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OPTIMUM SYSTEMS CHOICE FOR STRATEGIC RETALIATION: AN APPLICATION OF MAX-MIN THEORY

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ABSTRACT: An application of the mathematical theory of max-min developed by John M. Danskin has been made to the problem of optimum systems choice for strategic retaliation. One opponent seeks to maximize his strike-second capability while the other seeks to minimize it by pre-emptive blunting. Both are subject to total-resource limitations and both are fully cognizant of each other's systems options. The present report (a) reviews the mathematical model and its solution; (b) discusses the problem of parameter evaluation; (c) describes preliminary results obtained from sample calculations with semi-realistic parameter values; (d) undertakes the beginnings of a parametric sensitivity analysis; and (e) examines limitations of the model.

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In connection with Naval Ordnance Laboratory studies of possible future seabased candidate systems for strategic retaliation, as reported for example in the first draft of SINBAD ONE, Undersea Long-Range Missile (ULM) Systems, Submerged Intercontinental Ballistic Deterrent (U), Vols. 1 and 2, NOL(WO), 1 March 1967, the need emerged for determining objective methods of quantitatively comparing the survivability and cost/effectiveness of different strategic systems in the strike-second role. The present study, undertaken under Task No. MAT 03L 000/F099 01 01 Prob 001, reports initial results in an unclassified manner.

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REFERENCES

(a) "The Theory of Max-Min and its Application to Weapons Allocation Problems," J. M. Danskin, Springer, Berlin, 1967

(b) "Search and Screening," B. O. Koopman, Operations Evaluation Group Report, 1946

1. INTRODUCTION

The present report describes a simplified mathematical model of strategic retaliation wherein opponent #1 seeks to maximize by suitable choice of weapons systems his expected survivable strike-second capability, which opponent #2 seeks to minimize by striking first with weapons systems optimally chosen for the purpose. Both opponents are constrained in systems choice and procurement only by total expenditure limits. Both are fully aware of their mutual systems options.

This type of mathematical problem is a "game" only in special instances. More generally it is a "max-min" problem, as defined in reference (a). The particular model employed here is the one described in Chapter V of reference (a). Although it is illustrative of a wider class of mathematical models, we shall here be concerned not with its mathematical interest but with the insights it can offer concerning retaliatory systems choice.

We shall review the main features of the mathematical model, discuss parameter evaluation, and go on to describe the results of some semi-realistic sample calculations and their implications. Appendix A summarizes the main features of the mathematical solution.

2. VULNERABILITY CHARACTERISTICS OF STRATEGIC FORCES

The model must be introduced by a few qualitative remarks about vulnerability. All military systems are in some degree actually or potentially vulnerable to pre-emptive countermeasures. It is this circumstance that gives rise to perennial concern about the basing of strategic forces. For present purposes it will be convenient to distinguish with respect to basing vulnerability two idealized classes of systems, which we shall term "numerically vulnerable" (NV) and "percentage vulnerable" (PV). Numerically vulnerable systems are those like fixed ICBM silos that force no search effort on the blunting-minded opponent but may force him to considerable destructive effort, while the reverse is true of percentage vulnerable systems. The terminology reflects the definition that for a fixed level of attacker investment and an increasing level of the second-striker's investment a fixed percentage (hence an increasing number) of the PV systems come under attack; whereas an approximately fixed number of the NV systems (hence a decreasing percentage) come under attack. This is true because the PV systems characteristically require search, which subjects any newly-procured PV unit to the same probability of detection/destruction as any previously-deployed unit. NV units, as we shall see, find safety in numbers, so that a newly procured NV unit is on the average slightly safer than any of its predecessors were. NV units have to be procured in numbers, otherwise they are not safe; while the first PV unit is as safe as the hundredth.

No real strategic system exactly fits either of these two idealized vulnerability categories. That is, there is no system that when searched for and found requires zero additional effort to destroy, and no system that required zero search effort. However, a system like MINUTEMAN requires almost zero search effort, and a system like POLARIS requires negligible cost-to-destroy relative to cost-to-find. In fact the great majority of strategic striking systems thus far procured or proposed fall very nearly into one of these two categories or the other.

There are, however, two important types of systems that do not fit well into either category. These obey a law of vulnerability that approximates that of PV systems at low levels of blunting opposition, but departs from this at the higher levels. The first of these is typified by a mobile/concealed system deployed in an operating area (e.g., Montana, the Great Lakes, etc.) so confined that it may become possibly subject to saturation attack by area barrage of H-bombs. Area attack obviates search. The fractional vulnerability of the system is simply equal to the fraction of the deployment area that can be covered by H-bomb kill patterns.

The second example of a system not exactly suited to the NV, PV categories is one which consists of a small number of striking units, either fixed or mobile, that are concealed among a large number of decoys or false targets. The true targets must be genuinely indistinguishable from the false ones. If this condition is satisfied the would-be blunter must indiscriminately attack the entire system of decoys and true targets. As in the case of area barrage his military return for so doing, the expected number of true targets killed, is linearly proportional to blunting procurement investment, up to saturation (100% kill) of the target-plus-decoy system. Such striking systems — namely, those vulnerable to area barrage and those concealed among large numbers of decoys — may be termed "linearly vulnerable". At low levels of counterforce opposition they can be treated as PV systems, and will be so treated here. The model presented here considers only PV and NV systems. To treat linearly vulnerable systems more accurately the model should be expanded.

3. MATHEMATICAL MODEL

Let C_1 designate the total resources of opponent #1 (the second-striker) available for strategic retaliatory systems procurement, measured in billions of dollars, C_2 the corresponding blunting resources of the first-striker. Let X_i represent the amount (billions) invested by #1 in retaliatory system i, where i runs through the list of percentage vulnerable candidate systems (i.e., those involving mobility/concealment). It is assumed that the various PV systems are "diversified" in the sense that countermeasures investments against one are not effective against the others. In the absence of countermeasures (i.e., on first strike) it is assumed that the i^{th} system can deliver $V_i X_i$ million lbs. of nuclear "throw weight," where V_i is an effectiveness/cost parameter measuring (say) millions of lbs. of throw weight per billion dollars invested. In general V_i has a different numerical value for each of the available candidate systems. After attack by opponent #2 the fraction of #1's striking power surviving is of the form $\exp(-A_i Y_i)$, where Y_i is the amount of resources allocated by #2 to blunting the i^{th} system and A_i is a parameter measuring the vulnerability of the i^{th} system to countermeasures. This may be seen from the well-known law of random search (reference (b)), which is of exponential form, with $A_i Y_i$ a measure of search effort against the i^{th} system. Parameter evaluation will be discussed in the next section. The survivable throw weight for all of #1's systems is

$$\sum_i V_i X_i e^{-A_i Y_i} ,$$

summation being extended over all candidate PV systems.

Similarly for numerically vulnerable systems $U_j X_j$ is the number of millions of lbs. of throw weight deliverable by ready units on first strike and the surviving fraction after blunting is again of exponential form. If we consider the exponential as representing the probability that any particular "silo" survives, then the quantity in the exponent is the negative of the expected number of hits, proportional to the number of H-bombs (or other units of destruction) allocated by the attacker per silo. The latter number is proportional to the ratio (Y_j/X_j) of #2's to #1's resources allocated to the j^{th} system. That is, the total survivable throw weight is

$$\sum_i U_j X_j e^{-B_j Y_j / X_j} ,$$

summation being extended over all candidate NV systems. The total survivable throw weight for all systems, a "payoff" loosely related to the degree to which #1 can deter #2, is the sum of the two summations just noted. It is to be maximized by #1's optimum (maximizing) choice of the X_i, following #2's optimum (minimizing) choice of the Y_i,

$$\underset{X_{i,j}\ Y_{i,j}}{\mathrm{Max\ Min}} \left\{ \sum_i V_i X_i e^{-A_i Y_i} + \sum_j U_j X_j e^{-B_j Y_j / X_j} \right\}, \qquad (1)$$

subject to total resource limitations,

$$\sum_i X_i + \sum_j X_j = C_1,$$

$$\sum_i Y_i + \sum_j Y_j = C_2,$$

$$X_i \geq 0,\ Y_i \geq 0,\ X_j \geq 0,\ Y_j \geq 0$$

for all i, j.

Note that each system is characterized by only two parameters, a cost index and an invulnerability index.

The mathematical solution of this problem, involving a sort of modification of the Lagrange multiplier technique, has been given in detail by Danskin (reference (a)), and is summarized in Appendix A.

4. FORMULAS FOR PARAMETER EVALUATION

A model pitched on the level of generality characteristic of max-min theory requires a certain amount of intermediate-grade theorizing to evaluate the parameters involved. In particular it will be recalled that each system is characterized by two numbers, one measuring effectiveness/cost, the other measuring vulnerability. These must be made accessible to practical evaluation.

The effectiveness/cost parameters, V_i, U_j, may be defined, for example, as (first strike) ready throw weight (millions of lbs.) per billion dollars total investment in the system. In the numerical examples given in this report we shall arbitrarily consider only ready (30-minute delay) throw weight, not inventory, mobilizable, or eventual throw weight. Similarly we shall consider 10-year total system costs, excluding R&D costs or pro-rating them to procurement. It is obvious that such definitions are arbitrary and may be discriminatory. (For example systems of 20-year lifetime are not fairly compared with those of 10-year lifetime, if all are amortized over a 10-year period.)

We now examine the vulnerability indices. For purposes of simplifying our illustrative examples we hypothesize a first-striker's capability for the mid-1970's and later that depends on a single type of offensive weapon, variously applied to blunt the second-striker's systems of whatever nature. This assumed all-purpose threat is a ballistic missile, capable of lifting heavy payloads, which consist of a number of fairly low-yield separate nuclear warheads. It is assumed that the latter can be scattered around at random or placed in a pattern near the aim point. If such a concept were technically realizable without heavy weight penalties, the result would be a substantial improvement in area coverage over what could be achieved by the same throw weight devoted to a single high-yield nuclear weapon. Thus, this type of weapon might conceivably become a threat to limited-area PV systems as well as NV systems in the time frame in question. It would recommend itself to a first-striker because of its obvious adaptability to surprise attack.

a. Area Attack on PV Systems

For PV systems subject to area barrage attack of the kind just described, the formula

$$A_i = \frac{A'M\alpha}{A_o\ C_m} \qquad (2)$$

is useful for evaluating the parameter A_i that appears in the exponential of Eq. (1). Here A', a function of target hardness, is the incapacitation (mission abort) area against one deployed unit of the retaliatory system per effective individual bomb, α is the fraction of individual bombs effective, A_o is the system operating area, M is the number of individual bombs per missile, and C_m is the attacker's cost per missile in billions of dollars.

It is assumed that no tactics are available to the attacker that would permit approximate localization of the second-striker's PV units. Thus, the entire operating area A_o must be subjected to barrage. If the individual warheads are lobbed in at random or directed with CEP's comparable to or greater than the mean separation of burst points, the consequent law of random destruction is the Poisson law of Eq. (1) (first term), identical to the law of random search. In this case Eq. (1) applies rigorously and the system under attack is truly percentage-vulnerable, as we have defined the term.

