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TRAJECTORY PARAMETERS AND MISS COEFFICIENTS

for

FREE-FALL BALLISTIC TRAJECTORIES

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1 - Summary

It is assumed in this report that all trajectories are based upon an object being launched with an initial velocity at a specific launch angle. There is no powered flight. The earth is assumed to be spherical and non-rotating with a vacuum atmosphere.

For any desired earth range there is an infinite family of trajectory paths which will give that range. The main parameter chosen to describe these different paths is the eccentricity of the elliptical trajectory. In the curves showing the trajectory results the eccentricity is chosen as the independent variable. The following trajectory parameters are plotted as a function of range and eccentricity:

(a) initial launch velocity

(b) initial launch angle

(c) time of flight

(d) error in range due to an error in the initial magnitude of the velocity when the launch angle is held fixed.

(e) error in range due to an error in the velocity which is perpendicular to the initial velocity direction when the velocity magnitude is held fixed.

Curves are shown for earth ranges extending from 5400 nautical miles to 21,600 nautical miles.

2 - Symbols and nomenclature

The following symbols, some of which are illustrated in figures 1, 2, 3, and 4 were used throughout:

r = the distance from the earth's center (focus of elliptical trajectory) to the object.

μ = GM, where G is the universal gravitational constant and M is the mass of the earth.

h = angular momentum per unit mass

v_o = initial magnitude of launch velocity

α = initial launch angle (the angle between the velocity vector and the launch horizontal)

λ = the earth's geocentric range angle (the angle between the launch position vector and the impact position vector)

R = radius of the earth

θ = angle between the major axis and the position vector of the object

θ_o = the angle between the ellipse's major axis and the reference axis.

l = semi-latus rectum of the ellipse.

ϵ = eccentricity of the ellipse

E = total energy of the object per unit mass

$$E = \frac{\text{potential energy}}{\text{unit mass}} + \frac{\text{kinetic energy}}{\text{unit mass}} = -\frac{\mu}{r} + \frac{1}{2}v^2$$

γ = angle between the position vector and the velocity vector

a = semi-major axis of the ellipse

T = time of flight, launch to impact

E_1 = the eccentric anomaly at the launch site

E_2 = the eccentric anomaly at the impact point

3 - Discussion

An elliptical trajectory can be described in two ways. One way is in geometrical terms and the other way is in energy terms (ref. 1). In this report both forms are used to derive equations for velocity, launch angle, and miss coefficients.

The geometrical form is given by

$$\frac{1}{r} = \frac{1}{l}\left[1 + e\cos(\theta - \theta_o)\right]. \qquad (1)$$

The energy form is given by

$$\frac{1}{r} = \frac{\mu}{h^2}\left[1 + \sqrt{1 + \frac{2Eh^2}{\mu^2}}\cos(\theta - \theta_0)\right]. \tag{2}$$

It can be seen that by equating equal parts in equations (1) and (2)

$$\ell = \frac{h^2}{\mu} \tag{3}$$

and

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}. \tag{4}$$

The geometrical meaning of θ, θ_0, r, and ℓ are shown in Figure 1.

At a given launch site it is possible to send an object into any kind of desired trajectory by the proper choice of velocity magnitude, V_0, and launch angle, α. In Figure 2 are shown three possible trajectories. Trajectory (a) is an elliptical path which gives a earth range of $2\pi R$ nautical miles, if R is expressed in nautical miles. Trajectory (b) takes an object three-fourths of the way around the earth having a range of $\frac{3}{2}\pi R$ nautical miles. Trajectory (c) indicates an elliptical path which has an earth range of πR nautical miles.

Define λ as the earth's geocentric range angle from launch to impact. The earth range then equals λR. The relationship between θ_0 and λ is given by

$$\theta_0 = \frac{\lambda}{2} - 180^\circ. \tag{5}$$

It is possible to express the geometrical ellipse as a function of the desired angle,

$$\frac{1}{r} = \frac{1 + e\cos(\theta + 180^\circ - \frac{\lambda}{2})}{R[1 + e\cos(180^\circ - \lambda/2)]} \tag{6}$$

or

$$\frac{1}{r} = \frac{1 - e(\cos\theta\cos\lambda/2 + \sin\theta\sin\lambda/2)}{R[1 - e\cos\lambda/2]}. \tag{7}$$

3.1 - Derivation of the Velocity Equation

The launch velocity can be expressed as a function of range angle, λ, and eccentricity, e. Equating equal parts of equations (2) and (7) one gets

$$\frac{h^2}{\mu} = R\left[1 - e\cos\frac{\lambda}{2}\right] \tag{8}$$

and

$$\sqrt{1+\frac{2Eh^2}{\mu^2}}\cos(\theta-\theta_0) = e\left(-\cos\theta\cos\frac{\lambda}{2} - \sin\theta\sin\frac{\lambda}{2}\right). \tag{9}$$

Equation (8) can also be expressed as

$$\frac{h^2}{\mu} = \frac{R^2 v_0^2\sin^2\gamma}{\mu} = R\left[1-e\cos\frac{\lambda}{2}\right] \tag{10}$$

where γ is the angle between the launch velocity vector and the launch position vector. Solving for $v_0^2\sin^2\gamma$, equation (10) gives

$$v_0^2\sin^2\gamma = \frac{\mu}{R}\left[1-e\cos\frac{\lambda}{2}\right]. \tag{11}$$

Now using equations (4)

$$e^2 = 1+\frac{2Eh^2}{\mu^2} = 1 + \frac{2\left(-\frac{\mu}{R}+\frac{1}{2}v_0^2\right)v_0^2R^2\sin^2\gamma}{\mu^2} \tag{12}$$

and solving for $v_0^2\sin^2\gamma$ one obtains

$$v_0^2\sin^2\gamma = \frac{\mu^2(1-e^2)}{2\mu R - v_0^2R^2}. \tag{13}$$

From equations (11) and (13) one finally obtains an expression for v_0 as a function of e and λ.

$$\boxed{v_0^2 = \frac{\mu}{R}\left[2 - \frac{(1-e^2)}{1-e\cos\lambda/2}\right]} \tag{14}$$

3.2 - Derivation of the Launch Angle Equation

The launch angle, α, is related to γ' by the expression

$$\alpha = 90^\circ - \gamma'. \tag{15}$$

From equation (11) we get

$$\sin^2\gamma' = \frac{\mu}{R}\left[\frac{1-e\cos\lambda/2}{v_0^2}\right] \tag{16}$$

and since

$$\sin\gamma' = \sin(90^\circ-\alpha) = \cos\alpha \tag{17}$$

we obtain

$$\cos^2\alpha = \frac{\mu}{R}\left[\frac{1-e\cos\lambda/2}{v_0^2}\right]. \tag{18}$$

For the launch angle we have

$$\boxed{\alpha = \cos^{-1}\left[\frac{\sqrt{\frac{\mu}{R}(1-e\cos\lambda/2)}}{v_0}\right]}. \tag{19}$$

3.3 - Derivation of the Miss Coefficient, $d(R\lambda)/dv_o$

The change in range, $d(R\lambda)$, caused by a change in velocity, dv_o, when the launch angle, α, is held fixed can now be derived starting with equation (14).

Solving for $\cos\frac{\lambda}{2}$ in equation (14) gives

$$\cos\frac{\lambda}{2} = \frac{1}{e}\left[1 - \frac{\frac{\mu}{R}(1-e^2)}{\left(2\frac{\mu}{R} - v_o^2\right)}\right]. \tag{20}$$

Note that from equations (12) and (17) one obtains

$$e^2 = 1 - \frac{\left(2\frac{\mu}{R} - v_o^2\right)v_o^2 R^2 \cos^2\alpha}{\mu^2}. \tag{21}$$

Solving for $\left(2\frac{\mu}{R} - v_o^2\right)$ in equations (21) and substituting into equation (20) one obtains

$$\cos\frac{\lambda}{2} = \frac{1}{e}\left[1 - \frac{Rv_o^2\cos^2\alpha}{\mu}\right]. \tag{22}$$

Expressing the eccentricity, e, as given in equation (21) one finally gets an expression for $\cos\lambda/2$ as a function of v_o alone,

$$\cos\frac{\lambda}{2} = \frac{1 - \frac{R}{\mu}v_o^2\cos^2\alpha}{\sqrt{1 - \frac{\left(2\frac{\mu}{R} - v_o^2\right)v_o^2R^2\cos^2\alpha}{\mu^2}}}. \tag{23}$$

By differentiation of equation (23) one can obtain

$$\boxed{\frac{d(R\lambda)}{dv_o} = \frac{4R^2}{\mu v_o e \sin\lambda/2}\left[\frac{\mu}{R} + \left(\frac{v_o^2}{e} - \frac{e\mu}{R} - \frac{\mu}{Re}\right)\cos\lambda/2 + \left(\frac{\mu}{R} - v_o^2\right)\cos^2\frac{\lambda}{2}\right]} \tag{24}$$

after the elimination of the $\cos^2\alpha$ terms by use of equation (18).

If R is given in feet, μ in $(\text{feet})^3 / (\text{second})^2$, and v_o in feet/second, it will be necessary to include a multiplying factor of $\left(\frac{1}{6076.1}\right)$ to give the miss coefficient $d(R\lambda)/dv_o$ in nautical miles/feet/second.

3.4 - Derivation of the Miss Coefficient, $d(R\lambda)/dv_\perp$.

An intermediate miss coefficient $d(R\lambda)/d\alpha$ can be derived from equation (23) by differentiation. This miss coefficient can be converted to $d(R\lambda)/dv_\perp$ which gives the change in range caused by a change in the velocity perpendicular to the launch velocity vector. See Figure 3.

