2$ $\qquad$ $Z_{25\ 25} = -56m^2 - 56$

$Z_{25\ 26} = -56$ $\qquad$ $Z_{25\ 32} = 56$

$Z_{25\ 33} = 56$

$$Z_{25\ 37} = 28AC_{13} - 14AC_9 - 14AC_8m^2 + 14AC_7m^4 - 28AC_{11}m^4$$
$$+ 14AC_6m^6 + 21A_0b_9 + 7A_0b_8m^2 - 49A_0b_7m^4 - 35A_0b_6m^6$$
$$- 77A_1b_9 + 49A_1b_5m^2 + 35A_1b_7m^4 - 7A_1b_4m^4 - 21A_1b_3m^6$$
$$- 42A_1b_8m^2 + 7A_2b_8 + 7A_2b_5 - 35A_2b_4m^2 + 119A_2b_6m^4$$
$$- 91A_2b_3m^4 + 126A_2b_7m^2$$

$Z_{26\ 25} = -30m^2$ $\quad Z_{26\ 26} = -90m^2 - 30$

$Z_{26\ 27} = -90$ $\quad Z_{26\ 33} = 30$

$Z_{26\ 34} = 90$

$$Z_{26\ 37} = 36AC_{14} + 16AC_{13}m^2 - 26AC_9m^2 + 2AC_8m^4 - 20AC_{12}m^4 + 10AC_7m^6 + 37A_0b_9m^2 - 19A_0b_8m^4 - 25A_0b_7m^6 + A_1b_8m^4 + 35A_1b_5m^4 - 7A_1b_4m^6 - 110A_1b_9m^2 - 27A_2b_9 + 7A_2b_5m^2 + 85A_2b_7m^4 - 49A_2b_4m^4 - 35A_2b_3m^6 + 58A_2b_8m^2$$

$Z_{27\ 26} = -12m^2$ $\quad Z_{27\ 27} = -132m^2 - 12$

$Z_{27\ 28} = -132$ $\quad Z_{27\ 34} = 12$

$Z_{27\ 35} = 132$

$$Z_{27\ 37} = 32AC_{14}m^2 - 10AC_9m^4 - 12AC_{13}m^4 + 6AC_8m^6 + 11A_0b_9m^4 - 15A_0b_8m^6 - 33A_1b_9m^4 + 7A_1b_5m^6 + 51A_2b_8m^4 - 7A_2b_5m^4 - 21A_2b_4m^6 - 10A_2b_9m^2$$

$Z_{28\ 27} = -2m^2$ $\quad Z_{28\ 28} = -182m^2 - 2$

$Z_{28\ 35} = 2$ $\quad Z_{28\ 36} = 182$

$$Z_{28\ 37} = 17A_2b_9m^4 - 5A_0b_8m^6 - 7A_2b_5m^6 + 2AC_9m^6 - 4AC_{14}m^4$$

$Z_{29\ 29} = -2m^2 - 240$ $\quad Z_{29\ 30} = -2$

$Z_{29\ 37} = -9A_1b_6 - 2AC_{10}$

$Z_{30\ 29} = -182m^2$ $\quad Z_{30\ 30} = -12m^2 - 182$

$Z_{30\ 31} = -12$

$$Z_{30\ 37} = 14AC_{10}m^2 - 6AC_{11} + 5A_1b_7 - 49A_1b_6m^2 - 27A_2b_6$$

$Z_{31\ 30} = -132m^2$ $\qquad$ $Z_{31\ 31} = -30m^2 - 132$

$Z_{31\ 32} = -30$

$$Z_{31\ 37} = 2AC_{11}m^2 - 10AC_{12} + 34AC_{10}m^4 + 19A_1b_8 - 7A_1b_7m^2$$
$$- 71A_1b_6m^4 - 13A_2b_7 - 103A_2b_6m^2$$

$Z_{32\ 31} = -90m^2$ $\qquad$ $Z_{32\ 32} = -56m^2 - 90$

$Z_{32\ 33} = -56$

$$Z_{32\ 37} = 22AC_{11}m^4 - 14AC_{13} - 10AC_{12}m^2 + 18AC_{10}m^6 + 33A_1b_9$$
$$+ 35A_1b_8m^2 + A_2b_8 - 29A_1b_7m^4 - 31A_1b_6m^6 - 61A_2b_7m^2$$
$$- 125A_2b_6m^4$$

$Z_{33\ 32} = -56m^2$ $\qquad$ $Z_{33\ 33} = -90m^2 - 56$

$Z_{33\ 34} = -90$

$$Z_{33\ 37} = 10AC_{12}m^4 - 18AC_{14} - 22AC_{13}m^2 + 14AC_{11}m^6 + 77A_1b_9m^2$$
$$+ 13A_1b_8m^4 - 17A_1b_7m^6 + 1[illegible]A_2b_9 - 19A_2b_8m^2 - 83A_2b_7m^4$$
$$- 49A_2b_6m^6$$

$Z_{34\ 33} = -30m^2$ $\qquad$ $Z_{34\ 34} = -132m^2 - 30$

$Z_{34\ 35} = -132$

$$Z_{34\ 37} = 10AC_{12}m^6 - 34AC_{14}m^2 - 2AC_{13}m^4 + 55A_1b_9m^4 - 3A_1b_8m^6$$
$$+ 23A_2b_9m^2 - 41A_2b_8m^4 - 35A_2b_7m^6$$

$Z_{35\ 34} = -12m^2$ $\qquad$ $Z_{35\ 35} = -182m^2 - 12$

$Z_{35\ 36} = -182$

$$Z_{35\ 37} = 6AC_{13}m^6 - 14AC_{14}m^4 + 11A_1b_9m^6 + A_2b_8m^4 - 21A_2b_8m^6$$

$Z_{36\,35} = -2m^2$ $\qquad$ $Z_{36\,36} = -240m^2 - 2$

$Z_{36\,37} = 2AC_{14}m^6 - 7A_2b_9m^6$

All elements in the W-, X-, Y-, and Z-matrices which are not otherwise defined are zero.

The total volume rate of flow through the pipe, Q_T, is given by

$Q_T = W_0a^2(Q_0 + KQ_1 + K^2Q_2)$, where

$$Q_0 = 4 \int_{x=0}^{x=1} \int_{y=0}^{y=\frac{1}{m}\sqrt{1-x^2}} W_0 \, dy \, dx \qquad (4.1)$$

$$Q_1 = 4 \int_{x=0}^{x=1} \int_{y=0}^{y=\frac{1}{m}\sqrt{1-x^2}} W_1 \, dy \, dx \qquad (4.2)$$

$$Q_2 = 4 \int_{x=0}^{x=1} \int_{y=0}^{y=\frac{1}{m}\sqrt{1-x^2}} W_2 \, dy \, dx \quad . \qquad (4.3)$$

Since the equations are in nondimensional form, the limits in the x-direction are taken to be from $x = 0$ to $x = 1$ and the limits in the y-direction are from $y = 0$ to the boundary of the cross section.

From equation (3.3) it is noted that W_1 is a function odd in x and even in y integrated over an area symmetric about the origin. Therefore, $Q_1 = 0$ and the first approximation of W have no effect on the flow rate.

Since $Q_1 = 0$, the equation for the total rate of flow through the pipe is reduced to:

$$\begin{bmatrix} Z_{1,1} & \cdots & Z_{1,36} \\ \vdots & \ddots & \vdots \\ Z_{36,1} & \cdots & Z_{36,36} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{34} \\ d_{35} \end{bmatrix} = \begin{bmatrix} Z_{1,37} \\ \vdots \\ Z_{36,37} \end{bmatrix}$$

Z-Matrix

$$Q_T = W_0a^2[Q_0 - Q_2K^2] \quad .$$

The product $Q_0W_0a^2$ is the rate of flow through a straight pipe. Therefore, the total flow rate is expressed as:

$$Q_T = Q_0W_0a^2 \left[1 - \frac{Q_2}{Q_0} K^2\right] \quad . \tag{4.4}$$

The bracketed terms represent the reduction in flow rate due to the curvature of the pipe and are designated as:

$$\frac{F_c}{F_s} = \left[1 - \frac{Q_2}{Q_0} K^2\right] \quad , \tag{4.5}$$

where F_c/F_s is the ratio of the flux through a curved pipe to the flux through a straight pipe, both having the same cross section, length and inlet pressure.

Equations (3.1) and (3.5) are even in both x and y, and to simplify integration are written as follows:

$$W_0 = A - Ax^2 - Am^2y^2 \tag{4.6}$$

$$\begin{aligned} W_2 = {} & d_0 + (d_1 - d_0)x^2 + (d_2 - d_0m^2)y^2 + (d_3 - d_1)x^4 + (d_4 - d_2 - d_1m^2)x^2y^2 \\ & + (d_5 - d_2m^2)y^4 + (d_6 - d_3)x^6 + (d_7 - d_4 - d_3m^2)x^4y^2 \\ & + (d_8 - d_5 - d_4m^2)x^2y^4 + (d_9 - d_5m^2)y^6 + (d_{10} - d_6)x^8 \\ & + (d_{11} - d_7 - d_6m^2)x^6y^2 + (d_{12} - d_8 - d_7m^2)x^4y^4 + (d_{13} - d_9 - d_8m^2)x^2y^6 \\ & + (d_{14} - d_9m^2)y^8 + (d_{15} - d_{10})x^{10} + (d_{16} - d_{11} - d_{10}m^2)x^8y^2 \\ & + (d_{17} - d_{12} - d_{11}m^2)x^6y^4 + (d_{18} - d_{13} - d_{12}m^2)x^4y^6 \end{aligned} \tag{4.7}$$

$$+ (d_{19} - d_{14} - d_{13}m^2)x^2y^8 + (d_{20} - d_{14}m^2)y^{10} + (d_{21} - d_{15})x^{12}$$
$$+ (d_{22} - d_{16} - d_{15}m^2)x^{10}y^2 + (d_{23} - d_{17} - d_{16}m^2)x^8y^4$$
$$+ (d_{24} - d_{18} - d_{17}m^2)x^6y^6 + (d_{25} - d_{19} - d_{18}m^2)x^4y^8$$
$$+ (d_{26} - d_{20} - d_{19}m^2)x^2y^{10} + (d_{27} - d_{20}m^2)y^{12} + (d_{28} - d_{21})x^{14}$$
$$+ (d_{29} - d_{22} - d_{21}m^2)x^{12}y^2 + (d_{30} - d_{23} - d_{22}m^2)x^{10}y^4$$
$$+ (d_{31} - d_{24} - d_{23}m^2)x^8y^6 + (d_{32} - d_{25} - d_{24}m^2)x^6y^8$$
$$+ (d_{33} - d_{26} - d_{25}m^2)x^4y^{10} + (d_{34} - d_{27} - d_{26}m^2)x^2y^{12} + (d_{35} - d_{27}m^2)y^{14}$$
$$+ (-d_{28}x^{16}) + (-d_{29} - d_{28}m^2)x^{14}y^2 + (-d_{30} - d_{29}m^2)x^{12}y^4$$
$$+ (-d_{31} - d_{30}m^2)x^{10}y^6 + (-d_{32} - d_{31}m^2)x^8y^8 + (-d_{33} - d_{32}m^2)x^6y^{10}$$
$$+ (-d_{34} - d_{33}m^2)x^4y^{12} + (-d_{35} - d_{34}m^2)x^2y^{14} + (-d_{35}m^2)y^{16}$$

The integrated portion of each term of equations (4.6) and (4.7) is represented by:

$$4 \int_{x=0}^{x=1} \int_{y=0}^{y=\frac{1}{m}\sqrt{1-x^2}} x^i y^j \, dy \, dx \quad ,$$

where the proper i and j are taken from the term being integrated. Integration with respect to y simplifies this expression to:

$$\frac{4}{(j+1)m^{j+1}} \int_0^1 x^i (1 - x^2)^{\frac{j+1}{2}} dx \quad .$$

The evaluation of Q_0 and Q_2 is reduced to the evaluation of this expression for each term in the equations, multiplying it by the corresponding constant and summing the values obtained for each term in the equation.

A computer program was written for the preceding W-, X-, Y-, and Z-matrices and the double integration of the solutions. The following 20 cases are presented:

<u>Case 1</u> $m = 0.1$ $C = 2.02$

$$\frac{F_c}{F_s} = 1 - 0.785935 \times 10^{-9} K^2$$

$$Q_T = \frac{\pi W_0 a^2}{0.2} [1 - 0.785935 \times 10^{-9} K^2]$$

<u>Case 2</u> $m = 0.2$ $C = 2.08$

$$\frac{F_c}{F_s} = 1 - 0.140091 \times 10^{-7} K^2$$

$$Q_T = \frac{\pi W_0 a^2}{0.4} [1 - 0.140091 \times 10^{-7} K^2]$$

<u>Case 3</u> $m = 0.3$ $C = 2.18$

$$\frac{F_c}{F_s} = 1 - 0.623529 \times 10^{-7} K^2$$

$$Q_T = \frac{\pi W_0 a^2}{0.6} [1 - 0.623529 \times 10^{-7} K^2]$$

<u>Case 4</u> $m = 0.4$ $C = 2.32$

$$\frac{F_c}{F_s} = 1 - 0.136545 \times 10^{-6} K^2$$

$$Q_T = \frac{\pi W_0 a^2}{0.8} [1 - 0.136545 \times 10^{-6} K^2]$$

Case 5 m = 0.5 C = 2.50

$$\frac{F_c}{F_s} = 1 - 0.194319 \times 10^{-6}K^2$$

$$Q_T = \pi W_0 a^2 [1 - 0.194319 \times 10^{-6}K^2]$$

Case 6 m = 0.6 C = 2.72

$$\frac{F_c}{F_s} = 1 - 0.209387 \times 10^{-6}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{1.2} [1 - 0.209387 \times 10^{-6}K^2]$$

Case 7 m = 0.7 C = 2.98

$$\frac{F_c}{F_s} = 1 - 0.188178 \times 10^{-6}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{1.4} [1 - 0.188178 \times 10^{-6}K^2$$

Case 8 m = 0.8 C = 3.28

$$\frac{F_c}{F_s} = 1 - 0.150800 \times 10^{-6}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{1.6} [1 - 0.150800 \times 10^{-6}K^2]$$

Case 9 m = 0.9 C = 3.62

$$\frac{F_c}{F_s} = 1 - 0.112857 \times 10^{-6}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{1.8} [1 - 0.112857 \times 10^{-6}K^2]$$

Case 10 m = 1.0 C = 4.00

$$\frac{F_c}{F_s} = 1 - 0.813396 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{2} [1 - 0.813396 \times 10^{-7}K^2]$$

Case 11 m = 1.1 C = 4.42

$$\frac{F_c}{F_s} = 1 - 0.575702 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{2.2} [1 - 0.575702 \times 10^{-7}K^2]$$

Case 12 m = 1.2 C = 4.88

$$\frac{F_c}{F_s} = 1 - 0.404917 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{2.4} [1 - 0.404917 \times 10^{-7}K^2]$$

Case 13 m = 1.3 C = 5.38

$$\frac{F_c}{F_s} = 1 - 0.284980 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{2.6} [1 - 0.284980 \times 10^{-7}K^2]$$

Case 14 m = 1.4 C = 5.92

$$\frac{F_c}{F_s} = 1 - 0.201481 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{2.8} [1 - 0.201481 \times 10^{-7}K^2]$$

Case 15 $m = 1.5$ $C = 6.50$

$$\frac{F_c}{F_s} = 1 - 0.143396 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{3} [1 - 0.143396 \times 10^{-7}K^2]$$

Case 16 $m = 1.6$ $C = 7.12$

$$\frac{F_c}{F_s} = 1 - 0.102846 \times 10^{-7}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{3.2} [1 - 0.102846 \times 10^{-7}K^2]$$

Case 17 $m = 1.7$ $C = 7.78$

$$\frac{F_c}{F_s} = 1 - 0.743684 \times 10^{-8}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{3.4} [1 - 0.743684 \times 10^{-8}K^2]$$

Case 18 $m = 1.8$ $C = 8.48$

$$\frac{F_c}{F_s} = 1 - 0.542271 \times 10^{-8}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{3.6} [1 - 0.542271 \times 10^{-8}K^2]$$

Case 19 $m = 1.9$ $C = 9.22$

$$\frac{F_c}{F_s} = 1 - 0.398714 \times 10^{-8}K^2$$

$$Q_T = \frac{\pi W_0 a^2}{3.8} [1 - 0.398714 \times 10^{-8}K^2]$$

Case 20 m = 2.0 C = 10.00

$$\frac{F_c}{F_s} = 1 - 0.295576 \times 10^{-8} K^2$$

$$Q_T = \frac{\pi W_0 a^2}{4} [1 - 0.295576 \times 10^{-8} K^2]$$

S·'· tituting equation (4.6) into (4.1) and integrating yields:

$$Q_0 = \frac{\pi W_0 a^2 C}{4m(m^2 + 1)} , \qquad (4.8)$$

which is the rate of flow through a straight pipe with an elliptical cross section for any $C > 0$ and $m > 0$.

The computer program solves the problem for any $C > 0$, but for the examples presented $C = 2(m^2 + 1)$ by choice. If $m = 1$; $c = 4$, which is the value of C taken by Dean for his solution and equation (4.8) reduces to:

$$Q_0 = \frac{\pi W_0 a^2}{2m} \qquad (4.9)$$

CHAPTER V

DISCUSSION

After formulation of the present theory, a digital computer program was written to obtain the required solutions. The program is in Appendix A. To get the rate of flow through a pipe with a small curvature and an elliptical cross section requires the following input data: (1) the value of C for the fluid, (2) the value of m for the elliptical cross section, and (3) the value of K (Dean's number).

Twenty cases were considered for $C = 2(m^2 + 1)$, $m = 0.1, 0.2, 0.3, \ldots, 2.0$. The solutions were expressed in terms of K and the nondimensionalization constants W_0 and a. The results for these cases are presented in Chapter IV and Appendix B.

The graphs in this section are plotted for the cases $C = 2(m^2 + 1)$ and $m = 0.5, 1.0,$ and 1.5. The cross section of a pipe is shown in Figure 3 for each value of m.

