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# BIFILAR PENDULUM TECHNIQUE FOR DETERMINING MASS PROPERTIES OF DISCOS PACKAGES

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Johns Hopkins University

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A bifilar pendulum was used to determine the mass properties of the Discos packages on the Triad satellite. The pendulum design was unique in that it allowed the package to be rotated into six different angular orientations. Each angular position was located with respect to the center of mass of the package. Analysis and results are given that show that the inertia matrix can be completed using the bifilar pendulum.



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*Technical Memorandum*

# BIFILAR PENDULUM TECHNIQUE FOR DETERMINING MASS PROPERTIES OF DISCOS PACKAGES

by R.A. MATTEY

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8621 Georgia Avenue ◦ Silver Spring, Maryland ◦ 20910
Operating under Contract N00017-72-C-4401 with the Department of the Navy



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# 1. INTRODUCTION

The accuracy of the Navy navigation satellite, Transit, is dependent on precise orbit determination and orbit prediction. Improvements in orbit prediction have been limited by uncertainties in orbit disturbance from solar pressure and atmospheric drag. The Disturbance Compensation System (Discos) was designed at Stanford University under subcontract to the Applied Physics Laboratory of The Johns Hopkins University. It was flown on the Triad satellite, and it has opened possibilities for improvements in accuracy and operational convenience for the Transit system by freeing Triad's orbit of disturbances larger than $5 \times 10^{-12}$ g.

To eliminate the effect of forces on the Discos system because of the various satellite components, each main electronics package (shown in Fig. 1) had its mass properties measured. To accomplish this a bifilar pendulum was designed that could rotate a body into six distinct positions with respect to a coordinate system that had its origin fixed at the body's center of mass. With this rotational capability, six independent moment of inertia measurements were made. The data were then reduced with the aid of a computer, and the complete inertia matrix with respect to the package center of mass was determined.

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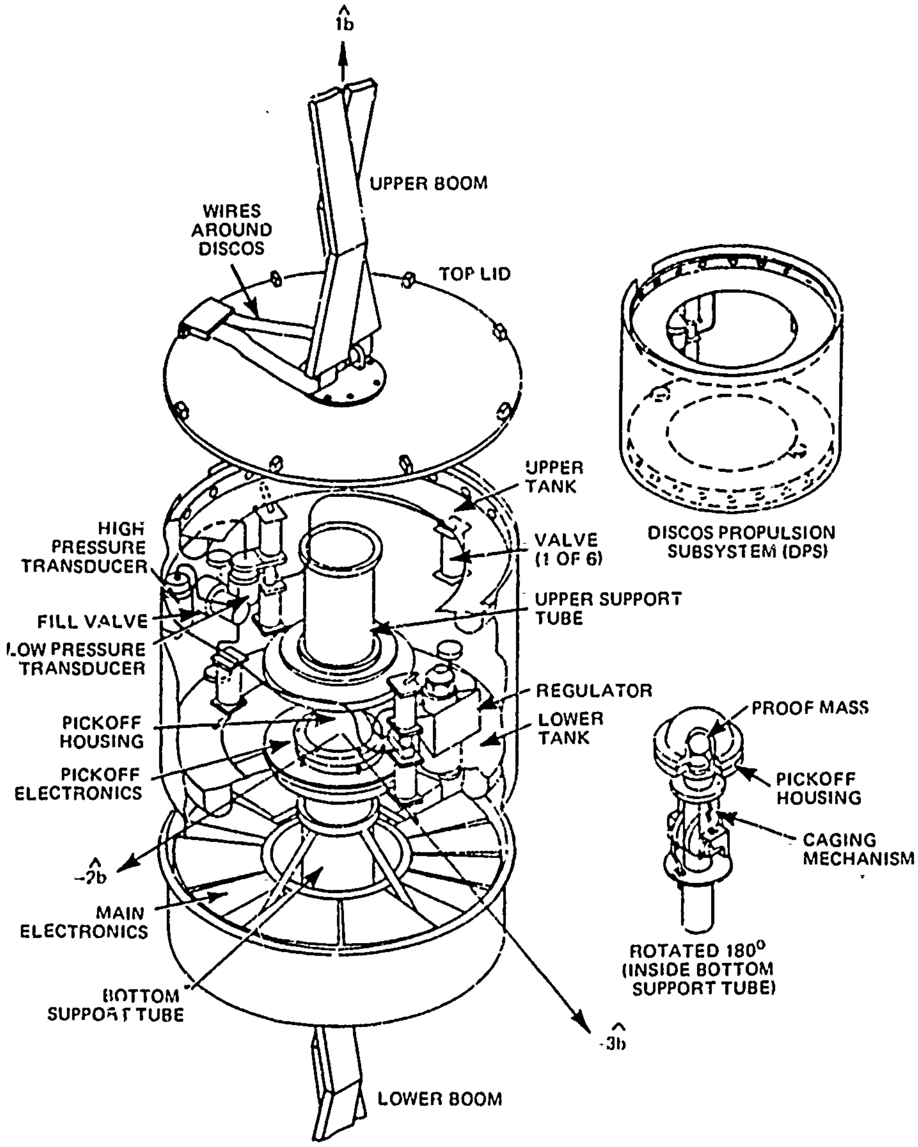



Fig. 1 DISTURBANCE COMPENSATION SYSTEM (DISCOS)

# 2. ANALYSIS

Figure 2 shows a line with respect to a mass particle $m_j$. The moment of inertia of the particle with respect to the line L ($\lambda$, $\mu$, $\gamma$) is

$$I_L = \lambda^2 Ixx + \mu^2 Iyy + \gamma^2 Izz - 2\mu\lambda Ixy - 2\lambda\gamma Ixz - 2\mu\gamma Iyz \, . \quad (1)$$

$\lambda$, $\mu$, and $\alpha$ are the direction cosines of a line L in space that passes through the origin of a fixed Cartesian system of coordinates.

In order to solve for the unknown cross products in Eq. (1), it is necessary that the body shown in Fig. 2 be moved to six different angular positions and that the moment of inertia is measured at each of those positions. Equation (1) was derived so that only three of the angular positions can be mutually perpendicular. Figure 3 shows three of the mutually perpendicular axes $x'$, $y'$, and $z'$. Each of these axes could be placed on the axis of rotation. Also, the three skew lines are shown in Fig. 3; their angular rotations and line numbers are indicated below:

| Line | Rotation |
|---|---|
| 1 | +5° |
| 2 | -5°,+45° |
| 3 | +45° |

To use the moment of inertia data obtained from the six different measurements, the equations are set up as shown in Eq. (2). Equation (1) is rewritten where the prime refers to the moment of inertia of the axis being measured:

