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THE EFFECT OF NOISE ANISOTROPY ON DETECTABILITY IN AN OPTIMUM ARRAY PROCESSOR

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ABSTRACT

The effect of localized noise sources on the performance of the optimum, i.e., likelihood ratio detector, is investigated.

Expressions were obtained for the performance loss of optimum detectors with noise which consists of (1) an isotropic part, and (2) a component from multiple point sources.

INTRODUCTION

The object of this paper is to investigate the effect of localized noise sources on the performance of the optimum (likelihood-ratio) detector.

In previous work[1] expressions for the performance loss of likelihood-ratio detectors when the noise consisted of an isotropic part and a component from a single point source were obtained. In the present analysis these results are extended to the case of more than one point source in an attempt to also get some estimate of the performance loss caused by anisotropy sources that are not strongly localized. Such sources can, presumably, be represented by a large number of closely spaced point sources.

NOMENCLATURE

The notation used is similar to that used by Edelblute, et al.[2] The detector is assumed to be a directional array consisting of M hydrophones, and the received signal at the ith hydrophone is $x_i(t)$. Then if the spectrum of $x_i(t)$ is limited to frequencies below W Hz, and the $x(t)$ are observed over an interval, T, such that $WT >> 1$, $x_i(t)$ can be expanded in a Fourier Series:

[1]Peter M. Schultheiss, "Passive Detection of a Sonar Target in a Background of Ambient Noise and Interference from a Second Target," Yale University Progress Report No. 17, submitted to General Dynamics Corporation, Electric Boat Division, September 1964.
This report is included in "Processing of Data from Sonar Systems, Volume III," Kanefsky, Levesque, Schultheiss, and Tuteur, General Dynamics Corporation, Electric Boat Division Report U417-65-033 (Aug. 23, 1965).
[2]David J. Edelblute, Joanne M. Fisk, and Gerald L. Kinnison, "Criteria for Optimum-Signal-Detection Theory for Arrays," J. Acoust. Soc. Am., 41, 199-205 (Jan. 1967).

$$x_i(t) = \sum_{n=-WT}^{W(T)} x_i(n)\, e^{j2\pi nt/T}, \tag{1}$$

where the $x_i(n)$ are complex Fourier coefficients satisfying $x_i(-n) = x_i^*(n)$ and where the asterisk stands for complex conjugate. All the available information about the signals received by the entire array is therefore contained in the set of vectors

$$\underline{X}(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_M(n) \end{bmatrix}. \tag{2}$$

It is assumed that $\underline{X}(n)$ and $\underline{X}(m)$ are statistically independent for $n \neq \pm m$. Suppose that the signal $x_i(t)$ received at the ith hydrophone consists of signal and noise; then the signal component is given by

$$s_i(t) = \sum_{n=-WT}^{WT} y_i(n)\, e^{j2\pi nt/T}, \tag{3}$$

so that the signal component at all hydrophones is represented by

$$\underline{Y}(n) = \begin{bmatrix} y_1(n) \\ \vdots \\ y_M(n) \end{bmatrix}. \tag{4}$$

Here again $\underline{Y}(n)$ are assumed to be independent from $\underline{Y}(m)$ for $n \neq \pm m$. Also, the signal is assumed to be independent from the noise. The normalized noise covariance matrix is defined by

$$\underline{Q}(n) = \frac{1}{N(n)} \langle \underline{X}^*(n)\, \underline{X}^T(n) \rangle_N, \tag{5}$$

where the superscript T refers to matrix transposition and the symbol $\langle\ \rangle_N$ means ensemble average subject to the noise-only hypothesis. $N(n)$ is the average noise power at frequency $2\pi n/T$ radians per second.

The normalized signal covariance matrix is

$$\underline{P}(n) = \frac{1}{S(n)} \langle \underline{Y}^*(n)\, \underline{Y}^T(n) \rangle_S, \tag{6}$$

where $S(n)$ is the average signal power at frequency $2\pi n/T$ radians per second. If the signal is a plane wave, the elements $y_i(n)$ of $\underline{Y}(n)$ are all delayed replicas of each other; thus

$$y_i(n) = c_i s(n)\, e^{j\frac{2\pi n \tau_i}{T}}, \tag{7}$$

where $s(n)$ is the nth Fourier coefficient of the signal wave form; the c_i are weighting factors to take into account that the signal strength or gain at different hydrophones may be different, and τ_i is the delay at the ith hydrophone. The c_i's are conveniently defined in such a way that

$$\langle s^*(n)\, s(n) \rangle = 1 \tag{8}$$

for all n. Hence

$$\underline{Y}(n) = s(n) \begin{bmatrix} c_1 e^{j\frac{2\pi n\tau_1}{T}} \\ \cdot \\ \cdot \\ \cdot \\ c_M e^{j\frac{2\pi n\tau_M}{T}} \end{bmatrix} = s(n)\, \underline{V}_0(n) \,, \tag{9}$$

and therefore

$$\underline{P}(n) = \frac{\langle s^*(n)\, s(n)\rangle}{S(n)}\, \underline{V}_0^*(n)\, \underline{V}_0^T(n) \tag{10}$$

$$\underline{P}(n) = \frac{1}{S(n)}\, \underline{V}_0^*(n)\, \underline{V}_0^T(n) \,. \tag{11}$$

$\underline{P}(n)$ is seen to be of rank 1. Because of the independence of signal and noise the covariance matrix of signal and noise together is $N(n)\, \underline{Q}(n) + S(n)\, \underline{P}(n)$.

The detection performance of the optimal processor is defined in terms of the standard detection index.[3]

$$d = \frac{\mu_1 - \mu_0}{\sigma_0} \,, \tag{12}$$

where μ_1 and μ_0 are the mean values of the output signal when signal is present, and absent, respectively, and where σ_0 is the standard deviation of the output under the condition that signal is absent.