If the individual bombs (or other weapons) are capable of being burst in a geometrical pattern that gives uniform area coverage, if they are 100% reliable, and if we ignore doubts about the state of human knowledge of weapon kill radii, then the exponential of Eq. (1) must be expanded

$$e^{-A_i Y_i} = 1 - A_i Y_i + \cdots ,$$

and only the linear term retained, so that the force requirement for 100% coverage of the operating area becomes

$$Y_i = \frac{1}{A_i} = \frac{A_O \, C_m}{A' \, M\alpha} .$$

For blunting expenditures Y_i less than this amount, the surviving fraction of the second-striker's retaliatory system is $1-A_i Y_i$. This is the case of "linear vulnerability" previously mentioned. It represents an idealized case, strictly valid only when $\alpha = 1$ (all missiles reliable). The PV and linear cases coincide except at heavy levels of attack. Our model will here be applied to linearly-vulnerable as well as true PV systems, with the understanding that whenever it predicts a small surviving fraction of a linearly-vulnerable system that fraction is over-[illegible]mated.

b. Ran[illegible] Search for Seabased PV Systems

For random seach by enemy ASW units a formula similar to Eq. (2) applies, with $A'\alpha M$ replaced by area searched out per search unit within A_O during some tactically significant time interval before or at H-hour and C_m replaced by cost per search unit in billions of dollars. The tactically significant time interval must presumably be short enough to avoid giving strategic warning of the attacker's intent.

c. Aimed Attacks on NV Systems

Knowledge or predictability of the locations of NV system units permits the first-striker to "engineer" his attack.

Evaluation of the vulnerability index B_j in Eq. (1) for NV systems subject to our previously-described ballistic missile attack proceeds from the well-known formula (reference (a)) for survival probability q of a point target subject to attack by k independently-aimed weapons,

$$q = e^{-\log 2 \left(\frac{R}{C}\right)^2 k},$$

where R = individual bomb incapacitation radius and C = CEP. If A', as before, is the incapacitation area per bomb, then $R^2 = A'/\pi$. The parameter k is equal to the ratio of total number of effective bombs employed against the j^{th} NV system, $\alpha M C_m^{-1} Y_j$, to the number of fixed point-targets ("silos") comprising the NV system in question--namely, $U_j X_j / W$, where W is the ready nuclear throw weight deliverable per silo. Thus the surviving fraction of silos is

$$q = \exp\left\{ - \frac{\log 2}{\pi C^2} \frac{MA'}{C_m} \frac{\alpha W}{U_j} \frac{Y_j}{X_j} \right\} = e^{-B_j Y_j / X_j},$$

whence B_j is evaluated as

$$B_j = \frac{0.221\, A' \alpha M W}{C^2 C_m U_j}. \qquad (3)$$

This expression shows the gain in invulnerability (B_j^{-1}) with increasing effectiveness/cost parameter U_j. It is this that gives NV systems their previously-mentioned characteristic of finding "safety in numbers." Equation (3) is valid strictly only when $A' \ll C^2$ and only for attack by individually aimed weapons. Where these conditions are violated the resulting formulas in general overestimate the effectiveness of the attack.

d. Attacks on Decoy Systems

Suppose that some true targets are hidden among a large number N_d of false targets or decoys from which they are indistinguishable. We now show that the (linearly vulnerable) system so concealed is approximately percentage vulnerable, whether it be fixed or mobile. The first-striker's system is as before.

If K is the average number of bombs placed in a pattern such as to cover the area accessible to the individual target (i.e., sufficient to cover any area of target position uncertainty caused by target data delays) and other quantities are as before, then the expected number of targets (true targets plus decoys) destroyed by the surprise attack is $\alpha MY_1/KC_m$, and the surviving fraction of all targets, equal to the surviving fraction of true targets, is approximately

$$1 - \frac{M Y_1 \alpha}{KC_m N_d} \approx e^{-\frac{\alpha M}{KC_m N_d} Y_1} = e^{-A_1 Y_1}$$

whence

$$A_1 = \frac{\alpha M}{KC_m N_d} . \qquad (4)$$

Thus, a PV type of vulnerability formula applies to the decoy problem, provided N_d is much greater than the number of true targets. As in the case of pattern barrage, we make the approximation of replacing a linear expression by an exponential, a procedure that causes underestimation of the effectiveness of a very heavy attack. The foregoing applies only to the case of pre-existing decoys. If decoys are purchased in fixed ratio to true targets, the system is NV unless mobile or concealed.

5. QUALITATIVE RESULTS FROM THE MODEL

Some qualitative results of the max-min model will now be examined. First, Danskin proves as a theorem that the optimum allocation of #1's resources for survivable striking power will involve in general the purchase of at most one NV system. This result is not surprising, since we noted that NV systems find safety in numbers, and any system that fails to earn top score in the retaliatory bangs-per-buck competition can for given resources be procured only in fewer numbers than the winning system. Hence its inclusion in a mix would, by reducing total numbers, detract from the safety of all.

The fact that a few PV units are as safe as a great many, on the other hand, means that no bonus results in general from excessive procurement of any particular PV system. In fact, over-procurement merely rewards the opponent's existing countermeasures investment. This is true whether that investment is for search effort, for weapons of the kind discussed above, or for any other form of blunting capability. For example, if #1 buys a certain number of retaliatory systems and puts them in operating area A, and #2 buys enough nuclear area-destruction capability to cover A completely, then if #1 proceeds to put any more systems similar to the original ones (mobile or not) into A his investment is completely wasted, for he is merely rewarding his opponent's prior investment by increasing his expected kills per dollar. Thus one would anticipate that among PV systems in general no single system should be over-procured, but that additional investment should go toward diversifying the threat by buying modest numbers of each of a number of non-trivially different systems comprising a <u>mix</u>.

This elementary expectation is confirmed by the mathematics. It is shown in Appendix A that for given values of the total investments, C_1, C_2, and of the fraction of #1's resources devoted to PV systems, there exists a number n such that the first n of the candidate PV systems, rank-ordered by strike-first bangs-per-buck, $V_1, V_2, V_3, \ldots, V_n$, should be included in an optimum mix, and no others. More significantly, the amount of funds devoted to procurement of any one of these chosen systems should be directly proportional to its invulnerability index, $1/A_i$, and should be completely independent of its effectiveness/cost parameter, V_i. Thus cost enters only in determining admissibility or non-admissibility to the mix. Once admission is granted, the proper procurement level is determined solely by proportionality to system invulnerability. This is perhaps the most arresting single result of the model.

The model fails to give any comparably simple and general answer to the question of optimum balance between investment in the mix of PV systems and the (at most) one NV system. It is necessary to examine limiting cases. If $C_1 >> C_2$, which represents the case of Goliath retaliating against Little David's surprise attack, Goliath's optimum policy tends to favor heavy procurement of the NV system, provided the unit cost is very low, so that overwhelming numbers can be bought. If the roles are reversed, $C_2 >> C_1$, so that Little David strikes second against Goliath, the model indicates that for realistic magnitudes of C_2 (though not in the hypothetical limit $C_2 \to \infty$ David must buy only PV systems and no NV system at all. This becomes all the more true the smaller David's resources are. Moreover, as those resources shrink in general the number n of candidate systems admissible to his optimum mix does not decrease. In other words, diversification of retaliatory PV systems is not a luxury of the rich, but a necessity of the poor.

This point has been widely missed or misunderstood in the U.S., where the suggestion to diversify strategic forces is often met by the observation, "We can't afford it." The max-min model results make it clear that the shoe is on the other foot: only the very rich (Goliath--the big spender in our first example) can afford not to diversify. Another implication for national policy may be worth noting: if the international nuclear arms race ever gives way to an arms control mode of austerity, the max-min model indicates that the U.S. will be forced toward a mix of diversified systems, no one of which is heavily procured. Given a low total budget and an opponent who may cheat, that is the only path of credible survivability.

6. SAMPLE QUANTITATIVE RESULTS

Sample calculations have been made with the aid of a computer program based on the model. For these we have deliberately chosen hypothetical parameter values and system characteristics in order to keep the discussion unclassified. When more realistic candidate system parameter values are employed more meaningful results will become available. Because of the observed tendency of the quantitative to take precedence over the qualitative (*i.e.*, for numbers to exclude reason), it is essential that the reader approach the results to be discussed with a clear understanding that they concern only semi-realistic systems.

The systems considered are of the two types previously discussed, numerically vulnerable and percentage vulnerable. All are considered to be subject to the most damaging type of surprise attack that opponent #2, the first striker, can procure with his limited budget, C_2. A simplification is assumed at the outset: that this most damaging attack is of the type discussed in Section 4. This is certainly plausible for the NV (fixed) targets, and also appears to be true for PV targets deployed in limited areas. For simplicity we have assumed that the first striker (opponent #2) can achieve a CEP of 0.1 n.mi. when needed and that he can deliver $M = 100$ individual bombs each of 100 KT yield, with 100% reliability ($\alpha = 1$), for a total cost of $15 million per large missile. (Such numbers obviously exaggerate the attacker's capabilities.) All systems of opponent #1 are considered subject to this same basic attack, modified for optimum pattern barrage against PV systems and for point-target attack against NV systems. No consideration is given to the effect of AICBM's on either side. (This could be roughly taken into account by raising the effective costs of all ballistic missile systems.)

The candidate retaliatory systems considered are listed in Table 6.1. The characteristics ascribed to them in that table lead, by the formulas of Section 4, to the parameter values listed in Table 6.2. These were used in obtaining the results to be described. All parameters of Section 4 have been discussed except W, the throw weight per silo of opponent #1's candidate NV systems. This was assumed to be 7000 lb. Since all weights in Equation (3) of Section 4 are expressed in millions of lb., the value of W used in that equation is $W = 7 \times 10^{-3}$.

Table 6.1. Vulnerability and cost characteristics of hypothetical candidate systems considered in sample max-min calculations.

System	System cost ($ billions) to obtain 4-million lb. ready throw weight capability	Area of system vulnerability to 100 KT optimum burst (sq. n. mi.)	Area of mobility (sq. n. mi.)
NV #1 ("Super-hardened fixed-base")	20	0.00196*	---
NV #2 ("Bargain fixed-base")	4	0.0176**	---
PV #1 ("Land-mobile, soft")	20	16	3×10^5
PV #2 ("Great Lakes submersible")	23	16	9×10^4
PV #3 ("CONUS offshore submersible")	24	16	1.8×10^6
PV #4 ("Forward-deployed submarine")	40	16	1.2×10^7
PV #5 ("All-ocean submarine")	60	16	1.2×10^8

* Hardened to survive 50-yd. miss of 100 KT.
**Hardened to survive 150-yd. miss of 100 KT.

Table 6.2. Parameter values, consistent with Table 6.1., assumed in sample max-min calculations.

System	B_j	U_j	A_i	V_i
NV #1	9.10	0.222	-	-
NV #2	37.0	1.00	-	-
PV #1	-	-	0.36	0.200
PV #2	-	-	1.20	0.174
PV #3	-	-	0.06	0.167
PV #4	-	-	0.009	0.100
PV #5	-	-	0.0009	0.067

Two types of numerically-vulnerable candidate systems are hypothesized, first a super-hardened fixed-based system in the continental U. S. (CONUS), capable of remaining operational after a 50-yard miss distance by a 100-KT bomb, and offered at the comparatively low price of \$20 B 10-year cost for 4 million lb. of ready, unopposed throw weight (on first strike). The second NV system considered is a fairly hard version of the same thing (150-yard survival distance) capable of delivering the same throw weight for a bargain 10-year price of \$4 B.

The percentage-vulnerable systems considered in Table 6.1. were for simplicity all assumed to have a 100-KT incapacitation vulnerable area of 16 sq. n. mi. Obviously more or less hardening of particular systems would strongly affect this important parameter. The first system considered is a mobile, land-based, soft (3 psi vulnerability criterion or less) system, costing \$20 B for 4 million lb. unopposed throw weight, deployed with uniform probability density in an area equivalent to three average-sized western U. S. states. The second PV system consists of rather costly (at \$23 B for the same throw weight) submersibles in the Great Lakes. The third system is technically similar but deployed in 1.8 million sq. n. mi. of CONUS offshore waters (0.25 million sq. n. mi. of which is continental shelf). It is arbitrarily priced somewhat higher at \$24 B. The fourth PV system resembles POLARIS in being forward-deployed in 12 million sq. n. mi. of ocean, but is priced at an unrealistically low \$40 B for the same throw weight capability as above. Finally, a hypothetical submarine system costing \$60 B for the same throw weight is considered as PV system #5. This system is supposed to be equipped with an 11,000-n mi. range missile capable of reaching targets from the antipodes and to be deployed in all oceans.

