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Ilan Adler, et al

California University

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by

Ilan Adler, Robert Levin and Robert M. Oliver
Operations Research Center
University of California, Berkeley

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ABSTRACT

Several budget planning models are presented that exploit the longitudinal stability of manpower cohorts. The budgetary planning process is described along with the problem of identifying and obtaining various types of longitudinal data. An infinite horizon linear program for calculating minimum cost manpower input plans is found to have a straightforward solution under the assumption of "nearly monotone" survivor fractions.

LONGITUDINAL BUDGET PLANNING MODELS

by

Ilan Adler, Robert Levin and Robert M. Oliver

I. INTRODUCTION

In an earlier report by Grinold, Marshall and Oliver [1973] the authors formulate a longitudinal manpower planning model in which manpower requirements and survivor fractions are known and given in future time periods. Most of these models and the associated data files have been implemented for interactive use and real-time simulation. See the later report by Grinold and Oliver [1973].

In this paper we formulate a longitudinal planning model to study budget rather than manpower needs. The models in this paper examine the current budget allocations due to the composition of existing manpower levels, the survivor fractions that will determine future behavior of new manpower and budget inputs and the budgetary restrictions in future time periods. See also the forthcoming book by Grinold and Marshall [1974].

A particular application of these budget planning models for a system consisting of many manpower skill categories is the enlisted force in the U.S. Navy. Interactive computer models of these budget formulations have also been developed to aid decision-makers who wish to test the effects of alternative policies on staffing requirements and future manpower budgets. These interactive budget planning models have a variety of uses:

(1) to predict the budgets that will be required by the current stock of manpower or continuation of existing budgets.

(2) to calculate unfulfilled requirements and the new budgets necessary to meet them.

(3) to identify bottlenecks in the budget planning process.

(4) to assist in preparation of future manpower budgets, to simulate the effects of policy changes on future budgets.

(5) to relate alternate personnel retention and performance assumptions to the need for future budget inputs, and

(6) to calculate maximum effectiveness manpower schedules when upper bounds on future budgets are given.

Individuals in any budget category can be identified by characteristics such as: rank, salary, number of years of experience in the skill category, length of service, and personal attributes such as age and measures of performance. The models presented in this paper are designed to assist in preparing manpower budgets and meeting aggregate strength requirements.

Section II of the paper describes the underlying budget models. In the application to budgets for manpower requirements the models are based on the assumption of longitudinal stability in the service lifetimes of different manpower cohorts. We show that the accession schedule that exactly meets future budgets is found by solving a set of lower triangular system of linear equations. Section III relates several methods of describing the longitudinal behavior of manpower budgets and shows how the flows can be estimated from existing data. We present an infinite horizon linear program for the calculation of future accessions subject to upper bound restrictions on discretionary budgets. We derive readily verifiable conditions on the inputs to the infinite horizon problem that guarantee that the equality solution, described in Section II, will be feasible and optimal. In cases where the equality solution is not optimal we obtain a simple nonlinear recursion which is optimal under the assumptions of "nearly monotone" budget survivor fractions.

The models presented in this paper examine the relationships between three factors: (i) the current budget allocations, (ii) the survivor fractions that determine the longitudinal behavior of future budgets and categories and (iii) the manpower requirements for future times. The size of our models, the type of calculations performed and the availability of interactive programs allow policy makers to quickly analyze the impact of various assumptions and policies.

II. THE UNDERLYING BUDGET MODELS

General Formulation

We consider an organization which is divided into many budget categories, and where a flow of funds over time may be associated with a particular budget category or group of categories. We borrow heavily from the notation of the earlier paper [1973], with the caution to the reader that, in the present paper, most decision variables have the dimensions of budgets rather than manpower.

We idealize the evolution of a budget program by analyzing its changes at discrete points in time $(i = \ldots -2,-1,0,1,2, \ldots)$. We say that period i is the interval between times $(i - 1)$ and i ; it is a future period if $i > 1$, a past period if $i \leq 0$ and the current period if $i = 1$. In period i a budget of size $\bar{x}_i$ is added to the category; that budget is called the *new program budget* for period i and $\bar{x}_i$ is its *size*. Let $\bar{\alpha}_{ij}$ be the fraction of the new budget in period i which is still funded in period $i + j$ $(j \geq 0)$. Let $\bar{z}_k$ be the total budget in the category at time k and let $(m + 1)$ be the maximum number of periods for which a program may be funded. Thus $\bar{\alpha}_{ij} = 0$ if $j > m$. For some future time k we have

$$(1) \qquad \bar{z}_k = \bar{x}_k\bar{\alpha}_{k,0} + \bar{x}_{k-1}\bar{\alpha}_{k-1,1} + \ldots + \bar{x}_{k-m}\bar{\alpha}_{k-m,m} .$$

Equation (1) simply says that the budget at time k is made up of costs surviving from programs initiated in earlier periods. Thus it is natural to call the $\bar{\alpha}_{ij}$'s the *cost survivor fractions* for the program which enters at time i .

At time 0 , the history of past budget programs is given by the vector $(\bar{x}_{-m},\bar{x}_{1-m}, \ldots \bar{x}_{-2},\bar{x}_{-1},\bar{x}_0)$. The current inventory of program budgets is given by $\bar{y}_1 = \bar{x}_0\bar{\alpha}_{0,1} + \bar{x}_{-1}\bar{\alpha}_{-1,2} + \ldots + \bar{x}_{1-m}\bar{\alpha}_{1-m,m}$. The quantity $\bar{y}_1$, which includes budgets supporting all programs begun during the past m periods, is called the current *cost legacy*. In a future period k the cost legacy $\bar{y}_k$ from past programs up to and including period 0 will be

$$\bar{y}_k = \begin{cases} \bar{x}_0\bar{\alpha}_{0,k} + \bar{x}_{-1}\bar{\alpha}_{-1,k+1} + \dots + \bar{x}_{k-m}\bar{\alpha}_{k-m,m} & \text{if } k \le m \\ 0 & \text{if } k > m . \end{cases} \tag{2}$$

Discretionary Budgets and Stationarity

Suppose we have a planning horizon of T periods with total budgets $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_T$. From Equations (1) and (2) we see that future program budgets must satisfy

$$\begin{aligned} &\bar{\alpha}_{1,0}\bar{x}_1 &&= \bar{z}_1 - \bar{y}_1 \\ &\bar{\alpha}_{1,1}\bar{x}_1 + \bar{\alpha}_{2,0}\bar{x}_2 &&= \bar{z}_2 - \bar{y}_2 \\ &\dots \\ &\bar{\alpha}_{1,T-1}\bar{x}_1 + \bar{\alpha}_{2,T-2}\bar{x}_2 + \dots + \bar{\alpha}_{T,0}\bar{x}_T &&= \bar{z}_T - \bar{y}_T \end{aligned} \tag{3}$$

Here we have assumed

A1: The *discretionary budgets* $\bar{z}_k - \bar{y}_k$ for periods $k = 1,2, \dots, T$ are met exactly.