By means of a trivial extension of the result of Edelblute et al.,[2] it can be shown that

$$d = \frac{\sum_{n=1}^{WT} K(n)\, S(n)\, G_0^2(n)}{\sqrt{\sum_{n=1}^{WT} K^2(n)\, N^2(n)\, G_0^2(n)}} \,, \tag{13}$$

where $G_0(n)$ is the maximum value of the array gain at frequency $2\pi n/T$, given by

$$G_0(n) = \underline{V}_0^T(n)\, \underline{Q}^{-1}(n)\, \underline{V}_0^*(n) \,, \tag{14}$$

and where

$$K(n) = \frac{S(n)/N^2(n)}{1 + S(n)\, G_0(n)/N(n)} \,. \tag{15}$$

Equation (13) can be simplified somewhat by using a small signal approximation: if

[3] F. Bryn, "Optimum Signal Processing of Three-Dimensional Arrays Operating on Gaussian Signals and Noise," J. Acoust. Soc. Am., 34, 289 (Mar. 1962).

$$S(n)\ G_0(n) \ll N(n)\,, \tag{16}$$

then

$$K(n) \approx \frac{S(n)}{N^2(n)} \tag{17}$$

and

$$d \approx \sqrt{\sum_{n=1}^{WT} \frac{S^2(n)\ G_0(n)^2}{N^2(n)}}\,. \tag{18}$$

Equation (18) will be employed in the sequel, under the assumption that inequality (16) usually holds.

EFFECT OF DIRECTIONAL INTERFERENCE

Suppose that the noise component of $x_i(t)$ consists of two parts, an isotropic part and an interference part. It is assumed that the interference is generated by R point sources. The rth point source is located at an azimuth angle θ_r, and its spectral density is $I_r(\omega)$; hence the interference power from the rth interference source at the frequency ω_n is $I_r(n)$. The desired target is at the azimuth angle $\theta_0 = 0$, and it is assumed that the array is steered in the target direction. The isotropic noise power at the frequency ω_n is $N_0(n)$. The isotropic noise component, the interference sources, and the target signal are all assumed to be mutually independent Gaussian processes with zero mean. Then the total noise power density is given by

$$N(n) = N_0(n) + \sum_{r=1}^{R} I_r(n)\,, \tag{19}$$

and the normalized noise covariance matrix has the form

$$\underline{Q}(n) = \frac{N_0(n)}{N(n)}\ \underline{Q}_0(n) + \sum_{r=1}^{R} \frac{I_r(n)}{N(n)}\ \underline{V}_r^*(n)\ \underline{V}_r^T(n)\,, \tag{20}$$

where $Q_0(n)$ is the normalized covariance matrix of the isotropic noise component and where each element of the summand results from one of the interference point sources. By direct analogy to Eqs. (9), (10), and (11)

$$\underline{V}_r(n) = \begin{bmatrix} c_1\, e^{j\frac{2\pi n \tau_1^{(r)}}{T}} \\ \cdot \\ \cdot \\ \cdot \\ c_M\, e^{j\frac{2\pi n \tau_M^{(r)}}{T}} \end{bmatrix} \tag{21}$$

where $\tau_i^{(r)}$ is the delay of the plane wave from the rth interference source at the ith hydrophone.

The matrix $Q(n)$ can be inverted by using the following matrix identity: if $\underline{A}$ is a non-singular matrix of dimension M and $\underline{B}$ is a matrix of M rows and R columns, then

$$(\underline{A} + \underline{B}^* \underline{B}^T)^{-1} = \underline{A}^{-1} - \underline{A}^{-1} \underline{B}^* (\underline{I} + \underline{B}^T \underline{A}^{-1} \underline{B}^*)^{-1} \underline{B}^T \underline{A}^{-1} \tag{22}$$

This identity is easily proved by multiplication.

In the present application let

$$\underline{A} = \frac{N_0(n)}{N(n)} \underline{Q}_0(n) \tag{23}$$

and

$$\underline{B}^* = \left[\sqrt{\frac{I_1(n)}{N(n)}} \underline{V}_1^*(n) \;\vdots\; \sqrt{\frac{I_2(n)}{N(n)}} \underline{V}_2^*(n) \cdots \vdots \cdots \vdots \cdots \sqrt{\frac{I_R(n)}{N(n)}} \underline{V}_R^*(n) \right] \tag{24}$$

Also, to simplify the notation let

$$K_r = K_r(n) = \frac{I_r(n)}{N(n)} \qquad \text{for} \quad r = 1 \ldots R \tag{25}$$

and

$$G_{rs} = G_{rs}(n) = \underline{V}_r^T(n)\, \underline{Q}_0^{-1}(n)\, \underline{V}_s^*(n) \,. \tag{26}$$

Note that $G_{rs}(n)$ has the general form of an array gain (see Eq. (19)); we can call it a "cross array gain." It is clear that

$$G_{rs}^*(n) = G_{sr}(n) \,. \tag{27}$$

In terms of this notation the matrix $\underline{I} + \underline{B}^T \underline{A}^{-1} \underline{B}^*$ of Eq. (22) becomes

$$\begin{bmatrix} 1 + K_1 G_{11} & \sqrt{K_1 K_2}\, G_{12} & \cdots & \sqrt{K_1 K_R}\, G_{1R} \\ \sqrt{K_2 K_1}\, G_{12}^* & 1 + K_2 G_{22} & \cdots & \sqrt{K_2 K_R}\, G_{2R} \\ \sqrt{K_R K_1}\, G_{1R}^* & \cdots & \cdots & 1 + K_R G_{RR} \end{bmatrix} = \underline{I} + \underline{G} \tag{28}$$

by an obvious definition of the "cross-array-gain" matrix $\underline{G}$. Note that G is square, of dimensionality R, and Hermitian.