Immediately after differentiating equation (23) one can obtain

$$\frac{d(R\lambda)}{d\alpha} = -\frac{2R^2 v_o^2 \sin 2\alpha}{\mu e \sin \lambda/2}\left[1 - \frac{R}{2\mu e}\left(2\frac{\mu}{R} - v_o^2\right)\cos\lambda/2\right]. \qquad (25)$$

Using

$$\sin 2\alpha = \frac{2}{v_o^2}\left[\left(\frac{\mu}{R} - \frac{\mu}{R} e\cos\lambda/2\right)\left(v_o^2 - \frac{\mu}{R} + \frac{\mu}{R} e \cos\lambda/2\right)\right]^{1/2} \qquad (26)$$

and

$$\left(2\frac{\mu}{R} - v_o^2\right) = \frac{\mu(1-e^2)}{R(1-e\cos\lambda/2)} \qquad (27)$$

one can obtain the simpler form

$$\frac{d(R\lambda)}{d\alpha} = \frac{2R\left[\left(\frac{1+e^2}{e}\right)\cos\frac{\lambda}{2} - 2\right]}{1 - e\cos\lambda/2}. \qquad (28)$$

Now referring again to Figure 3, it can be seen that

$$\frac{d(R\lambda)}{dv_\perp} = \frac{d(R\lambda)}{d\alpha}\cdot\frac{d\alpha}{dv_\perp} \approx \frac{d(R\lambda)}{d\alpha}\cdot\frac{\tan^{-1}\frac{\Delta v_\perp}{v_o}}{\Delta v_\perp} = \frac{d(R\lambda)}{d\alpha}\cdot\frac{\frac{\Delta v_\perp}{v_o}}{\Delta v_\perp}$$

for $\Delta v_\perp << v_o$. For convenience $\Delta v_\perp$ was chosen as 1 foot/second. The miss coefficient becomes,

$$\boxed{\frac{d(R\lambda)}{dv_\perp} = \frac{2R\left[\left(\frac{1+e^2}{e}\right)\cos\frac{\lambda}{2} - 2\right]}{v_o\left(1 - e\cos\frac{\lambda}{2}\right)}}. \tag{29}$$

If R is given in nautical miles, $d(R\lambda)/dv_\perp$ will be in units of nautical miles/feet/seconds.

3.5 - Expression for Time of Flight

The time of flight is given by the expression

$$T = \frac{E_2 - E_1 - e(\sin E_2 - \sin E_1)}{\sqrt{\frac{\mu}{a^3}}} \tag{30}$$

where

$$a = \frac{\ell}{1-e^2} = \frac{R\left[1 - e\cos\frac{\lambda}{2}\right]}{1-e^2} \tag{31}$$

and

$$\sin E_2 = -\frac{R\sin\lambda/2}{a\sqrt{1-e^2}} , \quad \sin E_1 = \frac{R\sin\lambda/2}{a\sqrt{1-e^2}} . \tag{32}$$

See Figure 4 for the description of angles E_1 and E_2.

Expressing the time of flight in terms of e , λ , and v_o gives

$$\boxed{T = \frac{\sin^{-1}\left[\frac{-\sqrt{1-e^2}\sin\lambda/2}{1-e\cos\lambda/2}\right] - \sin^{-1}\left[\frac{\sqrt{1-e^2}\sin\lambda/2}{1-e\cos\lambda/2}\right] + 2e\frac{\sqrt{1-e^2}\sin\lambda/2}{1-e\cos\lambda/2}}{\sqrt{\frac{\mu}{R^3}\left(2 - \frac{R}{\mu}v_o^2\right)^3}}} \tag{33}$$

4.0 - Results

The curves shown in Figures 5 through 12 show the trajectory parameters and miss coefficients plotted as a function of eccentricity for specific ranges.

5 - REFERENCES

1 - Principles of Mechanics - J. L. Synge and B.A. Griffith
McGraw-Hill Book Co., Inc., New York, 1942

5.2 Note on Minimum Energy Orbits

The minimum energy orbits have no practical significance for ranges exceeding 10,800 n. mi. The minimum energy orbit for these greater ranges is circular, e= 0. For circular orbits the miss coefficients are infinite.