Under A1 it is quite possible that for a given set of $\bar{z}_k$'s , $\bar{y}_k$'s and $\bar{x}_{ij}$'s some $\bar{x}_k$ could be negative. Such a result would say that in order to exactly meet total budgets in all periods $1,2, \dots, T$ it will be necessary to reduce program budgets in some period.

We concentrate on the equality solution (assumption A1) for several reasons. First, it is misleading to state the problem as if the new program budgets $(\bar{x}_k)$ are the only variables which the decision-maker can influence. The cost legacies $(\bar{y}_k)$, the total budgets $(\bar{z}_k)$, and to some extent the cost survivor fractions $(\bar{\alpha}_{ij})$ can all be changed or explicitly influenced. Second, plans that are eventually recommended will probably conform to the equality constraints since budget restrictions do not generally allow for slack in the system. Third, we intend to use the models in this section to test the effects of alternate policies on several objectives: (1) the departure of realistic budgets from ideal budgets, (2) the impact of policy changes on new program budgets, and (3) the costs associated with moving budgets from one category to another. In a later section we shall drop A1 and treat the discretionary budgets as upper bounds and look for an "optimal" schedule of new program budgets.

In the remainder of the paper we make an important second assumption.

A2: The cost survivor fractions $\bar{\alpha}_{i,j}$ are stationary from period to period. That is, $\bar{\alpha}_{i,j} = \bar{\alpha}_j$ independent of i and independent of $\bar{x}_i$. Under assumption A2 Equation (3) simplifies to

$$
(4) \qquad \begin{aligned}
&\bar{\alpha}_0\bar{x}_1 &= \bar{z}_1 - \bar{y}_1 \\
&\bar{\alpha}_1\bar{x}_1 + \bar{\alpha}_0\bar{x}_2 &= \bar{z}_2 - \bar{y}_2 \\
&\quad\cdots \\
&\bar{\alpha}_{T-1}\bar{x}_1 + \bar{\alpha}_{T-2}\bar{x}_2 + \ldots + \bar{\alpha}_0\bar{x}_T &= \bar{z}_T - \bar{y}_T
\end{aligned}
$$

The cost legacies are given by

$$
(5) \qquad \begin{aligned}
\bar{y}_1 &= \bar{\alpha}_1\bar{x}_0 + \bar{\alpha}_2\bar{x}_{-1} + \ldots + \bar{\alpha}_m\bar{x}_{1-m} \\
\bar{y}_2 &= \bar{\alpha}_2\bar{x}_0 + \bar{\alpha}_3\bar{x}_{-1} + \ldots + \bar{\alpha}_m\bar{x}_{2-m} \\
&\quad\cdots \\
\bar{y}_T &= \bar{\alpha}_T\bar{x}_0 + \ldots \qquad\qquad + \bar{\alpha}_m\bar{x}_{T-m}
\end{aligned}
$$

Equation (4) can be used in a number of ways. We have mentioned already that, given the total budgets, cost legacies, and cost survivor fractions, (4) can be used to calculate new program budgets for each period of the planning horizon T. Alternatively, given planned budget inputs over the next T periods the $\bar{z}_i$'s can be considered as the result of these inputs. Also, given total budgets and planned inputs, the cost legacies which satisfy Equation (4) can be determined.

Budgetary Planning in Manpower Systems

Our manpower model is presented in Grinold, Marshall and Oliver [1973]. Briefly, x_j is the size of the manpower cohort entering in period j , α_j is the j-period manpower survivor fraction, and c_j is the cost of supporting an individual in his j^{th} period of service.

In budgetary terms, the cost of cohort i in its first period is $\bar{x}_i = c_0 x_i$, and j periods later the cost is $\bar{\alpha}_j \bar{x}_i = c_j \alpha_j x_i$ so that in the notation of the earlier paper we identify $\bar{\alpha}_j = \frac{c_j \alpha_j}{c_0}$. With this equivalence, the interpretation of $\bar{x}_i$ is that of the initial cost of the new manpower cohort in period i . Then $\bar{\alpha}_j \bar{x}_i = \frac{c_j \alpha_j}{c_0} = \frac{c_j}{c_0} \alpha_j \bar{x}_i + 0(1 - \alpha_j)\bar{x}_i$ is the "expected" cost, including the possibility of growth in costs due to such things as promotions, salary increases, inflation, etc., j periods hence. Thus, a manpower accession x_i has an initial cost $\bar{x}_i = c_0 x_i$ and a cost j periods later equal to $\bar{\alpha}_j \bar{x}_i$

$$\bar{x}_i \bar{\alpha}_j = (c_0 x_i)\left(\frac{c_j \alpha_j}{c_0}\right) = x_i \alpha_j c_j \ . \tag{6}$$

The budget conservation Equations (4) become

$$\begin{aligned}
&c_0 \alpha_0 x_1 &&= \bar{z}_1 - \bar{y}_1 \\
&c_1 \alpha_1 x_1 + c_0 \alpha_0 x_2 &&= \bar{z}_2 - \bar{y}_2 \\
&\cdots \\
&c_{T-1} \alpha_{T-1} x_1 + \cdots + c_0 \alpha_0 x_T &&= \bar{z}_T - \bar{y}_T
\end{aligned} \tag{7}$$

and the cost legacy Equations (5) become

$$\bar{y}_k = \begin{cases} c_k \alpha_k x_0 + c_{k+1} \alpha_{k+1} x_{-1} + \dots + c_m \alpha_m x_{k-m} & k \le m \\ 0 & k > m \end{cases} \quad . \tag{8}$$