A. Streamlines in the Cross Sectional Plane of the Pipe and the Vorticity Centers of the Secondary Flow

The first-order approximation of the streamlines, ψ_1 = constant, was plotted by Thomas and Walters [1964] for a pipe with an elliptical cross section. The second-order approximation of the streamlines, ψ_2 = constant, has not

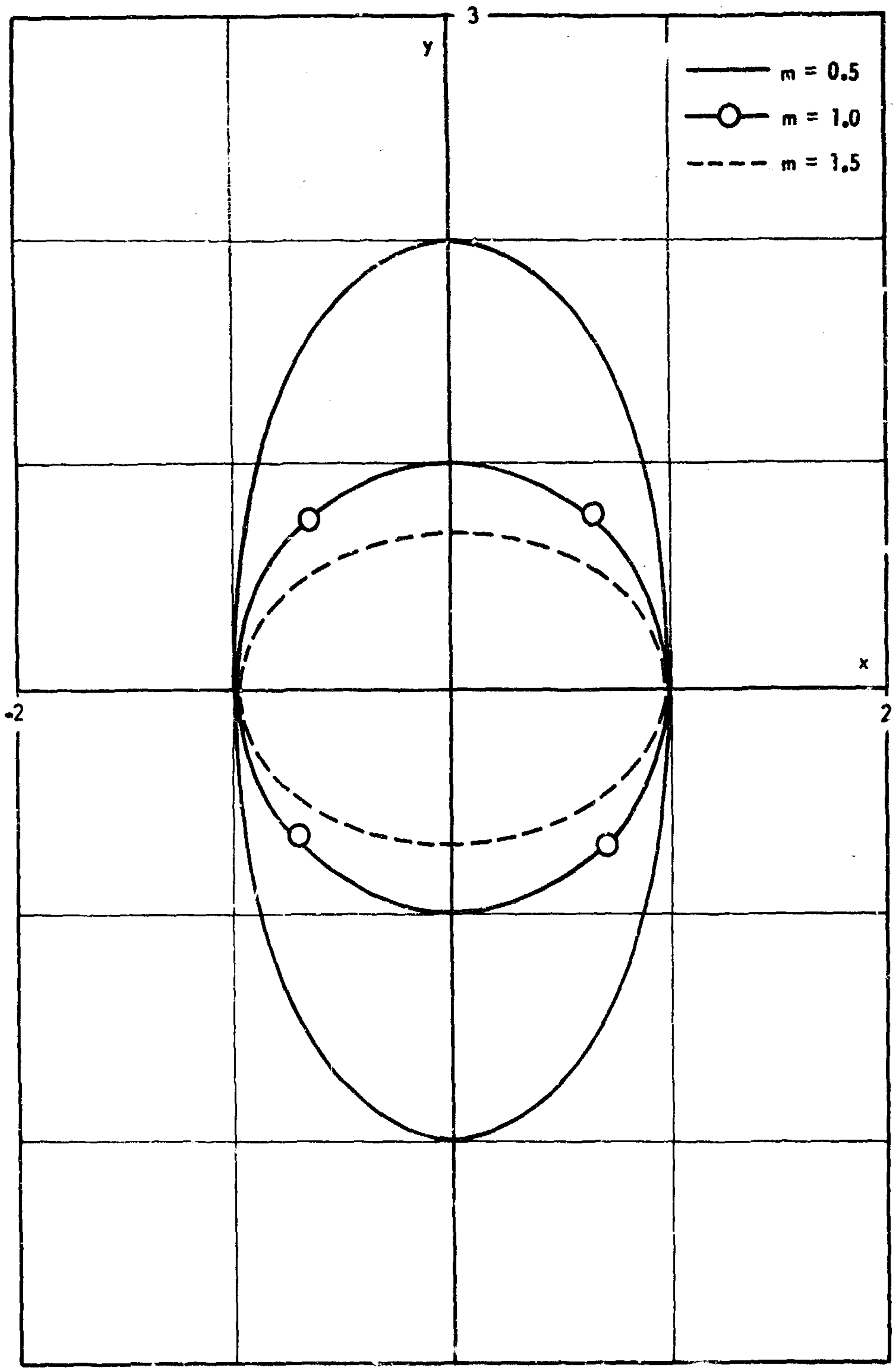

Figure 3. Cross Section of Pipe at m = 0.5, 1.0, 1.5

previously been plotted for a pipe with an elliptical cross section. The centers of secondary flow were presented by Dean [1927] for a pipe with a circular cross section. The following is an extension of the existing data and is in complete agreement with it.

1. First-Order Approximation of Streamlines on the Cross-Sectional Plane of the Pipe

To plot the streamlines, ψ_1 = constant, on a cross section of the pipe equation (3.2) is expressed as

$$\begin{aligned}[A_1y]x^6 &+ [(A_0 - 2A_1)y + (2A_1m^2 + A_2)y^3]x^4 \\ &+ [(A_1 - 2A_0)y + (2A_0m^2 - 2A_1m^2 - 2A_2)y^3 \\ &+ (A_1m^4 + 2A_2m^2)y^5]x^2 + [A_0y + (A_2 - 2A_0m^2)y^3 \\ &+ (A_0m^4 - 2A_2m^2)y^5 + A_2m^4y^7 - \psi_1] = 0 \quad .\end{aligned} \tag{5.1}$$

Constant values are taken for ψ_1 and y. This reduces the bracketed terms to constants resulting in a sixth-degree polynomial in x with constant coefficients. The data required for a plot of x versus y with ψ_1 = constant are calculated by subroutine PLOT of the digital computer program in Appendix A. These data are presented in Tables 1, 2, and 3 and plotted in Figures 4, 5, and 6. Figures 4, 5, and 6 show, as can be seen from equation (5.1), that the value of x is undetermined if y = 0. The streamlines, ψ_1 = constant, are symmetric with respect to the y-axis. Taking the negative value of the constant used to plot the streamlines in the first and second quadrants results in another curve in the third and fourth quadrants. The streamlines plotted for the two constants

Table 1. First-Order Approximation of Streamlines in the Cross Section of the Pipe at m = 0.5

y	x $\psi_1 = 5 \times 10^{-4}$		x $\psi_1 = 2 \times 10^{-3}$	
1.7	0.2037	-0.2037		
1.3	0.5673	-0.5673	0.2966	-0.2966
0.9	0.7129	-0.7129	0.4902	-0.4902
0.5	0.7551	-0.7551	0.4674	-0.4674
0.1	0.4538	-0.4538		
	$\psi_1 = -5 \times 10^{-4}$		$\psi_1 = -2 \times 10^{-3}$	
0				
-0.1	-0.4538	0.4538		
-0.5	-0.7551	0.7551	-0.4674	0.4674
-0.9	-0.7129	0.7129	-0.4902	0.4902
-1.3	-0.5673	0.5673	-0.2966	0.2966
-1.7	-0.2037	0.2037		

are symmetric about the x-axis and the origin. The x, y relation is dependent on the parameters C and m. In the cases plotted $C = 2(m^2 + 1)$ by choice. In loose terms, the streamlines may be thought of as projections of the paths of fluid elements on the cross section of the pipe. The secondary motion is caused by the pipe curvature and represents a loss of energy which retards the primary flow through the pipe for a given inlet pressure.

Table 2. First-Order Approximation of Streamlines in the Cross Section of the pipe at m = 1.0

y	x, $\psi_1 = 1 \times 10^{-4}$		x, $\psi_1 = 7 \times 10^{-4}$	
0.9	0.2175	-0.2175		
0.7	0.5905	-0.5905	0.3147	-0.3147
0.5	0.7483	-0.7483	0.5184	-0.5184
0.3	0.8165	-0.8165	0.5475	-0.5475
0.1	0.7610	-0.7610		
	$\psi_1 = -1 \times 10^{-4}$		$\psi_1 = -7 \times 10^{-4}$	
0				
-0.1	-0.7610	0.7610		
-0.3	-0.8165	0.8165	-0.5475	0.5475
-0.5	-0.7483	0.7483	-0.5184	0.5184
-0.7	-0.5905	0.5905	-0.3147	0.3147
-0.9	-0.2175	0.2175		

2. Second-Order Approximation of Streamlines on the Cross-Sectional Plane of the Pipe

To plot the streamlines (ψ_2 = constant) on a cross section of pipe, equation (3.4) is expressed as

$$[C_{10}y]x^{13} + [(-2C_{10} + C_6)y + (C_{11} + 2m^2C_{10})y^3]x^{11} + [(C_{10} - 2C_6 + C_3)y \tag{5.2}$$
$$+ (-2C_{11} - 2C_{10}m^2 + C_7 + 2m^2C_6)y^3 + (C_{12} + 2C_{11}m^2$$
$$+ m^4C_{10})y^5]x^9 + [(C_6 - 2C_3 + C_1)y + (C_{11} - 2C_7 - 2C_6m^2 + C_4$$

Table 3. First-Order Approximation of Streamlines in the Cross Section of the Pipe at m = 1.5

y	x $\psi_1 = 5 \times 10^{-5}$		x $\psi_1 = 3 \times 10^{-4}$	
0.5			0.1648	-0.1648
0.4	0.2394	-0.2394	0.4171	-0.4171
0.3	0.3538	-0.3538	0.5073	-0.5073
0.2	0.3195	-0.3195	0.5077	-0.5077
0.1			0.3158	-0.3158
	$\psi_1 = -5 \times 10^{-5}$		$\psi_1 = -3 \times 10^{-4}$	
0				
-0.1			-0.3158	0.3158
-0.2	-0.3195	0.3195	-0.5077	0.5077
-0.3	-0.3538	0.3538	-0.5073	0.5073
-0.4	-0.2394	0.2394	-0.4171	0.4171
-0.5			-0.1648	0.1648

$$+ 2m^2C_3)y^3 + (-2C_{12} - 2C_{11}m^2 + C_8 + 2m^2C_7 + m^4C_6)y^5 + (C_{13}$$

$$+ 2m^2C_{12} + m^4C_{11})y^7]x^7 + [(C_3 - 2C_1 + C_0)y + (C_7 - 2C_4$$

$$- 2C_3m^2 + C_{[illegible]} + 2m^2C_1)y^3 + (C_{12} - 2C_8 - 2C_7m^2 + C_{[illegible]} + 2m^2C_4$$

$$+ m^4C_3)y^5 + (-2C_{13} - 2C_{12}m^2 + C_9 + 2m^2C_8 + m^4C_7)y^7 + (C_{14}$$

$$+ 2m^2C_{13} + m^4C_{12})y^9]x^5 + [(C_1 - 2C_0)y + (C_4 - 2C_2 - 2[illegible]m^2$$

$$+ 2m^2C_0)y^3 + (C_8 - 2C_5 - 2C_4m^2 + 2m^2C_2 + m^4C_1)y^5 + ([illegible]_{13} - 2C_9$$

$$- 2C_8m^2 + 2m^2C_5 + m^4C_4)y^7 + (-2C_{11} - 2C_{13}m^2 + 2[illegible]$$

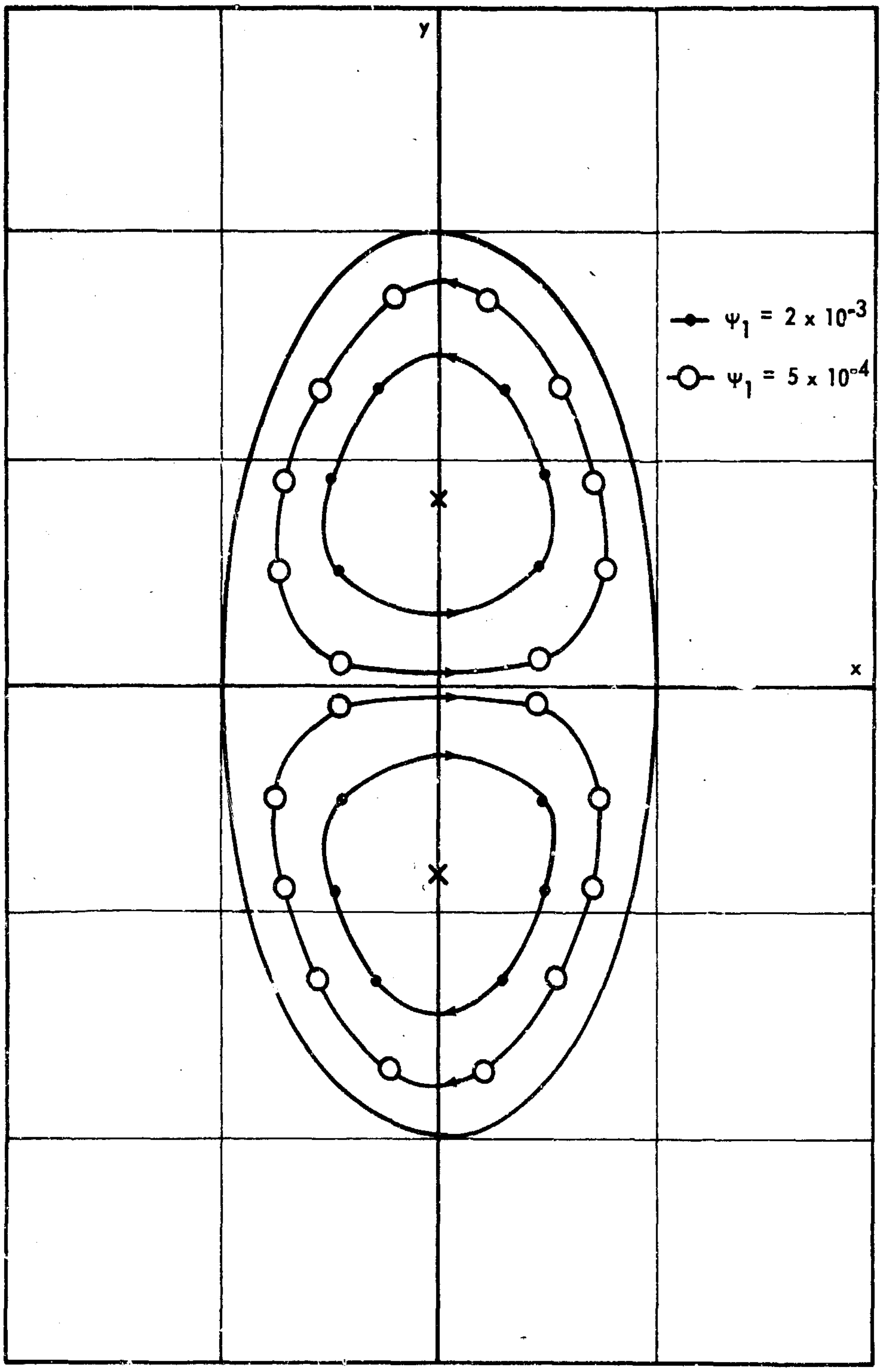

Figure 4. First-Order Approximation of the Streamlines on the Cross Section of the Pipe at m = 0.5

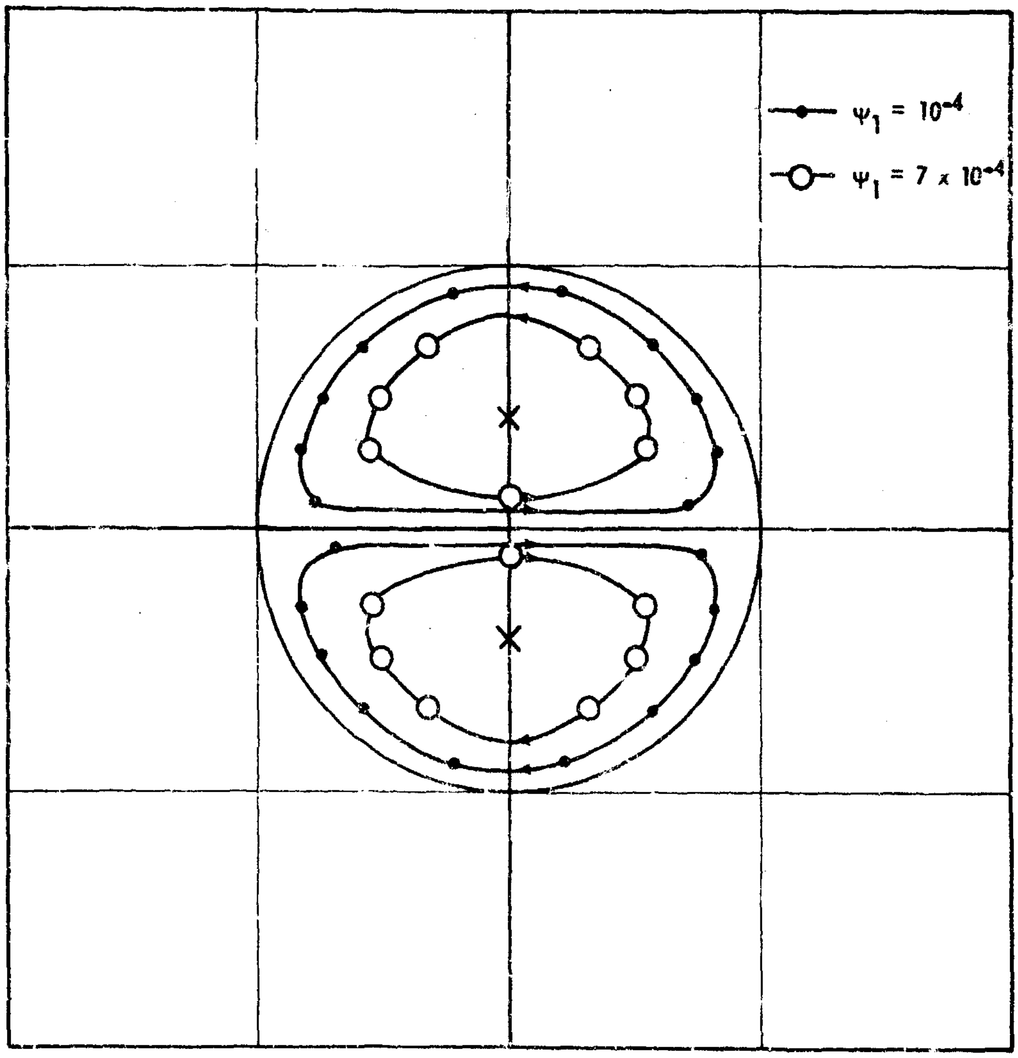

Figure 5. First-Order Approximation of the Streamlines on the Cross Section of the Pipe at m = 1.0

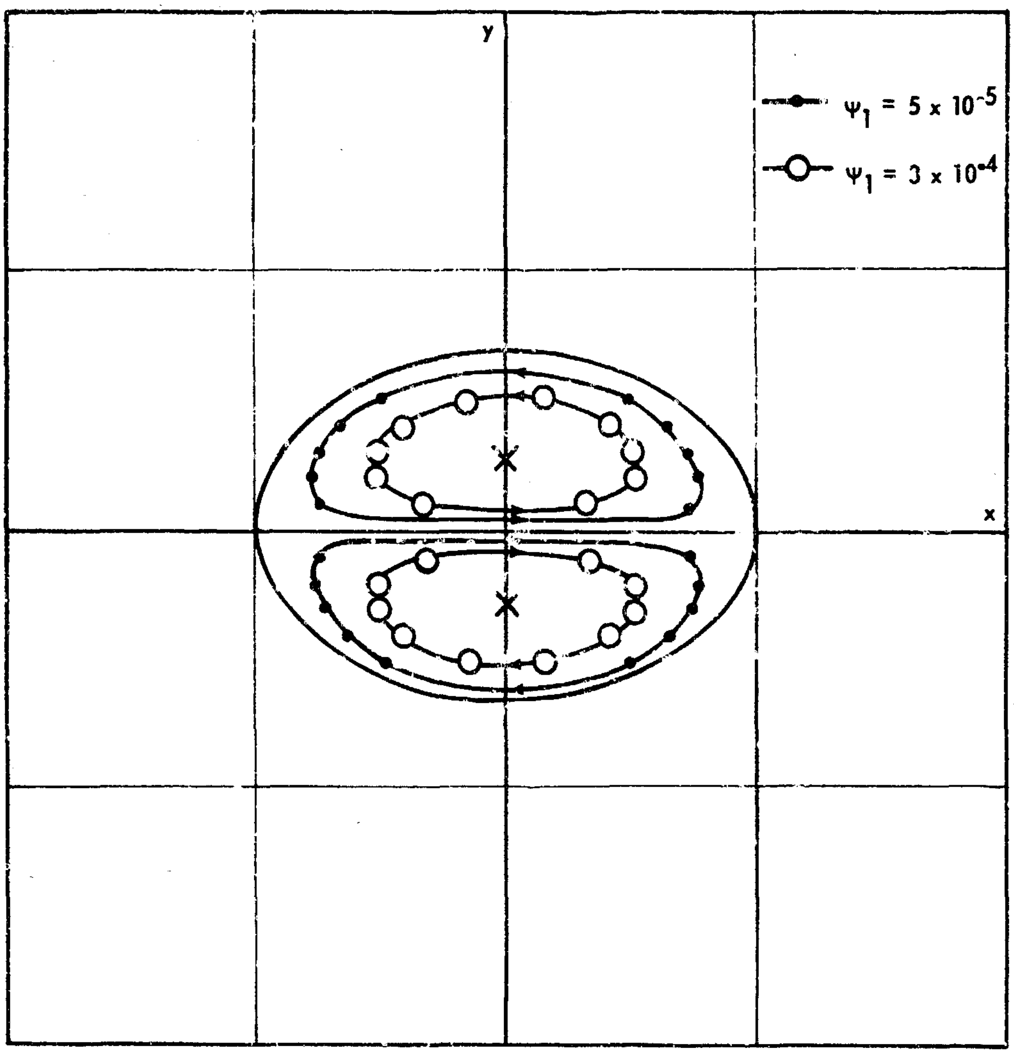

Figure 6. First-Order Approximation of the Streamlines on the Cross Section of the Pipe at m = 1.5

$$+ m^4C_8)y^9 + (2m^2C_{14} + m^4C_{13})y^{11}]x^3 + [C_0y + (C_2 - 2C_0m^2)y^3$$

$$+ (C_5 - 2C_2m^2 + m_4C_0)y^5 + (C_9 - 2C_5m^2 + C_2m^4)y^7 + (C_{14}$$

$$- 2C_9m^2 + m^4C_5)y^9 + (-2C_{14}m^2 + m^4C_9)y^{11} + m^4C_{14}y^{13}]x - \psi_2 = 0$$

Constant values are taken for ψ_2 and y. This reduces the bracketed terms to constants resulting in a thirteenth-degree polynomial in x with constant coefficients. The data required for a plot of x versus y with ψ_2 = constant are calculated by subroutine PLOT of the digital computer program in Appendix A. These data are presented in Tables 4, 5, and 6 and plotted in Figures 7, 8, and 9. Figures 7, 8, and 9 show, as can be seen from equation (5.2), that the value of x is undetermined if y = 0 and the value of y is undetermined if x = 0. The curves, ψ_2 = constant, are plotted in the first and third quadrants by taking a positive constant for ψ_2, and taking a negative value of the same constant produced the same curves in the second and fourth quadrants. The curves in the four quandrants are symmetric about the x-axis, y-axis, and the origin. The x,y relation is dependent on the parameters C and m. In the cases plotted $C = 2(m^2 + 1)$ by choice. It is clear from Figures 7, 8, and 9 that the contribution of ψ_2 to the secondary velocities, U and V, are the same for each of the four quadrants.