SYMBOLS OF THE GEOMETRICAL ELLIPSE

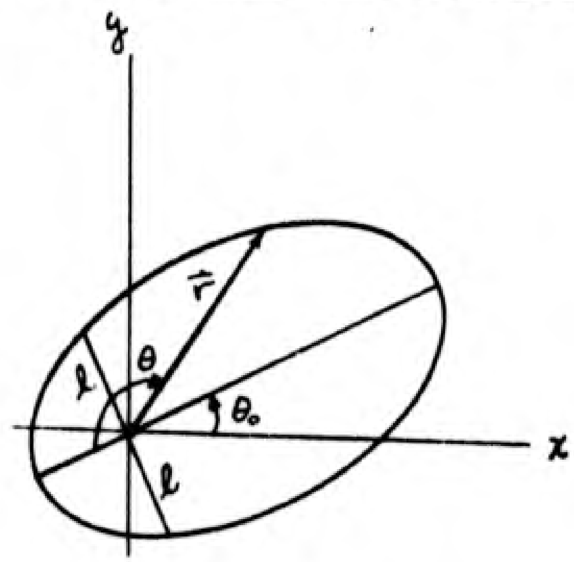


Fig. 1

TYPES OF BALLISTIC TRAJECTORIES

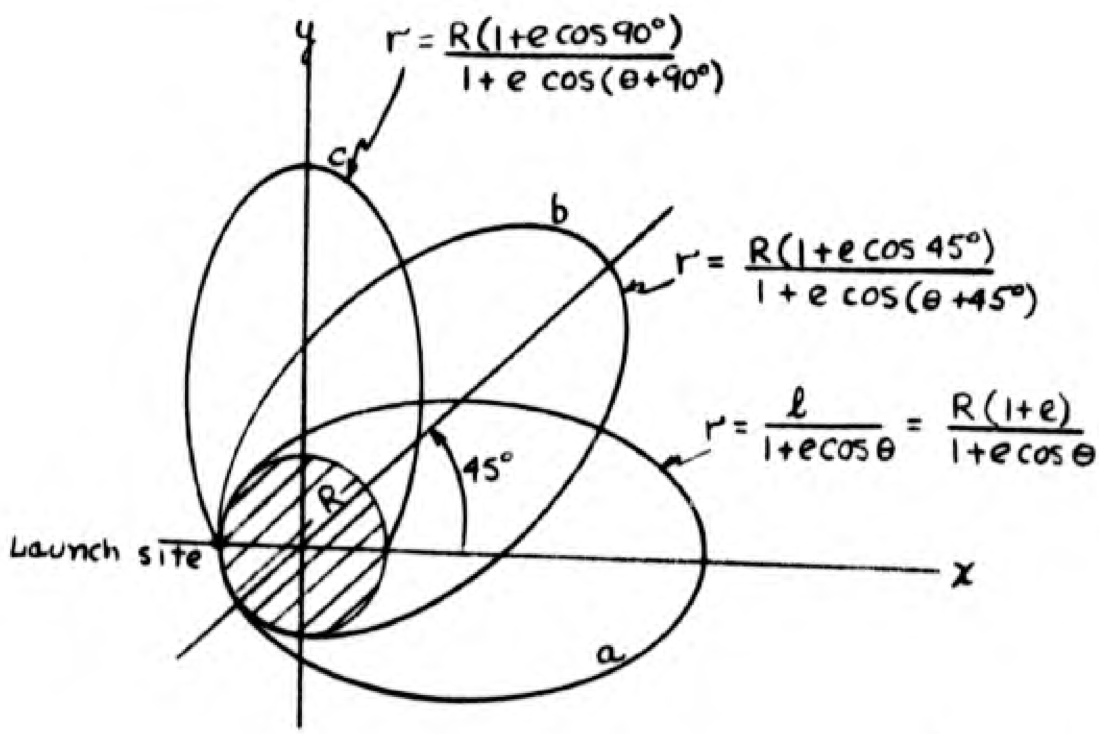


Fig. 2

VELOCITY COMPONENTS

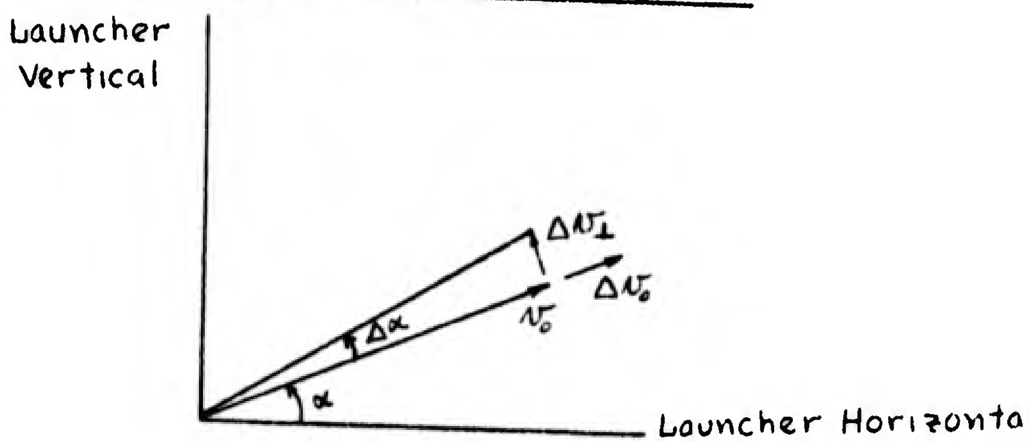


Fig. 3

ECCENTRIC ANOMALIES AT LAUNCH AND IMPACT

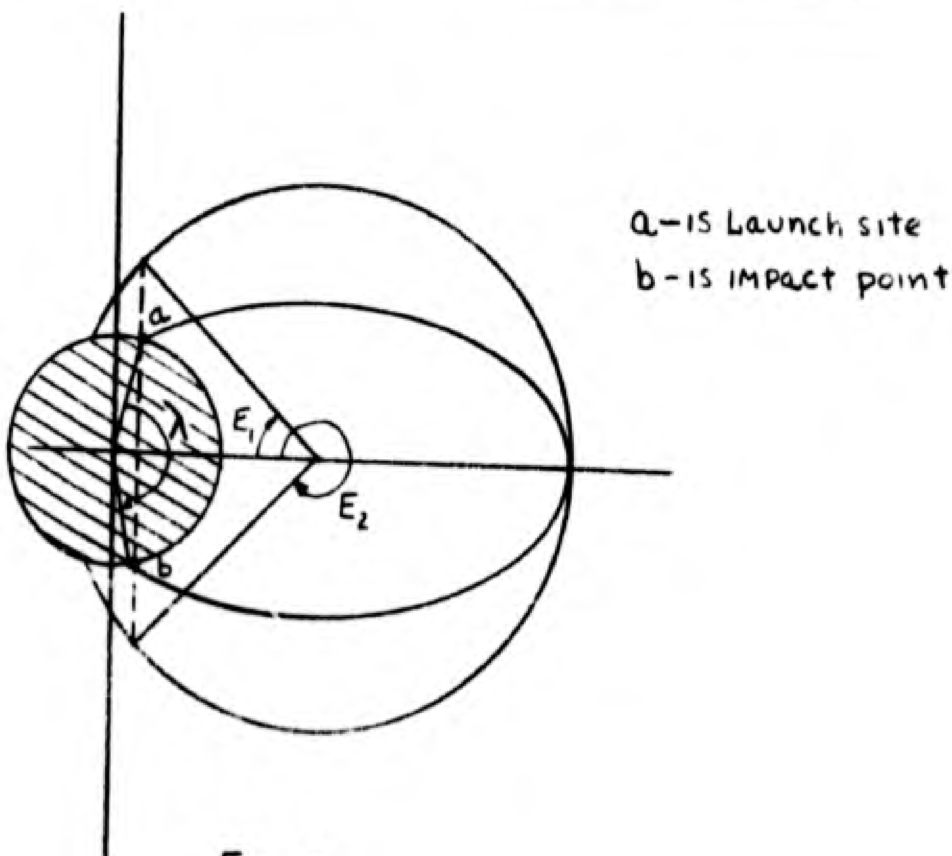
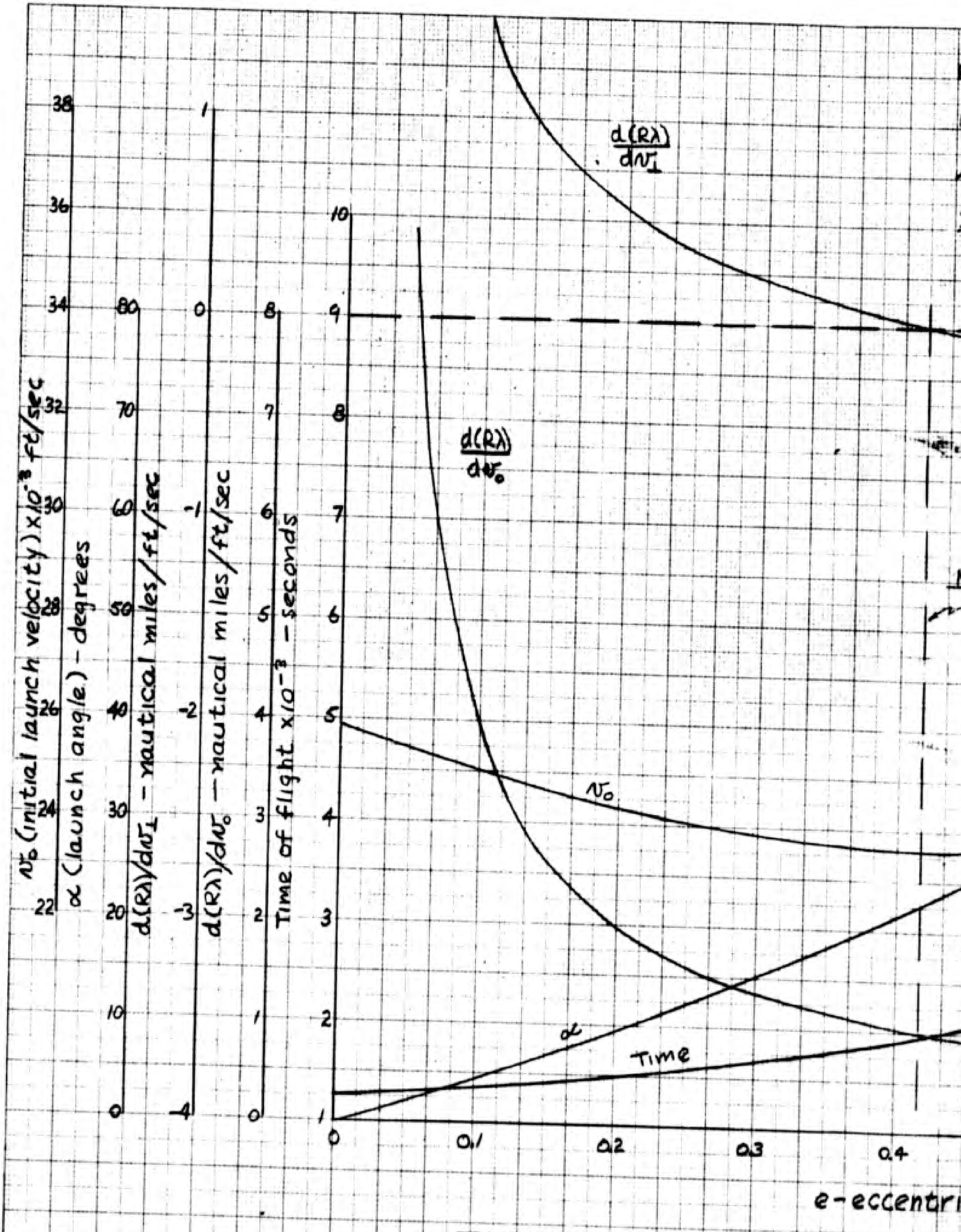


Fig. 4



TRAJECTORY PARAMETERS AND MISS COEF

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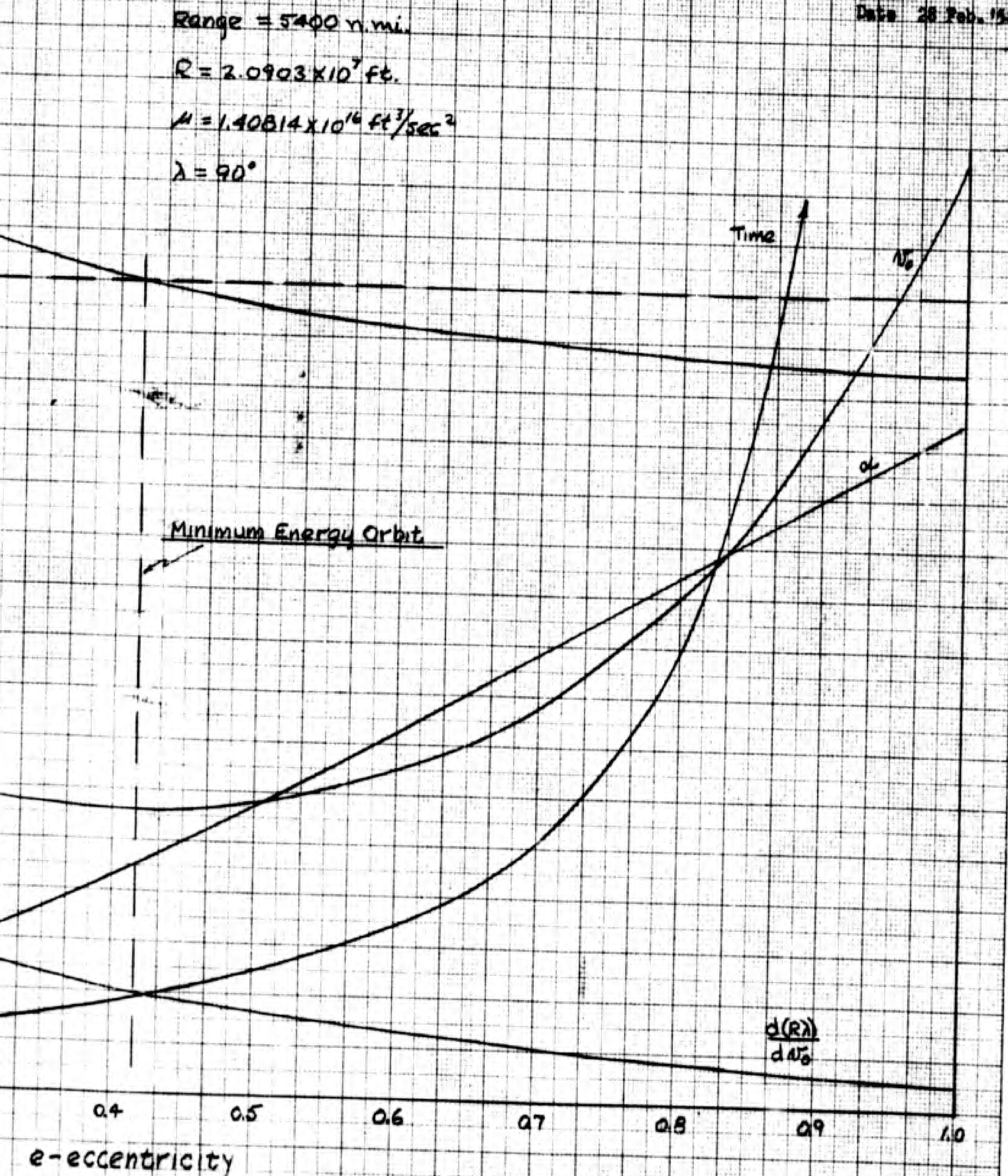
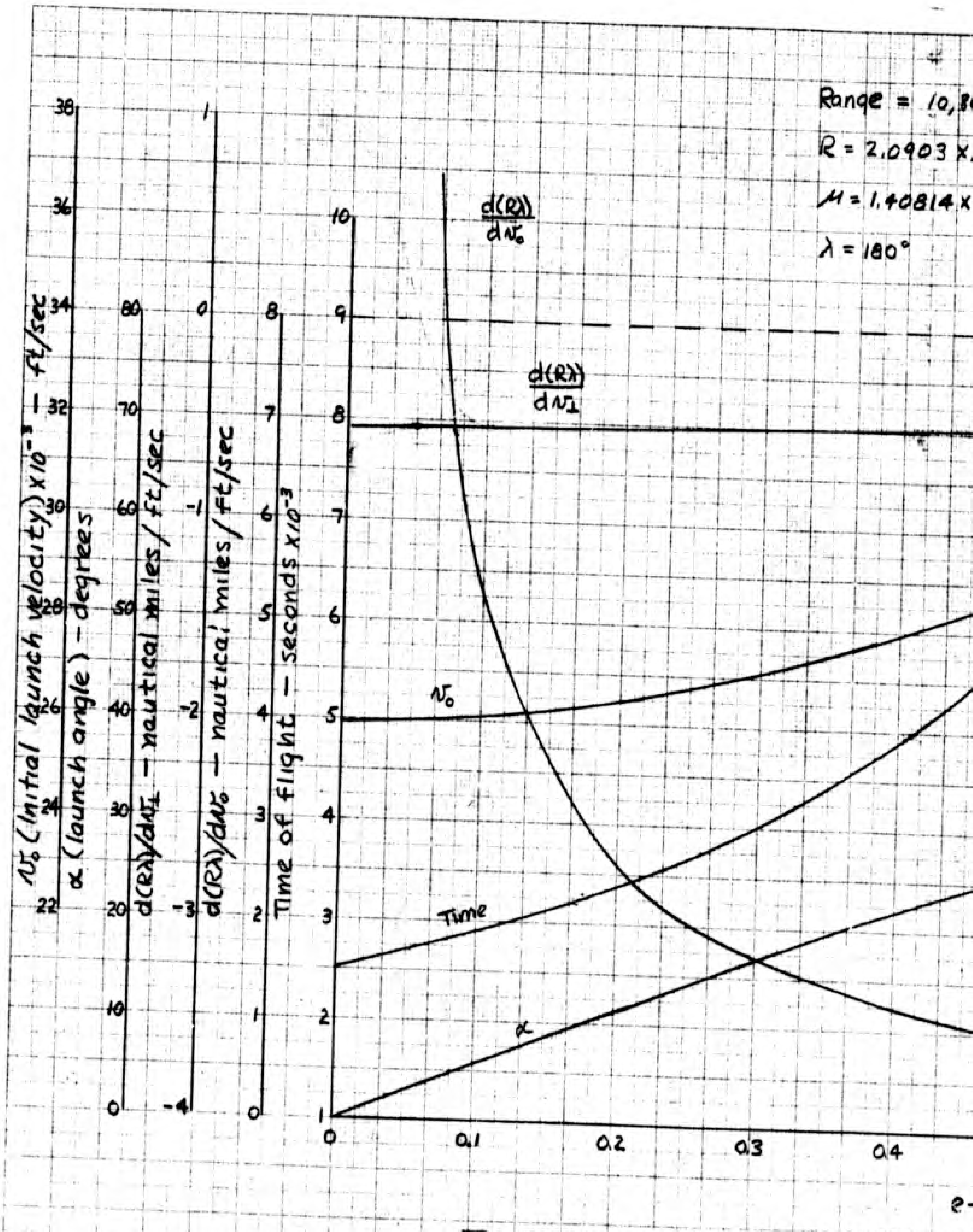