III. MAXIMIZING MANPOWER LEVELS

The manpower accession schedule that meets future discretionary budgets $\bar{z}_t - \bar{y}_t$ exactly is found by solving the equations

$$
\begin{aligned}
&\bar{\alpha}_0 x_1 && = \bar{z}_1 - \bar{y}_1 \\
&\bar{\alpha}_1 x_1 + \bar{\alpha}_0 x_2 && = \bar{z}_2 - \bar{y}_2 \\
&\quad\vdots \qquad \vdots && \quad \vdots \\
(1) \quad &\bar{\alpha}_{m-1} x_1 + \bar{\alpha}_{m-2} x_2 + \dots + \bar{\alpha}_0 x_m && = \bar{z}_m - \bar{y}_m \\
&\bar{\alpha}_m x_1 + \bar{\alpha}_{m-1} x_2 + \dots + \bar{\alpha}_1 x_m + \bar{\alpha}_0 x_{m+1} && = \bar{z}_{m+1} \\
&\qquad \bar{\alpha}_m x_2 + \dots + \bar{\alpha}_2 x_m + \bar{\alpha}_1 x_{m+1} + \bar{\alpha}_0 x_{m+2} && = \bar{z}_{m+2} \\
&\qquad\qquad \vdots \qquad \vdots \qquad \vdots && \quad \vdots
\end{aligned}
$$

where it is understood that $\bar{\alpha}_j = \alpha_j c_j$.

In this section we shall relax the assumption that future budget requirements are satisfied exactly. Instead we shall treat the variables $\bar{z}_t$ as upper bounds on the total budget level at time t . Moreover, we shall restrict the manpower accessions x_t to be nonnegative. Thus the equalities in (1) will be replaced by inequalities $(\leq)$. This leaves us with an infinite system of linear inequalities that will, in general, have a large number of possible solutions. To obtain a single accession schedule in this case we will specify a performance criterion and then select the accession schedule that optimizes that criterion.

In the analysis that follows we assume that the performance criterion is to maximize the present worth of the size of all future manpower levels. This objective is obtained by discounting future levels to today's levels and then summing over all future periods. We show that there are many conditions of practical interest when the equality solution discussed in earlier sections is indeed an optimal solution to the infinite-horizon program.

The Infinite Horizon Program

There are several approaches to the solution of (1) when the restriction of exactly meeting the budget requirements is relaxed. In describing a manpower maximization model in which the budget requirements are considered to be upper bounds the problem is to choose the nonnegative vector $(x_1, x_2, \ldots)$ that satisfies

(2)
$$\begin{aligned} &\bar{\alpha}_0 x_1 && \le \bar{z}_1 - \bar{y}_1 \\ &\bar{\alpha}_1 x_1 + \bar{\alpha}_0 x_2 && \le \bar{z}_2 - \bar{y}_2 \qquad x_t \ge 0 \\ &\bar{\alpha}_2 x_1 + \bar{\alpha}_1 x_2 + \bar{\alpha}_0 x_3 && \le \bar{z}_3 - \bar{y}_3 \\ &\vdots \quad \vdots \quad \vdots && \vdots \end{aligned}$$

As we have seen in the earlier paper the stock levels of manpower in period i having j years of service is

(3)
$$n_{ij} = x_{i-j}\alpha_j$$

so that the total stock of people in period i is

(4)
$$\sum_{j=0}^{m} n_{ij} = \sum_{j=0}^{\min(m,i-1)} x_{i-j}\alpha_j + y_i \; .$$

Multiplying (4) by δ^{i-1} and summing over all periods yields an objective of discounted manpower effectiveness or strength equal to

(5)
$$\sum_{i=1}^{\infty} \delta^{i-1} \sum_{j=0}^{m} n_{ij} = \sum_{i=1}^{\infty} \delta^{i-1} y_i + \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \delta^{i-1} x_{i-j}\alpha_j \; .$$

This measure makes the contribution of an individual more valuable the earlier it is available. Since the legacies y_i are usually known and fixed we will concentrate on maximizing the right-hand term in (5) with respect to the accession levels $(x_1, x_2, \ldots)$.

If we define the *expected discounted lifetime* of an individual as

$$\tau = \sum_{j=0}^{m} \alpha_j \delta^j = \alpha_0 + \alpha_1\delta + \alpha_2\delta^2 + \ldots + \alpha_m\delta^m \qquad \delta < 1 \tag{6}$$

the right-hand term in (5) can be written as

$$\sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \delta^{i-1} x_{i-j}\alpha_j = \sum_{j=1}^{\infty} \tau x_j \delta^{j-1} = \tau \sum_{j=1}^{\infty} x_j \delta^{j-1} .$$

It is obvious that the value of τ will not affect the optimal solution. Thus, without loss of generality, we can study the problem of maximizing the simpler expression

$$\sum_{j=1}^{\infty} x_j \delta^{j-1} \tag{7}$$

subject to the inequalities of (2). The dual program is to find nonnegative variables $\bar{u}_1, \bar{u}_2, \ldots$ which

$$\text{Minimize} \quad \bar{u}_1(\bar{z}_1 - \bar{y}_1) + \bar{u}_2(\bar{z}_2 - \bar{y}_2) + \ldots \tag{8}$$

subject to the inequalities

$$\begin{aligned} \bar{u}_1\bar{\alpha}_0 + \bar{u}_2\bar{\alpha}_1 + \bar{u}_3\bar{\alpha}_2 + \ldots &\geq 1 \\ \bar{u}_2\bar{\alpha}_0 + \bar{u}_3\bar{\alpha}_1 + \ldots &\geq \delta \\ \bar{u}_3\bar{\alpha}_0 + \ldots &\geq \delta^2 \\ \vdots \qquad & \quad \vdots \end{aligned} \tag{9}$$

A feasible solution of the dual program is given by

$$\tilde{u}_t = \eta\delta^{t-1} \quad \text{with} \quad \eta = \left(\sum_{j=0}^{m} \bar{\alpha}_j \delta^j\right)^{-1} \tag{10}$$

This solution always exists, is strictly positive and always satisfies each dual constraint as a strict equality.