By defining a vector of cross-array gains $\underline{g}$ such that

$$\underline{g}^T = \left[\sqrt{K_1}\, G_{01} \quad \sqrt{K_2}\, G_{02} \cdots \sqrt{K_R}\, G_{0R} \right] , \tag{29}$$

and detection index d, as given in Eq. (18) can be written in the form

$$d = \sqrt{\sum_{n=1}^{WT} \left[\frac{S(n)}{N_0(n)} F(n) \right]^2} \,, \tag{30}$$

where

$$F(n) = \underline{V}_0^T(n)\, \underset{\sim}{Q}^{-1}(n)\, \underline{V}_0^*(n)$$

$$= G_{00} - \underline{g}^T[\underline{I} + \underline{G}]^{-1}\, \underline{g}^* . \tag{31}$$

It is clear that G_{00} is the optimum array gain with isotropic noise and that $\underline{g}^T[\underline{I} + \underline{G}]^{-1}\underline{g}^*$ represents the effect of the interference. In general, the evaluation of the interference term in specific instances is difficult for two reasons:

1. The cross-array gains, G_{ij}, are quadratic forms involving the inverted $\underline{Q}_0(n)$ matrix.

2. Even if the G_{ij} are known the $R \times R$ matrix $[\underline{I} + \underline{G}]$ must be inverted. Thus it is necessary either to solve Eq. (31) by computer or to make approximations permitting an analytic result.

The standard simplification that has been used in most previous analyses and which eliminates part of the difficulty involved in evaluating the G_{ij} is to assume that there is no correlation between different hydrophones due to the isotropic noise component. The effect of correlation has been considered before and is not expected to alter the results obtained here in any significant way. With this assumption $\underline{Q}_0(n) = \underline{I}$, and

$$G_{00}(n) = G_{11}(n) = \ldots = G_{RR}(n) = M \tag{32}$$

$$G_{rs}(n) = \sum_{i=1}^{M} e^{j\omega_n(\tau_i^{(r)} - \tau_i^{(s)})} . \tag{33}$$

A further small simplification results from the assumption that the array is steered on target; this implies that $\tau_i^{(0)} = 0$, and therefore

$$G_{0r}(n) = \sum_{i=1}^{M} e^{-j\omega_n \tau_i^{(r)}} . \tag{34}$$

Equation (31) is now explicitly evaluated for a number of simple special cases.

SINGLE POINT INTERFERENCE

If there is only a single interference, Eq. (31) becomes

$$F(n) = G_{00} - \frac{K_1 |G_{01}|^2}{1 + K_1 G_{11}} , \tag{35}$$

and with the simplification $\underline{Q}_0(n) = \underline{I}$; this becomes

$$F(n) = M - \frac{K_1 |G_{01}|^2}{1 + K_1 M} , \tag{36}$$

by Eq. (34)

$$|G_{n1}|^2 = \sum_{i=1}^{M}\sum_{k=1}^{M} e^{j\omega_n(\tau_k - \tau_i)} = \sum_{i=1}^{M}\sum_{k=1}^{M} \cos \omega_n(\tau_k - \tau_i)$$

$$= M + 2\sum_{i=1}^{M-1}\sum_{k=i+1}^{M} \cos \omega_n(\tau_i - \tau_k) , \tag{37}$$

where, for simplicity the superscript on τ_i has been omitted; i.e., $\tau_i \equiv \tau_i^{(1)}$. Hence, the term representing the loss of detectability due to interference in Eq. (31) becomes

$$\frac{K_1 M}{1 + K_1 M}\left[1 + \frac{2}{M}\sum_{i=1}^{M-1}\sum_{k=i+1}^{M} \cos \omega (\tau_i - \tau_k)\right] . \tag{38}$$

This result is essentially identical to Eq. (29) of Ref. (1).

The double summation term in this result can be represented as the sum of a large number of phasors, which, for large M and sufficiently large ω_n appear at almost random angles. Hence, to a first approximation this term makes a negligible contribution and therefore

$$F(n) \approx M - \frac{K_1 M}{1 + K_1 M} \approx M - 1 , \tag{39}$$

where the second approximation is permissible if $K_1 M >> 1$. Thus one obtains the well-known result that asymptotically the cost of a single point interference is equivalent to no more than the loss of one hydrophone from the array.

TWO POINT INTERFERENCES

With two interferences $F(n)$ becomes

$$F(n) = G_{00}\left[- \frac{K_1(1 + K_2 G_{22})|G_{01}|^2 + K_2(1 + K_1 G_{11})|G_{02}|^2 - 2K_1 K_2 \mathrm{Re}\,(G_{01} G_{12} G_{02}^*)}{(1 + K_1 G_{11})(1 + K_2 G_{22}) - K_1 K_2 |G_{12}|^2}\right] , \tag{40}$$

where Re ( ) means real part. As before, the term in brackets represents the effect of the interference. If, as before,

$$\underset{\approx}{Q}_0(n) = \underline{I} ,$$

then

$$G_{00} = G_{11} = G_{22} = M ,$$

$$|G_{0s}|^2 = \sum_{i=1}^{M-1}\sum_{k=1}^{M} \cos \omega_n(\tau_k^{(s)} - \tau_i^{(s)}) = M + 2\sum_{i=1}^{M}\sum_{k=i+1}^{M} \cos \omega_n(\tau_i^{(s)} - \tau_k^{(s)}) ,$$

$$|G_{12}|^2 = M + \sum_{i=1}^{M-1}\sum_{k=i+1}^{M} \cos \omega_n(\tau_i^{(1)} - \tau_i^{(2)} - \tau_k^{(1)} + \tau_k^{(2)}) ,$$

and

$$\mathrm{Re}\,(G_{01}G_{12}G_{02}^{*}) = \sum_{i=1}^{M}\sum_{k=1}^{M}\sum_{\ell=1}^{M} \cos \omega_n(\tau_i^{(1)} - \tau_k^{(1)} + \tau_k^{(2)} - \tau_\ell^{(2)}) . \tag{41}$$