AND MISS COEFFICIENTS FOR A RANGE OF 5400 NAUTICAL MILES

FIG. 5

B



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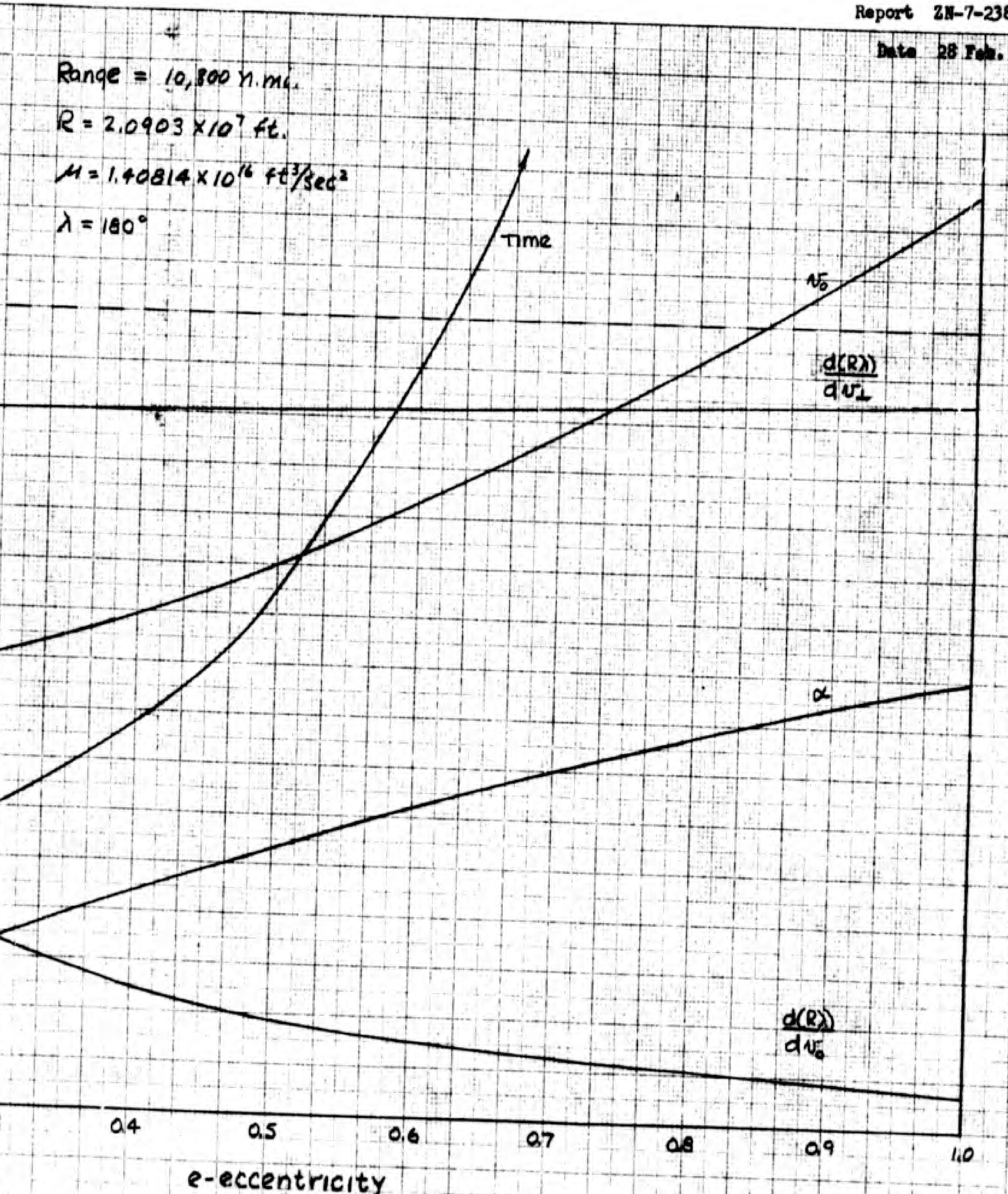
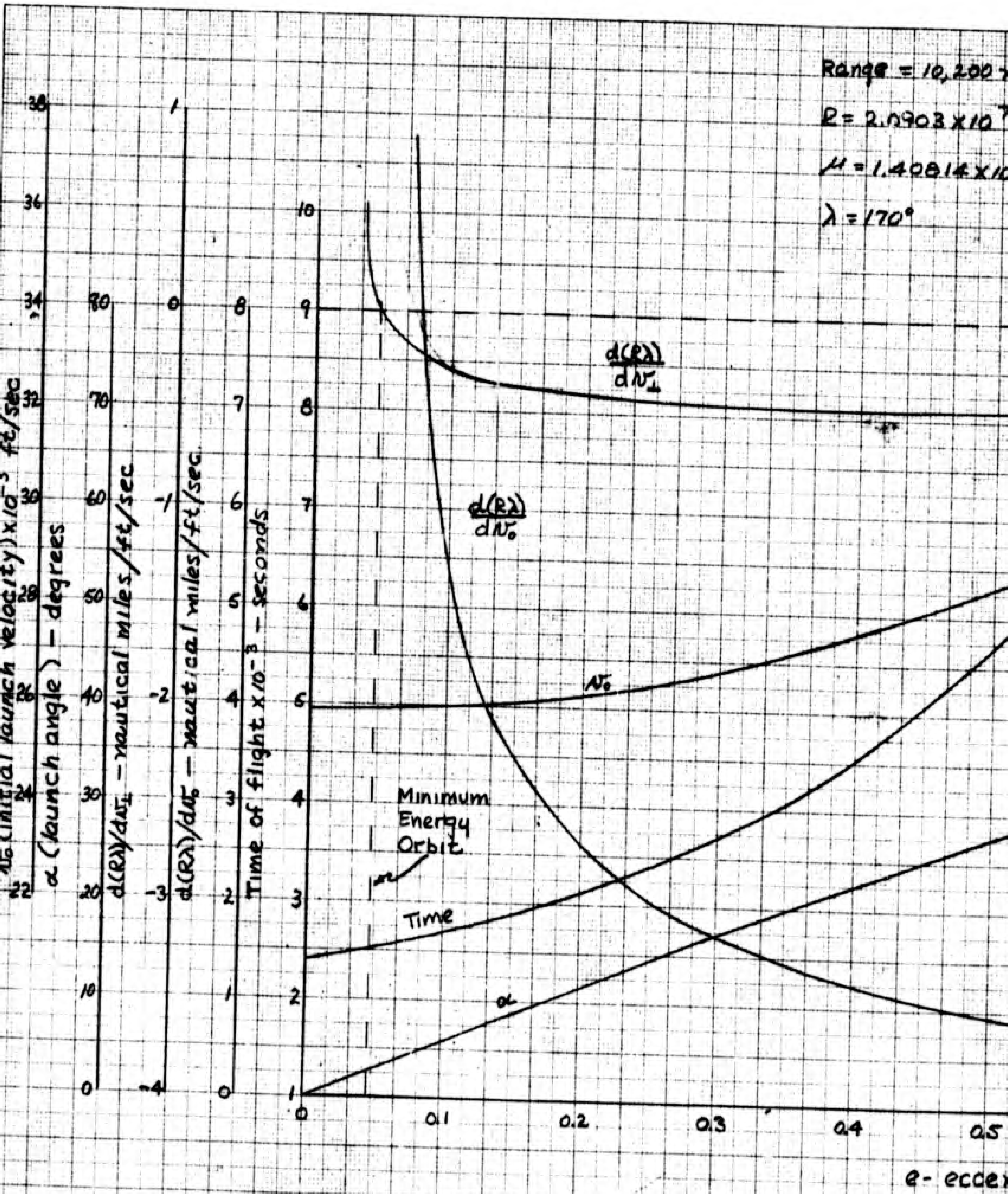