We may use $\tilde{u}_t$ to obtain an upper bound to the primal objective as follows: Let $(x_1, x_2, \ldots)$ be any solution to the inequalities (2). Then we may write

$$\eta^{-1} \sum_{t=1}^{\infty} x_t \delta^{t-1}$$

$$= \sum_{j=0}^{m} \bar{\alpha}_j \delta^j \sum_{t=1}^{\infty} x_t \delta^{t-1} = \sum_{t=1}^{\infty} (\bar{\alpha}_{t-1} x_1 + \ldots + \bar{\alpha}_0 x_t)\delta^{t-1}$$

$$\leq \sum_{t=1}^{\infty} (\bar{z}_t - \bar{y}_t)\delta^{t-1}, \quad \text{so that}$$

$$\sum_{t=1}^{\infty} x_t \delta^{t-1} \leq \eta \sum_{t=1}^{\infty} (\bar{z}_t - \bar{y}_t)\delta^{t-1} = \sum_{t=1}^{\infty} (\bar{z}_t - \bar{y}_t)\tilde{u}_t .$$

Note that the solution $(\tilde{x}_1, \tilde{x}_2, \ldots)$ to the Equations (1) attains this upper bound, i.e.

$$\sum \tilde{x}_t \delta^{t-1} = \eta \sum_{t=1}^{\infty} (\bar{z}_t - \bar{y}_t)\delta^{t-1} .$$

Thus if the equality solution $(\tilde{x}_1, \tilde{x}_2, \ldots)$ is feasible (nonnegative), it is optimal.

<u>Optimality of Equality Solution</u>

Intuition suggests the equality solution of (2) will often be an optimal solution of the infinite horizon planning problem posed in (5). Identification of the conditions which must exist for the equality solution to be nonnegative and thus optimal gives considerable insight into the structure of the problem. These conditions place bounds on the magnitude of allowable changes in future budgets. Moreover, optimality of the equality solution gives us an analytic expression for total system performance as a function of the continuation rates, discretionary budgets, and costs.

To begin the analysis of this section we look for solution of a reduced system of linear equations

$$
\begin{aligned}
&x_1 &&= 1 \\
&\bar{\beta}_1 x_1 + x_2 &&= \bar{\phi}_1 \\
(11)\qquad &\bar{\beta}_1\bar{\beta}_2 x_1 + \bar{\beta}_1 x_2 + x_3 &&= \bar{\phi}_1\bar{\phi}_2 \\
&\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 x_1 + \bar{\beta}_1\bar{\beta}_2 x_2 + \bar{\beta}_1 x_3 + x_4 &&= \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 \\
&\vdots &&\vdots
\end{aligned}
$$

which is equivalent to normalizing the equality system in (1) by the constant $\frac{\bar{z}_1 - \bar{y}_1}{\bar{\alpha}_0}$. In (11) $\bar{\phi}_j$ is defined to be the ratio of successive discretionary budgets, i.e. $\bar{z}_{j+1} - \bar{y}_{j+1}/\bar{z}_j - \bar{y}_j$. Alternatively, one can view (11) as the equality system in (2) with $\bar{\alpha}_0 = 1$, $\bar{\beta}_1\bar{\beta}_2 \ldots \bar{\beta}_j = \bar{\alpha}_j$, $\bar{z}_1 - \bar{y}_1 = 1$ and $(\bar{z}_{j+1} - \bar{y}_{j+1}) = \bar{\phi}_1\bar{\phi}_2 \ldots \bar{\phi}_j$. In either case, multiplication of a solution vector $x = (x_1, x_2, \ldots)$ of (11) by $(\bar{z}_1 - \bar{y}_1)/\bar{\alpha}_0$ yields the equality solution of (1). It should be noted by the reader that just as the cost continuation rates $\bar{\beta}_1, \bar{\beta}_2, \ldots \bar{\beta}_j$ measure the period to period increases or decreases in the cost survivor fraction, the growth rates $\bar{\phi}_1, \bar{\phi}_2, \ldots \bar{\phi}_j$ measure the period to period increases or decreases in the discretionary budgets available for hiring new accessions.

With this definition of terms it is tempting to believe that so long as $\bar{\phi}_j \geq \bar{\beta}_j$, i.e. the one period increase (decrease) in discretionary budgets exceeds the one period change in the cost survivor fractions, nonnegative accessions will meet new budgets in the next period. However, the simple numerical example having data

(12a)

$$(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, \bar{\phi}_4) = (5 \quad 0.2 \quad 1 \quad 1)$$

$$(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3, \bar{\beta}_4) = (2 \quad 0.05 \quad 1 \quad 1)$$

yields a solution

(12b)
$$(x_1, x_2, x_3, x_4, x_5) = (1 \quad 3 \quad {}^{-}5.1 \quad 10.8 \quad {}^{-}20.49)$$

with negative components. This example shows that the condition $\bar{\phi}_j - \bar{\beta}_j \geq 0$ is insufficient to guarantee nonnegative accessions in all periods if there are periods of very large budgets followed by a period of small budgets.

It is not difficult to show by a direct substitution of unknowns in the first three quations of (11) that an equality solution x_j satisfies

(13)

$$x_1 = 1$$

$$x_2 = \bar{\phi}_1 - \bar{\beta}_1$$

$$x_3 = (\bar{\phi}_1\bar{\phi}_2 - \bar{\beta}_1\bar{\beta}_2) - \bar{\beta}_1(\bar{\phi}_1 - \bar{\beta}_1)$$

$$\vdots$$

By a reversion of the series in (11) it can be shown that in general x_j satisfies the j^{th} order linear, homogeneous difference equation,

(14)
$$x_{j+1} = \sum_{i=1}^{j} x_{j+1-i}(\bar{\phi}_j - \bar{\beta}_i) \prod_{k=0}^{i-1} \bar{\beta}_k \, . \qquad j \geq 1$$

With (14) it is now possible to obtain x_{j+1} recursively in terms of $x_1, x_2, \ldots x_j$. As we have already pointed out the solution of x_j only depends on $\bar{\beta}_i$'s and $\bar{\phi}_i$'s with $i \leq j$. In other words changing the budgets on periods beyond j does not affect the accessions in time periods on or before j. Since cost continuation rates are nonnegative it is simple to show that the sequence of inequalities

$$\bar{\phi}_j \geq \text{Max } \{\bar{\beta}_1, \bar{\beta}_2, \ldots \bar{\beta}_j\} \qquad j \geq 1 \tag{15}$$

is sufficient to ensure that $x_j \geq 0$. The system of inequalities in (15) is, of course, much more restrictive than the one period inequalities $\bar{\phi}_j \geq \bar{\beta}_j$ (all j) as it compares the growth rate in one period to *all previous* continuation rates. The proof is straightforward: $x_1 = 1$ is nonnegative and by (13) $\bar{\phi}_1 \geq \bar{\beta}_1$ also implies $x_2 \geq 0$. If we now assume that $x_1, x_2, \ldots x_j$ are all nonnegative and that (15) holds, then x_{j+1} in (14) is a sum of nonnegative terms. Thus, by induction on n we see that (15) is sufficient to guarantee that all accessions x_j are nonnegative.