If the two interference sources are widely separated in angle from each other and from the target, then $\tau_k^{(1)}$ and $\tau_k^{(2)}$ differs substantially for all k and are not close to zero. Under these conditions the coefficient of K_1K_2 in both the numerator and denominator of Eq. (40) increases with M while the other terms increase with M^2. Hence, for large M these coefficients become negligible with the result:

$$F(n) \approx M - \frac{K_1|G_{01}|^2}{1 + K_1 M} - \frac{K_2|G_{02}|^2}{1 + K_2 M} , \tag{42}$$

and

$$F(n) \approx M - \frac{K_1 M}{1 + K_1 M} - \frac{K_2 M}{1 + K_2 M} , \tag{43}$$

where the second approximation involves neglect of the oscillating terms in $|G_{01}|^2$ and $|G_{02}|^2$. Thus the effect of interferences is seen to be additive under these conditions. For small interference-to-ambient-noise ratios, where K_1M and K_2M are very much less than unity, $F(n)$ is reduced roughly by $(K_1 + K_2)M$; while for very large interference-to-ambient-noise ratio, the reduction is no greater than 2. Thus for small interference-to-noise ratio, the detection index d decreases roughly with the first power of interference-to-noise ratio (see Eq. (29)), but the maximum effect is no greater than the loss of two hydrophones.

Suppose next that the two interference sources are sufficiently close together so that for all frequenices of interest, and for all i,

$$\omega_n|\tau_i^{(1)} - \tau_i^{(2)}| \approx 0 , \tag{44}$$

then

$$|G_{02}|^2 \approx |G_{01}|^2 ,$$

$$|G_{12}|^2 \approx M^2 ,$$

and

$$\mathrm{Re}\,(G_{01}G_{12}G_{02}^{*}) \approx M|G_{01}|^2 . \tag{45}$$

Then

$$F(n) \approx G_{00} - \frac{K_1(1 + K_2M)|G_{01}|^2 + K_2(1 + K_1M)|G_{01}|^2 - 2MK_1K_2|G_{01}|^2}{1 + (K_1 + K_2)M} = G_{00} - \frac{(K_1 + K_2)|G_{01}|^2}{1 + (K_1 + K_2)M} . \tag{46}$$

Thus, the result converges to the case of a single plane-wave interference of strength $K_1 + K_2$ in this case.

For a linear array of M elements spaced d feet apart we can set

$$\left|\tau_i^{(1)} - \tau_i^{(2)}\right| = \left| |M/2 - i| \frac{d}{c} |\sin\theta_1 - \sin\theta_2| \right| , \tag{47}$$

where the delay at the central array element is arbitrarily given the value zero. For small $\theta_2 - \theta_1$,

$$\left|\tau_i^{(1)} - \tau_i^{(2)}\right| = 2\,|M/2 - i|\,\frac{d}{c}\sin\left|\frac{\theta_1-\theta_2}{2}\right|\cos\left(\frac{\theta_1+\theta_2}{2}\right) \approx |M/2 - i|\,\frac{d}{c}\,|\theta_1-\theta_2|\,\cos\theta_m, \tag{48}$$

where $\theta_m = (\theta_1 + \theta_2)/2$ is the angle halfway between the two interferences. Then, since $\omega_{n_{max}}$ is $2\pi W$

$$\omega_n\left|\tau_i^{(1)} - \tau_i^{(2)}\right|_{max} = \pi WM\,\frac{d}{c}\,|\theta_1-\theta_2|\,\cos\theta_m \tag{49}$$

and the two interferences are close enough together so that Eq. (46) holds if

$$\pi WM\,\frac{d}{c}\,|\theta_1-\theta_2|\,\cos\theta_m \ll 1$$

or

$$|\theta_1-\theta_2| \ll \frac{1}{\pi WM\,\frac{d}{c}\cos\theta_m}. \tag{50}$$

As an example let $W = 5000$ Hz, $d = 2$ ft, $c = 5000$/ft, and $M = 10$, then if $|\theta_1 - \theta_2| \ll 0.016/\cos\theta_m$ radians the two interference points have the same effect as a single one with a higher power level and therefore the maximum detectability loss can be no greater than a single hydrophone as shown by Eq. (39).

Note that for $\theta_m = \pi/2$ the two interferences are located symmetrically to the end-fire axis of the array; therefore their effect is always that of a single interference. This, however, is due to the symmetry of the linear array and does not hold in other cases.

Note further that Eq. (50) is a rather conservative limit since neither the effect of integration over frequency or over hydrophone spacing has been considered. Depending on the exact form of the power spectrum these integrations should result in increasing the value of $|\theta_1 - \theta_2|$ by a factor of 4 or 5 over that given in Eq. (50).

If the interference-to-noise ratio is small enough so that $(K_1 + K_2)\,M \ll 1$, then Eq. (46) and Eq. (43) are approximately the same; thus under this condition the effect of two interferences on the detectability is proportional to the interference power, and independent of the spacing of the two interference sources from each other. Note, however, that approximating $|G_{01}|^2$ and $|G_{02}|^2$ by M still implies that the interference direction is substantially different from the target direction.