ERS AND MISS COEFFICIENTS FOR A RANGE OF 10,800 NAUTICAL MILES FIG. 7

B



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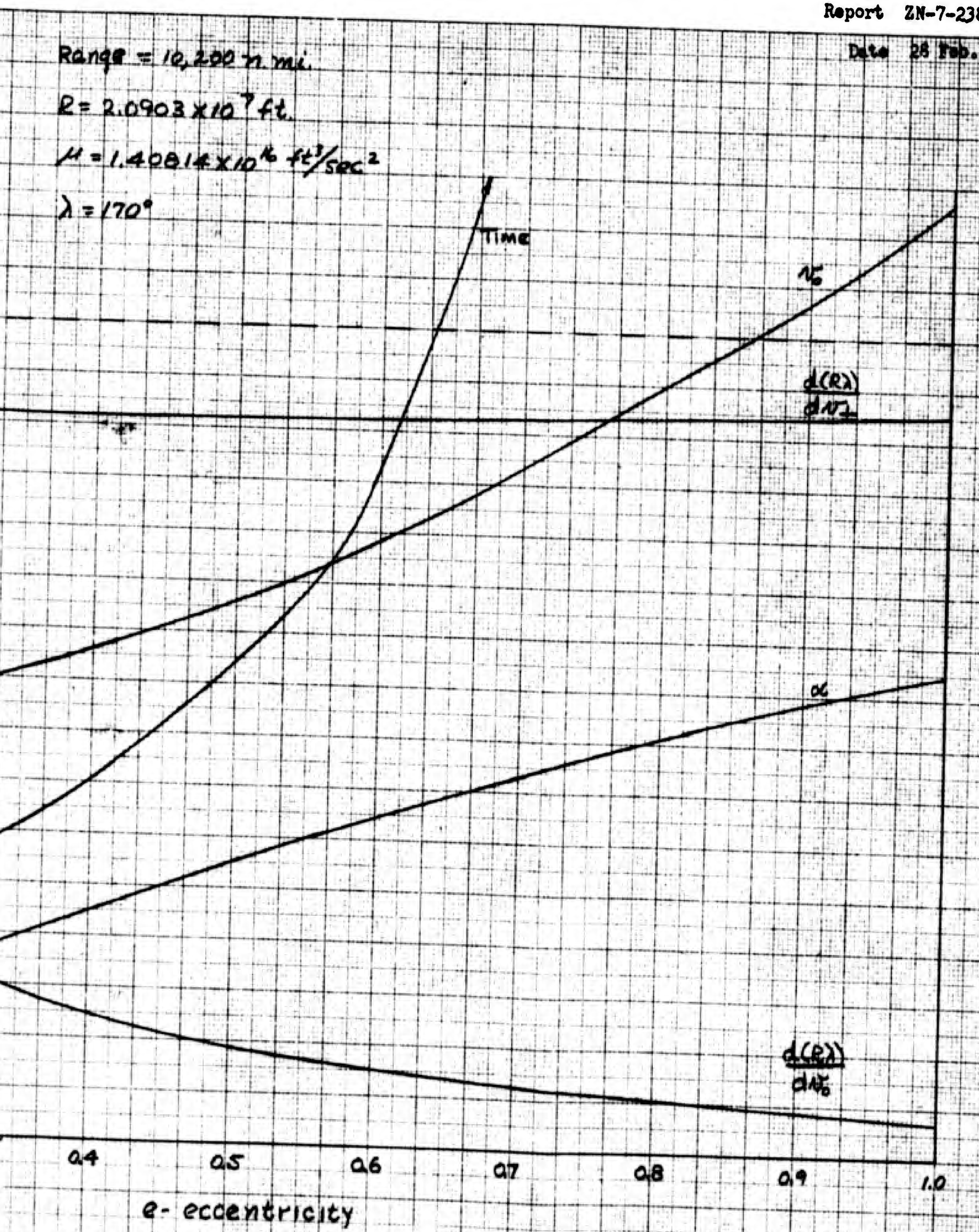
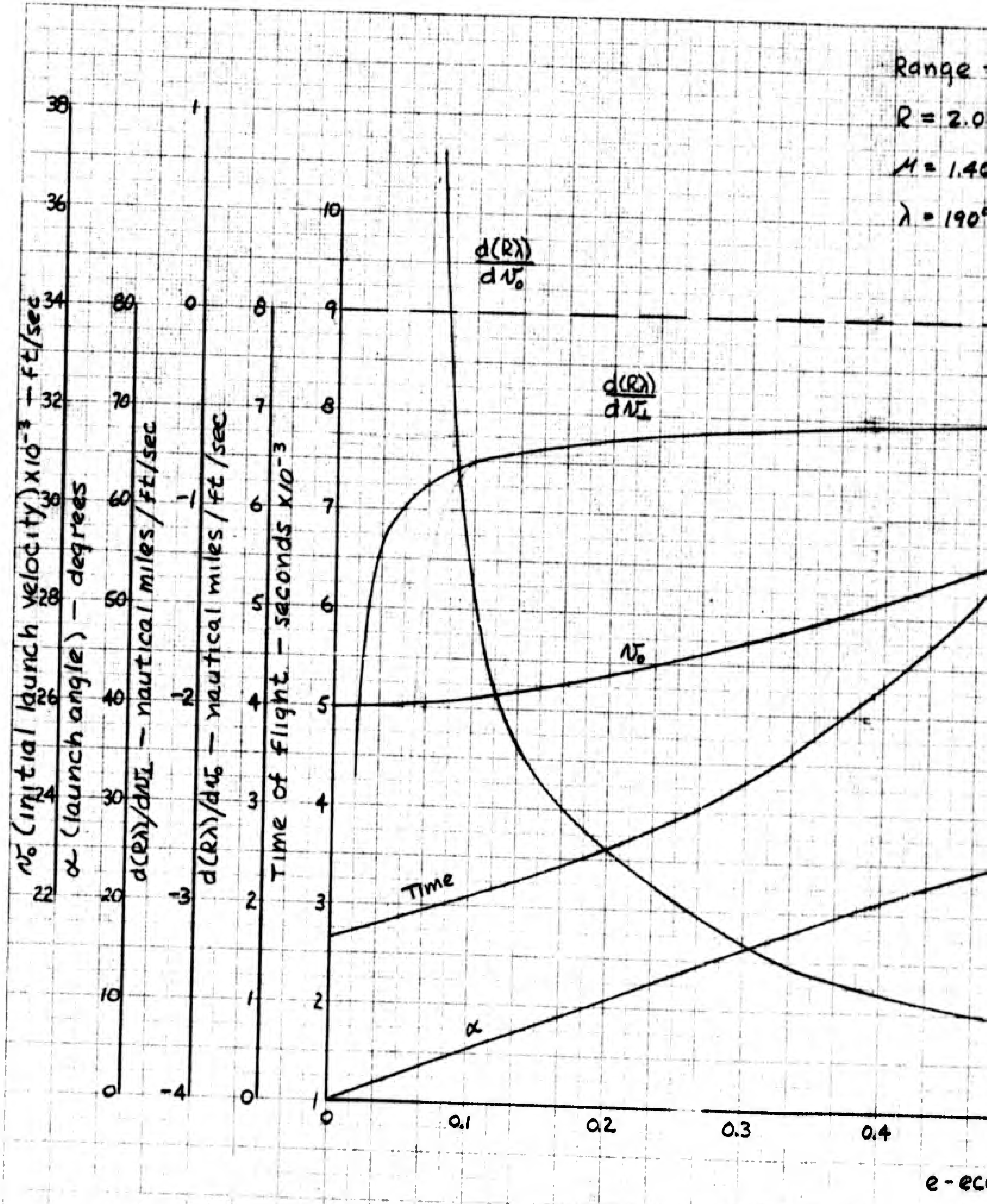