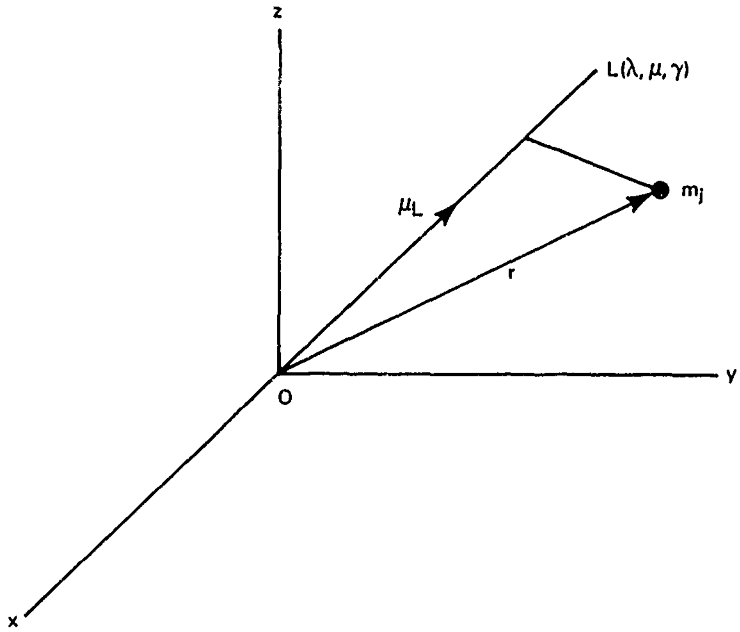



Fig. 2 LINE WITH RESPECT TO A MASS PARTICLE

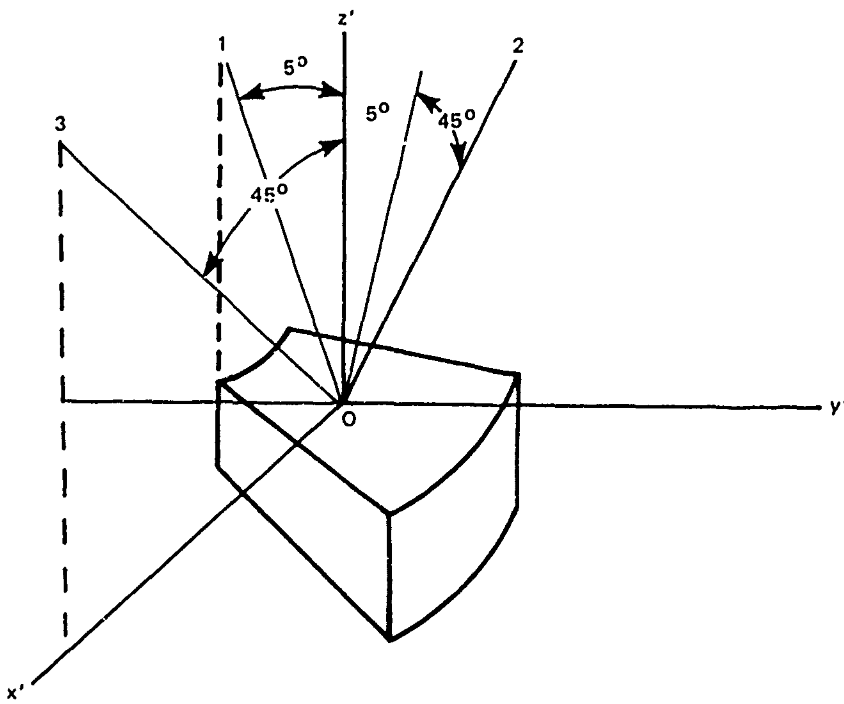



Fig. 3 SYSTEM MEASUREMENT AXES

$$\mu\lambda Ixy + \lambda\gamma Ixz + \mu\gamma Iyz = \tfrac{1}{2}[\lambda^2 I'xx + \mu^2 I'yy + \gamma^2 I'zz - I_L] . \quad (2)$$

Now, setting up a matrix of the left-hand side of Eq. (2) in consideration of Fig. 4 where $S\alpha$ = sin $\alpha$ and $C\alpha$ = cos $\alpha$ (also for three skew lines) we obtain the following:

$$\begin{bmatrix} S\beta_1 S\alpha_1 C\beta_1 S\alpha_1 & C\beta_1 S\alpha_1 C\alpha_1 & S\beta_1 S\alpha_1 C\alpha_1 \\ S\beta_2 S\alpha_2 C\beta_2 S\alpha_2 & C\beta_2 S\alpha_2 C\alpha_2 & S\beta_2 S\alpha_2 C\alpha_2 \\ S\beta_3 S\alpha_3 C\beta_3 S\alpha_3 & C\beta_3 S\alpha_3 C\alpha_3 & S\beta_3 S\alpha_3 C\alpha_3 \end{bmatrix} \begin{bmatrix} I'xy \\ I'xz \\ I'yz \end{bmatrix} . \quad (3)$$

In setting up the right-hand side in a matrix form, we obtain:

$$\begin{bmatrix} (C\beta_1 S\alpha_1)^2 I'xx & (S\beta_1 S\alpha_1)^2 I'yy & C^2\alpha I'zz - I'_L \\ (C\beta_2 S\alpha_2)^2 I''xx & (S\beta_2 S\alpha_2)^2 I''yy & C\alpha_2 I''zz - I''_L \\ (C\beta_3 S\alpha_3)^2 I'''xx & (S\beta_3 S\alpha_3)^2 I'''yy & C\alpha_3 I'''zz - I'''_L \end{bmatrix} . \quad (4)$$

Using matrix notation on the left- and right-hand side,

$$[Cij] \begin{bmatrix} Ixy \\ Ixz \\ Iyz \end{bmatrix} = [Di - ILi] \; (\tfrac{1}{2}) , \quad (5)$$

therefore letting $\begin{bmatrix} Ixy \\ Ixz \\ Iyz \end{bmatrix}$ = Eij. Solving for the cross products

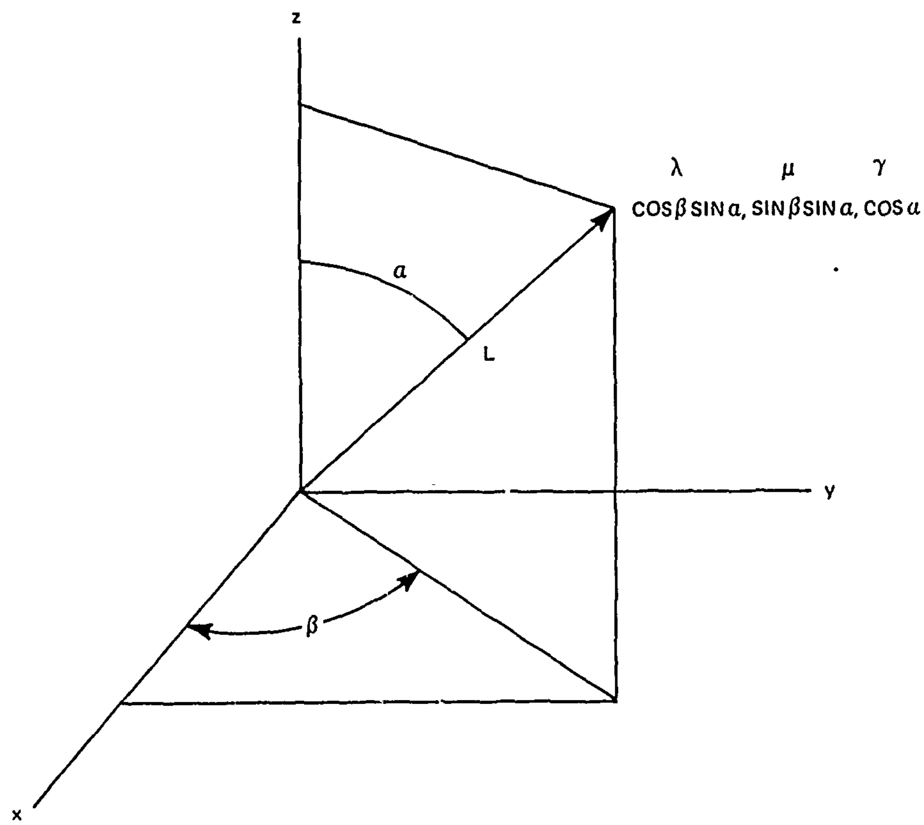



Fig. 4 LINE WITH REFERENCE DESIGNATIONS

we have

$$[Eij] = \frac{[Cij]^{-1}\,[Dij - Iij]}{2} \quad . \qquad (6)$$

Knowing all the cross products we can set up the following matrix equation to solve for the principal axis:

$$\begin{bmatrix} (Ixx - I) & Ixy & Ixz \\ Iyx & (Iyy - I) & Iyz \\ Izx & Izy & (Izz - I) \end{bmatrix} = 0 \quad . \qquad (7)$$

If we let $I = \lambda$ we can solve the above as follows:

$$\begin{aligned} &(IxxIyyIzz - I^2zyIxx - I^2zxIyy - I^2yxIzz + 2IyzIzxIyx) \\ &+ \lambda(IxxIyy + IyyIzz + IxxIzz - I^2zy + I^2zx + I^2yx) \qquad (8) \\ &+ \lambda^2(Ixx + Iyy + Izz) - \lambda^3 = 0 \quad . \end{aligned}$$

The three roots of the above will be the moments of inertia about the principal axis of the package.