Such are the competitors, and the question is, which one or ones to buy? To answer this question the max-min model requires one more datum: the total blunting investment C_2 by the first-striker. This information is not readily forthcoming because of its speculative nature. The blunting investment must

Table 6.3. Second-striker's max-min optimum allocation policies for spending $C_1=\$20$ B to maximize survivable throw weights against opponents of various degrees of "toughness" (various C_2-values). Figures given are in billions, C_2 regions as shown in Figure 6.1. Systems are those identified in Table 6.1.

	Range of Blunting Total Investment C_2 by First-Striker.			
System	Region 1 $C_2 < \$1$ B	Region 2 $\$1 \text{ B} < C_2 < \$11$ B	Region 3 $\$11 \text{ B} < C_2 < \$60$ B	Region 4 $C_2 > \$60$ B
NV #1	-	-	-	-
NV #2	20	-	-	-
PV #1	-	2.7	*	*
PV #2	-	0.8	*	*
PV #3	-	16.4	2.5	*
PV #4	-	-	16.9	1.8
PV #5	-	-	-	17.9

*Optimum solution calls for an amount less than \$0.5 B.

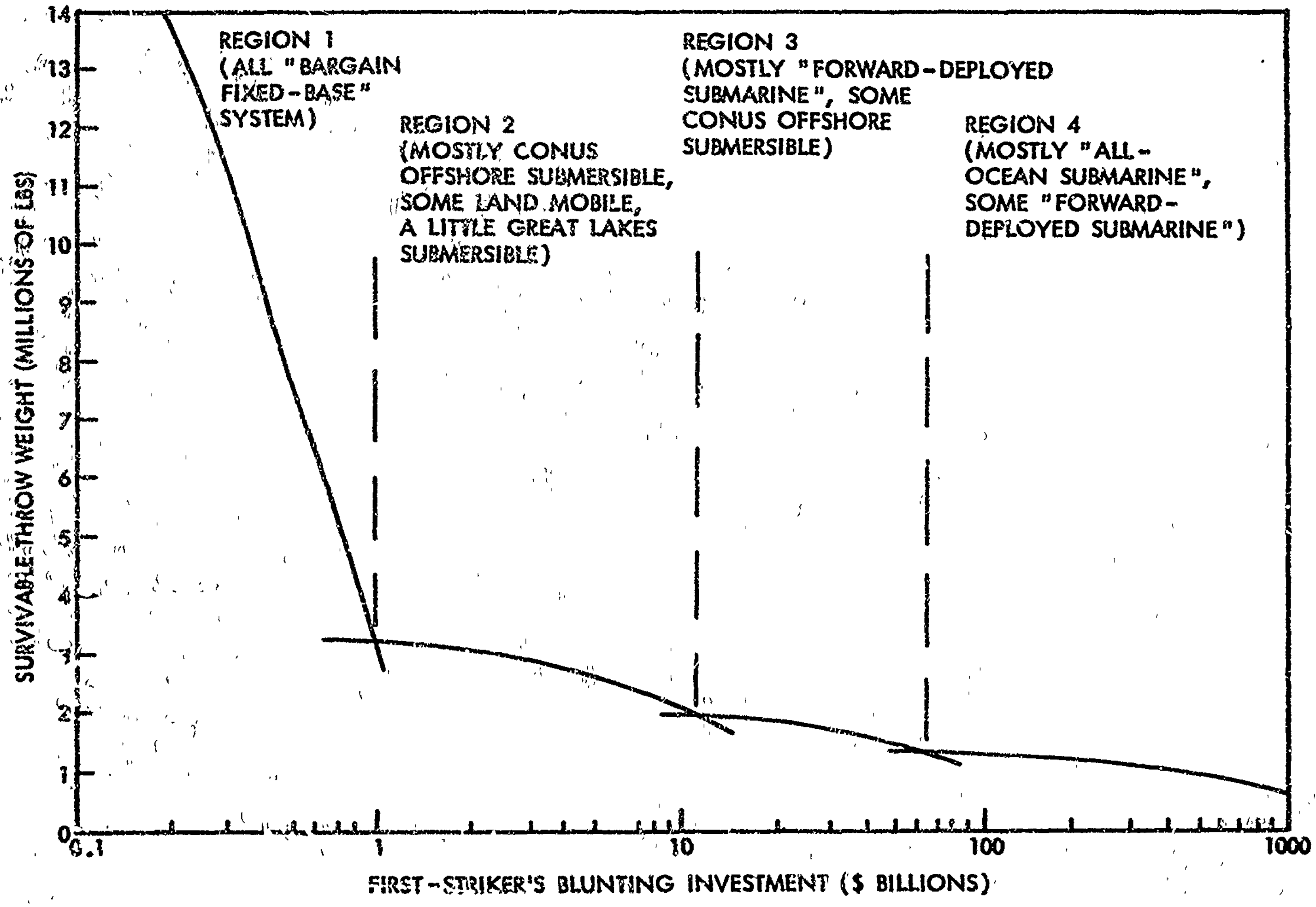

FIG. 6.1 SAMPLE CALCULATIONS SHOWING SURVIVABLE THROW WEIGHT FOR OPTIMUM STRATEGIC MIXES VS. LEVEL OF BLUNTING INVESTMENT, AS DETERMINED BY MAX-MIN MODEL. SECOND-STRIKER'S INVESTMENT FIXED AT $20B. MIX COMPOSITIONS SPECIFIED IN TABLE 6.3.

therefore be parameterized. All answers will be treated as functions of C_2. For the parameter values listed in Table 6.2., the max-min model gives the results indicated in Figure 6.1. and Table 6.3. Throughout all present calculations it is assumed that the second-striker commits a fixed total investment, C_1 = \$20 billion. This, too, could be parameterized in a more extensive study.

The data are observed to fall into four separate regions determined by values of the parameter C_2, as shown in Figure 6.1. In each region a given optimum mix composition prevails. If the first-striker's blunting expenditure is less than about \$1 B (Region 1), NV System #2, the cheap fixed-base system, should be procured exclusively. For blunting in the 1-11 billion dollar range (Region 2) NV systems, being too vulnerable to the assumed attack, drop out entirely in favor of PV systems, primarily #3, the CONUS offshore system, and a little of the land-mobile system, with a token procurement of a few missiles for the Great Lakes. In a tougher environment represented by 11-60 billion dollar blunting a small amount of the CONUS offshore system survives in the optimum mix, but the major investment should be in the POLARIS-type system, or one deployed in an operating area of comparable size. Finally, in the extremely adverse environment of over 60 billion-dollar blunting, excellent residual survivable capability of the order of 1 million lb. throw weight up to and beyond a C_2-value equal to the GNP of any single nation is offered by the "all-ocean submarine" system.

The slow decrease of survivable force over three decades of a logarithmic scale in Figure 6.1. is testimony to the inherent "toughness" of PV systems in general and seabased systems in particular. To be sure, specific ASW countermeasures have not been considered in the present calculations, but this is not due to oversight. It reflects the fact that in the absence of some now unforeseen ASW breakthrough the type of ballistic missile attack considered here appears by far the surest and most cost/effective form of surprise blunting of submersible systems.

One system is conspicuous by its absence from Figure 6.1. and Table 6.3: the "super-hardened fixed-base" system (NV #1). This candidate is the only one considered here that fails to make the grade in any circumstances. At low levels of opposition it is beaten out by its cheaper, less-hardened counterpart, NV #2; while at high levels of blunting it is beaten out by the PV systems. From this indicator, hardening would appear to be at a dead end.

The region boundaries and optimum mix compositions shown in Figure 6.1. and Table 6.1. will be altered when more realistic parameter values are used, but it is believed that otherwise many of the qualitative features of the present hypothetical example may be preserved. The model can be improved upon, as will be indicated in the next section, but even in its present crude form it offers some guide to intuition. Its shortcomings have to be compensated by judgment, as is true of all reductions of experience to numbers. If the need for such compensation is clearly understood, it would appear that the max-min type of model can be of some assistance to decision-makers concerned with strategic systems choice.

7. PARAMETER SENSITIVITY

The sensitivity of results to parameter variations could most profitably be explored in the context of more realistic parameter values than those considered here. Therefore, we confine ourselves to indicating the type of results that might emerge from a more meaningful study.

Figure 7.1 shows the result of picking a particular system (PV #3, the CONUS offshore submersible) and varying its cost and vulnerability parameters. It will be observed that, for the case of a blunting investment of C_2=\$20B, neither (a) reducing system 10-year cost from \$24 B to \$17 B, nor (b) doubling the system operating area, has much effect on the capabilities of the mix. But if both improvements are made together the expected survivable throw weight increases by a substantial 45%. This is typical of the non-obvious results that can emerge

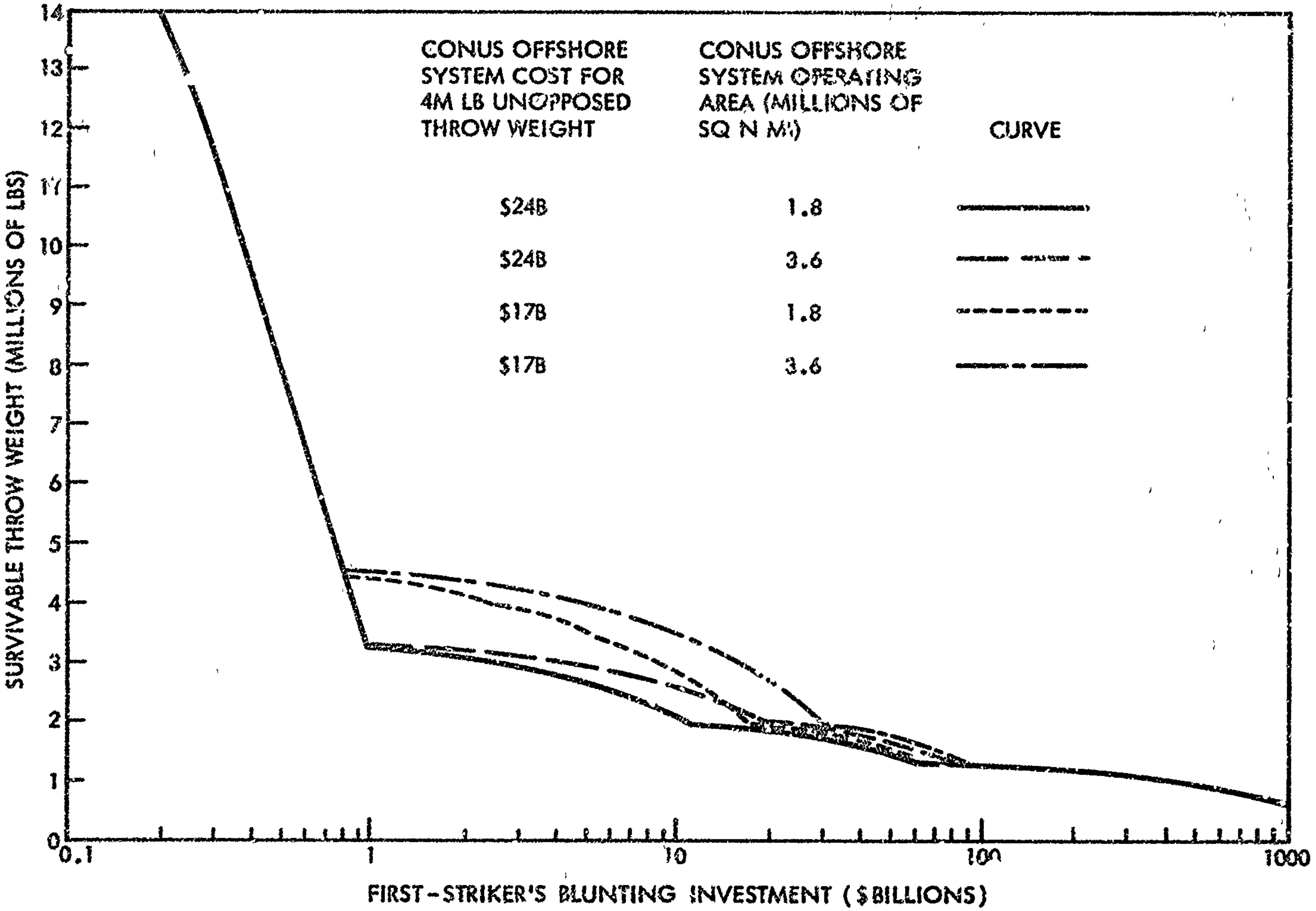

FIG. 7.1 SAMPLE CALCULATIONS SHOWING EFFECT OF VARYING THE COST AND VULNERABILITY PARAMETERS OF THE CONUS OFFSHORE SUBMERSIBLE SYSTEM, PV#3. OTHER CONDITIONS, SYSTEMS, AND PARAMETERS AS BEFORE (TABLE 6.2)

from detailed quantitative study. Tables 7.1, 7.2, 7.3 list the optimum allocations associated with the dotted curves of Figure 7.1.