D MISS COEFFICIENTS FOR A RANGE OF 10,200 NAUTICAL MILES

FIG. 6

B



TRAJECTORY PARAMETERS AND MISS COEF

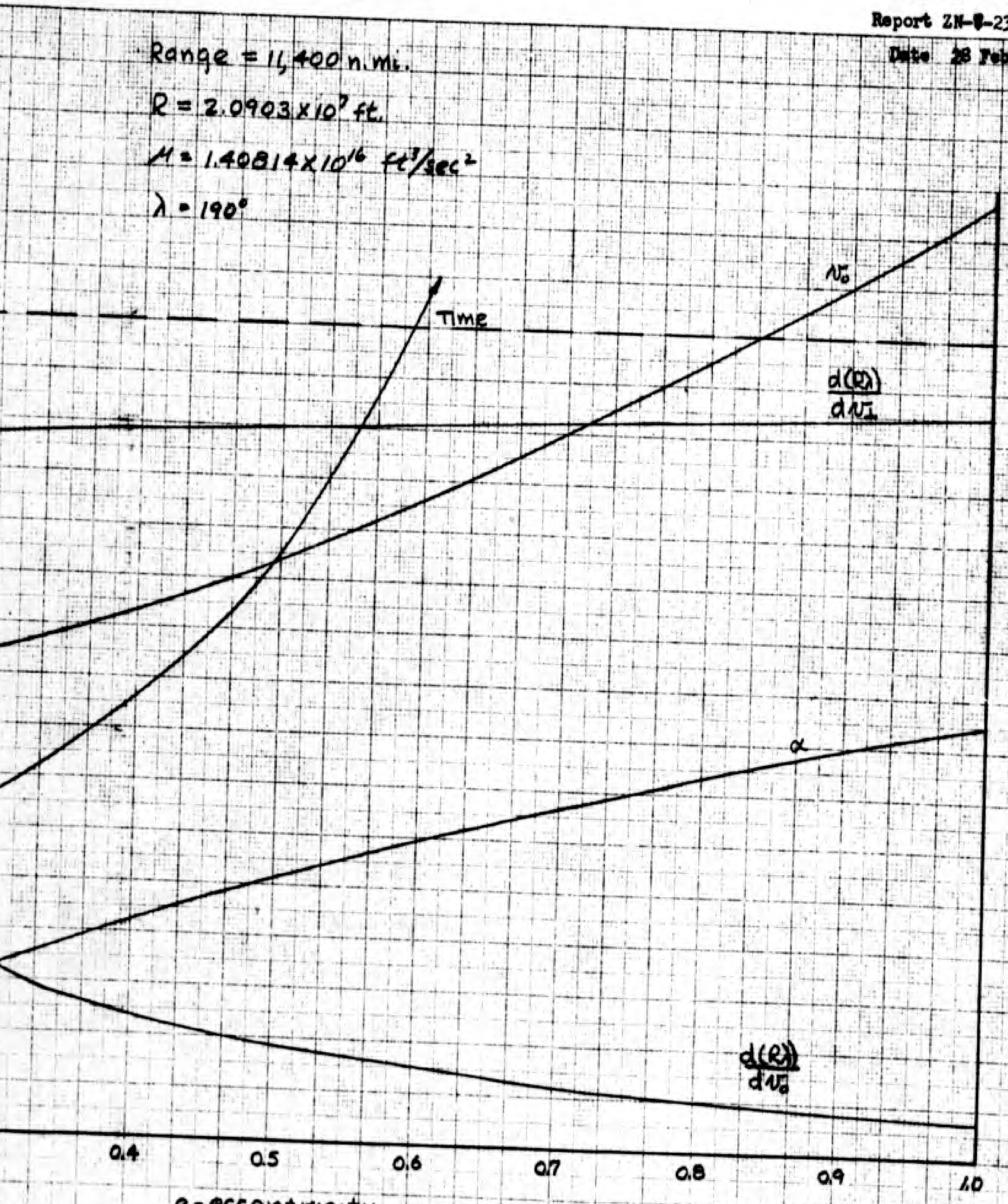
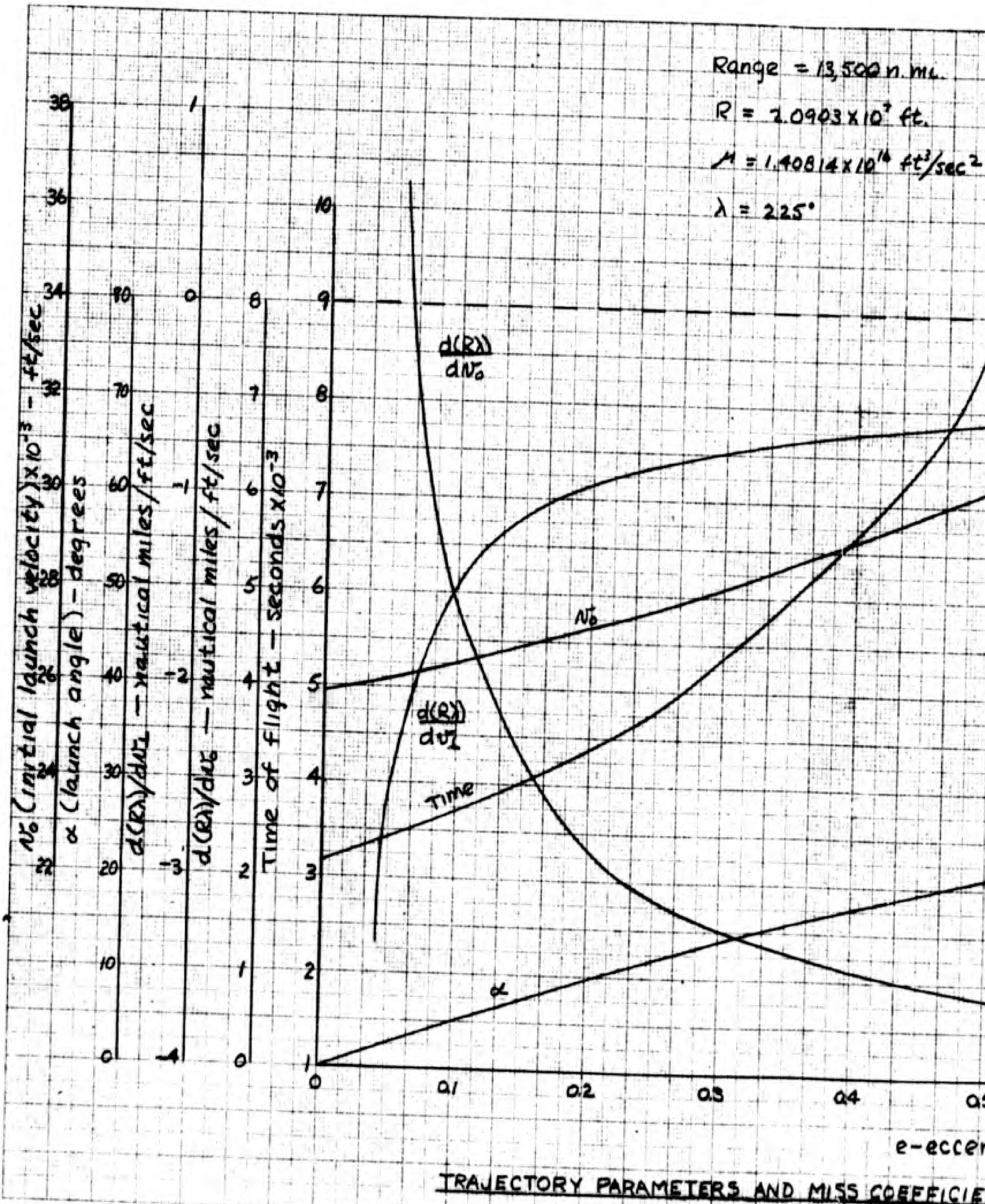


FIG. 8

RS AND MISS COEFFICIENTS FOR A RANGE OF 11,400 NAUTICAL MILES



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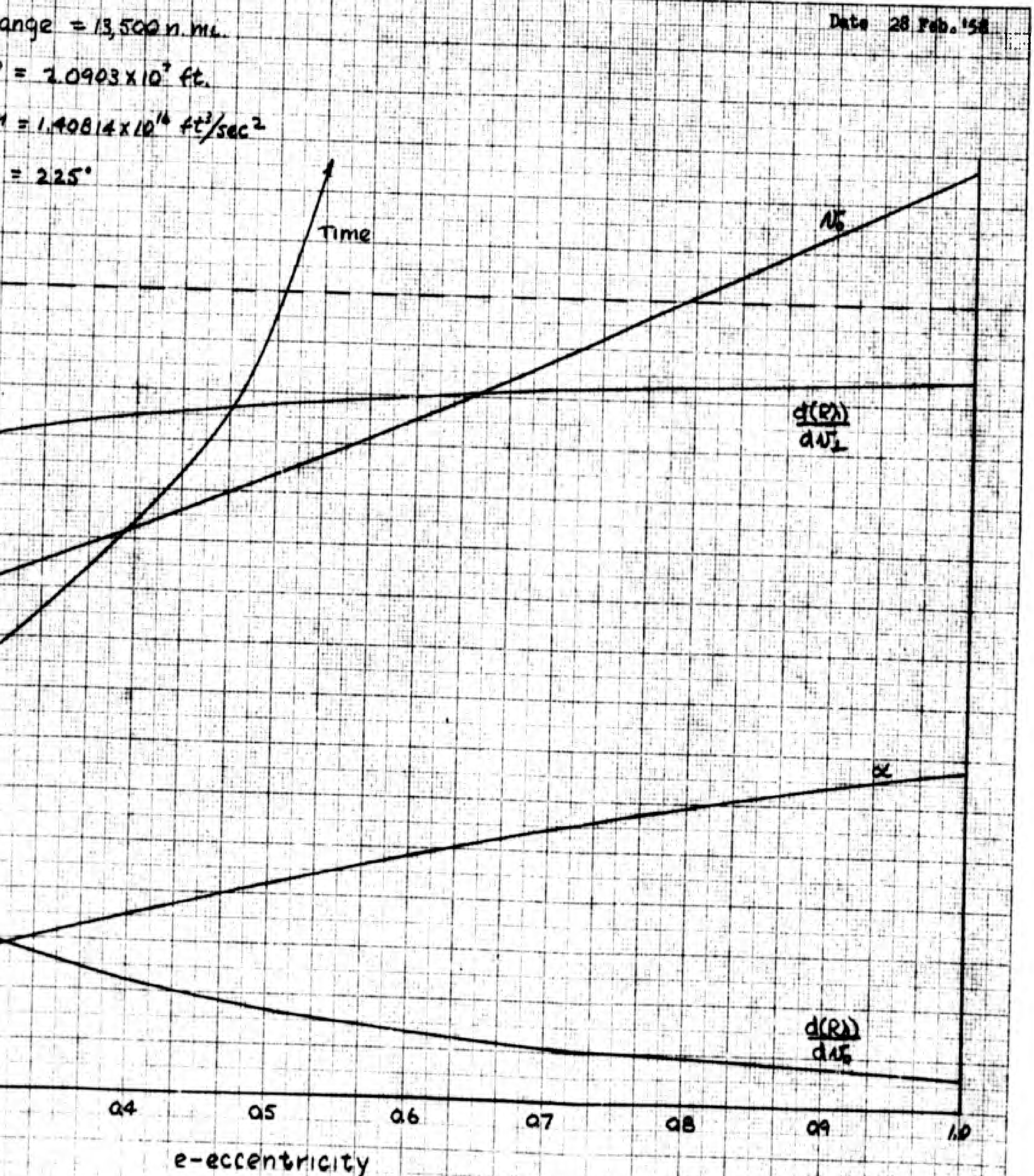
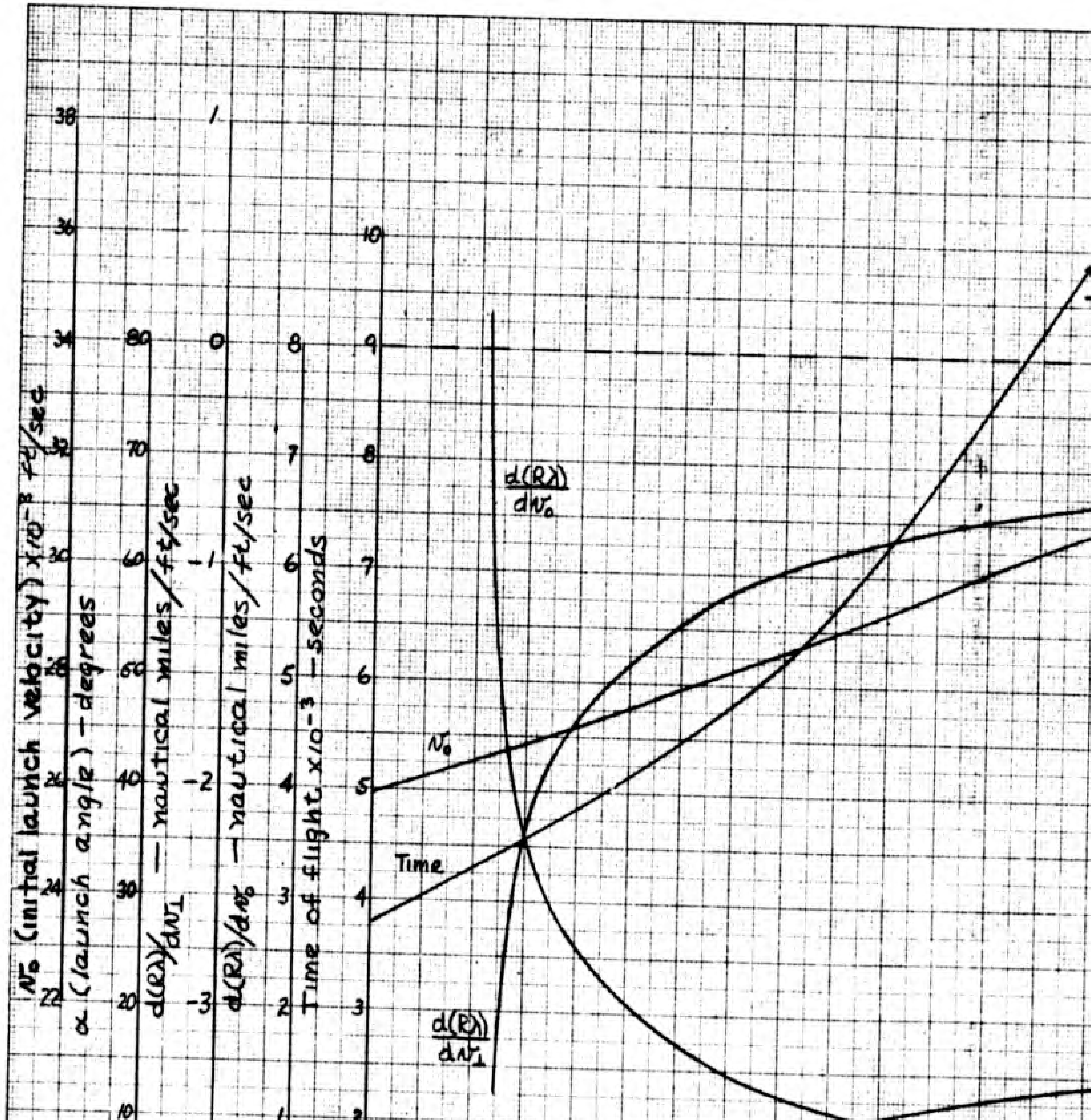