Again we use the data in (12a) to indicate why the equality solution in (12b) fails to be optimal. Notice that

$$\begin{aligned} \bar{\phi}_1 &= 5 \geq \text{Max } \{2\} = 2 \\ \bar{\phi}_2 &= 0.2 < \text{Max } \{2, 0.05\} = 2 \\ \bar{\phi}_3 &= 1 < \text{Max } \{2, 0.05, 1\} = 2 \\ \bar{\phi}_4 &= 1 < \text{Max } \{2, 0.05, 1, 1\} = 2 \,. \end{aligned} \tag{16}$$

Necessary conditions for the x_j in (14) to be nonnegative are obtained by noting that we can always rewrite the $(j + 1)^{st}$ equation in (11) as

$$\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 \ldots \bar{\phi}_j - \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \ldots \bar{\beta}_j = \sum_{i=2}^{j+1} x_i \prod_{k=0}^{j+1-i} \bar{\beta}_k \,. \tag{17}$$

Since the right-hand side is nonnegative if all $x_i \geq 0$, it follows that the inequalities on cumulative products,

(18)
$$\bar{\phi}_1 \geq \bar{\beta}_1$$
$$\bar{\phi}_1\bar{\phi}_2 \geq \bar{\beta}_1\bar{\beta}_2$$
$$\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 \geq \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \text{ , etc.}$$

must hold. While $\phi_j \geq \beta_j$ imply (18) the converse is not true as it is quite possible that $\bar{\phi}_j < \bar{\beta}_j$ while $\bar{\phi}_1\bar{\phi}_2 \ldots \bar{\phi}_j \geq \bar{\beta}_1\bar{\beta}_2 \ldots \bar{\beta}_j$. Thus, the simple and local test of whether the growth rate in net budgets exceeds the cost continuation rate in *each period* lies somewhere between the necessary conditions of (18) and the more global sufficiency conditions in (15).

When the equality solution is nonnegative (hence optimal), our original objective, the discounted sum of future manpower levels (neglecting legacies) is

$$\text{(19)} \qquad \tau \sum_{t=1}^{\infty} x_t \delta^{t-1} = \left(\sum_{j=0}^{m} \alpha_j \delta^j \Bigg/ \sum_{j=0}^{m} c_j \alpha_j \delta^j \right) \sum_{t=1}^{\infty} (\bar{z}_t - \bar{y}_t)\delta^{t-1}$$

(see pp. 9-11). This formula has a reasonable interpretation in the case where $\alpha_0 = 1$ and the α_j are nonincreasing. Let $\alpha_j - \alpha_{j+1}$ be the probability that an individual's lifetime is equal to j . With this stochastic interpretation of the survivor fractions, we define two random variables: T the individual's lifetime and K the total support cost of an individual. When δ is equal to 1 , the term in parentheses in (19) is simply $E[T]/E[K]$. Thus, we can increase our objective by keeping $E[T]$ fixed and reducing costs. Notice that if we attempt to increase expected lifetime by changing the α_j , then the cost will change also. We can get a more accurate estimate of the impact of possible changes by rewriting the first term of (19) with $\delta = 1$ and α_j expressed in terms of continuation rates

$$\frac{E[T]}{E[K]} = \left(\sum_{j=0}^{m} \prod_{k=0}^{j} \beta_k\right) \Big/ \left(\sum_{j=0}^{m} c_j \prod_{k=0}^{j} \beta_k\right) . \tag{20}$$

The derivative of the above expression with respect to β_ℓ is

$$\frac{\sum_{j=\ell}^{m} \left\{1 - c_j \frac{E[T]}{E[K]}\right\} \alpha_j}{\beta_\ell E[K]} .$$

If we let $\bar{c} = E[K]/E[T]$ denote the "average cost," then we see that if the cost c_j in the periods following ℓ is greater than average, the derivative is negative and increasing β_ℓ will decrease our objective. On the other hand, if the downstream costs are less than average, then increasing β_ℓ will increase manpower levels. This agrees with our intuitive expectation.

Special Cases

This section examines several special cases and derives tighter and more easily verified conditions under which the equality solution is optimal.

First, if the $\bar{\beta}_j$ are nondecreasing, then $\bar{\phi}_j \geq \bar{\beta}_j$ implies $\bar{\phi}_j \geq \bar{\beta}_i$ for $i \leq j$, thus the local conditions $\bar{\phi}_j \geq \bar{\beta}_j$ are sufficient for the equality solution to be nonnegative.

In a second case, if the $\bar{\alpha}_j$ are nonincreasing, then $\bar{\beta}_j \leq 1$ for all j . Moreover, nonincreasing $\bar{\alpha}_j$ imply, see Equation (5), Section 2, that the legacy $\bar{y}_t$ is nonincreasing. If $\bar{z}_t$ is nondecreasing, it follows that $\bar{z}_t - \bar{y}_t$ is nondecreasing and thus that $\bar{\phi}_j \geq 1$ for all j . Therefore, we have optimality of the equality solution under the readily verified conditions $\bar{z}_t$ nondecreasing and $\bar{\alpha}_j$ nonincreasing.