Figure 7.2 shows the effect of changing the mix composition by introducing another FV system candidate, a clandestine surface "Q-" ship that is assumed indistinguishable from 9000 other ships to be found at sea on the average at any given time. The assumption is made that the disguise of these ships is so effective that no alternative remains to the would-be blunter but to attack all 9000 ships, or what fraction he can, at an assumed cost of \$3 M per ship incapacitated. (The resulting value of A_1 from Eq. (4) is 0.037.) It is further assumed that the surface ship system is rather cheap, at \$16 B 10-year cost for 4 M lb. unopposed throw weight. As always, we fix the second-striker's investment at \$20 B. Other systems and parameters are as in Table 6.2, except that the CONUS offshore submersible system is priced at \$17 B for 4 M lb. unopposed throw weight and is assumed deployed in a 3.6 M sq. n. mi. operating area, implying patrols to about 600 n. mi. offshore.

Comparison with the corresponding dashed curve of Figure 7.1 shows that inclusion of an extra system in the mix in most cases produces a worthwhile improvement. For instance, for the case of first-striker's blunting investment C_2 = \$20 B, a 30% increase in expected survivable throw weight from 2.7 M lb. to 3.5 M lb., results. The optimum mix composition for the case corresponding to Figure 7.2 is shown in Table 7.4.

It is of some interest to inquire what the max-min model implies about the optimum allocation of the first-striker's resources to blunting the various systems that compose an optimum retaliatory mix. The answer depends, of course, on both C_1 and C_2, as well as the nature of the mix in question. To take a specific example, suppose Goliath, with C_2 = \$100 B to spend on blunting, wishes to attack our hero, who has only C_1 = \$20 B to spend. Table 7.5 shows the optimum resource allocations on both sides. Most of the second-striker's resources (63%) should be spent on the

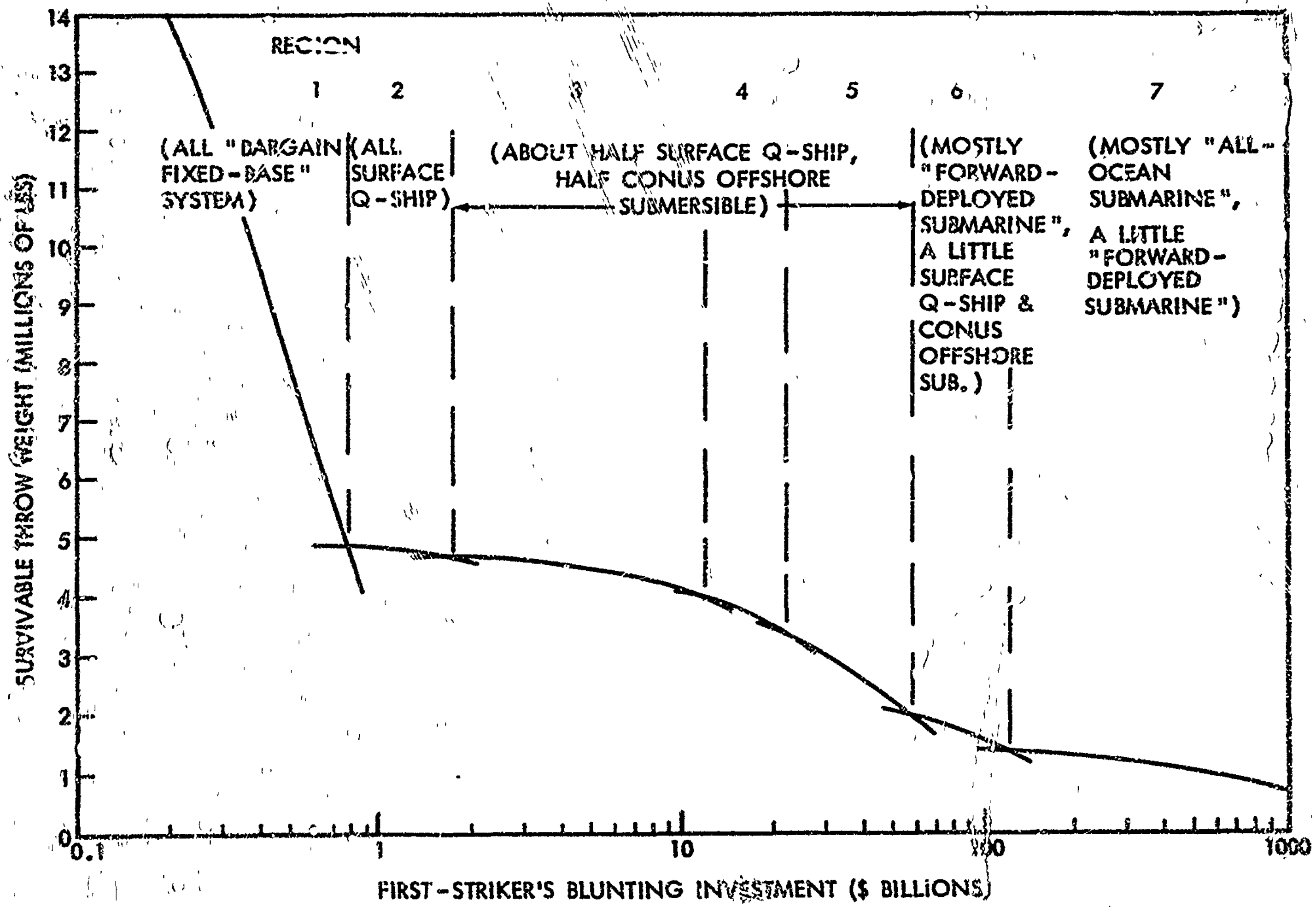

FIG. 7.2. SAMPLE CALCULATIONS SHOWING EFFECT OF ADDING A NEW CANDIDATE SYSTEM, A SURFACE Q-SHIP AT $16B FOR 4M LB. UNOPPOSED THROW WEIGHT, HIDDEN AMONG 9000 FALSE TARGETS. ALL OTHER SYSTEMS AND PARAMETERS AS BEFORE, WITH CONUS OFFSHORE SUBMERSIBLE SYSTEM @ $17B, DEPLOYED IN 3.6M SQ. N. MI.

Table 7.1. Effect of parameter variation: CONUS offshore submersible deployed in 3.6×10^6 sq. n. mi., system cost assumed \$24 B for 4 M lb. unopposed throw weight. Second-striker's max-min optimum allocation policies shown for expenditure of $C_1 = \$20$ B to maximize survivable throw weights against opponents of various degrees of "toughness". Figures in billions.

System	Range of Blunting Investment C_2 by First-Striker.			
	Region 1 $C_2 < \$1$ B	Region 2 $\$1 \text{ B} < C_2 < \$20$ B	Region 3 $\$20 \text{ B} < C_2 < \$90$	Region 4 $C_2 > \$90$ B
NV #1 ("Super-hardened fixed-base")	---	---	---	---
NV #2 ("Bargain fixed-base")	20	---	---	---
PV #1 ("Land-mobile soft")	---	1.5	*	*
PV #2 ("Great Lakes submersible")	---	*	*	*
PV #3 ("CONUS offshore submersible")	---	10.0	4.5	0.5
PV #4 ("Forward-deployed submarine")	---	---	15.0	1.8
PV #5 ("All-ocean submarine")	---	---	---	17.6

*Less than \$0.5 B.

Table 7.2. Effect of parameter variations: CONUS offshore submersible deployed in 1.8 M sq. n. mi., system cost assumed \$17B for 4 M lb. unopposed throw weight. Second-striker's max-min optimum allocation policies shown for expenditure of C_1=\$20B to maximize survivable throw weight against opponents of various degrees of "toughness". Figures in billions.

System	Range of Blunting Total Investment C_2 by First-Striker.					
	Region 1 $C_2 < \$0.8B$	Region 2 $\$0.8B < C_2 < \$2.5B$	Region 3 $\$2.5B < C_2 < \$6B$	Region 4 $\$6B < C_2 < \$17B$	Region 5 $\$17B < C_2 < \$70B$	Region 6 $C_2 > \$70B$
NV #1 ("Super-hardened fixed-base")	---	---	---	---	---	---
NV #2 ("Bargain fixed-base")	20	---	---	---	---	---
PV #1 ("Land-mobile soft")	---	---	2.9	2.7	*	*
PV #2 ("Great Lakes submersible")	---	---	---	0.8	*	*
PV #3 ("CONUS off-shore submersible")	---	20	17.1	16.4	2.5	*
PV #4 ("Forward-deployed submarine")	---	---	---	---	16.9	1.8
PV #5 ("All-ocean submarine")	---	---	---	---	---	17.9

*Less than \$0.5 B.

Table 7.3. Effect of parameter variation: CONUS offshore submersible operating area assumed 3.6×10^6 sq. n. mi., system cost assumed \$17B for 4 M lb. unopposed throw weight. Second-striker's max-min optimum allocation policies shown for expenditure of C_1=\$20B to maximize survivable throw weight against opponents of various degrees of "toughness". Figures in billions.

System	Range of Blunting Total Investment C_2 by First-Striker.					
	Region 1 $C_2 < \$0.8B$	Region 2 $\$0.8B < C_2 < \$5B$	Region 3 $\$5B < C_2 < \$10B$	Region 4 $\$10B < C_2 < \$30B$	Region 5 $\$30B < C_2 < \$90B$	Region 6 $C_2 > \$90B$
NV #1 ("Super-hardened fixed-base")	---	---	---	---	---	---
NV #2 ("Bargain fixed-base")	20	---	---	---	---	---
FV #1 ("Land-mobile soft")	---	---	1.5	1.5	*	*
FV #2 ("Great Lakes submersible")	---	---	---	0.5	*	*
FV #3 ("CONUS off-shore sub-mersible")	---	20	18.5	18.0	4.5	0.5
FV #4 ("Forward-deployed submarine")	---	---	---	---	15.0	1.8
FV #5 ("All-ocean submarine")	---	---	---	---	---	17.6

*Less than \$0.5 B.

Table 7.4. Effect of adding another PV system competitor; namely, a surface Q-ship system costing \$16B for 4M lb. unopposed throw weight, hidden among 9000 false targets** at sea. Other candidate systems include \$17B CONUS offshore submersible deployed in 3.6M sq. n. mi., others as before. Second striker's max-min optimum allocation policies shown for expenditure of C_1=\$20B to maximize survivable throw weights against opponents of various degrees of "toughness". Figures in billions.