S AND MISS COEFFICIENTS FOR A RANGE OF 13,500 NAUTICAL MILES

FIG. 9

B



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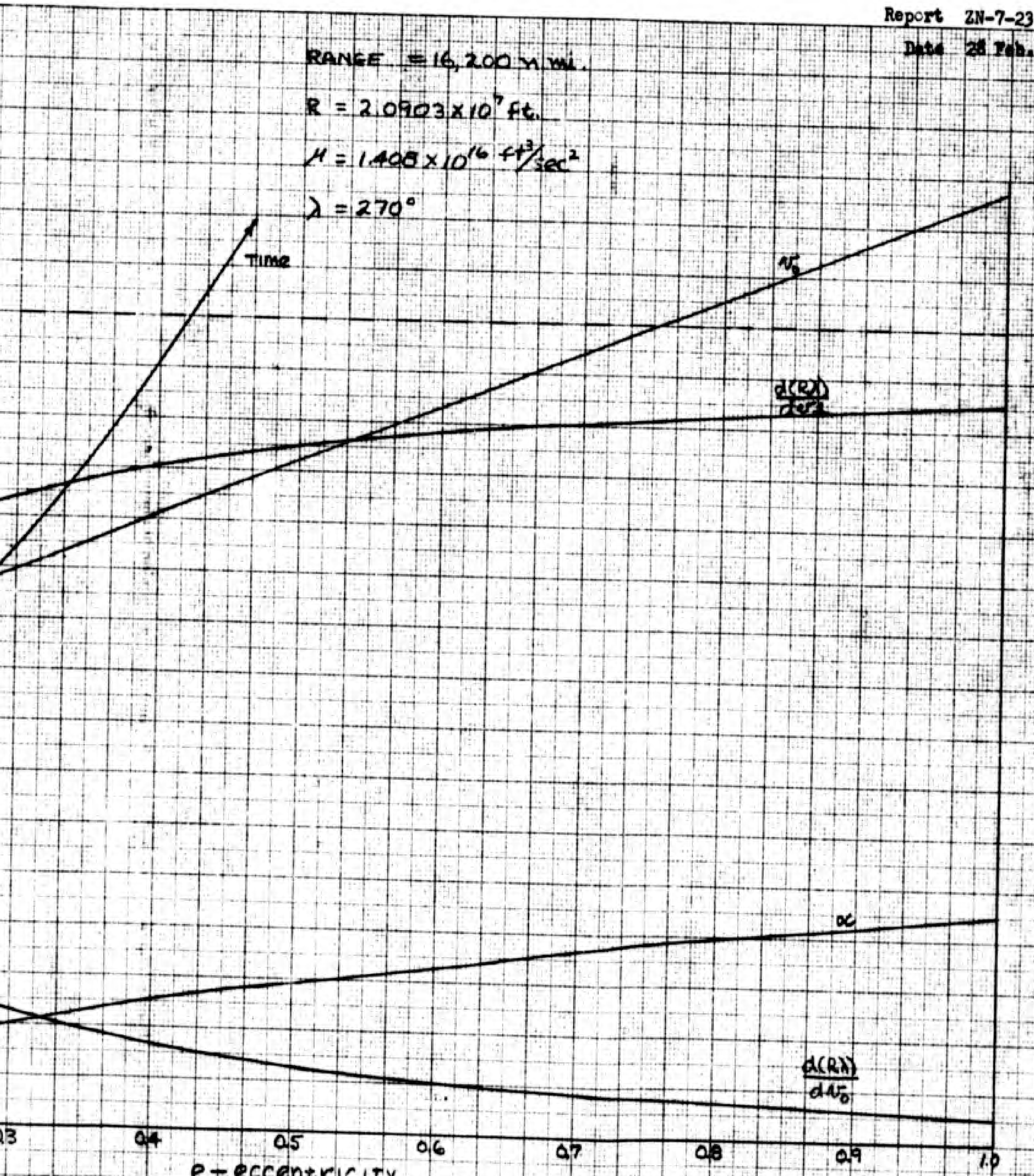

ERS AND MISS COEFFICIENTS FOR A RANGE OF 16,200 NAUTICAL MILES

FIG. 10

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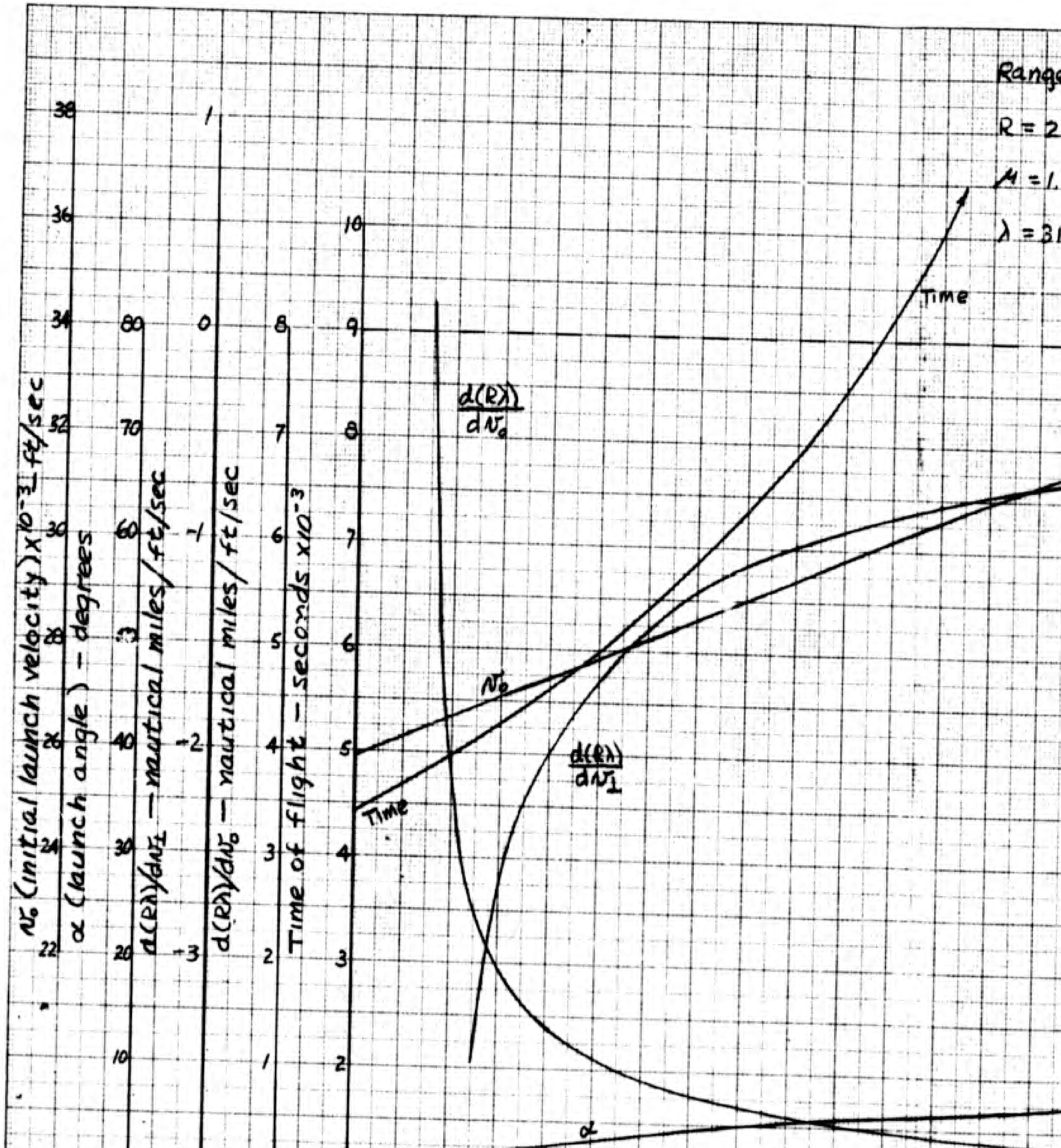