3. Vorticity Centers of the Secondary Flow

The centers of secondary flow are points in the cross-sectional plane of the pipe where the secondary velocity vanishes. These points are called vorticity centers. For the secondary velocity to vanish both components,

Table 4. Second-Order Approximation of Streamlines in the Cross Section of the Pipe at m = 0.5

y	x $\psi_2 = 3 \times 10^{-7}$		x $\psi_2 = 5 \times 10^{-8}$	
1.7				
1.3				0.2427
0.9	0.1902	0.5709	0.0284	0.7883
0.5	0.1275	0.7250	0.0205	0.8837
0.1	0.4286	0.4286	0.0740	0.8243
0				
-0.1	-0.4286	-0.4286	-0.0740	-0.8243
-0.5	-0.1276	-0.7250	-0.0205	-0.8837
-0.9	-0.1902	-0.5709	-0.0284	-0.7883
-1.3				-0.2427
-1.7				
	$\psi_2 = -3 \times 10^{-7}$		$\psi_2 = -5 \times 10^{-8}$	
1.7				
1.3				-0.2427
0.9	-0.1902	-0.5709	-0.0284	-0.7883
0.5	-0.1276	-0.7250	-0.0205	-0.8837
0.1	-0.4286	-0.4286	-0.0740	-0.8243
0				
-0.1	0.4286	0.4286	0.0740	0.8243
-0.5	0.1276	0.7250	0.0205	0.8837
-0.9	0.1902	0.5709	0.0284	0.7883
-1.3				0.2427
-1.7				

Table 5. Second-Order Approximation of Streamlines in the Cross Section of the Pipe at m = 1.0

y	x $\psi_2 = 5 \times 10^{-8}$		x $\psi_2 = 1 \times 10^{-7}$	
0.9	0.1998			
0.7	0.5778	0.5778	0.1519	0.4633
0.5	0.0387	0.7409	0.0787	0.6787
0.3	0.0400	0.8164	0.0812	0.7492
0.1	0.0989	0.7576		0.6070
0				
-0.1	-0.0989	-0.7576		-0.6070
-0.3	-0.0400	-0.8164	-0.0812	-0.7492
-0.5	-0.0387	-0.7409	-0.0787	-0.6787
-0.7	-0.5778	-0.5778	-0.1519	-0.4633
-0.9	-0.1998			
	$\psi_2 = -5 \times 10^{-8}$		$\psi_2 = -1 \times 10^{-7}$	
0.9	-0.1998			
0.7	-0.5778	-0.5778	-0.1519	-0.4633
0.5	-0.0387	-0.7409	-0.0787	-0.6787
0.3	-0.0400	-0.8164	-0.0812	-0.7492
0.1	-0.0989	-0.7576		-0.6070
0				
-0.1	0.0989	0.7576		0.6070
-0.3	0.0400	0.8164	0.0812	0.7492
-0.5	0.0387	0.7409	0.0787	0.6787
-0.7	0.5778	0.5778	0.1519	0.4633
-0.9	0.1998			

Table 6. Second-Order Approximation of Streamlines in the Cross Section of the Pipe at m = 1.5

y	x $\psi_2 = 5 \times 10^{-8}$		x $\psi_2 = 2 \times 10^{-8}$	
0.5		0.2703	0.1056	0.4569
0.4	0.1927	0.4585	0.0678	0.6180
0.3	0.1644	0.5572	0.0517	0.6921
0.2	0.1705	0.5902	0.0356	0.7254
0.1	0.4611		0.4786	
0				
-0.1	-0.4611		-0.4786	
-0.2	-0.1705	-0.5902	-0.0356	-0.7254
-0.3	-0.1644	-0.5572	-0.0517	-0.6921
-0.4	-0.1927	-0.4585	-0.0678	-0.6180
-0.5		-0.2703	-0.1056	-0.4569
	$\psi_2 = -5 \times 10^{-8}$		$\psi_2 = -2 \times 10^{-8}$	
0.5		-0.2703	-0.1056	-0.4569
0.4	-0.1927	-0.4585	-0.0678	-0.6180
0.3	-0.1644	-0.5572	-0.0517	-0.6921
0.2	-0.1705	-0.5902	-0.0356	-0.7254
0.1	-0.4611		-0.4786	
0				
-0.1	0.4611		0.4786	
-0.2	0.1705	0.5902	0.0356	0.7254
-0.3	0.1644	0.5572	0.0517	0.6921
-0.4	0.1927	0.4585	0.0678	0.6180
-0.5		0.2703	0.1056	0.4569

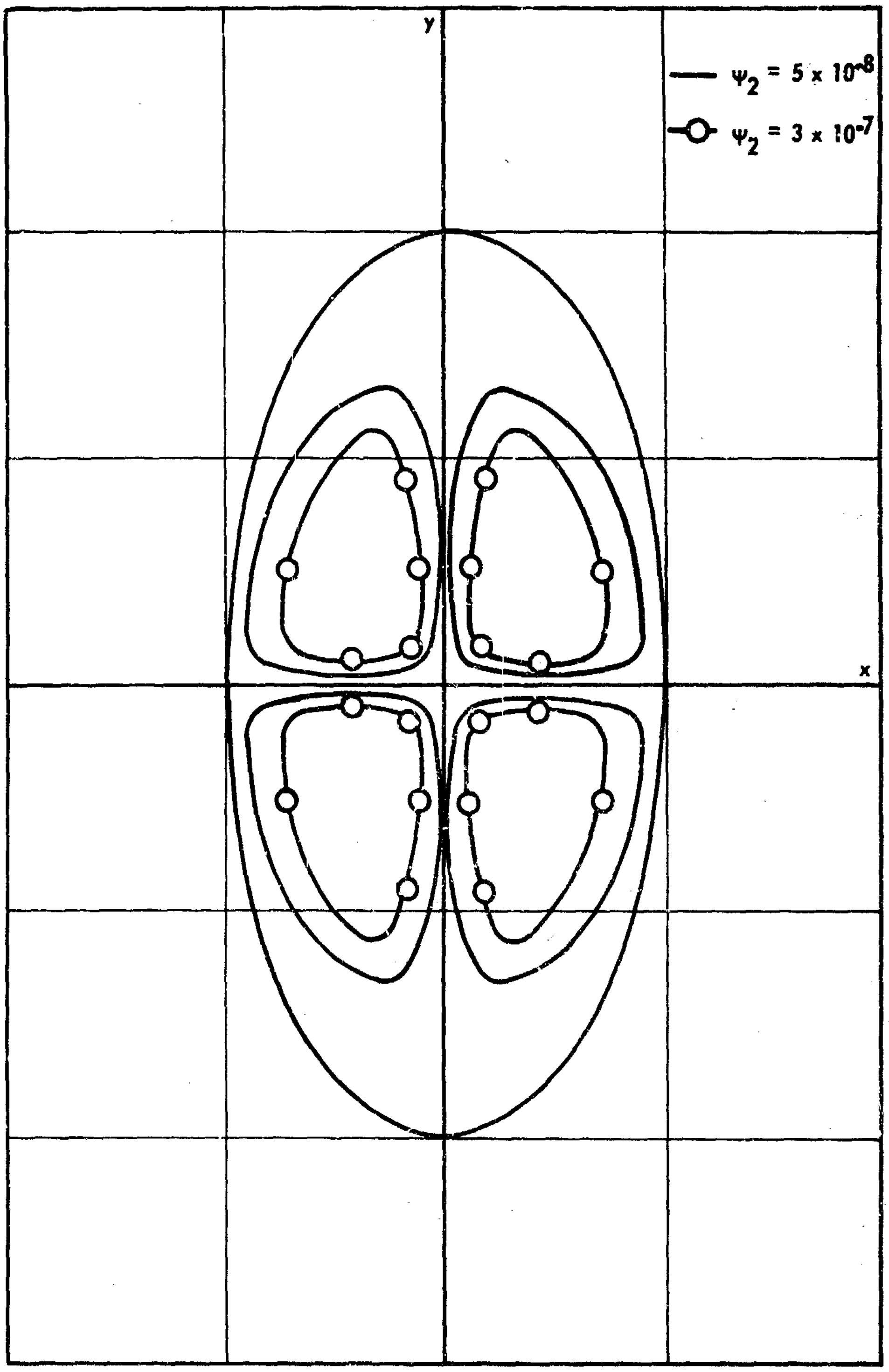

Figure 7. Second-Order Approximation of the Streamlines on the Cross Section of the Pipe at m = 0.5

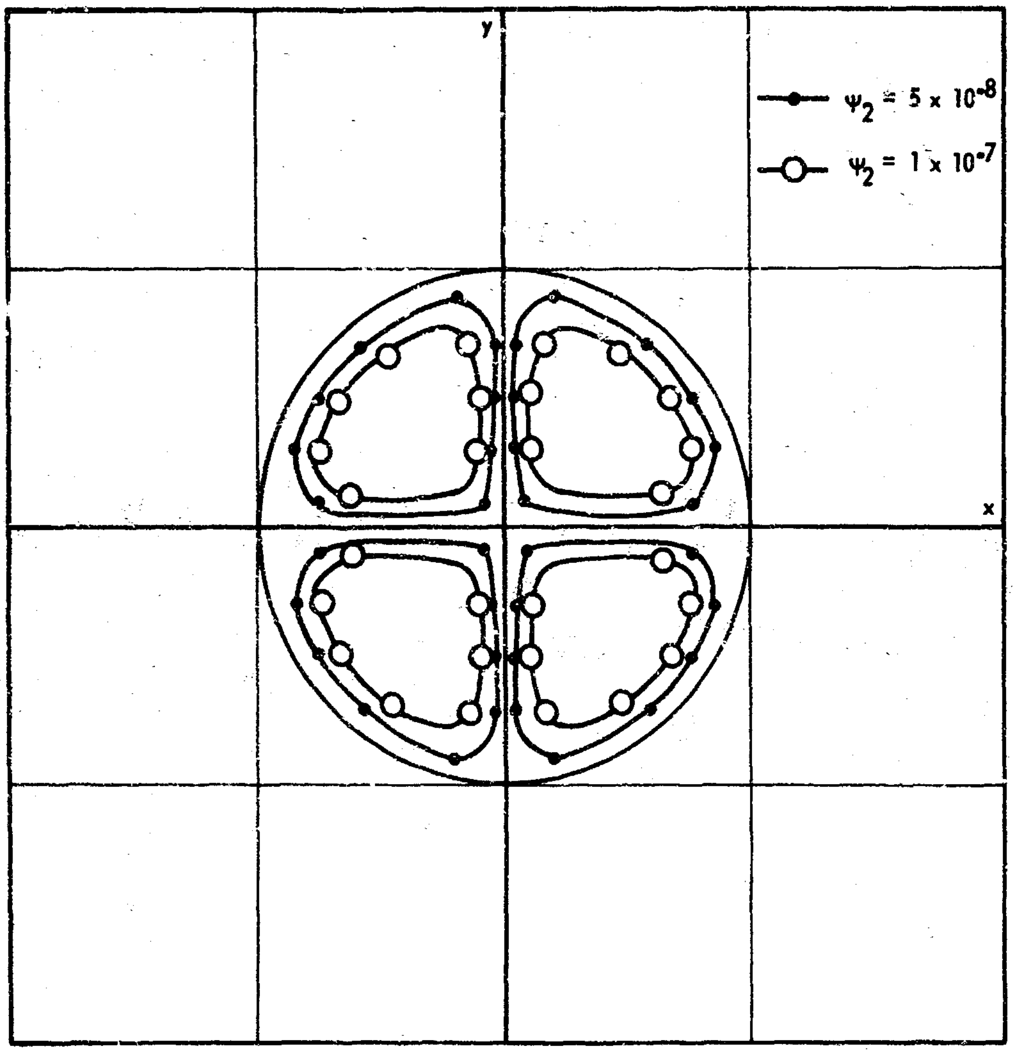

Figure 8. Second-Order Approximation of the Streamlines on the Cross Section of the Pipe at m = 1.0

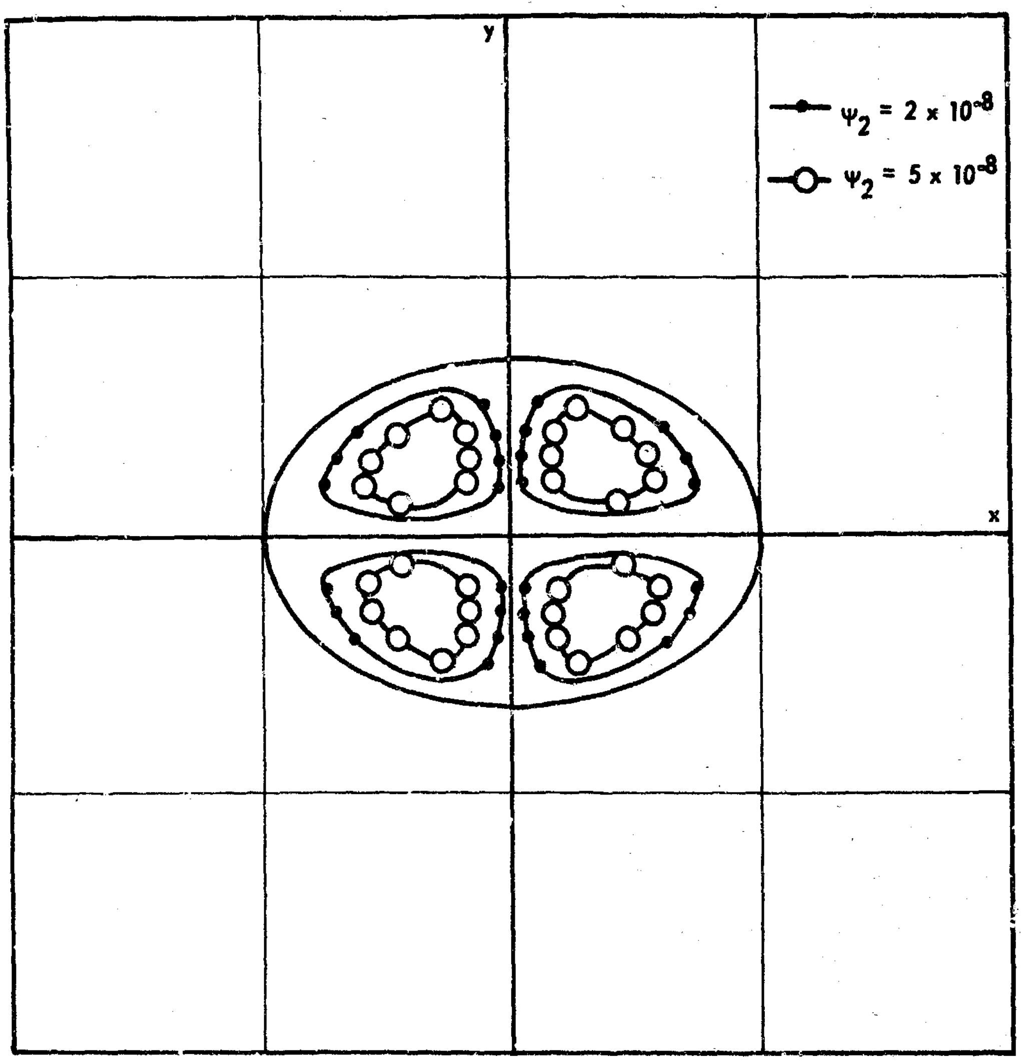

Figure 9. Second-Order Approximation of the Streamlines on the Cross Section of the Pipe at m = 1.5

U and V, must vanish. Sufficient accuracy for location of the vorticity centers is retained after the second-order approximation of the stream function is neglected. Applying equations (2.34) and (2.37), and differentiating equation (3.2), yields:

$$V = \frac{\partial\psi_1}{\partial x} = (1 - x^2 - m^2y^2)\,2xy[(2A_0 - A_1) + 3A_1x^2 + (A_1m^2 + 2A_2)y^2] \tag{5.3}$$

$$U = \frac{\partial\psi_1}{\partial y} = (1 - x^2 - m^2y^2)\,[A_0 + (A_1 - A_0)x^2 + (3A_2 - 5A_0m^2y^2 \tag{5.4}$$

$$+ (-A_1)x^4 + (-5A_1m^2 - 3A_2)x^2y^2 + (-7A_2m^2)y^4]$$

Inspection of equation (5.3) reveals that V vanishes for all values of y at x = 0. Setting U = 0 in equation (5.4) dividing by the boundary conditions and substituting x = 0 yields:

$$7A_2m^2y^4 + (5A_0m^2 - 3A_2)y^2 - A_0 = 0 \quad . \tag{5.5}$$

Solving equation (5.5) by the quadratic formula and taking the square root of both sides yields:

$$y = \pm\sqrt{\frac{3A_2 - 5A_0m^2 + \sqrt{(5A_0m^2 - 3A_2)^2 + 28A_0A_2m^2}}{14A_2m^2}} \quad . \tag{5.6}$$

The minus sign on the small radical is removed because it results only in imaginary values of y, which are not applicable to this problem. The data required for a plot of m versus y are calculated by the digital computer program in Appendix A. These data are presented in Table 7 and plotted in