System	Range of Blunting Investment C_2 by First-Striker. Region 1 $C_2 < \$0.8B$	Region 2 $\$0.8B < C_2 < \$2B$	Region 3 $\$2B < C_2 < \$12B$	Region 4 $\$12B < C_2 < \$22B$	Region 5 $\$22B < C_2 < \$58B$	Region 6 $\$58B < C_2 < \$120B$	Region 7 $C_2 > \$120B$
NV #1 ("Super-hardened fixed-base")	---	---	---	---	---	---	---
NV #2 ("Bargain fixed-base")	20	---	---	---	---	---	---
PV #1 ("Land-mobile, soft")	---	---	---	0.9	0.9	*	*
PV #2 ("Great Lakes submersible")	---	---	---	---	*	*	*
PV #3 ("CONUS offshore submersible")	---	---	11.0	10.6	10.4	3.8	0.5
PV #4 ("Forward-deployed submarine")	---	---	---	---	---	12.7	1.7
PV #5 ("All-ocean submarine")	---	---	---	---	---	---	17.3
PV #6 ("Surface Q-ship")	---	20	9.0	8.6	8.5	3.1	0.5

* Less than \$0.5B.

** It is assumed that the first-striker must attack all 9000 ship targets and that these can be killed for an average cost of \$3M each.

least-vulnerable system admitted to the mix, in accordance with the principle discussed in Section 5. Yet the remaining 37% of his investment draws 71% of the attacker's countermeasure investment. In effect, Goliath gives up on the toughest system in the mix and, being a proper bully, beats up on the weaker systems. For all his efforts, though, Figure 7.2 shows that an expected 1.56M lb. of retaliatory throw weight survives.

These results clarify the significance of the mix concept: systems admitted to the mix in even small amounts perform a vital function in "drawing fire" (or countermeasures resources) away from the principal system (s) in the mix. Thus mutual support is the key concept responsible for success of the optimum mix, as contrasted with any single "optimum" system.

Unopposed on first strike, the second-striker's mix in Table 7.5 could deliver 3.01 M lb. of ready throw weight, as shown by the fourth column. After $100 B-worth of optimally-applied countermeasures, 52% of this survives. This might suggest that a mix of this kind would constitute a sound deterrent to a rational opponent, since 48% attrition does not look like very good return on a $100 B investment. To quantify the advantage of a mix, Table 7.5 includes in the right-hand column information on the survivable throw weight if each of the mix systems were bought alone at the total investment level of $20 B, and if each were subject to the entire $100 B-worth of blunting. (Note that the numbers in the right-hand column are non-additive.) The best survival, of course, is shown by the "toughest" system in the mix, PV #4. But this survival amounts to only 0.81 M lb. of second-strike throw weight, as contrasted with 1.56M lb. for the mix. That is, the optimum mix shows 1.92 times better "survivability" than the best single component system in it.

It is of interest to observe in Table 7.5 the ratio of blunting dollars to procurement dollars for optimum allocations on both sides. These ratios exceed the over-all spending ratio of 5:1 for each of the lesser-procured PV systems.

Table 7.5. Optimum allocations for the same mix considered in Figure 7.2 and Table 7.4, given that the first-striker spends C_2=\$100B on blunting, while the second-striker spends C_1=\$20B on retaliatory forces.

System	Second-striker's Optimum Procurement Allocation ($B)	First-striker's Optimum Blunting Allocation ($B)	Ratio: Blunting Dollars per Procurement Dollar	Throw Weight Deliverable by this System on Unopposed First Strike (M lb.)	Surviving Throw Weight of this System after Optimum Blunting of the Mix (M lb.) (% Survival)	Throw Weight that Would Survive if this System Alone were Bought for $20B with $100B Blunting (M lb.) (% Survival)
PV #4 ("Forward-deployed submarine")	12.7	29.0	2.3:1	1.270	0.98 (77%)	0.81 (43%)
PV #3 ("CONUS offshore submersible")	3.8	36.5	9.6:1	0.893	0.30 (34%)	0.24 (5%)
PV #6 ("Surface Q-ship")	3.1	31.2	10.0:1	0.775	0.25 (32%)	0.12 (2.5%)
PV #1 ("Land-mobile, soft")	0.3	2.6	8.7:1	0.060	0.024 (39%)	Negligible
PV #2 ("Great Lakes submersible")	0.1	0.7	7.0:1	0.017	0.008 (43%)	Negligible
Totals	$20B	$100B	Over-all :5:1	3.01	1.56	

This suggests that even rather slightly-procured or minor systems components of a mix often more than "pull their weight" in terms of the countermeasure expenditures they force on the attacker. Even the tiny Great Lakes system, worth only $0.1B of procurement money (which could buy only a few units) would appear to pay for itself on this basis. Where a mix philosophy is being applied, it would seem, the enemy seldom gets a free ride -- a few units oppose him in even the unlikeliest places. But the main dictum of optimum PV system procurement policy remains: don't over-procure any single component system beyond what its invulnerability warrants.

Table 7.5 illustrates in its last two columns another characteristic feature of a PV systems mix: the surviving percentage of the system in each case is greater if the system is bought as part of a mix than if it is bought alone. This means that any given PV unit is that much safer in the mix. By sharing the countermeasures burden, all component systems profit in survivability.

Finally, in view of the dominant influence of costs on much of the thinking about systems choice, an investigation was made of the benefits that might ensue from really substantial cost reductions. While all other parameters were held the same as for Figure 7.2, the surface Q-ship cost for 4M lb. unopposed throw weight was arbitrarily reduced to $9B, and the CONUS offshore submersible system cost to $10B. Since these two competitors thus remain on about an equal footing, they continue to share the honors of main procurement over a wide range of blunting investments. Cost reduction produces a notable quantitative increase in survivable striking power over most of the range. At $20B blunting, 6M lb. of second-strike throw weight survives, given our usual $20B investment by the second-striker. Thus in purely quantitative terms cost-reduction pays off handsomely. This is shown in Figure 7.3 and Table 7.6. However, the rather steep rate of fall of the survivability curve, Figure 7.3, between blunting investments of $10B and

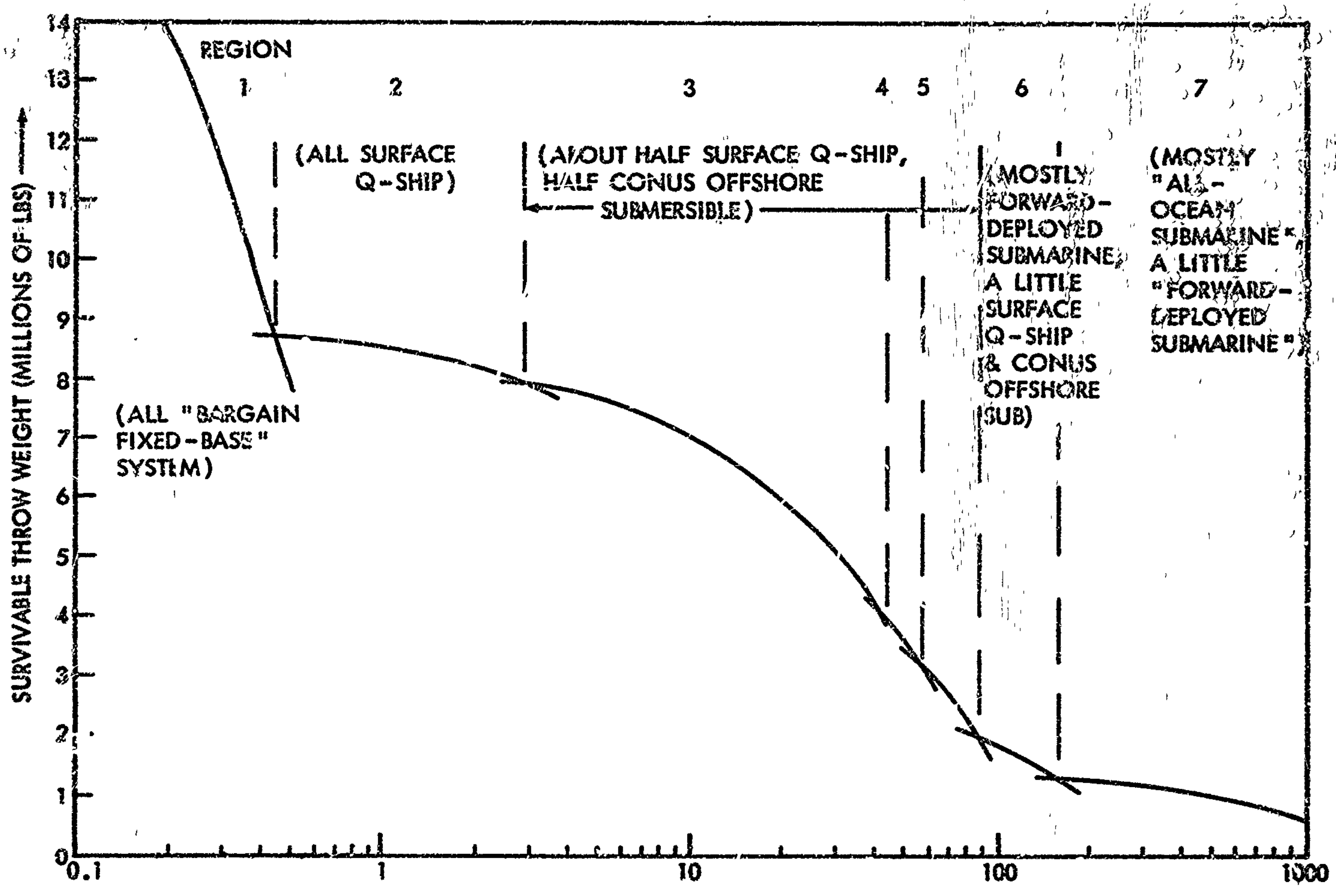

FIG. 7.3 SAMPLE CALCULATIONS SHOWING EFFECT OF COST-CUTTING, SURFACE Q-SHIP AT $9B FOR 4M LB. UNOPPOSED THROW WEIGHT, CONUS OFFSHORE SUBMERSIBLE SYSTEM AT $10B, DEPLOYED IN 3.6M SQ. N. MI. ALL OTHER SYSTEMS AND PARAMETERS AS IN FIGURE 7.2.

$100B suggests a disquieting feature of the cost-reduction approach: the steeper the rate of fall of this curve, the more reward the first-striker receives for additional blunting investments, and the more he is presumably tempted to engage in an arms race. Those who believe in "spending our opponents to death" as a profitable U. S. national policy will approve of steep survivability curves, while those who dislike a nuclear arms race will disapprove. The max-min model is neutral on such subjects, because it is concerned merely with maximum survivability of blunting, not with deeper questions of deterrent stability. Having thus encountered a weakness of the max-min model, we might appropriately turn now to a more general consideration of its limitations.

8. LIMITATIONS OF THE MODEL

The following are some of the shortcomings of the max-min model as a description of reality or tool for decision-making.

(1) It is a static, expected-value model only, based on average weight of retaliation as sole figure of merit. Confidence levels are readily calculated from a knowledge of expected values and of the relevant probability law (Poisson), but they are not a feature of the model itself.

(2) For this reason and the fact that it leaves time entirely out of consideration, the model has little <u>directly</u> to do with deterrence. Many subtle but important considerations are ignored by it. For example, the ability to react non-suicidally to strategic warning might generate a requirement for an overtly and rapidly <u>mobilizable</u> force component in the mix. If so, the model is unaware of the fact. Similarly a system's ability to survive more than 30 minutes and to exhibit staying power for a long war against a dug-in opponent might be useful, but earns no bonus from the model.

(3) For similar reasons the model, as noted in Section 7, gives no extra credit for arms-race reduction to those systems least susceptible to blunting

Table 7.6. Effect of substantial cost reductions, pricing surface Q-ship** at $9B for 4M lb. unopposed throw weight and CONUS offshore submersible system at $10B in 2.6M sq. n. mi. deployment area. Other parameters as before. Second-striker's max-min optimum allocation policies shown for expenditure of C_1=$20B to maximize survivable throw weights against opponents of various degrees of "toughness". Figures in billions.