Since for rotation about a principal axis the angular velocity vector coincides with this axis, the set of numbers $\omega_x'$, $\omega_y'$, $\omega_z'$, which satisfies the following corresponding to $I_1$ of Eq. (8), consists of the direction numbers for axis $I_1$.

$$\begin{aligned} &(Ixx - I_1)\omega_x^{(1)} - Ixy\omega_y^{(1)} - Ixz\omega_z^{(1)} = 0 \\ &-Iyx\omega_x^{(1)} + (Iyy - I_1)\omega_y^{(1)} - Iyz\omega_z^{(1)} = 0 \qquad (9) \\ &-Izx\omega_x^{(1)} - Izy\omega_y^{(1)} + (Izz - I_1)\omega_z^{(1)} = 0 \quad . \end{aligned}$$

Solving Eq. (9) will result in the ratios $\omega_x^{(1)} : \omega_y^{(1)} : \omega_z^{(1)}$, and the axis associated with $I_1$ is thereby defined relative to the original coordinate system. To solve the ratios, let one direction number equal 1.0. Repeat this procedure for the other two roots of Eq. (8) to define the other principal axes with respect to the original coordinate system.

By using the above analysis and by knowing the moment of inertia of a package for three perpendicular axes as well as three skew axes, the inertia matrix can be completed. Once this inertia matrix is completed, the principal moments of inertia of the body can be determined as well as the orientation of the principal axes with respect to the coordinate axis system used for the original measurements.

## 3. DEVELOPMENT OF THE MEASURING TECHNIQUE

The design goals for the measurement of the mass properties were as follows:

1. Error in the moment of inertia ±0.5% of the sum of the three inertias about any set of three mutually perpendicular axes;

2. Mass center location determined within ±0.001 inch of mounting surfaces and hole patterns; and

3. Mass measurements of all packages to within ±0.050 gram.

Of all the requirements, the third was obviously the one that would be the easiest to meet. To determine the inertia matrix and center of mass it was apparent that extraordinary precision would have to be obtained.

There were two basic proposals put forth by APL to obtain the complete inertia matrix:

1. Obtain a large A-frame and suspend a torsional pendulum at least 6 feet long from it, and

2. Develop a pendulum similar to a trifilar pendulum.

The second proposal was selected for the following reasons:

1. Package orientation and subsequent dynamic unbalance would have a minimal effect on the system.

2. A preliminary calculation of the error allowed on the smallest package showed it to be on the order of $2.5 \times 10^{-5}$ slug ft$^2$. It was apparent, therefore, that air

drag on any system used would affect the period measurement and should be eliminated if possible. By using the second proposal the entire system could be put efficiently into a bell jar.

3. Other considerations would be the weight and, therefore, the configuration of the basic system. A bifilar pendulum lends itself to efficient mass distribution. Most of the mass can be concentrated at the center of oscillation, thereby eliminating large $Mr^2$ terms.

Figures 5 and 6 show the preliminary bifilar pendulum arrangement. Part of the testing on this bifilar pendulum concerned the problem of determining the best suspension system wire diameter and material. The figures show the pendulum supported by two 0.007 inch BeCu wires. Subsequent to the use of the 0.007 inch BeCu wire, music wire and hypodermic needle tubing suspension systems were tested. The final suspension system chosen was stainless steel hypodermic needle tubing with a 0.042 inch O.D. and a 0.006 inch wall.

The basic equation for determining the moment of inertia on a bifilar pendulum (knowing the period) is

$$I = \frac{T^2 W D^2}{16\pi^2 L}, \tag{10}$$

where I = moment of inertia, T = period, D = distance between the supporting wires, W = weight on the pendulum, and L = wire length.

Figure 6 shows the pendulum with a test mass in place and small pins that could be selectively removed and relocated to produce a change in its moment of inertia. This test mass proved that a moment of inertia change of approximately ±0.5% of the sum of three perpendicular axes could be measured. After it was proven that the bifilar pendulum would give the accuracy desired, the design of a more sophisticated system was begun.