MORE THAN TWO INTERFERENCES

Extension of the results obtained so far to more interferences is difficult and involves further approximations. Consider first the large interference-to-noise ratio case, with all interferences widely separated. We assume as before that $Q_0 = \underline{I}$ and that therefore $G_{rr} = M$ for $r = 0, 1, 2, \ldots, R$. It can be seen from Eq. (41) that the off-diagonal elements of $\underline{G}$ are of order $\sqrt{M}$. It can be shown in general (see Appendix) that for the purpose of approximate inversion an n-dimensional matrix whose diagonal elements are of order k relative to the off-diagonal elements can be approximated by a diagonal matrix if $k \gg n$. Thus if $\sqrt{M} \gg R$ the off-diagonal elements of the matrix $[\underline{I} + \underline{G}]$ can be neglected in forming the inverse, with the result that

$$F(n) \approx G_{00} - \sum_{r=1}^{R} \frac{K_r |G_{r0}|^2}{1 + K_r G_{rr}} \tag{51}$$

$$= M - \sum_{r=1}^{R} \frac{K_r M}{1 + K_r M} \left[1 + \frac{2}{M} \sum_{i=1}^{M-1} \sum_{k=i+1}^{M} \cos \omega_n \left(\tau_i^{(r)} - \tau_k^{(r)} \right) \right], \tag{52}$$

where $\tau_i^{(r)}$ is the interference delay from the rth interference source at the ith hydrophone. Equation (51) is the direct extension of Eq. (43) and indicates that for widely spaced point interference sources the detectability loss is approximately equivalent to the loss of one hydrophone per interference source. The approximation is good only for $R \ll \sqrt{M}$.

EFFECT OF DISTRIBUTED INTERFERENCE SOURCE

A distributed interference source can be represented by a large number of closely spaced point sources. Suppose that the interference source has a spectral density $I(n)$ and that the interference power is uniform for angles inside the interval $\theta_1 < \theta < \theta_2$ and zero outside. Then the interference can be represented by R points of spectral density $I(n)/R$ equally spaced in the interval, where R is a large number. Initially it will be assumed that the interference-to-ambient-noise ratio is small. Although the result obtained under this assumption is somewhat academic (since the interference effect is very small in any case) it is possible to obtain an analytic result which is probably applicable with some mofidications to larger interference-to-ambient-noise ratio as well. Under this assumption, the elements of the matrix $\underline{G}$ are all very small, and it is approximately true that

$$\underline{I} + \underline{G} \approx \underline{I}. \tag{53}$$

Then the matrix inversion is, of course, trivial. The precise condition for Eq. (53) to be a good approximation may be deduced from Ref. (4); a simple sufficient condition is that

$$\sum_{s=1}^{R} \sqrt{K_r K_s}\, |G_{rs}| \ll 1 \quad \text{for all} \quad r = 1, \ldots, R. \tag{54}$$

In the present discussion $K_r = K_I/R$ for all $r = 1, \ldots, R$, where $K_I = I(n)/N_0(n)$ is the total interference-to-ambient noise ratio. A conservative upper bound on K_I such that Eqs. (53) and (54) are good approximations is obtained by letting $|G_{rs}| = M$ for all r, s (see Eq. (49)). Hence, if

$$K_I M \ll 1, \tag{55}$$

Eq. (53) is a good approximation, and under these conditions Eq. (31) becomes

$$F(n) = G_{00} - \underline{g}^T \underline{g}^*$$

$$= G_{00} - \frac{K_I}{R} \sum_{r=1}^{R} |G_{0r}|^2, \tag{56}$$

and by use of Eq. (41) this becomes:

[4]B. Freeman, Principles and Techniques of Applied Mathematics (John Wiley and Sons, New York, 1957), p. 34.

$$F(n) = G_{00} - K_I M - \frac{2K_I}{R} \sum_{i=1}^{M-1} \sum_{k=i+1}^{M} \sum_{r=1}^{R} \cos \omega_n \left(\tau_i^{(r)} - \tau_k^{(r)}\right) . \tag{57}$$

We assume now that the azimuth angle subtended by the interference is small enough so that the $\tau_i^{(r)}$ do not differ very greatly as r goes from 1 to R. Then, it is possible to expand $\tau_i^{(r)}$ in a Taylor series in r as follows:

$$\tau_i^{(r)} = \tau_i^{(m)} + (r - m)\,\Delta\tau_i , \tag{58}$$

where $m = R/2$ is used as the point about which the expansion is performed; $\tau_i^{(m)}$ is effectively the mean delay of the interference wavefront.

As R is allowed to go to infinity the summation in r can be converted into an integral. Hence Eq. (57) becomes, after some reduction

$$F(n) = G_{00} - K_I \left[M + 2 \sum_{i=1}^{M-1} \sum_{k=i+1}^{M} \frac{\sin \frac{\omega_n R}{2} (\Delta\tau_i - \Delta\tau_k)}{\frac{\omega_n R}{2} (\Delta\tau_i - \Delta\tau_k)} \cos \omega_n \left(\tau_i^{(m)} - \tau_k^{(m)}\right) \right] \tag{59}$$

$$= G_{00} - K_I \left[M + 2 \sum_{i=1}^{M-1} \sum_{k=i+1}^{M} \frac{\sin \omega_n \left(\tau_i^{(m)} - \tau_k^{(m)}\right)}{\omega_n \left(\tau_i^{(m)} - \tau_k^{(m)}\right)} \cos \omega_n \left(\tau_i^{(m)} - \tau_k^{(m)}\right) \right] , \tag{60}$$

where, in going from Eq. (59) to (60) we have used the fact that

$$\frac{R}{2} (\Delta\tau_i - \Delta\tau_k) = m(\Delta\tau_i - \Delta\tau_k) = \tau_i^{(m)} - \tau_k^{(m)} . \tag{61}$$

As before, the term representing the loss of detectability is the bracketed term in Eq. (60). Except for the $\sin x/x$ term the form of the double summation is the same as that which would be obtained for a single interference, (Eq. 54) with mean delay $\tau_i^{(m)}$ at the ith hydrophone, under the condition $K_I M \ll 1$. In fact, the argument that the summation of the oscillating terms tends to become negligible applies here with ever greater force, because of the $\sin x/x$ term. One can conclude, therefore, that for interference-to-ambient noise ratio small enough to satisfy Eq. (55), and if the angle subtended by the interference is relatively small, a distributed interference source affects the performance in essentially the same way as a single point interference.

