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The resulting receiver structures estimate the reference waveform directly from the received process; whereas, the well-known matched filter or correlation receiver uses a stored reference waveform. In this report the Doppler performance of two such structures is investigated, the results indicating that such structures can be relatively insensitive to Doppler. The characteristic function for the "decision statistic" is computed for the discrete-time systems and numerically inverted to obtain the cumulative distribution function. Chernoff bounds are computed to evaluate the behavior of the "tail probabilities."

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COMMUNICATION RECEIVER STUDIES

Final Report

by

David C. Farden/Frank N. Cornett/Louis L. Scharf

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Abstract

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Acknowledgment

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Energy-Saving Policy

In concert with the national move to implement energy-saving policy Colorado State University is operating at daytime temperatures of 65° and reducing the level of nighttime lighting. The faculty and student body have traditionally been pedestrian-and cycle-oriented as evidenced by the absence of through-traffic on campus. Airline travel has been reduced to the minimum level consistent with the tasks spelled out in the contract.

List of Figures

I Introduction

In this report we investigate the performance of several different communications systems. The receiver structures for all of these systems belong to a family which we will call the family of predistorted-replica correlation receivers. This terminology arises because of the differences between a "predistorted-replica correlation receiver" and the well known correlation receiver (or matched filter). The latter uses a stored reference signal and correlates this with the received process, whereas the former derives its "reference waveform" directly from the received process.

We will begin our discussion by considering the conventional binary antipodal pulse-amplitude modulation (PAM) system, for which the transmitted signal is assumed to be of the form:

$$x(t) = \sum_{k=0}^{\infty} m_k \, s(t - kT), \tag{1.1}$$

where $m_k \in \{-1, 1\}$ and $s(\alpha) \equiv 0$ for all $\alpha \notin [0,T)$. For coherent communication, the receiver's task is to estimate the sequence $\{m_k\}_{k=0}^{\infty}$. As is well known, for the case that the received process is given by

$$r(t) = \beta \, x(t) + n(t) \tag{1.2}$$

where β is the channel attenuation constant, and $n(t)$ is a white Gaussian noise process, the optimal receiver is the correlation receiver shown in Figure 1.1, with $S(t) = x(t)$, $s_{ref}((t)_T) = s((t)_T)$, where $(t)_T \overset{\Delta}{=} t$ modulo T. The term "optimal" is used here as "the best that can be done for any reasonable criterion of goodness without making use of the source dynamics." The reader is referred to ([1] - [4]) for a more lucid discussion of these points.

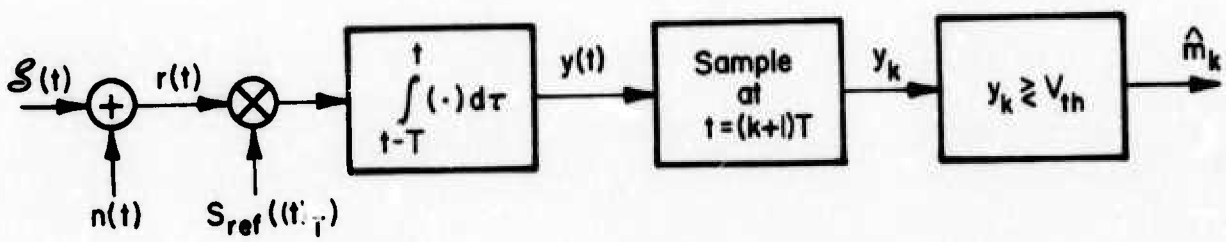


Figure 1.1 The correlation receiver

Now, if the channel is such that the signal component, $S(t)$, of the received process, $r(t)$, can be written as

$$S(t) = \sum_{k=0}^{\infty} m_k \, s_d(t - kT), \tag{1.3}$$

where $s_d(\alpha) \equiv 0$ for all $\alpha \notin [0,T)$, it is easily seen that the receiver of Figure 1.1 retains its optimality with $s_{ref}((t)_T)$, for the case that $s_d(.)$ is unknown, one can either use $s_{ref}((t)_T) = s((t)_T)$ and suffer the resulting degradation in performance, or resort to a different receiver structure, thus leading us to the so-called family of predistorted replica correlation receivers.

II Fixed Lag Autocorrelation Receiver

A. Continuous Time

We begin our discussion of predistorted replica correlation receivers with the Fixed Lag AutoCorrelation (FLAC) receiver. Consider the receiver structure illustrated in Figure 2.1, which is a continuous-time version of the system proposed by Farden [5]. The output of the correlator, y(t), can be written as

$$y(t) = \int_{t-T}^{t} r(\tau)r(\tau - T)d\tau$$

$$= \int_{t-T}^{t} S(\tau)S(\tau - T)d\tau + \int_{t-T}^{t} S(\tau)n(\tau - T)d\tau$$

$$+ \int_{t-T}^{t} S(\tau - T)n(\tau)d\tau + \int_{t-T}^{t} n(\tau)n(\tau - T)d\tau. \qquad (2.1)$$

If the signal component, $S(t)$, of the received process, r(t), can be written as

$$S(t) = m_{[\frac{t}{T}]} \, s_d((t)_T),$$

where [x] denotes the "largest integer contained in x," $(t)_T = t$ modulo T, $s_d(\alpha) \equiv 0$ for all $\alpha \notin [0,T)$, and n(t) is Gaussian White Noise (GWN), we can easily obtain

$$z(t) \overset{\Delta}{=} E\{y(t)\} = \int_{t-T}^{t} m_{[\frac{\tau}{T}]} \, m_{[\frac{\tau-T}{T}]} \, s_d((\tau)_T)s_d((\tau-T)_T)d\tau. \qquad (2.3)$$

The functions $(\xi t)_T$ and $[\frac{\xi t}{T}]$ are illustrated in Figure 2.2 for $\xi \in \{.75, 1. 1.25\}$. Finally, at $t = (k+1)T$, we have

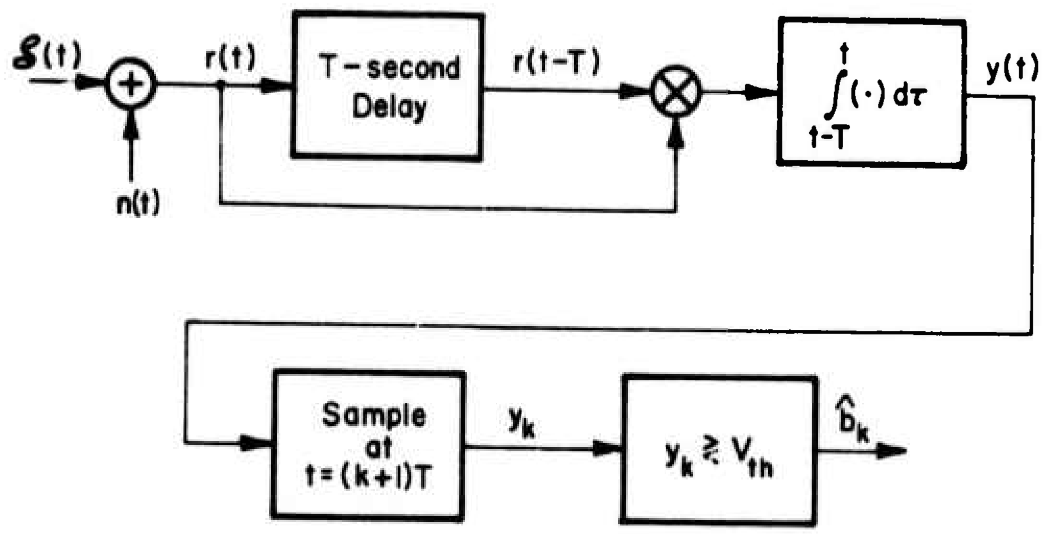


Figure 2.1 The continuous time fixed-lag autocorrelation receiver

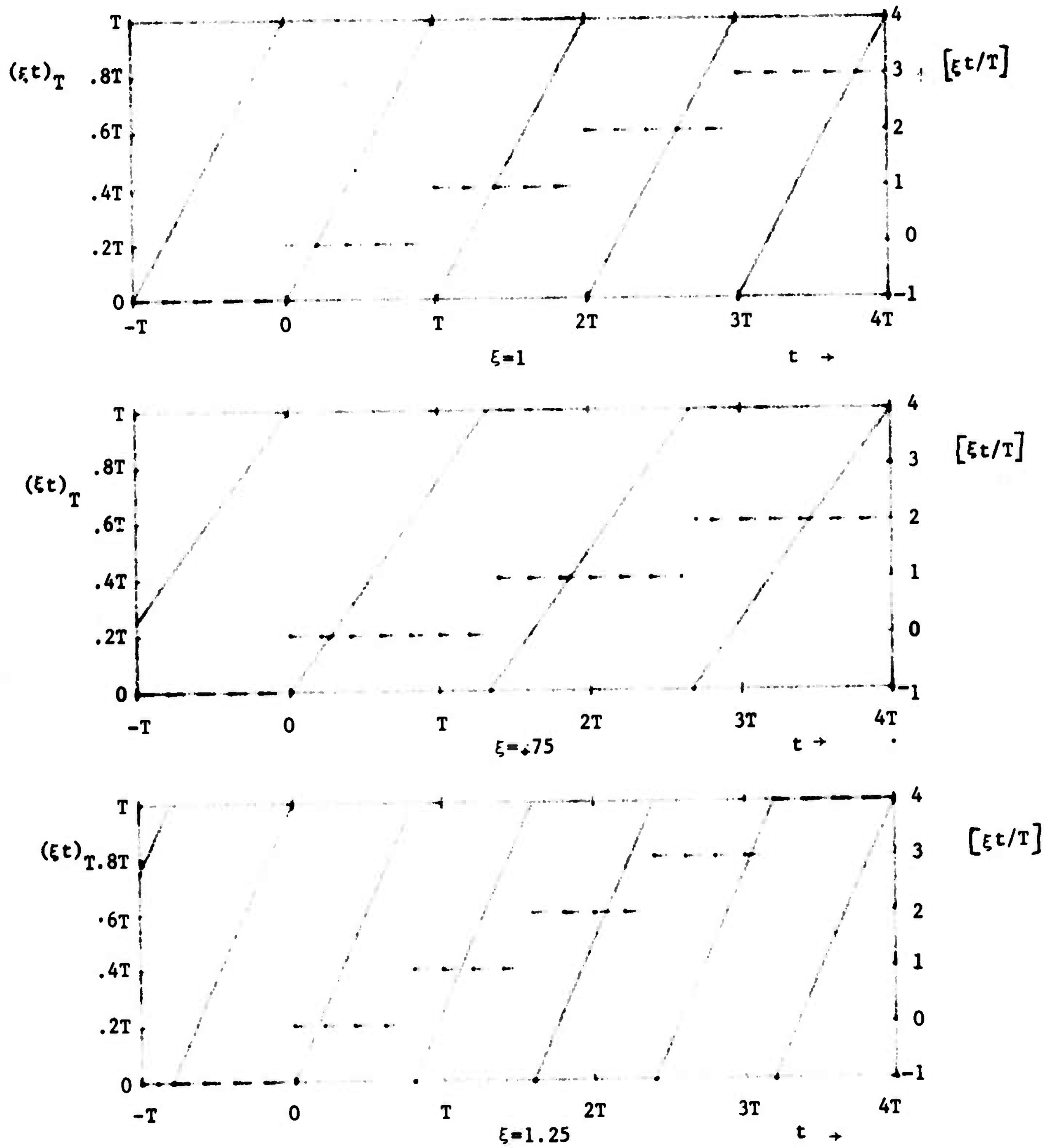


Figure 2.2 $[\xi t/T]$ and $(\xi t)_T$

$$z_k \triangleq E\{y(t)\}\Big|_{t=(k+1)T} = m_k\, m_{k-1} \int_{kT}^{(k+1)T} s_d^2((\tau)_T)d\tau$$

$$= m_k\, m_{k-1}\, C_{s_d}(0), \tag{2.4}$$

where

$$C_{s_d}(\gamma) \triangleq \int_0^T s_d(\tau)\, s_d(\tau-\gamma)d\tau . \tag{2.5}$$

Carefully note that $C_{s_d}(.)$ is not a cyclic correlation and that $s_d(\alpha) \equiv 0$ for all $\alpha \notin [0,T)$. This result suggests that if the sequence $\{m_k\}_{k=0}^{\infty}$ is a differentially encoded version of some information bearing sequence $\{b_k\}_{k=1}^{\infty}$, the FLAC system has some chance of success, at least for high signal-to-noise ratios. The differential encoding scheme to be used at the transmitter is the following (for $m_o = 1$, and $k = 1,2,3,\ldots$).

$$m_k = \begin{cases} m_{k-1}, \text{ if } b_k = 1 \\ -m_{k-1}, \text{ if } b_k = -1, \end{cases} \tag{2.6a}$$

or conversely,

$$b_k = \begin{cases} 1, \text{ if } m_k\, m_{k-1} = 1 \\ -1, \text{ if } m_k\, m_{k-1} = -1 \end{cases} \tag{2.6b}$$

Now, suppose that the signal component of the received signal has suffered a Doppler shift, i.e.,

$$S(t) = m_{[\frac{\xi t}{T}]}\, s_d((\xi t)_T). \tag{2.7}$$

The mean of the correlator output (2.3) becomes

$$z_\xi(t) = \int_{t-T}^{t} m_{[\frac{\xi\tau}{T}]}\, m_{[\frac{\xi(\tau-T)}{T}]}\, s_d((\xi\tau)_T) s_d((\xi\tau-\xi T)_T)d\tau. \tag{2.8}$$

Using the change of variable $u = \frac{\xi\tau}{T}$, and defining $x = \frac{\xi t}{T}$ and $z_\xi'(x) = z_\xi(t)$, we have

$$z'_\xi(x) = \frac{T}{\xi}\int_{x-\xi}^{x} m_{[u]}\, m_{[u-\xi]}\, s_d((u)_1 T)\, s_d((u-\xi)_1 T)du.$$

Partitioning the interval $[x-\xi,x]$ as:

$$a'_o = x-\xi < a'_1 < a'_2 < \ldots < a'_L = x,$$

so that $[u] = k+n_\ell$, $[u-\xi] = k+P_\ell$ for all $u \varepsilon [a'_{\ell-1}, a'_\ell)$ for $\ell = 1,2,\ldots,L$, where $k = [x]$ we may write:

$$z'_\xi(x) = \frac{T}{\xi}\sum_{\ell=1}^{L} m_{k+n_\ell}\, m_{k+P_\ell}\int_{a'_{\ell-1}}^{a'_\ell} s_d((u)_1 T)\, s_d((u-\xi)_1 T)du.$$

Furthermore, defining $a_\ell = a'_{\ell-1} - k - n_\ell$, $b_\ell = a'_\ell - k - n_\ell$, and $q_\ell = n_\ell - P_\ell$, we obtain:

$$z'_\xi(x) = \frac{T}{\xi}\sum_{\ell=1}^{L} m_{k+n_\ell}\, m_{k+P_\ell}\int_{a_\ell}^{b_\ell} s_d(uT)s_d((u+q_\ell-\xi)T)du. \quad (2.9)$$

It is easily shown that for $\xi \varepsilon (1/2, 3/2)$, we have $n_\ell \varepsilon \{-2, -1, 0\}$, and $P_\ell \varepsilon \{-3, -2, -1, 0\}$. Table 2.1 shows all of the parameters necessary for the computation of (2.9) for any known signal $s_d(\cdot)$ and for any value of $\xi \varepsilon (1/2, 3/2)$.

Whereas in the absence of Doppler shift, information was being sent and received at the rate of one bit/T seconds, with a Doppler shift, information is being received at the rate of one bit/$\frac{1}{\xi}$ T seconds; hence, the receiver must observe the correlator output and make a decision once every T/ξ seconds. Consequently, we will now investigate when the maxima and minima of (2.9) should occur. Hopefully, the maxima and minima of (2.9) will occur for the values of x for which the quantity

$$\beta_x = \max_\ell (b_\ell - a_\ell) \quad (2.10)$$

is maximum with the constraint that $q_\ell = 1$. Applying this criteria with

Table 2.1

Parameter values for computing output of FLAC receiver

(a) $1/2 < \xi < 1$

$(x)_1 \in \{.\}$	ℓ	a_ℓ	b_ℓ	n_ℓ	p_ℓ	q_ℓ
$[0, 2\xi - 1)$	1	$(x)_1 + 1 - \xi$	ξ	-1	-2	1
	2	ξ	1	-1	-1	0
	3	0	$(x)_1$	0	-1	1
$[2\xi - 1, \xi)$	1	$(x)_1 + 1 - \xi$	1	-1	-1	0
	2	0	$(x)_1$	0	-1	1
$[\xi, 1)$	1	$(x)_1 - \xi$	ξ	0	-1	1
	2	ξ	$(x)_1$	0	0	0

(b) $\xi = 1$

$(x)_1 \in \{.\}$	ℓ	a_ℓ	b_ℓ	n_ℓ	p_ℓ	q_ℓ
$[0, 1)$	1	$(x)_1$	1	-1	-2	1
	2	0	$(x)_1$	0	-1	1

(c) $1 < \xi < 3/2$

$(x)_1 \in \{.\}$	ℓ	a_ℓ	b_ℓ	n_ℓ	p_ℓ	q_ℓ
$[0, \xi - 1$	1	$(x)_1 + 2 - \xi$	1	-2	-3	1
	2	0	$\xi - 1$	-1	-3	2
	3	$\xi - 1$	1	-1	-2	1
	4	0	$(x)_1$	0	-2	2
$[\xi - 1, 2\xi - 2)$	1	$(x)_1 + 1 - \xi$	$\xi - 1$	-1	-3	2
	2	$\xi - 1$	1	-1	-2	1
	3	0	$\xi - 1$	0	-2	2
	4	$\xi - 1$	$(x)_1$	0	-1	1
$[2\xi - 2, 1)$	1	$(x)_1 + 1 - \xi$	1	-1	-2	1
	2	0	$\xi - 1$	0	-2	2
	3	$\xi - 1$	$(x)_1$	0	-1	1

the aid of Table 2.1, we find that (i) for $1/2 < \xi < 1$, β_x attains its maximum value of ξ for $(x)_1 = \xi$, (ii) for $\xi = 1$, β_x attains its maximum value of 1 for $(x)_1 = 0$, and (iii) for $1 < \xi < 3/2$, β_x attains its maximum value of $2 - \xi$ whenever $(x)_1 \in [0, 2\xi - 2]$. Using these results, we find that for $1/2 < \xi < 1$ and $x = k + \xi$:

$$z'_\xi(x) = \frac{T}{\xi} m_k m_{k-1} \int_0^\xi s_d(uT)s_d((u + 1 - \xi)T)du$$

$$= \frac{1}{\xi} m_k m_{k-1} C_{s_d}((\xi - 1)T), \tag{2.11}$$

where the last equality follows from (2.5) and the fact that $s_d(\alpha) \equiv 0$ for all $\alpha \notin [0, T)$. Similarly, we find that for $1 < \xi < 3/2$ and $(x)_1 \in [0, \xi - 1)$:

$$z'_\xi(x) = \frac{T}{\xi} m_{k-2} m_{k-3} \int_{(x)_1 + 2 - \xi}^{1} s_d(uT)s_d((u+1-\xi)T)du$$

$$+ \frac{1}{\xi} m_{k-1} m_{k-3} C_{s_d}((\xi - 2)T) + \frac{1}{\xi} m_{k-1} m_{k-2} C_{s_d}((\xi - 1)T)$$

$$+ \frac{T}{\xi} m_k m_{k-2} \int_0^{(x)_1} s_d(uT)s_d((u + 2 - \xi)T)du, \tag{2.12}$$

whereas, for $1 < \xi < 3/2$ and $(x)_1 \in [\xi - 1, 2\xi - 2]$, we obtain:

$$z'_\xi(x) = \frac{T}{\xi} m_{k-1} m_{k-3} \int_{(x)_1 + 1 - \xi}^{\xi - 1} s_d(uT)s_d((u+2-\xi)T)du$$

$$+ \frac{1}{\xi} m_{k-1} m_{k-2} C_{s_d}((\xi - 1)T) + \frac{1}{\xi} m_k m_{k-2} C_{s_d}((\xi - 2)T)$$

$$+ \frac{T}{\xi} m_k m_{k-1} \int_{\xi - 1}^{(x)_1} s_d(uT)s_d((u+1-\xi)T)du. \tag{2.13}$$

Finally, a worst case analysis of both (2.12) and (2.13) ($m_{k-1} m_{k-2} = 1$, $m_{k-2} m_{k-3} = m_{k-1} m_{k-3} = m_k m_{k-2} = -1$ in (2.12) and $m_{k-1} m_{k-2} = 1$, $m_{k-1} m_{k-3} = m_k m_{k-2} = m_k m_{k-1} = -1$ in (2.13)) yields

$$z'_\xi(x) \geq \frac{1}{\xi} C_{s_d}((\xi - 1)T) - \frac{1}{\xi} \{(\xi - 1)P_{max} + |C_{s_d}((\xi - 2)T)|\}, \tag{2.14}$$

where we have defined

$$P_{max} = T \cdot \max_{u \in [0,1)} s_d^2(uT),$$

so that for $|b - a| < 1$, we have

$$\left| \int_a^b s_d(uT) s_d((u - \gamma)T) du \right| \leq \frac{|b - a|}{T} P_{max}.$$

Now define the gain of the system against Doppler to be $G = |z'_\xi(x)|/C_{s_d}(0)$. From (2.11) with $1/2 < \xi < 1$, $x = k + \xi$ and $m_k m_{k-1} = 1$, we have

$$G = \frac{1}{\xi} |C_{s_d}((\xi - 1)T)|/C_{s_d}(0). \tag{2.15}$$

From (2.14), for $1 < \xi < 3/2$:

$$G \geq \frac{1}{\xi \cdot C_{s_d}(0)} \{ |C_{s_d}((\xi - 1)T)| - (\xi - 1)P_{max} - |C_{s_d}((\xi - 2)T)| \} \tag{2.16}$$

Suppose for a first approximation that $s_d(.)$ has time bandwidth product TW and that $C_{s_d}(\gamma)$ is given by:

$$C_{s_d}(\gamma) = \begin{cases} C_{s_d}(0)\,(1 - |\gamma|W), & \text{for } |\gamma| < \frac{1}{W} \\ 0 & \text{, elsewhere} \end{cases} \tag{2.17}$$

For this choice of $C_{s_d}(.)$, (2.16) becomes, for $P_{max} = 2C_{s_d}(0)$, $TW \geq 2$, and $1 < \xi \leq 1 + \frac{1}{TW}$

$$G \geq \frac{3 - 2\xi - (\xi - 1)\,TW}{\xi}; \tag{2.18}$$

whereas (2.15) becomes, for $1 - \frac{1}{TW} \leq \xi < 1$,

$$G = \frac{1 - (1 - \xi)TW}{\xi}. \tag{2.19}$$

Figure 2.3 illustrates equations (2.18) and (2.19) for $TW \in \{2, 5, 10, 20, 50\}$.

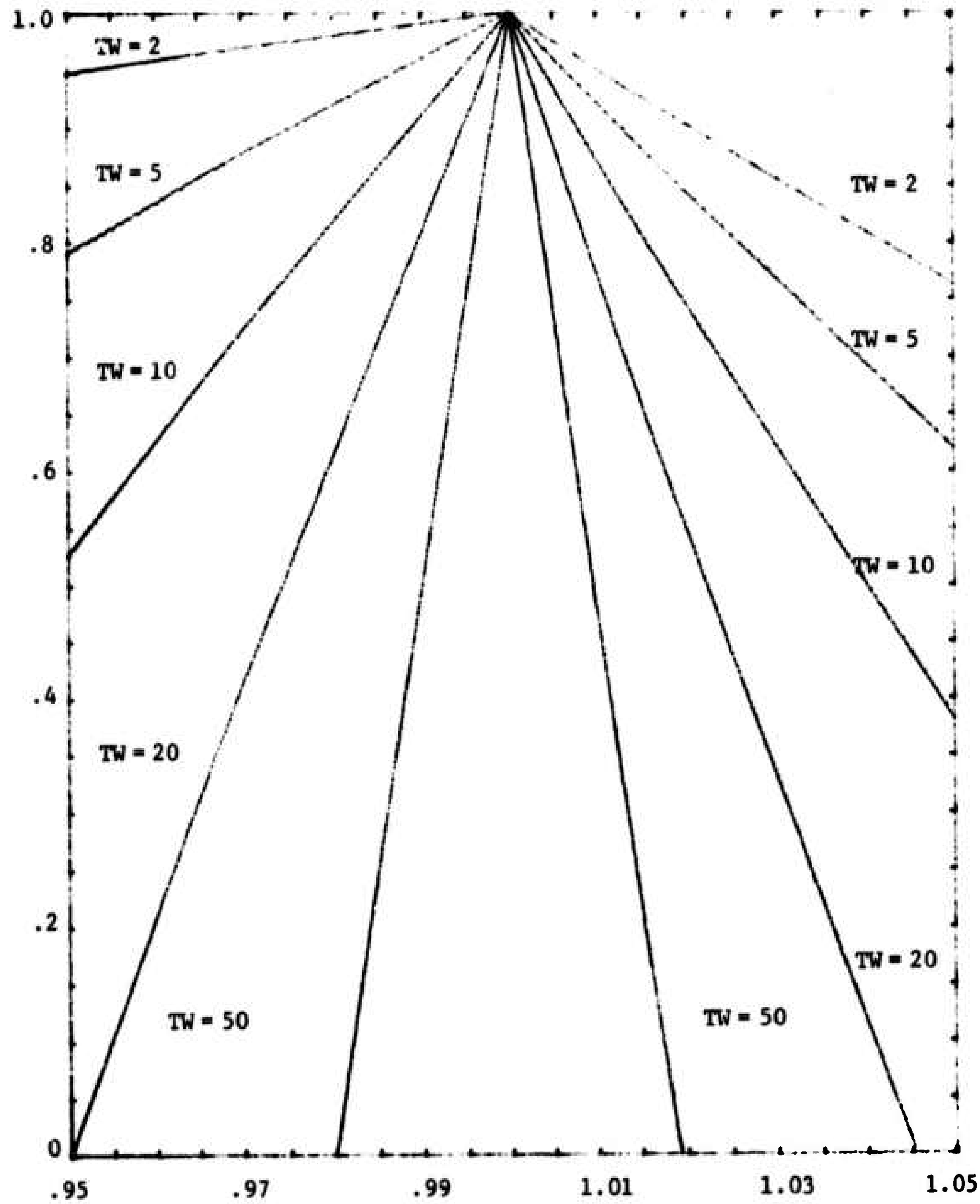


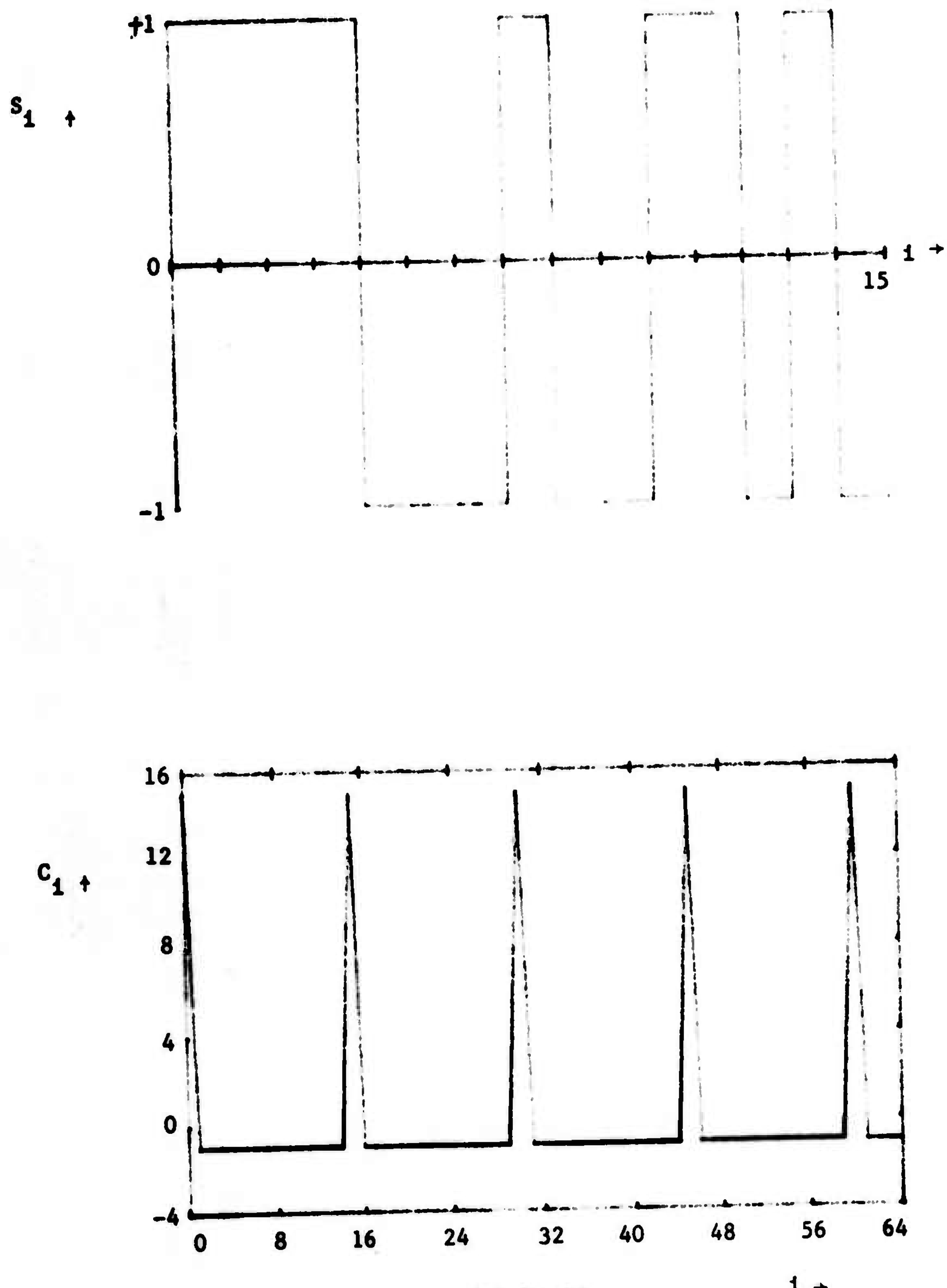
Figure 2.3 FLAC Doppler Gain, G, vs. Doppler Shift, ξ

B. The Discrete Time FLAC System

We will now discuss a discrete time version of the FLAC system. The carrier signal $s(t)$ will be extended for all $t \geq 0$ via the relationship $s(t) = s((t)_T)$. For definiteness $s(t)$ will be considered to be a sampled and held version of a pseudorandom noise sequence (PRS) which assumes the binary values of ± 1. Such sequences are easily generated by simple shift register circuts, and have desirable autocorrelation characteristics [6]. The value of $s(t)$ for $kNT \leq T < (k + 1)\, NT$, $k = 0, 1, 2, \ldots$, will be denoted as s_k, where N and T are positive numbers whose significance will become apparent momentarily. With this notation, the autocorrelation of s is defined as

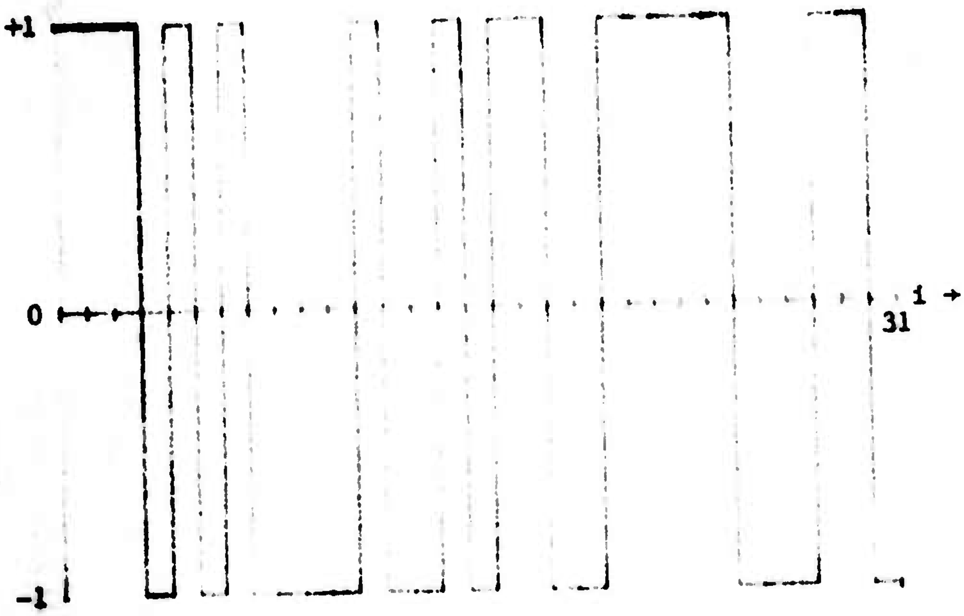
$$C_i \equiv \sum_{k=0}^{k-1} s_k\, s_{k+i} \begin{cases} \cong 0, & i \neq nK,\ n = 0, \pm 1, \pm 2, \ldots \\ = K, & i = nK \end{cases} \qquad (2.20)$$

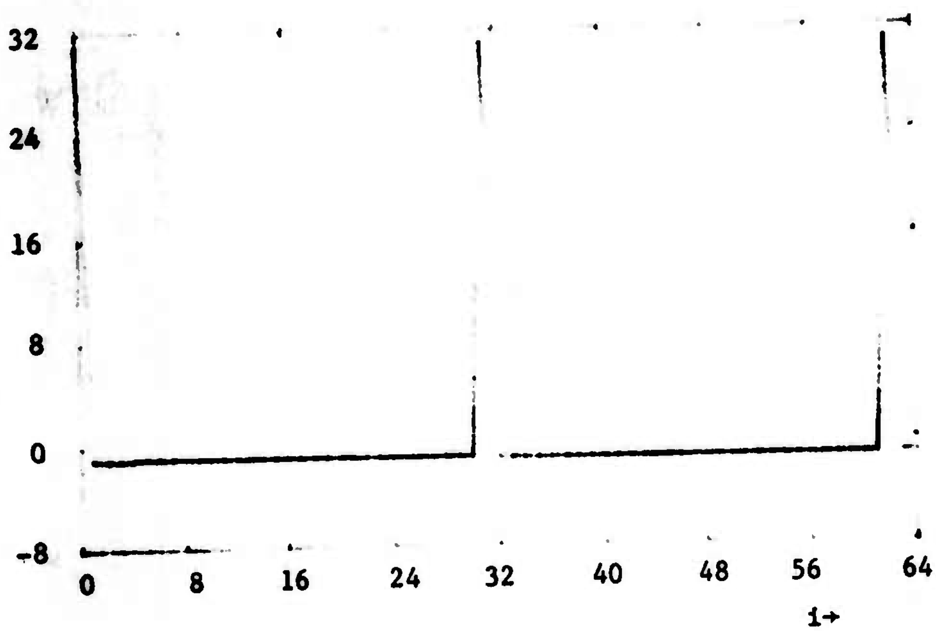
where K is the period of the PRS, i.e., $s_k = s_{k+K}$ for all $k \geq 0$. The actual value of K is determined by the bit length, BL, of the shift register which is used to generate the PRS: $K = 2^{BL} - 1$; hence, a typical PRS might have a period of 3, 7, 15,..., 1023, 2047, 4095, etc. Figure 2.4 illustrates two typical pseudorandom sequences of period 15 and 31, along with their respective autocorrelation sequences. Note that, for $i \neq nK$, for integral n, $C_i = -1$.



(a) K = 15

Figure 2.4 Typical PRS and autocorrelation





(b) K = 31

Figure 2.4 Typical PRS and autocorrelation

Construction of the FLAC transmitter is shown schematically in Figure 2.5.

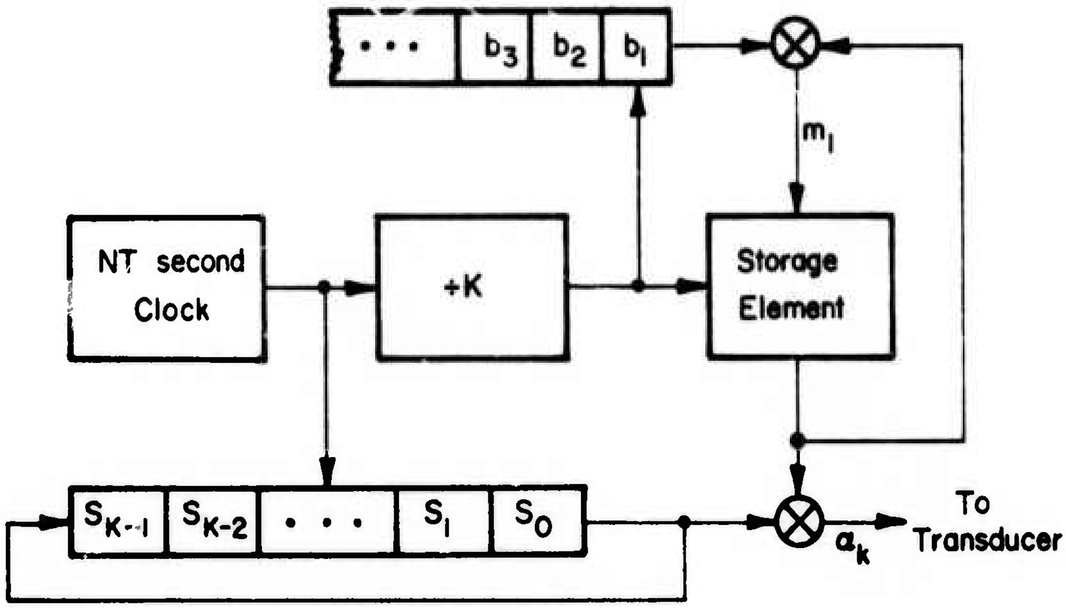


Figure 2.5 Discrete-time FLAC transmitter

The output α_k is seen to be

$$\alpha_k = m_{[\frac{k}{K}]} s_k, \; k = 0, 1, 2, \ldots$$

where the sequences $\{b_k\}$ and $\{m_k\}$ are the original information bearing sequence and the resultant encoded sequence, respectively, both of which were introduced earlier.