System	Range of Blunting Investment C_2 by First-Striker.						
	Region 1 C_2 < $0.5B	Region 2 $0.5B < C_2 < $3B	Region 3 $3B < C_2 < $40B	Region 4 $40B < C_2 < $60B	Region 5 $60B < C_2 < $90B	Region 6 $90B < C_2 < $160B	Region 7 C_2 > $160B
NV #1 ("Super-hardened fixed-base")	---	---	---	---	---	---	---
NV #2 ("Bargain fixed-base")	20	---	---	---	---	---	---
PV #1 ("Land-mobile, soft")	---	---	---	0.9	0.9	*	*
PV #2 ("Great Lakes submersible")	---	---	---	---	*	*	*
PV #3 ("CONUS offshore submersible")	---	---	11.0	10.6	10.4	3.8	0.5
PV #4 ("Forward-deployed submarine")	---	---	---	---	---	12.7	1.7
PV #5 ("All-ocean submarine")	---	---	---	---	---	---	17.3
PV #6 ("Surface Q-ship")	---	20	9.0	8.6	8.4	3.1	0.5

*Less than $0.5B.

**It is assumed as in Table 7.4 that the Q-ships are hidden among 9000 ship targets, all of which must be attacked indiscriminately and killed at a cost of $3M each.

countermeasures, no credit for drawing fire from the homeland (which all seabased systems under a ballistic missile attack of the type described in Section 4 would do), etc.

(4) The model gives no consideration to protection against technical "breakthroughs" in countermeasures against a single system — the reason most commonly advanced in favor of a "mix".

(5) R & D costs are neglected or subsumed in system procurement costs. This is a significant weakness, since it casts legitimate doubt on the desirability of procuring any of those systems that the model solution says should be procured in very small amounts. Where the optimum-solution procurement cost gets "lost in the R & D noise", it is probable that procurement should be made only in special circumstances - e.g., if the system can "ride piggyback" on some other system scheduled for substantial procurement. (For example, a Great Lakes system, if such were a realistic candidate might use the same hardware as a CONUS offshore submersible system, thus eliminating separate R & D costs).

(6) At-target (ABM) countermeasures are ignored or subsumed in effectiveness/cost parameters. Only blunting countermeasures are considered directly. Thus the first-striker's problem of optimum allocation between the two types of countermeasures is passed over entirely in favor of a sub-optimization.

(7) The model, as presently constituted, fails to match exactly the vulnerability characteristics of such systems as may be approximately "linearly vulnerable", in the terminology of Section 2. The approximation of treating such systems as percentage-vulnerable breaks down at heavy levels of attack. This deficiency would require changes in the mathematical formulation. This is not considered worthwhile at present because "linear vulnerability" is itself an idealization that ignores realistic factors, such as weapon unreliability, which act to force the actual course of warfare back toward the Poisson random process assumed in the model.

(8) The assumption that the PV candidate systems are "diversified" (in the sense that no two of them are simultaneously vulnerable to the same countermeasure) is not always realistic. An example of simultaneous vulnerability would be that incurred by a forward-deployed submarine system if its units transited through the operating area of a CONUS offshore submersible system.

(9) In general the omission of all judgmental factors, a characteristic of any mathematical model, implies that max-min theory can never provide more than a partial, qualitative guide to decision making.

The sub-optimization mentioned in (6), above, appears at first glance to be the major weakness of the model. However, there is a question of philosophy involved. There are two ways an enemy can allocate his resources for minimizing the effective survivability of our retaliatory missiles. One of these ways (blunting) can be very damaging to us, the other (ABM) does us no direct damage. To split off the blunting part, as our model does, and look only at that most-damaging-to-us part is obviously not looking at the whole problem. On the other hand, to look at the whole problem on the assumption that our opponent follows the optimum course of self-preservation for himself is hardly a conservative view for us to take. Our enemy might elect instead to follow the most damaging course for us. And it is this we must watch out for. That is, the course of self-preserving behavior for us does not necessarily consist in the assumption of self-preserving behavior by our enemies.

To give a concrete example: suppose we assumed self-preserving behavior by our enemy, and suppose the effectiveness of ABM's were such that his optimum allocation policy between blunting and ABM's, on the criterion of minimizing our number of deliverable warheads, would be to buy all ABM's and no blunting. Suppose our theorizing were based on the assumption that our opponent did just that. Then, since we assume that he buys no blunting forces, we have nothing to worry about in the area of blunting survivability, and are free to buy the most vulnerable systems. If we acted on this assumption, though, it could prove to be wrong. Our enemy might be less interested in self-preservation than we thought and more interested in damaging us. In short, we might have misjudged his values. In this case our decision to buy vulnerable systems would actually encourage degenerative tendencies in the value system of our opponent, by rewarding blunting efforts with a more visible payoff.

For this reason, it is not at all incorrect in principle to look separately at the most-damaging-to-us part of the problem, as the present model does. When the two parts of the problem, blunting and ABM, are linked together by an assumption of "optimum" behavior on the part of the opponent, the linkage is always weak and questionable, because the optimizing of behavior must be defined relative to human values. Values may or may not be known for one's own side, but can never conservatively be assumed known for an enemy. This circumstance provides a pitfall for attempts to build all-encompassing gamelike models of strategic behavior. Superoptimizing can be more fallacious than suboptimizing.

9. CONCLUSIONS

(1) Despite numerous minor shortcomings the max-min model discussed herein provides a useful frame of reference in which to place the selection of strategic systems for credible survivability.

(2) Until realistic effectiveness/cost and vulnerability parameter values are available on a comparable basis for all candidate retaliatory systems it is premature to discuss specific conclusions. The following must therefore be considered subject to revision in the light of experience.

(3) Against realistic levels of opposition the so-called numerically-vulnerable (fixed-base) systems appear non-competitive with percentage-vulnerable (mobile and/or concealed) systems.

(4) The latter are in general best procured as components of a "mix." Cost features importantly in determining the admissibility of a system to the optimum

mix; but once admission is granted the optimum procurement allocation to each system is determined solely by its degree of invulnerability.

(5) Invulnerability depends secondarily on target hardness, primarily on area of mobility/concealment or number of decoy/false-targets.

(6) For this reason seabased systems compete well not only for admission to an optimum mix but for a major share of procurement allocations within the mix.

APPENDIX A

REVIEW OF THE MATHEMATICAL SOLUTION

A-1. Preliminary Remark

The mathematics of max-min theory is presented thoroughly by J. M. Danskin in his book, "The Theory of Max-Min," Springer, Berlin, 1967. Even the non-mathematician interested in weapons choice, allocation and related problems will profit from examining the original text. However, as with other classics such as von Neumann's books, the original is apt to be more honored than read. Hence we devote this section to giving a sort of layman's birds-eye-view of those aspects of the theory that relate to the problem in hand, treated in Chapter V of the book. In so doing we shall minimize plagiarism by doing deliberate violence to the niceties of real-variable theory.

A-2. The Gibbs Lemma

The theory begins with what Danskin calls the "Gibbs Lemma," characterized by him as "the fundamental lemma of mathematical operations research." It has obvious analogues in calculus of variations and Lagrange multiplier theory.

Gibbs Lemma. If the set $(x_1^o, x_2^o, \ldots, x_n^o)$ maximizes $\Sigma_i f_i(x_i)$ for differentiable functions f_i, subject to side conditions $\Sigma_i x_i$ = constant, $x_i \geq 0$, $i = 1, 2, \ldots, n$, then there exists a λ such that

$$f_i'(x_i^\circ) = \lambda \text{ if } x_i^\circ > 0$$
$$\leq \lambda \text{ if } x_i^\circ = 0 .$$

The word "maximizes", above, may be replaced by "minimizes," if $f_i'(x_i^\circ)$ is replaced by $-f'(x_i^0)$.

Proof. Suppose $x_i^0 > 0$. Let $x_i^0 > \epsilon \geq 0$ and put

$$F(\epsilon) = f_i(x_i^0 - \epsilon) + f_j(x_j^0 + \epsilon) + \sum_{k \neq i, j} f_k(x_k^0)$$

for some $j \neq i$. The set of x-arguments thus modified still satisfies the side conditions $\Sigma_i x_i$ = constant, $x_i \geq 0$. F is a differentiable function of ϵ. Since $F(\epsilon)$ is by the hypothesis of the lemma a maximum at $\epsilon = 0$, the slope of $F(\epsilon)$ there is either zero or negative, $F'(0) \leq 0$. Performing the differentiation, we have

$$f_i'(x_i^0) \geq f_j'(x_j^0) .$$

The only hypothesis used in reaching this conclusion was $x_i^0 > 0$. If now $x_j^0 > 0$ as well, the reverse of the above inequality holds, and equality may be inferred. It follows that all $f_i'(x_i^0)$ with $x_i^0 > 0$ have a common value, which may be taken equal to λ. Those x_j^0 for which the above inequality cannot be reversed (namely, those that vanish) are of course those for which it holds,

$$f_i'(x_i^0) = \lambda \geq f_j'(x_j^0) \text{ if } x_j^0 = 0 .$$

The sign reversal of $f_i'(x_i^0)$ takes care of the case in which "minimum" replaces "maximum," with $F'(0) \geq 0$.

The lemma cannot be used in practice until it is known that the problem does indeed possess a maximizing (or minimizing) set of indpependent variables x_i. However, this can usually be determined by common sense from the nature of the problem or proven from considerations of concavity and convexity of the functions $f_i(x_i)$. Once its applicability is established, the lemma permits simultaneous determination of which x_i^o are non-vanishing and of what the values of these favored $x_i^o > 0$ are in the optimum (maximizing or minimizing) case. In simple problems the x_i^o can often be determined by solving for them explicitly in the equations

$$f_i'(x_i^o) = \lambda .$$

This yields the x_i^o as functions of λ. The numerical value of λ, as in more general Lagrange multiplier problems, must be evaluated from the side condition,

$$\Sigma_i \; x_i^o(\lambda) = \text{constant} .$$

Since one needs to know λ in order to know which x_i^o are non-vanishing, and vice versa, there are subtleties lurking in the procedure just sketched, but in most practical problems with some help from a computer one can get quite rapidly to the desired answer.

Danskin points out that this lemma is useful not only in operations research but also in economics, where it represents the marginal utility principle. Its continuous analogue has been applied by Koopman in search theory. The breadth of its applicability derives in part from the fact that the functions f_i need not depend on the x_i as their only arguments, but may involve other independent variables y_i, z_i, etc., with respect to which reapplications of the lemma may be possible. Thus simultaneous maximizings with respect to some of the variables and minimizings with respect to others become feasible. This circumstance gives its name to "max-min theory" and reveals it as being in a sense the extended study of the implications of the Gibbs lemma. About the connection with game theory, Danskin has this to say:

> "Max-Min theory is not a part of game theory in the usually understood sense. If in a two-step problem Max-Min = Min-Max, the pure-strategy solution of game theory is the solution to the Max-Min problem. But if Max-Min < Min-Max, there is no pure-strategy solution. In this case game theory moves on to mixed strategies. These have no meaning for us. The first player cannot hide his move, and the second obviously need not mix."

There is, of course, one sticky point about the max-min theory in applications to strategic allocation problems: the first player, to act intelligently, needs to know the total resources the second player will commit, though not the detailed allocation of those resources. Thus one of the elements that is supposed to be "known" (the total-resource constant in the Gibbs lemma) is in fact not known.

In general the first player must parameterize this quantity and study it as a variable in reaching what must finally be a judgmental rather than a deterministic choice of moves.