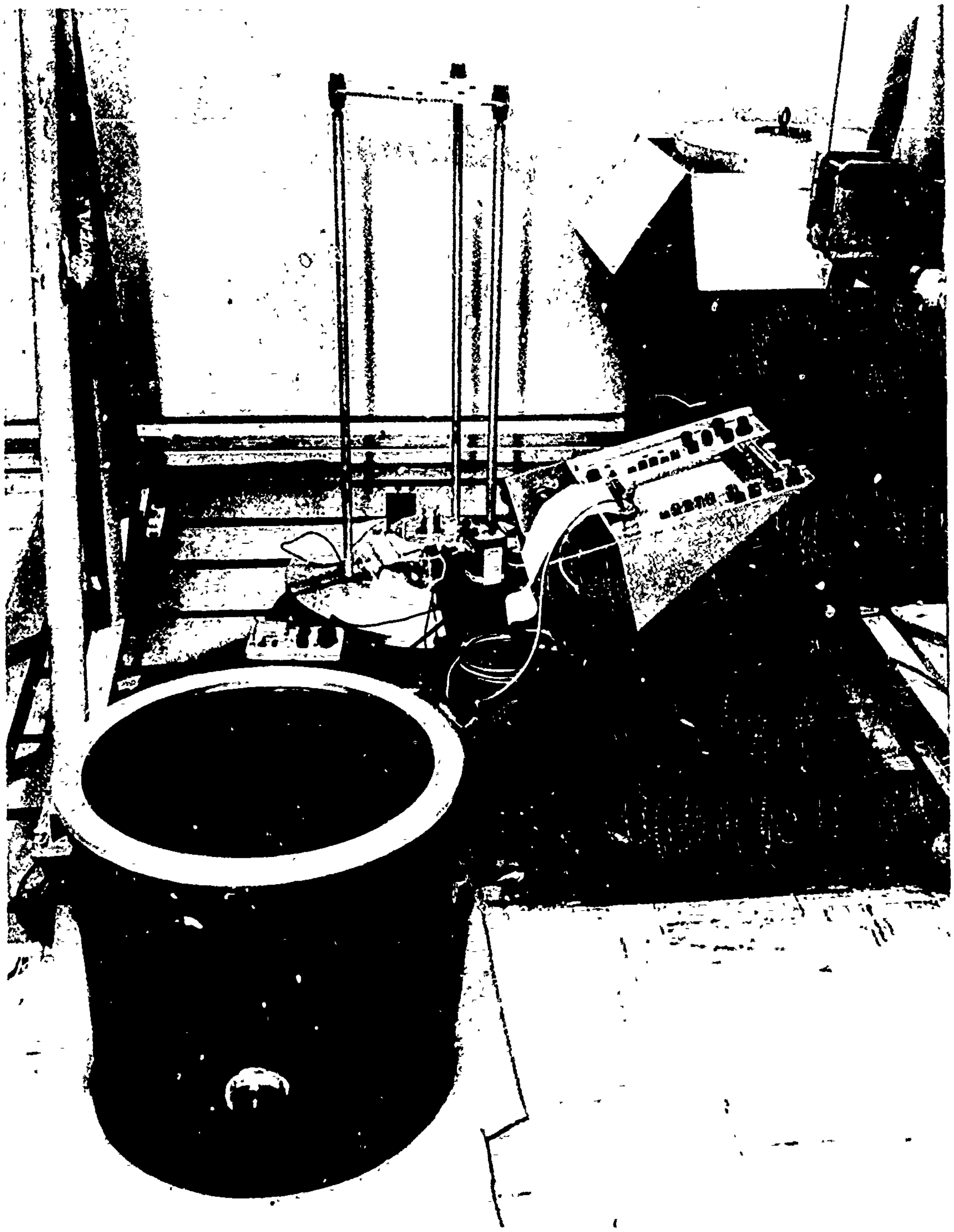


Fig. 5 PROTOTYPE MOMENT OF INERTIA DESIGN

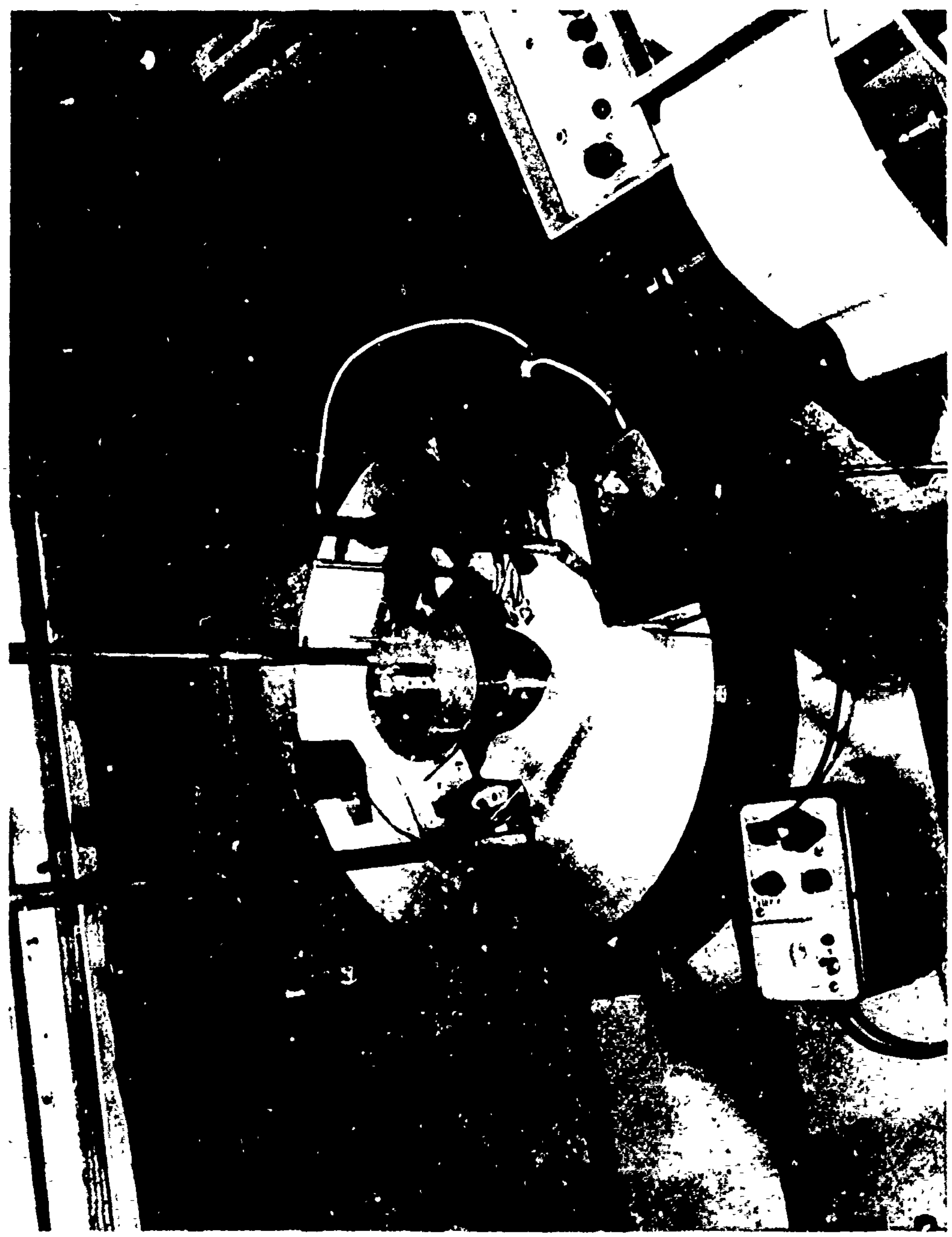


Fig. 6 PROTOTYPE SETUP, CLOSEUP VIEW

# 4. FINAL PENDULUM DESIGN

Based on the analysis and the preliminary experiments, a platform was designed that could rotate to three perpendicular axes and that could also be offset to locate the package into three skew positions. Figures 7 and 8 show a package in two of the orthogonal axis positions, and Fig. 9 shows a package in one of the skew positions. A counterweight was mounted beneath the package on the platform frame to compensate for the unbalanced moment caused by the location of the package in its various positions.

Inherent in the platform design was the necessity to design it so that the package, when attached to the platform, could be positioned at its center of mass. In Fig. 7 the arrows show the adjustment methods as indicated below:

| Arrow | Function (see Fig. 7) |
|---|---|
| 1 | This shaft was adjusted by a screw on the bottom that allowed the package to be adjusted in the vertical direction. |
| 2 | This shaft was adjustable to the right or left. |
| 3 | This screw adjusted the platform toward or away from the viewer. |

All of the above functions were adjusted in a specially designed fixture using precision measuring devices that measured to 0.0001 inch.

Figures 10c and 10d show the fixture being used with Discos package No. 5. The dial indicator (used for the measurement of the vertical motion of the platform) as well as the height gauge (used to locate the package counterweight) are shown. Figure 10 also shows the micrometers used for the lateral positioning of the package.