Table 7. Position of the Vorticity Centers of the Secondary Flow

m	x	y	y
0.1	0	3.798	-3.798
0.2	0	1.927	-1.927
0.3	0	1.315	-1.315
0.4	0	1.011	-1.011
0.5	0	0.826	-0.826
0.6	0	0.699	-0.699
0.7	0	0.605	-0.605
0.8	0	0.533	-0.533
0.9	0	0.475	-0.475
1.0	0	0.429	-0.429
1.1	0	0.391	-0.391
1.2	0	0.359	-0.359
1.3	0	0.332	-0.332
1.4	0	0.308	-0.308
1.5	0	0.288	-0.288
1.6	0	0.270	-0.270
1.7	0	0.254	-0.254
1.8	0	0.240	-0.240
1.9	0	0.228	-0.228
2.0	0	0.216	-0.216

Figure 10. The curves in Figure 10 are symmetric about the m-axis. The m,y relation is dependent on the parameters C and m. In the cases plotted $C = 2(m^2 + 1)$ by choice. It is evident from Figure 10 that y approaches zero as m becomes infinite and y approaches infinity as m approaches zero. This is to be expected because the axis of the ellipse coinciding with the y-axis, B, is

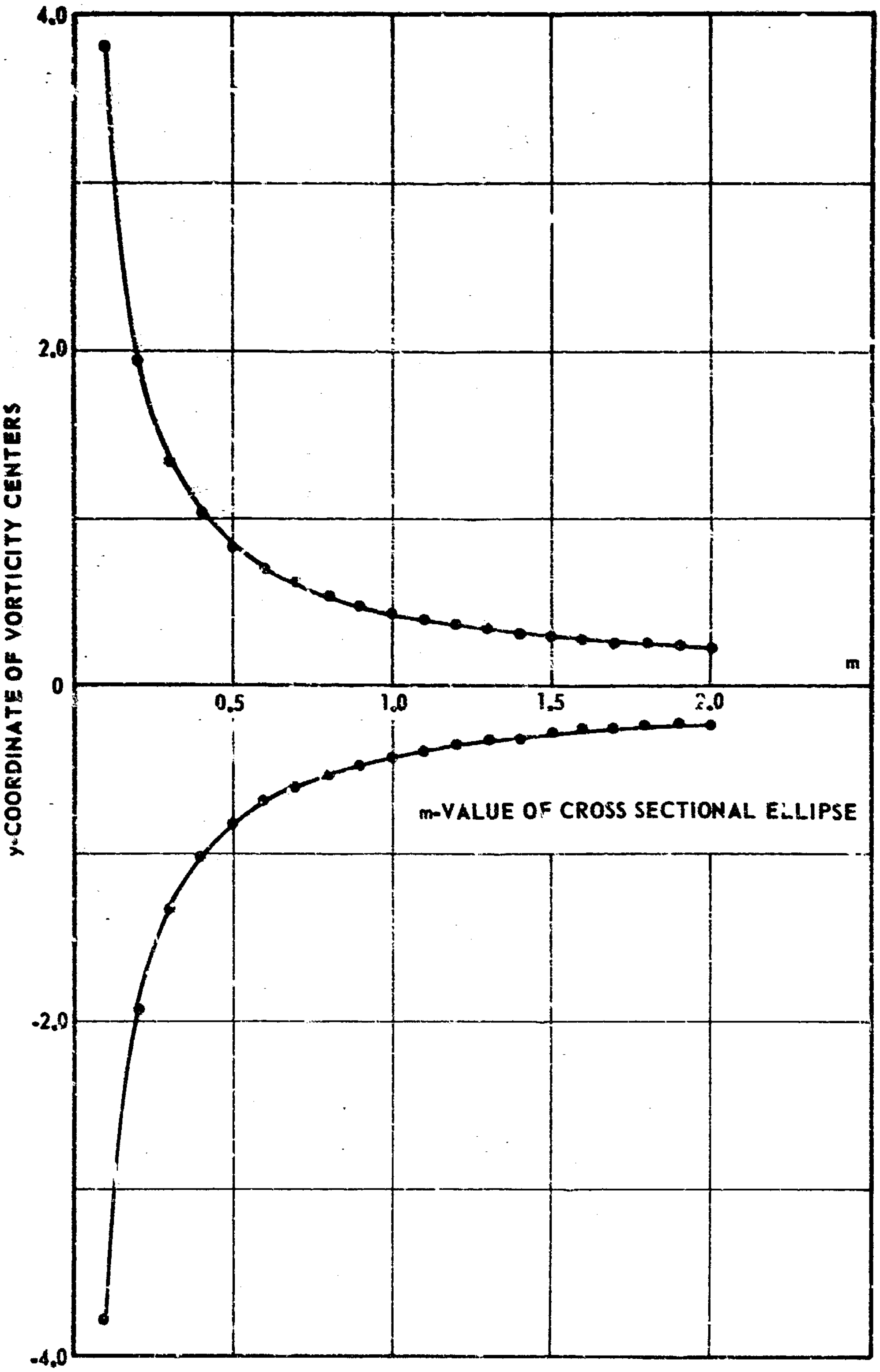

Figure 10. m Versus the y-Coordinate of the Vorticity Points at x =0

determined by the relation $B = 1/m$ and the y coordinate of the vorticity centers increases with an increase in B. For $m = 1$ (circular case) the position of the vorticity center is the same as that obtained by Dean [1928].

B. Streamlines in the Central Plane

The dimensional differential equation for the streamlines in the central plane is:

$$\frac{(R \pm x')\,d\theta}{q_\theta} = \frac{dx'}{q_x} \quad . \tag{5.7}$$

Sufficient accuracy is retained if only the zero-order approximation of W and the first-order approximation of ψ are considered. Also, x' is negligible in comparison with R for small curvatures. At the central plane $y = 0$, and from equations (2.24), (2.31), and (2.33) the terms become:

$$q_x = \frac{\nu U}{a} = \frac{\nu U}{a}\frac{\partial \psi_1}{\partial y} = \frac{\nu K}{a}(1 - x^2)^2(A_0 + A_1 x^2) \quad \text{at} \quad y = 0 \tag{5.8}$$

$$dx' = a\,dx$$

$$q_\theta = W_0(1 - x^2) \quad \text{at} \quad y = 0$$

where $C = 2(m^2 + 1)$ by choice. Substituting equations (5.8) into equation (5.7) and solving for $d\theta$ in nondimensional terms yields:

$$d\theta = \frac{1}{\dfrac{2W_0 a}{\nu}}\left[\frac{dx}{(1 - x^2)(A_0 + A_1 x^2)}\right] \quad . \tag{5.9}$$

Integration of equation (5.9) yields:

$$\theta = \frac{1}{\frac{4W_0a}{\nu}(A_0 + A_1)}\left[\log\frac{1+x}{1-x} + \sqrt{-\frac{A_1}{A_0}}\log\frac{1+x\sqrt{-\frac{A_1}{A_0}}}{1-x\sqrt{-\frac{A_1}{A_0}}}\right] \quad (5.10)$$

when $\frac{A_1}{A_0} < 0.$

$$\theta = \frac{1}{\frac{4W_0a}{\nu}(A_0 + A_1)}\left[\log\frac{1+x}{1-x} + 2\sqrt{\frac{A_1}{A_0}}\tan^{-1}\left(x\sqrt{\frac{A_1}{A_0}}\right)\right] \quad (5.11)$$

when $\frac{A_1}{A_0} > 0.$

The data required for a polar plot of θ versus x are calculated by subroutine PLOT of the digital computer program in Appendix A. The term θ increases steadily with x and approaches infinity as x approaches one or a minus one. Therefore, the streamline approaches but never reaches the inside surface of the pipe. The relation between θ and x does not involve $\frac{a}{R}$ and is, therefore, independent of the curvature of the pipe. Since A_0 and A_1 depend on the parameters C and m, the variation of θ with x depends on C, m, and the Reynolds number. In Table 8 the calculated values of θ versus x are given for m = 0.5, 1.0, and 1.5, taking the value of $4W_0a/\nu$ to be 130. The form of streamlines for ordinary flow are shown in Figure 11. In Figure 11 $\frac{a}{R}$ has been assumed, for graphical illustration, to have a large value of $\frac{1}{2}$ since the x, θ relation is independent of $\frac{a}{R}$, but the data would not be valid for a pipe with such large curvature. From Figure 11 it can be seen that the curvature of the

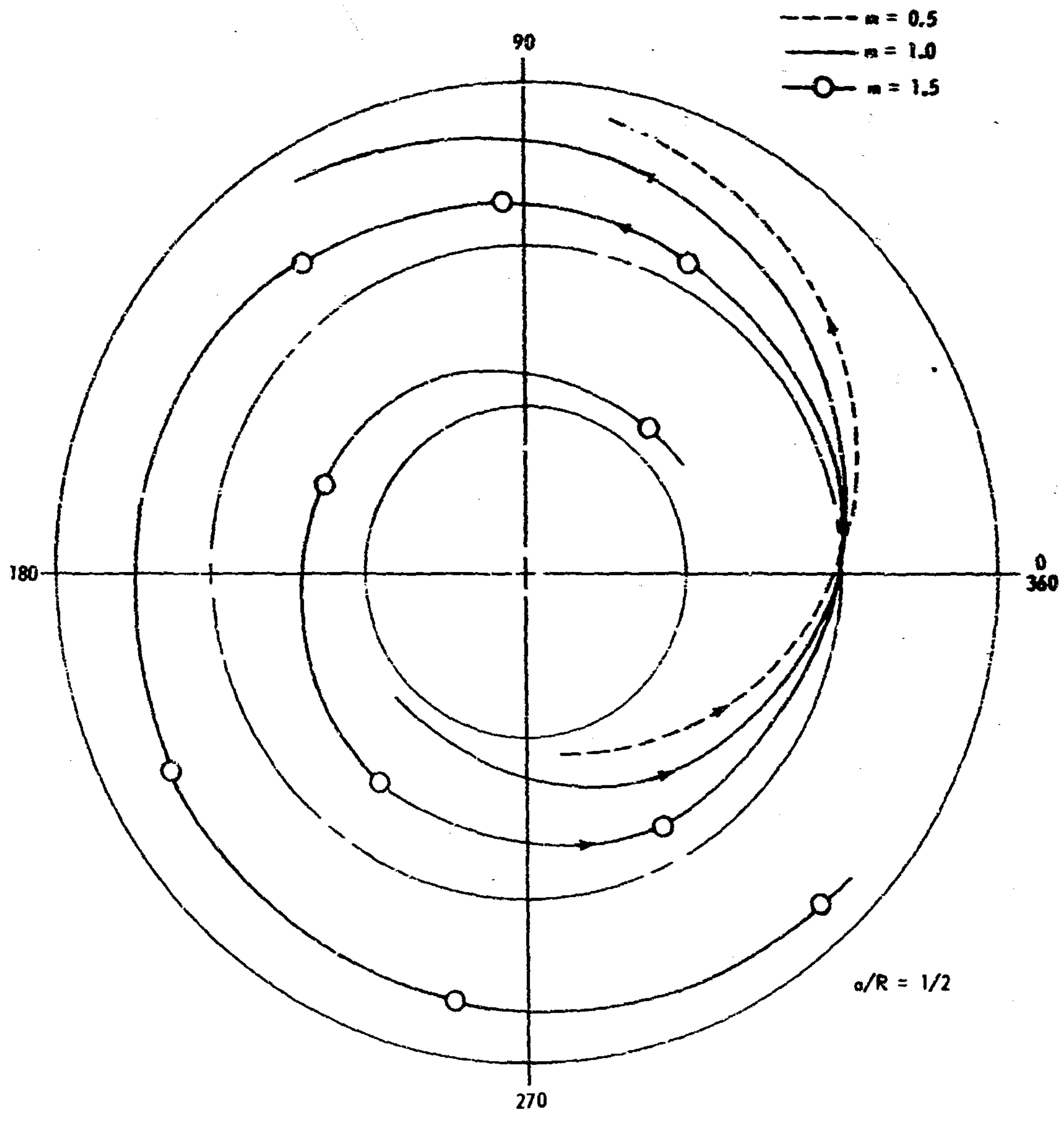

Figure 11. Streamlines in the Central Plane

Table 8. Streamlines in the Central Plane

x	θ m = 0.5	θ m = 1.0	θ m = 1.5
1.0	∞	∞	∞
0.8	69.1	110.9	312.0
0.6	44.1	71.7	204.0
0.4	27.1	44.4	126.2
0.2	13.0	21.4	60.9
0	0	0	0
-0.2	-13.0	- 21.4	- 60.9
-0.4	-27.1	- 44.4	-126.2
-0.6	-44.1	- 71.7	-204.0
-0.8	-69.1	-110.9	-312.0
-1.0	∞	∞	∞

streamline in the central plane decreases as m increases.

C. Effect of Pipe Curvature on the Flow Rate

The effect of pipe curvature on the flow rate is given by equation (4.5). It is

$$\frac{F_c}{F_s} = \left[1 - \frac{Q_2}{Q_0} K^2 \right]$$

where F_c/F_s is the ratio of the flux through a curved pipe to the flux through a straight pipe, both having the same inlet pressure, length, and cross section. The ratios, $\frac{Q_2}{Q_0}$, are calculated for 20 values of m by the digital computer program in Appendix A. The ratios, F_c/F_s, are calculated for values of K ranging from

0 to 1000 by use of a desk calculator. These data are presented in Table 9 and plotted in Figure 12 for m = 0.5, 1.0, and 1.5. The F_c/F_s, K relation is dependent on the parameters C and m since Q_2 and Q_0 are dependent on C and m. In the cases plotted $C = 2(m^2 + 1)$ by choice. The F_c/F_s values taken from Figure 12 are corrections factors for the pipe curvature. The F_c/F_s values are used with the equation

$$Q_T = \frac{F_c}{F_s} Q_0 \tag{5.12}$$

where Q_T is the total flow rate through a curved pipe and Q_0 is the flow rate through a straight pipe with the same inlet pressure, length, and cross section. From Figure 12 it is noted that F_c/F_s is approximately one for an extremely small Dean's number and the curvature effect is negligible. As the Dean's number is increased, the curvature effect rapidly increases. This is expected since the Dean's number is proportional to the velocity squared and the energy loss is much greater. The curvature effect decreases with an increase in the value of m. Figure 12 also shows that the effect of pipe curvature on the rate of flow through the pipe is greater for a pipe with the major axis of the cross-sectional ellipse perpendicular to the plane of the bend ($m < 1$) than when the major axis coincides with the radius of curvature ($m > 1$).

D. First-Order Approximation of the Primary Velocity in the Central Plane

The first-order approximation of the primary velocity is given by equation (3.3) as

Table 9. Flux Through a Curved Pipe/Flux Through a Straight Pipe for Various Dean's Numbers

K	F_c/F_s m = 0.5	F_c/F_s m = 1.0	F_c/F_s m = 1.5
0	1.000	1.000	1.000
10	0.999	0.999	0.999
100	0.998	0.999	0.999
200	0.992	0.997	0.999
300	0.983		
400	0.968	0.987	0.997
500	0.951		
600	0.930	0.971	0.995
700	0.905		
800	0.876	0.948	0.991
900	0.843		
1000	0.806	0.919	0.986

$$W_1 = (1 - x^2 - m^2y^2)x[b_0 + b_1x^2 + b_2y^2 + b_3x^4 + b_4x^2y^2 + b_5y^4 + b_6x^6 + b_7x^4y^2 + b_8x^2y^4 + b_9y^6]$$

In the central plane y = 0 and equation (3.3) becomes:

$$W_1 = b_0x + (b_1 - b_0)x^3 + (b_3 - b_1)x^5 + (b_6 - b_3)x^7 - b_6x^9 \qquad (5.13)$$

The data required for a plot of x versus W_1 are calculated by subroutine PLOT of the digital computer program in Appendix A. The factor 10^4 is just an amplification factor for the curves. These data are presented in Table 10 and plotted for m = 0.5, 1.0, and 1.5 in Figure 13. Figure 13 shows, as can be

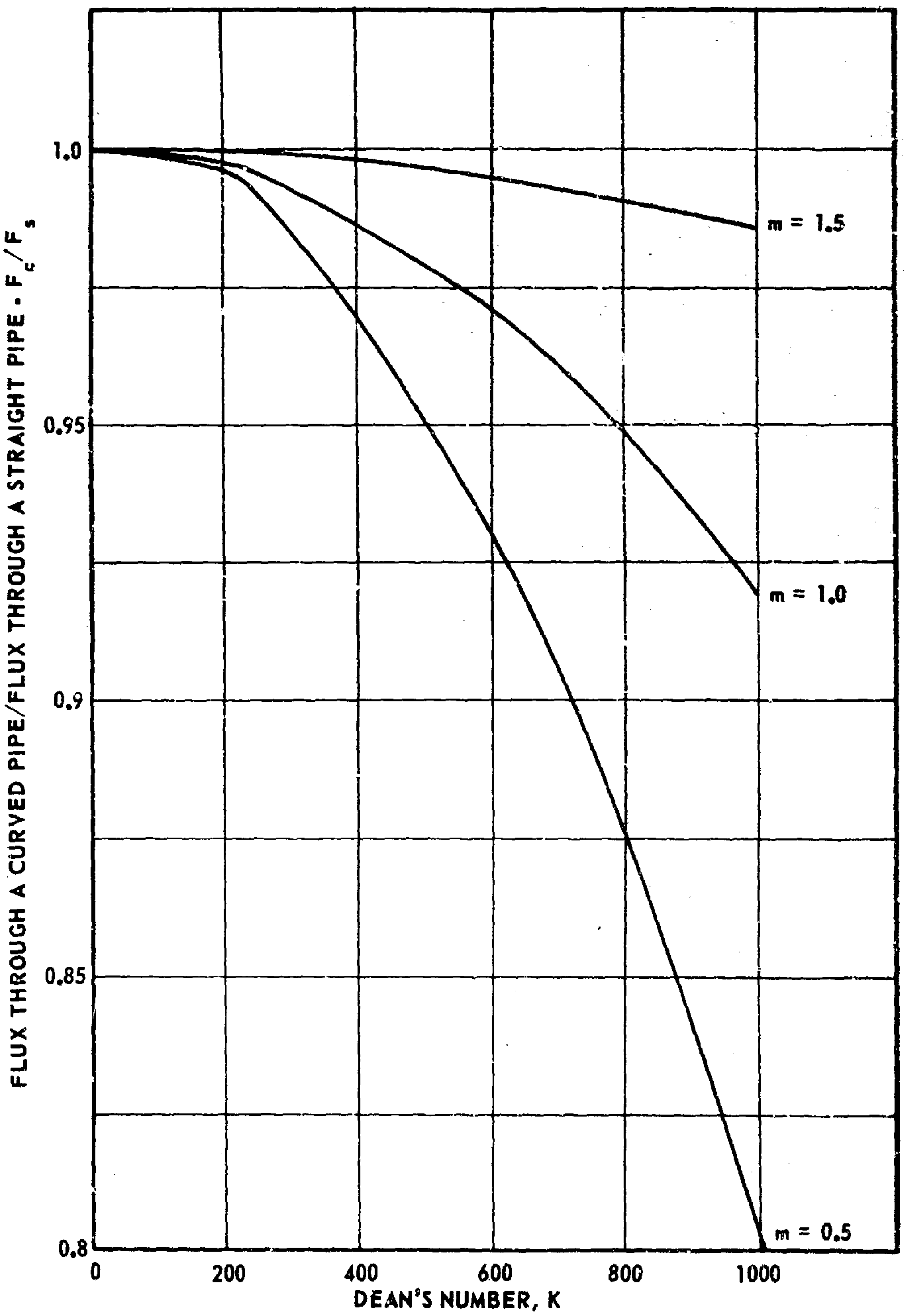

Figure 12. The Effect of Pipe Curvature Versus Dean's Number with m = 0.5, 1.0, 1.5

Table 10. First Order of Approximation of the Primary Velocity at $y = 0$

x	$W_1 \times 10^4$ $m = 0.5$	$W_1 \times 10^4$ $m = 1.0$	$W_1 \times 10^4$ $m = 1.5$
1.0	0	0	0
0.8	1.897	1.123	0.431
0.6	3.336	2.093	0.877
0.4	3.546	2.314	1.026
0.2	2.271	1.515	0.693
0	0	0	0
-0.2	-2.271	-1.515	-0.693
-0.4	-3.546	-2.314	-1.026
-0.6	-3.336	-2.093	-0.877
-0.8	-1.897	-1.123	-0.431
-1.0	0	0	0

seen from equation (5.13), that the function is odd and symmetric about the origin. The x, W_1 relation is dependent on the parameters C and m. In the cases plotted $C = 2(m^2 + 1)$ by choice. Figure 13 shows that the magnitude of the first-order approximation of the primary velocity in the central plane decreases with an increase in the value of m.