A-3. Strategic rataliation model

The mathematical problem, in slightly different notation from that of the tert, is to solve

$$\max_{x_i,\, x_j} \; \min_{y_i,\, y_j} \; F(x, y), \tag{1}$$

where

$$F(x, y) = \sum_i v_i x_i e^{-a_i y_i} + \sum_j u_j x_j e^{-b_j y_j / x_j}, \tag{2}$$

subject to the resource constraint equations,

$$\sum_i x_i + \sum_i x_j = C_1$$
$$\sum_i y_i + \sum_i x_j = C_2 \tag{3}$$

$$x_i \geq 0,\; y_i \geq 0,\; x_j \geq 0,\; y_j \geq 0 \quad \text{for all } i, j. \tag{4}$$

We now proceed to sketch a method of solution. The first term in (2) represents the surviving retaliatory strength for any numerically vulnerable system(s) the second-striker (opponent #1, the x-player) may procure. The reader can rather easily convince himself by elementary considerations that if this term alone were present the best strategy for the x-player would be to buy always a single system, viz., the one for which

$$u_j \exp\left[-\left(\frac{b_j}{C_1}\right) C_2\right] \tag{A}$$

is a maximum, where C_1 represents the total resources available to the second-striker and C_2 represents the corresponding sum available to the first-striker for blunting. In this way the effective vulnerability parameter, (b_j/C_1), is minimized, and this turns out to be the dominant consideration. Thus the second-striker should buy no more than one NV system.

We attempt no proof of this assertion, but refer the interested reader to Danskin's book. (The result should be rather obvious, since when x_j is made small, through an attempt to buy more than one NV system, the second term in (2) becomes small in two ways--first because of the factor x_j in front of the exponential and secondly because of the x_j in the denominator of the negative exponent--hence the x-player can only lose by subdividing his NV system investment.) The virtue of Danskin's rather deep methods is that they permit him to conclude that these results remain valid also in the case in which both terms are present in Eq. (2). That is, when the spectrum of choice includes both percentage-vulnerable and numerically vulnerable systems, the second-striker should still avoid buying more than a single "optimum" NV system. This is plausible but not obvious. The same NV system that is the "winner" in the competition among NV systems alone need not win out over all other NV systems when both NV and PV types are available candidates. This follows from the fact that when only portions, x_{NV}, y_{NV}, of the resources of each opponent are allocated to the NV system the criterion quantity

$$u_j x_{NV} e^{-\left(\frac{bj}{x_{NV}}\right) y_{NV}}$$

may no longer be greatest for the same system that maximizes this quantity when $x_{NV} = C_1$, $y_{NV} = C_2$. In practice, however, this point proves to be academic, since in most realistic cases, *if* any NV system is to be bought at all, that system turns out to be the clear winner over all competitors, both NV and PV, and thus the criterion (A) applies. This of course happens only when C_2 is sufficiently small.

Applying the Gibbs lemma to (2) for minimization with respect to y_i, and omitting superscript zeros denoting optimization, we have

$$\begin{aligned} -\frac{\partial F}{\partial y_i} = v_i a_i x_i e^{-a_i y_i} &= \mu \text{ if } y_i > 0 \\ &= \mu \text{ if } y_i = 0 \,. \end{aligned} \tag{5}$$

Similarly for y_j provided $x_j > 0$,

$$\begin{aligned} -\frac{\partial F}{\partial y_i} = u_j b_j e^{-b_j y_j / x_j} &= \mu \text{ if } x_j > 0,\ y_j > 0 \\ &\leqq \mu \text{ if } x_j > 0,\ y_j = 0 \,. \end{aligned} \tag{6}$$

Note that the same μ appears in (5) and (6), since simultaneous minimizing with respect to both the y_i and y_j is being performed. Of course if $x_j = 0$ the left-hand side of (6) vanishes (or the corresponding term is absent in (2)), so

$$0 = \mu \text{ if } x_j = 0,\ y_j > 0$$
$$0 = \mu \text{ if } x_j = 0,\ y_j = 0\ . \qquad (7)$$

From (3) some x_i or x_j is positive; hence from (5) or (6) $\mu > 0$. From (7) it follows that $x_j = 0$ implies $y_j = 0$, and from (5) that $x_i = 0$ implies $y_i = 0$. The solution may therefore be expressed as

$$y_i = \frac{1}{a_i} \log \frac{v_i a_i x_i}{\mu}, \text{if } v_i a_i x_i > \mu,$$
$$= 0 \text{ otherwise}, \qquad (8)$$

$$y_j = \frac{x_j}{b_j} \log \frac{u_j b_j}{\mu}, \text{if } u_j b_j > \mu,$$
$$= 0 \text{ otherwise}\ . \qquad (9)$$

By the result mentioned above, x_j and y_j are non-vanishing for at most a single j-value. We may therefore introduce new variables $S = x_j/C_1$, $D = y_j/C_2$, to represent the fraction of each opponent's resources that he devotes to the optimum NV system or to its blunting. Thus (9) becomes

$$D = \frac{C_1 S}{C_2 b_j} \log \frac{u_j b_j}{\mu}, \text{if } u_j b_j > \mu,$$
$$= 0 \text{ otherwise}\ . \qquad (10)$$

We shall treat S as an independent variable in what follows, and study the result of letting it range over $0 \leq S \leq 1$. Our payoff function (2) becomes

$$\sum_i v_i x_i\, e^{-a_i y_i} + v_j C_1 S\, e^{\frac{-b_j C_2 D}{C_1 S}} \qquad (11)$$

Denoting the Max of this quantity with respect to x_i-variation as $H(S)$, we now turn our attention to the study of $H(S)$, treating S initially as a fixed parameter.

Applying the Gibbs lemma to (11), this time for maximization with rezpect to the x_i, we have

$$\begin{aligned} \frac{\partial F}{\partial x_i} = v_i\, e^{-a_i y_i} &= \lambda \text{ if } x_i > 0 \\ &= \lambda \text{ if } x_i = 0 \; . \end{aligned} \qquad (12)$$

With the help of (5) this may be re-written as

$$\begin{aligned} \frac{\mu}{a_i x_i} &= \lambda \text{ if } v_i a_i x_i > \mu \\ v_i &= \lambda \text{ if } 0 < v_i a_i x_i \leqq \mu \\ v_i &\leqq \lambda \text{ if } x_i = 0 \; . \end{aligned} \qquad (13)$$

The results (8), (10), (13), along with the side conditions,

$$\begin{aligned} \sum_i x_i &= (1 - S)\, C_1 \, , \\ \sum_i y_i &= (1 - D)\, C_2 \, , \end{aligned} \qquad (14)$$

essentially comprise the solution of the problem. It remains to work it out more explicitly.

By (13) it is clear that whethar system i should be bought or not depends on the magnitude of its effectiveness/cost parameter v_i relative to some criterion level λ. Those systems having $v_i > \lambda$ will be "in", those with $v_i < \lambda$ will be "out". (The case $v_i = \lambda$ requires special treatment, but may be ignored for the moment.) It is therefore convenient to arrange the system numbering so that

$$v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_n \quad , \tag{15}$$

where n is the number of candidate PV systems. For simplicity we shall here suppose that degeneracies can be removed, so that strict inequalities hold in (15). This permits (14) to be written more explicitly as

$$\sum_{i=1}^{n_s} x_i = (1 - s)\ C_1 \quad , \tag{16}$$

where n_s is an implicit function of $\lambda = \lambda_s$ and s, such that n_s is the greatest integer satisfying

$$\lambda_s \leq v_{n_s} \quad . \tag{17}$$

By some manipulation of the foregoing equations we obtain

$$\begin{aligned} &\text{For } i \leq n_s\ (\lambda_s < v_i),\ x_i = \frac{\mu}{a_i \lambda_s} \\ &\qquad\qquad y_i = \frac{1}{a_i} \log \frac{v_i}{\lambda_s} \\ &\text{For } i = n_s, \text{ given } v_{n_s} = \lambda_s:\ x_i \leq \frac{\mu}{a_i \lambda_s} \\ &\qquad\qquad y_i = 0 \\ &\text{For } i > n_s:\ x_i = y_i = 0 \quad . \end{aligned} \tag{18}$$

For the moment, we shall ignore the case $v_{n_s} = \lambda_s$. Then from (16)

$$\sum_{i=1}^{n_s} \frac{\mu}{\lambda_s a_i} = (1 - s)\ C_1 \quad . \tag{19}$$

Similarly from (14) and (18)

$$\sum_{i=1}^{n_s} \frac{1}{a_i} \log \frac{v_i}{s} = (1 - D)\, C_2 \quad . \tag{20}$$

Let

$$w_s = \sum_{i=1}^{n_s} \frac{1}{a_i} \quad . \tag{21}$$

From (19),

$$\mu = \frac{\lambda_s (1 - S)\, C_1}{w_s} \tag{22}$$

From (10) and (22),

$$D = \frac{C_1 S}{C_2 b_j} \log \frac{u_j b_j w_s}{\lambda_s (1 - S)\, C_1} \quad \text{if} \quad u_j b_j > \frac{\lambda_s (1 - S)\, C_1}{w_s} \quad . \tag{23}$$

From (20) and (23),

$$\sum_{i=1}^{n_s} \frac{1}{a_i} \log v_i - w_s \log \lambda_s = C_2 - \frac{C_1 S}{b_j} \log \frac{u_j b_j w_s}{(1 - S)\, C_1} + \frac{C_1 S}{b_j} \log \lambda_s \quad ,$$

or

$$\lambda_s = \exp \left\{ \frac{\left(\sum_{i=1}^{n_s} \frac{1}{a_i} \log v_i \right) - C_2 + \frac{C_1 S}{b_j} \log \frac{u_j b_j w_s}{(1 - S)\, C_1}}{\frac{C_1 S}{b_j} + w_s} \right\} . \tag{24}$$

Eqs. (17) and (24) may be regarded as two simultaneous conditions for the evaluation of λ_s and n_s. The λ_s, n_s values thus obtained may be used, with the help of (16), (20), and (22), to evaluate the optimum allocations for the two opponents,

$$x_i = \frac{C_1 (1 - S)}{a_i \sum_{i=1}^{n_s} \frac{1}{a_i}}, \quad i = 1, 2, \ldots, n_s$$

$$y_i = \frac{1}{a_i} \log \frac{v_i}{\lambda} \tag{25}$$

$$D = 1 - \frac{1}{C_2} \sum_{i=1}^{n_s} \frac{1}{a_i} \log \frac{v_i}{\lambda} . \tag{26}$$

These relations are valid without modification through most of the range of S-values. Near the ends of the S-range and in certain intermediate regions special considerations apply. In Eq. (10) the condition D = 0 if $u_j b_j \leqq \mu$ translates into

$$u_j b_j \leqq \frac{\lambda_s (1 - S) C_1}{w_s} ,$$

or (27)

$$S \leqq 1 - \frac{u_j b_j w_s}{C_1 \lambda_s} .$$

By (23), D must vanish at S = 0. If the right-hand side of (27) is negative at S = 0, the inequality cannot be satisfied and consequently D cannot vanish for S > 0. Danskin shows that both D and λ_s are non-decreasing functions of S. If the right-hand side of (27), evaluated at S = 0, is positive, namely,

$$G = 1 - \frac{u_j b_j w_o}{C_1 \lambda_o} > 0,$$

where (28)

$$w_o = \sum_{i=1}^{n_o} \frac{1}{a_i} ,$$

and, from (24) for $S = 0$,

$$\lambda_o = \exp\left\{\frac{\left(\sum_{i=1}^{n_o} \frac{1}{a_i} \log v_i\right) - C_2}{w_o}\right\} \tag{29}$$

n_o being the greatest integer satisfying

$$\lambda_o \leqq v_{n_o} \tag{30}$$

Danskin shows that D vanishes on the closed interval $0 \leq S \leq G$. On this interval, by (26), λ retains the constant value λ_o and n_s the constant value n_o. More generally, wherever D is constant, λ_s and n_s are also constant and equal to their values at the left-hand (minimum-S) end of the constant-D interval.