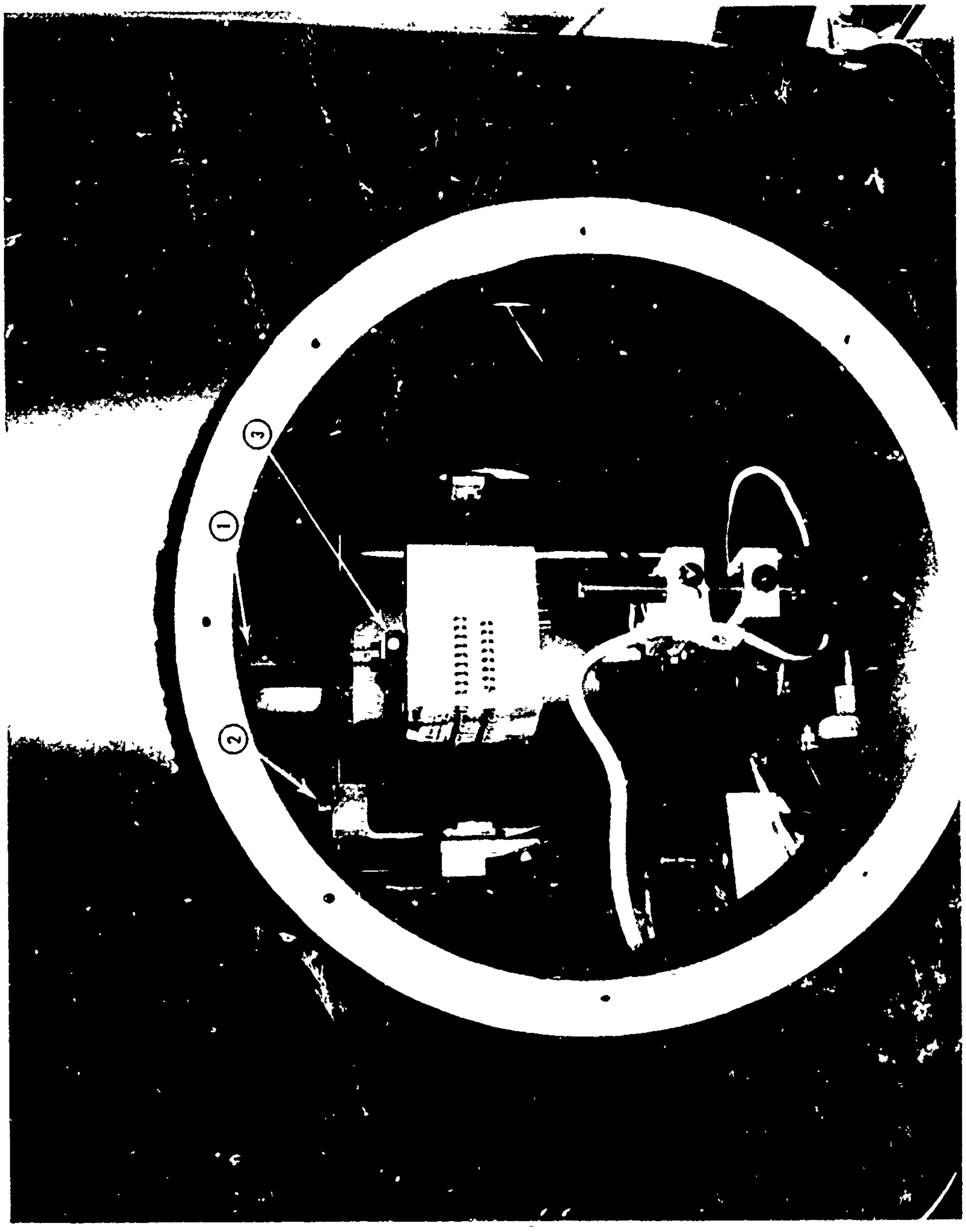



Fig. 7 PACKAGE ON BIFILAR PENDULUM WITH ARROWS SHOWING ADJUSTMENT METHODS

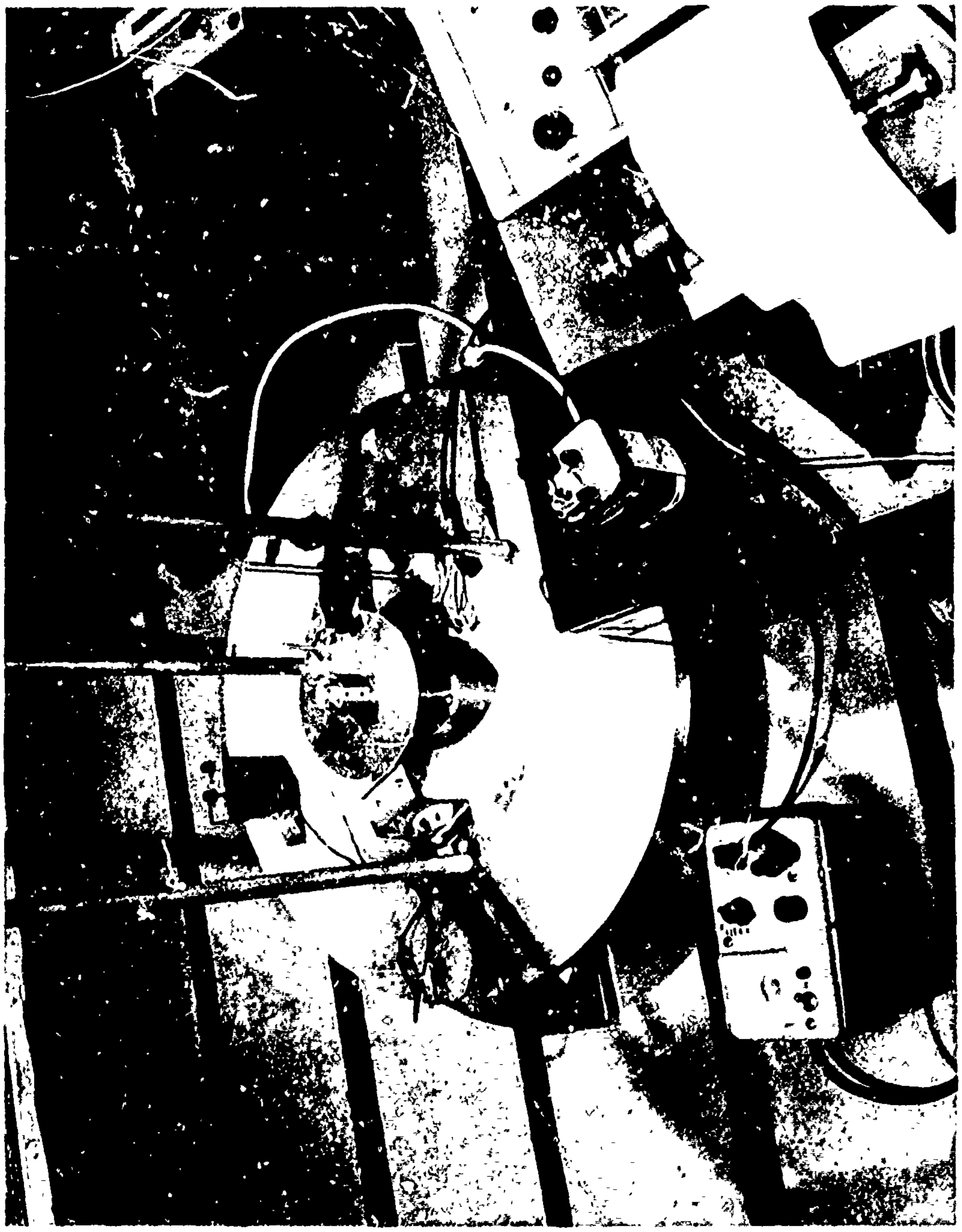


Fig. 8 PACKAGE ON BIFILAR PENDULUM IN ONE OF THE ORTHOGONAL AXIS POSITIONS

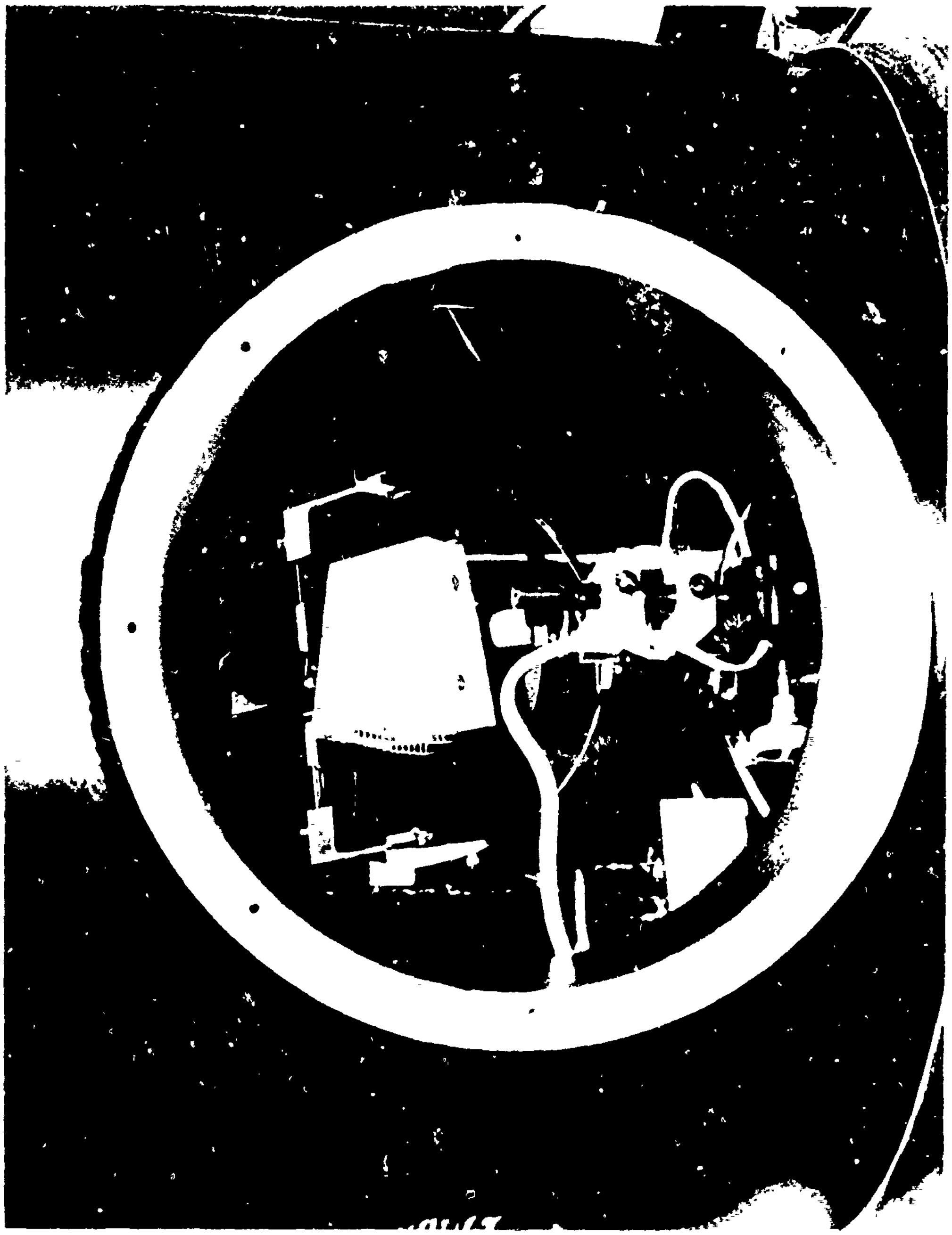
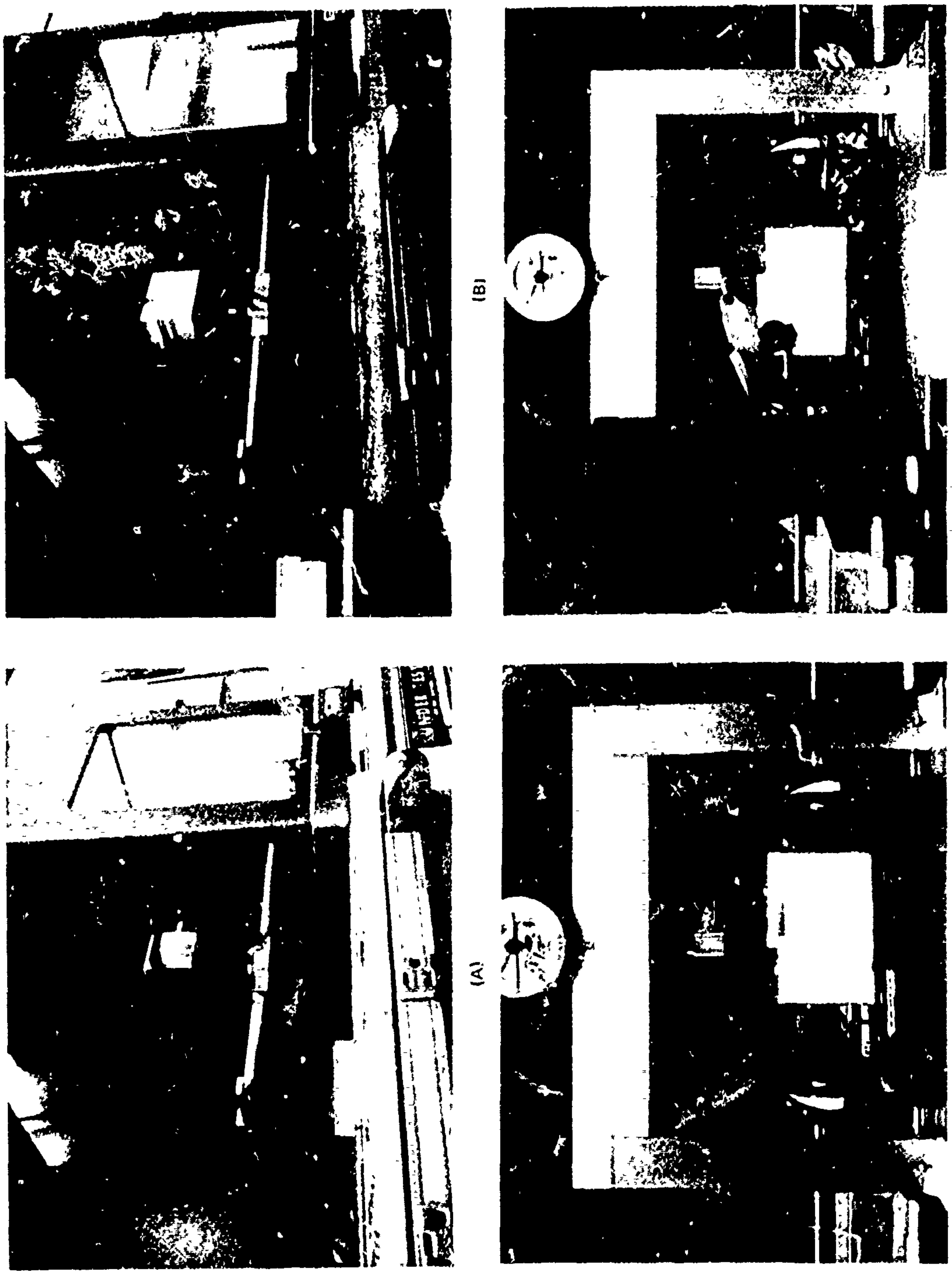


Fig. 9 PACKAGE IN ONE OF THE SKEW POSITIONS



(A) (B) (C) (D)

Fig. 10 CENTER OF MASS MEASUREMENT

All of the adjustments were made after the center of mass was measured (see Figs. 10a and 10b). The beam used for the center of mass determination was designed so that the moment of inertia platform was registered in a known location on the beam. This beam was supported by a 0.375 inch diameter ball in the center of the weighing pan on a precision balance and two 0.375 inch diameter balls at the other end.