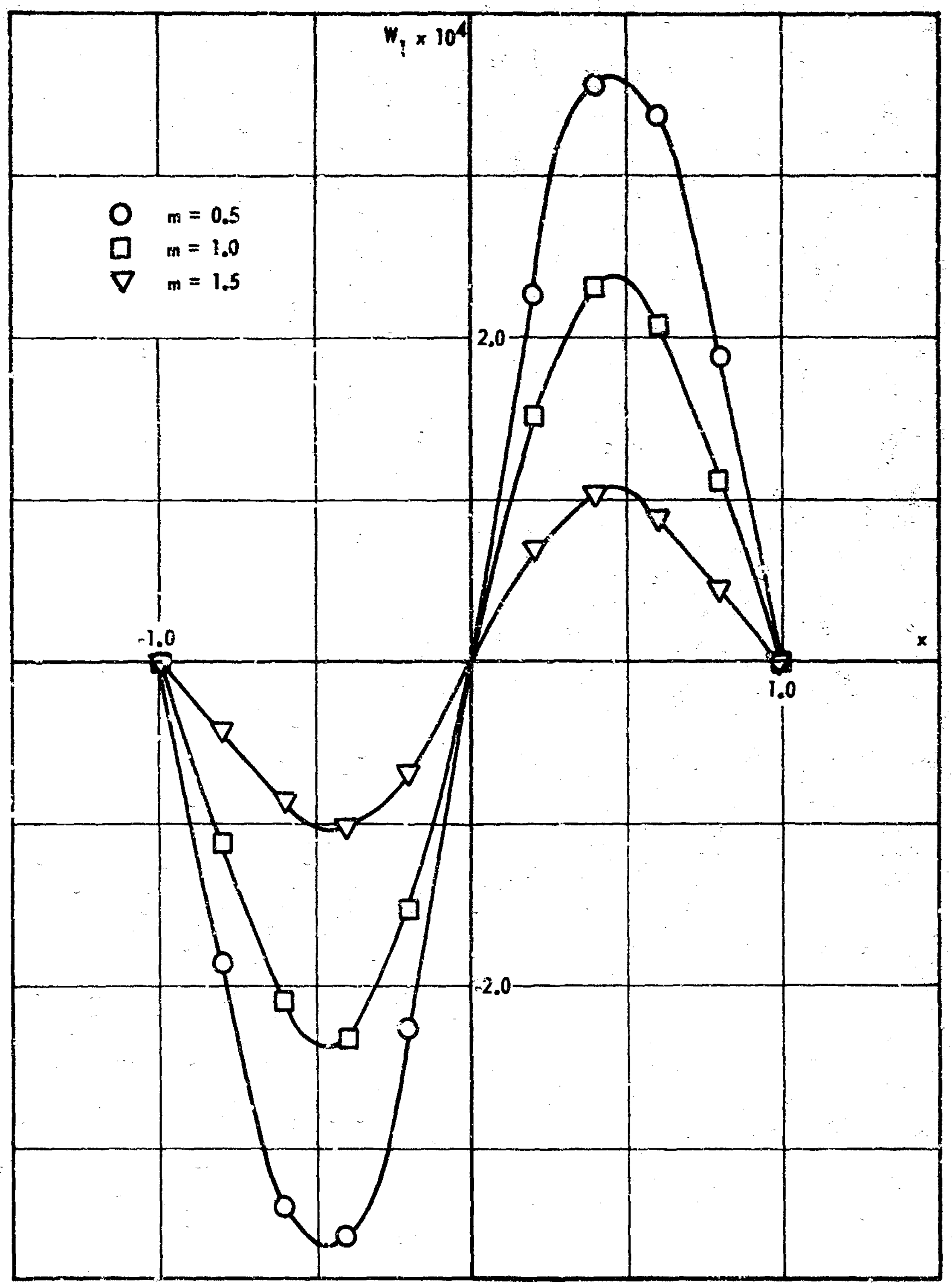

Figure 13. First-Order Approximation of the Primary Velocity at y = 0

CHAPTER VI

SUMMARY AND CONCLUSIONS

The graphical illustrations presented show that the streamline flow of an incompressible fluid through a curved pipe of elliptical cross section is similar to that of a curved pipe of circular cross section. It consists of a primary flow and a secondary flow. The primary flow is along and parallel to the center line of the pipe. The secondary flow is in the plane of the cross section of the pipe with the form of the streamlines as shown in Figures 7 through 9. The motion of the secondary flow ineffectively expends energy which results in a decreased rate of flow through the pipe. The secondary flow and centrifugal forces acting on the fluid are not present in a straight pipe. Therefore, the rate of flow through a straight pipe is an upper bound for the rate of flow through a curved pipe with the same inlet pressure, length and cross section.

The assumption made in the derivation of the governing differential equations are equivalent to those made by Dean [1928] for a pipe with a circular cross section. The governing differential equations of this study are equivalent to those obtained by Dean [1928] and the accuracy is the same. However, the solutions to the equations obtained in this study are exact and applicable to a pipe with an elliptical cross section. Dean's [1928] solutions are approximate because the unsymmetrical terms were neglected in the equation for the second-order approximation of the primary velocity.

For the circular case ($m = 1$) the solutions to the first four of the governing differential equations (2.38) through (2.41) are exactly equivalent to the corresponding solutions presented by Dean [1928] and are symmetric. The solution to the fifth governing equation (2.42) differs from Dean's [1928] approximate solution by the value represented by the unsymmetrical terms. The error in Dean's [1928] second-order approximation of the flow rate, due to neglect of the unsymmetrical terms, is approximately 0.17 per cent and 1.2 per cent for K values of 400 and 1000, respectively. The flow rate given by the equations of the present study differs from that of Dean's [1928] fourth-order approximation by 0.12 per cent and 0.0044 per cent for K values of 400 and 1000, respectively. Therefore, the equations obtained in the present study have about the same accuracy as Dean's [1928] fourth-order approximation for the case of a circular cross section.

Figure 12 shows that the effect of pipe curvature on the rate of flow is greater for a pipe with the major axis of the cross-sectional ellipse perpendicular to the plane of the bend ($m < 1$) than when the major axis coincides with the radius of curvature ($m > 1$). However, the fabrication is generally more difficult for pipe bends with $m > 1$.

The rate of flow of fluid through a curved pipe with an elliptical cross section is a function of C, m, and K. Therefore, the rate of flow of the fluid is dependent on the fluid properties, cross section of the pipe bend and the Dean's number.

The pipe curvature reduces the rate of flow through the pipe if the inlet pressure, pipe length, radius of curvature, and cross section of the pipe are identical. The magnitude of this reduction decreases as the value of m increases (Figure 12). Therefore, the accuracy of the solutions presented is dependent upon m, but the governing equations are probably not valid for any m when the Dean's number exceeds 1000.

CHAPTER VII

RECOMMENDATIONS FOR FUTURE RESEARCH

The following recommendations are made for future research:

1. A more accurate solution can probably be obtained by transferring the governing equations (2.35) and (2.36) into finite difference form and solving them directly on a computer.

2. The accuracy of this solution can be improved by extending it to a fourth-order approximation, but considerable work will be involved.

3. The solutions presented could be generalized to cover elastico-viscous fluid or magnetohydrodynamic flow.

4. The solutions presented could be further verified by experimentation.

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APPENDIX A

OPERATING INSTRUCTIONS AND COMPUTER PROGRAM

This program is for use on the IBM 7094 Computer. To solve for the rate of flow through the pipe and generate the data required to plot the graphs presented in Chapter V, numerical values for the case number, XM, C, XK, PSI 11, PSI 21, PSI 12, PSI 22, PSI 13, PSI 23, PSI 14, PSI 24, Y_0, and DEL Y must be included as input data. The input data are defined as follows:

XM = m

C = C

XK = K = Dean's number

PSI 11, PSI 12, PSI 13, and PSI 14 = four constants taken for ψ_1

PSI 21, PSI 22, PSI 23, and PSI 24 = four constants taken for ψ_2

Y_0 = initial value of Y for which values of X are determined

DEL Y = increment of Y added to Y_0

If the data to plot the graphs are not desired, the CALL PLOT card with its two continuation cards at call number 1004 can be removed from the deck and the program ends after solving for the rate of flow through the pipe. However, values for all of the input variables must be included as input data. Since the only input variables used to determine the flow rate are XM, C, and XK, any constant can be used for the other input data.

```
IBJOB VERSION 5 HAS CONTROL.
$IBJOB ELIF     MAP,DLOGIC
$IBFTC MAIN     LIST,REF

      DOUBLE PRECISION W,X,Y,Z,DET
      DOUBLE PRECISION XM,XM2,A0,A1,A2,B0,B1,B2,B3,B4,B5,B6,B7,B8,B9,C0,
     1C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14,D0,D1,D2,D3,D4,D5,D
     26,D7,D8,D9,D10,D11,D12,D13,D14,D15,D16,D17,D18,D19,D20,D21,D22,D23
     3,D24,D25,D26,D27,D28,D29,D30,D31,D32,D33,D34,D35,C,A
      DIMENSION W(36,37)
      DIMENSION X(36,37)
      DIMENSION Y(36,37)
      DIMENSION Z(36,37)
      COMMON XI,XJJ
      EQUIVALENCE (W(1,1),X(1,1),Y(1,1),Z(1,1))
      INTEGER CASE
      NAMELIST/INPUT/CASE,XM,C,PSI11,PSI21,PSI12,PSI22,PSI13,PSI23,PSI14
     1,PSI24,YC,XK,DELY
    5 READ(5,INPUT)
      WRITE(6,INPUT)
      C = C.C
      XM2 = XM*XM
      A=C/(2.*(XM**2+1.))
      W(1,1)=5.*XM**4+2.*XM**2+1.
      W(1,2)=-2.*XM**2-2.
      W(1,3)=-10.*XM**2-2.
      W(1,4)=C**2*XM**2/(48.*(XM**2+1.)**2)
      W(2,2)=10.*XM**4+6.*XM**2
      W(2,3)=105.*XM**4+20.*XM**2+3.
      W(2,4)=-C**2*XM**4/(16.*(XM**2+1.)**2)
      W(3,2)=5.*XM**4+12.*XM**2+15.
      W(3,3)=10.*XM**2+6.
      W(3,4)=-C**2*XM**2/(48.*(XM**2+1.)**2)
      W(2,1) = 0.
      W(3,1) = 0.
      WRITE (6,10)
   10 FORMAT(1X ,29X,3H***,10X,36HW    -    M    A    T    R    I    X,1
     10X,3H***///)
      CALL SESCMI (W,3,1,0,36,37,DET,RANK,SOLN)
      A0 = W(1,1)
      A1 = W(2,1)
      A2 = W(3,1)
      WRITE (6,30) A,A0,A1,A2
   30 FORMAT (1X,5HA  = ,E15.8,5X,5HA0 = ,E15.8,5X,5HA1 = ,E15.8,5X,5HA2
     1 = ,E15.8)
      DO 35 I = 1,10
      DO 35 J = 1,11
   35 X(I,J) = 0.
      X(1,1)=XM**2+3.
      X(1,2)=-3.
      X(1,3)=-1.
      X(1,11)=A0*A
      X(2,2)=XM**2+10.
      X(2,3)=+1.
      X(2,4)=-10.
      X(2,5)=-1.
      X(2,11)=A*A1-2.*A0*A
      X(3,2) = 3.*XM**2
      X(3,3) = 3. + 6.*XM**2
```

```
      X(3,5)=-3.
      X(3,6)=-6.
      X(3,11)=3.*A*A2-2.*A*A0*XM**2-2.*A*A1*XM**2
      X(4,4)=XM**2+21.
      X(4,5)=1.
      X(4,7)=-21.
      X(4,8)=-1.
      X(4,11)= A0*A -2.*A*A1
      X(5,4)=5.*XM**2
      X(5,5)=3.*XM**2+5.
      X(5,6)=+3.
      X(5,8)=-5.
      X(5,9)=-3.
      X(5,11)=A0*A*XM**2+A*A1*XM**2-3.*A*A2
      X(6,5)=3.*XM**2
      X(6,6)=15.*XM**2+3.
      X(6,9)=-3.
      X(6,10)=-15.
      X(6,11)=A0*A*XM**4+4.*A*A1*XM**4-6.*A*A2*XM**2
      X(7,7)=XM**2+36.
      X(7,8)=+1.
      X(7,11)=A*A1
      X(8,7)=21.*XM**2
      X(8,8)=6.*XM**2+21.
      X(8,9)=+6.
      X(8,11)=3.*A*A2
      X(9,8)=10.*XM**2
      X(9,9)=15.*XM**2+10.
      X(9,10)=+15.
      X(9,11)=-3.*A1*A *XM**4+6.*A*A2*XM**2
      X(10,9)=3.*XM**2
      X(10,10)=28.*XM**2+3.
      X(10,11)=3.*A*A2*XM**4-2.*A*A1*XM**6
      WRITE (6,40)
   40 FORMAT(1X ,29X,3H***,10X,36HX    -    M    A    T    R    I    X,1
     10X,3H***///)
      CALL SESCMI (X,10,1,0,36,37,DET,RANK,SOLN)
      B0 = X(1,1)
      B1 = X(2,1)
      B2 = X(3,1)
      B3 = X(4,1)
      B4 = X(5,1)
      B5 = X(6,1)
      B6 = X(7,1)
      B7 = X(8,1)
      B8 = X(9,1)
      B9 = X(10,1)
      WRITE (6,55) B0,B1,B2,B3,B4,B5,B6,B7,B8,B9
   55 FORMAT(1X,5HB0 = ,E15.8,5X,5HB1 = ,E15.8,5X,5HB2 = ,E15.8,5X,5HB3
     1= ,E15.8,5X,5HB4 = ,E15.8//1X,5HB5 = ,E15.8,5X,5HB6 = ,E15.8,5X,5H
     2B7 = ,E15.8,5X,5HB8 = ,E15.8//1X,5HB9 = ,E15.8)
      DO 60 I = 1,15
      DO 60 J = 1,16
   60 Y(I,J) = 0.
      Y(1,1)=120.*XM**4+144.*XM**2+120.
      Y(1,2)=-144.*XM**2-240.
```

```
Y(1,3)=-240.*XM**2-144.
Y(1,4)=+120.
Y(1,5)=+72.
Y(1,6)=+120.
Y(2,2)=120.*XM**4+480.*XM**2+840.
Y(2,3)=240.*XM**2+240.
Y(2,4)=-480.*XM**2-1680.
Y(2,5)=-240.*XM**2-480.
Y(2,6)=-240.
Y(2,7)=+840.
Y(2,8)=+240.
Y(2,9)=+120.
Y(3,2)=240.*XM**4+240.*XM**2
Y(3,3)=840.*XM**4+480.*XM**2+120.
Y(3,4)=-240.*XM**2
Y(3,5)=-480.*XM**2-240.
Y(3,6)=-1680.*XM**2-480.
Y(3,8)=+120.
Y(3,9)=+240.
Y(3,10)=+840.
Y(4,4)=120.*XM**4+1008.*XM**2+3024.
Y(4,5)=240.*XM**2+504.
Y(4,6)=+120.
Y(4,7)=-1008.*XM**2-6048.
Y(4,8)=-240.*XM**2-1008.
Y(4,9)=-240.
Y(4,11)=+3024.
Y(4,12)=+504.
Y(4,13)=+120.
Y(5,4)=800.*XM**4+1680.
Y(5,5)=840.*XM**4+1600.*XM**2+840.
Y(5,6)=1680.*XM**2+800.
Y(5,7)=-1680.*XM**2
Y(5,8)=-1600.*XM**2-1680.
Y(5,9)=-1680.*XM**2-1600.
Y(5,10)=-1680.
Y(5,12)=+840.
Y(5,13)=+800.
Y(5,14)=+840.
Y(6,4)=120.*XM**4
Y(6,5)=504.*XM**4+240.*XM**2
Y(6,6)=3024.*XM**4+1008.*XM**2+120.
Y(6,8)=-240.*XM**2
Y(6,9)=-1008.*XM**2-240.
Y(6,10)=-6048.*XM**2-1008.
Y(6,13)=+120.
Y(6,14)=+504.
Y(6,15)=+3024.
Y(7,7)=120.*XM**4+1728.*XM**2+7920.
Y(7,8)=240.*XM**2+864.
Y(7,9)=+120.
Y(7,11)=-1728.*XM**2-15840.
Y(7,12)=-240.*XM**2     -1728.
Y(7,13)=-240.
Y(8,7)=1680.*XM**4+6048.*XM**2
Y(8,8)=840.*XM**4+3360.*XM**2+3024.
```

```
      Y(8,9)=1680.*XM**2+1680.
      Y(8,10)=+840.
      Y(8,11)=-6048.*XM**2
      Y(8,12)=-3360.*XM**2-6048.
      Y(8,13)=-1680.*XM**2-3360.
      Y(8,14)=-1680.
      Y(9,7)=840.*XM**4
      Y(9,8)=1680.*XM**4+1680.*XM**2
      Y(9,9)=3024.*XM**4+3360.*XM**2+840.
      Y(9,10)=6048.*XM**2+1680.
      Y(9,12)=-1680.*XM**2
      Y(9,13)=-3360.*XM**2-1680.
      Y(9,14)=-6048.*XM**2-3360.
      Y(9,15)=-6048.
      Y(10,8)=120.*XM**4
      Y(10,9)=864.*XM**4+240.*XM**2
      Y(10,10)=7920.*XM**4+1728.*XM**2+120.
      Y(10,13)=-240.*XM**2
      Y(10,14)=-1728.*XM**2-240.
      Y(10,15)=-15840.*XM**2-1728.
      Y(11,11)=120.*XM**4+2640.*XM**2+17160.
      Y(11,12)=240.*XM**2+1320.
      Y(11,13)=+120.
      Y(12,11)=2880.*XM**4+15840.*XM**2
      Y(12,12)=840.*XM**4+5760.*XM**2+7920.
      Y(12,13)=1680.*XM**2+2880.
      Y(12,14)=+840.
      Y(13,11)=3024.*XM**4
      Y(13,12)=3528.*XM**4+6048.*XM**2
      Y(13,13)=3024.*XM**4+7056.*XM**2+3024.
      Y(13,14)=6048.*XM**2+3528.
      Y(13,15)=+3024.
      Y(14,12)=840.*XM**4
      Y(14,13)=2880.*XM**4+1680.*XM**2
      Y(14,14)=7920.*XM**4+5760.*XM**2+840.
      Y(14,15)=15840.*XM**2+2880.
      Y(15,13)=120.*XM**4
      Y(15,14)=1320.*XM**4+240.*XM**2
      Y(15,15)=17160.*XM**4+2640.*XM**2+120.
      Y(1,16)=8.*A0**2-24.*A0**2*XM**2-32.*A0*A1-4.*A1**2-12.*A1*A2+4.*A
     1*B0*XM**2-2.*A*B2
      Y(2,16)=80.*A0*A1-48.*A0*A1*XM**2+16.*A0**2+48.*A0**2*XM**2+16.*A1
     1**2+48.*A1*A2-2.*A*B4+4.*A*B2+4.*A*B1*XM**2-4.*A*B0*XM**2
      Y(3,16)=240.*A0*A1*XM**2+16.*A0*A1*XM**4+32.*A0*A2-256.*A0*A2*XM**
     12-80.*A0**2*XM**2+144.*A0**2*XM**4-112.*A1*A2+192.*A1*A2*XM**2-48.
     2*A2**2+32.*A1**2*XM**2-4.*A*B5+8.*A*B2*XM**2-4.*A*B0*XM**4
      Y(4,16)=-24.*A0**2-24.*A0**2*XM**2+96.*A0*A1*XM**2-48.*A1**2-24.*A
     11**2*XM**2-72.*A1*A2-2.*A*B7+4.*A*B4-2.*A*B2+4.*A*B3*XM**2-4.*A*B1
     2*XM**2
      Y(5,16)=-640.*A0*A1*XM**2+256.*A0*A1*XM**4+64.*A0*A2+512.*A0*A2*XM
     1**2+400.*A1*A2-832.*A1*A2*XM**2+48.*A0**2*XM**2-144.*A0**2*XM**4-1
     26.*A1**2*XM**2+16.*A1**2*XM**4-4.*A*B8+8.*A*B5+8.*A*B4*XM**2-8.*A*
     3B2*XM**2-4.*A*B1*XM**4
      Y(6,16)=72.*A0**2*XM**4-120.*A0**2*XM**6-384.*A0*A1*XM**4-96.*A0*A
     11*XM**6-192.*A0*A2*XM**2+768.*A0*A2*XM**4+480.*A1*A2*XM**2-552.*A1
     2*A2*XM**4-72.*A1**2*XM**4+24.*A2**2-24.*A2**2*XM**2-6.*A*B9+12.*A*
```

```
 3B5*XM**2-6.*A*B2*XM**4
  Y(7,16)=-48.*A0*A1-48.*A0*A1*XM**2+96.*A1**2+48.*A1**2*XM**2+48.*A
 11*A2+4.*A*B7-2.*A*B4+4.*A*B6*XM**2-4.*A*B3*XM**2
  Y(8,16)=336.*A0*A1*XM**2-272.*A0*A1*XM**4-96.*A0*A2-256.*A0*A2*XM*
 1*2-144.*A1**2*XM**2+112.*A1**2*XM**4-336.*A1*A2+1088.*A1*A2*XM**2-
 2144.*A2**2+8.*A*B7*XM**2+8.*A*B8-4.*A*B5-8.*A*B4*XM**2-4.*A*B3*XM*
 3*4
  Y(9,16)=560.*A0*A1*XM**4-144.*A0*A1*XM**6+64.*A0*A2*XM**2-768.*A0*
 1A2*XM**4-1280.*A1*A2*XM**2+1872.*A1*A2*XM**4-16.*A1**2*XM**4-96.*A
 21**2*XM**6+48.*A2**2+48.*A2**2*XM**2+12.*A*B9+12.*A*B8*XM**2-12.*A
 3*B5*XM**2-6.*A*B4*XM**4
  Y(10,16)=176.*A0*A1*XM**6+80.*A0*A1*XM**8+160.*A0*A2*XM**4-512.*A0
 1*A2*XM**6-624.*A1*A2*XM**4+512.*A1*A2*XM**6+64.*A1**2*XM**6-112.*A
 22**2*XM**2+352.*A2**2*XM**4+16.*A*B9*XM**2-8.*A*B5*XM**4
  Y(11,16)=-60.*A1**2-24.*A1**2*XM**2-12.*A1*A2-2.*A*B7-4.*A*B6*XM**
 12
  Y(12,16)=96.*A1**2*XM**2-128.*A1**2*XM**4+48.*A1*A2-448.*A1*A2*XM*
 1*2+48.*A2**2-4.*A*B8-8.*A*B7*XM**2-4.*A*B6*XM**4
  Y(13,16)=192.*A1**2*XM**4-24.*A1**2*XM**6+672.*A1*A2*XM**2-1320.*A
 11*A2*XM**4-72.*A2**2-24.*A2**2*XM**2-6.*A*B9-12.*A*B8*XM**2-6.*A*B
 27*XM**4
  Y(14,16)=16.*A1**2*XM**6+80.*A1**2*XM**8+980.*A1*A2*XM**4-1024.*A1
 1*A2*XM**6+16.*A2**2*XM**2-352.*A2**2*XM**4-16.*A*B9*XM**2-8.*A*B8*
 2XM**4
  Y(15,16)=256.*A1*A2*XM**6-140.*A1*A2*XM**8+88.*A2**2*XM**4-280.*A2
 1**2*XM**6-20.*A1**2*XM**8-10.*A*B9*XM**4
  WRITE (6,65)
65 FORMAT(1X ,29X,3H***,10X,26HY    -    M    A    T    R    I    X,1
 10X,3H***///)
  CALL SESCMI(Y,15,1,0,36,37,DET,RANK,SOLN)
  C0 = Y(1,1)
  C1 = Y(2,1)
  C2 = Y(3,1)
  C3 = Y(4,1)
  C4 = Y(5,1)
  C5 = Y(6,1)
  C6 = Y(7,1)
  C7 = Y(8,1)
  C8 = Y(9,1)
  C9 = Y(10,1)
  C10 = Y(11,1)
  C11 = Y(12,1)
  C12 = Y(13,1)
  C13 = Y(14,1)
  C14 = Y(15,1)
  NAMELIST/NAM3/C0,C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14
  WRITE(6,NAM3)
  DO 80 I = 1,36
  DO 80 J = 1,37
80 Z(I,J) = 0.
  Z(1,1)=-2.*XM**2-2.
  Z(1,2)=+2.
  Z(1,3)=+2.
  Z(2,2)=-2.*XM**2-12.
  Z(2,3)=-2.
  Z(2,4)=+12.
```

```
Z(2,5)=+2.
Z(3,2)=-2.*XM**2
Z(3,3)=-12.*XM**2-2.
Z(3,5)=+2.
Z(3,6)=+12.
Z(4,4)=-2.*XM**2-30.
Z(4,5)=-2.
Z(4,7)=+30.
Z(4,8)=+2.
Z(5,4)=-12.*XM**2
Z(5,5)=-12.*XM**2-12.
Z(5,6)=-12.
Z(5,8)=+12.
Z(5,9)=+12.
Z(6,5)=-2.*XM**2
Z(6,6)=-30.*XM**2-2.
Z(6,9)=+2.
Z(6,10)=+30.
Z(7,7)=-2.*XM**2-56.
Z(7,8)=-2.
Z(7,11)=+56.
Z(7,12)=+2.
Z(8,7)=-30.*XM**2
Z(8,8)=-12.*XM**2-30.
Z(8,9)=-12.
Z(8,12)=+30.
Z(8,13)=+12.
Z(9,8)=-12.*XM**2
Z(9,9)=-30.*XM**2-12.
Z(9,10)=-30.
Z(9,13)=+12.
Z(9,14)=+30.
Z(10,9)=-2.*XM**2
Z(10,10)=-56.*XM**2-2.
Z(10,14)=+2.
Z(10,15)=+56.
Z(11,11)=-2.*XM**2-90.
Z(11,12)=-2.
Z(11,16)=+90.
Z(11,17)=+2.
Z(12,11)=-56.*XM**2
Z(12,12)=-12.*XM**2-56.
Z(12,13)=-12.
Z(12,17)=+56.
Z(12,18)=+12.
Z(13,12)=-30.*XM**2
Z(13,13)=-30.*XM**2-30.
Z(13,14)=-30.
Z(13,18)=+30.
Z(13,19)=+30.
Z(14,13)=-12.*XM**2
Z(14,14)=-56.*XM**2-12.
Z(14,15)=-56.
Z(14,19)=+12.
Z(14,20)=+56.
Z(15,14)=-2.*XM**2
```

```
Z(15,15)=-90.*XM**2-2.
Z(15,20)=+2.
Z(15,21)=+90.
Z(16,16)=-2.*XM**2-132.
Z(16,17)=-2.
Z(16,22)=+132.
Z(16,23)=+2.
Z(17,16)=-90.*XM**2
Z(17,17)=-12.*XM**2-90.
Z(17,18)=-12.
Z(17,23)=+90.
Z(17,24)=+12.
Z(18,17)=-56.*XM**2
Z(18,18)=-30.*XM**2-56.
Z(18,19)=-30.
Z(18,24)=56.
Z(18,25)=+30.
Z(19,18)=-30.*XM**2
Z(19,19)=-56.*XM**2-30.
Z(19,20)=-56.
Z(19,25)=+30.
Z(19,26)=+56.
Z(20,19)=-12.*XM**2
Z(20,20)=-90.*XM**2-12.
Z(20,21)=-90.
Z(20,26)=+12.
Z(20,27)=+90.
Z(21,20)=-2.*XM**2
Z(21,21)=-132.*XM**2-2.
Z(21,27)=+2.
Z(21,28)=+132.
Z(22,22)=-2.*XM**2-182.
Z(22,23)=-2.
Z(22,29)=182.
Z(22,30)=+2.
Z(23,22)=-132.*XM**2
Z(23,23)=-12.*XM**2-132.
Z(23,24)=-12.
Z(23,30)=+132.
Z(23,31)=+12.
Z(24,23)=-90.*XM**2
Z(24,24)=-30.*XM**2-90.
Z(24,25)=-30.
Z(24,31)=+90.
Z(24,32)=+30.
Z(25,24)=-56.*XM**2
Z(25,25)=-56.*XM**2-56.
Z(25,26)=-56.
Z(25,32)=+56.
Z(25,33)=+56.
Z(26,25)=-30.*XM**2
Z(26,26)=-90.*XM**2-30.
Z(26,27)=-90.
Z(26,33)=30.
Z(26,34)=90.
Z(27,26)=-12.*XM**2
```

```
Z(27,27)=-132.*XM**2-12.
Z(27,28)=-132.
Z(27,34)=12.
Z(27,35)=132.
Z(28,27)=-2.*XM**2
Z(28,28)=-182.*XM**2-2.
Z(28,35)=+2.
Z(28,36)=+182.
Z(29,29)=-2.*XM**2-240.
Z(29,30)=-2.
Z(30,29)=-182.*XM**2
Z(30,30)=-12.*XM**2-182.
Z(30,31)=-12.
Z(31,30)=-132.*XM**2
Z(31,31)=-30.*XM**2-132.
Z(31,32)=-30.
Z(32,31)=-90.*XM**2
Z(32,32)=-56.*XM**2-90.
Z(32,33)=-56.
Z(33,32)=-56.*XM**2
Z(33,33)=-90.*XM**2-56.
Z(33,34)=-90.
Z(34,33)=-30.*XM**2
Z(34,34)=-132.*XM**2-30.
Z(34,35)=-132.
Z(35,34)=-12.*XM**2
Z(35,35)=-182.*XM**2-12.
Z(35,36)=-182.
Z(36,35)=-2.*XM**2
Z(36,36)=-240.*XM**2-2.
Z(1,37)=A0*B0
Z(2,37)=-2.*A*C0+3.*A0*B1-5.*A0*B0+A1*B0
Z(3,37)=2.*A*C0*XM**2+A0*B2-7.*A0*B0*XM**2+3.*A2*B0
Z(4,37)=-2.*A*C1+4.*A*C0+5.*A0*B3-11.*A0*B1+7.*A0*B0+3.*A1*B1-5.*A
11*B0
Z(5,37)=-6.*A*C2+6.*A*C1*XM**2+3.*A0*B4-5.*A0*B2-21.*A0*B1*XM**2+1
18.*A0*B0*XM**2-3.*A1*B2-3.*A1*B0*XM**2+9.*A2*B1-15.*A2*B0+8.*A0*B2
Z(6,37)=2.*A*C2*XM**2-4.*A*C0*XM**4+A0*B5-7.*A0*B2*XM**2+11.*A0*B0
1*XM**4+3.*A2*B2-13.*A2*B0*XM**2
Z(7,37)=-2.*A*C3+4.*A*C1-2.*A*C0+7.*A0*B6-17.*A0*B3+13.*A0*B1-3.*A
10*B0+5.*A1*B3-11.*A1*B1+7.*A1*B0
Z(8,37)=-6.*A*C4+12.*A*C2-8.*A*C1*XM**2-2.*A*C0*XM**2+10.*A*C3*XM*
1*2+5.*A0*B7-3.*A0*B4-35.*A0*B3*XM**2-9.*A0*B2-11.*A0*B0*XM**2+46.*
2A0*B1*XM**2-A1*B4+15.*A1*B2-17.*A1*B1*XM**2+10.*A1*B0*XM**2+15.*A2
3*B3-33.*A2*B1+21.*A2*B0
Z(9,37)=-10.*A*C5+8.*A*C2*XM**2+6.*A*C4*XM**2-12.*A*C1*XM**4+2.*A*
1C0*XM**4+3.*A0*B8+11.*A0*B5-21.*A0*B4*XM**2+33.*A0*B1*XM**4-13.*A0
2*B0*XM**4+2.*A0*B2*XM**2-7.*A1*B5+9.*A1*B2*XM**2+3.*A1*B0*XM**4+9.
3*A2*B4-7.*A2*B2-39.*A2*B1*XM**2+38.*A2*B0*XM**2
Z(10,37)=2.*A*C5*XM**2-4.*A*C2*XM**4+2.*A*C0*XM**6+A0*B9+11.*A0*B2
1*XM**4-5.*A0*B0*XM**6-7.*A0*B5*XM**2+3.*A2*B5-13.*A2*B2*XM**2+17.*
2A2*B0*XM**4
Z(11,37)=-2.*A*C6+4.*A*C3-2.*A*C1-23.*A0*B6+19.*A0*B3-5.*A0*B1+7.*
1A1*B6-17.*A1*B3+13.*A1*B1-3.*A1*B0
Z(12,37)=-6.*A*C7+12.*A*C4-16.*A*C3*XM**2+2.*A*C1*XM**2-6.*A*C2+14
1.*A*C6*XM**2-25.*A0*B1*XM**2-9.*A0*B7-49.*A0*B6*XM**2-3.*A0*B4+5.*
```

```
2A0*B2+74.*A0*B3*XM**2+A1*B7+9.*A1*B4-31.*A1*B3*XM**2-21.*A1*B2-7.*
3A1*BC*XM**2+38.*A1*B1*XM**2+21.*A2*B6-51.*A2*B3+39.*A2*B1-9.*A2*B0
 Z(13,37)=-10.*A*C6+20.*A*C5-10.*A*C2*XM**2+10.*A*C1*XM**4+10.*A*C7
1*XM**2-20.*A*C3*XM**4+5.*A0*B8-35.*A0*B7*XM**2-25.*A0*B5+5.*A0*B2*
2XM**2+55.*A0*B3*XM**4-35.*A0*B1*XM**4+30.*A0*B4*XM**2-5.*A1*B8+35.
3*A1*B5-5.*A1*B4*XM**2+25.*A1*B1*XM**4-5.*A1*B0*XM**4-30.*A1*B2*XM
4**2+15.*A2*B7-25.*A2*B4-65.*A2*B3*XM**2+5.*A2*B2-25.*A2*B0*XM**2+9
50.*A2*B1*XM**2
 Z(14,37)=-14.*A*C9+16.*A*C5*XM**2-2.*A*C2*XM**4+6.*A*C8*XM**2-12.*
1A*C4*XM**4+6.*A*C1*XM**6+15.*A0*B9-21.*A0*B8*XM**2+33.*A0*B4*XM**4
2-5.*A0*B2*XM**4-15.*A0*B1*XM**6-14.*A0*B5*XM**2-11.*A1*B9+21.*A1*B
35*XM**2-9.*A1*B2*XM**4-A1*B0*XM**6+9.*A2*B8+A2*B5-39.*A2*B4*XM**2+
451.*A2*B1*XM**4-23.*A2*B0*XM**4+22.*A2*B2*XM**2
 Z(15,37)=2.*A*C9*XM**2-4.*A*C5*XM**4+2.*A*C2*XM**6-7.*A0*B9*XM**2+
111.*A0*B5*XM**4-5.*A0*B2*XM**6+3.*A2*B9-13.*A2*B5*XM**2+17.*A2*B2*
2XM**4-7.*A2*B0*XM**6
 Z(16,37)=-2.*A*C10+4.*A*C6-2.*A*C3-25.*A0*B6-7.*A0*B3-23.*A1*B6+19
1.*A1*B3-5.*A1*B1
 Z(17,37)=-6.*A*C11+12.*A*C7-24.*A*C6*XM**2-6.*A*C4+6.*A*C3*XM**2+1
18.*A*C10*XM**2+3.*A0*B7+3.*A0*B4-39.*A0*B3*XM**2+102.*A0*B6*XM**2+
23.*A1*B7-45.*A1*B6*XM**2-15.*A1*B4+9.*A1*B2-21.*A1*B1*XM**2+66.*A1
3*B3*XM**2-69.*A2*B6+57.*A2*B3-15.*A2*B1
 Z(18,37)=-10.*A*C12+20.*A*C9-8.*A*C7*XM**2-10.*A*C5-6.*A*C4*XM**2+
118.*A*C3*XM**4+14.*A*C11*XM**2-28.*A*C6*XM**4-19.*A0*B8+13.*A0*B5-
29.*A0*B4*XM**2+77.*A0*B6*XM**4-57.*A0*B3*XM**4+58.*A0*B7*XM**2+29.
3*A1*B8-19.*A1*B7*XM**2-49.*A1*B5+21.*A1*B2*XM**2+47.*A1*B3*XM**4-2
47.*A1*B1*XM**4-2.*A1*B4*XM**2-43.*A2*B7-91.*A2*B6*XM**2+23.*A2*B4-
5A2*B2-51.*A2*B1*XM**2+142.*A2*B3*XM**2
 Z(19,37)=-14.*A*C13+26.*A*C9+8.*A*C8*XM**2-18.*A*C5*XM**2+6.*A*C4*
1XM**4+10.*A*C12*XM**2-20.*A*C7*XM**4+10.*A*C3*XM**6-41.*A0*B9+21.*
2A0*B5*XM**2+55.*A0*B7*XM**4-27.*A0*B4*XM**4-25.*A0*B3*XM**6+14.*A0
3*B8*XM**2+55.*A1*B9+7.*A1*B8*XM**2+13.*A1*B4*XM**4+15.*A
41*B2*XM**4-11.*A1*B1*XM**6-70.*A1*B5*XM**2-17.*A2*B8-65.*A2*B7*XM*
5*2-11.*A2*B5-9.*A2*B2*XM**2+85.*A2*B3*XM**4-57.*A2*B1*XM**4+74.*A2
6*B4*XM**2
 Z(20,37)=-18.*A*C14+24.*A*C9*XM**2-6.*A*C5*XM**4+6.*A*C13*XM**2-12
1.*A*C8*XM**4+6.*A*C4*XM**6+33.*A0*B8*XM**4+3.*A0*B5*XM**4-15.*A0*B
24*XM**6-30.*A0*B9*XM**2+33.*A1*B9*XM**2+28.*A1*B5*XM**4+3.*A1*B2*X
3M**6+9.*A2*B9-39.*A2*B8*XM**2+51.*A2*B4*XM**4-15.*A2*B2*XM**4-21.*
4A2*B1*XM**6+6.*A2*B5*XM**2