At the receiver, the analog input is sampled every T seconds to produce an input sequence $\{R_n\}$. Thus, under Dopplerless conditions each transmitted bit $m_{[\frac{k}{K}]} s_k$ is sampled N times (N will usually be taken as an integer), with each sample shifted into a shift register train (Figure 2.6) composed of two individual shift registers, each of length L. At the P^{th} bit position of each register the product $a_n^P \equiv a^P(nT) = R_{n-P} R_{n-L-P}$ is calculated, and the receiver output Y_n is taken to be

$$Y_n = \sum_{P=0}^{L-1} a_n^P = \sum_{P=0}^{L-1} R_{n-P} R_{n-L-P} \tag{2.21}$$

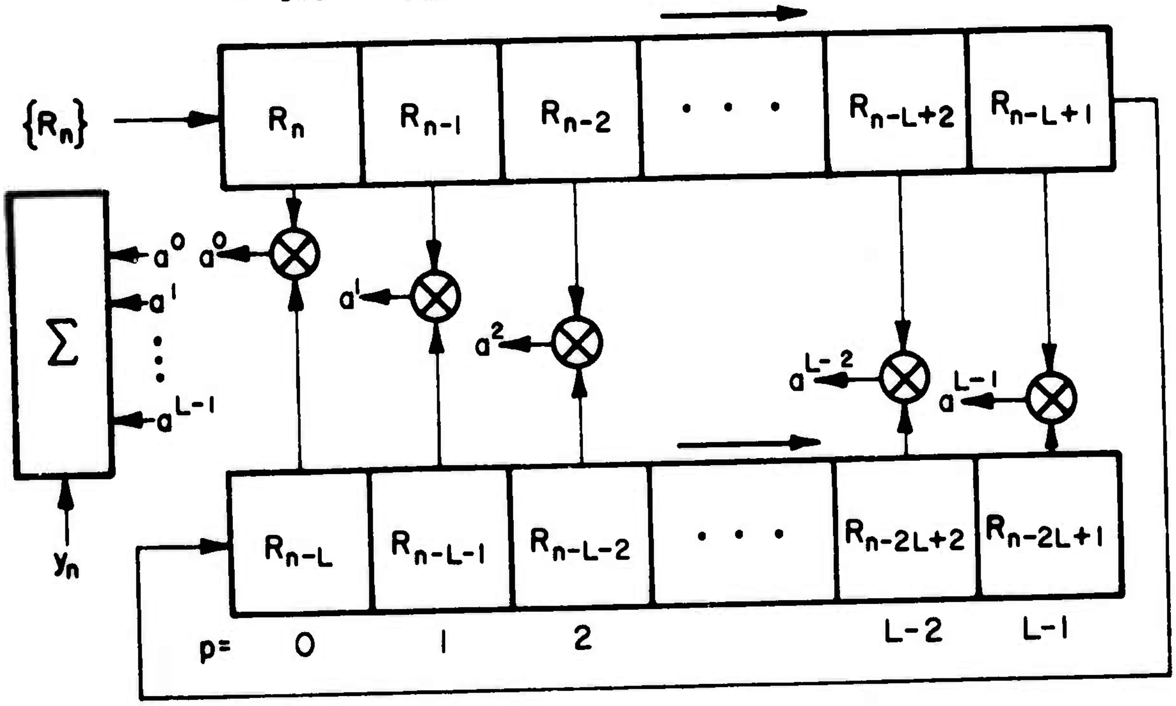


Figure 2.6 Discrete-time FLAC receiver

In order to obtain an expression for Y in terms of s, note that, with Doppler ξ, and under noiseless conditions, (recall that $\alpha_n \equiv \alpha(nNT)$),

$$R_n \equiv R(nT) = \alpha(\xi nT) = m_{\left[\frac{[\frac{\xi n}{N}]}{K}\right]} s(\xi nT) \tag{2.22}$$

Since $K > 1$ the whole part of $\xi n/N$ may be used in the subscript of m, and, for the signal being considered, $s(\xi nT) = s_{[\frac{\xi n}{N}]}$, so that (2.22) may be written as

$$R_n = m_{[\frac{\xi n}{NK}]} s_{[\frac{\xi n}{N}]} \tag{2.23}$$

By defining m_k and s_k as zero for $k < 0$, equation (2.21) may be written as

$$Y_n = \sum_{P=0}^{L-1} m_{[\frac{\xi(n-P)}{KN}]} m_{[\frac{\xi(n-P)-\xi L}{KN}]} s(\xi(n-P)T)s(\xi(n-P-L)T) \tag{2.24}$$

for all $n \geq 0$. Equation (2.24) is the general form for the receiver output; for the PRS-type signals being used (2.24) becomes

$$Y_n = \sum_{P=0}^{L-1} m_{[\frac{\xi(n-P)}{KN}]} \, m_{[\frac{\xi(n-P)-\xi L}{KN}]} \, s_{[\frac{\xi(n-P)}{N}]} \, s_{[\frac{\xi(n-P)-\xi L}{N}]} \tag{2.25}$$

If the m's in (2.25) are equal for $0 \leq P \leq L - 1$, it is desirable to maximize Y_n for $\xi = 1$, in which case the shift register length L should be chosen as $L = KN$. For if L is chosen in this way, the subscripts of s in (2.25), for $\xi = 1$, become $[\frac{n-P}{N}]$ and $[\frac{n-P}{N} - K] = [\frac{N-P}{N}] - K$, thus assuming a peak, as can be seen from (2.20).

A typical output sequence is illustrated in Figure 2.7 for $K = 15$, $N = 5$, $L = KN = 75$, and $\xi = 1$, with the information sequence b_i as indicated thereon. A feeling for the effects of Doppler (for the same values of K and L) can be obtained from Figure 2.8, a graph of $\frac{1}{N} Y_{2L}$ (the $2L$ was chosen arbitrarily) versus ξ, with m_k assumed to be +1 for all k. Although any plot of Y_n versus ξ is very dependent upon the PRS employed, and upon the shift register length L, some generalizations can be gleaned from Figure 2.8. First, note that the value of Y_{2L} becomes very erratic and unpredictable for ξ outside the range of approximately $(K - 1)/K$ to $(K + 1)/K$, and secondly, the value of N does not seem to have a drastic effect upon the values in the graph. Specifically, the larger value of N seems to offer no improvement in system performance. In fact, suppose N is taken to the limit in

$$Y'_n = \frac{1}{N} Y(nT) = \frac{1}{N} \sum_{P=0}^{NK-1} s(\xi(n-P)T)s(\xi(n-P-L)T)$$

For convenience, the product NT may be assumed constant at $NT = 1$, corresponding to the transmission of one bit of the PRS per second, so

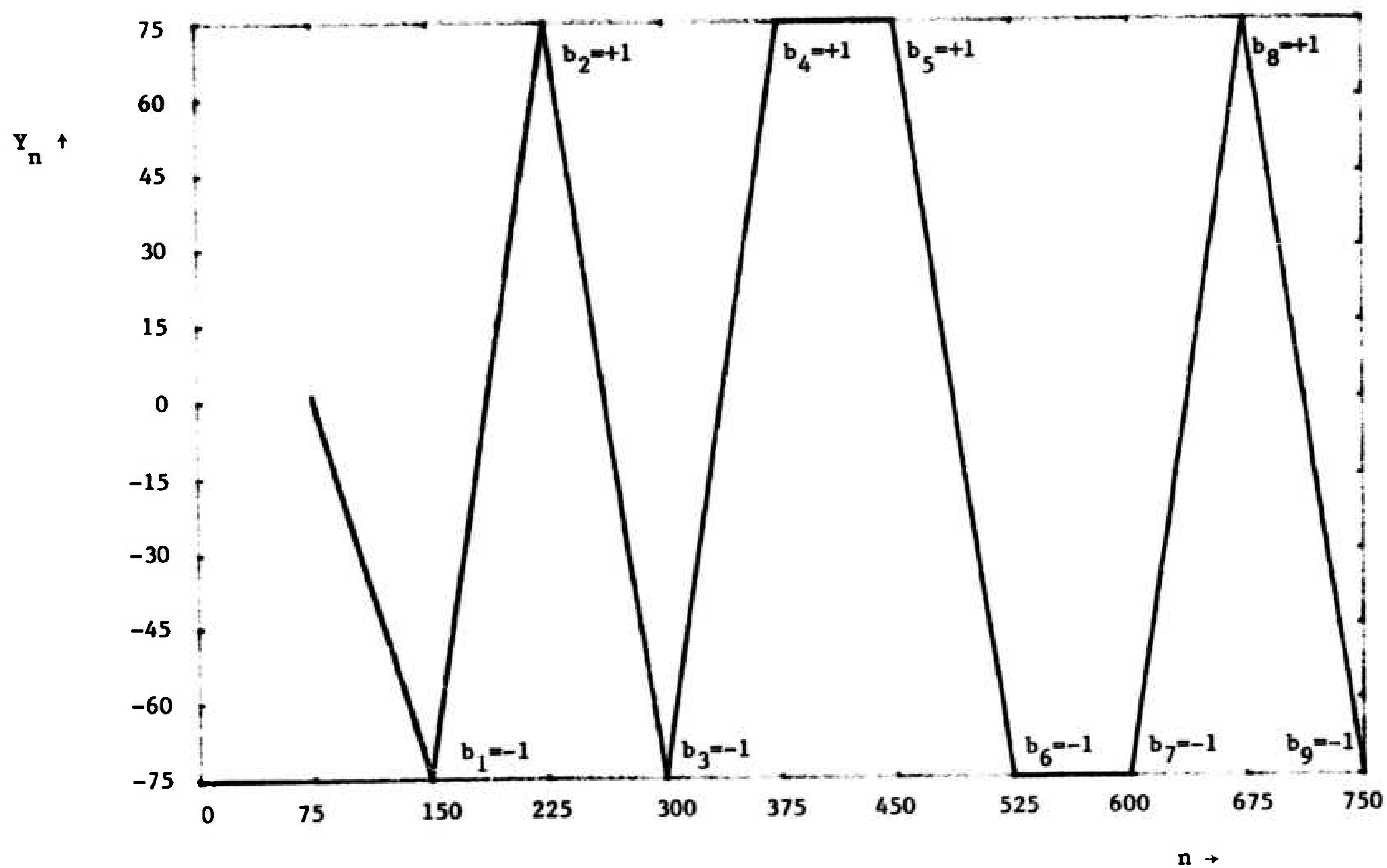


Figure 2.7 Typical FLAC output

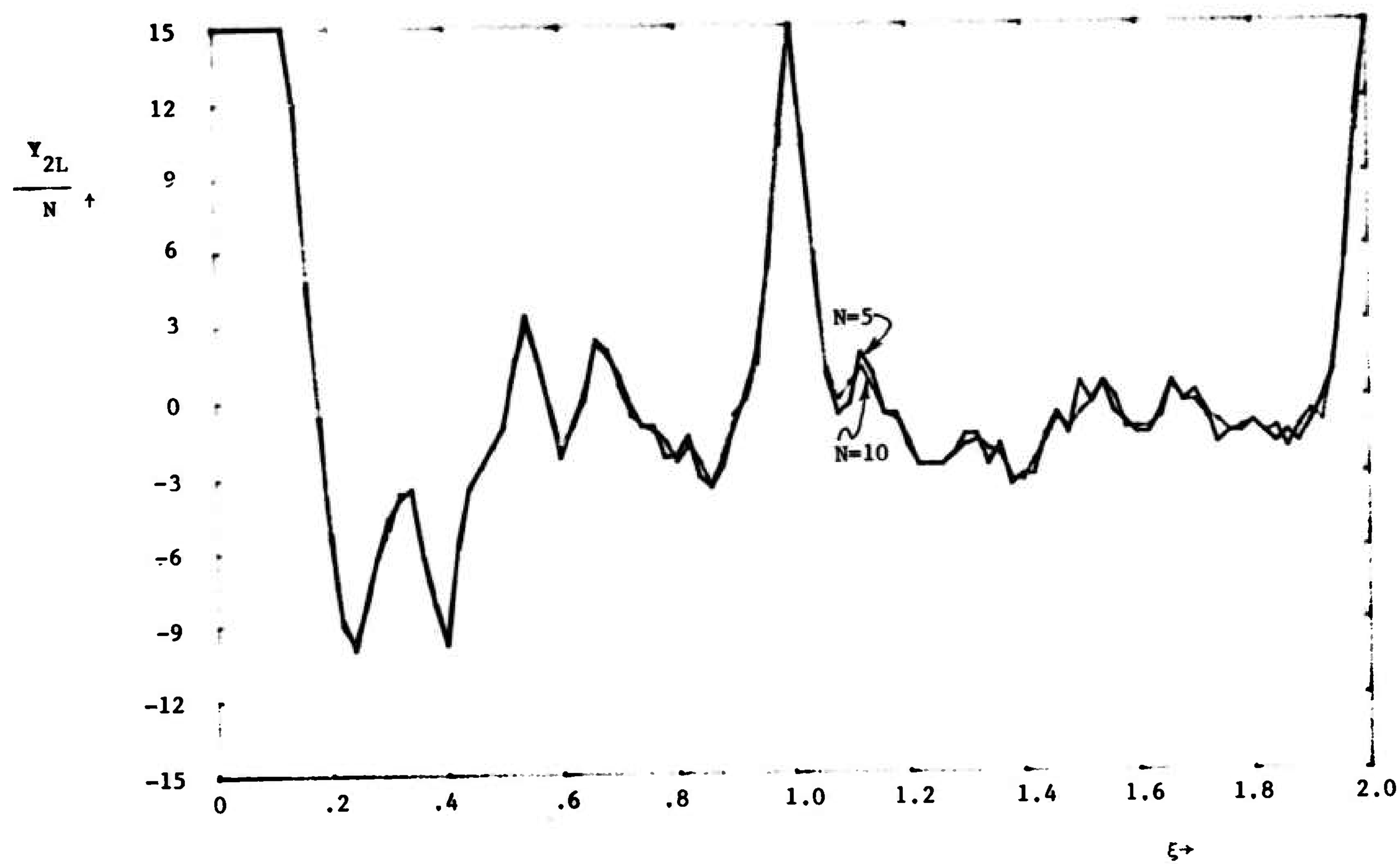


Figure 2.8 FLAC output versus ξ

that, letting $t = nT$, there obtains [7]

$$Y'(t) = \lim_{N \to \infty} \frac{1}{N} y(nT) = \int_0^K s(\xi(t-\tau))s(\xi(t-\tau) - \xi K)d\tau \qquad (2.26)$$

which is the counterpart of equation (2.8), with $m_k = +1$ for all k. If $\xi = (K-1)/K$, or $\xi = (K+1)/K$, the value of the integral in (2.26) is approximately zero, which indicates that the system is usable only for $1 - 1/K << \xi << 1 + 1/K$, a region which shrinks with increasing K.

The upshot of the above argument is that performance of the FLAC receiver with regard to Doppler cannot be dependably improved by sampling at a higher rate at the receiver, a result which is to be expected from the analysis undertaken previously.

C. Interference Sensitivity: The FLAC System

Suppose there are two signals impinging upon the FLAC receiver, one of which it is desired to detect, and the other of which is considered to be interference. Denote by ξ_d, K_d, and N_d the parameters of the desired signal, and by ξ_u, K_u, and N_u the parameters of the undesired signal, where these symbols play the same role as their unsubscripted counterparts in Section II-B. Further, let the desired signal, as transmitted, be given by

$$\alpha_k = m_{[\frac{k}{K_d}]} s_k = m_{[\frac{k}{K_d}]} s(t) \quad \text{for}$$

$kN_dT \leq t < (k + 1)N_dT$, $k = 0, 1, 2, \ldots$, and let the undesired signal be given by

$$\beta_k = J_{[\frac{k}{K_u}]} X_k = J_{[\frac{k}{K_u}]} X(t) \quad \text{for}$$

$kN_uT \leq t < (k + 1)\ N_uT$. Then the received sequence $\{R_n\}$ is given by

$$R_n = m_{\left[\frac{\xi_d n}{N_d K_d}\right]} s_{\left[\frac{\xi_d n}{N_d}\right]} + J_{\left[\frac{\xi_u n}{N_u K_u}\right]} X_{\left[\frac{\xi_u n}{N_u}\right]}, \text{ so that with inter-}$$

ference (2.25) becomes

$$\begin{aligned}
Y_n = &\sum_{P=0}^{L-1} m_{\left[\frac{\xi_d(n-P)}{K_d N_d}\right]} m_{\left[\frac{\xi_d(n-P-L)}{K_d N_d}\right]} s_{\left[\frac{\xi_d(n-P)}{N_d}\right]} s_{\left[\frac{\xi_d(n-P-L)}{N_d}\right]} \\
&+ \sum_{P=0}^{L-1} m_{\left[\frac{\xi_d(n-P)}{K_d N_d}\right]} J_{\left[\frac{\xi_u(n-P-L)}{K_u N_u}\right]} s_{\left[\frac{\xi_d(n-P)}{N_d}\right]} X_{\left[\frac{\xi_u(n-L-P)}{N_u}\right]} \\
&+ \sum_{P=0}^{L-1} m_{\left[\frac{\xi_d(n-P-L)}{K_d N_d}\right]} J_{\left[\frac{\xi_u(n-P)}{K_u N_u}\right]} s_{\left[\frac{\xi_d(n-P-L)}{N_d}\right]} X_{\left[\frac{\xi_u(n-P)}{N_u}\right]} \\
&+ \sum_{P=0}^{L-1} J_{\left[\frac{\xi_u(n-P)}{K_u N_u}\right]} J_{\left[\frac{\xi_u(n-P-L)}{K_u N_u}\right]} X_{\left[\frac{\xi_u(n-P)}{N_u}\right]} X_{\left[\frac{\xi_u(n-P-L)}{N_u}\right]}
\end{aligned} \tag{2.27}$$

The first and fourth terms in (2.27) are autocorrelation sums, whereas the second and fourth terms are crosscorrelations of the two signals s and X. It is a reasonable assumption that the periods K_d and K_u of s and X are unequal, and that the crosscorrelation of the two signals will be approximately zero. Therefore, the second and third terms in (2.27) will hereafter be neglected, and the only term which will constitute interference will be the autocorrelation of x.

Since K_u and K_d are of the form $2^{BL} - 1$, for integral BL, the relation between K_u and K_d must be either

$$K_d = \ell K_u + \ell - 1, \qquad \ell = 2, 3, \ldots \tag{2.28}$$

or

$$K_d = \frac{K_u - \ell + 1}{\ell}, \qquad \ell = 2, 3, \ldots \tag{2.29}$$

Supposing this relation is given by (2.28), and that $L = K_d N_d$, a contribution from the fourth term in (2.27) requires that the subscripts of X differ by kK_u, $k = 0, \pm 1, \pm 2, \ldots$, which is satisfied for

$$\xi_u = \frac{k\, N_u K_u}{N_d(\ell K_u + \ell - 1)} \cong \frac{kN_u}{\ell N_d}, \; k = 0, 1, 2, \ldots \tag{2.30}$$

where the approximation holds whenever $\ell K_u >> \ell - 1$. Thus, if N_u, N_d, and ℓ are chosen such that $\frac{N_u}{\ell N_d} = 2$, for example, then (2.30) becomes $\xi_u = 0, 2, 4, \ldots$, which is impractical, so that appreciable interference will not be observed.

If K_d and K_u are related by (2.29), the autocorrelation of the interference signal contributes to the output for $\xi_u = \frac{k\ell N_u K_u}{N_d\,(K_u - \ell + 1)} \cong \frac{k\ell N_u}{N_d}$. If $\frac{N_u}{\ell N_d}$ is chosen as $\frac{N_u}{\ell N_d} = 2$, then $\frac{\ell N_u}{N_d} = 2\ell^2 \geq 8$, and once again the values of ξ_u required for appreciable interference are impractical.

From the above discussion it is apparent that, if the bit rates NT and the periods K of all transmitters operating within a contiguous region can be controlled, then the FLAC receiver can be "tuned" to a particular transmitter by properly selecting the receiver shift register length L, and the sampling period T.

III The Alternating Forward/Reverse Sequence Autocorrelation Receiver

A. Continuous Time

In this section, the Alternating Reverse Sequence AutoCorrelation (ARSAC) receiver is discussed. The ARSAC system, conceived by Cornett [8], assumes the transmitted signal, $x(t)$, is given by

$$x(t) = m_{[\frac{t}{T}]}\, s(\overline{t}), \tag{3.1}$$

where $\overline{t} = (t)_{2T} + 2\left\lfloor \frac{(t)_{2T}}{T} \right\rfloor (T - (t)_{2T})$, and $\{m_k\}_{k=0}^{\infty}$ is a differentially encoded version of some information sequence $\{b_k\}_{k=1}^{\infty}$, as in (2.6). The functions $\overline{\xi t}$ and $[\frac{\xi t}{T}]$ are illustrated in Figure 3.1 for $\xi \in \{.75, 1, 1.25\}$. Note that the use of $\overline{t}$ as the argument for $s(.)$ in the transmitter has the effect of transmitting $s(.)$ alternately forwards and backwards. Denoting the received process by $r(t)$, the ARSAC receiver output, $y(t)$, is given by

$$y(t) = \int_0^T r(t - \tau)r(t - 2T + \tau)d\tau. \tag{3.2}$$

Suppose that the signal component, $\mathcal{S}(t)$, of the received process, $r(t) = \mathcal{S}(t) + n(t)$, is given by $\mathcal{S}(t) = x(t)$, and that $n(t)$ is GWN. Defining $z(t) = E\{y(t)\}$, we obtain, for $t = kT$:

$$z(t{=}kT) = m_{k-1}\, m_{k-2} \int_0^T s((\overline{kT - \tau}))s((\overline{kT - 2T + \tau}))d\tau$$

$$= m_{k-1}\, m_{k-2}\, T\int_0^1 s((\overline{(k - u)T})s((\overline{(k - 2 + u)T})du \tag{3.3}$$

Suppose that there exists an a, b such that for all $\alpha \in [a, b)$ we have $[\alpha] = h$, h a constant. Then if h is even, we have $\overline{\alpha T} = (\alpha - h)T$; and, if h is odd, we have $\overline{\alpha T} = (1 + h - \alpha)T$. Consequently, we find

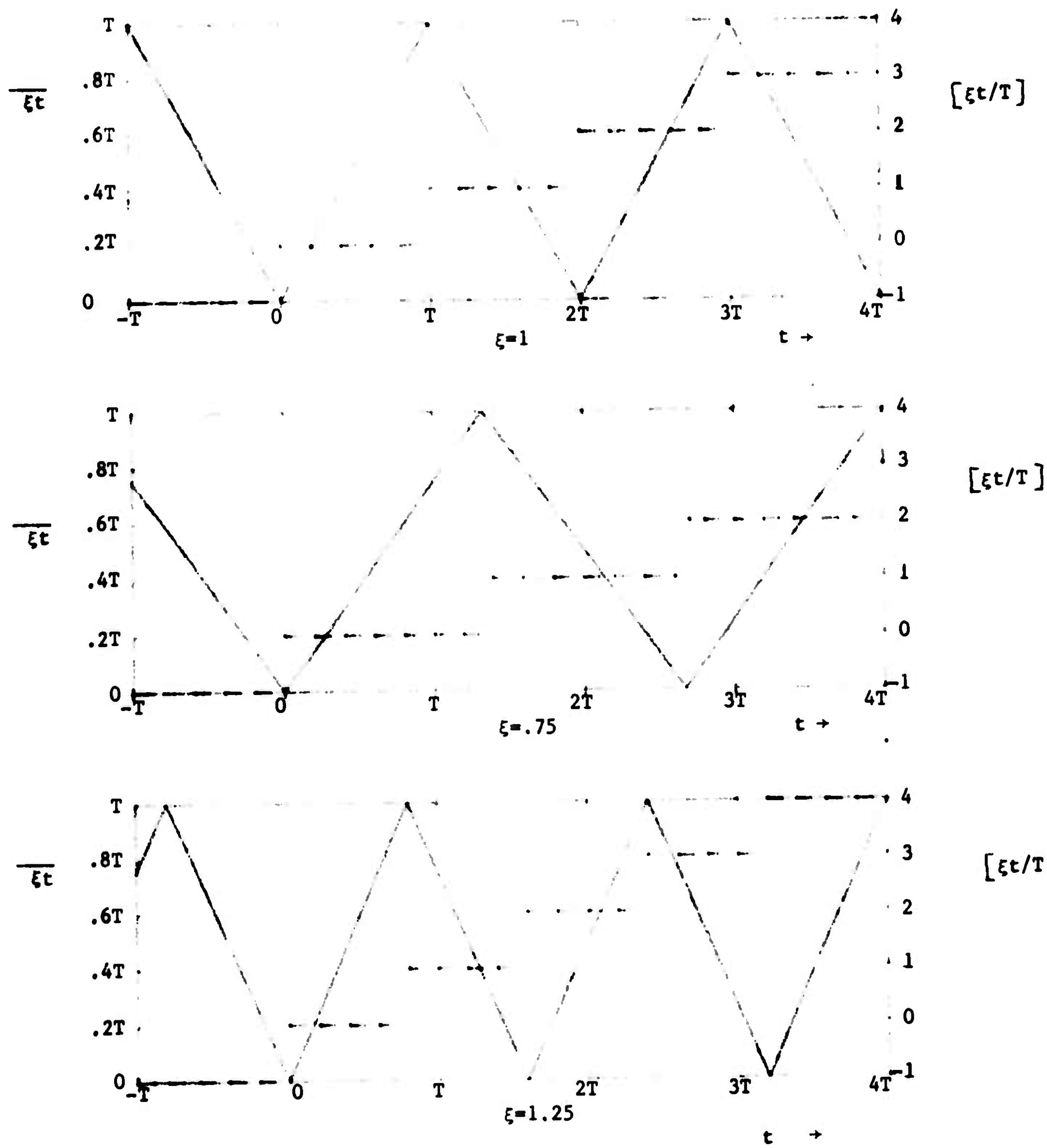


Figure 3.1 $\overline{\xi t}$ and [ξt/T] vs. t

from (3.3) that for k even, $\overline{(k - u)T} = \overline{(k - 2 + u)T} = uT$ and for k odd, $\overline{(k - u)T} = \overline{(k - 2 + u)T} = (1 - u)T$, for all $u \in [0, T)$. Finally, we find that (3.3) may be written as

$$z(t = kT) = m_{k-1}\, m_{k-2}\, C_s(0), \tag{3.4}$$

where

$$C_s(\gamma) = \int_0^T s(\tau)s(\tau - \gamma)d\tau.$$

We note that (3.4) and (2.4) are identical, indicating that the operation of the FLAC system and the ARSAC system are identical at $t = kT$, when there is no Doppler shift.

Now, suppose that the signal component of the received process is given by

$$S(t) = m_{[\frac{\xi t}{T}]}\, s_d(\overline{\xi t}). \tag{3.5}$$

Using (3.2), with $r(t) = S(t) + n(t)$, $n(t)$ GWN, and $z_\xi(t) = E\{y(t)\}$, we obtain

$$z_\xi(t) = \int_0^T m_{[\frac{\xi(t-\tau)}{T}]}\, m_{[\frac{\xi(t-2T+\tau)}{T}]}\, s_d(\overline{\xi(t - \tau)})s_d(\overline{\xi(t-2T+\tau)})d\tau. \tag{3.6}$$

Making the change of variable $u = \frac{\xi\tau}{T}$, and defining $x = \frac{\xi t}{T}$ and $z'_\xi(x) = z_\xi(t)$, we obtain

$$z'_\xi(x) = \frac{T}{\xi}\int_0^\xi m_{[x-u]}\, m_{[x-2\xi+u]}\, s_d((\overline{x-u)T})\; s_d((\overline{x-2\xi+u)}T)du. \tag{3.7}$$

Partitioning the interval $[0, \xi]$ as

$$0 = a_0 < a_1 < a_2 < \ldots < a_L = \xi$$

such that, with $x = k + (x)_1$, we have $[x - u] = k + n_\ell$, and $[x - 2\xi + u] = k + P_\ell$ for all $u \in [a_{\ell-1}, a_\ell)$, we see that (3.7) may be written as

$$z'_\xi(x) = \frac{T}{\xi} \sum_{\ell=1}^{T} m_{k+n_\ell} m_{k+P_\ell} \int_{a_{\ell-1}}^{a_\ell} s_d(\overline{(x-u)T})\, s_d(\overline{(x-2\xi+u)T})\,du, \quad (3.8)$$

where

$$\overline{(x-u)T} = \begin{cases} ((x)_1 - u - n_\ell)T, \text{ if } k+n_\ell \text{ is even} \\ (1-(x)_1+n_\ell+u)T, \text{ if } k+n_\ell \text{ is odd;} \end{cases} \quad (3.9a)$$

and

$$\overline{(x-2\xi+u)T} = \begin{cases} ((x)_1 - 2\xi+u-P_\ell)T, \text{ if } k+P_\ell \text{ is even} \\ (1-(x)_1+2\xi+P_\ell-u)T, \text{ if } k+P_\ell \text{ is odd} \end{cases} \quad (3.9b)$$

Table 3.1 gives all the parameters needed for the computation of (3.8) and (3.9) for $1/2 < \xi < 3/2$. The maxima and minima of $z'_\xi(x)$ should occur for those values of x for which $\overline{(x-u)T} = \overline{(x-2\xi+u)T}$ in (3.8). After investigating the possibilities with the aid of (3.9), it is found that the maxima and minima will occur when

$$(x)_1 = \begin{cases} \xi, & 1/2 < \xi < 1 \\ \xi - 1, & 1 < \xi < 3/2 \\ 0, & \xi = 1 \end{cases}$$

Using Table 3.1, for $1/2 < \xi < 1$ and $(x)_1 = \xi$, we find

$$z'_\xi(x) = \frac{T}{\xi} m_k m_{k-1} \begin{cases} \int_0^\xi s_d^2((\xi-u)T)du, \text{ for } k \text{ even} \\ \int_0^\xi s_d^2((1-\xi+u)T)du, \text{ for } k \text{ odd} \end{cases} \quad (3.10)$$

Table 3.1

Parameter values for the computation of the output of the ARSAC receiver

(a) $1/2 < \xi < 1$

$(x)_1 \in \{.\}$	ℓ	n_ℓ	P_ℓ	$a_{\ell-1}$	a_ℓ
$[0, \xi - 1/2)$	1	0	-2	0	$(x)_1$
	2	-1	-2	$(x)_1$	$2\xi - (x)_1 - 1$
	3	-1	-1	$2\xi - (x)_1 - 1$	ξ
$[\xi - 1/2, 2\xi - 1)$	1	0	-2	0	$2\xi - (x)_1 - 1$
	2	0	-1	$2\xi - (x)_1 - 1$	$(x)_1$
	3	-1	-1	$(x)_1$	ξ
$[2\xi - 1, \xi)$	1	0	-1	0	$(x)_1$
	2	-1	-1	$(x)_1$	ξ
$[\xi, 1)$	1	0	-1	0	$2\xi - (x)_1$
	2	0	0	$2\xi - (x)_1$	ξ

(b) $\xi = 1$

$(x)_1 \in \{.\}$	ℓ	n_ℓ	P_ℓ	$a_{\ell-1}$	a_ℓ
$[0, 1/2)$	1	0	-2	0	$(x)_1$
	2	-1	-2	$(x)_1$	$1 - (x)_1$
	3	-1	-1	$1 - (x)_1$	1
$[1/2, 1)$	1	0	-2	0	$1 - (x)_1$
	2	0	-1	$1 - (x)_1$	$(x)_1$
	3	-1	-1	$(x)_1$	1

Table 3.1

Parameter values for the computation of the output of the ARSAC receiver

(c) $1 < \xi < 3/2$

$(x)_1 \in \{.\}$	ℓ	n_ℓ	P_ℓ	$a_{\ell - 1}$	a_ℓ
$[0, \xi - 1)$	1	0	-3	0	$(x)_1$
	2	-1	-3	$(x)_1$	$2\xi - (x)_1 - 2$
	3	-1	-2	$2\xi - (x)_1 - 2$	$(x)_1 + 1$
	4	-2	-2	$(x)_1 + 1$	ξ
$[\xi - 1, 2\xi - 2)$	1	0	-3	0	$2\xi - (x)_1 - 2$
	2	0	-2	$2\xi - (x)_1 - 2$	$(x)_1$
	3	-1	-2	$(x)_1$	$2\xi - (x)_1 - 1$
	4	-1	-1	$2\xi - (x)_1 - 1$	ξ
$[2\xi - 2, \xi - 1/2)$	1	0	-2	0	$(x)_1$
	2	-1	-2	$(x)_1$	$2\xi - (x)_1 - 1$
	3	-1	-1	$2\xi - (x)_1 - 1$	ξ
$[\xi - 1/2, 1)$	1	0	-2	0	$2\xi - (x)_1 - 1$
	2	0	-1	$2\xi - (x)_1 - 1$	$(x)_1$
	3	-1	-1	$(x)_1$	ξ

Similarly, for $1 < \xi < 3/2$ and $(x)_1 = \xi - 1$, we obtain

$$z'_\xi(x) = \frac{T}{\xi} m_k m_{k-3} \begin{cases} \displaystyle\int_0^{\xi-1} s_d^2((\xi - 1 - u)T)du, & \text{for } k \text{ even} \\ \displaystyle\int_0^{\xi-1} s_d^2((2 - \xi + u)T)du, & \text{for } k \text{ odd} \end{cases} \tag{3.11}$$

$$+ \frac{T}{\xi} m_{k-1} m_{k-2} \begin{cases} \displaystyle\int_{\xi-1}^{\xi} s_d^2((1 - \xi + u)T)du, & \text{for } k \text{ even} \\ \displaystyle\int_{\xi-1}^{\xi} s_d^2((\xi - u)T)du, & \text{for } k \text{ odd} \end{cases}$$

Now, defining the gain of the system against Doppler, G, as

$$G = |z'_\xi(x)|/C_{s_d}(0),$$

and

$$P_{max} = T \cdot \max_{u\varepsilon[0,1)} s_d(uT),$$

we obtain, for $1/2 < \xi < 1$, and $(x)_1 = \xi$,

$$G \geq \frac{1}{\xi} + \frac{(\xi - 1)}{\xi} \frac{P_{max}}{C_{s_d}(0)}; \tag{3.12}$$

whereas, for $1 < \xi < 3/2$, $(x)_1 = \xi - 1$, and $m_k m_{k-3} = - m_{k-1} m_{k-2}$,

$$G \geq \frac{1}{\xi} + \frac{(1 - \xi)}{\xi} \frac{P_{max}}{C_{s_d}(0)}. \tag{3.13}$$

The lower bounds of (3.12) and (3.13) are illustrated in Figure 3.2 for $1/2 < \xi < 3/2$ and $P_{max}/C_{s_d}(0) \ \varepsilon \ \{1.5, 2\}$.

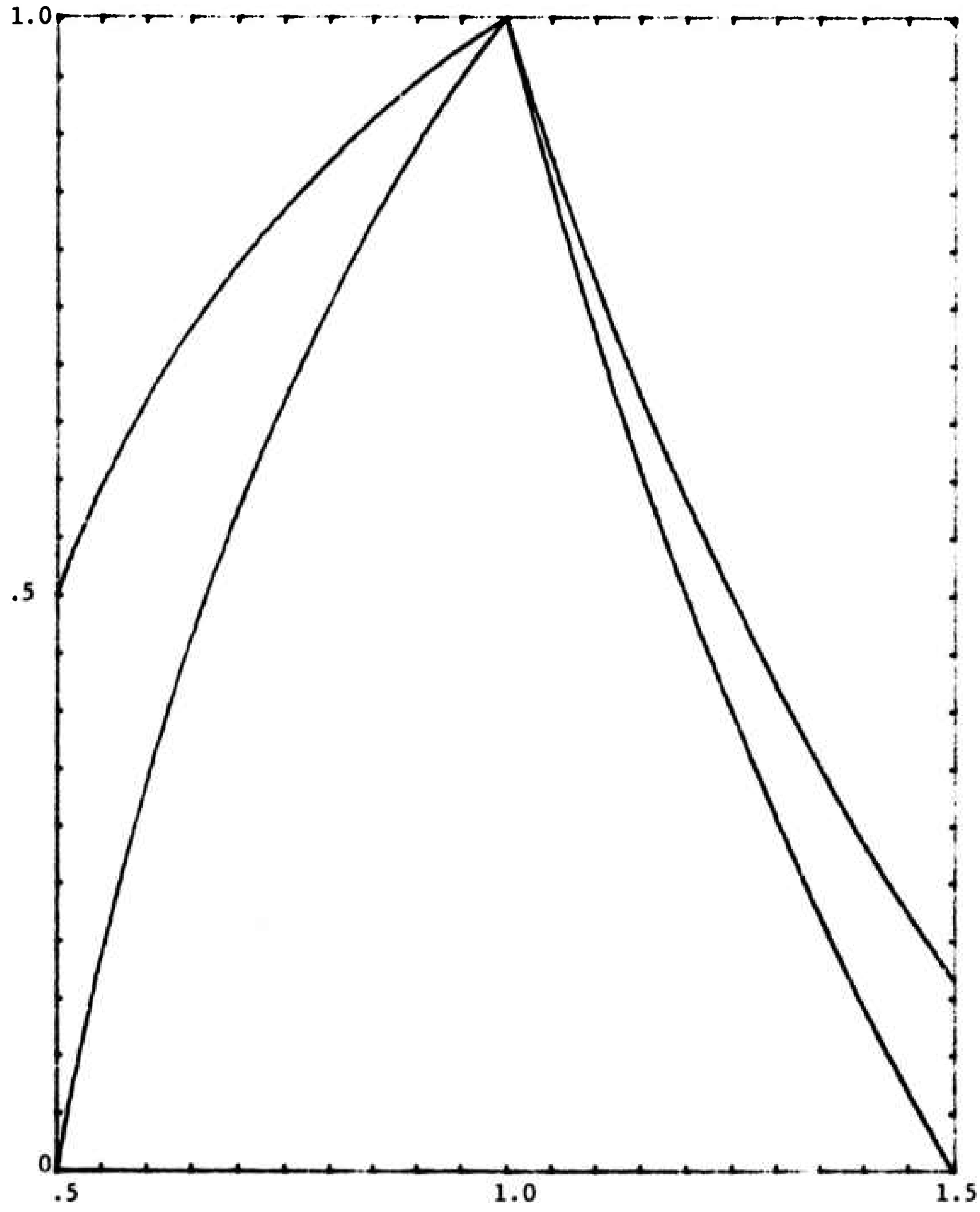


Figure 3.2 Lower bound for ARSAC Doppler gain, G, vs. ξ, for $P_{max} = 1.5\ C_{s_d}(0)$ (upper curve), and $\bar{P}_{max} = 2\ C_{s_d}(0)$ (lower curve).