If we further specialize the first case so that $\bar{\beta}_j = \bar{\beta} < \bar{\phi}_j$ for all j , then we can write

(21) $$x_{j+1} = \bar{\phi}_{j-1}\left(\frac{\bar{\phi}_j - \bar{\beta}}{\bar{\phi}_{j-1} - \bar{\beta}}\right)x_j \qquad j = 2,3, \ldots$$

with $x_1 = 1$, and $x_2 = \bar{\phi}_1 - \bar{\beta}$. If $\bar{\phi}_j = \bar{\phi}$ $j = 1,2, \ldots, T$, and $\bar{\phi}_j = 1$ thereafter, we obtain

(22) $$x_{j+1} = \begin{Bmatrix} \bar{\phi}^{j-1}(\bar{\phi} - \bar{\beta}) & j \leq T \\ \bar{\phi}^{T}(1 - \bar{\beta}) & j > T \end{Bmatrix} .$$

Optimality for "Nearly Monotone" Survivors

In this section, we shall present a very efficient solution method for our maximization problem under an additional assumption.

A3: ("Nearly Monotone" Survivors) $\bar{\alpha}_{j-1}/\bar{\alpha}_j \geq \delta \quad j = 1, \ldots, m$.

This assumption is quite simple and is justified in many practical cases (since $\delta < 1$ and we can expect that $\bar{\alpha}_{j-1} \geq \bar{\alpha}_j$) .

Our method is given and explained in the following theorem.

Theorem 1:

Let $\bar{x}$ be defined by

(23) $$\bar{x}_1 = \min_{1 \leq i \leq m+1} \left\{\frac{\bar{z}_i - \bar{y}_i}{\bar{\alpha}_{i-1}}\right\}$$
$$\bar{x}_j = \min_{j \leq i \leq m+j} \left\{\frac{\bar{z}_i - \bar{y}_i - \sum_{k=1}^{j-1} \bar{\alpha}_{i-k}\bar{x}_k}{\bar{\alpha}_{i-j}}\right\} .$$

Under assumption A3, $\bar{x}$ is an optimal solution of the linear program defined by maximizing the objective (7), subject to the constraints (2).

Proof:

Obviously, $\bar{x}$ is a feasible solution of (7). Suppose $\tilde{x}$ is an optimal solution. (Such an optimal solution exists due to the discounting factor δ presented in the objective function.)

We shall show inductively that $\tilde{x}_j$ can be replaced by $\bar{x}_j$ $(j = 1,2, \ldots)$ without changing the value of the objective function.

Let $k = \min\{j \mid \tilde{x}_j > 0 ; j \geq 2\}$. (If $\tilde{x}_j = 0$ $j = 2, \ldots, m+1$, then the optimality of $\tilde{x}$ implies that $\tilde{x} = \bar{x}$.)

Let us define a new solution $\hat{x}$ by

(24)
$$\hat{x}_j = \begin{cases} \tilde{x}_1 + \Delta_1 & j = 1 \\ \tilde{x}_k - \Delta_k & j = k \\ \tilde{x}_j & j \neq 1, k \end{cases}$$

where Δ_1, Δ_k are determined such that

(25)
$$\Delta_k = \frac{\bar{\alpha}_{p-1}}{\bar{\alpha}_{p-k}} \Delta_1 \quad \left(\text{where } \frac{\bar{\alpha}_{p-1}}{\bar{\alpha}_{p-k}} = \max_{k \leq i \leq m+1} \frac{\bar{\alpha}_{i-1}}{\bar{\alpha}_{i-k}} \right)$$

$$\Delta_1 \leq \bar{x}_1 - \tilde{x}_1$$

$$\Delta_k \leq \tilde{x}_k$$

Thus, if $\bar{x}_1 \neq \tilde{x}_1$, we can find Δ_1, Δ_k satisfying (25) with $\Delta_1 > 0$.

In fact, we can define Δ_1 by

$$\Delta_1 = \min \left\{ x_1 - \tilde{x}_1 \; ; \; x_k \frac{\bar{\alpha}_{p-k}}{\bar{\alpha}_{p-1}} \right\}.$$

Obviously, $\hat{x}$ is a feasible solution. Moreover, evaluating the values of the objective function for $\hat{x}$ and $\tilde{x}$, we have

$$(26) \quad \sum_{j=1}^{\infty} \delta^{j-1}\hat{x}_j - \sum_{j=1} \delta^{j-1}\tilde{x}_j = \Delta_1 - \delta^{k-1}\Delta_k = \Delta_1\left(1 - \frac{\bar{\alpha}_{p-1}}{\bar{\alpha}_{p-k}}\delta^{k-1}\right) \geq 0$$

where the last inequality resulted from assumption A3 since

$$\frac{\bar{\alpha}_{p-1}}{\bar{\alpha}_{p-k+1}} = \frac{\bar{\alpha}_{p-1}}{\bar{\alpha}_{p-2}}\frac{\bar{\alpha}_{p-2}}{\bar{\alpha}_{p-3}} \cdots \frac{\bar{\alpha}_{p-k+1}}{\bar{\alpha}_{p-k}} \leq \delta^{1-k} .$$

We can repeat this process of increasing $\tilde{x}_1$ by decreasing some $\tilde{x}_j$ $(j > k)$ until $\hat{x}_1$ becomes equal to $\bar{x}_1$ without changing the value of the objective function.

Now, assuming $\tilde{x}_j = \bar{x}_j$ for $j = 1,2, \ldots, r$, we can prove using the same arguments as above that $\tilde{x}_{r+1}$ can be replaced by $\bar{x}_{r+1}$ while maintaining the optimality of $\tilde{x}$.

Hence, $\bar{x}$ is an optimal solution of (7).||

Note that under assumption A3 the computation of $\bar{x}$ is done recursively starting with $\bar{x}_1$. Thus, if one is interested in computing the programs of only the first n periods, it can be done easily by simply applying (23) for the first n periods. For a numerical example, we refer the reader to Section IV.

IV. A NUMERICAL EXAMPLE

We present an example below illustrating applications of our techniques to budgetary planning in a manpower system.

Consider a manpower system in which a group of men (cohort) enters each year, and from each cohort, a fraction α_j survives at least j years. The cost of support of a man in his $(j + 1)$st year of service is C_j .

Specifically, we let $\alpha_0 = 1$, $\alpha_1 = .9$, $\alpha_2 = .8 \ldots \alpha_9 = .1$, $\alpha_{10} = \alpha_{11} = \ldots = 0$, i.e. we assume that 1/10 of the original cohort leaves the system each year, for 10 years.