The advantage of measuring the center of mass in this manner was that the package (once installed on the inertia platform) was not removed until all of its mass properties were determined.

To complete the system, the pendulum was suspended in a bell jar (Fig. 11), which rested on a collar. The collar had two ports that could be used for viewing, and when required the port cover could be disassembled for system adjustment without disturbing the bell jar.

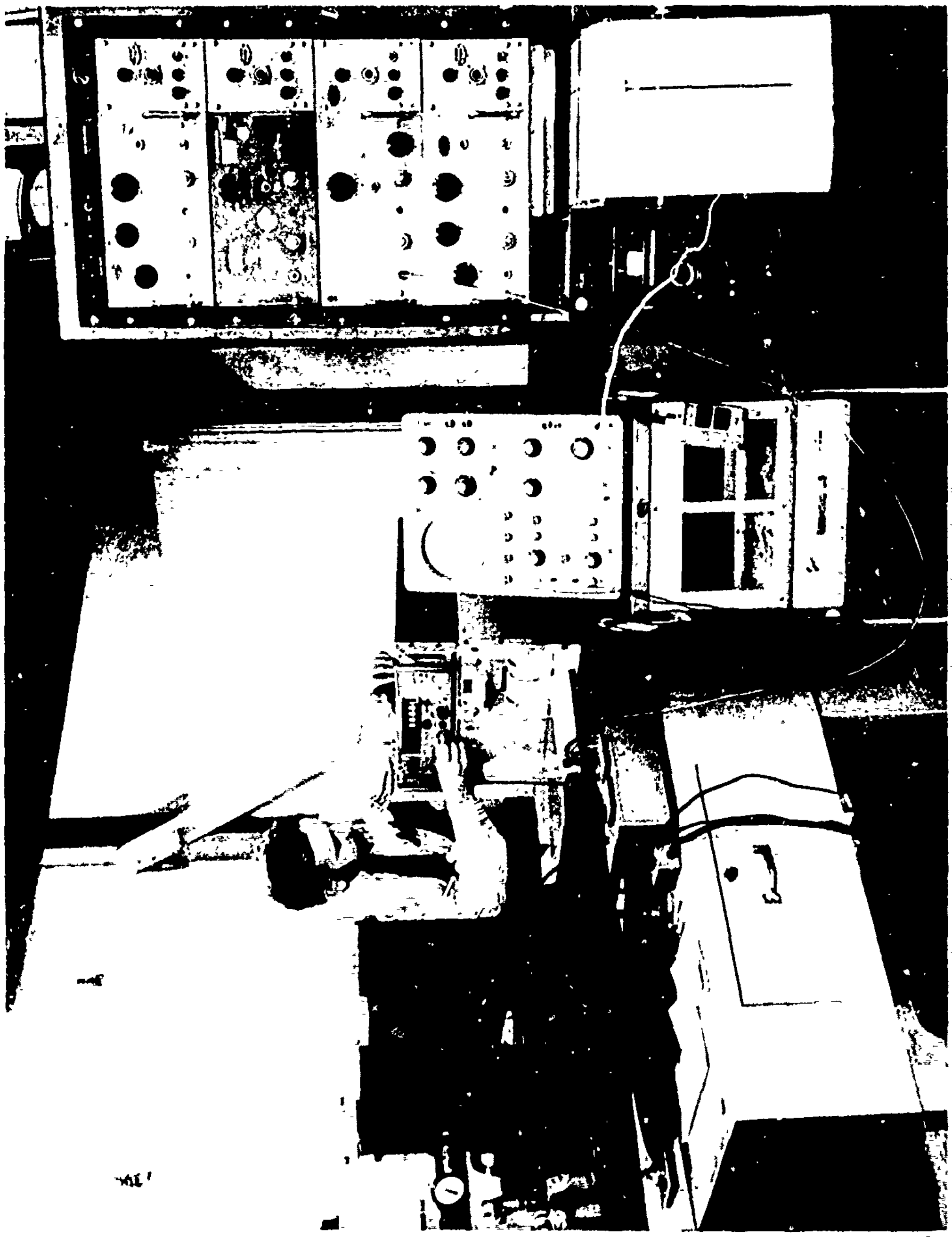


Fig. 11 COMPLETE SYSTEM LAYOUT

# 5. ELECTRONICS INSTRUMENTATION

It was obvious in the development of the measuring technique that the bifilar pendulum was going to have to be carefully instrumented to be successful. Figure 11 shows all of the instruments external to the bell jar, and Fig. 8 is a view that shows the instrumentation inside the bell jar.

A shaft was attached to the bottom of the moment of inertia platform. It was guided by a nylon bearing approximately 0.010 inch larger than its O.D. To start the pendulum, a timing motor was actuated that rotated a cam-operated wire that "locked" onto the pendulum shaft. After the pendulum was rotated 20°, the wire was abruptly released from the inertia pendulum shaft, allowing the pendulum to swing freely.

A fan-shaped disk with a small hole in it was located directly above the pendulum shaft. The "fan" rotated between a light and an N-P-N planar silicon light sensor. When the hole passed over the light sensor, the light illuminating the sensor triggered the external electronics.

The following is a list of the external electronics and their use:

| Instrument | Function |
| --- | --- |
| Oscilloscope | To determine if the light was centered over the hole in the "fan." |
| Sanborn recorder | To check for pendulum damping. |
| Printer | To record the period of the pendulum. |
| Counter | To check if the average of 10 periods was remaining constant. |

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# 6. TEST RESULTS

On 23 November 1971 the final calibration tests and package testing began on the bifilar pendulum. During all of the tests a vacuum of 30 inches of mercury was maintained.

To account for any measurement variation owing to the weight of the package being measured, a calibration weight was made for each package. Also, to determine if the measured moments of inertia were reasonable, a computer program was written that calculated the moment of inertia of homogeneous packages the same size as Discos (see Appendix A). Figure 12 shows the program printout.

To do the final calculations a computer program was written. This program calculated the test mass moments of inertia as well as the final package moment of inertia. In this way the program was continually checked for accuracy by a known mass. Figure 13 is a sample of the program printout. The printout notation is as follows:

$I_{ij}P$ = Final package moment of inertia and cross product,

IL(ij) = Moment of inertia of skew line,

Ii = Principal moment of inertia,

i = 1, 2, 3, and

IPRO(ij) = Final moment of inertia matrix check.