B. The Discrete Time ARSAC Receiver

As before, the carrier signal $s(t)$ will be assumed to be a sampled and held version of a PRS of period K, but with this system, the first K bits of the sequence must be transmitted alternately forwards and backwards. By defining $\overline{t}_{K,N}$ as $\overline{t}_{K,N} \equiv J + [\frac{J}{K}](2K - 2J - 1)$, with $J = [\frac{t}{N}]$ modulo $2K$, the carrier signal may be taken as $s(t) = s_{\overline{(t/T)}_{K,N}}$ which will be shortened to $s_{\overline{t/T}}$ when no confusion will result. Here, N and T have the same meaning as in the previous section, so that one bit of the PRS is transmitted every NT seconds. A graph of the function $\overline{\xi t}$ versus t can be seen in Figure 3.3, for $\xi \in \{.75, 1, 1.25\}$.

The ARSAC transmitter is illustrated schematically in Figure 3.4. The transmitter output is $\alpha_k = m_{(t/T)_k} \overline{s_{t/T}}$, where α_k denotes $\alpha(t)$ for $kNT \leq t < (k + 1)NT$, $k = 0, 1, 2\ldots$, and t_K denotes $[\frac{t}{KN}]$.

At the receiver, the input is sampled every T seconds, providing an input sequence $\{R_n\}$, with each sample entered into a shift register train composed of two shift registers, each of length L, one of which shifts to the right, and the other to the left, as is illustrated in Figure 3.5.

The receiver output Y_n is

$$Y_n = \sum_{P=0}^{L-1} a_n^P = \sum_{P=0}^{L-1} R_{n-P} R_{n-2L+P+1}$$

With Doppler ξ, $R_n = \alpha(\xi nT) = \alpha_{[\frac{\xi n}{N}]} = m_{(\xi n)_K} s_{\overline{\xi n}}$, so that Y_n is given by

$$Y_n = \sum_{P=0}^{L-1} m_{(\xi(n-P))_K} m_{(\xi(n-2L+P+1))_K} s_{\overline{\xi(n-P)}} s_{\overline{\xi(n-2L+P+1)}} \tag{3.14}$$

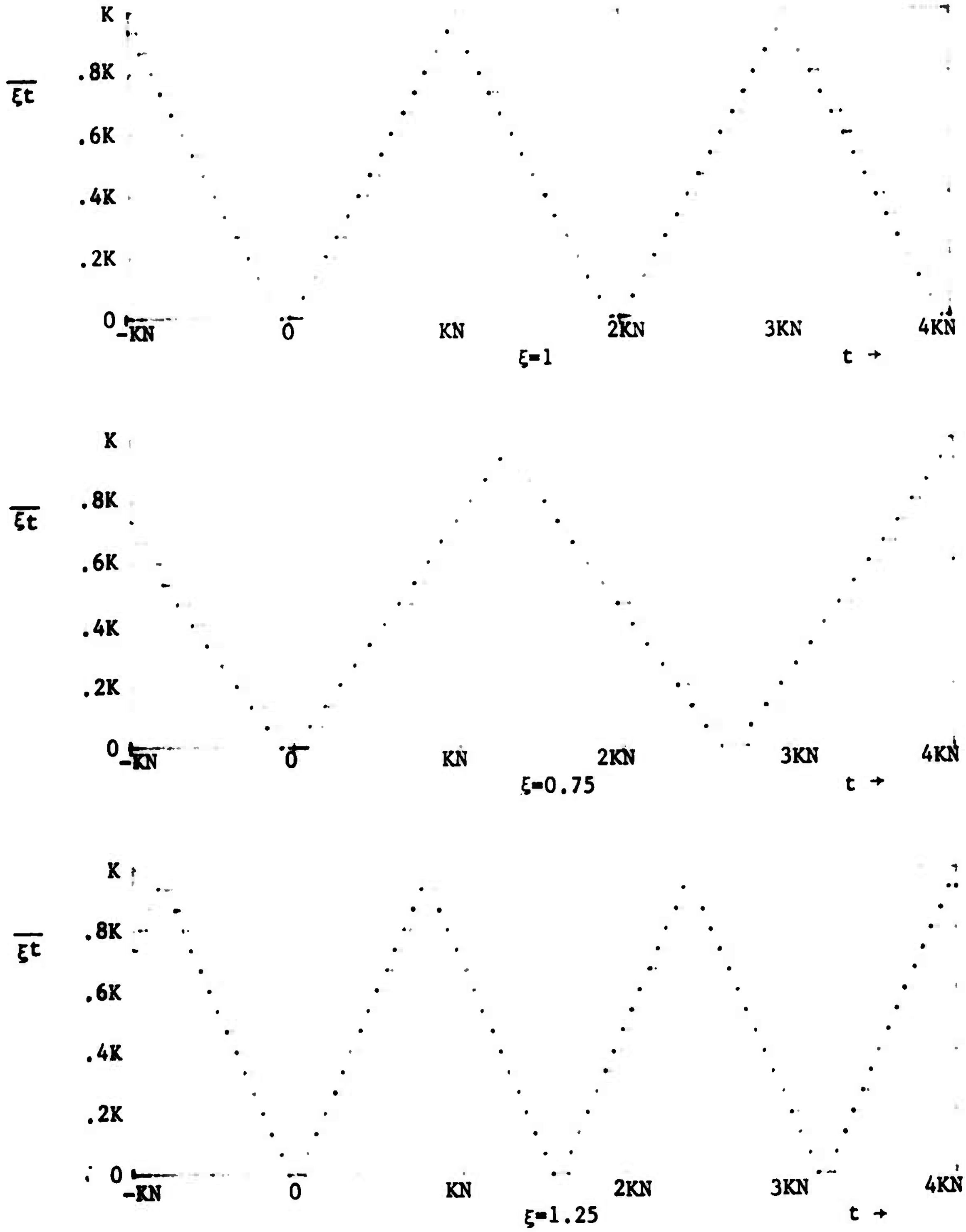


Figure 3.3 $(\xi t)_{K,N}$ (K = 15) vs. t

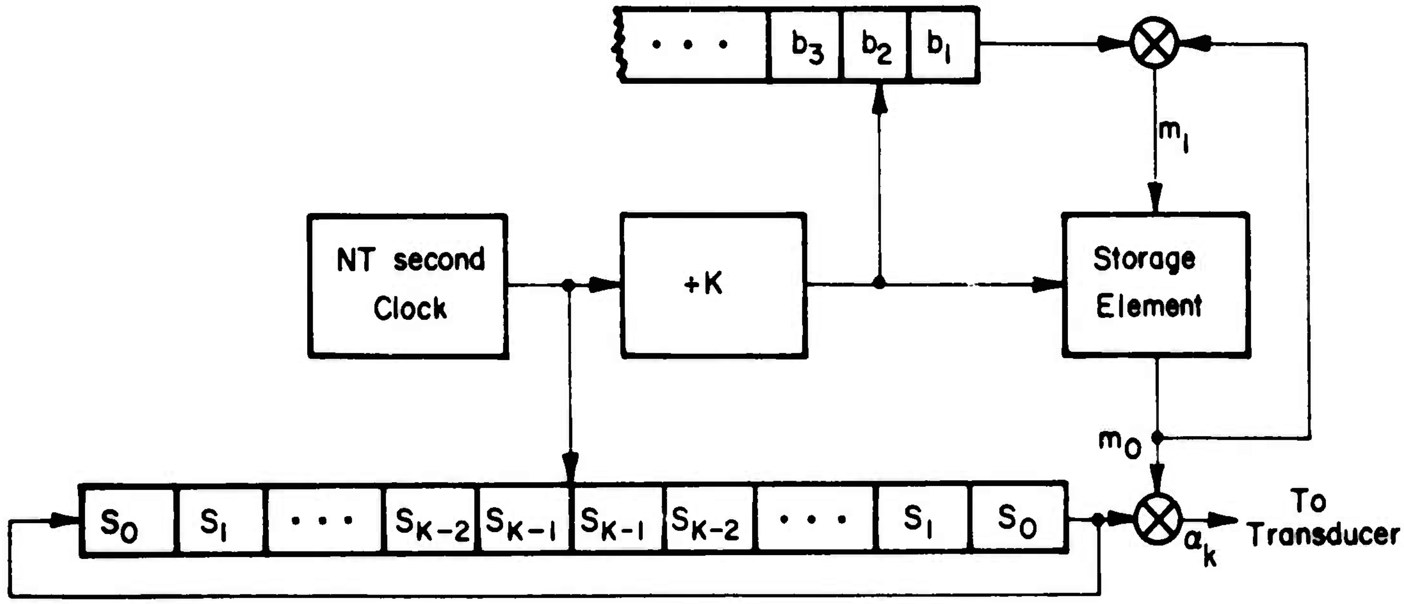


Fig. 3.4 Discrete-time ARSAC transmitter

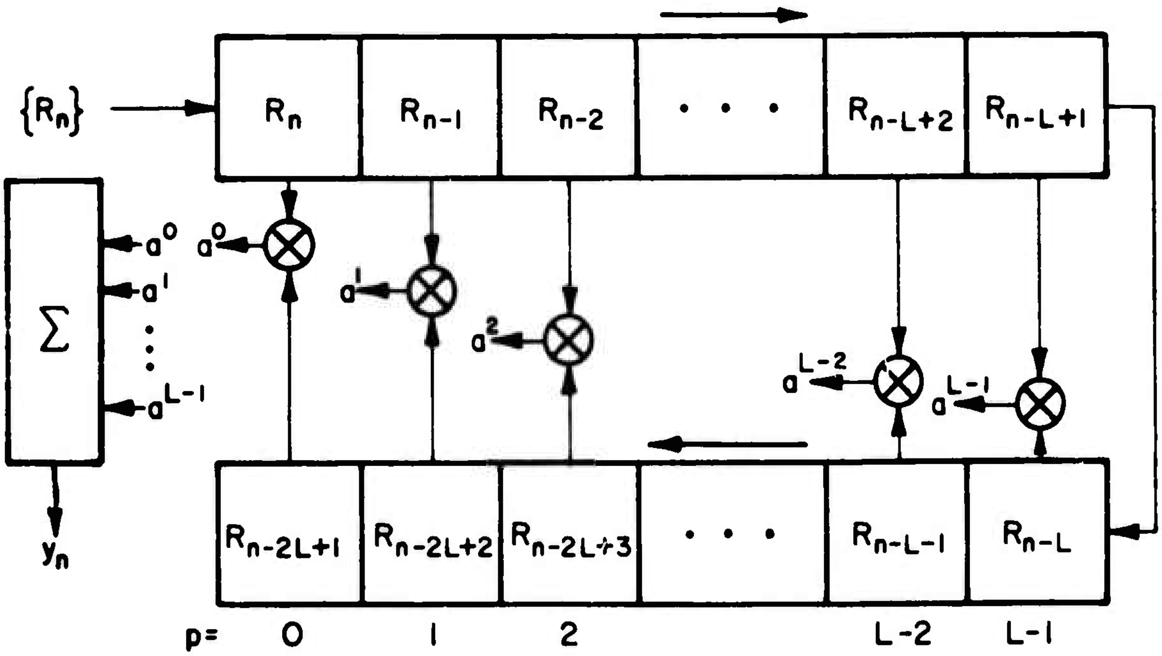


Fig. 3.5 Discrete-time ARSAC receiver

Equation (3.14) is valid for all $n \geq 0$ if it is agreed that $m_k = s_k = 0$ for $k < 0$.

Figure 3.6 illustrates a typical output sequence for $K = 15$, $N = 5$, $L = KN = 75$, and $\xi = 0.81$. Note that the output becomes decorrelated between the peaks, which are manifested distinctly above what might be termed correlation "noise". This represents another advantage over the FLAC system, in that estimates of the information bits may be made directly from the output amplitude, whereas such estimates must be made at specific times when using the FLAC receiver.

The positions of the peaks in time, and the relative spacing between them may be determined somewhat heuristically by the following argument: After operation of the receiver has begun, the first data sample reaches the $(L - 1)$st position of the first shift register after $L - 1$ steps. Thereafter, an entire PRS group of K bits passes through this position and into the second shift register approximately every NK/ξ steps. Thus, one might expect that a peak will occur in the output sequence Y_n whenever n is an integer close to $L - 1 + kNK/\xi$, $k = 1, 2, 3, \ldots$. It will be shown below that this integer is actually $n = \{L - 1 + \frac{kNK}{\xi}\}$, where $\{x\}$ denotes the smallest integer $\geq x$.

In order to visualize the effects of Doppler, assume for the moment that $m_k = +1$ for all k and define Δn as

$$\Delta n = L - 1 + \frac{kNK}{\xi} - n, \quad k = 1, 2, 3, \ldots,$$

where n is taken as $n = \{L - 1 + kNK/\xi\}$. Then $-1 < \Delta n \leq 0$, and the subscripts of s in (3.14) become

$$\overline{\xi(n - P)} = \overline{\xi(L - 1 + \frac{kNK}{\xi} - \Delta n - P)} = \overline{kNK + \xi(L - P) - \xi(1 + \Delta n)}$$

and

$$\overline{\xi(n - 2L + P + 1)} = \overline{\xi(L - 1 + \frac{kNK}{\xi} - \Delta n - 2L + P + 1)} = \overline{kNK - (\xi(L - P) + \xi\Delta n)}.$$

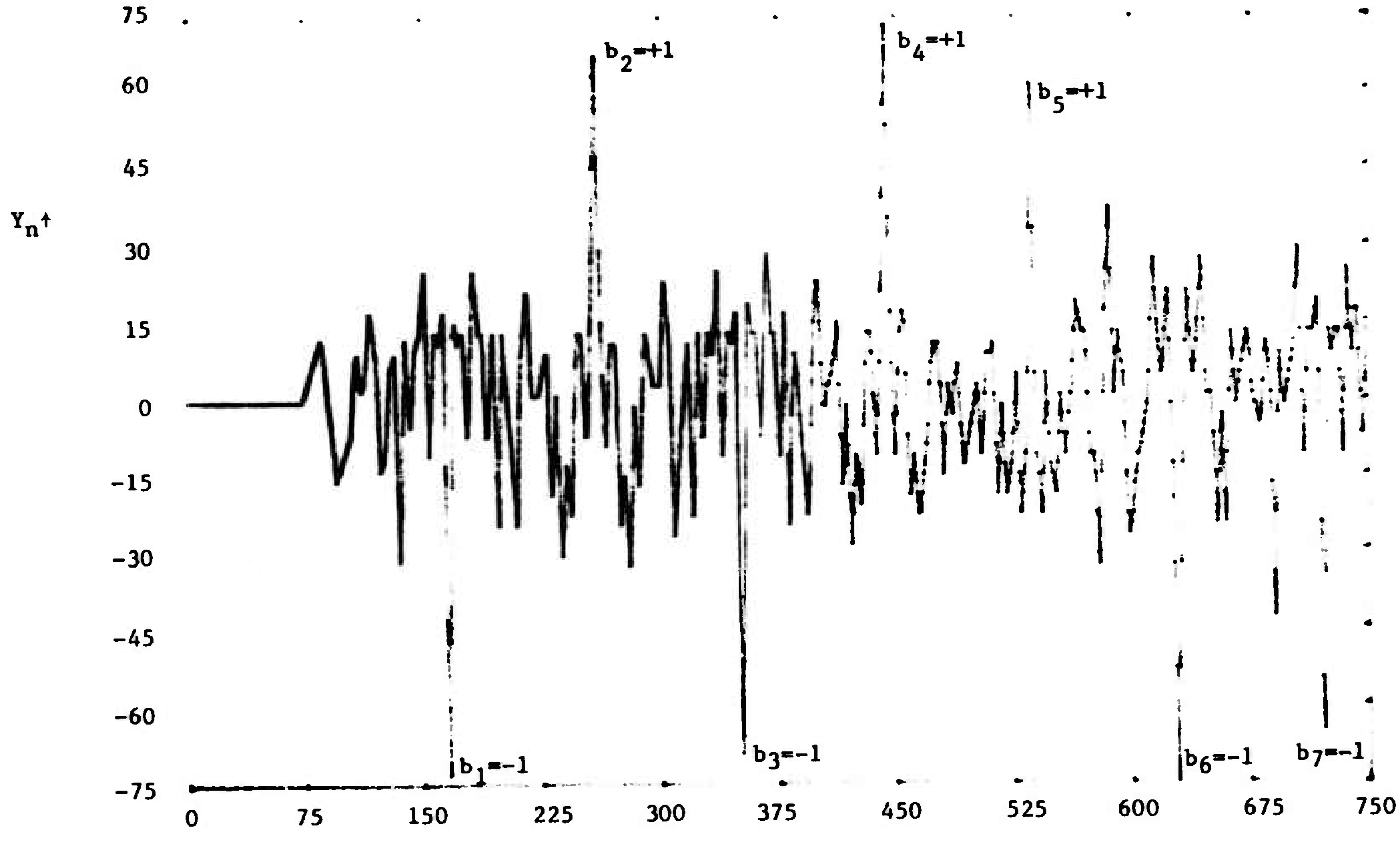


Figure 3.6 Typical ARSAC Output

Figure 3.3 indicates that

$$\overline{kNK - x} = \overline{kNK + x}$$

for all x such that

$$mN < x < (m + 1)N, \quad m = 0, 1, 2, \ldots,$$

(equality may hold on the left for $m = 0$), so that the subscripts in (3.14) will be equal for P such that

$$mN < \xi(L - P) - \xi(1 + \Delta n) < (m + 1)N$$

and

$$mN < \xi(L - P) + \xi\Delta n < (m + 1)N.$$

Choosing the tighter upper and lower bounds for $\xi(L - P)$ begets

$$mN + \xi(1 + \Delta n) < \xi(L - P) < (m + 1)N - \xi\Delta n, \quad -1/2 < \Delta n \leq 0$$

and

$$mN - \xi\Delta n < \xi(L - P) < (m + 1)N + \xi(1 + \Delta n), \quad -1 < \Delta n \leq -1/2$$

whence

$$L + \Delta n - (m + 1)\frac{N}{\xi} < P < L - 1 - \Delta n - m\frac{N}{\xi}, \quad -1/2 < \Delta n \leq 0 \tag{3.15}$$

$$L - 1 - \Delta n - (m + 1)\frac{N}{\xi} < P < L + \Delta n - m\frac{N}{\xi}, \quad -1 < \Delta n \leq -1/2 \tag{3.16}$$

The rationale for the particular choice of n may be interjected here. For consideration of equations (3.15) and (3.16) indicates that the greatest possibility of satisfying these inequalities occurs whenever $\Delta n = -1/2$. Since the difference between the maximum and minimum values of Δn must be at least one, the optimum range of Δn is $-1 < \Delta n \leq 0$, as was chosen.

To insure the existence of an integer between the bounds in (3.15) and (3.16) for every m requires that the segment width defined by these bounds be greater than 1, which in turn requires that $N > 2\xi$. Thus, if the maximum possible value of ξ is denoted as ξ_{max}, it is necessary that $N > 2\xi_{max}$ for reasonably good operation of the system.

Note that the larger the value of N, the more integers will lie inside the segments, with resulting better system performance.

The above discussion indicates that bit misalignments will occur at the Pth position of the correlator shift registers whenever P satisfies

$$L-1-\Delta n-\frac{mN}{\xi} \le P \le L+\Delta n-m\frac{N}{\xi}, \qquad -1/2 < \Delta n \le 0 \tag{3.17}$$

or

$$L+\Delta n-m\frac{N}{\xi} \le P \le L-1-\Delta n-m\frac{N}{\xi}, \qquad -1 < \Delta n \le -1/2 \tag{3.18}$$

for $m = 1, 2, 3, \ldots, [\frac{\xi}{N}(L+\Delta n)]$, $-1/2 \le \Delta n \le 0$. The lengths of the intervals in (3.17) and (3.18) are given by

$$0 < 1 + 2\Delta n \le 1, \qquad -1/2 < \Delta n \le 0$$

and

$$0 \le -1-2\Delta n < 1, \qquad -1 < \Delta n \le -1/2$$

Hence, at most two integers may lie within the interval defined by (3.17), so that, for the worst possible case, a total of

$$2[\frac{\xi}{N}(L+\Delta n)] \quad (\text{for } -\frac{1}{2} \le \Delta n \le 0)$$

bit misalignments may exist in the shift registers.

Peak degradation will occur due to these misalignments only if the appropriate adjacent bits of s are of opposite polarity, as is shown in Figure 3.7, an illustration of a small section of the correlator shift registers.

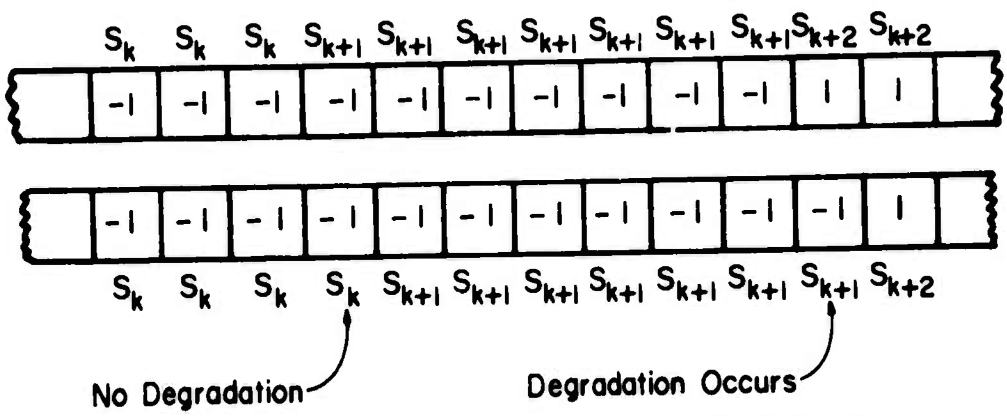


Figure 3.7 Bit misalignment causing peak degradation

If degradation does occur, the magnitude of the peak will be lessened by 2 (+1 is subtracted instead of added) for each bit which is misaligned and of polarity opposite that of an adjacent bit. Letting $\Delta_K(R)$ denote the maximum number of polarity reversals in R bits of a PRS of period K (with successive groups of K bits transmitted alternately forwards and backwards), the maximum amount of degradation ΔY_d which may occur due to Doppler is (2 bits misaligned per reversal) x (2 subtracted per misaligned bit) x (number reversals), so that

$$\Delta Y_d = 4\Delta_K(\{\xi L/N\}) \tag{3.19}$$

The value $\{\xi L/N\}$ is simply the number of PRS bits which "fit" into each of the shift registers whenever a peak occurs.

As an example, suppose $K = 7$, and that the PRS sequence is transmitted as

$$\overbrace{\underset{s_0}{-1}, \underset{s_1}{1}, -1, -1, 1, 1, \underset{s_6}{1}}^{7}, \overbrace{\underset{s_6}{1}, \underset{s_5}{1}, \underset{\cdot}{1}, \underset{\cdot}{-1}, \underset{\cdot}{-1}, 1, -1}^{7}, \overbrace{-1, 1, -1, -1, \ldots}^{7}$$

Supposing further that $L = NK$, and $\xi = 0.8$, then (3.19) is evaluated as

$$\Delta Y_d = 4\Delta_7\,(\{0.8 \cdot 7\}) = 4\Delta_7\,(\{5.6\}) = 4\Delta_7\,(6).$$

In the sequence $s_0, s_1, \ldots, s_5$ there are 3 polarity reversals, whereas in the $s_6, s_5, \ldots s_1$ there are only 2 such reversals. Thus $\Delta_7(6) = 3$, and (3.19) evaluates to $\Delta Y_d = 12$.

In order to prove the advantages of the ARSAC receiver with regard to Doppler, let the normalized peak amplitude Y' be defined as

$$Y' = \frac{L - \Delta Y_d}{N}.$$

Now, if L is proportional to N, i.e., $L = BN$, then Y' becomes

$$Y' = B - \frac{\Delta Y_d}{N} = B - \frac{4\Delta_K(\{\xi B\})}{N}$$

which tends to B as N tends to infinity, independently of ξ. Thus, relative peak degradation due to Doppler (with $m_k = +1$ for all k, as was assumed earlier) can be made arbitrarily small in the ARSAC system by increasing the sampling rate at the receiver.

With the restriction that $m_k = +1$ removed, the above analysis may have shortcomings, since the value of Y_n, for $n = \{L - 1 + kNK/\xi\}$, may depend not only upon m_k and m_{k-1}, but also upon m_{k+1} and m_{k-2}. The continuous time analog of this problem was treated in Section IV-B.

An analytical description of the situation is achieved by considering the subscripts of m in (3.14), the first of which becomes, for n as given above,

$$(\xi(n-P))_K = (kNK + \xi(L-P) - \xi(1+\Delta n))_K$$

Employing the definition of t_K, this equation becomes

$$(\xi(n-P))_K = [\frac{kNK + \xi(L-P) - \xi(1+\Delta n)}{KN}] = k + [\frac{\xi(L-P) - \xi(1+\Delta n)}{KN}]$$

Hence, the subscript under consideration remains constant at k if

$$0 \leq \frac{\xi(L-P) - \xi(1+\Delta n)}{KN} < 1,$$

which is satisfied for $L - 1 - \Delta n - K\frac{N}{\xi} < P \leq L - 1$.

Similar treatment for the second subscript of m indicates that $(\xi(n - 2L + P + 1))_K$ remains constant at $k - 1$ if

$$L + \Delta n - K\frac{N}{\xi} \leq P \leq L - 1,$$

so that, since $-1 < \Delta n \leq 0$, both inequalities are satisfied for

$$\{L - K\frac{N}{\xi}\} \leq P \leq L - 1 \tag{3.20}$$

For P outside this range the two subscripts become $k + 1$, and $k - 2$, respectively. Therefore, under the worst possible conditions, which correspond to the products $m_k\, m_{k-1}$ and $m_{k+1}\, m_{k-2}$ being of opposite polarity, and to the exact alignments of all PRS bits in the Pth position of the shift registers, for $0 \leq P < \{L - K\frac{N}{\xi}\} \equiv j$, the peak amplitude is degraded by

$$\Delta Y_m = 2\sum_{P=0}^{j} 1 = \begin{cases} 2(j+1), & j > 0 \\ 0, & j \leq 0 \end{cases}$$

Consideration of equation (3.20) shows that, if L is such that $L - KN/\xi \leq 0$, then the degradation ΔY_m will be zero. Thus, L can be chosen so that $L \leq KN/\xi$, which is satisfied for $L = [KN/\xi_{max}]$. If L is chosen in this manner, only the degradation due to bit misalignments will contribute to peak amplitude reduction.

In the above analysis it has been assumed that the received sequence $\{R_n\}$ is composed of signals emanating from a single transmitter, and that no interference is present from other systems in operation near the receiver. If such is not the case, system operation may be impaired to the point that the desired signal is totally undetectable. Consideration of this problem and of methods for achieving its remedy will be undertaken in the following section.

C. Interference Sensitivity: The ARSAC System

For convenience, let it be assumed at the outset that the encoded message bits are +1 for all time for both the desired and the undesired signal. Then the transmitted signals are given by

$$s_{\overline{(t/T)}_{K_d,N_d}} \quad \text{and} \quad x_{\overline{(t/T)}_{K_u,\,N_u}}$$

and (3.14) becomes

$$\begin{aligned} Y_n &= \sum_{P=0}^{L-1} s_{\overline{(\xi_d(n-P))}_{K_d,N_d}} \, s_{\overline{(\xi_d(n-2L+P+1))}_{K_d,N_d}} \\ &+ \sum_{P=0}^{L-1} s_{\overline{(\xi_d(n-P))}_{K_d,N_d}} \, x_{\overline{(\xi_u(n-2L+P+1))}_{K_u,N_u}} \\ &+ \sum_{P=0}^{L-1} s_{\overline{(\xi_d(n-2L+P+1))}_{K_d,\,N_d}} \, x_{\overline{(\xi_u(n-P)}_{K_u,N_u}} \\ &+ \sum_{P=0}^{L-1} x_{\overline{(\xi_u(n-P))}_{K_u,N_u}} \, x_{\overline{(\xi_u(n-2L+P+1))}_{K_u,\,N_u}} \end{aligned} \quad (3.21)$$

Again it will be assumed that the second and third sums are negligible, so that only the first and fourth terms contribute to Y_n. No further analysis is required; from the results of Section III-B one sees that the fourth term, which corresponds to the interference, peaks at

$$n = \{L - 1 + k\, K_u N_u/\xi_u\},$$

for $k = 1, 2, 3, \ldots$. Furthermore, the maximum possible value of this term, L, is the same as that for the first, desired term. Hence, the peak due to the desired signal and the peak due to interference may be indistinguishable in amplitude, and since ξ_d and ξ_u are not in general controllable, the relative position in time of these two peaks cannot be dependably established.

Although the above results were deduced with the assumption that the message bits for each signal were constant at +1, an unreasonable assumption, it is apparent that an equivalent situation may occur if a string of +1's or -1's occurs in the message bit sequence corresponding to the interference signal. Therefore, it must be concluded that the ARSAC system exhibits poor interference characteristics.

IV. Noise Performance

In this section we investigate the noise performance for the family of predistorted replica correlation receivers. We will be concerned with the computation of the cumulative distribution function (cdf) of the statistic z, given by

$$z = \frac{1}{N} R_1^T R_2, \tag{4.1}$$

where $R_i = S_i + N_i$

$$R_i^T = (r_{i,1}, r_{i,2}, \ldots, r_{i,N}),$$

$$S_i^T = (s_{i,1}, s_{i,2}, \ldots, s_{i,N}),$$

$$N_i^T = (n_{i,1}, n_{i,2}, \ldots, n_{i,N}),$$

$n_{i,\ell}$, $i \in \{1, 2\}$, $\ell \in \{1, 2, \ldots, N\}$ is normally distributed with mean zero and variance σ_i^2 (we will use the notation $n_{i,\ell} \sim N(0, \sigma_i^2)$), and $E\{n_{i,\ell}\, n_{j,k}\} = \sigma_i^2\, \delta_{i,j}\, \delta_{k,l}$. S_i is the (deterministic) signal component of the received data vector, R_i. Several solutions to similar problems are available in the engineering literature (e.g. see [9] - [11]); however, since the results treated there are not applicable to the problem at hand, they will not be dealt with here.

We begin our analysis by computing the characteristic function $\phi_z(\xi)$, defined by

$$\phi_z(\xi) = E\{e^{i\xi z}\}, \tag{4.2}$$

where $E\{\cdot\}$ denotes statistical expectation. Feller [12] suggests that if x_1 and x_2 are independent random variables with cdf's F_1 and F_2, and characteristic functions (ch.f.'s) ϕ_1 and ϕ_2, respectively, then $y = x_1 x_2$ has the ch.f. ϕ_y given by

$$\phi_y(\xi) = \int_{-\infty}^{\infty} \phi_2(\xi x)dF_1(x) = \int_{-\infty}^{\infty} \phi_1(\xi x)dF_2(x). \tag{4.3}$$

Making use of (4.3) and the fact that $\phi_{r_{i,k}}(\xi) = \exp\{j\ s_{i,k}\xi - \frac{1}{2}\ \sigma_i^2\ \xi^2\}$,

we find that the ch. f. for the random variable $y_k = r_{1,k}\ r_{2,k}$ is

given by

$$\phi_{y_k}(\xi) = \exp\left\{\frac{j\ s_{1,k}\ s_{2,k}\xi - \frac{1}{2}\ (\sigma_2^2\ s_{1,k}^2 + \sigma_1^2\ s_{2,k}^2)\xi^2}{1+\sigma_1^2\ \sigma_2^2\ \xi^2}\right\}$$

$$\cdot \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{-(1+\sigma_1^2\ \sigma_2^2\ \xi^2)}{2\sigma_1^2} \left(x - \frac{s_{1,k} + j\ s_{2,k}\ \sigma_1^2\ \xi}{1+\sigma_1^2\ \sigma_2^2\ \xi^2}\right)^2\right\}dx$$

or

$$\phi_{y_k}(\xi) = (1+\sigma_1^2\ \sigma_2^2\ \xi^2)^{-1/2} \exp\left\{\frac{j\ s_{1,k}\ \xi - \frac{1}{2}\ (\sigma_2^2\ s_{1,k}^2 + \sigma_1^2\ s_{2,k}^2)\xi^2}{1+\sigma_1^2\ \sigma_2^2\xi^2}\right\}. \tag{4.4}$$

Now, defining

$$C_1 = S_1^T S_2,$$

$$C_2 = \sigma_2^2\ S_1^T\ S_1 + \sigma_1^2\ S_2^T S_2, \tag{4.5}$$

and noting that our independence assumption implies that

$\phi_{Nz}(\xi) = \prod_{k=1}^{N} \phi_{y_k}(\xi)$, we have

$$\phi_z(N\xi) = (1+\sigma_1^2\ \sigma_2^2\ \xi^2)^{-\frac{N}{2}} \exp\left\{\frac{j\ C_1\xi - \frac{1}{2}\ C_2\xi^2}{1+\sigma_1^2\ \sigma_2^2\ \xi^2}\right\}, \tag{4.6}$$

since $\phi_{Nz}(\frac{\xi}{N}) = \phi_z(\xi)$. Defining the standardized random variable, η, by

$$\eta = \frac{z - \mu_z}{\sigma_z}, \tag{4.7}$$

where

$$\mu_z = E\{z\} = \frac{C_1}{N},$$

and

$$\sigma_z^2 = E\{z^2\} - \mu_z^2 = \frac{\sigma_1^2\,\sigma_2^2}{N} + \frac{C_2}{N^2},$$

we obtain

$$\phi_\eta(\tfrac{\xi}{\beta}) = (1+\xi^2)^{-\frac{N}{2}} \exp\left\{\frac{-j\,\beta_1\xi^3 - \frac{1}{2}\beta_2\,\xi^2}{1+\xi^2}\right\}, \tag{4.8}$$

where $\beta = \sigma_1\,\sigma_2/N\sigma_z$, $\beta_1 = C_1/\sigma_1\sigma_2$, and $\beta_2 = C_2/\sigma_1^2\sigma_2^2$. Unfortunately, the ch.f. (4.8) is not easily inverted to obtain the denisty function; however, all of the information about the density function is contained in (4.8). From (4.8) we can obtain all of the moments, $\mu_k = E\{\eta^k\}$, using the relationship

$$\mu_k = (-j)^k \left.\frac{d^k\phi_\eta(t)}{dt^k}\right|_{t=0}. \tag{4.9}$$

Computationally, a somewhat less tedious approach is to first compute the cumulants K_k, using the relationship

$$\log\,\phi_\eta(t) = \sum_{k=1}^{\infty} K_k \frac{(jt)^k}{k!}, \tag{4.10}$$

and noting that the moments, μ_k, and cumulants, K_k, are formally related by [13, p. 318]

$$\exp\,\{\sum K_k \frac{t^k}{k!}\} = \sum \mu_k \frac{t^k}{k!} \tag{4.11}$$

Kendall and Stuart [13, p. 69] have tabulated the first ten moments in terms of the first ten cumulants, which, for $K_1 = 0$ and $K_2 = 1$ are repeated in (4.12)

$$\begin{aligned}
\mu_1 &= 0\\
\mu_2 &= 1\\
\mu_3 &= K_3\\
\mu_4 &= K_4 + 3\\
\mu_5 &= K_5 + 10K_3
\end{aligned} \tag{4.12}$$

$$\mu_6 = K_6 + 15K_4 + 10K_3^2 + 15$$
$$\mu_7 = K_7 + 21K_5 + 35K_4K_3 + 105K_3$$
$$\mu_8 = K_8 + 28K_6 + 56K_5K_3 + 35K_4^2 + 210K_4 + 280K_3^2 + 105$$
$$\mu_9 = K_9 + 36K_7 + 84K_6K_3 + 126K_5K_4 + 378K_5 + 1260K_4K_3 + 280K_3^3 + 1260K_3$$
$$\mu_{10} = K_{10} + 45K_8 + 120K_7K_3 + 210K_6K_4 + 630K_6 + 126K_5^2 + 2520K_5K_3 + 1575K_4^2 + 2100K_4K_3^2 + 3150K_4 + 6300K_3^2 + 945 \quad (4.12)$$

The reader will recall that for a standardized normal random variable, the only nonzero cumulant is $K_2' = 1$ and the nonzero moments are given by

$$\mu'_{2\ell} = \frac{(2\ell)!}{2^\ell \ell!}, \quad \ell = 1, 2, 3, \ldots . \quad (4.13)$$

Knowledge of the first few moments of η will enable us to compute an Edgeworth expansion [14] for the probability density function f_η, from which an approximation to the cdf F_η is easily obtained. We now proceed in developing an expression for the cumulants K_k. Defining $h(\xi) = \log \phi_\eta(\xi/\beta)$, we have

$$h(\xi) = -\frac{N}{2}\log(1+j\xi) - \frac{N}{2}\log(1-j\xi) - \frac{j\beta_1\xi^3 + \frac{1}{2}\beta_2\xi^2}{1+\xi^2}. \quad (4.14)$$

Using standard partial fraction techniques, and noting that

$$\log(1 \pm j\xi) = \sum_{k=1}^{\infty} \frac{(\pm j\xi)^k (-1)^{k+1}}{k}, \quad (4.15)$$

we obtain

$$h(\xi) = \sum_{k=1}^{\infty} [(j\xi)^{2k} (\frac{N}{2k} + \frac{1}{2}\beta_2) + \beta_1 (j\xi)^{2k+1}]. \quad (4.16)$$

It follows that the cumulants, K_k, are given by

$$K_k = \begin{cases} \beta^k k!(\frac{N}{k} + \frac{1}{2}\beta_2), & k = 2, 4, 6, \ldots \\ \beta_1 \beta^k k!, & k = 3, 5, 7, \ldots \end{cases}, \quad (4.17)$$

with $K_1 = 0$.