We assume a first year cost per man of \$10,000 and assume that the cost rises by 10% for every year a man remains in the system, so that $C_1 = \$10{,}000 \times 1.1 = \$11{,}000$, $C_2 = (\$10{,}000) \times (1.1)^2 = \$12{,}100$, and in general $C_j = (\$10{,}000) \times (1.1)^j$, $j \leq 9$. We emphasize that in this model the cost of supporting a man depends only on his length of service, and is independent of the calendar year.

Let us assume that, as of December 31, 1974, we have $n_0 = 100$ men with 1 year of service in the system, $n_1 = 90$ men with 2 years of service, $n_2 = 80$, ... $n_9 = 10$. Using the survivor fractions α_j given above, we may compute what our past accessions to the system must have been in order to account for current manpower levels. Thus if 1975 is year 1, the number of men entering in 1974 is $x_0 = \frac{n_0}{\alpha_0} = \frac{100}{1} = 100$. Similarly the 1973 cohort size was $x_{-1} = \frac{n_1}{\alpha_1} = \frac{90}{.9} = 100$. We see that with this data we must have $x_0 = x_{-1} = \ldots = x_{-9} = 100$, i.e. 100 men entered each year from 1965 to 1974.

Suppose it is desired to know how much money will be required in future years to support survivors from the current manpower stocks. In 1975 the cost will be

$$\bar{y}_1 = C_1\alpha_1x_0 + C_2\alpha_2x_{-1} + \ldots + C_9\alpha_9x_{-8}$$

$$= (\$10{,}000) \times (1.1 \times .9 + (1.1)^2 \times .8 + \ldots + (1.1)^9 \times .1) \times (100)$$

$$= \$6{,}531{,}000 .$$

Similarly in 1980 the cost will be

$$\bar{y}_6 = C_6\alpha_6 x_0 + C_7\alpha_7 x_{-1} + C_8\alpha_8 x_{-2} + C_9\alpha_9 x_{-3}$$

$$= \$1,958,000$$

and in general

$$\bar{y}_j = C_j\alpha_j x_0 + C_{j+1}\alpha_{j+1} x_{-1} + \ldots + C_9\alpha_9 x_{j-9} \ .$$

The quantities $\bar{y}_j$ are what we call cost legacies.

The cost legacies for 1975–1984 are tabulated below:

1975	$\bar{y}_1 = \$6,531,000$	1980	$\bar{y}_6 = \$1,958,000$
1976	$\bar{y}_2 = \$5,541,000$	1981	$\bar{y}_7 = \$1,249,000$
1977	$\bar{y}_3 = \$4,573,000$	1982	$\bar{y}_8 = \$665,000$
1978	$\bar{y}_4 = \$3,641,000$	1983	$\bar{y}_9 = \$236,000$
1979	$\bar{y}_5 = \$2,763,000$	1984	$\bar{y}_{10} = \$0$

The budget legacy for 1984 is zero because in that year there will be no survivors from current manpower stocks.

Suppose it is desired to know what future budgets will be required in order to continue an input of $x_j = 100$ men each year. The required budget for 1975 is $\bar{z}_1 = \bar{y}_1 + C_0\alpha_0 x_1 = \$6,531,000 + \$10,000 \times 1 \times 100 = \$7,531,000$. In 1980 the required budget will be $\bar{z}_6 = \bar{y}_6 + C_5\alpha_5 x_1 + C_4\alpha_4 x_2 + \ldots + C_0\alpha_0 x_6 = \$7,531,000$. In general, if we know the accessions x_j , we can compute the budgets $\bar{z}_j$ by $\bar{z}_j = \bar{y}_j + C_{j-1}\alpha_{j-1} x_1 + C_{j-2}\alpha_{j-2} x_2 + \ldots + C_0\alpha_0 x_j$. This merely says that the total budget for a given year is made up of the cost of all accessions before the base year 1975 (cost legacies) plus the cost of all accessions in 1975 and after. In this example it is hardly surprising that the $\bar{z}_j$ should all be equal, since future accessions are maintained at the constant level of past accessions.

We now change the emphasis to computing future accessions given future budget constraints. In particular suppose we are given annual budgets $\bar{z}_j$ for $j = 1, 2, \ldots, 10$ (1975-1984) and we wish to compute the number of men x_j that must be input each year in order to exactly exhaust the budgets. We solve the following equations

$$
\begin{aligned}
C_0\alpha_0 x_1 \qquad\qquad\qquad\qquad &= \bar{z}_1 - \bar{y}_1 \\
C_1\alpha_1 x_1 + C_0\alpha_0 x_2 \qquad\qquad &= z_2 - \bar{y}_2 \\
\vdots \qquad\qquad\qquad\qquad &\quad \vdots \\
C_9\alpha_9 x_1 + \cdots \quad + C_0\alpha_0 x_{10} &= \bar{z}_{10} - \bar{y}_{10}
\end{aligned}
$$

Consider the following cases:

<u>Case 1</u>:

If the annual budget $\bar{z}_j$ is \$7,531,000 for each year, 1975-1984, the above equations give the solution $x_1 = x_2 = \ldots = x_{10} = 100$, which is what we would expect from previous results.

<u>Case 2</u>:

Suppose in 1978 and 1981 we obtain budget increases of 25%, so that $\bar{z}_1 = \bar{z}_2 = \bar{z}_3 =$ \$7,531,000, $\bar{z}_4 = \bar{z}_5 = \bar{z}_6 =$ \$9,414,000, and $\bar{z}_7 = \bar{z}_8 = \bar{z}_9 = \bar{z}_{10} =$ \$11,767,000. Then we obtain

$x_1 = 100$	$x_6 = 104$
$x_2 = 100$	$x_7 = 342$
$x_3 = 100$	$x_8 = 113$
$x_4 = 288$	$x_9 = 119$
$x_5 = 102$	$x_{10} = 128$

Note the sharp increases in accessions for years 4 (1978) and 7 (1981). In the subsequent years the accessions drop back, even though the budget continues at an increased level. This is a result of the large cost legacies from the 1978 and 1981 cohorts.