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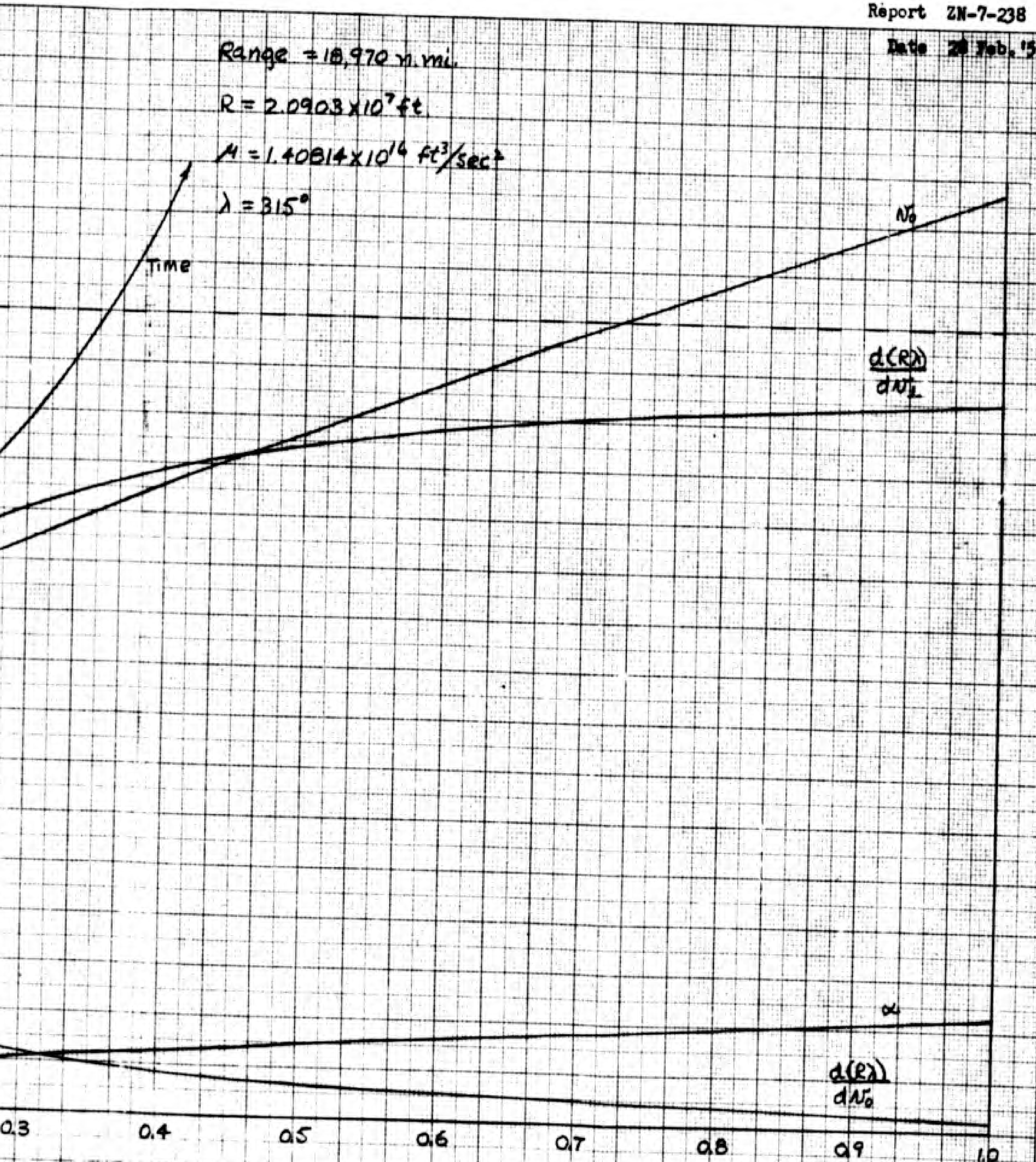

...ERS AND MISS COEFFICIENTS FOR A RANGE OF 18,970 NAUTICAL MILES

FIG. 11

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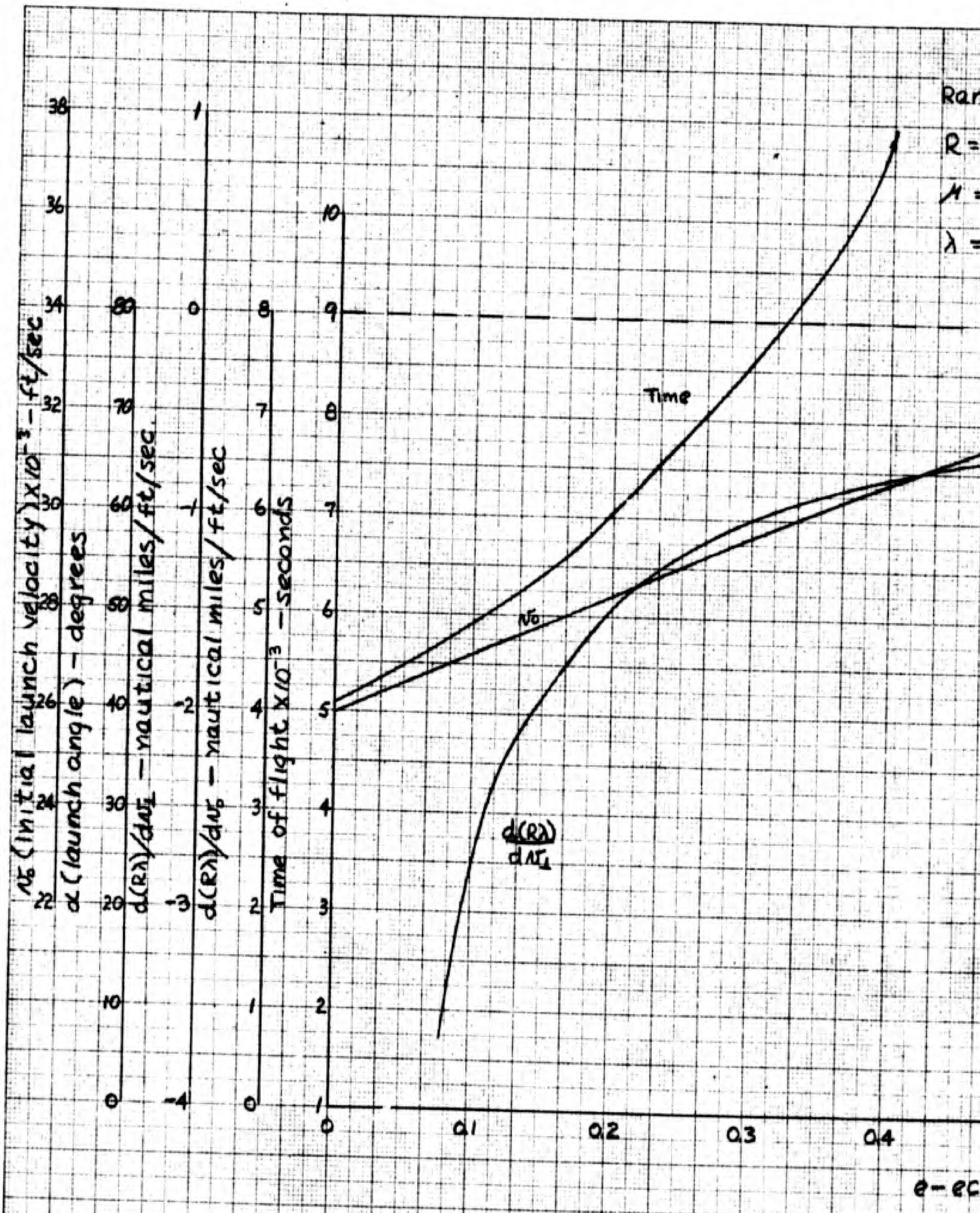

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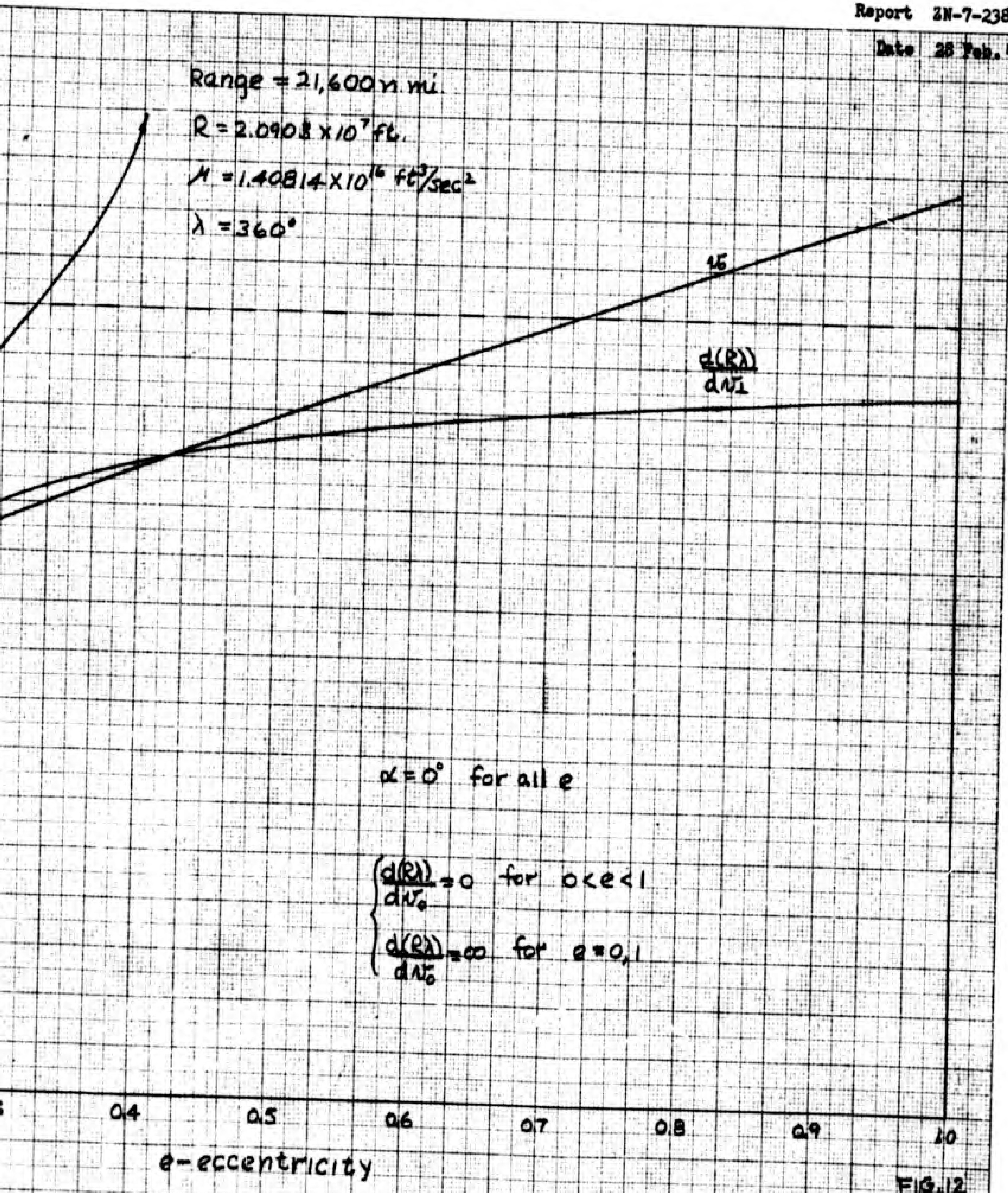

FIG. 12

RS AND MISS COEFFICIENTS FOR A RANGE OF 21,600 NAUTICAL MILES

C