Substituting (4.17) into (4.12) we obtain

$$
\begin{aligned}
\mu_1 &= 0 \\
\mu_2 &= 1 \\
\mu_3 &= 3!\beta_1\beta^3 \\
\mu_4 &= 4!\beta^4(\tfrac{N}{4} + \tfrac{1}{2}\beta_2) + 3 \\
\mu_5 &= 5!\beta_1\beta^5 + \tfrac{5!}{2}\beta_1\beta^3 \\
\mu_6 &= 6!\beta^6(\tfrac{N}{6} + \tfrac{1}{2}\beta_2 + \tfrac{1}{2}\beta_1^2) + \tfrac{6!}{2}\beta^4(\tfrac{N}{4} + \tfrac{1}{2}\beta_2) + 15 \\
\mu_7 &= 7!\beta_1\beta^7(1 + \tfrac{N}{4} + \tfrac{1}{2}\beta_2) + \tfrac{7!}{2}\beta_1\beta^5 + \tfrac{7!}{8}\beta_1\beta^3 \qquad (4.18) \\
\mu_8 &= 8!\beta^8[(1 + \tfrac{N}{4})(\tfrac{N}{8} + \tfrac{1}{2}\beta_2) + \tfrac{1}{8}\beta_2^2 + \tfrac{1}{6}\beta_1^2] \\
&\quad + \tfrac{8!}{2}\beta^6(\tfrac{N}{6} + \tfrac{1}{2}\beta_2 + \tfrac{1}{4}\beta_1^2) + \tfrac{8!}{8}\beta^4(\tfrac{N}{4} + \tfrac{1}{2}\beta_2) + 105 \\
\mu_9 &= 9!\beta_1\beta^9(1 + \tfrac{5}{12}N + \beta_2 + \tfrac{1}{6}\beta_1^2) + \tfrac{9!}{2}\beta_1\beta^7(1 + \tfrac{N}{4} + \tfrac{1}{2}\beta_2) \\
&\quad + \tfrac{9!}{8}\beta_1\beta^5 + \tfrac{9!}{48}\beta_1\beta^3 \\
\mu_{10} &= 10!\beta^{10}[(\tfrac{N}{4} + \tfrac{1}{2}\beta_2)(\tfrac{N}{6} + \tfrac{1}{2}\beta_2 + \beta_1^2) + \tfrac{N}{10} + \tfrac{1}{2}\beta_2 + \tfrac{3}{2}\beta_1^2] \\
&\quad + \tfrac{10!}{4}\beta^8[(\tfrac{N}{4} + \tfrac{1}{2}\beta_2)^2 + \tfrac{N}{4} + \beta_2 + 2\beta_1^2] + \tfrac{10!}{8}\beta^6(\tfrac{N}{6} + \tfrac{1}{2}\beta_2 + \tfrac{1}{2}\beta_1^2) \\
&\quad + \frac{10!\beta^4}{48}(\tfrac{N}{4} + \tfrac{1}{2}\beta_2) + 945
\end{aligned}
$$

The Edgeworth expansion is a series representation of a standardized probability density function in terms of Tchebycheff-Hermite polynomials, the first term of which is the standard normal density:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} . \qquad (4.19)$$

The motivation for considering an Edgeworth expansion rather than the formally equivalent Gram-Charlier Type A series lies in practical applications for which only a small number of terms are used [14]. In standardized measure, the first few terms of the Edgeworth expansion are found from [14, p. 229] to be

$$f(x) \approx \psi(x) - [\frac{1}{3!} K_3 \psi^{(3)}_{(x)}] + [\frac{1}{4!} K_4 \psi^{(4)}_{(x)} + \frac{10}{6!} K_3^2 \psi^{(6)}_{(x)}]$$

$$- [\frac{1}{5!} K_5 \psi^{(5)}_{(x)} + \frac{35}{7!} K_3 K_4 \psi^{(7)}_{(x)} + \frac{280}{9!} K_3^3 \psi^{(9)}_{(x)}] \quad (4.20)$$

An added benefit in using expansions such as (4.20) is the ease in obtaining an approximation to the cdf, F(x), in fact, with

$$\Psi_{(x)} = \int_{-\infty}^{x} \psi(x')dx' ,$$

$$F(x) = \int_{-\infty}^{x} f(x')dx' \approx \Psi_{(x)} - [\frac{1}{3!} K_3 \psi^{(2)}_{(x)}]$$

$$+ [\frac{1}{4!} K_4 \psi^{(3)}_{(x)} + \frac{10}{6!} K_3^2 \psi^{(5)}_{(x)}]$$

$$- [\frac{1}{5!} K_5 \psi^{(4)}_{(x)} + \frac{35}{7!} K_3 K_4 \psi^{(6)}_{(x)} + \frac{280}{9!} K_3^3 \psi^{(8)}_{(x)}],$$

$$(4.21)$$

or, after "simplification,"

$$F(x) \approx \Psi_{(x)} + \psi_{(x)}[(\beta_1\beta^3 - 3\beta_1\beta^5 + 15\beta_1\beta^7(\frac{N}{4} + \frac{1}{2}\beta_2) - \frac{35}{12}\beta_1^2\beta^6)$$

$$+ x(3\beta^4(\frac{N}{4} + \frac{1}{2}\beta_2) - \frac{15}{2}\beta_1^2\beta^6)$$

$$+ x^2(-\beta_1\beta^3 + 6\beta_1\beta^5 - 45\beta_1\beta^7(\frac{N}{4} + \frac{1}{2}\beta_2) + 70\beta_1^3\beta^9)$$

$$+ x^3(\beta^4(\frac{N}{4} + \frac{1}{2}\beta_2) + 5\beta_1^2\beta^6)$$

$$+ x^4(-\beta_1\beta^5 + 15\beta_1\beta^7(\frac{N}{4} + \frac{1}{2}\beta_2) - 35\beta_1^3\beta^9)$$

$$+ x^5(-\frac{1}{2}\beta_1^2\beta^6) + x^6(-\beta_1\beta^7(\frac{N}{4} + \frac{1}{2}\beta_2) + \frac{14}{3}\beta_1^3\beta^9)$$

$$+ x^8(-\frac{1}{6} \beta_1^3\beta^9)]. \quad (4.22)$$

At this point, it seems worthwhile to interpret the parameters we have been using. We will assume that the signal energy is the same in each channel of the correlator, or $S_1^T S_1 = S_2^T S_2$. Interpreting the input signal energy, $E_{s_{in}}$, as $\frac{1}{N} S_1^T S_1$, we find that β is given by:

$$\beta = \frac{1}{N^{\frac{1}{2}} \sqrt{1 + \gamma_1 \, SNR_i}} , \tag{4.23}$$

where we have defined $\gamma = \sigma_2^2/\sigma_1^2$, $\gamma_1 = (1 + \gamma)/\gamma$, and the input signal-to-noise ratio as $SNR_i = E_{s_{in}}/\sigma_1^2$. We now define a correlation factor, CORR, by

$$CORR = \frac{S_1^T S_2}{S_1^T S_1} \tag{4.24}$$

which is easily shown to satisfy:

$$-1 \leq CORR \leq 1.$$

Finally, we obtain

$$\beta_1 = \frac{N \cdot CORR \cdot SNR_i}{\sqrt{\gamma}} ,$$

and (4.25)

$$\beta_2 = N\gamma_1 SNR_i .$$

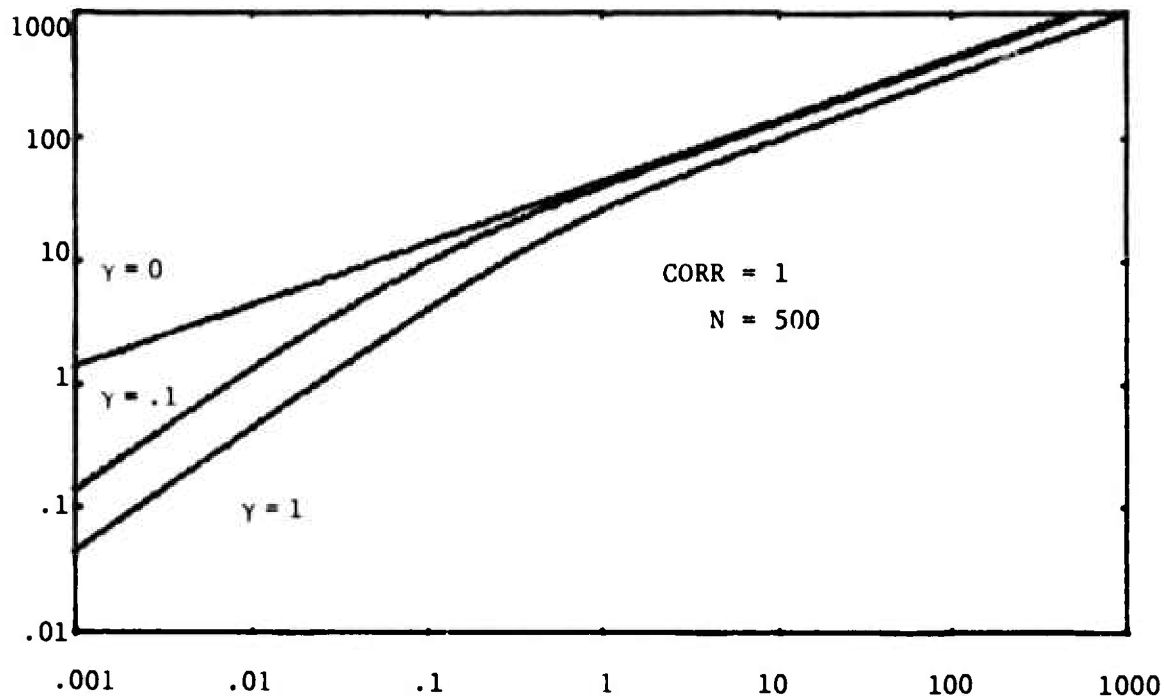
A suitable performance index for binary antipodal signaling is

$$\rho(\gamma,\ CORR,\ SNR_i, N) = \frac{2|\mu_z|}{\sigma_z} = \frac{2|CORR| \cdot N^{1/2} \cdot SNR_i}{\sqrt{\gamma + (1 + \gamma)\, SNR_i}} . \tag{4.26}$$

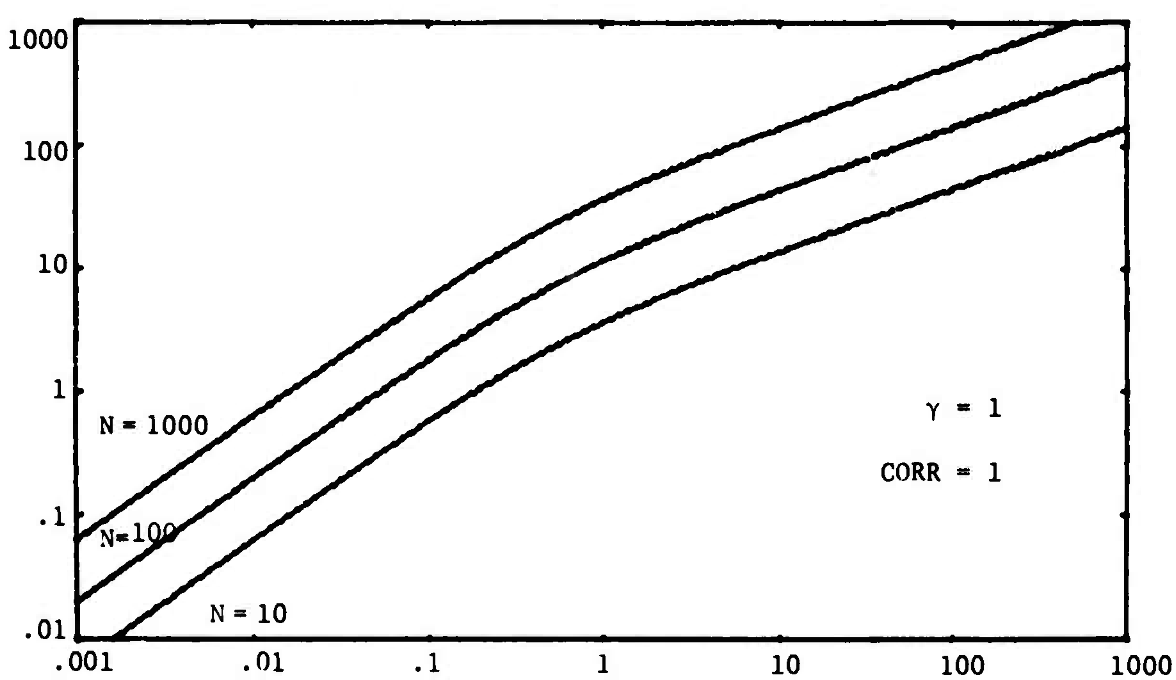
The above performance index (4.26) is illustrated in Figure 4.1. For a fixed reference correlation receiver, with $\gamma = 0$, and reference signal S_{ref}, such that $S_{ref}^T S_{ref} = S_1^T S_1$ and

$$\alpha = \frac{S_1^T S_{ref}}{S_1^T S_2}, \tag{4.27}$$

our performance index becomes



(a)



(b)

Figure 4.1 The performance index, ρ, vs. SNR_i

$$\rho_\alpha(0,\ \mathrm{CORR},\ \mathrm{SNR}_i, N) = 2\alpha|\mathrm{CORR}|\sqrt{N \cdot \mathrm{SNR}_i}. \tag{4.28}$$

We can now define an efficiency factor, EF, as

$$\mathrm{EF}(\gamma,\ \alpha,\ \mathrm{SNR}_i) = \frac{\rho(\gamma,\ \mathrm{CORR},\ \mathrm{SNR}_i, N)}{\rho_\alpha(0,\ \mathrm{CORR},\ \mathrm{SNR}_i,\ N)} = \frac{\sqrt{\mathrm{SNR}_i}}{\alpha\sqrt{\gamma + (1+\gamma)\mathrm{SNR}_i}} \tag{4.29}$$

The above efficiency factor (4.29) is shown in Figure 4.2. $\mathrm{EF}(\gamma,\ \alpha,\ \mathrm{SNR}_i)$ provides us with a strong indication of the tradeoffs involved in choosing between a predistorted replica correlation receiver and a fixed-reference correlation receiver.

With the above interpretations in mind, we return to the Edgeworth approximation of the cdf given in (4.22). Since $\Psi(x)$ is the well-tabulated standard normal integral, we need only concern ourselves with the non-normal component of the cdf, H(x), defined by

$$H(x) = F(x) - \Psi(x). \tag{4.30}$$

The Edgeworth approximation of (4.30) given by (4.22) is shown in Figure 4.3 for $|x| \leq 5$ and several choices of N, SNR_i, γ, and CORR. Only positive values of CORR were used since

$$H(x)\Big|_{-\mathrm{CORR}} = -H(-x)\Big|_{\mathrm{CORR}}. \tag{4.31}$$

Figure 4.3 suggests that H(x), the non-normal component of the cdf F(x) is not of large magnitude, and tends to decrease with increasing N. Consequently, an investigation of the behavior of $\phi_\eta(\xi)$ for large N would seem appropriate.

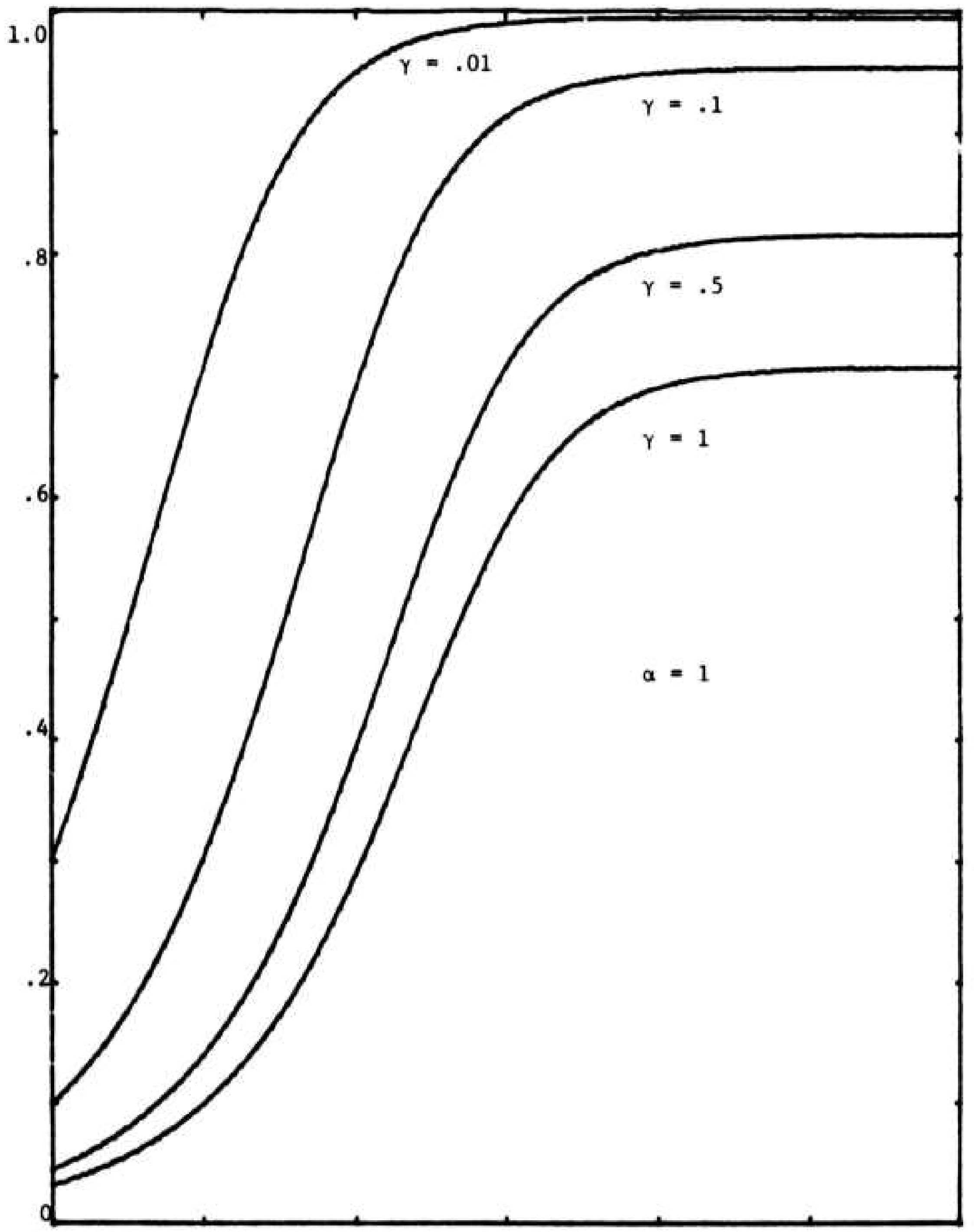
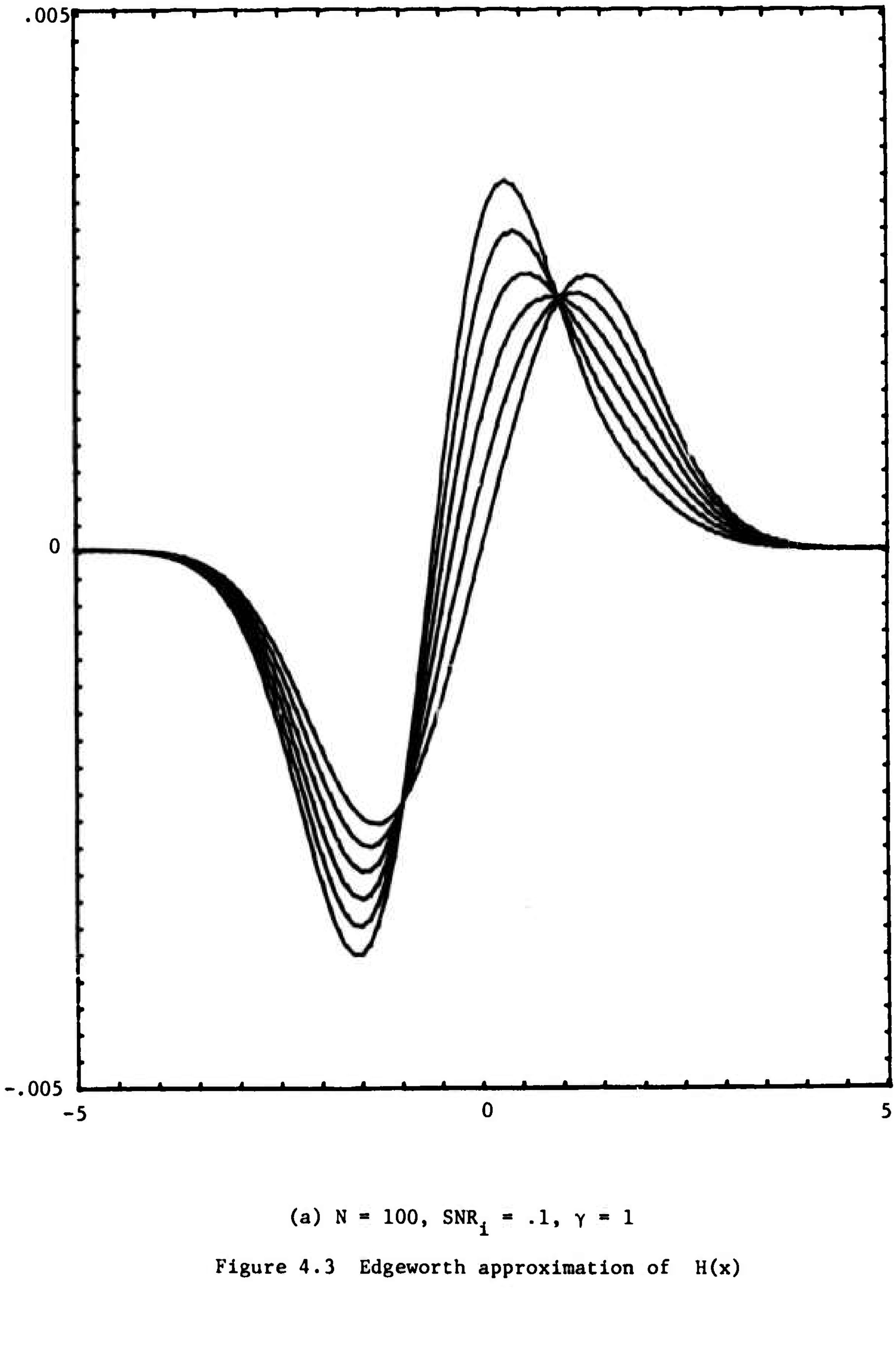
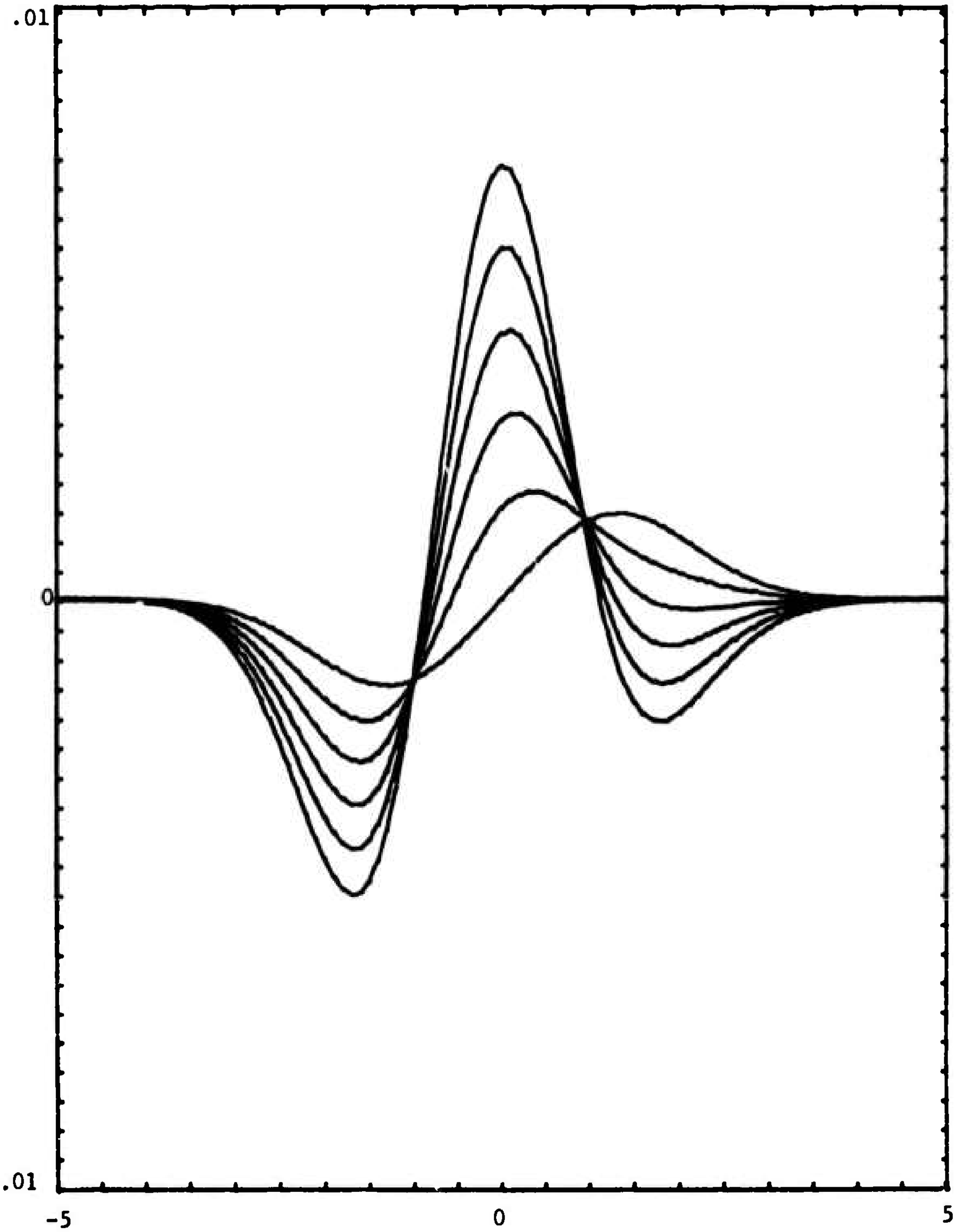


Figure 4.2 The efficiency factor, EF, vs. SNR_1

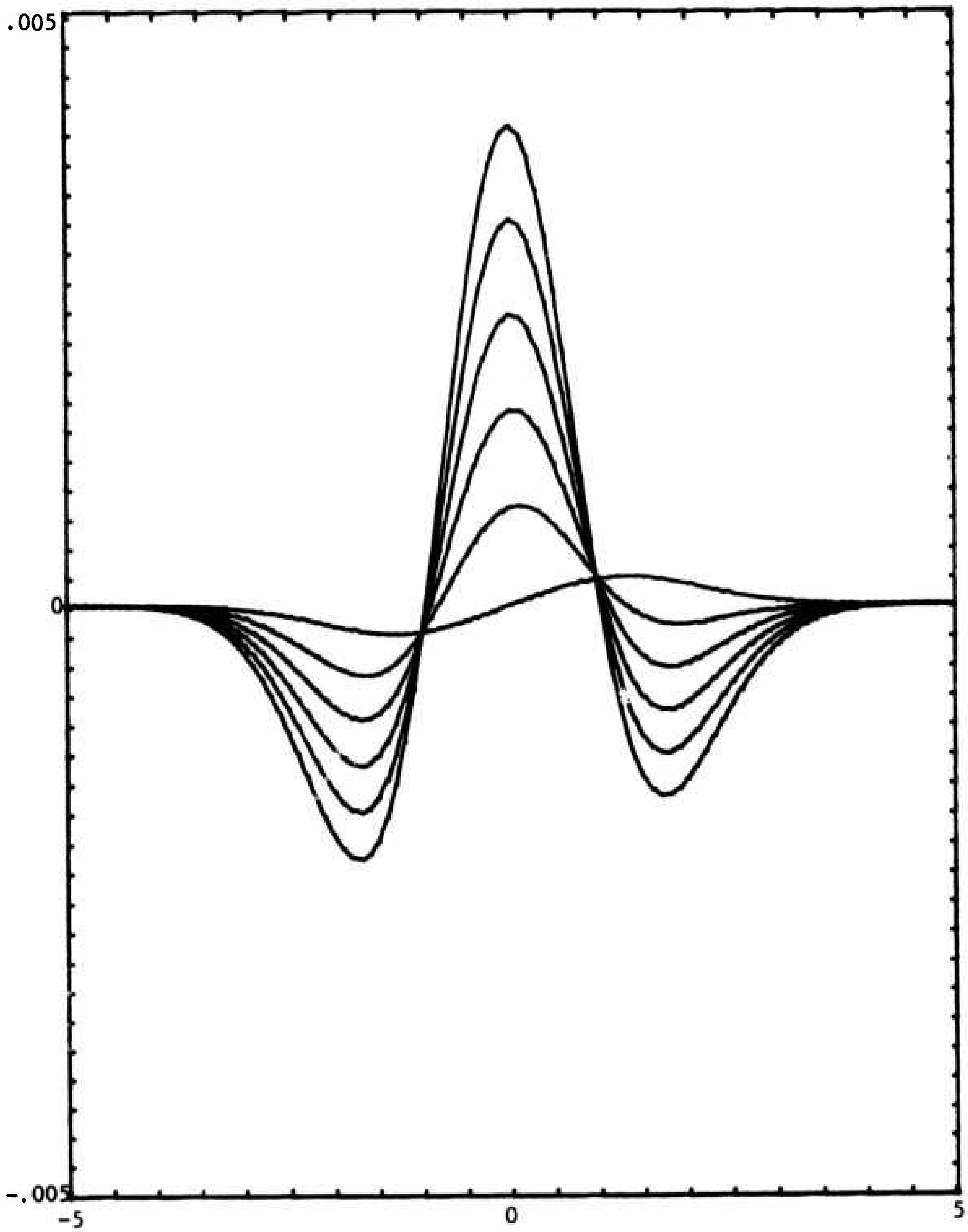


(a) $N = 100$, $SNR_i = .1$, $\gamma = 1$

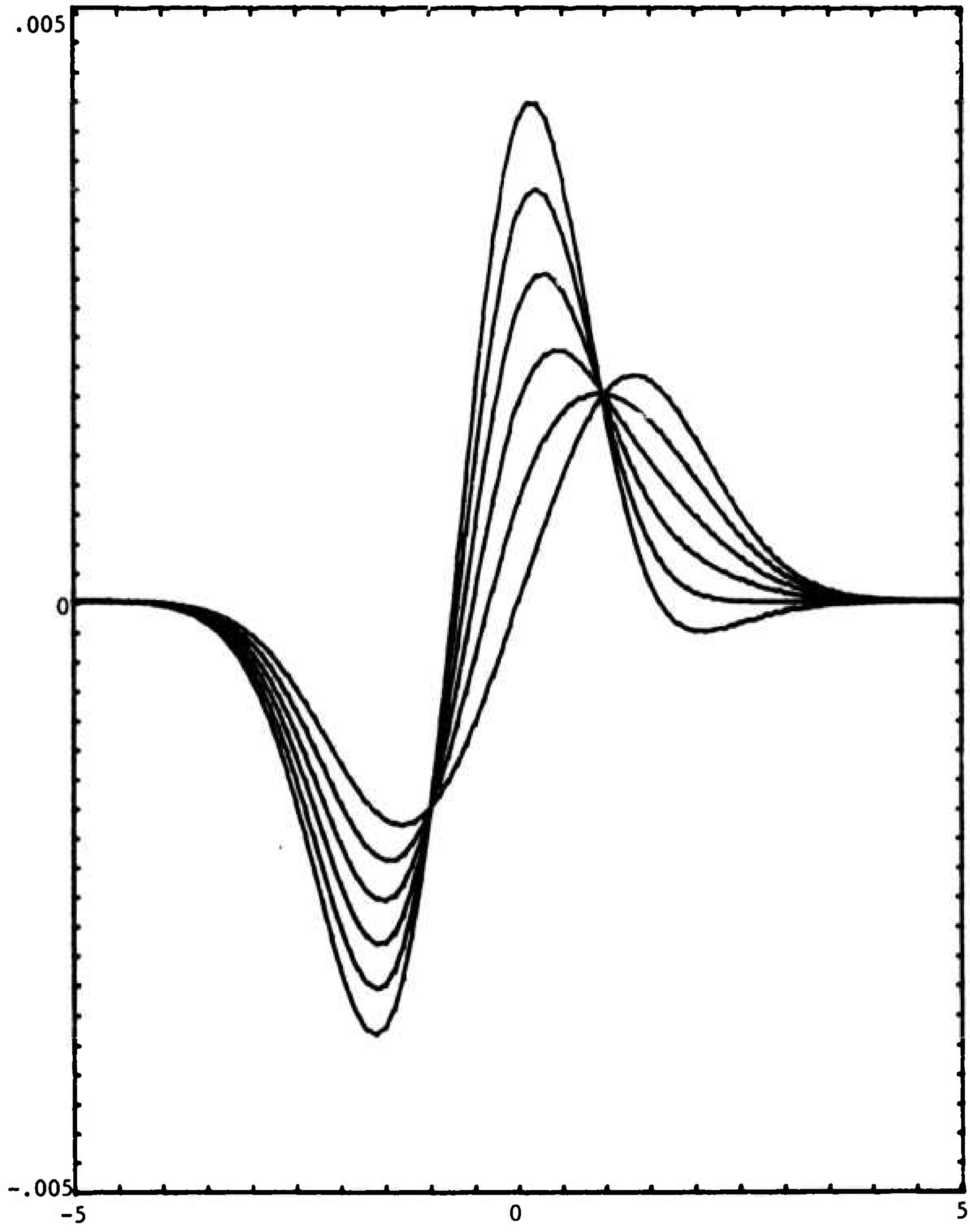
Figure 4.3 Edgeworth approximation of $H(x)$



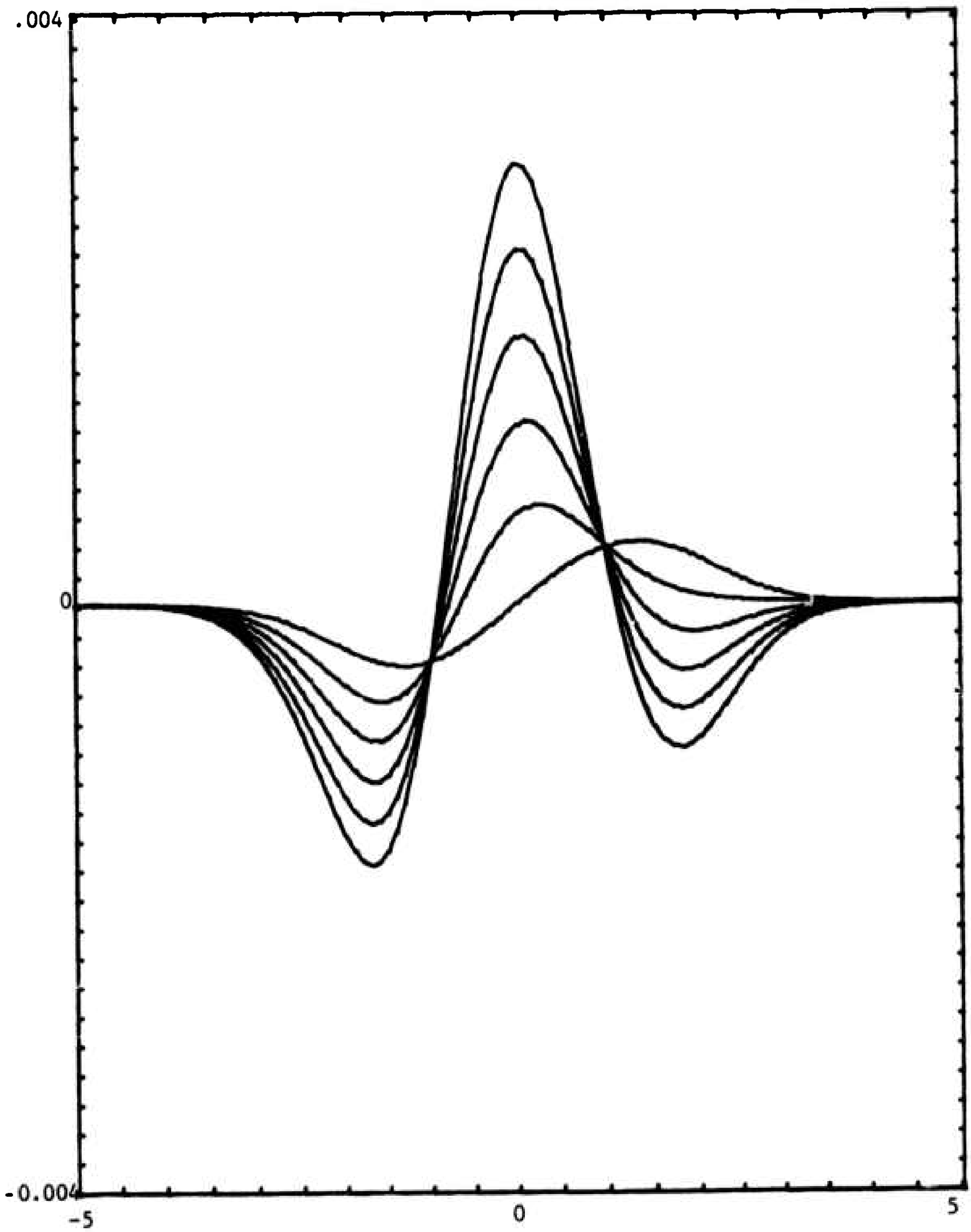
(b) $N = 100$, $SNR_i = 1$, $\gamma = 1$