In order to obtain an estimate of the magnitude of azimuth angle that can be considered "small," consider a linear array with M hydrophones spaced d feet apart. For such an array

$$\tau_i^{(r)} = i \frac{d}{c} \sin \theta_r , \tag{62}$$

where θ_r is the azimuth angle of the rth interference point. Assume that the interference power is uniform over the range $\theta_1 \le \theta \le \theta_2$ and is zero outside this range. The center of the interference is at the angle

$$\theta_m = \frac{1}{2} (\theta_1 + \theta_2) . \tag{63}$$

Then, by analogy with Eq. (58) we expand $\sin \theta_r$ about θ_m; i.e.,

$$\sin \theta_r \approx \sin \theta_m + (\theta_r - \theta_m) \cos \theta_m . \tag{64}$$

All the other steps leading to Eq. (60) can then be performed in exactly the same way, with the summation over R replaced by an integration over θ_r. The final result can be put into the form

$$F(n) = G_{00} - K_I \left[M + 2 \sum_{k=1}^{M} (M-k) \cos\left(\frac{k\omega_n d}{c} \sin\theta_m\right) \frac{\sin\left\{\frac{k\omega_n d}{c} \cos\theta_m \left(\frac{\theta_2-\theta_1}{2}\right)\right\}}{\frac{k\omega_n d}{c}\cos\theta_m\left(\frac{\theta_2-\theta_1}{2}\right)} \right] . \tag{65}$$

Except for the $\sin x/x$ term in the summation, this is again the expression that one would have obtained for a single point interference location at the angle θ_m. It is clear that the accuracy of this expression depends on the accuracy of Eq. (64) which in turn is a fairly good approximation for $\theta_2 - \theta_1$ less than about 1 radian. Thus we conclude that an interference source spread over no more than one radian affects the detectability essentially like a single point interference provided that $K_I M \ll 1$.

Since the effect of interference for $K_I M \ll 1$ is very small, the result just obtained is somewhat academic and it would be desirable to extend it somehow to the case of $K_I M \gg 1$. Unfortunately this is quite difficult; in fact, the only simple result that has been obtained is an extension of Eq. (46) to more than two interference sources. As in the case of two interferences, it is assumed that the interference points are close enough together so that for all frequencies of interest

$$\max_{r,s} \omega_n \left| \tau_i^{(r)} - \tau_i^{(s)} \right| \approx 0 . \tag{66}$$

The extension to R interference points is then quite straightforward, and the result is that the detectability loss is again equivalent to that of a single interference source of strength K_I. For a linear array having M equally spaced hydrophones the maximum value of $\theta_2 - \theta_1$ for which this result holds is given by Eq. (50).

COMPUTATIONAL RESULTS

Since it has not been possible to obtain meaningful analytic results for cases in which the approximations made in the above work are not applicable, F(n) has been evaluated on a digital computer for a number of different array and interference patterns, and for specific frequencies. The results of some of these computations are presented in Figs. 1 through 6. In all computations it is assumed that the array is steered on target at an angle $\theta = 0$ and that an interference exists at some angle θ_1. The curves are then plots of F(n) as θ_1 is varied. Thus, if θ_1 is near 0 the interference is near the target in azimuth, and F(n) is small. Also, the assumption that correlation of ambient noise waveforms between different hydrophones is zero has not been used; instead the exact form of the $\underline{Q}_0(n)$ matrix as given by Bryn[3] was used. As a result $F(n) \neq M$ in the absence of interference as would be inferred from equations such as (36), (43), and (52). In fact, $F(n) < M$ in all cases; however, this is a coincidence. It is possible for $F(n) > M$ as is shown by Bryn.[3] In all cases the interference-to-ambient noise ratio is large.

Figure 1 shows the effect of a single point interference with a small circular array. It shows that if the interference directions differ by more than about 40 degrees from the target direction the effect on F(n) is essentially negligible. It must be borne in mind, however, that this is only demonstrated for a single frequency (5000 rad/sec). The picture looks different at other frequencies, and the integrated effect of all frequencies therefore has the effect of the loss of one hydrophone as is predicted by the analysis in the section on single point interference.

Figures 2 and 3 are similar to Fig. 1 except that the interference consists, respectively, of two and of four points, separated by 0.1 radian. Since the interference covers a large azimuth segment, the effect on F(n) covers a larger angle; however, it is still true that for interference sources at angles far removed from zero the effect on F(n) is small. A similar result is shown