 Z(21,37)=2.*A*C14*XM**2-4.*A*C9*XM**4+2.*A*C5*XM**6+11.*A0*B9*XM**
14-5.*A0*B5*XM**6-13.*A2*B9*XM**2+17.*A2*B5*XM**4-7.*A2*B2*XM**6
 Z(22,37)=4.*A*C10-2.*A*C6-9.*A0*B6+25.*A1*B6-7.*A1*B3
 Z(23,37)=12.*A*C11-32.*A*C10*XM**2-6.*A*C7+10.*A*C6*XM**2+A0*B7-53
1.*A0*B6*XM**2-9.*A1*B7+7.*A1*B4-35.*A1*B3*XM**2+94.*A1*B6*XM**2+75
2.*A2*B6-21.*A2*B3
 Z(24,37)=20.*A*C12-16.*A*C11*XM**2-10.*A*C8-2.*A*C7*XM**2+26.*A*C6
1*XM**4-35.*A*C10*XM**4+11.*A0*B8-23.*A0*B7*XM**2-79.*A0*B6*XM**4-4
23.*A1*B8+21.*A1*B5+7.*A1*B4*XM**2+69.*A1*B6*XM**4-49.*A1*B3*XM**4+
326.*A1*B7*XM**2+41.*A2*B7-7.*A2*B4-77.*A2*B3*XM**2+194.*A2*B6*XM**
42
 Z(25,37)=28.*A*C13-14.*A*C9-14.*A*C8*XM**2+14.*A*C7*XM**4-28.*A*C1
11*XM**4+14.*A*C6*XM**6+21.*A0*B9+7.*A0*B8*XM**2-49.*A0*B7*XM**4-35
2.*A0*B6*XM**6-77.*A1*B9+49.*A1*B5*XM**2+35.*A1*B7*XM**4-7.*A1*B4*X
3M**4-21.*A1*B3*XM**6-42.*A1*B8*XM**2+7.*A2*B8+7.*A2*B5-35.*A2*B4*X
4M**2+119.*A2*B6*XM**4-91.*A2*B3*XM**4+126.*A2*B7*XM**2
```

```
 Z(26,37)=36.*A*C14+16.*A*C13*XM**2-26.*A*C9*XM**2+2.*A*C8*XM**4-20
1.*A*C12*XM**4+10.*A*C7*XM**6+37.*A0*B9*XM**2-19.*A0*B8*XM**4-25.*A
2C*E7*XM**6+A1*B8*XM**4+35.*A1*B5*XM**4-7.*A1*B4*XM**6-110.*A1*
3B9*XM**2-27.*A2*B9+7.*A2*B5*XM**2+65.*A2*B7*XM**4-49.*A2*B4*XM**4-
435.*A2*B3*XM**6+58.*A2*BE*XM**2
 Z(27,37)=32.*A*C14*XM**2-1C.*A*C9*XM**4-12.*A*C13*XM**4+6.*A*C8*XM
1**6+11.*A0*B9*XM**4-15.*AC*B8*XM**6-33.*A1*B9*XM**4+7.*A1*B5*XM**6
2+51.*A2*B8*XM**4-7.*A2*B5*XM**4-21.*A2*B4*XM**6-10.*A2*B9*XM**2
 Z(28,37)=-5.*AC*B9*XM**6+17.*A2*B9*XM**4-7.*A2*B5*XM**6+2.*A*C9*XM
1**6-4.*A*C14*XM**4
 Z(29,37)=-9.*A1*B6-2.*A*C1C
 Z(30,37)=-6.*A*C11+14.*A*C10*XM**2+5.*A1*B7-49.*A1*B6*XM**2-27.*A2
1*B6
 Z(31,37)=-1C.*A*C12+2.*A*C11*XM**2+34.*A*C10*XM**4+19.*A1*B8-7.*A1
1*B7*XM**2-71.*A1*B6*XM**4-13.*A2*B7-103.*A2*B6*XM**2
 Z(32,37)=-14.*A*C13-1C.*A*C12*XM**2+22.*A*C11*XM**4+18.*A*C10*XM**
16+33.*A1*B9+35.*A1*B8*XM**2-29.*A1*B7*XM**4-31.*A1*B6*XM*
2*6+A2*B8-61.*A2*B7*XM**2-125.*A2*B6*XM**4
 Z(33,37)=-18.*A*C14-22.*A*C13*XM**2+10.*A*C12*XM**4+14.*A*C11*XM**
16+77.*A1*B9*XM**2+13.*A1*B8*XM**4-17.*A1*B7*XM**6+15.*A2*B9-19.*A2
2*B8*XM**2-83.*A2*B7*XM**4-49.*A2*B6*XM**6
 Z(34,37)=-34.*A*C14*XM**2-2.*A*C13*XM**4+10.*A*C12*XM**6+55.*A1*B9
1*XM**4-3.*A1*B8*XM**6+23.*A2*B9*XM**2-41.*A2*B8*XM**4-35.*A2*B7*XM
2**6
 Z(35,37)=-14.*A*C14*XM**4+6.*A*C13*XM**6+11.*A1*B9*XM**6+A2*B9*XM*
1*4-21.*A2*B8*XM**6
 Z(36,37)=2.*A*C14*XM**6-7.*A2*B9*XM**6
 WRITE (6,85)
85 FORMAT(1X ,29X,3H***,10X,36HZ     -     M     A     T     R     I     X,1
10X,3H***///)
 CALL SESCMI(Z,36,1,0,36,37,DET,RANK,SOLN)
 DC  = Z(1  ,1)
 D1  = Z(2  ,1)
 D2  = Z(3  ,1)
 D3  = Z(4  ,1)
 D4  = Z(5  ,1)
 D5  = Z(6  ,1)
 D6  = Z(7  ,1)
 D7  = Z(8  ,1)
 D8  = Z(9  ,1)
 D9  = Z(10 ,1)
 D10 = Z(11 ,1)
 D11 = Z(12 ,1)
 D12 = Z(13 ,1)
 D13 = Z(14 ,1)
 D14 = Z(15 ,1)
 D15 = Z(16 ,1)
 D16 = Z(17 ,1)
 D17 = Z(18 ,1)
 D18 = Z(19 ,1)
 D19 = Z(20 ,1)
 D20 = Z(21 ,1)
 D21 = Z(22 ,1)
 D22 = Z(23 ,1)
 D23 = Z(24 ,1)
 D24 = Z(25 ,1)
```

```
      D25 = Z(26 ,1)
      D26 = Z(27 ,1)
      D27 = Z(28 ,1)
      D28 = Z(29 ,1)
      D29 = Z(30 ,1)
      D30 = Z(31 ,1)
      D31 = Z(32 ,1)
      D32 = Z(33 ,1)
      D33 = Z(34 ,1)
      D34 = Z(35 ,1)
      D35 = Z(36 ,1)
      NAMELIST/NAM4/D0,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,D11,D12,D13,D14,D1
     15,D16,D17,D18,D19,D20,D21,D22,D23,D24,D25,D26,D27,D28,D29,D30,D31,
     2D32,D33,D34,D35
      WRITE (5,NAM4)
      Z(1,1) = D0
      Z(3,1) = D1 - D0
      Z(5,1) = D3 - D1
      Z(1,3) = D2 - D0*XM2
      Z(3,3) = D4 - D2 - D1*XM2
      Z(1,5) = D5 - D2*XM2
      Z(7,1) = D6 - D3
      Z(5,3) = D7 - D4 - D3*XM2
      Z(3,5) = D8 - D5 - D4*XM2
      Z(1,7) = D9 - D5*XM2
      Z(9,1) = D10 - D6
      Z(7,3) = D11 - D7 - D6*XM2
      Z(5,5) = D12 - D8 - D7*XM2
      Z(3,7) = D13 - D9 - D8*XM2
      Z(1,9) = D14 - D9*XM2
      Z(11,1) = D15 - D10
      Z( 9,3) = D16 - D11 - D10*XM2
      Z( 7,5) = D17 - D12 - D11*XM2
      Z( 5,7) = D18 - D13 - D12*XM2
      Z( 3,9) = D19 - D14 - D13*XM2
      Z( 1,11)= D20 - D14*XM2
      Z( 13,1)= D21 - D15
      Z(11,3) = D22 - D16 - D15*XM2
      Z( 9,5) = D23 - D17 - D16*XM2
      Z( 7,7) = D24 - D18 - D17*XM2
      Z( 5,9) = D25 - D19 - D18*XM2
      Z(3,11) = D26 - D20 - D19*XM2
      Z(1,13) = D27 - D20*XM2
      Z(15,1) = D28 - D21
      Z(13,3) = D29 - D22 - D21*XM2
      Z(11,5) = D30 - D23 - D22*XM2
      Z(9,7)  = D31 - D24 - D23*XM2
      Z(7,9)  = D32 - D25 - D24*XM2
      Z(5,11) = D33 - D26 - D25*XM2
      Z(3,13) = D34 - D27 - D26*XM2
      Z(1,15) = D35 - D27*XM2
      Z(17,1) = -D28
      Z(15,3) = -D29 - D28*XM2
      Z(13,5) = -D30 - D29*XM2
      Z(11,7) = -D31 - D30*XM2
      Z(9,9)  = -D32 - D31*XM2
```

```
      Z(7,11) = -D33 - D32*XM2
      Z(5,13) = -D34 - D33*XM2
      Z(3,15) = -D35 - D34*XM2
      Z(1,17) = -D35*XM2
      DO 100 L=2,18,2
      I = L-2
      XI = I
      LK = 18- I
      DO 100 K= 2,LK,2
      J = K-2
      XJ = J
      XJJ = (XJ + 1.)/2.
      CC1= 4.0/((XJ + 1.)*(XM**(XJ+1.)))
      CALL ICRAT (0.0,1.0,.01,XINT,1)
      IF(I.EQ.0.AND.J.EQ.2) XINT02 = XINT*CC1
      IF(I.EQ.2.AND.J.EQ.0) XINT20 = XINT*CC1
      IF(I.EQ.0.AND.J.EQ.0) XINT00 = XINT*CC1
      SUM = Z(I+1,J+1)*CC1 *XINT
      C = C + SUM
      C2 = C
  100 CONTINUE
      QO= A*(XINT00 - XINT20 - XM2*XINT02)
      VORP = SQRT((3.*A2-5.*A0*XM2+SQRT((5.*A0*XM2-3.*A2)**2+28.*A0*A2*X
     1M2))/(14.*A2*XM2))
      VORM = - VORP
      WRITE (6,1001)QO,VORP,VORM,Q2
 1001 FORMAT (1X,5HQO = ,E15.8,5X,7HVORP = ,E15.8,5X,7HVORM = ,E15.8,5X,
     15HQ2 = ,E15.8)
      Q2DQO = Q2/QO
      QT = QO*(1. - Q2DOO*XK*XK)
      WRITE (6,1004) Q2DOO,QT
 1004 FORMAT(1X,8HQ2/QO = ,E15.8,5X,11HQ(TOTAL) = ,E15.8)
      CALL PLOT (XM,XM2,A0,A1,A2,B0,B1,B2,B3,B4,B5,B6,B7,B8,B9,C0,C1,C2,
     1C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14,PSI11,PSI21,PSI12,PSI22,P
     2SI13,PSI23,PSI14,PSI24,YC,DELY)
      GO TO 5
      END
```

```
$IBFTC SESC3B                    SIM. EQN. SOLVR OR MATRIX INVRTR

      SUBROUTINE SESOMI(X,N,NB,MS,MN1,MN2,D,R,E)
      DOUBLE PRECISION WORK,SAVR,X,Y,D,SUM,SAVEB
      DOUBLE PRECISION XM,XM2,AC,A1,A2,B0,B1,B2,B3,B4,B5,B6,B7,B8,B9,C0,
     1C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14
      DIMENSION X(MN1,MN2),WORK(99),SAVR(50)
      DIMENSION IHLD(50)
      E=0.
      R=0.
      DO 27 I=1,N
   27 SAVR(I)=X(1,I)
      DO 21 I=1,N
   21 IHLD(I)=I
      IF(MS)6,4,6
    6 NN=N+N
      NB=N
      MN=N+1
      DO 14 I=1,N
      DO 14 J=MN,NN
   14 X(I,J)=0.D0
      DO 15 I=1,N
      J=I+N
   15 X(I,J)=1.D0
      GO TO 16
    4 NN=N+NB
   16 JJ=NN
      SAVEB=X(1,N+1)
      NNN=N-1
      D=1.D0
      DO 5 I=1,N
      KK=N-I
      IF(KK)10,10,26
   26 LL=KK+1
      IJJ=1
      L=I
      WORK=X
      DO 17 II=1,LL
      DO 17 J=1,LL
      IF(ABS(WORK)-ABS(X(II,J)))18,17,17
   18 WORK=X(II,J)
      L=J+I-1
      IJJ=J
   17 CONTINUE
      IF(IJJ-1)2,2,19
   19 DO 20 II=1,N
      Y=X(II,1)
      X(II,1)=X(II,IJJ)
   20 X(II,IJJ)=Y
      IY=IHLD(I)
      IHLD(I)=IHLD(L)
      IHLD(L)=IY
      D=-D
    2 DO 1 L=1,KK
      IF(ABS(X)-ABS(X(L+1,1)))7,1,1
    7 D=-D
      DO 9 J=1,JJ
```

```
      Y=X(I,J)
      X(I,J)=X(L+1,J)
    9 X(L+1,J)=Y
    1 CCNTINLE
   10 JJ=JJ-1
      IF(X)11,8,11
   11 C=C*X
      R=R+1.
      DC 12 J=1,JJ
   12 WCRK(J)=X(1,J+1)/X
      KK=JJ+1
      CC 3 K=1,NNN
      DO 3 J=2,KK
    3 X(K,J-1)=X(K+1,J)-X(K+1,1)*WORK(J-1)
      DO 5 J=1,JJ
    5 X(N,J)=WCRK(J)
      NN=N-1
      DC 22 I=1,NN
      L=I+1
      CC 22 J=L,N
      IF(IHLC(I)-IHLD(J))22,22,23
   23 IY=IHLC(I)
      IHLD(I)=IHLD(J)
      IHLD(J)=IY
      CC 25 K=1,NB
      Y=X(I,K)
      X(I,K)=X(J,K)
   25 X(J,K)=Y
   22 CCNTINLE
      SUM=C.CO
      DC 28 I=1,N
   29 SUM=SUM+X(I,1)*SAVR(I)
      TEST=ABS((SAVEB-SUM)/SAVEB)
      IF(TEST-.00001)13,13,8
   13 RETURN
    8 F=1.
      GO TC 13
      END
```

```
$IBFTC IGRT

      SUBROUTINE IGRAT(LL,UL,DELTA,ANS,NOEQ)
      DIMENSION F(6),U(6),R(3)
      DATA (U(I), I=1,6) / .11930959,-.11930959,.33060469,-.33060469,
     A .46623476,-.46623476 /
      DATA (R(I), I=1,3) / .23395697,.18038079,.85662246E-1 /
      REAL LLIM,MULT,LL
      ULIM = UL
      LLIM = LL
      MULT = 1.0
      IF(ULIM .GE. LLIM)   GO TO 5
      TMP1 = LLIM
      LLIM = ULIM
      ULIM = TMP1
      MULT =-1.0
    5 A = LLIM
      DEL = ABS(DELTA)
      LAST = 1
      ANS = 0.0
      IF(ABS (ULIM-LLIM) - .00001)80,80,10
   10 B = A + 1.0*DEL
      IF(B-ULIM)40,30,20
   20 B = ULIM
   30 LAST = 2
   40 DO 50    I=1,6
      X = (B-A)*U(I) + .5*(A+B)
      CALL INTEQS(X,F(I),NOEQ)
   50 CONTINUE
   60 ANS = ANS + (B-A)*(R(1)*(F(1)+F(2)) + R(2)*(F(3)+F(4)) + R(3)*
     A (F(5)+F(6)))
      GO TO (70,80),LAST
   70 A = B
      GO TO 10
   80 ANS = ANS * MULT
      RETURN
      END


$IBFTC INTESQ

      SUBROUTINE INTEQS(X,Y,N)
      COMMON XI,XJJ
      Y = (X**XI)*(1.-X*X)**XJJ
      RETURN
      END
```

```
$IBFTC PLOTT   LIST,REF

      SUBROUTINE PLOT (  XM,XM2,A0,A1,A2,B0,B1,B2,B3,B4,B5,B6,B7,B8,B9,C
     10,C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,C14,SX1,SY1,SX2,SY2,S
     2X3,SY3,SX4,SY4,YC,    DELY)
      DIMENSION AA(16),BB(14),CC(14),RR(13),RI(13),AA1(14),BB1(14),AA2(1
     14),BB2(14)
      DOUBLE PRECISION BB,CC,RR,RI,AA1,BB1,AA2,BB2
      L = 1
      XM4 = XM2*XM2
      YF = YC + 4.*DELY
    2 GO TO (5,10,15,20),L
    5 CONTINUE
      X=SX1
      Y = SY1
      GO TO 25
   10 CONTINUE
      X = SX2
      Y = SY2
      GO TO 25
   15 CONTINUE
      X = SX3
      Y = SY3
      GO TO 25
   20 CONTINUE
      X = SX4
      Y = SY4
   25 CONTINUE
      AA(L) = X
      AA(L + 4) = Y
      L = L + 1
      IF( L.GT.4) GO TO 30
      GO TO 2
   30 CONTINUE
      N = 1
   32 GO TO (35,40) ,N
   35 M = 1
  351 CONTINUE
      Y = YC
   36 Y2 = Y*Y
      G0 =     Y*(A0+Y2*(Y2*((A2*XM4)*Y2+(A0*XM4-2.*A2*XM2))+(A2-2.*A0*XM2
     1)))
      G1 = Y*((A1-2.*A0) +Y2*((A1*XM4+2.*A2*XM2)*Y2 + (2.*A0*XM2-2.*A1*X
     1M2-2.*A2)))
      G2 = Y*((A0-2.*A1) + Y2*(2.*A2*XM2 + A2))
      G3 = Y*(A1)
      BB(1)= G3
      BB(2)= 0.0
      BB(3)= G2
      BB(4)= 0.0
      BB(5)= G1
      BB(6)= 0.0
      BB(7)= G0 - AA(M)
      DO 361 I =1,14
  361 CC(I) = 0.0D0
      CALL ROOTS(BB,CC,6,6,RR,RI,IERR,AA1,BB1,AA2,BB2)
      Y = Y + DELY
```

```
      IF(Y.GT.YF) GO TO 38
      GO TO 36
   38 M = M + 1
      IF (M.GT.4) GO TO 60
      GO TO 351
   40 M = 1
  401 CONTINUE
      Y = Y0
   41 Y2 = Y*Y
      AF0  = C0
      AF1  = C2 - 2.*C0*XM2
      AF2  = C5 - 2.*C2*XM2 + XM4*C0
      AF3  = C9 - 2.*C5*XM2 + XM4*C2
      AF4  = C14- 2.*C9*XM2 + XM4*C5
      AF5  =     -2.*C14*XM2 + XM4*C9
      AF6  =                   XM4*C14
      AF8  = C1 - 2.*C0
      AF9  = C4 - 2.*C2 - 2.*C1*XM2 + 2.*XM2*C0
      AF10 = C8 - 2.*C5 - 2.*C4*XM2 + 2.*XM2*C2 + XM4*C1
      AF11 = C13- 2.*C9 - 2.*C8*XM2 + 2.*XM2*C5 + XM4*C4
      AF12 = -2.*C14 - 2.*C13*XM2 + 2.*XM2*C9 + XM4*C8
      AF13 = 2.*XM2*C14 + XM4*C13
      AF14 = C3 - 2.*C1 + C0
      AF15 = C7-2.*C4-2.*C3*XM2+C2+2.*XM2*C1
      AF16 = C12 -2.*C8-2.*C7*XM2 + C5 + 2.*XM2*C4 + XM4*C3
      AF17 = -2.*C13-2.*C12*XM2 +C9 + 2.*XM2*C8 + XM4*C7
      AF18 = C14 + 2.*XM2*C13 + XM4*C12
      AF19 = C6 -2.*C3 + C1
      AF20 = C11 -2.*C7 -2.*C6*XM2 + C4 + 2.*XM2*C3
      AF21 = -2.*C12 - 2.*C11*XM2 +C8 +2.*XM2*C7 + XM4*C6
      AF22 = C13 + 2.*XM2*C12 + XM4*C11
      AF23 = C10 - 2.*C6 + C3
      AF24 = -2.*C11 - 2.*C10*XM2 + C7 + 2.*XM2*C6
      AF25 = C12 + 2.*C11*XM2 + XM4*C10
      AF26 = - 2.*C10 + C6
      AF27 = C11 + 2.*XM2*C10
      BB(1)= Y*(C10)
      BB(2)=0.0
      BB(3)= Y*(AF26 + Y2*(AF27))
      BB(4)=0.0
      BB(5)= Y*(AF23 + Y2*(AF25*Y2 + AF24))
      BB(6)= 0.0
      BB(7)= Y*(AF19 + Y2*(Y2*(AF22*Y2+AF21)+AF20))
      BB(8)= 0.0
      BB(9) =Y*(AF14+Y2*(Y2*(Y2*(AF18*Y2+AF17)+AF16)+AF15))
      BB(10) = 0.0
      BB(11)=Y*(AF8+Y2*(Y2*(Y2*(Y2*(AF13*Y2+AF12)+AF11)+AF10)+AF9))
      BB(12)= 0.0
      BB(13)=Y*(AF0+Y2*(Y2*(Y2*(Y2*(Y2*(AF6*Y2+AF5)+AF4)+AF3)+AF2)+AF1))
      BB(14)= -AA(M +4)
      DO 411 I = 1,14
  411 CC(I) = 0.0D0
      CALL ROOTS(BB,CC,13,13,RR,RI,IERR,AA1,BB1,AA2,BB2)
      Y= Y +DELY
      IF(Y.GT.YF) GO TO 43
      GO TO 41
```

```
43 M = M + 1
   IF( M.GT.4) GO TO 60
   GO TO 401
60 N = N+1
   IF(N.GT.2) GO TO 65
   GO TO 32
65 CONTINUE
   DELX = 0.19
   X = -0.99
69 CONTINUE
   T1 = (1.0 + X)/(1.0 - X)
   T2 = A1/A0
   T3 = 0.5/(A0 + A1)
   T4 = A0/A1
   IF (T4.GE.0.0)  GO TO 70
   T5 = SQRT(-T2)
   THETL = T3*(ALOG(T1)+T5*ALOG((1.0+T5*X)/(1.0-T5*X)))
   GO TO 75
70 CONTINUE
   T5 = SQRT(T2)
   THETL = T3*(ALOG(T1) + 2.0*T5*ATAN(T5*X))
75 CONTINUE
   X2 = X*X
   W1 = (1.0-X2)*X*(X2*(X2*(B6*X2 + B3) + B1) + B0)
   X = X + DELX
   DELX = 0.20
   IF(X.GT.0.8.AND.X.LT.1.1) X = 0.99
   IF(X.GT.1.0) GO TO 80
   GO TO 69
80 CONTINUE
   RETURN
   END
```

```
$IBFTC ROOST

      SUBROUTINE ROOTS (A,B,NN,NR,RR,RI,IERR,A1,A2,B1,B2)
      DIMENSION A(1), B(1), RR(1), RI(1), A1(1), B1(1), A2(1), B2(1)
      DOUBLE PRECISION A,B,A1,B1,A2,B2,RR,RI,C,D,C1,D1,C2,D2,F,G,F1,G1,F
     12,G2,FH,FK,X,Y,FM
      IERR=0
      L=1
      N=NN
      FM=1.D0
      IF (NN-1) 25,25,1
    1 N1=N+1
      JA=NR
      D1=DMAX1(DABS(A(N+1)),DABS(B(N+1)))
      D2=DMAX1(DABS(A),DABS(B))
      FK=N
      FM=D2**(1.D0/FK)/D1**(1.D0/FK)
      IF (DABS(FM).LT.1.D0) GO TO 27
      K=N
      DO 2 I=1,N
      A(I)=A(I)/(D1*FM**K)
      B(I)=B(I)/(D1*FM**K)
    2 K=K-1
      A(N1)=A(N1)/D1
      B(N1)=B(N1)/D1
    3 DO 4 I=1,N1
      A1(I)=A(I)
      B1(I)=B(I)
      A2(I)=A(I)
    4 B2(I)=B(I)
      JA=JA-1
      NW=N-JA
      NW1=NW+1
      JR=JA+1
      IF (JA.EQ.0) GO TO 6
      DO 5 J=1,JA
      C=N1-J
      DO 5 I=1,NW1
      D=N-J-I+2
      A1(I)=D*A1(I)/C
    5 B1(I)=D*B1(I)/C
      IF (A1(NW+1).EQ.0.D0.AND.B1(NW+1).EQ.0.D0) GO TO 3
    6 X=1.01
      Y=.999
      IL=1
      LL=1
      IF (NW.EQ.1) GO TO 15
    7 C=A1
      D=B1
      C1=A1
      C2=-B1
      D1=B1
      D2=A1
      DO 8 I=1,NW
      F=X*C-Y*D+A1(I+1)
      G=X*D+Y*C+B1(I+1)
      IF (I.EQ.NW) GO TO 8
```

```
      F1=X*C1-Y*D1+F
      F2=X*C2-Y*D2-G
      G1=X*D1+Y*C1+G
      G2=X*D2+Y*C2+F
      C=F
      D=G
      C1=F1
      C2=F2
      D1=G1
    8 D2=G2
      C=DMAX1(DABS(F1),DABS(G1))
      D=DMAX1(DABS(F2),DABS(G2))
      IF (C.LT.D) GO TO 10
      IF (C.EQ.DABS(F1)) GO TO 9
      C1=F2-F1*(G2/G1)
      IF (C1.EQ.0.D0) GO TO 28
      FK=(-F+F1*(G/G1))/C1
      FH=-G/G1-FK*G2/G1
      GO TO 12
    9 C1=G2-G1*(F2/F1)
      IF (C1.EQ.0.D0) GO TO 28
      FK=(-G+G1*(F/F1))/C1
      FH=-F/F1-FK*F2/F1
      GO TO 12
   10 IF (D.EQ.DABS(G2)) GO TO 11
      C1=G1-G2*(F1/F2)
      IF (C1.EQ.0.D0) GO TO 28
      FH=(-G+G2*(F/F2))/C1
      FK=-F/F2-FH*F1/F2
      GO TO 12
   11 C1=F1-F2*(G1/G2)
      IF (C1.EQ.0.D0) GO TO 28
      FH=(-F+F2*(G/G2))/C1
      FK=-G/G2-FH*G1/G2
   12 X=X+FH
      Y=Y+FK
      IF ((X**2+Y**2).EQ.((X+FH)**2+(Y+FK)**2)) GO TO 16
      FH1=SNGL(X+FH)-SNGL(X)
      FK1=SNGL(Y+FK)-SNGL(Y)
      IF (FH1.NE.0..OR.FK1.NE.0.) GO TO 14
      GO TO (13,14), IL
   13 LL=194
      IL=2
   14 LL=LL+1
      IF (LL.GT.200) GO TO 16
      GO TO 7
   15 D=A1**2+B1**2
      X=(-A1*A1(2)-B1*B1(2))/D
      Y=(-A1*B1(2)+B1*A1(2))/D
   16 IF (JA.EQ.0) GO TO 18
      DO 17 I=1,N
      A2(I+1)=X*A2(I)-Y*B2(I)+A2(I+1)
   17 B2(I+1)=X*B2(I)+Y*A2(I)+B2(I+1)
      IF (DABS(A2(N+1)).GT.1.D-8.OR.DABS(B2(N+1)).GT.1.D-8) GO TO 21
   18 DO 20 IJ=1,JR
      DO 19 I=1,N
```

```
      A(I+1)=X*A(I)-Y*B(I)+A(I+1)
   19 B(I+1)=X*B(I)+Y*A(I)+B(I+1)
      N=N-1
      N1=N+1
      RR(L)=X/FM
      RI(L)=Y/FM
   20 L=L+1
      IF (N-1) 26,25,21
   21 IF (NW-1) 3,3,22
   22 DO 23 I=1,NW
      A1(I+1)=X*A1(I)-Y*B1(I)+A1(I+1)
   23 B1(I+1)=X*B1(I)+Y*A1(I)+B1(I+1)
      NW=NW-1
      DO 24 I=1,N1
      A2(I)=A(I)
   24 B2(I)=B(I)
      GO TO 6
   25 D=A**2+B**2
      RR(L)=(-A*A(2)-B*B(2))/(D*FM)
      RI(L)=(-A*B(2)+B*A(2))/(D*FM)
   26 RETURN
   27 FM=1.DO
      GO TO 3
   28 IERR=1
      RETURN
      END
```