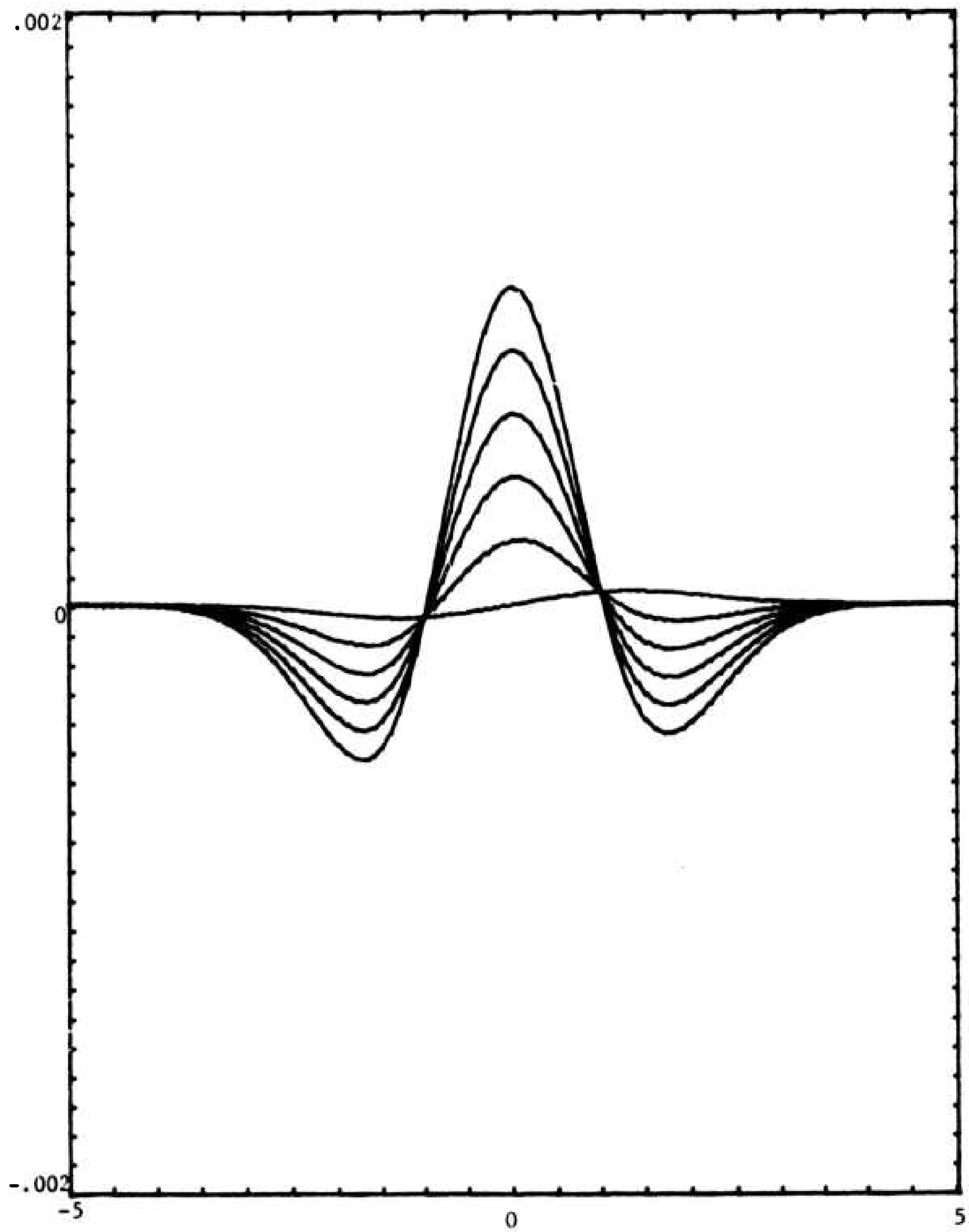
(c) N = 100, SNR_i = 10, γ = 1



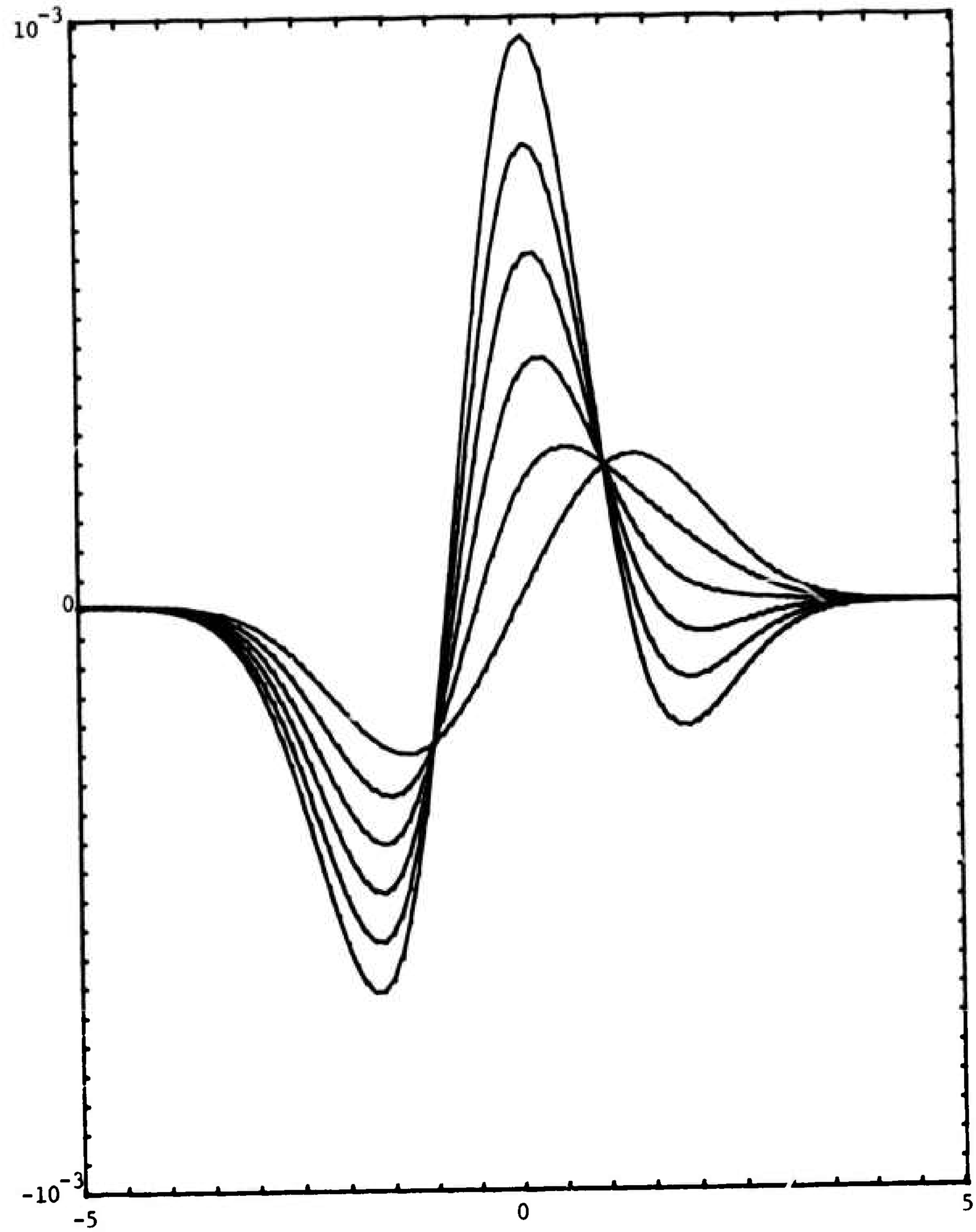
(d) N = 100, SNR_1 = .1, γ = .1



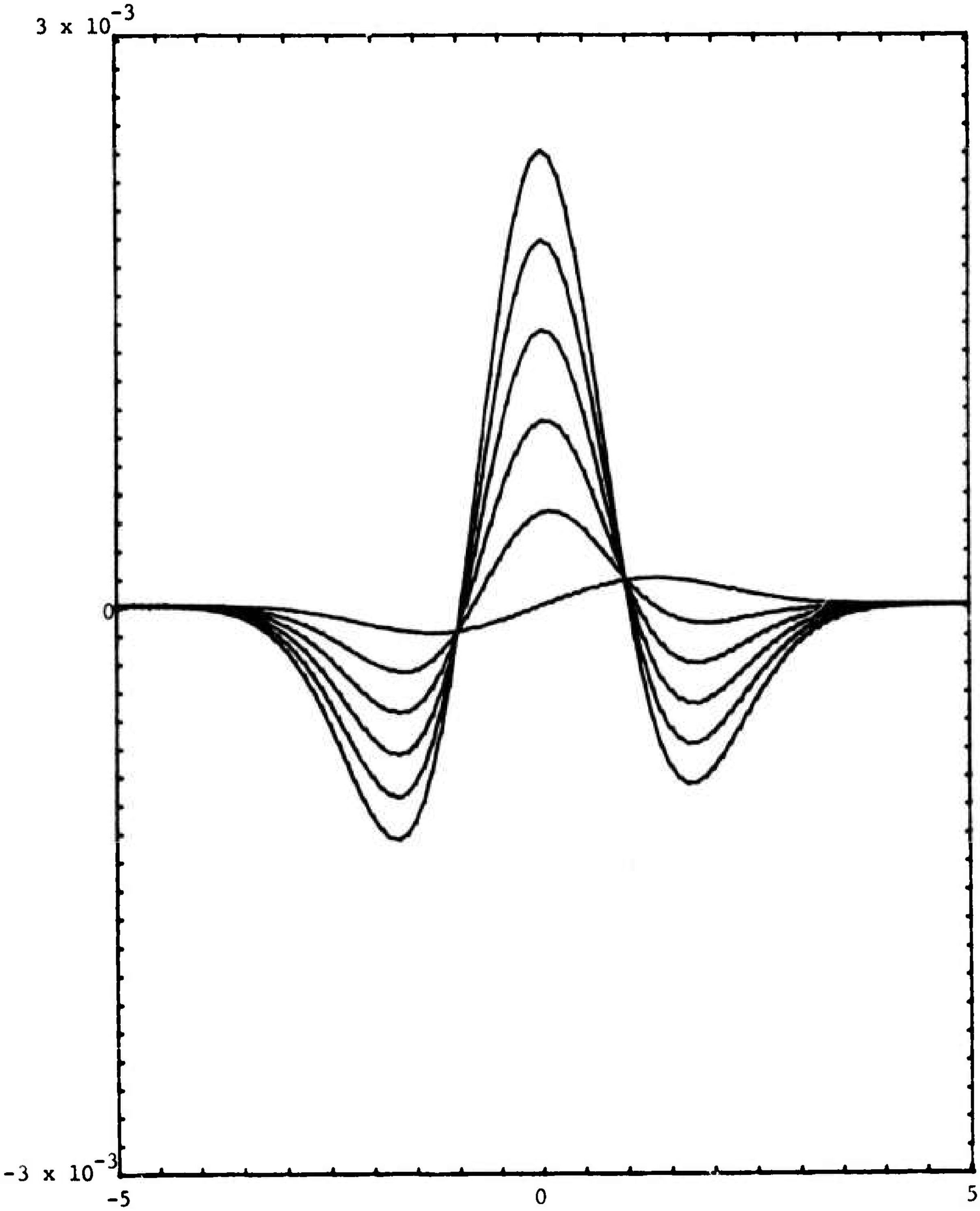
(e) N = 100, SNR_i = 1, γ = .1



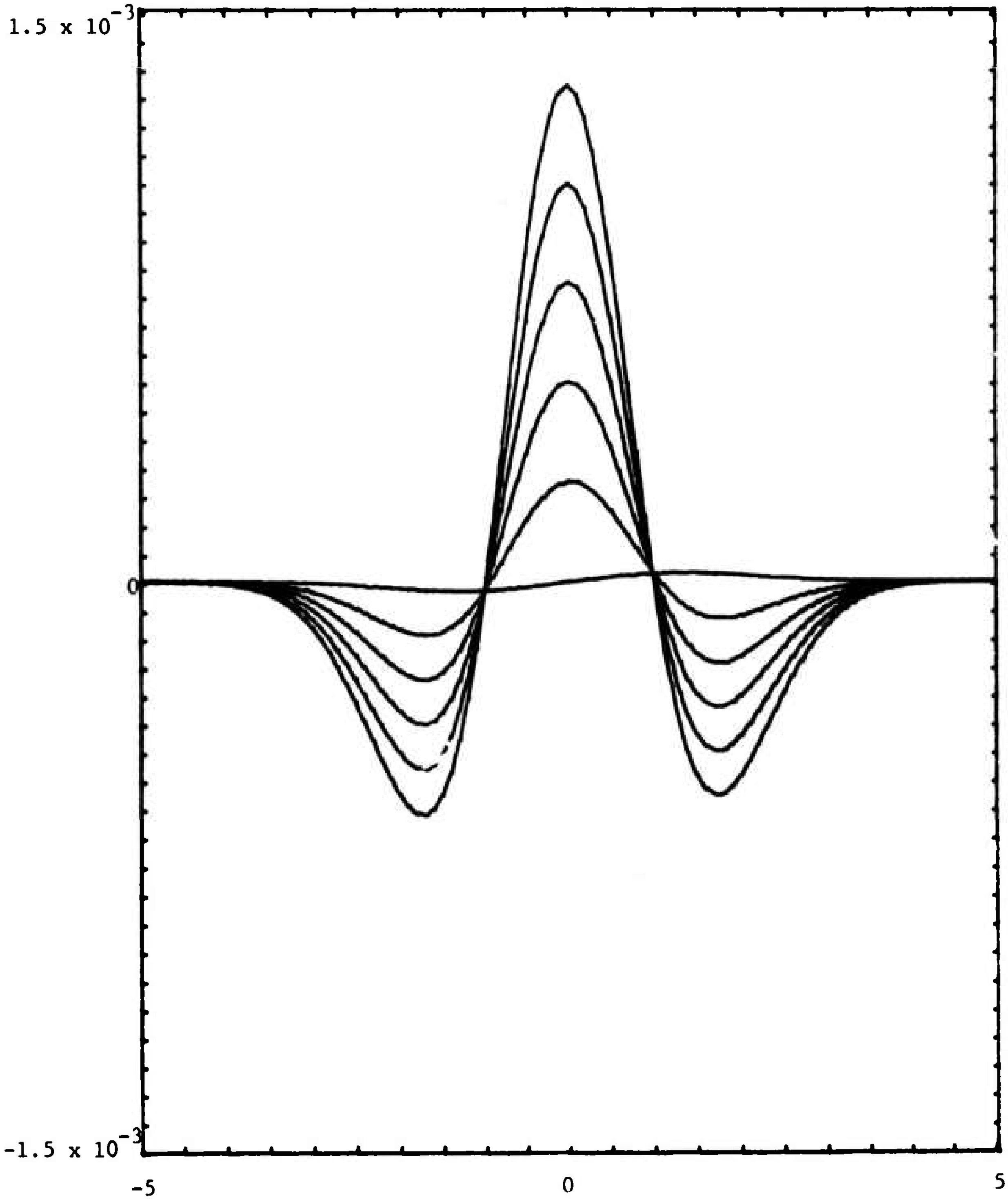
(f) N = 100, SNR_i = 10, γ = .1



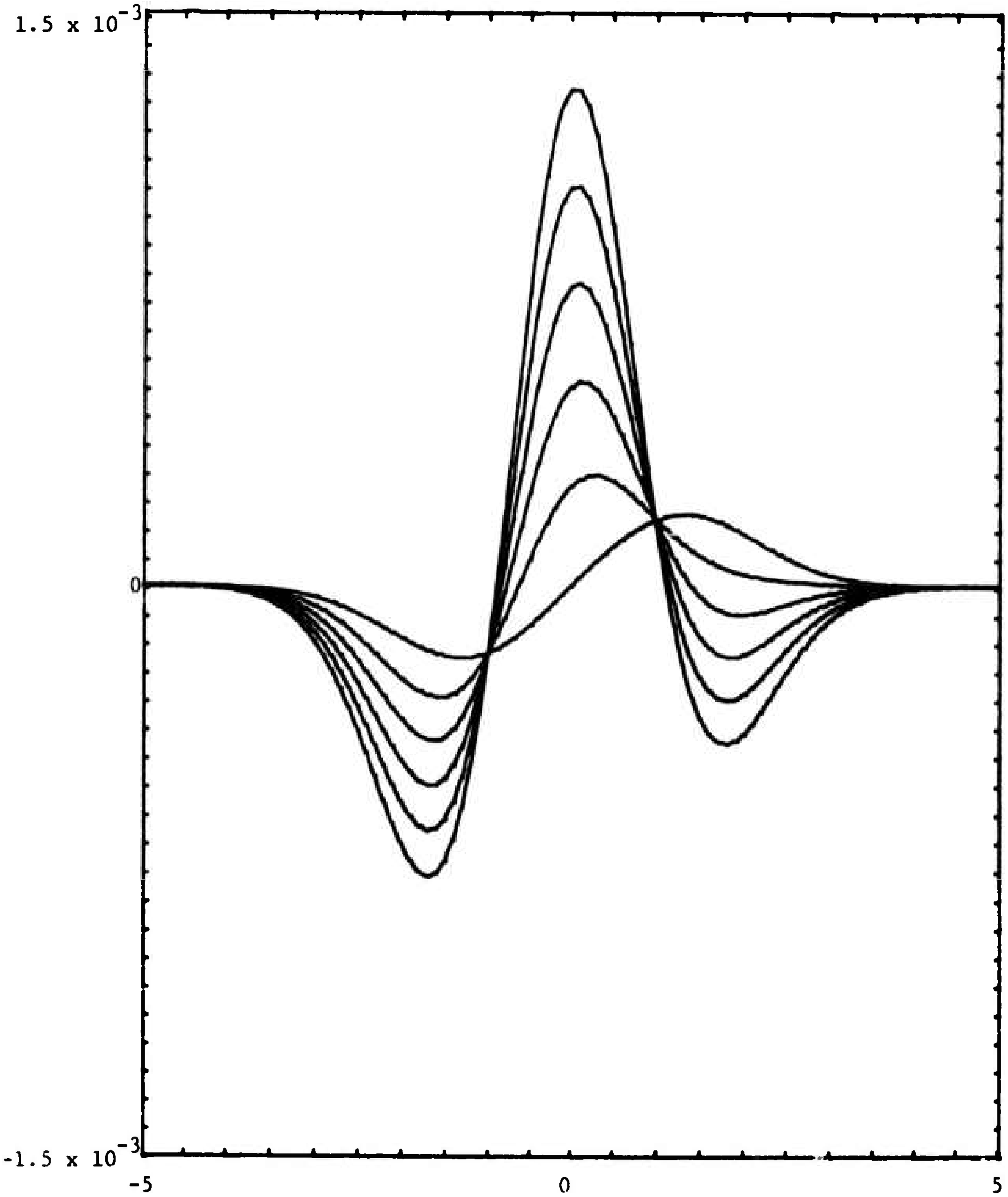
(g) N = 1000, SNR_i = .1, γ = 1



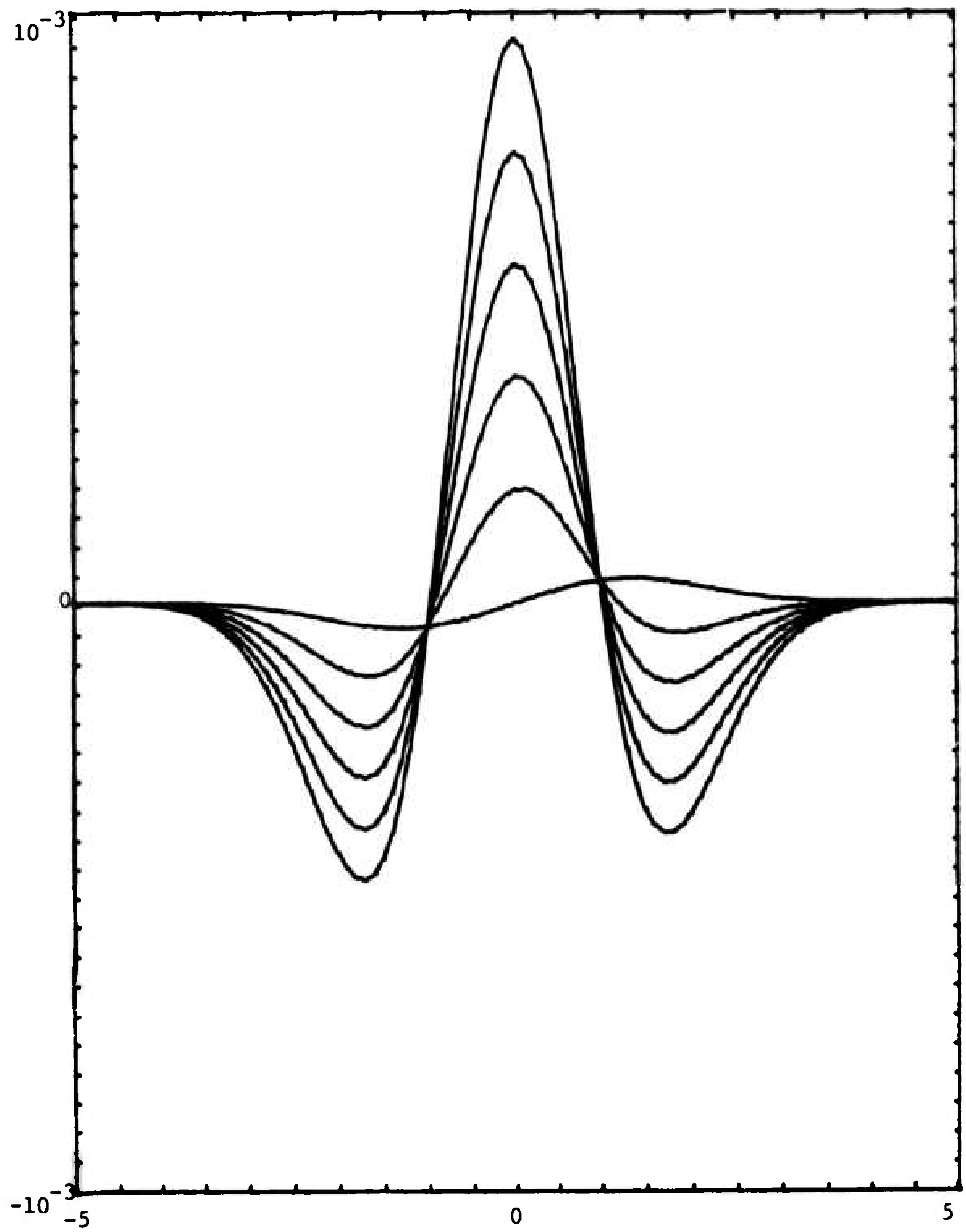
(h) N = 1000, SNR_i = 1, γ = 1



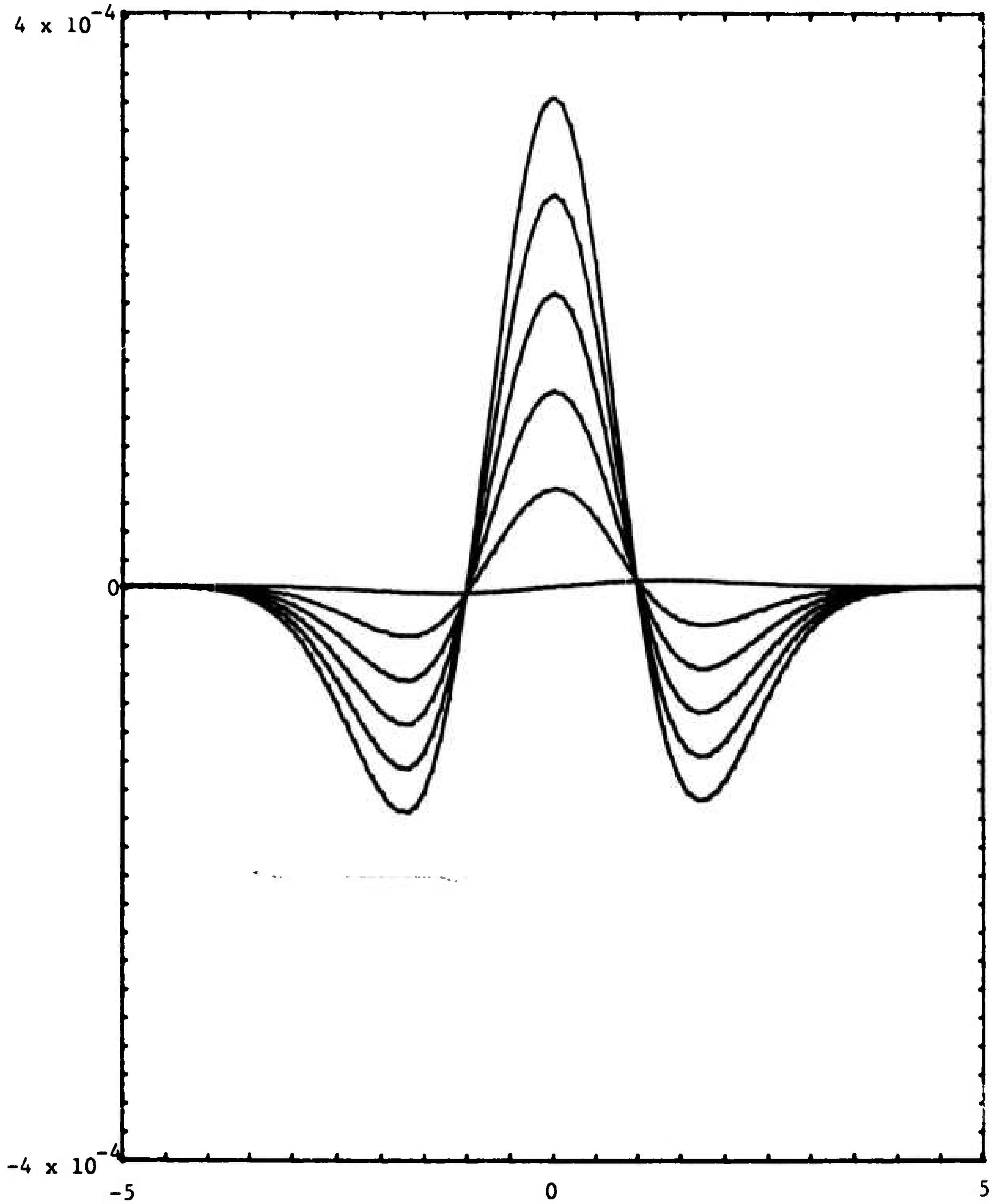
(i) N = 1000, SNR_i = 10, γ = 1



(j) $N = 1000$, $SNR_i = .1$, $\gamma = .1$



(k) $N = 1000$, $SNR_i = 1$, $\gamma = .1$



(1) N = 1000, SNR_i = 10, γ = .1

From (4.8), we have

$$\phi_\eta(\xi) = (1+\beta^2\xi^2)^{-\frac{N}{2}} \exp\left\{\frac{-j\beta_1\beta^3\xi^3 - \frac{1}{2}\ \beta_2\beta^2\xi^2}{1+\beta^2\xi^2}\right\}, \tag{4.32}$$

where

$$\beta^2 = \frac{1}{N(1+\gamma_1 \mathrm{SNR}_i)} \xrightarrow[N\to\infty]{} 0,$$

$$\beta_1\beta^3 = \frac{\mathrm{CORR}\cdot \mathrm{SNR}_i}{N^{1/2}\sqrt{\gamma}\ (1+\gamma_1\ \mathrm{SNR}_i)^{3/2}} \xrightarrow[N\to\infty]{} 0,$$

and

$$\beta_2\beta^2 = \frac{\gamma_1 \mathrm{SNR}_i}{1+\gamma_1\ \mathrm{SNR}_i}\ .$$

Noting that $(1+\beta^2\xi^2)^{-\frac{N}{2}} \xrightarrow[N\to\infty]{} \exp\{-\frac{1}{2}\frac{\xi}{(1+\gamma_1\ \mathrm{SNR}_i)}\}$, we find that $\lim_{N\to\infty} \phi_\eta(\xi) = e^{-\frac{1}{2}\xi^2}$, i.e., η is approximately normal for large N. With this important result, we note that the ordering of the bracketed terms in the Edgeworth expansion (4.20) and (4.21) are arranged such that successive terms decrease as increasing integral powers of $N^{-1/2}$, as shown in [14]. Even with these important remarks, we still have no feel for the error in our approximation. (In fact, the Edgeworth series does not always converge [14].) Consequently, we will now consider a method for the numerical computation of H(x) directly from the characteristic function.

A well-known result from probability theory is that if a and b are continuity points of a cdf F(.) having ch.f. $\phi(.)$, then

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-jta} - e^{-jtb}}{jt}\phi(t)dt. \tag{4.33}$$

The similarity between (4.33) and the well-known Fourier integral suggests that a Fast Fourier Transform (FFT) algorithm might be used to provide an efficient numerical computation of (4.33). With $x = b$ and $a = 0$, we may rewrite (4.33) as

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi} \left[\int_0^T \frac{-\phi(-t) + \phi(t)}{jt}\, dt + \int_0^T \frac{\phi(-t)e^{jxt} - \phi(t)e^{-jxt}}{jt}\, dt \right]. \tag{4.34}$$

Now, assuming that $\rho(\cdot)$ is absolutely integrable and defining

$$I_T = \frac{1}{2\pi} \int_0^T \frac{-\phi(-t) + \phi(t)}{jt}\, dt,$$

we have

$$\frac{dF(x)}{dx} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jxt} dt, \tag{4.35}$$

and

$$I_T = \frac{1}{2\pi} \int_0^T \frac{2j \ \mathrm{Im}\ \phi(t)}{jt}\, dt$$

$$= \frac{1}{\pi} \int_0^T \frac{1}{t}\, \mathrm{Im} \left[\int_{-\infty}^{\infty} e^{jxt}\, f(x) dx \right] dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \left[\int_0^T \frac{\sin xt}{t}\, dt \right] dx.$$

Noting that

$$\int_0^\infty \frac{\sin xt}{t}\, dt = \begin{cases} -\frac{\pi}{2}, & x < 0 \\ 0, & x = 0 \\ \frac{\pi}{2}, & x > 0 \end{cases},$$

we find that

$$\lim_{T\to\infty} I_T = \frac{1}{2} - F(0),$$

so that we may write

$$F(x) = \frac{1}{2} + \lim_{T\to\infty} \frac{1}{2\pi} \int_0^T \frac{\phi(-t)e^{jxt} - \phi(t)e^{-jxt}}{jt}\, dt,$$

or

$$F(x) = \frac{1}{2} - \lim_{T\to\infty} \frac{1}{\pi} \int_0^T \frac{\mathrm{Im}\,\{\phi(t)e^{-jxt}\}}{t}\, dt. \tag{4.36}$$

Equation (4.36) can also be found in [13]. If we could remove the Im(.) operation from inside the integral, we could easily implement an FFT algorithm. Following Nutall [15], we alleviate this problem by using "The old give and take trick." Choose $a(t)$ such that

i) $$F_a(x) = \frac{1}{2} - \lim_{T\to\infty} \frac{1}{\pi} \int_0^T \frac{\mathrm{Im}\,\{a(t)e^{-jxt}\}}{t}\, dt$$

and

ii) $$\frac{\phi(t) - a(t)}{t} \in L^1\ (0, \infty).$$

For such an $a(.)$, we have

$$F(x) = F_a(x) - \lim_{T\to\infty} \frac{1}{\pi}\, \mathrm{Im}\, \{ \int_0^T \frac{(\phi(t) - a(t))e^{-jxt}}{t} dt\}. \tag{4.37}$$

In particular, choosing $a(t) = e^{-\frac{1}{2}t^2}$, we have $F_a(x) = \Psi(x)$ and

$$H(x) = -\lim_{T\to\infty} \frac{1}{\pi}\, \mathrm{Im}\, \{ \int_0^T \frac{(\phi(t) - a(t))e^{-jxt}}{t}\, dt\}. \tag{4.38}$$

Defining

$$g(x) = \Delta t \sum_{m=1}^{M} \frac{(\phi(t_m) - a(t_m))e^{-jxt_m}}{t_m}$$

we may approximate $H(x)$ as

$$H(x_\ell) \simeq -\frac{1}{\pi}\, \mathrm{Im}\; g(x_\ell), \tag{4.39}$$

where, for $t_m = (m-1)\Delta T$, $m = 1, 2, \ldots, M$, and $x_\ell = \frac{2\pi(\ell - 1)}{M\Delta t}$, $\ell = 1, 2, \ldots, M$, we have

$$g(x_\ell) = \Delta t \sum_{m=1}^{M} \frac{(\phi(t_m) - a(t_m))}{t_m} \exp\{\frac{-j2\pi(\ell - 1)(m - 1)}{M}\}, \tag{4.40}$$

which is precisely in the form for which the FFT is designed. We now need only to choose a Δt and an $M = T/\Delta t$ so that the approximation (4.39) is a good one. First, we note that $g(x_\ell)$ is periodic with period $\frac{2\pi}{\Delta t}$, i.e., $g(x_\ell \pm \frac{2\pi}{\Delta t}) = g(x_\ell)$. In particular, we have, with $\Delta x = 2\pi/T$:

$$g(m\Delta x) = \begin{cases} g(x_{M+m+1}), & m = -[\frac{M-1}{2}],\ -[\frac{M-1}{2}]+1,\ \ldots,\ -1 \\ g(x_{m+1}), & m = 0,\ 1,\ 2,\ \ldots,\ [\frac{M}{2}], \end{cases} \tag{4.41}$$

so that we have obtained an approximation of $H(x)$ for $|x| \le \frac{M}{2}\Delta x$.

The approximation of (4.39) and (4.40) involves errors due to truncating the integration limit as well as a sampling error due to approximating the truncated integral with a Riemann sum. Standard numerical analysis techniques could, of course, be applied to approximate and/or bound these errors. Alternatively, in view of the fact that we may rewrite (4.38) as

$$H(x) = \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{(\phi(t) - a(t))e^{-jxt}}{t} dt, \tag{4.42}$$

we find that we are simply approximating a Fourier integral with the FFT. Consequently, the results of Cooley, Lewis, and Welch [16], for example, can be exploited to provide appropriate error bounds.

Following [16], and defining:

$$d(t) = \frac{\phi(t) - a(t)}{t}, \tag{4.43}$$

$$d_p(t) = \sum_{k=-\infty}^{\infty} d(t + kT),$$

and

$$H_p(x) = \sum_{k=-\infty}^{\infty} H(x + kX), \tag{4.43}$$

it is easily shown that

$$H(x_\ell) = \frac{j}{2\pi} \int_0^T d_p(t) \exp\{-\frac{j2\pi(\ell - 1)t}{T}\}dt, \tag{4.44}$$

and

$$d(t_m) = -j \int_0^X H_p(u)e^{j(m - 1)u\Delta t}du.$$

Furthermore,

$$\{d_p(t_m)\}_{m=1}^{M} \quad \text{and} \quad \{H_p(x_\ell)\}_{\ell=1}^{M} \quad \text{are related by:}$$

$$H_p(x_\ell) = \frac{j\Delta t}{2\pi}\sum_{m=1}^{M} d_p(t_m)\exp\{\frac{-j2\pi(m-1)(\ell - 1)}{M}\}, \tag{4.45}$$

and

$$b_p(t_m) = -j\Delta x \sum_{\ell=1}^{M} H_p(x_\ell)\exp\{\frac{j2\pi(m - 1)(\ell - 1)}{M}\}.$$

As noted previously, we are making the approximation

$$H(x_\ell) \simeq \hat{H}(x_\ell) = \frac{j\Delta t}{2\pi} \sum_{m=1}^{M} d(t_m)\exp\{\frac{-j2\pi(m - 1)(\ell - 1)}{M}\}. \tag{4.46}$$

We will bound the error in (4.46) via the inequality:

$$|H(x_\ell) - \hat{H}(x_\ell)| \leq |H(x_\ell) - H_p(x_\ell)| + |H_p(x_\ell) - \hat{H}(x_\ell)|. \tag{4.47}$$

Noting that

$$|H_p(x_\ell) - \hat{H}(x_\ell)| \leq \frac{\Delta t}{2\pi} \sum_{m=1}^{M} |d_p(t_m) - d(t_m)|, \tag{4.48}$$

appropriate bounds for $|H(x_\ell) - H_p(x_\ell)|$ and $|d_p(t_m) - d(t_m)|$ will be found and used in (4.47). The well-known Markov inequality states that

$$P[|n| \geq \varepsilon] \leq \frac{E\{|n|^\nu\}}{\varepsilon^\nu}. \tag{4.49}$$

Note that for ν an even positive integer the bound in (4.49) involves the νth moment of η, and that the bound decreases more rapidly with increasing ε for larger ν. Since μ_{10} is available from (4.18), the following bound (4.50) may be computed.

$$P[|\eta| \geq \varepsilon] \leq \mu_{10}/\varepsilon^{10} \tag{4.50}$$

Using (4.50) and an inequality given in [17, p. 39], we find that

$$|H(x)| \leq \frac{\mu_{10}}{\varepsilon^{10}} + \frac{1}{\varepsilon} e^{-\frac{1}{2}\varepsilon^2}, \tag{4.51}$$

for all x such that $|x| > \varepsilon$.