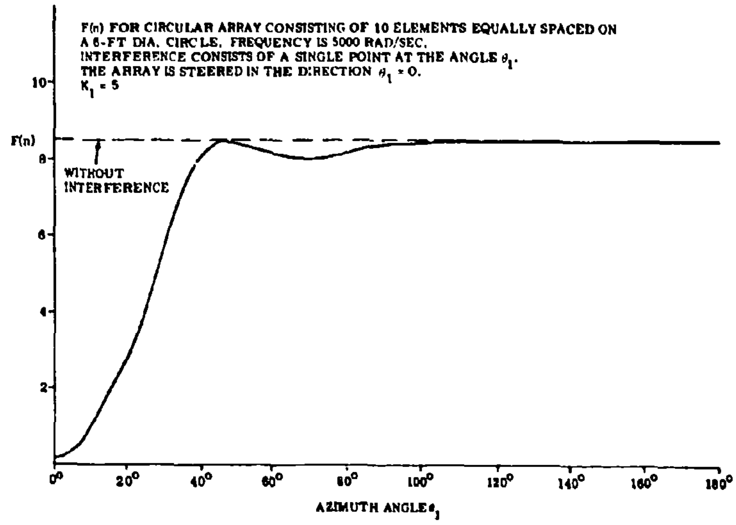

Fig. 1. Effect of single point interference

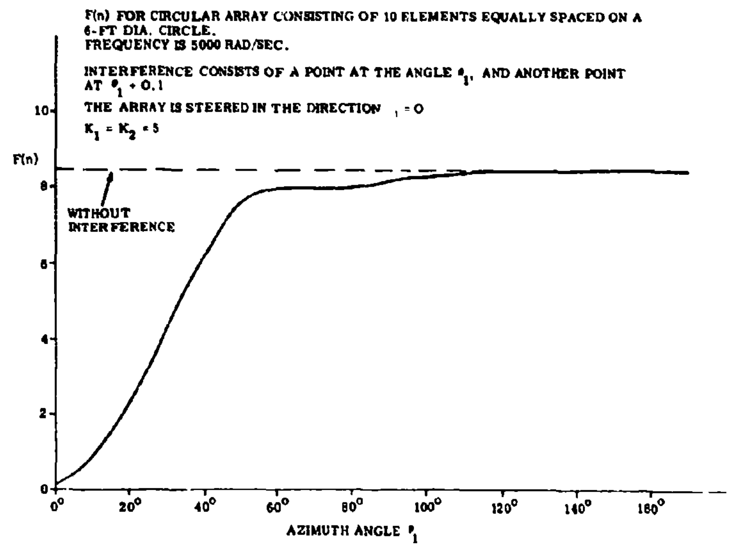

Fig. 2. Effect of two interference points spaced 0.1 radians apart in azimuth

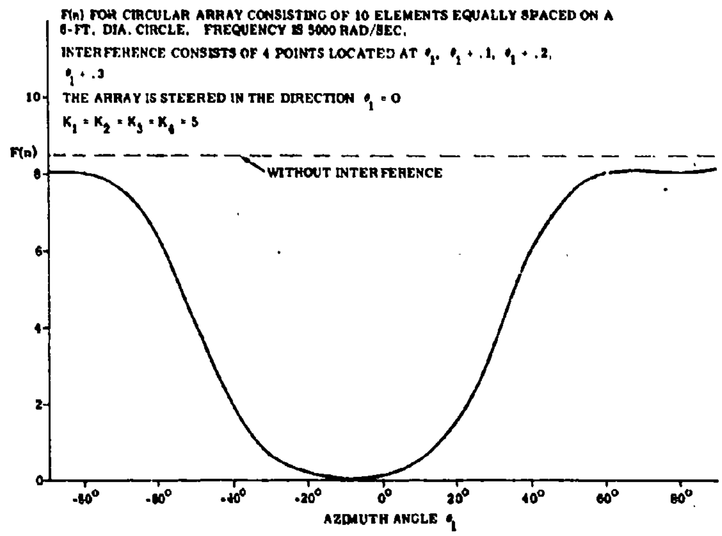

Fig. 3. Effect of four interference points spaced 0.1 radians apart in azimuth

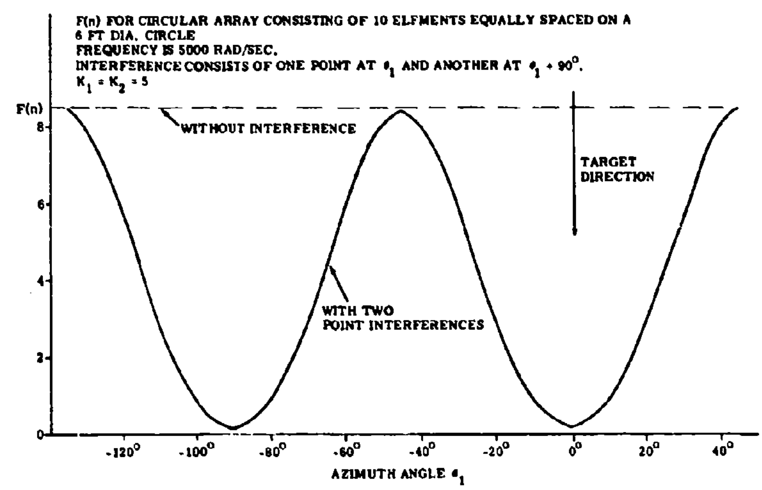

Fig. 4. Effect of two interference points spaced 90 degrees apart in azimuth

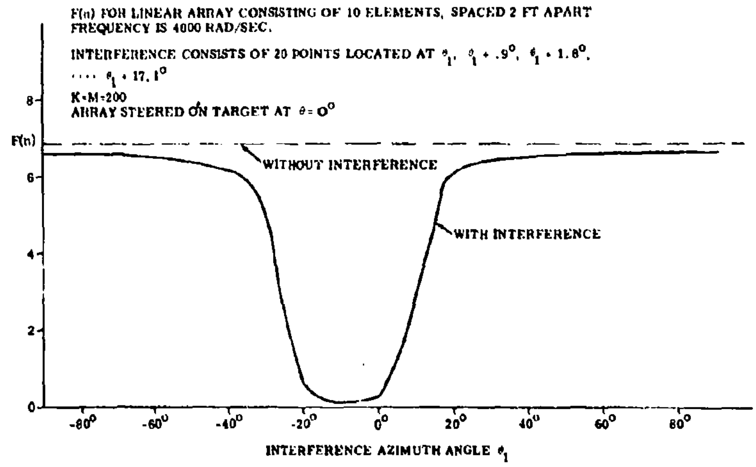

Fig. 5. Effect of a distributed interference source approximated by 20 point interferences spaced 0.9 degree apart. Frequency is 4000 rad/sec.