APPENDIX B

RESULTS

C = 2.50

CASE 5

XM = 0.40000000E 00

A = 0.10000000E 01

A0 = 0.81325744E-02	A1 = -0.68609736E-03
A2 = -0.97284416E-03	
B0 = 0.12275631E-02	B1 = -0.11393347E-02
B2 = -0.72499039E-03	B3 = 0.40119893E-03
B4 = 0.53608607E-03	B5 = 0.15275917E-03
B6 = -0.15927414E-04	B7 = -0.10873222E-03
B8 = -0.64739868E-04	B9 = -0.11241284E-04
C0 = 0.69227686E-05	C1 = -0.15786941E-05
C2 = -0.57445541E-05	C3 = 0.53272846E-06
C4 = 0.22511617E-06	C5 = 0.13238618E-05
C6 = -0.13322194E-06	C7 = -0.18405079E-06
C8 = -0.18304866E-06	C9 = [illegible].11645324E-06
C10 = 0.78165178E-08	C11 = 0.37296582E-07
C12 = 0.29131246E-07	C13 = 0.13214829E-07
C14 = 0.44572204E-08	

D0 = -0.78940970E-06

D1 = 0.27640558E-05

D2 = 0.12408060E-05

D3 = -0.040000082E-05

D4 = -0.36843782E-05

D5 = -0.71110050E-06

D6 = 0.29335931E-05

D7 = 0.40737273E-05

D8 = 0.19504065E-05

D9 = 0.24398656E-06

D10 = -0.11768777E-05

D11 = -0.20369105-05

D12 = -0.17012411E-05

D13 = -0.51375617E-06

D14 = -0.51517571E-07

D15 = 0.25271631E-06

D16 = 0.48715251E-06

D17 = 0.61277059E-06

D18 = 0.32555807E-06

D19 = 0.93122271E-07

D20 = 0.66277360E-08

D21 = -0.18897032E-07

D22 = -0.63921863E-07

D23 = -0.10283680E-06

D24 = -0.83927032E-07

D25 = -0.34505248E-07

D26 = 0.67632013-08

D27 = -0.49788629E-09

D28 = 0.46206709E-09

D29 = 0.14275470E-08

D30 = 0.75433856E-08

D31 = 0.88623727E-08

D32 = 0.50265173E-08

D33 = 0.15429692E-08

D34 = 0.24248057E-09

D35 = 0.15090297E-10

Q0 = 0.31415923E 01

Q2 = -0.61047082E-06

Q2/Q0 = -0.19431892E-06

VORP = 0.82588796E 00

VORM = -0.82588796E 00

C = 6.50

CASE 15

XM = 0.15000000E 01

A = 0.10000000E 01

A0 = 0.41087188E-02

A1 = -0.19523771E-02

A2 = -0.19677409E-02

B0 = 0.38130281E-03

B1 = -0.51625305E-03

B2 = -0.55811986E-03

B3 = 0.27185692E-03

B4 = 0.56902572E-03

B5 = 0.20099952E-03

B6 = -0.48034968E-04

B7 = -0.11503959E-03

B8 = 0.55882562E-04

B9 = 0.21533363E-03

C0 = 0.19831074E-05

C1 = -0.20306899E-05

C2 = -0.89426459E-06

C3 = 0.83055011E-06

C4 = 0.68938810E-06

C5 = -0.20637122E-06

C6 = -0.17443536E-06

C7 = -0.39693449E-07

C8 = 0.39080392E-06

C9 = 0.31098098E-06

C10 = 0.11950084E-07

C11 = -0.39021902E-07

C12 = -0.24146492E-06

C13 = -0.40512033E-06

C14 = -0.24739573E-06

D0 = -0.52650429E-07

D1 = 0.38062327E-06

D2 = 0.23159584E-06

D3 = -0.76367925E-06

D4 = -0.14998904E-05

D5 = -0.73885116E-06

D6 = 0.78764179E-06

D7 = 0.28239710E-05

D8 = 0.35036516E-05

D9 = 0.13379614E-05

D10 = -0.47970180E-06

D11 = -0.25918261E-05

D12 = -0.53988222E-05

D13 = -0.42535094E-05

D14 = -0.12328861E-05

D15 = 0.17762290E-06

D16 = 0.13072291E-05

D17 = 0.38765723E-05

D18 = 0.50280913E-05)

D19 = 0.35096484E-05

D20 = 0.38015629E-06

D21 = -0.36900270E-07

D22 = -0.34301843E-06

D23 = -0.13097306E-05

D24 = -0.25300329E-05

D25 = -0.24191602E-05

D26 = -0.84063161E-06

D27 = 0.91408365E-07

D28 = 0.32482919E-08

D29 = 0.36867079E-07

D30 = 0.17688326E-06

D31 = 0.45445078E-06

D32 = 0.64932289E-06

D33 = 0.46538070E-06

D34 = 0.84701632E-07

D35 = -0.52653376E-07

Q0 = 0.10471974E 01

Q2 = -0.15016421E-07

Q2/Q0 = -0.14339628E-07

VORP = 0.28793975E 00

VORM = -0.28793975E 00

C = 4.00

CASE 10

XM = 0.10000000E 01

A = 0.10000000E 01

A0 = 0.69444444E-02 | A1 = -0.17361111E-02

A2 = -0.17361111E-02

B0 = 0.82465277E-03 | B1 = -0.91145833E-03

B2 = -0.91145833E-03 | B3 = 0.39062500E-03

B4 = 0.78125000E-03 | B4 = 0.39062500E-03

B6 = -0.43402777E-04 | B7 = -0.13020833E-03

B8 = -0.13020833E-03 | B9 = -0.43402777E-04

C0 = 0.53172977E-05 | C1 = -0.30337271E-05

C2 = -0.30337271E-05 | C3 = 0.86769225E-06

C4 = 0.14931815E-05 | C5 = 0.86769225E-06

C6 = 0.14424534E-06 | C7 = -0.43273602E-06

C8 = -0.43273602E-06 | C9 = -0.14424534E-06

C10 = 0.53822889E-08 | C11 = 0.21529155E-07

C12 = 0.32293733E-07 | C13 = 0.21529155E-07

C15 = 0.53822889E-08

D0 = -0.27081060E-06	D1 = 0.14618229E-05
D2 = 0.85993357E-06	D3 = -0.24770679E-05
D4 = -0.38892866E-05	D5 = -0.14446701E-05
D6 = 0.21444442E-05	D7 = 0.55700687E-05
D8 = 0.51091772E-05	D9 = 0.13936700E-05
D10 = -0.10670710E-05	D11 = -0.38699997E-05
D12 = -0.57397364E-05	D13 = -0.30777047E-05
D14 = -0.76396422E-06	D15 = 0.31178924E-06
D16 = 0.14493480E-05	D17 = 0.28736014E-05
D18 = 0.23597631E-05	D19 = 0.16021832E-05
D20 = 0.22322972E-06	D21 = -0.46686016E-07
D22 = -.26815118E-06	D23 = -0.64046568E-06
D24 = -0.81407119E-06	D25 = -0.58064111E-06
D26 = -0.22029153E-06	D27 = -0.34721103E-07
D28 = 0.26964840E-08	D29 = 0.18191922E-07
D30 = 0.52525373E-07	D31 0.84124962E-07
D32 = 0.80707635E-07	D33 = 0.463[illegible]185E-07
D34 = 0.14774596E-07	D35 = 0.20130[illegible]7E-08

Q0 = 0.15707962E 01

Q2 = -0.12776789E-06

Q2/Q0 = -0.81339574E-07

VORP = 0.42923705E 00

VORM = -0.42923705E 00

APPENDIX C

LIST OF SYMBOLS

A = constant $= \dfrac{C}{2(m^2 + 1)}$

A_i = constants $\quad i = 0,1,2$

b_i = constants $\quad i = 0,1,2,\ldots,9$

C = constant $= \dfrac{Ga^2}{\mu W_0}$

C_i = constants $\quad i = 0,1,2,\ldots,14$

d_i = constants $\quad i = 0,1,2,\ldots,35$

e = eccentricity of the cross-sectional ellipse

e_1 = constant

f_r, f_θ, f_y = components of the body forces

G = constant = mean pressure gradient

K = constant $= \dfrac{2W_0^2 a^3}{R\nu}$

m = constant related to the eccentricity by $e = \dfrac{1}{m}\sqrt{m^2 - 1}$ when $m \geq 1$ and $e = \dfrac{1}{m}\sqrt{1 - m^2}$ when $0 < m \leq 1$

P = pressure

q_r, q_θ, q_y = dimensionalized velocity components

q_x = dimensionalized velocity in the x-direction

r, θ, y = cylindrical coordinates

R = radius of curvature of the center of the pipe

W = nondimensionalized velocity in the θ-direction

W_0 = velocity along the central axis for the flow through a straight elliptic pipe

W_1 = first-order approximation of W

W_2 = second-order approximation of W

μ = absolute viscosity

$\nu = \frac{\mu}{\rho}$ = kinematic viscosity

ρ = fluid density

ψ = stream function

ψ_1 = first-order approximation of the stream function

ψ_2 = second-order approximation of the stream function

The ' superscripts with parameters denote dimensionalized variables.

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13. ABSTRACT

This study concerns the laminar flow of an incompressible fluid through a curved pipe with an elliptical cross section. The governing equations are derived by applying the Navier-Stokes and continuity equations in cylindrical coordinates and the method of successive approximations to get the five partial differential equations. These equations are solved by the perturbation method, and twenty numerical examples are presented. For each example, arbitrary numerical parameters are assumed for input into an IBM 7094 Computer to obtain solutions for the simultaneous algebraic equations. For an eccentricity of one, the ellipse degenerates to a circle and Dean's solution for the streamline flow of an incompressible fluid through a curved pipe with a circular cross section is obtained.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Newtonian						
Viscous						
Streamline						
Visco-Elastic						
Nonlinear						
Incompressible						
Eccentric						

END

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