With the help of (5) and (19), the payoff function H (S) of Eq. (11) may be re-written as

$$H(S) = \lambda_s (1 - S) C_1 + v_j C_1 S e^{-\frac{b_j C_2 D}{C_1 S}} . \tag{31}$$

Consequently on the initial S-interval where D vanishes (if any), we have

$$H(S) = \lambda_o (1 - S) C_1 + v_j C_1 S \;. \tag{32}$$

Thus in the interval $0 \leq S \leq S_1 = G$ the payoff is just a linear function of S, which has its maximum at either $S = 0$ or $S = S_1$. In practice G generally turns out to be negative, so the "interval" $0 \leq S \leq S_1$ is of zero length.

At the right-hand $(S \longrightarrow 1)$ end of the interval $0 \leq S \leq 1$ similar conditions prevail. Obviously $D = y_j/C_2$ can be no greater than 1, yet by formula (23) the logarithm to which

D is proportional becomes infinite as $S \to 1$. Beyond a critical value $S = S_2$ (which always exists in the open interval $0 < S_2 < 1$) D becomes equal to the total resources of the first-striker, C_2, and remains equal to this value for $S_2 \leq S \leq 1$. From (23), S_2 is given by

$$1 = \frac{C_1 S_2}{C_2 b_j} \log \frac{u_j b_j w_{S_2}}{\lambda_{S_2} (1 - S_2) C_1} \quad . \qquad (33)$$

To evaluate this we need to know something about the behavior of λ_s. As S increases λ_s rises from $\lambda_o > 0$, given by (29). This rise is steady, in accordance with (24), as long as the strict inequality in (17) is satisfied. At each of the v_i values, however, $i = 1, 2, \ldots n_o$, where λ_s becomes equal to v_i, Danskin shows that there is a finite S-interval over which λ_s must remain constant, equal to v_i, before beginning to rise again. The indication that this occurs is that λ_s, straightforwardly calculated from (24), <u>drops</u> in value with increasing S. This happens because one comes to a point at which it becomes natural, in conformity with (17), to drop a term out of the sum w_s. Such a drop in w_s causes λ_s to drop discontinuously. (To verify this possibility, consider C_2 in (24) very large.) Danskin has shown that this must not be allowed to happen. Eq. (29) is temporarily invalidated, and λ_s, n_s, w_s, and D all hold their values constant over a limited S-interval. As S increases further, λ_s, as calculated from (24) for the decreased n_s-value, rises again to the v_i level. At this point (24) is reinstated. It remains valid for increasing S until λ_s rises to the next (larger) v_i - value, at which point the above considerations reapply.

The optimum solution, to repeat, requires that whenever λ_s, calculated by (24), seeks to decrease for increasing S, which occurs at successive values $\lambda_s = v_i$, Eq. (24) must be set aside and both λ_s and w_s must maintain the constant values they had at the left-hand end of the S-interval thus encountered. Proceeding from left to right in this fashion through decreasing subscripts i in v_i, we come finally to the region where λ_s, obedient to (17), has increased through all v_i values except the greatest, v_1. Our point $S = S_2$ is precisely the point at which $\lambda_s = \lambda_{s_2} = v_1$ for the first time, for S increasing from zero. To the right of $S = S_2$ on a plot of λ_s vs. S, i. e., for $S_2 \leq S \leq 1$, λ_s must hold its constant value of v_1. By (17), S can never exceed v_1, the greatest of the v's. Since all but the most cost-effective of the PV systems (system #1) have dropped out by the time S reaches S_2, the sum in Eq. (21) is reduced to a single term, $w_{s_1} = 1/a_1$, and this w-value holds throughout $S_2 \leq S \leq 1$. Using these values of λ_{s_2} and w_{s_2} in (33), we have

$$(1 - S_2)e^{\frac{C_2 b_1}{C_1 S_2}} = \frac{u_j b_j}{C_1 a_1 v_1} \qquad (34)$$

as a transcendental equation whose unique root between 0 and 1 determines S_2. For $S_2 \leq S \leq 1$ the payoff function from (31) has the value

$$H(S) = C_1\, v_1 (1 - S) + C_1\, u_j\, S\, e^{-\frac{b_j\, C_2}{S\, C_1}} , \qquad (35)$$

inasmuch as $D = 1$, $\lambda_s = v_1$ on this interval.

In the intermediate region, $S_1 \leq S \leq S_2$, the payoff function may with the help of (31) and (26) be written as

$$H(S) = \lambda_s (1 - S) C_1 \qquad (36)$$

$$+ u_j\, C_1\, S \exp\left\{\frac{b_j}{C_1\, S}\left(-C_2 + \sum_{i=1}^{n_s} \frac{1}{a_i} \log \frac{v_i}{\lambda_s}\right)\right\} .$$

In calculations using this formula, as previously explained, it is essential to instruct the computer to use a non-decreasing sequence of λ_s-values as S increases.

We now have all information needed for plotting H(S) vs. S. Inspection of this curve will show where the maximum payoff H_{max} lies. In most practical problems it lies at one end of the range or the other; that is, $S = 0$ or $S = 1$. All agonizing about the nature of the intermediate curve goes for naught in 99% of the realistic cases.

Having done the work of plotting H(S) vs. S for one j-value (one NV system arbitrarily selected), we must by brute strength repeat the chore for each other j-value and compare all results to find the highest attainable H-value. No theorem is available to eliminate a priori all but a single NV contender. That regrettable fact poses no practical problem in this day of high-speed computers, unless the number of NV contenders is very large--an unlikely case.

A-4. Summary of Results

The mathematical solution may proceed as follows:

(1) An arbitrary ζ-value (NV system) is selected from among the NV candidates. It is treated as the only NV system in the competition.

(2) The PV systems are numbered in rank order of effectiveness/cost according to Eq. (15).

(3) The parameters λ_o and n_o are evaluated by simultaneous solution of Eqs. (29) and (30).

(4) The quantity G in (28) is calculated. If $G \leq 0$, the quantity S_1 is put equal to zero. If $G > 0$, then $S_1 = G$.

(5) A quantity S_2 is evaluated by numerical solution of the transcendental equation (34).

(6) The quantity H(0) is evaluated (from Eq. (32)) as $H(0) = C_1\lambda_o$. The quantity $H(S_1)$ is evaluated from (32). On a plot of H(S) vs. S the points H(0) and $H(S_1)$ are connected by a straight line for $0 \leq S \leq S_1$. (Eq. (32) is the equation of this line.) The interval $0 \leq S \leq S_1$ will be termed "Region One."

(7) Divide "Region Two," $S_1 < S < S_2$, into a number of small sub-intervals. At each interval, working from left to right (small to large S), calculate λ_s and n_s from Eqs. (24) and (17). Ensure that λ_s is a non-decreasing function by maintaining λ_s and n_s constant at the values they had at the left-hand end point of any interval in which the calculated value of λ_s decreases below any previously-

calculated value. With λ_s thus determined, calculate $H(S)$ from (36) and plot against S in Region Two.

(8) In "Region Three," $S_2 \leq S \leq 1$, use (35) to complete the plot of H(S) vs. S. If plotting is not desired, have the computer record the maximum value H_M encountered in the above calculations, and the value of $S = S_M$ at which this maximum occurs. Otherwise determine H_M and S_M by inspection of the curves or data.

(9) Repeat the above for each of the other j-values. Evaluate H_M, S_M for each j and determine which j-value yields the maximum H_M. This determines the NV system that should be bought. If the highest value of H_M occurs for $S_M = 0$, a commonly encountered case, no NV system should be bought.

(10) With the proper j-value (if any) thus determined, and the maximum H_M, S_M known, evaluate the optimum mix composition as follows:

(a) If $0 \leq S_M \leq S_1$, i. e., the maximum lies in Region One, (usually $S_M = 0$) then

$$x_i = \frac{C_1 (1 - S_M)}{a_i \sum_{i=1}^{n_o} \frac{i}{a_i}} \qquad \text{for } i = 1, 2, \cdots, n_o \tag{37}$$

$$y_i = \frac{1}{a_i} \log \frac{v_i}{\lambda_o}$$

$$x_i = y_i = 0 \qquad \text{for } i > n_o.$$

$$r = yj/C_2 = 0.$$

The vanishing of D implies that in this region, even if $S_M > 0$ (S_M being the optimum fraction of the second-striker's resources devoted to procurement of the NV system), the first striker should not attempt to blunt the NV system.

(b) If $S_1 < S_M < S_2$, the maximum lies in Region Two. Let λ_M and n_M represent the values of λ_S and n_S at $S = S_M$. Then

$$\left.\begin{aligned} x_i &= \frac{C_1 (1 - S_M)}{a_i \sum_{i=1}^{n_M} \frac{1}{a_i}} \\ y_i &= \frac{1}{a_i} \log \frac{v_i}{\lambda_M} \end{aligned}\right\} \quad \text{for } i = 1,2,\ldots,n_M \qquad (38)$$

$$x_i = y_i = 0 \text{ for } i > n_M.$$

$$D = 1 - \frac{1}{C_2} \sum_{i=1}^{n_M} \frac{1}{a_i} \log \frac{v_i}{\lambda_M}$$

$$= \frac{C_1 S_M}{C_2 b_j} \log \left[\frac{u_j b_j \sum_{i=1}^{n_M} \frac{1}{a_i}}{\lambda_M (1-S_M) C_1} \right].$$

These two alternative expressions for D, obtained from (23) and (26), provide a useful check on the computer program.

(c) If $S_2 \leq S_M \leq 1$, the maximum lies in Region Three. Actually, because of concavity of function, Eq. (35), H_M cannot be at an interior point of Region Three, but (as in Region One and in the other intervals of constant D and λ_S) must lie on the boundary. If it lies at $S_M = 1$, then

$x_i = y_i = 0$ for all PV systems, and only the optimum-J NV system should be bought. If it lies at $S_M = S_1$, then, as previously discussed, only a single PV system, #1, should be bought, and

$$x_1 = C_1 (1 - S_M)$$
$$y_1 = 0 \qquad (39)$$
$$x_i = y_i = 0 \text{ for } i > 1.$$

In either case D = 1, so the first striker concentrates all his blunting on the NV system, ignoring the PV system, if any.

This completes our description of a straightforward and uninspired way of handling the problem. For further inquiry into the mathematics, treated in both more rigorous and elegant fashion, and for alternative expressions of many of quantities that may serve as cross-checks, Danskin's book must be consulted. For practical numerical results, based on the solution described above, a computer tape programmed in BASIC is available from NOL on request. It provides adequate computational accuracy for practical purposes.

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13. ABSTRACT

An application of the mathematical theory of max-min developed by John M. Danskin has been made to the problem of optimum systems choice for strategic retaliation. One opponent seeks to maximize his strike-second capability while the other seeks to minimize it by pre-emptive blunting. Both are subject to total-resource limitations and both are fully cognizant of each other's systems options. The present report (a) reviews the mathematical model and its solution; (b) discusses the problem of parameter evaluation; (c) describes preliminary results obtained from sample calculations with semi-realistic parameter values; (d) undertakes the beginnings of a parametric sensitivity analysis; and (e) examines limitations of the model. (U)

14 KEY WORDS	LINK A		LINK B		LINK C	
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Strategic Retaliation						
Game Theory						

Naval Ordnance Laboratory, White Oak, Md.
(NOL technical report 67-59)
OPTIMUM SYSTEMS CHOICE FOR STRATEGIC RETALIATION: AN APPLICATION OF MAX-MIN THEORY, by T. E. Phipps, Jr. 20 April 1967. v.p. charts, tables. NOSC task MAT 03L 000/F099 01 01.

UNCLASSIFIED

An application of the mathematical theory of max-min developed by John M. Danskin has been made to the problem of optimum systems choice for strategic retaliation. This report (a) reviews the mathematical model and its solution; (b) discusses the problem of parameter evaluation; (c) describes preliminary results obtained from sample calculations with semi-realistic parameter values; (d) undertakes the beginnings of a parametric sensitivity analysis; and (e) examines limitations of the model.

1. Games - Theory
2. Warfare - Analysis
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