Noting that for all x such that $|x| \leq \frac{X}{2}$ we have $|x \pm kX| \geq \frac{2k-1}{2} X$, and making use of (4.43) and (4.51) we find that

$$|H(x) - H_p(x)| \leq \sum_{k=1}^{\infty} |H(x-kX)| + |H(x+kX)|$$

(for $X \geq 2$)

$$\leq 2\mu_{10}\{(\tfrac{1}{2}X)^{-10} + (\tfrac{3}{2}X)^{-10}\} + \frac{1}{X} e^{-\frac{1}{8}X^2} + \frac{1}{3X} e^{-\frac{9}{8}X^2}$$

$$+ 2\mu_{10} \sum_{k=3}^{\infty} (\frac{2k-1}{2} X)^{-10} + 4 \sum_{k=3}^{\infty} \exp\{-\tfrac{1}{2}(\frac{2k-1}{2}X)^2\}.$$

The series are easily bounded as

$$\sum_{k=3}^{\infty} (\frac{2k-1}{2}X)^{-10} \leq \frac{1}{X} \int_{\frac{3}{2}X}^{\infty} x^{-10} dx = \frac{1}{6}(\tfrac{3}{2}X)^{-10},$$

and

$$\sum_{k=3}^{\infty} \exp\{-\tfrac{1}{2}(\frac{2k-1}{2} X)^2\} \leq \frac{1}{X} \int_{\frac{3}{2}X}^{\infty} e^{-\frac{1}{2}x^2} dx$$

$$\leq \frac{2}{3X^2} e^{-\frac{9}{8}X^2}.$$

We have:

$$|H(x) - H_p(x)| \leq 2\mu_{10}[(\tfrac{1}{2}X)^{-10} + \tfrac{7}{6}(\tfrac{3}{2}X)^{-10}] + \frac{1}{X}[e^{-\frac{1}{8}X^2} + (\tfrac{1}{3} + \tfrac{1}{3X})e^{-\frac{9}{8}X^2}], \tag{4.52}$$

which would be the only bound necessary if $\{d_p(t_m)\}_{m=1}^{M}$, the "pre-aliased" version [16] of $\{d(t_m)\}_{m=1}^{M}$ were available. Similarly, noting that for all t such that $|t| \leq T/2$, we have $|t \pm kT| \geq \frac{2k-1}{2}T$, from (4.43) we obtain

$$|d_p(t) - d(t)| \leq \sum_{k=1}^{\infty} \frac{|\phi(t-kT)| + |a(t-kT)| + |\phi(t+kT)| + |a(t+kT)|}{\frac{2k-1}{2}T}.$$

Now, since $|\phi(t)|$ and $|a(t)|$ are strictly decreasing with $|t|$, we have

$$|d_p(t) - d(t)| \leq 2\sum_{k=1}^{\infty} \frac{|\phi(\frac{2k-1}{2}T)| + |a(\frac{2k-1}{2}T)|}{\frac{2k-1}{2}T}$$

$$\leq \frac{1}{T}\left(|\phi(\tfrac{1}{2}T)| + |a(\tfrac{1}{2}T)| + \tfrac{1}{3}|\phi(\tfrac{3}{2}T)| + \tfrac{1}{3}|a(\tfrac{3}{2}T)|\right) + \frac{2}{T}\int_{\frac{3}{2}T}^{\infty} \frac{|\phi(u)|}{u}\,du + \frac{2}{T}\int_{\frac{3}{2}T}^{\infty} \frac{|a(u)|}{u}\,du. \tag{4.53}$$

Recall that for the problem at hand, $\phi(t)$ and $a(t)$ are given by:

$$\phi(t) = (1 + \beta^2 t^2)^{-\frac{N}{2}} \exp\left\{\frac{-j\beta_1\beta^3 t^3 - \frac{1}{2}\beta_2\beta^2 t^2}{1 + \beta^2 t^2}\right\} \tag{4.54}$$

and

$$a(t) = e^{-\frac{1}{2}t^2}.$$

We have:

$$\int_T^{\infty} \frac{|\phi(t)|}{t}\,dt \le \int_{\beta T}^{\infty} u^{-1}(1+u^2)^{-\frac{N}{2}} \exp\left\{\frac{-\frac{1}{2}\beta_2 u^2}{1+u^2}\right\} du$$

$$\le \exp\left\{\frac{-\frac{1}{2}\beta_2\beta^2T^2}{1+\beta^2T^2}\right\} \int_{\beta T}^{\infty} u^{-1}(1+u^2)^{-\frac{N}{2}} du$$

(for $N > 2$)

$$= \exp\left\{\frac{-\frac{1}{2}\beta_2\beta^2T^2}{1+\beta^2T^2}\right\} \frac{1}{2} \int_{1+\beta^2T^2}^{\infty} (x-1)^{-1} x^{-\frac{N}{2}}\,dx$$

$$\le \exp\left\{\frac{-\frac{1}{2}\beta_2\beta^2T^2}{1+\beta^2T^2}\right\} \frac{1}{2\beta^2T^2} \int_{1+\beta^2T^2}^{\infty} x^{-\frac{N}{2}}\,dx$$

$$= \frac{\exp\left\{\frac{-\frac{1}{2}\beta_2\beta^2T^2}{1+\beta^2T^2}\right\}}{(N-2)\beta^2T^2(1+\beta^2T^2)^{\frac{N}{2}-1}} \tag{4.55}$$

$$\triangleq b_1(T),$$

and

$$\int_T^{\infty} \frac{|a(t)|}{t}\,dt \le \frac{1}{T}\int_T^{\infty} e^{-\frac{1}{2}t^2}\,dt \le \frac{1}{T^2}\, e^{-\frac{1}{2}T^2} \triangleq b_2(T). \tag{4.56}$$

Combining (4.47), (4.48), and (4.52) through (4.56), we finally obtain:

$$|H(x_\ell) - \hat{H}(x_\ell)| \le 2\mu_{10}[(\tfrac{1}{2}X)^{-10} + \tfrac{7}{6}(\tfrac{3}{2}X)^{-10}]$$
$$+ \frac{1}{X}[e^{-\frac{1}{8}X^2} + (\tfrac{1}{3} + \tfrac{1}{3X})\, e^{-\frac{9}{8}X^2}]$$
$$+ \frac{M\Delta t}{2\pi T}[|\phi(\tfrac{1}{2}T)| + |a(\tfrac{1}{2}T)| + \tfrac{1}{3}|\phi(\tfrac{3}{2}T)| + \tfrac{1}{3}|a(\tfrac{3}{2}T)|]$$
$$+ \frac{M\Delta t}{\pi T}[b_1(\tfrac{3}{2}T) + b_2(\tfrac{3}{2}T)]. \tag{4.57}$$

Note that $X = 2\pi M/T$, and $M\Delta t = T$. For any range yielding suitably small errors, we have:

$$\frac{7}{6}(\frac{3}{2}X)^{-10} << (\frac{1}{2}X)^{-10}$$

$$(\frac{1}{3} + \frac{1}{3X})e^{-\frac{9}{8}X^2} << e^{-\frac{1}{8}X^2}$$

$$\frac{1}{3}|\phi(\frac{3}{2}T)| << |\phi(\frac{1}{2}T)|$$

$$\frac{1}{3}|a(\frac{3}{2}T)| << |a(\frac{1}{2}T)|$$

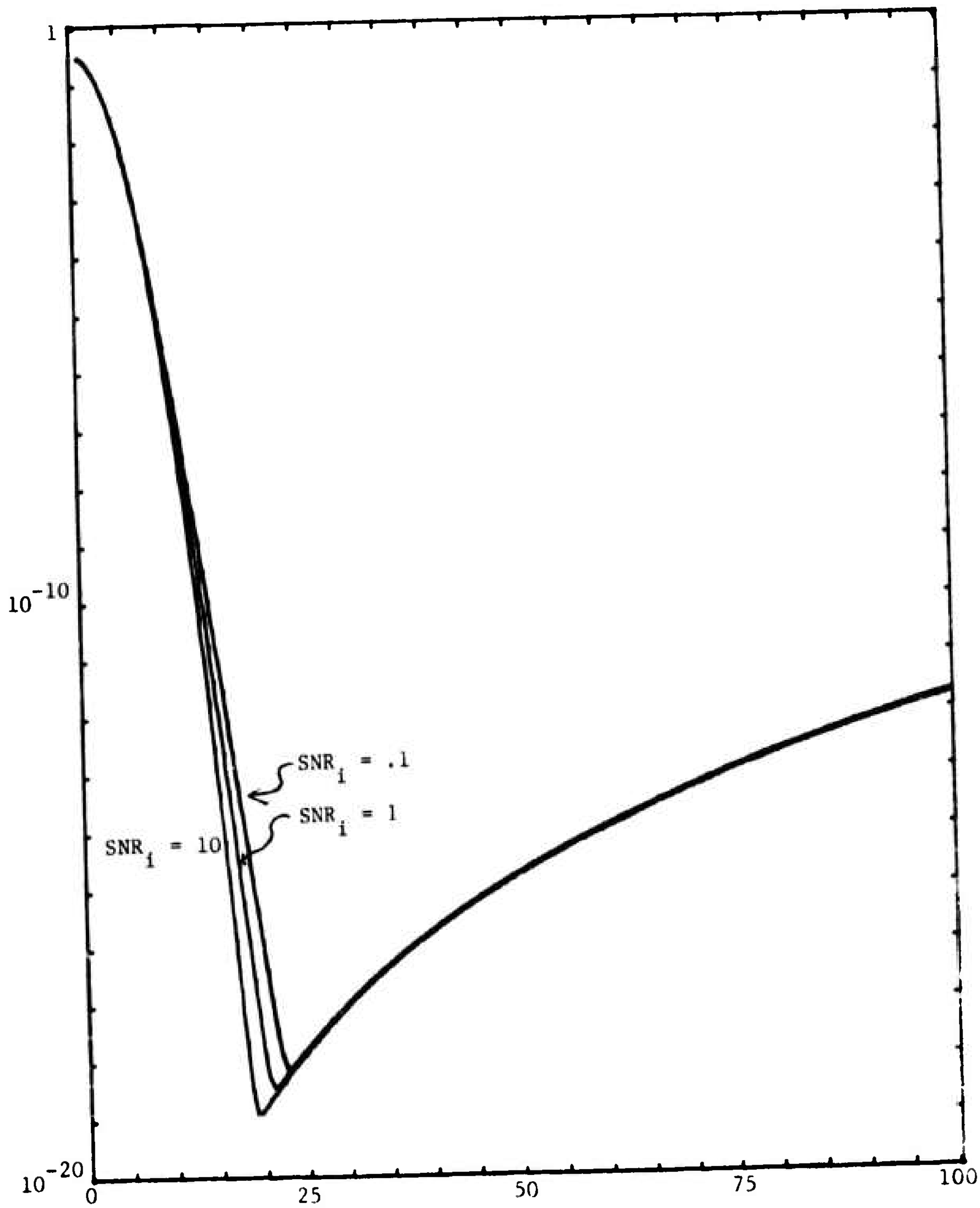
$$b_1(\frac{3}{2}T) << |\phi(\frac{1}{2}T)|$$

$$b_2(\frac{3}{2}T) << |a(\frac{1}{2}T)|$$

Hence, a very good approximation to (4.47) is

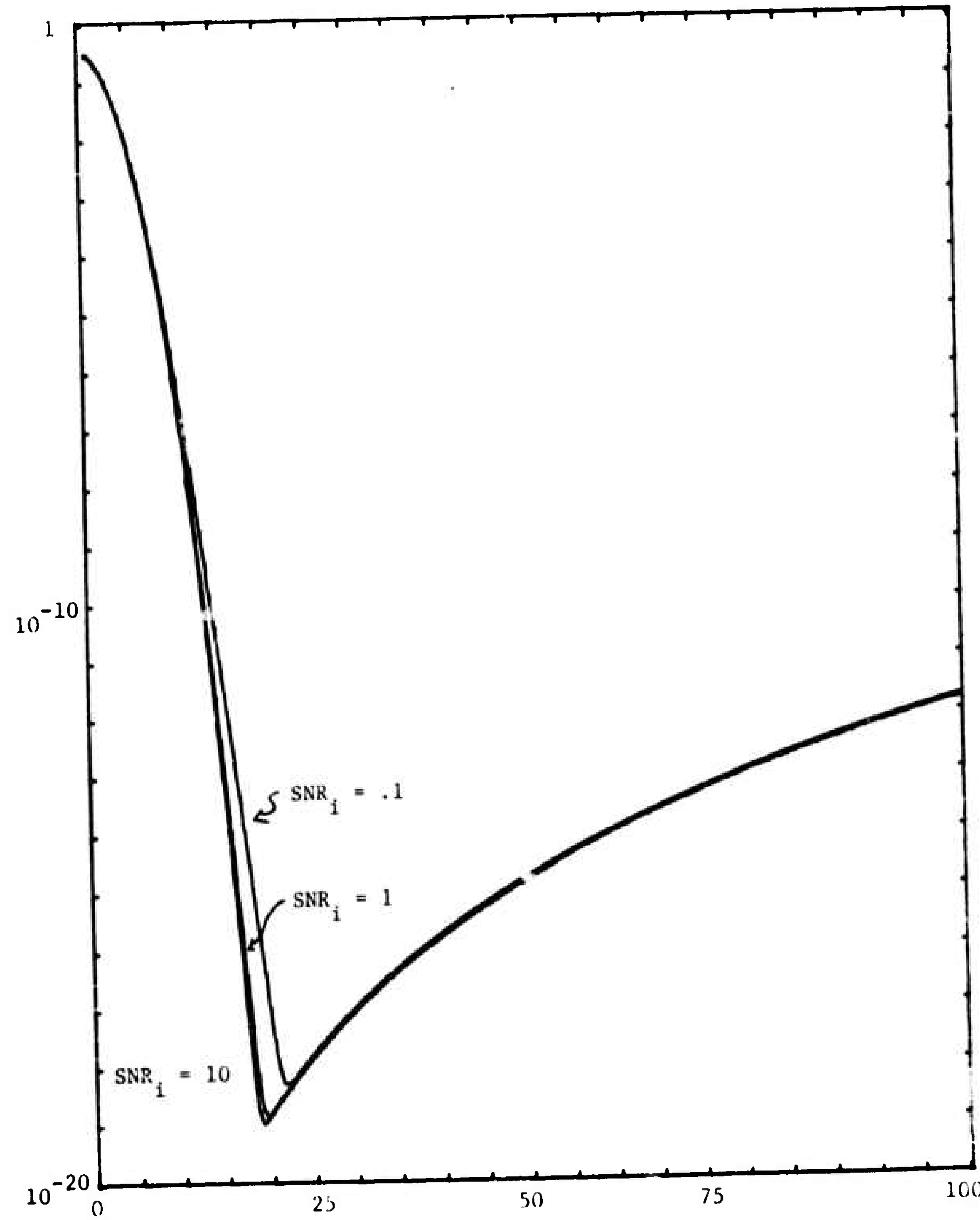
$$|H(x_\ell) - \hat{H}(x_\ell)| \leq 2\mu_{10}(\frac{T}{\pi M})^{10} + \frac{T}{2\pi M}\exp\{-\frac{1}{8}\frac{4\pi^2 M^2}{T^2}\} + \frac{1}{2\pi}[|\phi(\frac{1}{2}T)| + e^{-\frac{1}{8}T^2}]. \tag{4.58}$$

The approximate bound of (4.58) is shown in Figure 4.4 for $N = 100$, $CORR = 1$, $SNR_i \in \{.1, 1, 10\}$, $\gamma\in\{.1, 1\}$, and several values of M. Since the bound is decreasing with increasing N and decreasing $|CORR|$, Figure 4.4 suggests that for $T = 40\pi$, $N \geq 100$, $|CORR| \leq 1$, $.1 \leq SNR_i \leq 10$, $.1 \leq \gamma \leq 1$, and $M = 1024$, an error of less than 10^{-10} will result. For this choice of T, we have $\Delta x = .05$, which should provide both acceptable resolution and error in approximating $H(x)$ by $\hat{H}(x_\ell)$ in (4.46). Figure 4.5 illustrates the numerical results for parameter choices corresponding to those of Figure 4.3. A careful comparison of Figures 4.3 and 4.5 reveals several discrepancies, indicating that the numerical inversion technique with its associated error bound is indeed worthwhile.

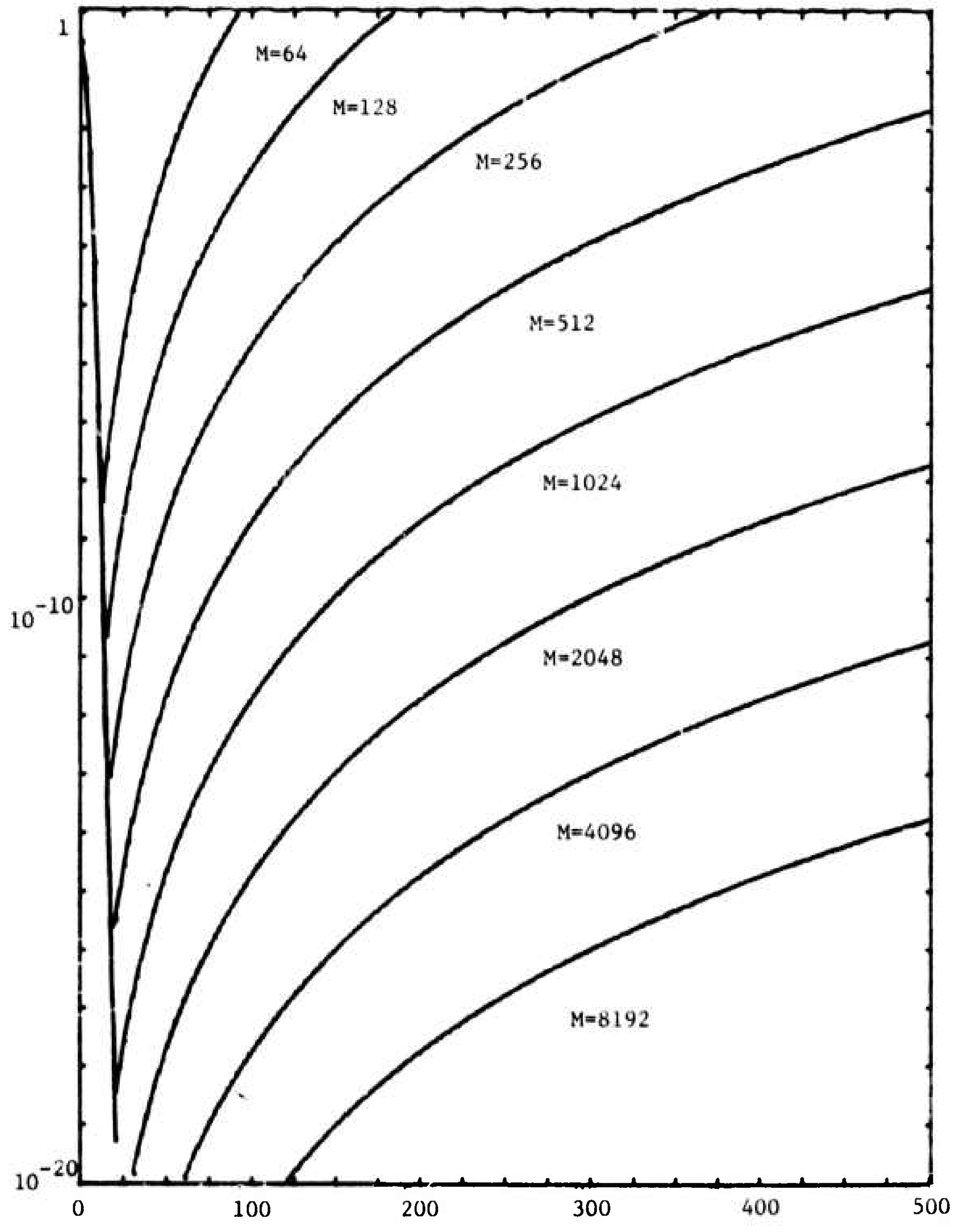


(a) N = 100, γ = 1, M = 1024

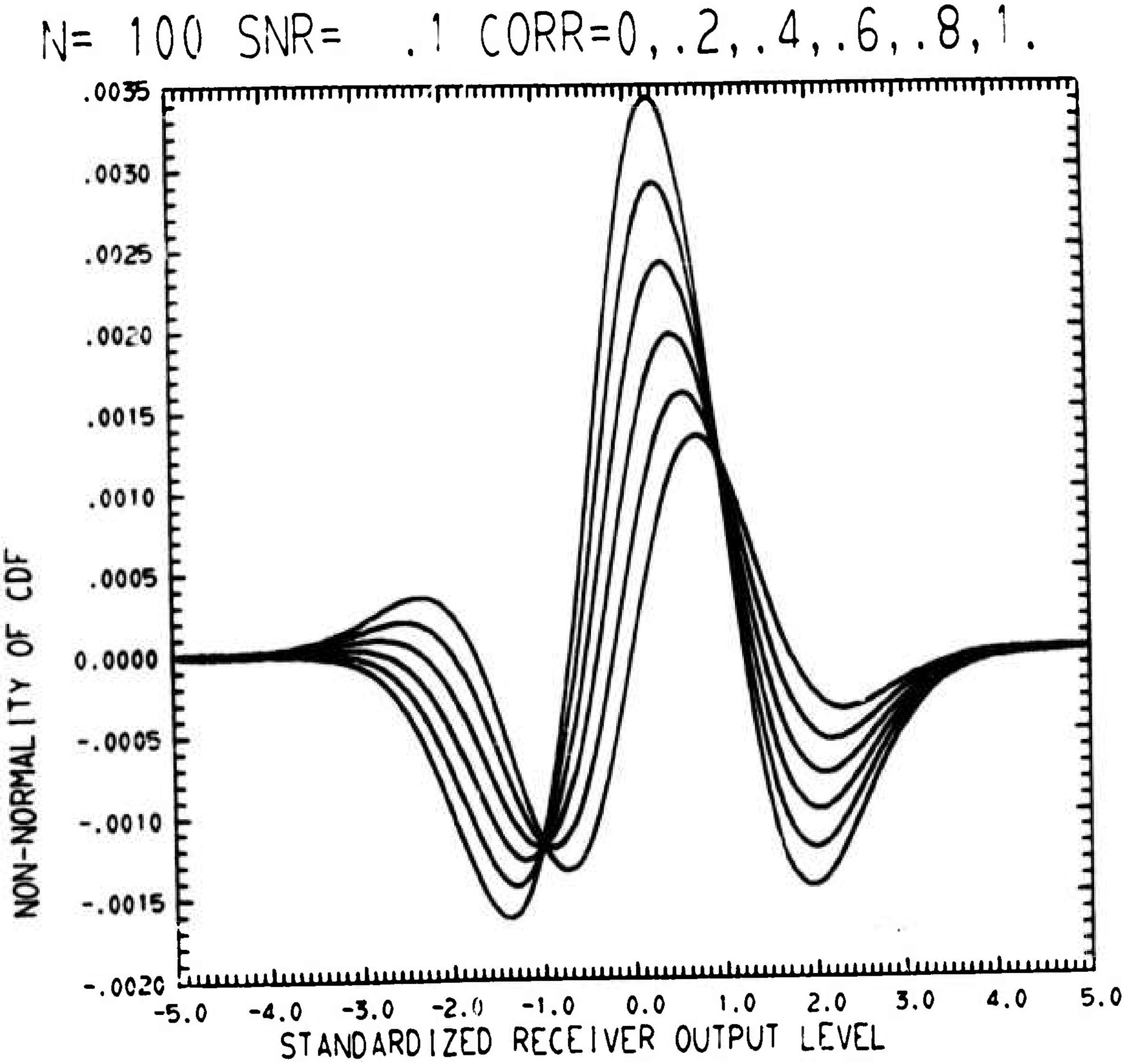
Figure 4.4 Approximate error bound vs. T



(b) N = 100, γ = .1, M = 1024

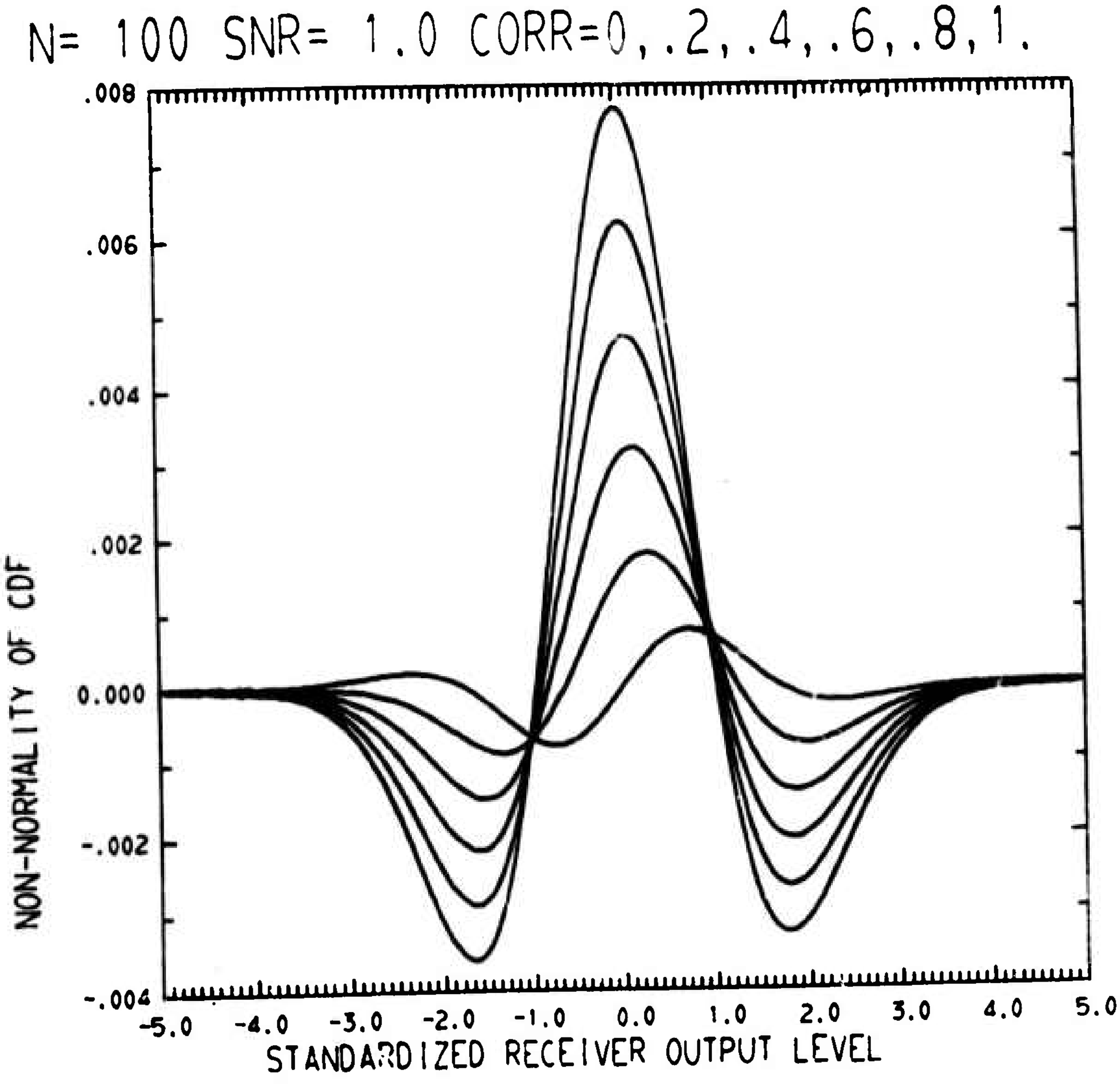


(c) N = 100, SNR_i = 1, γ = 1

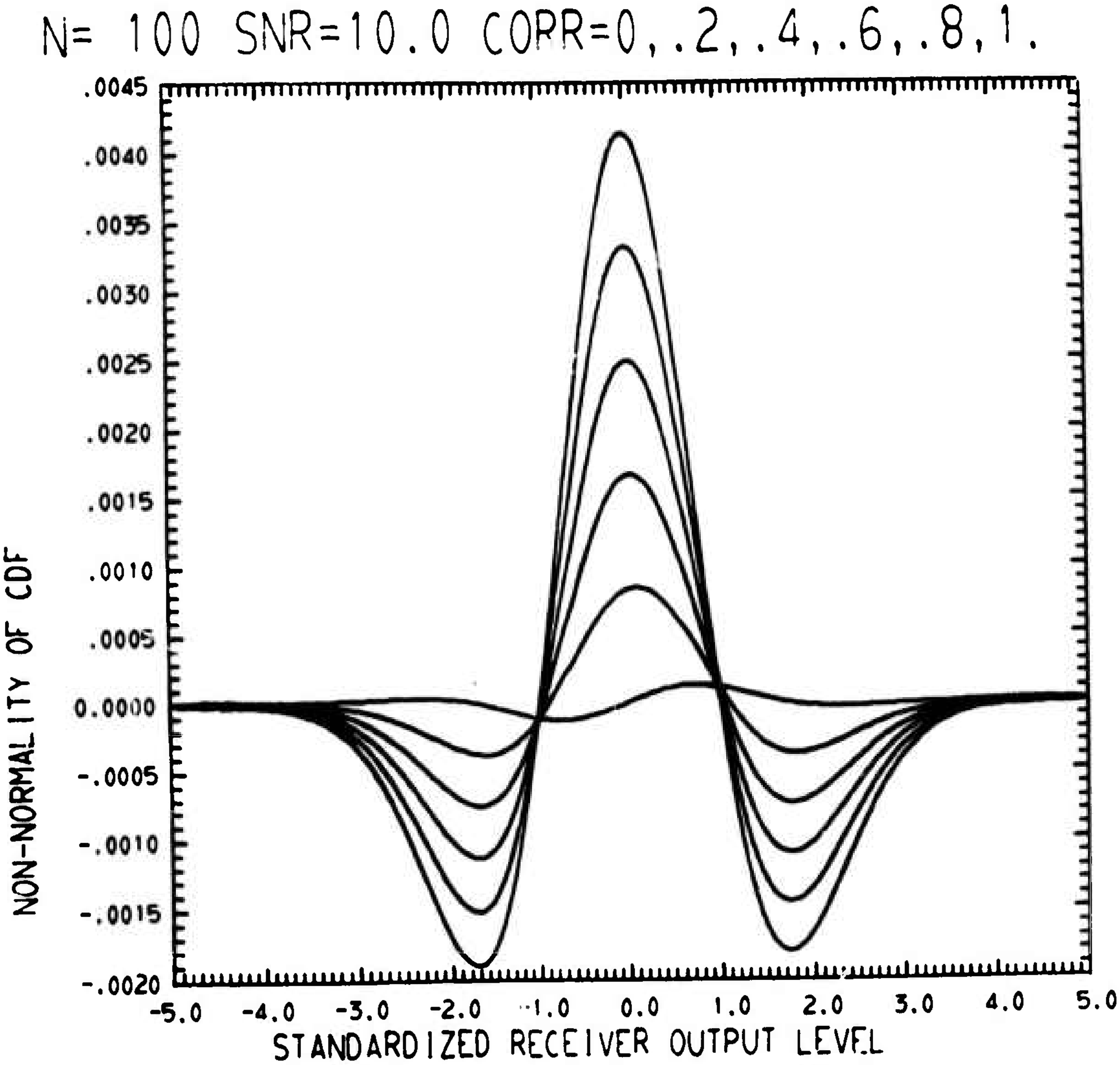


(a) $\gamma = 1$

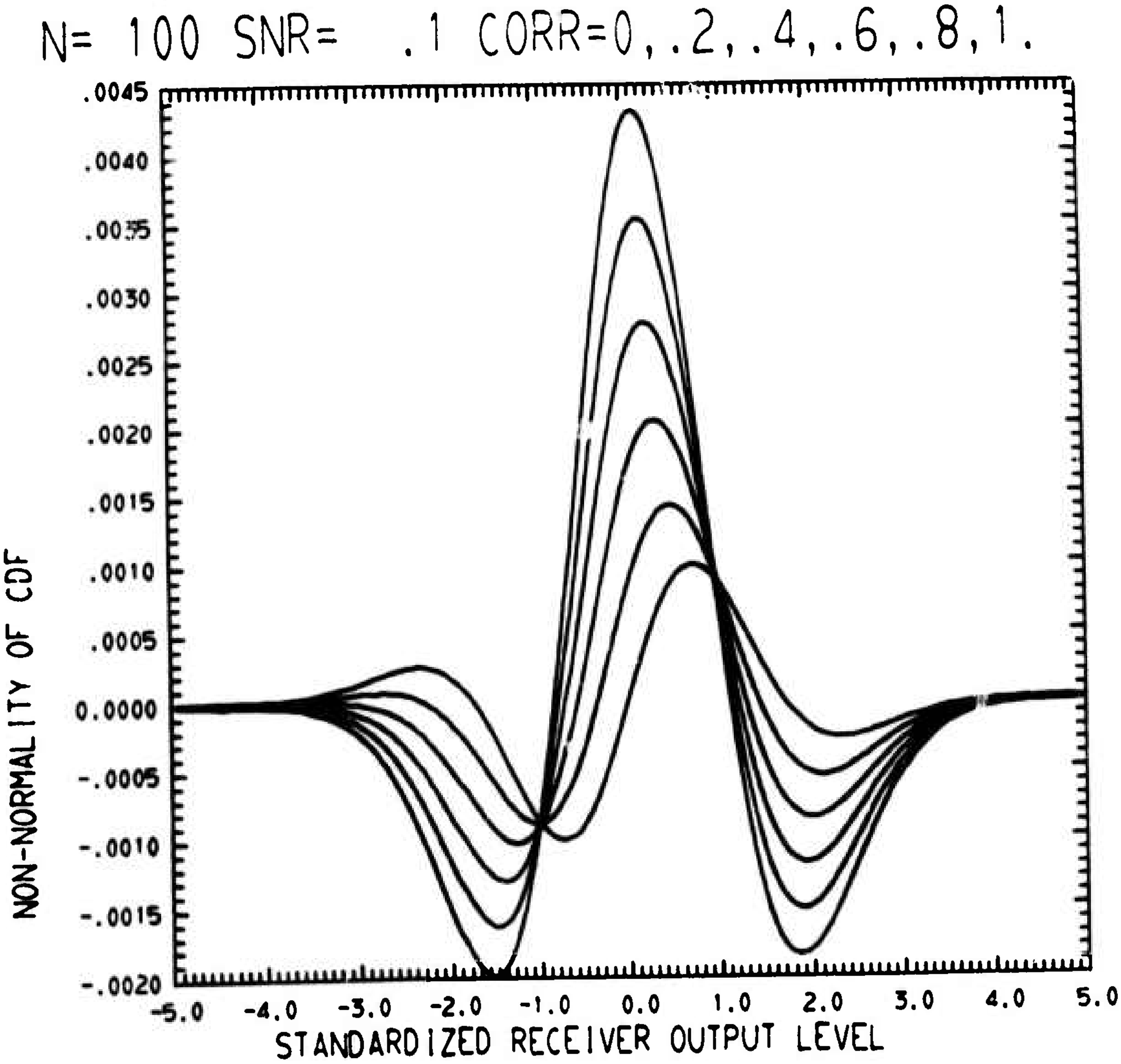
Figure 4.5 Non-normal component of cdf


N= 100 SNR= 1.0 CORR=0,.2,.4,.6,.8,1.
.008
.006
.004
.002
0.000
-.002
-.004
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