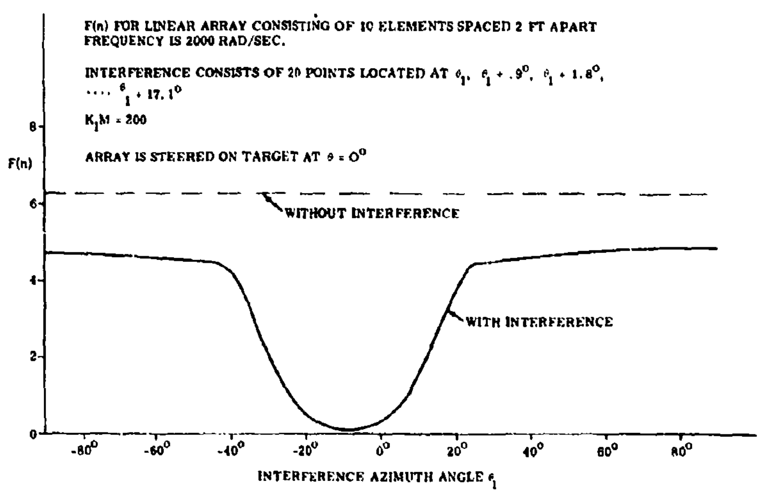

Fig. 6. Effect of a distributed interference source approximated by 20 point interferences spaced 0.9 degree apart. Frequency is 2000 rad/sec.

in Fig. 4 which shows the effect of two interferences separated by a large angle (90 degrees). The last two figures show the effect of a strong interference ($K_1 = 200$) distributed over a relatively large angle (17 degrees). Again the effect at angles far removed from the target angle is small, but, as is shown in Fig. 6, the relative effect is quite different at different frequencies, as has already been pointed out.

The computations leading to the results shown in Figs. 1 through 6 were quite time consuming, with computing times on the order of several minutes on the IBM 7094 for the cases with large numbers of interference points. For this reason no attempt was made to compute the complete detection index, since this would have required summation of $F(n)$ over a large number of frequencies. The computer results therefore still do not conclusively answer the question of how serious is the effect of large distributed interferences. The indications are, however, that the results of the previous section are valid under considerably wider conditions than those assumed there to produce analytical approximations. In fact, it appears that loss of detectability for rather widely distributed interference is equivalent to, at most, a few hydrophones.

CONCLUSIONS

The major difficulty in obtaining general estimates of the effect of directional noise on the detectability in an array processor is that the mathematical manipulations required to obtain answers are quite complex. Results have therefore been obtained only in a restricted number of simple cases.

The general tenor of these results is that if the anisotropic-to-isotropic noise ratio is small the effect of a number of local noise sources is additive; that is, the loss of detectability resulting from two noise sources of equal strength is twice that resulting from a single source. For large anisctropic-to-ambient noise ratio the effect depends on whether the directional noise sources are close together or not. For a single point source it has been shown previously and corroborated here that the loss in detectability is approximately equivalent to the loss of one hydrophone from the array. If there are R noise sources, widely separated from each other and from the target direction, the loss is approximately equivalent to the loss of R hydrophones, provided that $R \ll \sqrt{M}$, where M is the number of hydrophones.

Point noise sources that are close together affect the system like a single distributed noise source, and the indications are that if such an anisotropy is spread over a relatively small angle, its effect is essentially that of a single point noise. Unfortunately this has not been conclusively demonstrated, even by use of a digital computer, and only a rather conservative estimate of azimuth angle that can be considered to be "small" has been obtained.

Appendix

APPROXIMATE INVERSION OF A MATRIX WHOSE DIAGONAL TERMS ARE LARGE RELATIVE TO THE OFF-DIAGONAL TERMS

Let the $n \times n$ nonsingular matrix $\underline{A}$ be given by

$$\underline{A} = \underline{D} + \underline{B}, \tag{A-1}$$

where $\underline{D}$ is diagonal and $\underline{B}$ is a matrix with zero diagonal elements. It is assumed that all the non-diagonal elements of $\underline{B}$ are of about the same order of magnitude, and that the elements of $\underline{D}$ are of about K times that magnitude, with $K \gg 1$.

The inverse of $\underline{A}$ is given by

$$\underline{A}^{-1} = (\underline{D} + \underline{B})^{-1} = \underline{D}^{-1}(\underline{I} + \underline{BD}^{-1})^{-1} = \underline{D}^{-1}\left[\underline{I} - \underline{BD}^{-1} + (\underline{BD}^{-1})^2 \ldots\right]. \tag{A-2}$$

Since the elements of $\underline{D}$ are of order K relative to $\underline{B}$, the elements of $\underline{BD}^{-1}$ are order $1/K$ relative to unity.

It can be shown[5] that a sufficient condition for convergence of Eq. (A-2) is

$$\sum_{j=1}^{n} |b_{ij}| < 1 , \quad i = 1, \ldots, n , \tag{A-3}$$

where b_{ij} are the elements of BD^{-1}. Assuming all of these elements to be of about the same order of magnitude, condition (A-3) can be expressed in the approximate form

$$nb_0 < 1 , \tag{A-4}$$

where b_0 is a representative element of BD^{-1}. This element is of order $1/K$; therefore convergence requires

$$n/K < 1 . \tag{A-5}$$

The convergence will clearly be more rapid if this inequality is sharper; hence one can approximately neglect the matrix $\underline{B}$ in the inversion of $\underline{A}$ if $n/K \ll 1$.

ACKNOWLEDGMENTS

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[5] U. N. Faddeeva, Computational Methods in Linear Algebra (Dover Publications, New York, 1959).