Case 3A:

Suppose we solve the same problem as in Case 2 except we have 25% decreases instead of increases, so that $\bar{z}_1 = \bar{z}_2 = \bar{z}_3 = \$7,531,000$, $\bar{z}_4 = \bar{z}_5 = \bar{z}_6 = \$5,648,000$, and $\bar{z}_7 = \bar{z}_8 = \bar{z}_9 = \bar{z}_{10} = \$4,236,000$. Solving the equations we obtain

$x_1 = 100 \qquad x_6 = 96$

$x_2 = 100 \qquad x_7 = -48$

$x_3 = 100 \qquad x_8 = 88$

$x_4 = -88 \qquad x_9 = 83$

$x_5 = 98 \qquad x_{10} = 76$

We note that x_4 and x_7 are negative. This means that we must discharge 88 new men in 1978 and 48 in 1981 in order to exactly meet budgets.

Case 3B:

Suppose we have the same conditions as Case 3A, but are not allowed to discharge anyone, i.e. we require $x_j \geq 0$ for all j. To stay within budgets, we must have

$$
\begin{aligned}
C_0\alpha_0 x_1 & \leq \bar{z}_1 - \bar{y}_1 \\
C_1\alpha_1 x_1 + C_0\alpha_0 x_2 & \leq \bar{z}_2 - \bar{y}_2 \\
\vdots \qquad & \qquad \vdots \\
C_9\alpha_9 x_1 + C_8\alpha_8 x_2 + \ldots + C_0\alpha_0 x_{10} & \leq \bar{z}_{10} - \bar{y}_{10} .
\end{aligned}
$$

In general there are many solutions to these inequalities. To choose a particular "optimal" solution we assume that an accession j years in the future has present value δ^j, and seek to maximize the total present value of all future accessions, which is $\sum_{j=1}^{10} x_j \delta^{j-1}$. Here δ is a discount factor satisfying

$0 < \delta < 1$. A common value for δ is .95 .

We may compute such a solution using the algorithms of Section III. The solution we obtain is

$$\begin{array}{ll} x_1 = 100 & x_6 = 44 \\ x_2 = 100 & x_7 = 0 \\ x_3 = 11 & x_8 = 86 \\ x_4 = 0 & x_9 = 80 \\ x_5 = 97 & x_{10} = 72 \end{array}$$

The results of Cases 3A and 3B are compared in Figure 1. Note that in periods 3 and 6 we must reduce our accessions in anticipation of budget cuts to come in periods 4 and 7.

Conclusion

We see from this example that in order to avoid premature discharge of personnel, we must anticipate future budget cuts and reduce accessions appropriately.

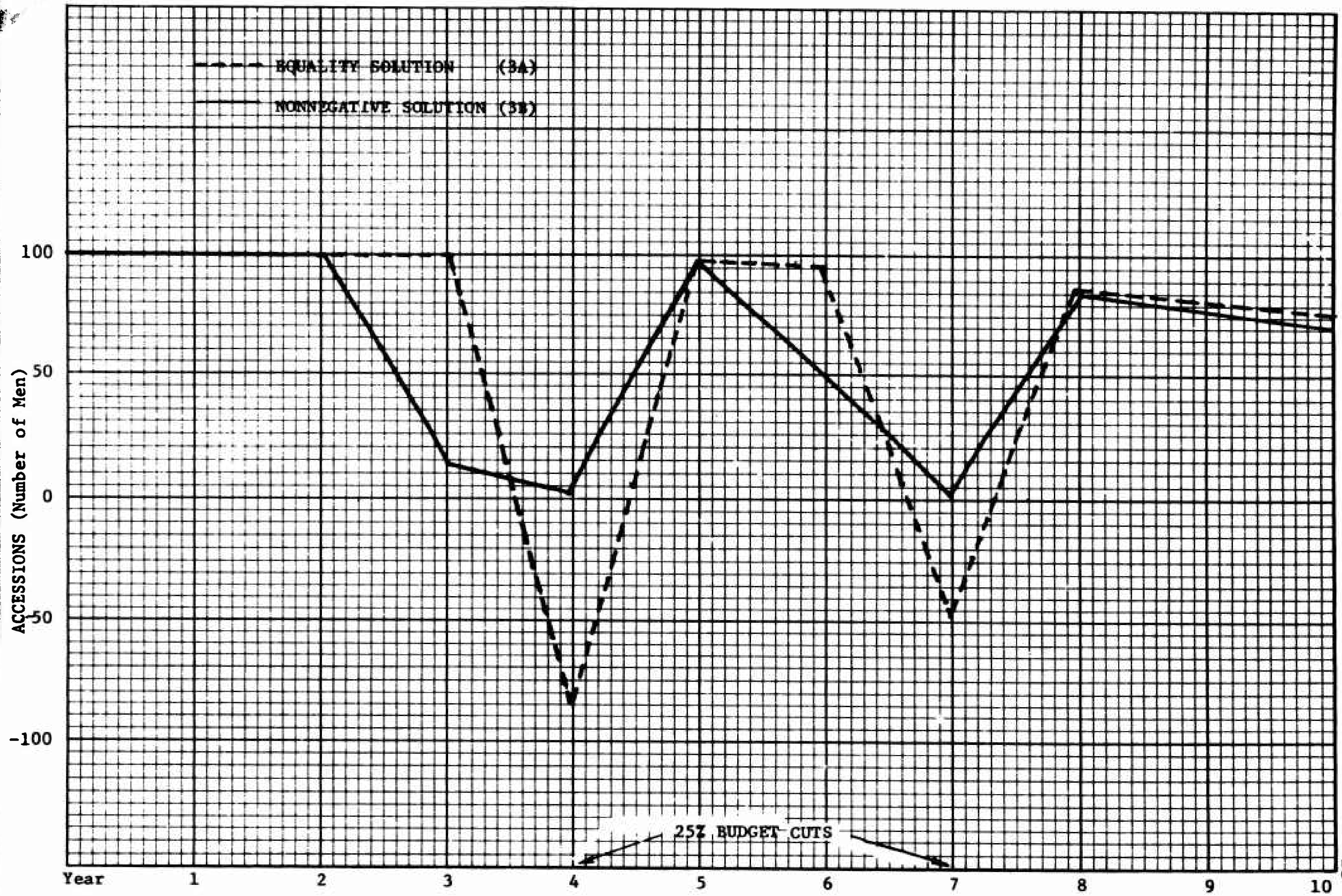

FIGURE 1: ACCESSIONS, YEARS 1-10, CASES 3A AND 3B.

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