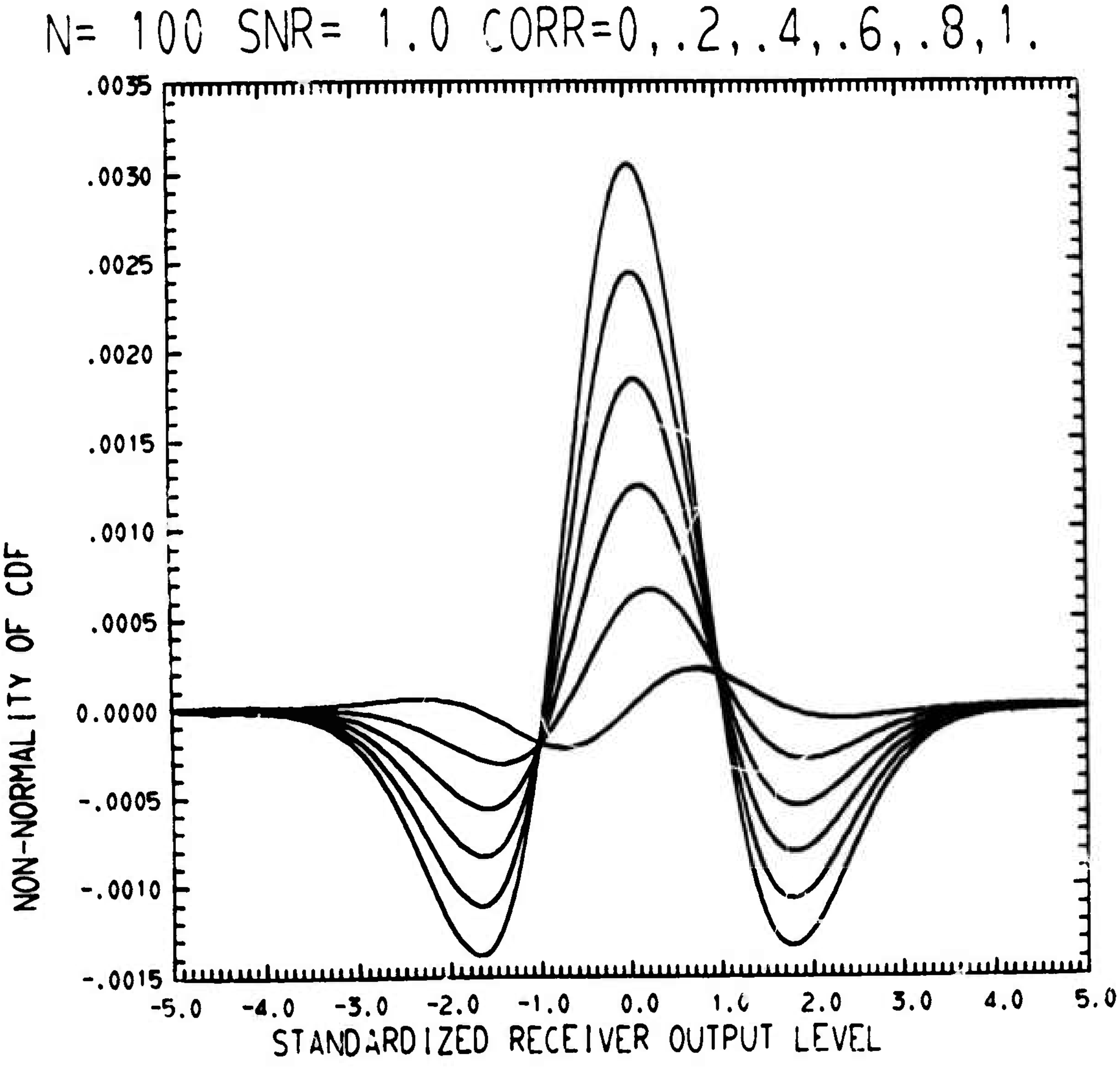
(b) $\gamma = 1$


N= 100 SNR=10.0 CORR=0,.2,.4,.6,.8,1.
.0045
.0040
.0035
.0030
.0025
.0020
.0015
.0010
.0005
0.0000
-.0005
-.0010
-.0015
-.0020
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

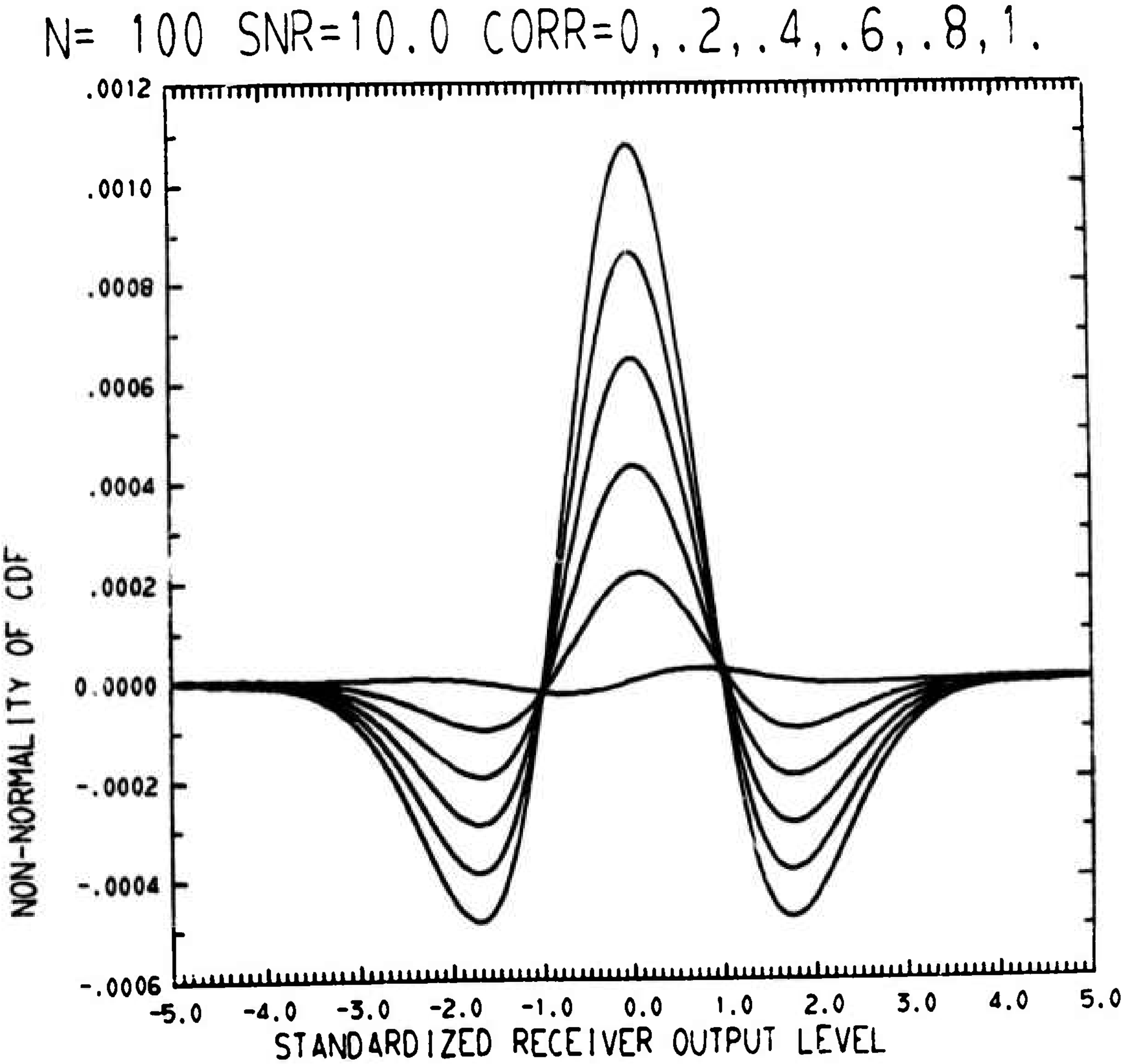
(c) γ = 1


N= 100 SNR= .1 CORR=0,.2,.4,.6,.8,1.
NON-NORMALITY OF CDF
STANDARDIZED RECEIVER OUTPUT LEVEL
.0045
.0040
.0035
.0030
.0025
.0020
.0015
.0010
.0005
0.0000
-.0005
-.0010
-.0015
-.0020
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0

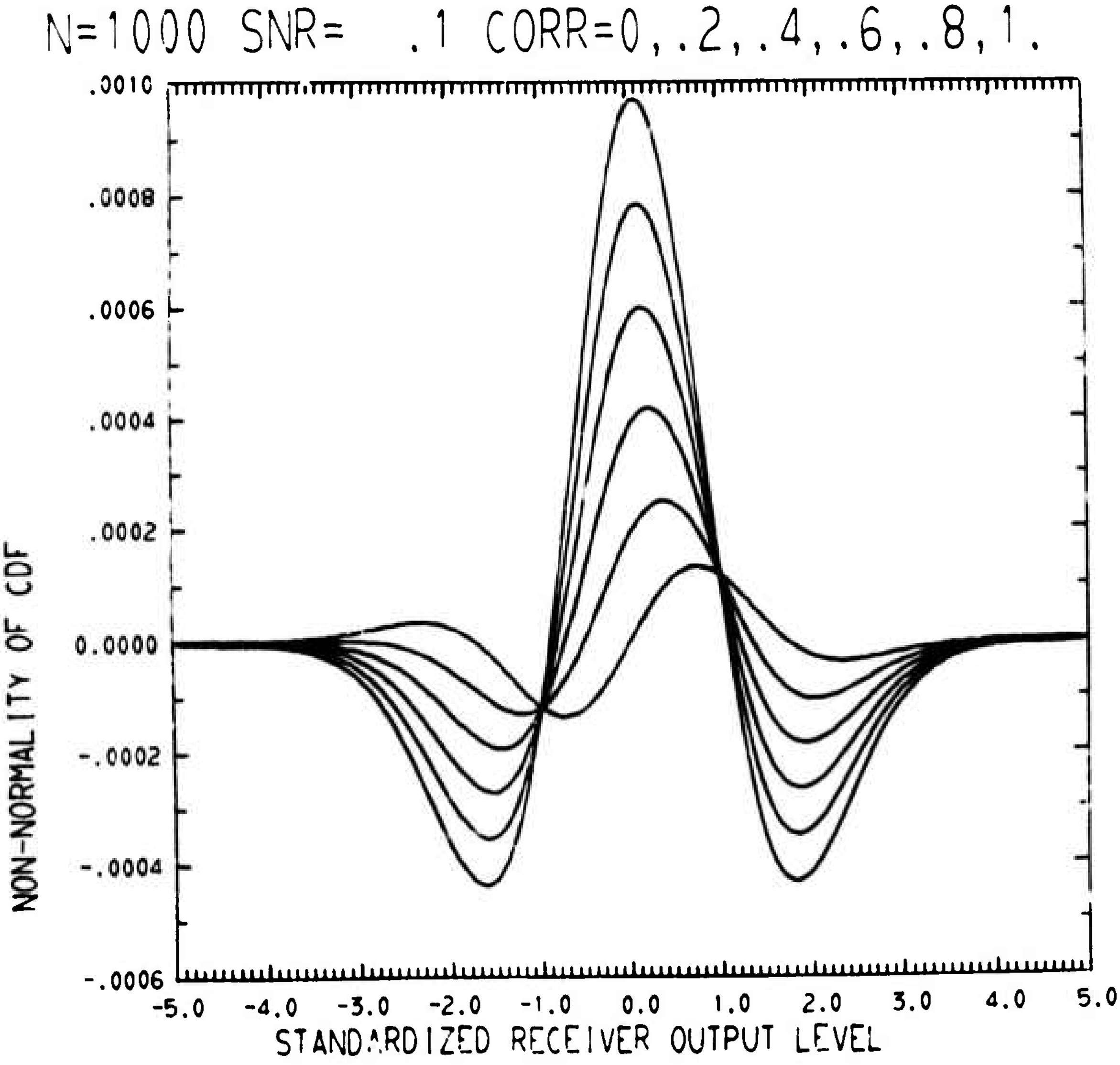
(d) γ = .1


N= 100 SNR= 1.0 CORR=0,.2,.4,.6,.8,1.
.0035
.0030
.0025
.0020
.0015
.0010
.0005
0.0000
-.0005
-.0010
-.0015
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

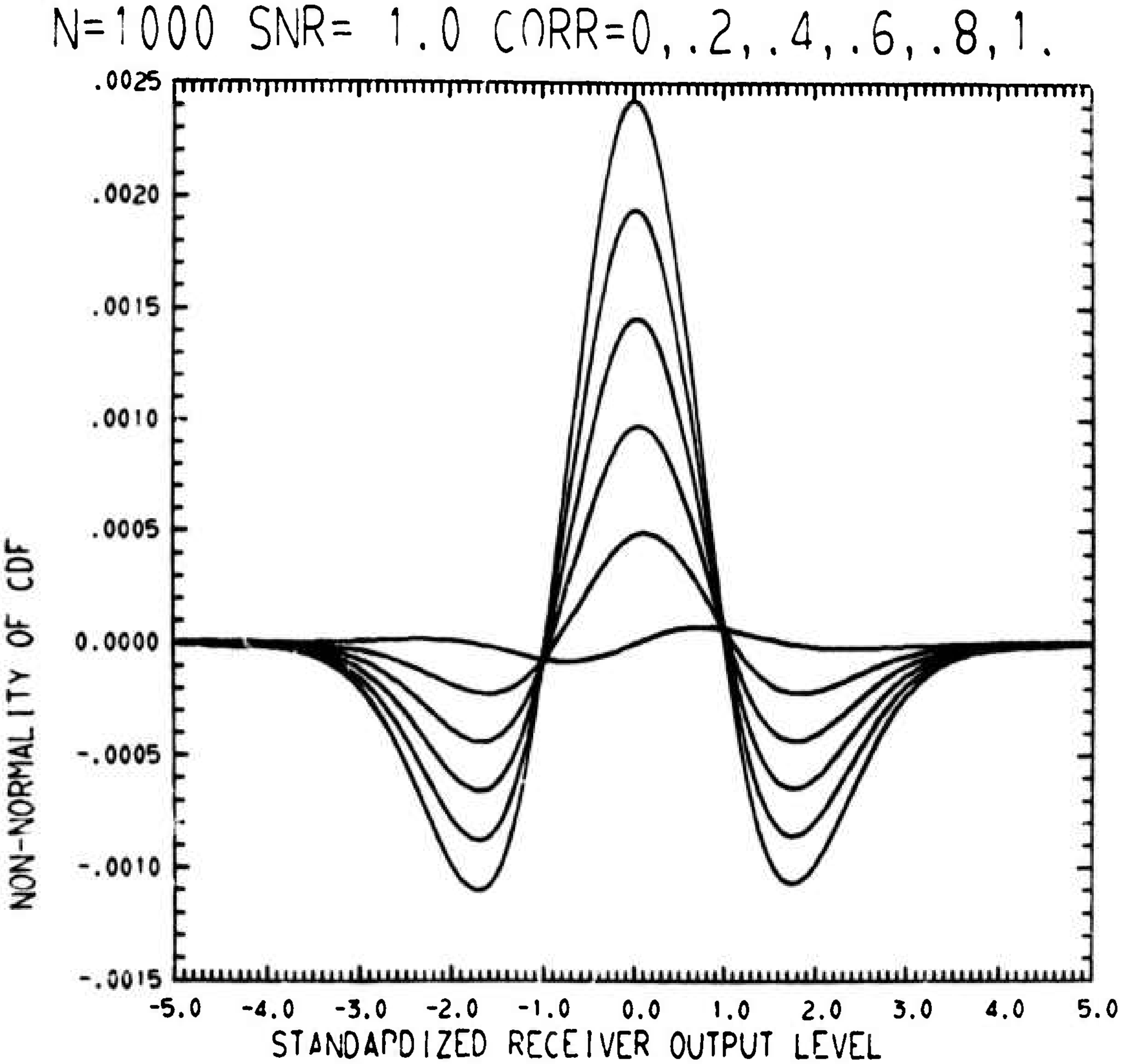
(e) γ = .1


N= 100 SNR=10.0 CORR=0,.2,.4,.6,.8,1.
.0012
.0010
.0008
.0006
.0004
.0002
0.0000
-.0002
-.0004
-.0006
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

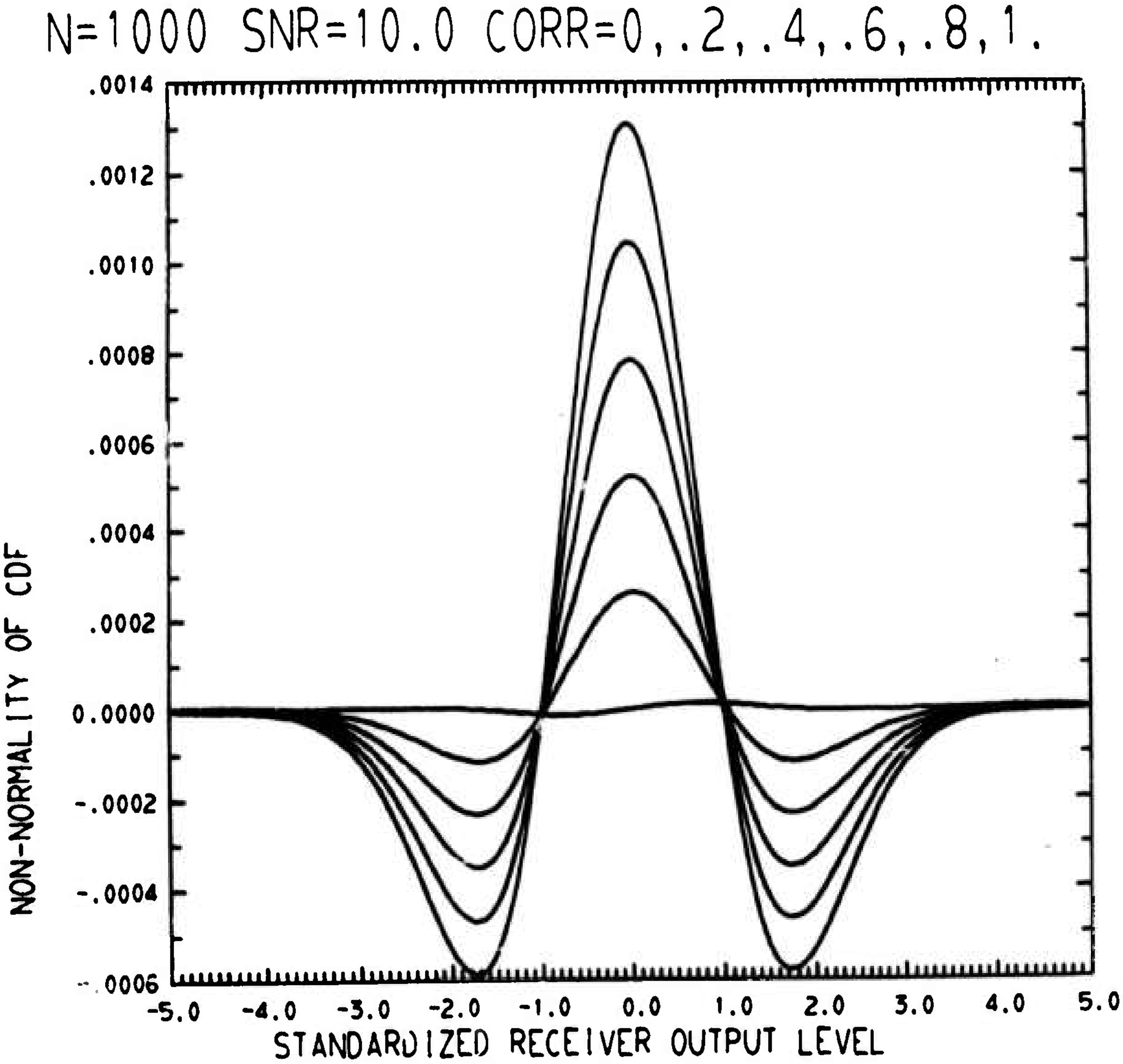
(f) γ = .1


N=1000 SNR= .1 CORR=0,.2,.4,.6,.8,1.
.0010
.0008
.0006
.0004
.0002
0.0000
-.0002
-.0004
-.0006
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

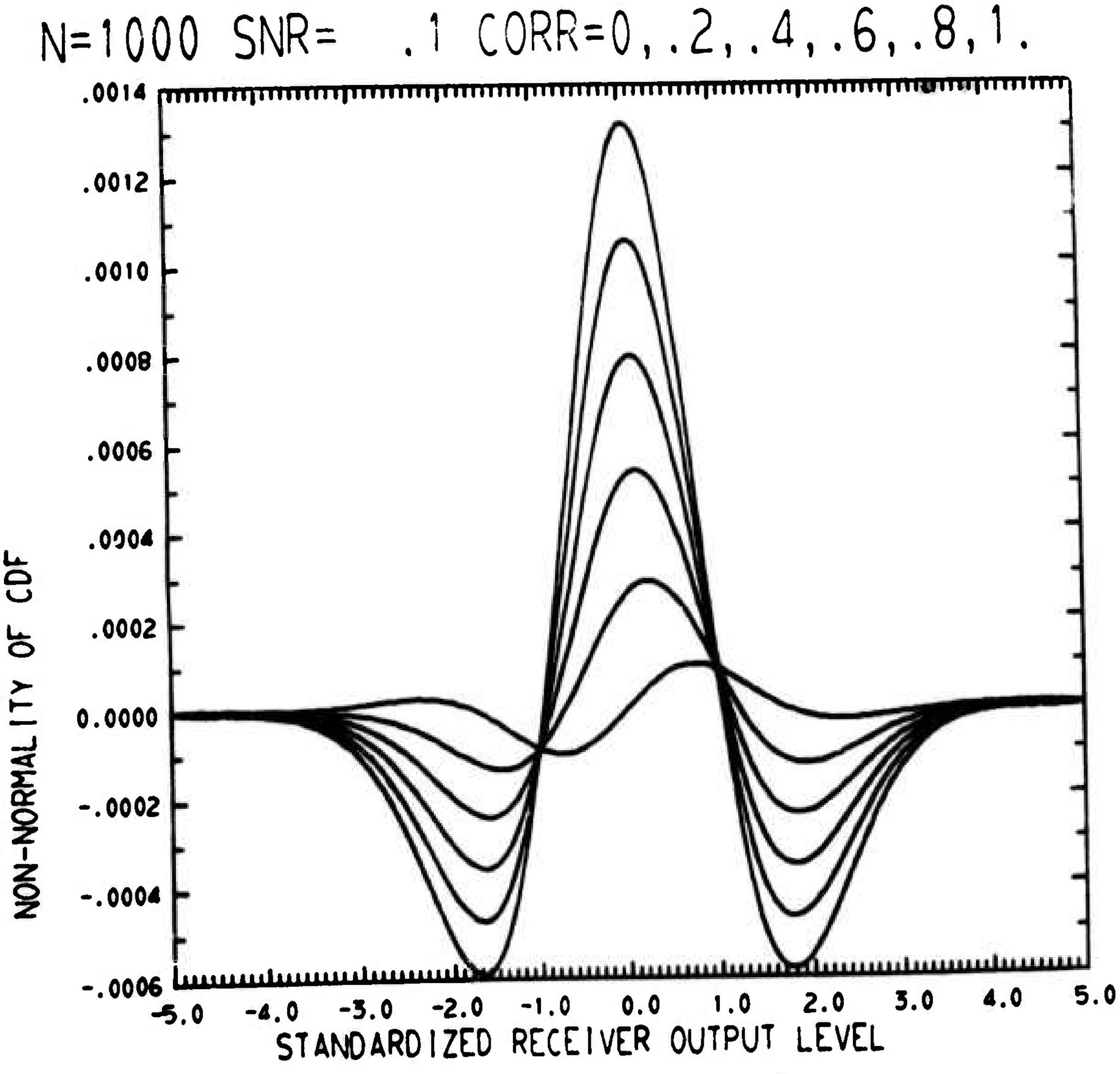
(g) $\gamma = 1$


N=1000 SNR= 1.0 CORR=0,.2,.4,.6,.8,1.
.0025
.0020
.0015
.0010
.0005
0.0000
-.0005
-.0010
-.0015
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL
NON-NORMALITY OF CDF

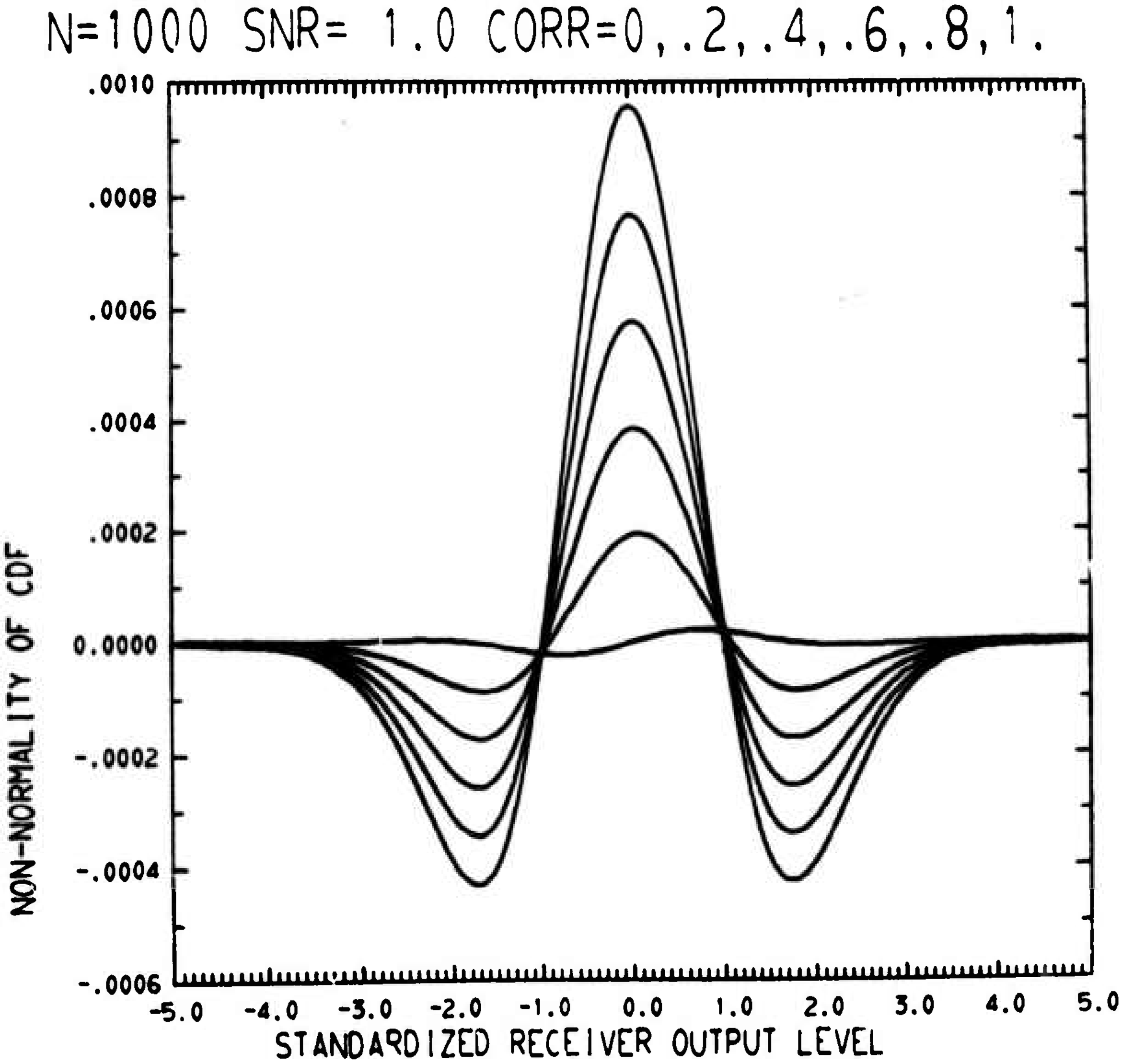
(h) $\gamma = 1$


N=1000 SNR=10.0 CORR=0,.2,.4,.6,.8,1.
.0014
.0012
.0010
.0008
.0006
.0004
.0002
0.0000
-.0002
-.0004
-.0006
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
NON-NORMALITY OF CDF
STANDARDIZED RECEIVER OUTPUT LEVEL

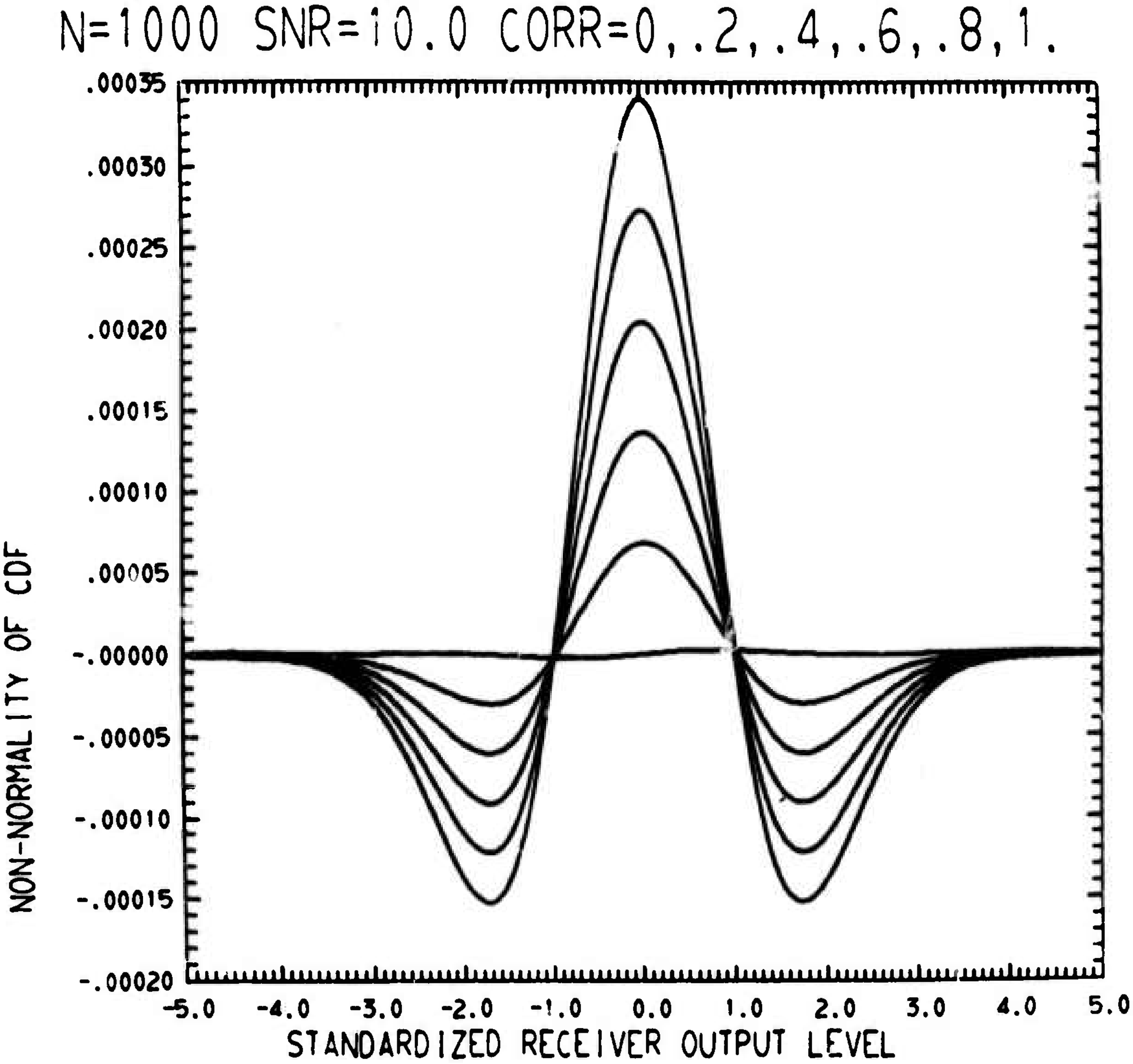
(i) $\gamma = 1$


N=1000 SNR= .1 CORR=0,.2,.4,.6,.8,1.
.0014
.0012
.0010
.0008
.0006
.0004
.0002
0.0000
-.0002
-.0004
-.0006
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

(j) $\gamma = .1$


N=1000 SNR= 1.0 CORR=0,.2,.4,.6,.8,1.
.0010
.0008
.0006
.0004
.0002
0.0000
-.0002
-.0004
-.0006
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

(k) γ = .1


N=1000 SNR=10.0 CORR=0,.2,.4,.6,.8,1.
.00035
.00030
.00025
.00020
.00015
.00010
.00005
-.00000
-.00005
-.00010
-.00015
-.00020
NON-NORMALITY OF CDF
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
STANDARDIZED RECEIVER OUTPUT LEVEL

(1) $\gamma = .1$

Although both the Edgeworth approximation (4.22) and the numerical inversion computation (4.46) are useful, their usefulness decreases when considering the tails of the cdf $F(x)$. Consequently, a "tight" bound on the tails of $F(x)$ would be desirable. The Markov inequality (4.59) can, of course, be used, but the so-called Chernoff bound [17, pp. 118-122] decreases exponentially, whereas (4.49) decreases only as $\varepsilon^{-\nu}$. Also, the distribution under consideration here is skewed (since $K_3 \neq 0$), a property which is lost when using bounds such as (4.49).

Consider for $\lambda \geq 0$:

$$P\{\eta > \varepsilon\} = \int_{\varepsilon}^{\infty} dF_\eta(x)$$

$$\leq \int_{-\infty}^{\infty} e^{\lambda(x - \varepsilon)} dF_\eta(x)$$

$$= e^{-\lambda\varepsilon} \int_{-\infty}^{\infty} e^{\lambda x}\, dF_\eta(x)$$

$$= e^{-\lambda\varepsilon}\, \phi_\eta(-j\lambda),$$

for all $\lambda \geq 0$, such that $\phi_\eta(-j\lambda)$ exists. From the development of (4.8) it is easily seen that $\phi_\eta(-j\lambda)$ exists only for $\lambda^2\beta^2 < 1$.

Similarly, for $\lambda \leq 0$:

$$P\{\eta < \varepsilon\} = \int_{-\infty}^{\varepsilon} dF_\eta(x) \leq \int_{-\infty}^{\infty} e^{\lambda(x - \varepsilon)} dF_\eta(x)$$

$$= e^{-\lambda\varepsilon} \int_{-\infty}^{\infty} e^{\lambda x} dF_\eta(x)$$

$$= e^{-\lambda\varepsilon}\, \phi_\eta(-j\lambda),$$

for all $\lambda \leq 0$ such that $\phi_\eta(j\lambda)$ exists.

Defining $h(\lambda) = \log \phi_\eta(-j\lambda)$ $(\lambda^2\beta^2 < 1)$, we have

$$P\{\eta > \varepsilon\} \le e^{-\lambda\varepsilon + h(\lambda)}, \text{ for } 0 \le \lambda < 1/\beta,$$

and

$$P\{\eta < \varepsilon\} \le e^{-\lambda\varepsilon + h(\lambda)}, \text{ for } -1/\beta < \lambda \le 0.$$

We now seek to minimize the quantity $b(\lambda) = \exp\{h(\lambda) - \lambda\varepsilon\}$ with respect to λ. We have:

$$b'(\lambda) = (h'(\lambda) - \varepsilon)\, b(\lambda) = 0$$

iff $h'(\lambda) = \varepsilon$, where $h'(\lambda) = \varepsilon$ will result in the minimum $b(\lambda)$ if

i) there exists a λ such that $h'(\lambda) = \varepsilon$

and

ii) $b''(\lambda)\Big|_{\varepsilon = h'(\lambda)} > 0$.

We have:

$$b''(\lambda)\Big|_{\varepsilon=h'(\lambda)} = h''(\lambda)b(\lambda)\Big|_{\varepsilon=h'(\lambda)} > 0$$

iff

$$h''(\lambda)\Big|_{\varepsilon=h'(\lambda)} > 0 \ ,$$

and

$$h''(\lambda)\Big|_{\varepsilon=h'(\lambda)} = e^{-h(\lambda)} \int_{-\infty}^{\infty} (x-\varepsilon)^2 e^{\lambda x} dF_\eta(x) > 0,$$

implying that $\varepsilon = h'(\lambda)$ does lead to the minimum $b(\lambda)$. For the problem at hand,

$$h(\lambda) = -\frac{N}{2}\log(1-\beta^2\lambda^2) + \frac{\beta_1\beta^3\lambda^3 + \frac{1}{2}\beta_2\beta^2\lambda^2}{1-\beta^2\lambda^2}$$

and

$$h'(\lambda) = \frac{(N+\beta_2)\beta^2\lambda + 3\beta_1\beta^3\lambda^2 - N\beta^4\lambda^3 - \beta_1\beta^5\lambda^4}{(1-\beta^2\lambda^2)^2} \ . \tag{4.59}$$

The bounds of interest become

$$P\{\eta > h'(\lambda)\} \leq e^{h(\lambda) - \lambda h'(\lambda)}, \text{ for } 0 \leq \lambda < \frac{1}{\beta}, \tag{4.60}$$

and

$$P\{\eta < h'(\lambda)\} \leq e^{h(\lambda) - \lambda h'(\lambda)}, \text{ for } -\frac{1}{\beta} < \lambda \leq 0.$$

We note that $\{h'(\lambda): 0 \leq \lambda < \frac{1}{\beta}\} = \{x: 0 \leq x < \infty\}$ and $\{h'(\lambda): -\frac{1}{\beta} < \lambda \leq 0\} = \{x: -\infty < x \leq 0\}$. The bounds of (4.60) are illustrated in Figure 5.6 for $|h'(\lambda)| \leq 10$, CORR = 1, $N\epsilon\{2, 10, 100, 1000\}$, $SNR_i \epsilon\{.1, 1, 10\}$, and $\gamma \epsilon \{.1, 1\}$. Note that the bounds of (4.60) are decreasing with |CORR| and that

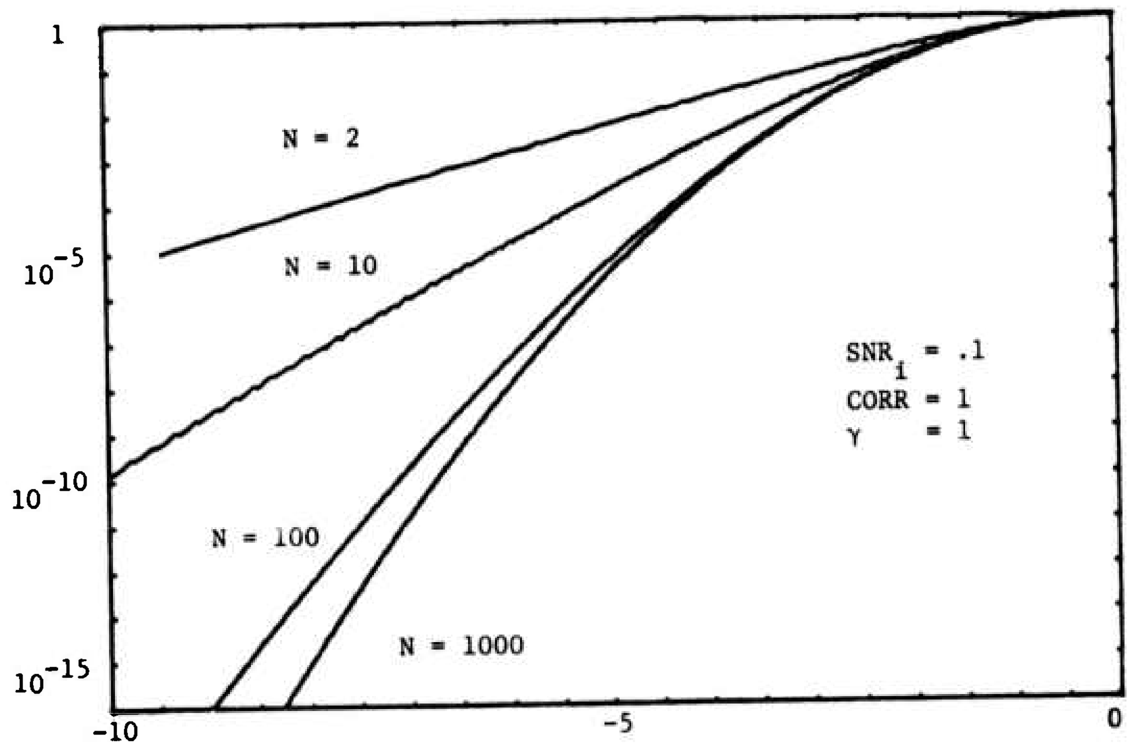
$$P\{\eta < \epsilon\}\Big|_{-CORR} = P\{\eta > -\epsilon\}\Big|_{CORR}, \tag{4.61}$$

and

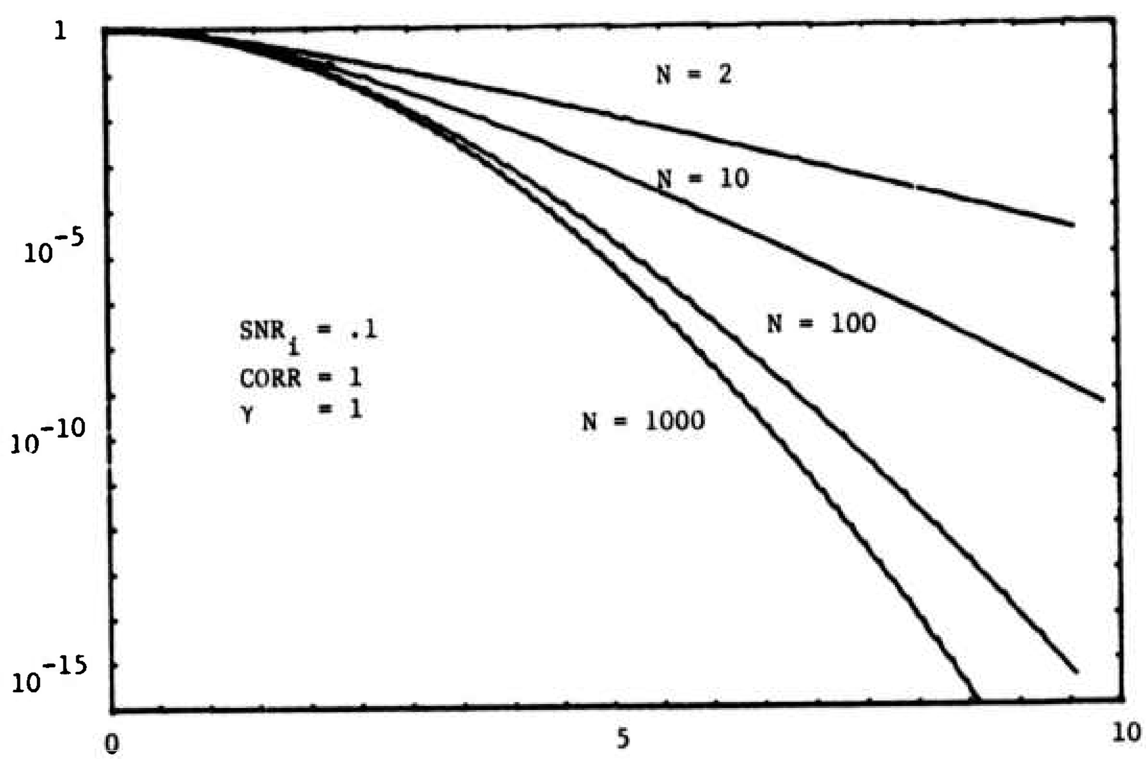
$$P\{\eta > \epsilon\}\Big|_{-CORR} = P\{\eta < -\epsilon\}\Big|_{CORR}.$$

We now have a relatively complete characterization of the cdf $F_\eta(x)$, where η is the standardized r.v. $\eta = \frac{z - \mu_z}{\sigma_z}$. In order to use these results for the r.v. z, we note that

$$F_z(x) = F_\eta\left(\frac{x - \mu_z}{\sigma_z}\right). \tag{4.62}$$

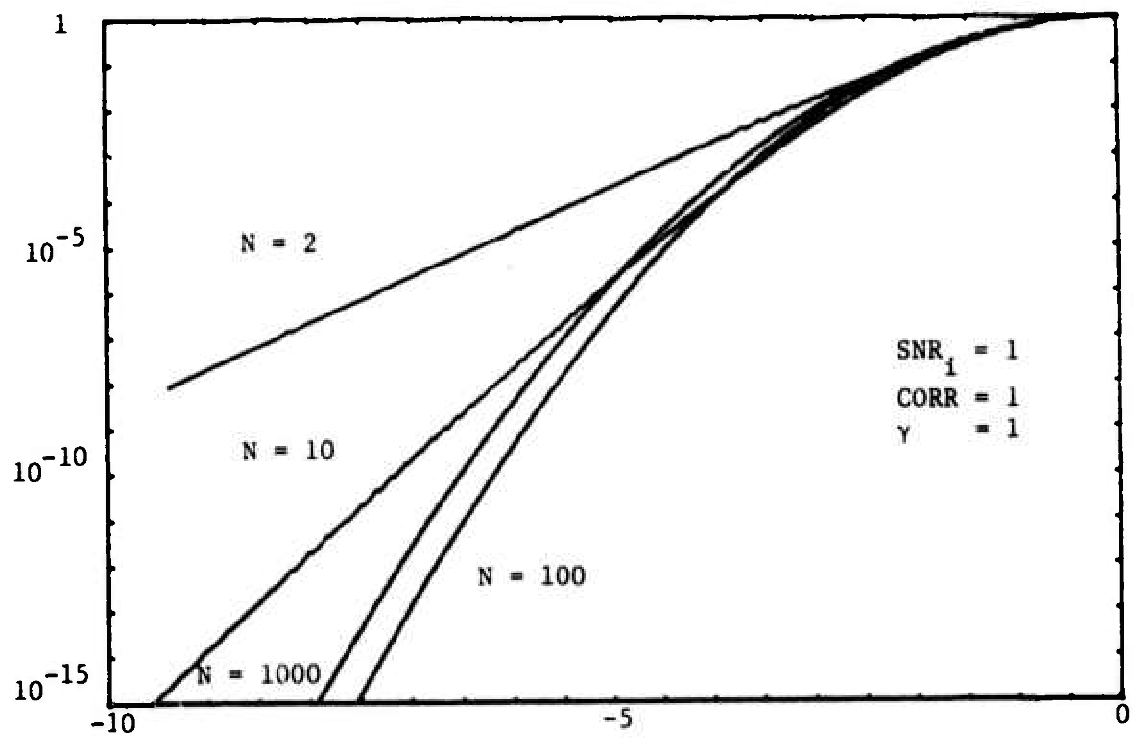


(a)

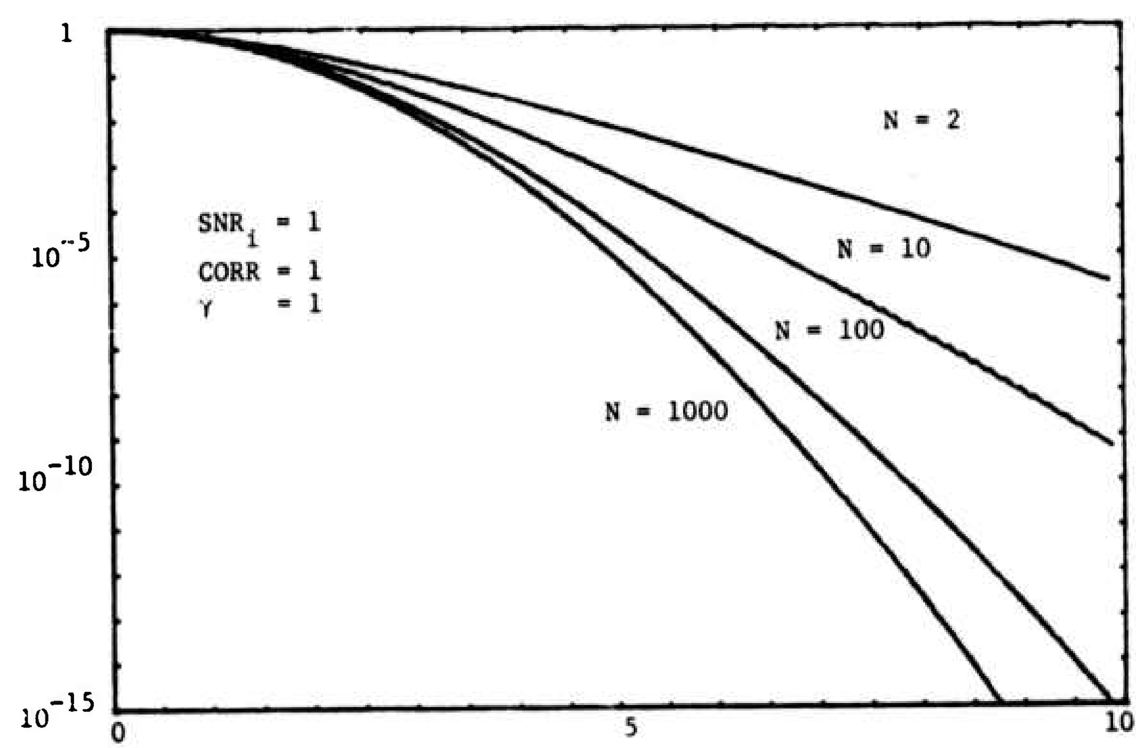


(b)

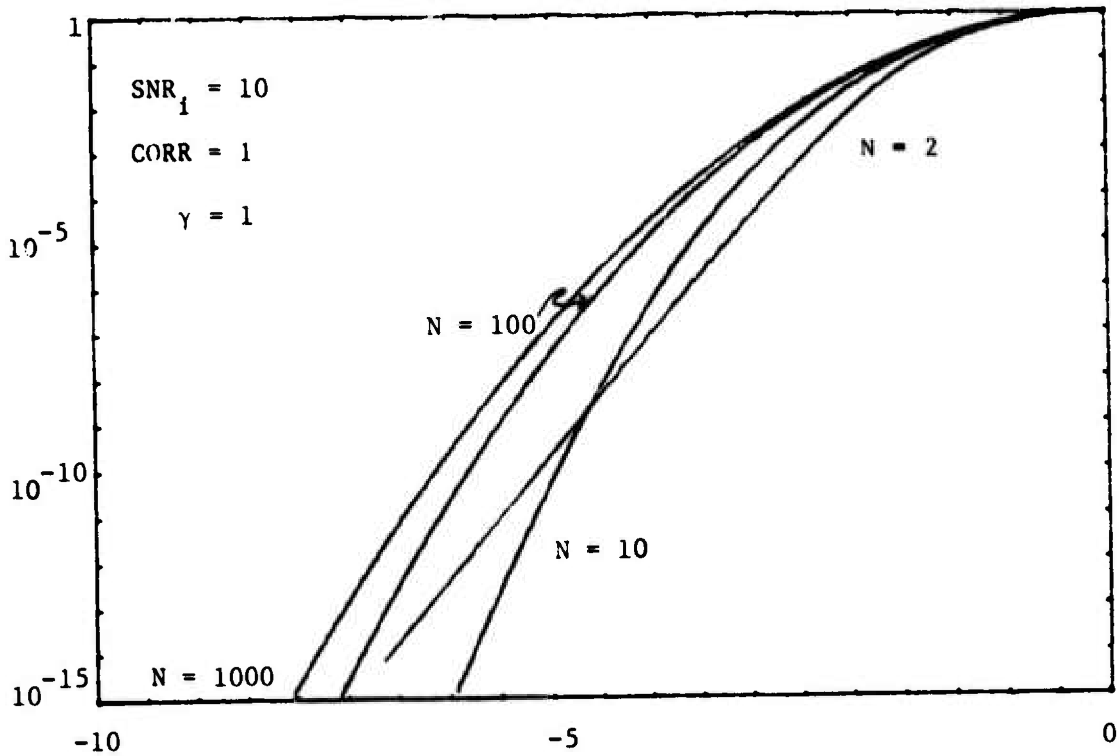
Figure 4.6 Chernoff bounds for cdf tails


1
10^{-5}
10^{-10}
10^{-15}
-10
-5
0
N = 2
N = 10
N = 100
N = 1000
SNR_i = 1
CORR = 1
γ = 1

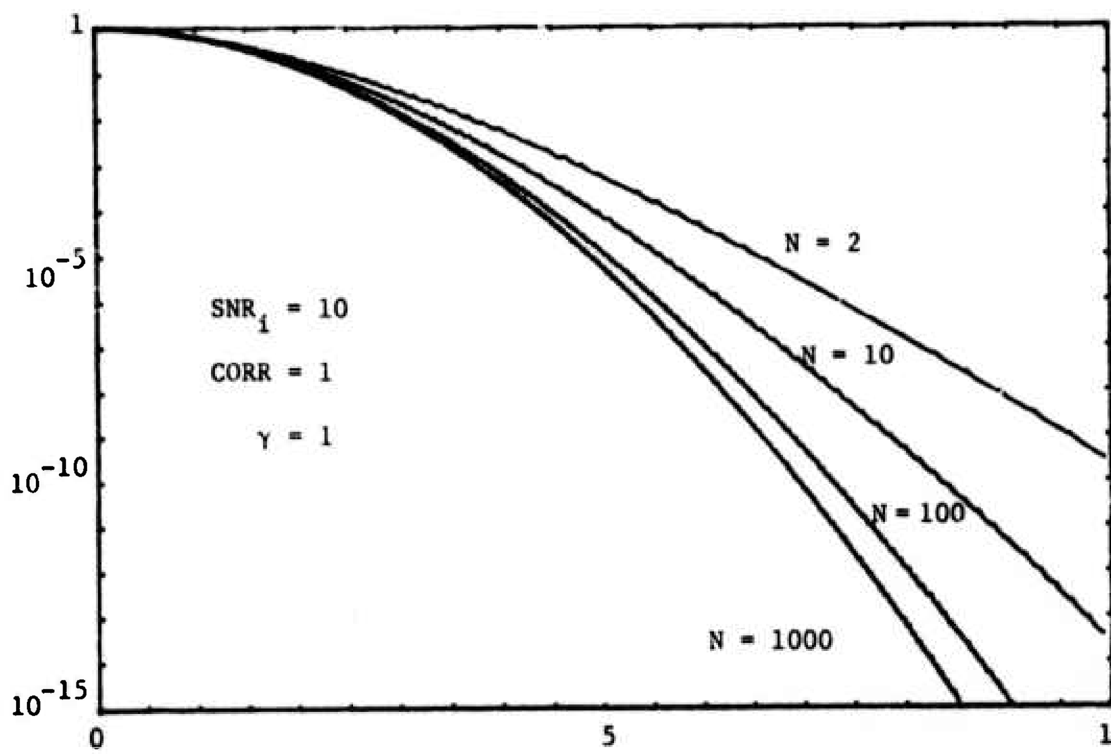
(c)


1
10^{-5}
10^{-10}
10^{-15}
0
5
10
N = 2
N = 10
N = 100
N = 1000
SNR_i = 1
CORR = 1
γ = 1

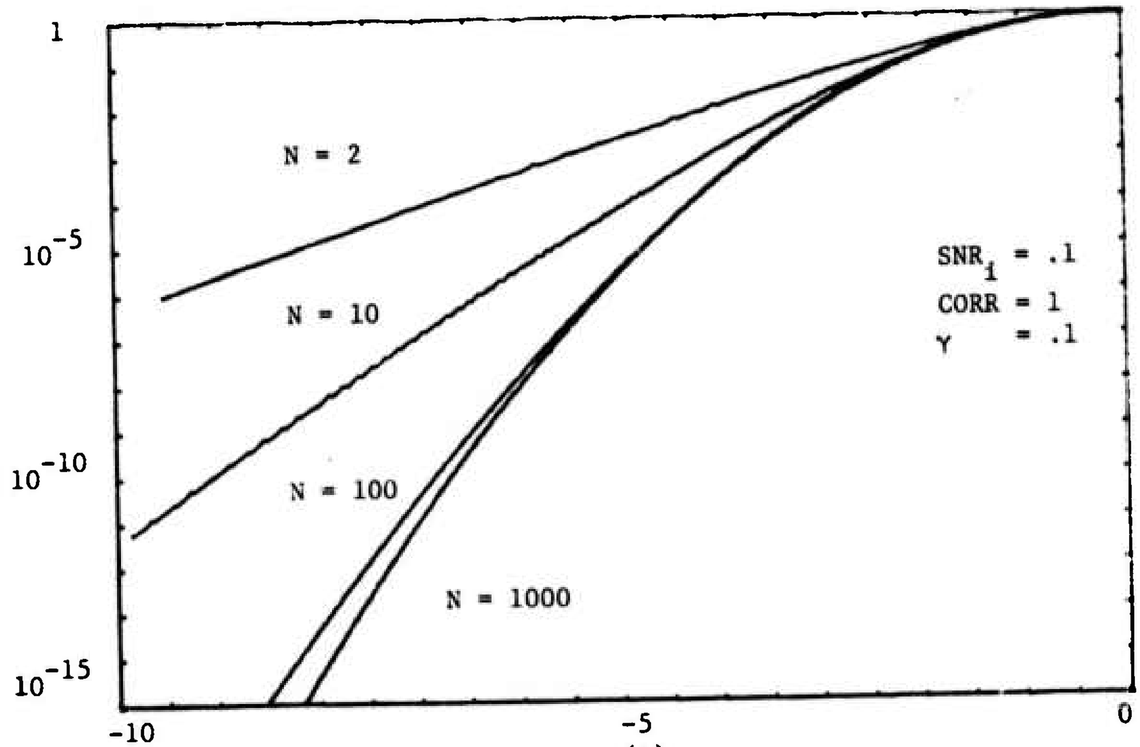
(d)


1
10^{-5}
10^{-10}
10^{-15}
-10
-5
0
SNR_i = 10
CORR = 1
γ = 1
N = 2
N = 100
N = 10
N = 1000

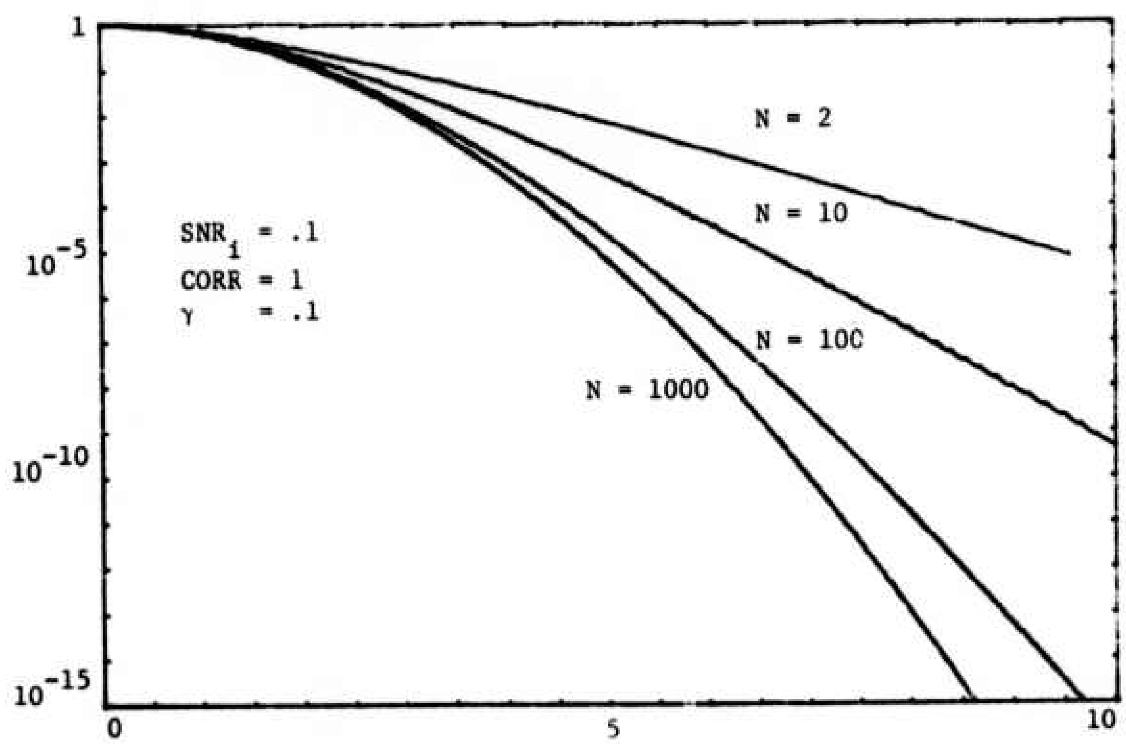
(e)


1
10^{-5}
10^{-10}
10^{-15}
0
5
10
SNR_i = 10
CORR = 1
γ = 1
N = 2
N = 10
N = 100
N = 1000

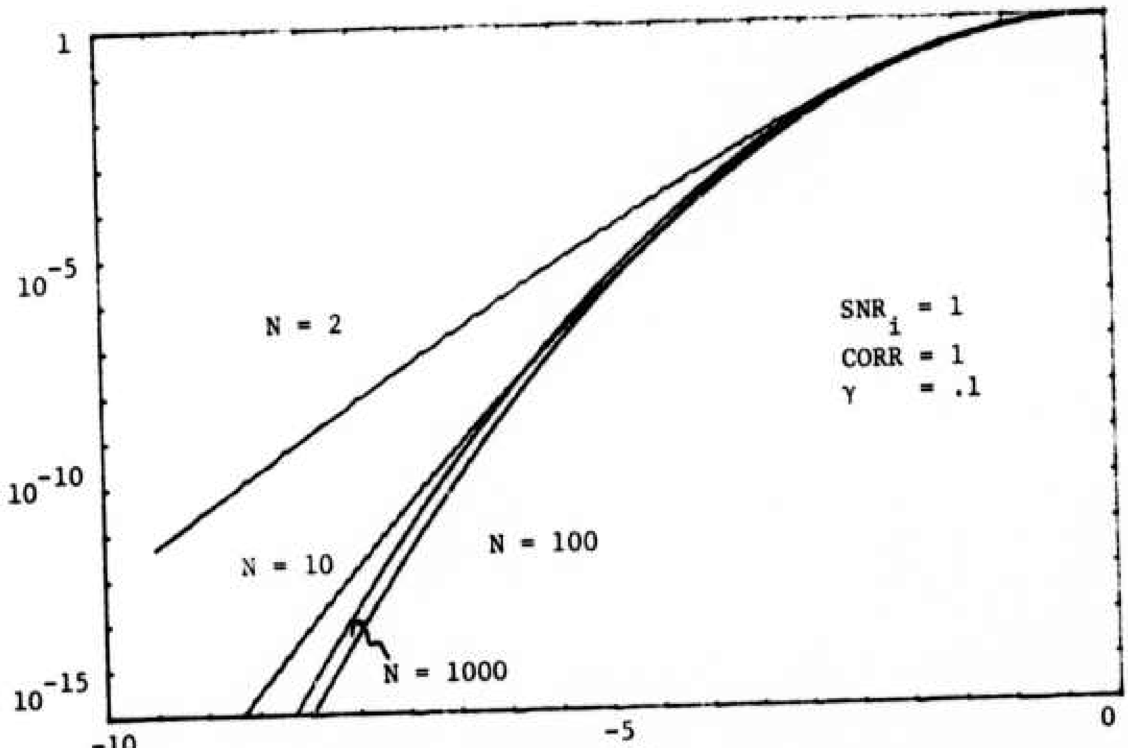
(f)


1
10^-5
10^-10
10^-15
-10
-5
0
N = 2
N = 10
N = 100
N = 1000
SNR_i = .1
CORR = 1
γ = .1

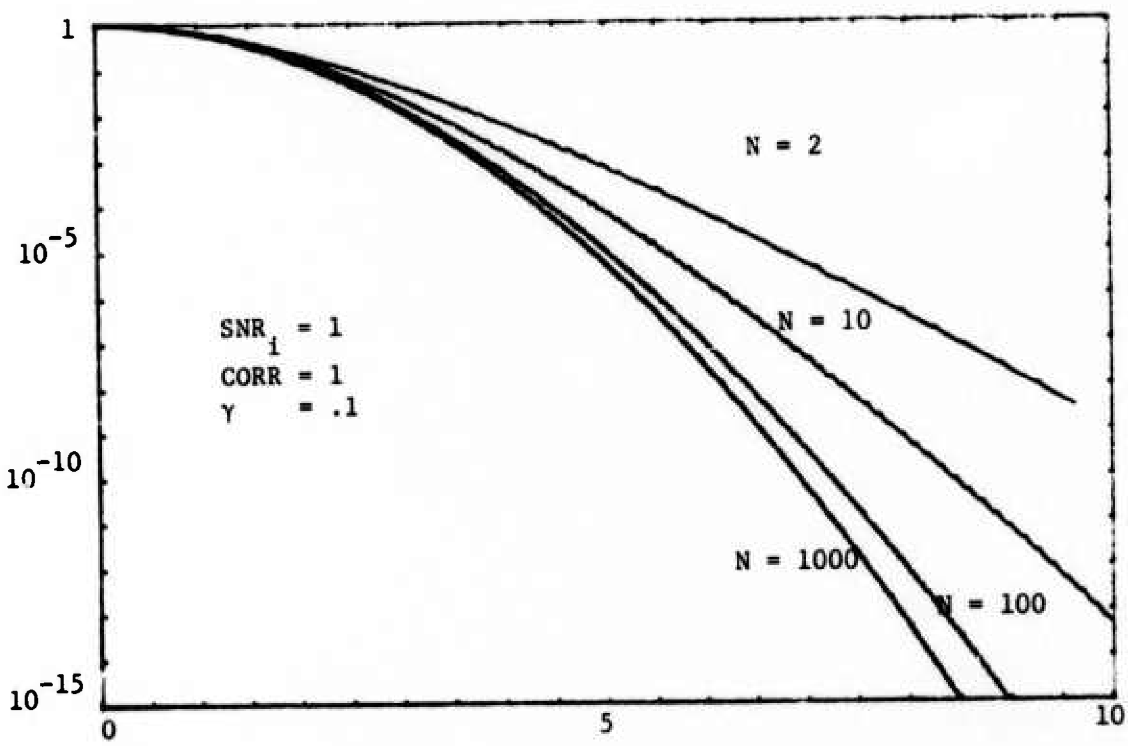
(g)


1
10^-5
10^-10
10^-15
0
5
10
N = 2
N = 10
N = 10C
N = 1000
SNR_i = .1
CORR = 1
γ = .1

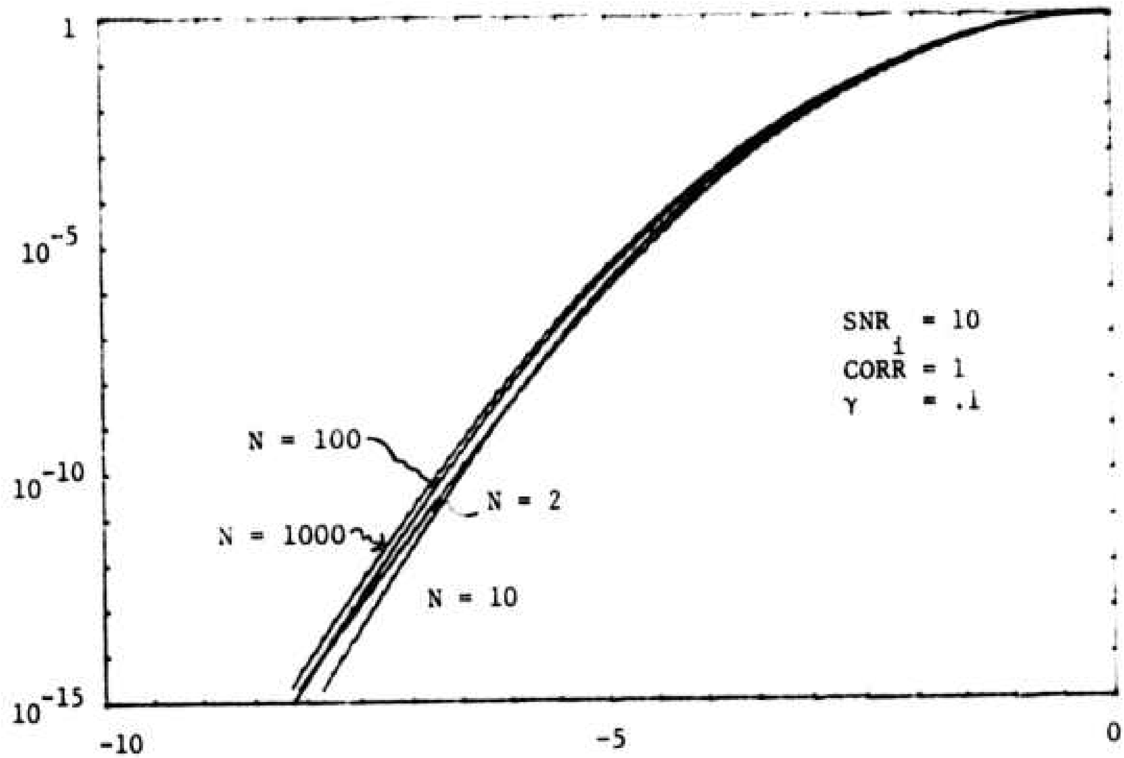
(h)


1
10^{-5}
10^{-10}
10^{-15}
-10
-5
0
N = 2
N = 10
N = 100
N = 1000
SNR_i = 1
CORR = 1
γ = .1

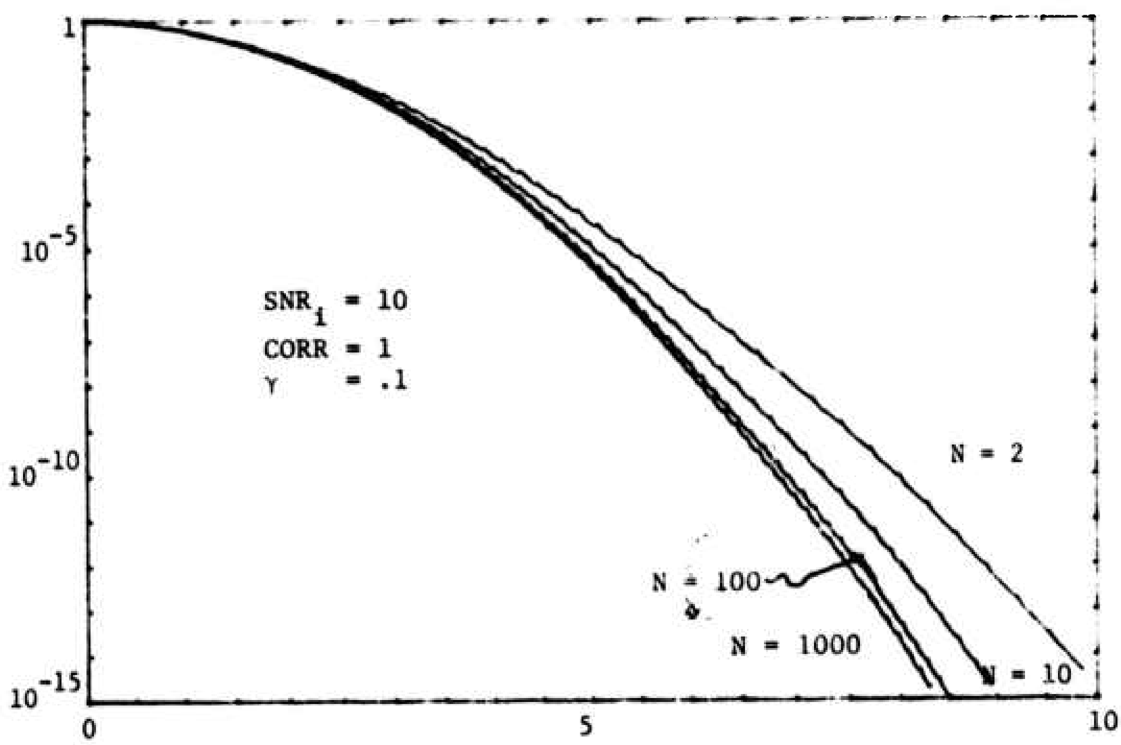
(i)


1
10^{-5}
10^{-10}
10^{-15}
0
5
10
N = 2
N = 10
N = 100
N = 1000
SNR_i = 1
CORR = 1
γ = .1

(j)


1
10^{-5}
10^{-10}
10^{-15}
-10
-5
0
SNR_i = 10
CORR = 1
γ = .1
N = 100
N = 2
N = 1000
N = 10

(k)


1
10^{-5}
10^{-10}
10^{-15}
0
5
10
SNR_i = 10
CORR = 1
γ = .1
N = 2
N = 100
N = 1000
N = 10

(1)

V Conclusion

In this report, we have studied two predistorted-replica correlation receivers: The FLAC receiver and the ARSAC receiver. A Doppler analysis was performed for each, the results of which indicate that for signal time bandwidth products TW such that

$$|\xi - 1| < \frac{1}{2TW},$$

the difference in sensitivity is not dramatic; however, for larger time-bandwidth products the ARSAC receiver vastly outperforms the FLAC receiver.

Sections II-C and III-C pointed out that the FLAC concept can perform satisfactorily in a multiple user environment; whereas the ARSAC concept will not.

Sections II-B and III-B dealt with discrete time structures for the FLAC and the ARSAC systems, respectively. In fact, a digital realization of the FLAC receiver can be made requiring only two additions and one multiply per output sample. The output, y_n, for the FLAC receiver is, from Section II-B:

$$y_n = \sum_{p=0}^{L-1} R_{n-p} R_{n-L-p}, \tag{5.1}$$

or

$$y_{n+1} = \sum_{p=-1}^{L-2} R_{n-p} R_{n-L-p}$$

$$= R_{n-L+1}(R_{n+1} - R_{n-2L+1}) + y_n, \tag{5.2}$$

indicating that if $y_\ell = 0$, and $R_n = 0$ for all $n \le \ell$, (5.2) is equivalent to (5.1) for all $n = \ell, \ell+1, \ell+2, \ldots$.

It is to be emphasized that the assumptions made about the channel imply that there is no intersymbol interference, an assumption which is certainly restrictive. For the known signal case, several authors have shown that the optimal structure is a matched filter followed by a transversal filter [3]. It is conjectured here that similar performance improvement can be obtained by following the predistorted replica correlation receiver by a transversal filter.

Finally, the noise performance for the family of predistorted replica correlation receivers was dealt with in Section IV. The analysis there assumed that the noise component of the received process was additive Gaussian noise, such that the samples of the received process were uncorrelated.

References

[1] D.W. Tufts, "Nyquist's problem - the joint optimization of trnasmitter and receiver in pulse amplitude modulation," Proc. IEEE, vol. 53, pp. 248-259, Mar. 1965.

[2] G.D. Forney, "Maximum-likelihood sequence estimation of digital sequences in the presence of intersymbol interference," IEEE Trans. Inform. Theory, vol. IT-18, pp. 363-378, May 1972.

[3] T. Ericson, "Structure of optimum receiving filters in data transmission systems," IEEE Trans. Inform. Theory, vol. IT-17, pp. 352-353, May 1971.

[4] J.M. Wozencraft and I.M. Jacobs, Principles of Communication Engineering, John Wiley, New York, 1965.

[5] Research proposal, May 1973.

[6] D. Graupe, Identification of Systems, pp. 80-84, Van Nostrand Reinhold, 1972.

[7] W. Rudin, Principles of Mathematical Analysis, p. 112, McGraw-Hill, 1964.

[8] F.N. Cornett, D.C. Farden and L.L. Scharf, "Predistorted replica correlation receiver," Status Report, Dec. 1973.

[9] L.C.Andrews, "The probability density function for the output of a cross-correlator with bandpass inputs," IEEE Trans. Inform. Theory, vol. IT-19, pp. 13-19, Jan. 1973.

[10] N.P. Murarka, "The probability density function for correlated noisy reference inputs," IEEE Trans. Communication Technology, vol. COM-19, pp. 711-714, Oct. 1971.

[11] L.E. Miller and J.S. Lee, "The probability density function for the output of an analog cross-correlator with correlated bandpass inputs," IEEE Trans. Inform. Theory, vol. IT-20, pp. 433-440, July 1974.

[12] W. Feller, An Introduction to Probability Theory and Its Applications, vol. II, John Wiley, New York, 1966.

[13] M.G. Kendall and A. Stuart, The Advanced Theory of Statistics, vol. 1, Charles Griffin & Company, London, 1963.

[14] H. Cramér, Mathematical Methods of Statistics, Princeton University Press, Princeton, 1957.

[15] A.H. Nutall, "Alternate forms for numerical evaluation of cumulative probability distributions directly from characteristic functions," Proc. IEEE (Letters), vol. 58, pp. 1872-1873, Nov. 1970.

[16] J.W. Cooley, P.A.W. Lewis and P.D. Welch, "Application of the Fast Fourier Transform to computation of Fourier integrals, Fourier series, and convolution integrals," IEEE Trans. Audio Electroacoust., vol. AU-15, pp. 79-84, June 1967.

[17] H.L. Van Trees, Detection, Estimation, and Modulation Theory, Part I, John Wiley, New York, 1968.