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```
 DISCOS PACKAGE NUMBER ONE
H=.16666,
WPG=101.8346,
AL=11.00;

IXXA= 4.7198426378328l2E-05          IYYA= 2.699342784094399E-05          IZZA= 4.140520797936360E-05;
ILC5= 4.129573431766529E-05          ILC45= 4.430181717884578E-05         IL545= 4.424708034799662E-05;

 DISCOS PACKAGE NUMBER TWO
H=.16666,
WPG=101.8337,
AL=11.00;

IXXA= 4.719846061678672E-05          IYYA= 2.699319894301918E-05          IZZA= 4.140475625397196E-05;
ILC5= 4.129528431169351E-05          ILC45= 4.430160644088026E-05         IL545= 4.424687240424104E-05;

 DISCOS PACKAGE NUMBER THREE
H=.16666,
WPG=101.5781,
AL=11.00;

IXXA= 4.707955134283555E-05          IYYA= 2.692544183431676E-05          IZZA= 4.130079376302114E-05;
ILC5= 4.119159681499870E-05          ILC45= 4.419017255317628E-05         IL545= 4.413557407891706E-05;

 DISCOS PACKAGE NUMBER FOUR
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WPG=108.3349,
AL=11.00;

IXXA= 5.021130058946486E-05          IYYA= 2.871649245775591E-05          IZZA= 4.404817156769240E-05;
ILC5= 4.393171053772418E-05          ILC45= 4.712973622867855E-05         IL545= 4.707150556359444E-05;

 DISCOS PACKAGE NUMBER FIVE
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WPG=60.6268,
AL=11.00;

IXXA= 3.746195022730431E-05          IYYA= 2.142485281790145E-05          IZZA= 3.286356105618586E-05;
ILC5= 3.277667129374319E-05          ILC45= 3.516275564274499E-05         IL545= 3.511931076052306E-05;

 DISCOS PACKAGE NUMBER SEVEN
.166666,
WPG=180.00,
AL=11.00;
```

Fig. 12 HOMOGENEOUS PACKAGE PRINTOUT

```
  PACKAGE NUMBER SIX FINAL DATA 2-15-72
-7.05,
45.2166,
0.0,
45.00,
-90.0,
5.0,

X=-4.831955529144152E-03           Y= 2.570223416640093E-03           Z=-4.954684735454512E-03;

IP(1,1)= 4.044166623875318E-05     IP(2,1)= 1.881503462172664E-05     IP(3,1)= 3.381574333227712E-05;

IP(1,1)= 4.022774155679168E-05     IP(2,1)= 1.848615314131246E-05     IP(3,1)= 3.361011598701578E-05;

IL(1,1)= 3.624146002467435E-05     IL(2,1)= 3.695635121103181F-05     IL(3,1)= 3.353488764238925E-05;

IXYP=-4.910484676656778E-06        IXZP= 0.000000000000000E+00        IYZP= 0.000000000000000E+00;

IXXP= 4.022774155679168E-05        IYYP= 1.848615314131246E-05        IZZP= 3.361011598701578E-05;

IXYP=-4.910484676656778E-06        IXZP= 0.000000000000000F+00        IYZP= 0.000000000000000E+00;
COEF OF CUBIC ECUATION

CUT(0)= 1.000000000000000E+00      CUT(1)=-9.232401068511993E-05      CUT(2)= 2.692924142041021E-09
CUT(3)=-2.415393481892260E-14;
REAL ROOTS OF CUPIC     I1= 1.742840686347335F-05           I2= 3.361080668214738E-05
I3= 4.128500601445416E-05;
IMAGINARY ROOTS OF CUBIC
 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00

  PACKAGE NUMBER SEVEN 2-2-72
-172.95,
45.2166,
0.0,
-45.00,
-90.00,
+5.00,

X=-8.933069690661783F-03           Y=-7.995794401827752E-03           Z= 1.702257025093406E-02;

IP(1,1)= 9.208205032408225E-05     IP(2,1)= 6.097784073533316E-05     IP(3,1)= 5.811223503431335E-05;

IP(1,1)= 8.804226563168035E-05     IP(2,1)= 5.598031693818939E-05     IP(3,1)= 5.691277068409604E-05;

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IXYP=-1.554105256218463E-05        IXZP=-1.140152289735851E-05        IYZP=-8.164013707415531E-06;
COFF OF CUBIC EQUATION
```

Fig. 13 FINAL PROGRAM PRINTOUT

## 7. CONCLUDING REMARKS

On 11 April 1972 Figs. 14a and 14b were sent to Stanford University, thereby completing the package mass property documentation for the Discos electronic packages.

After insertion into orbit, Discos operated successfully for 1 year until it was commanded off. It exceeded its design requirements for orbital perturbations of $10^{-11}$ g.

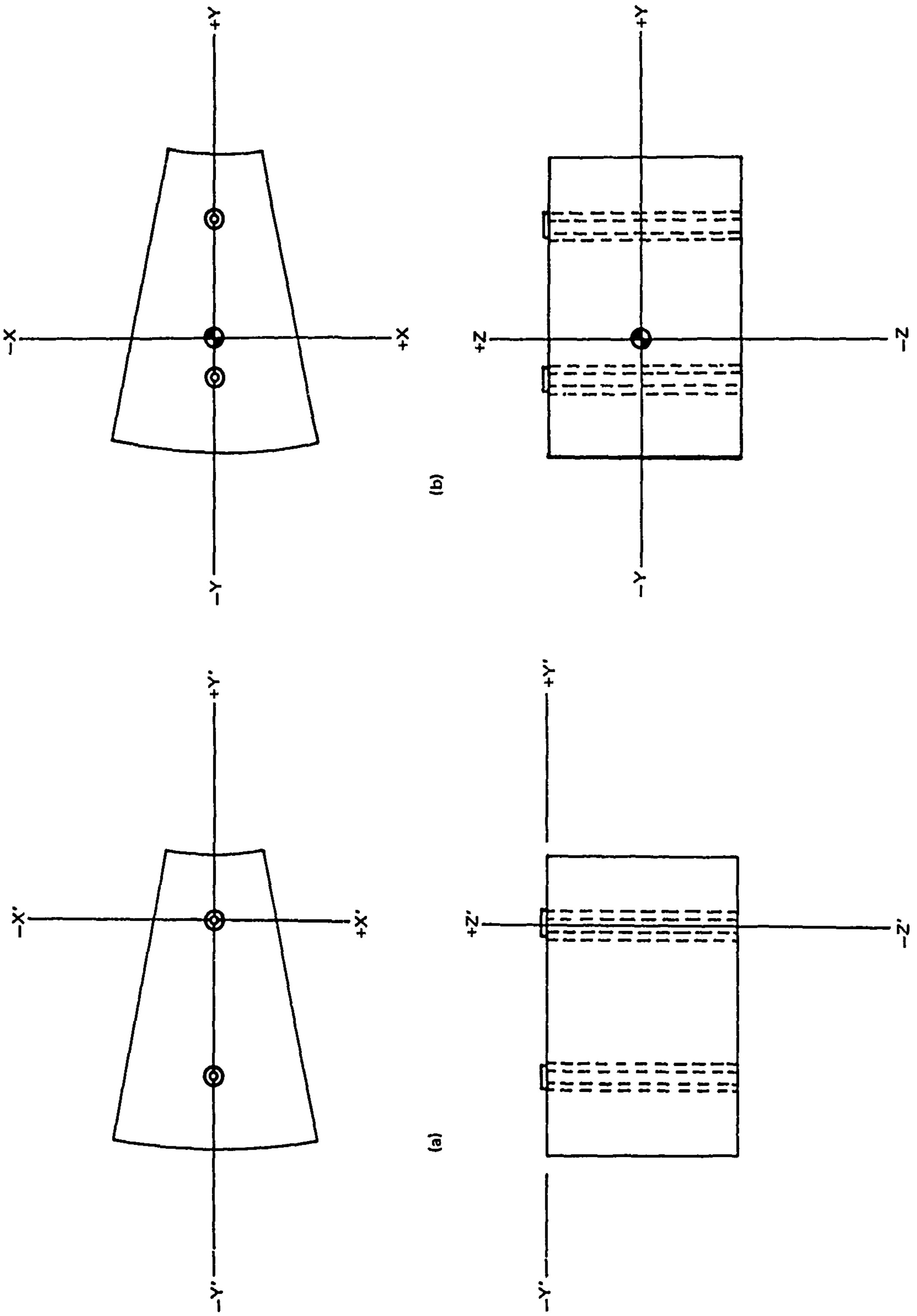


Fig. 14a and b PACKAGE MASS PROPERTIES

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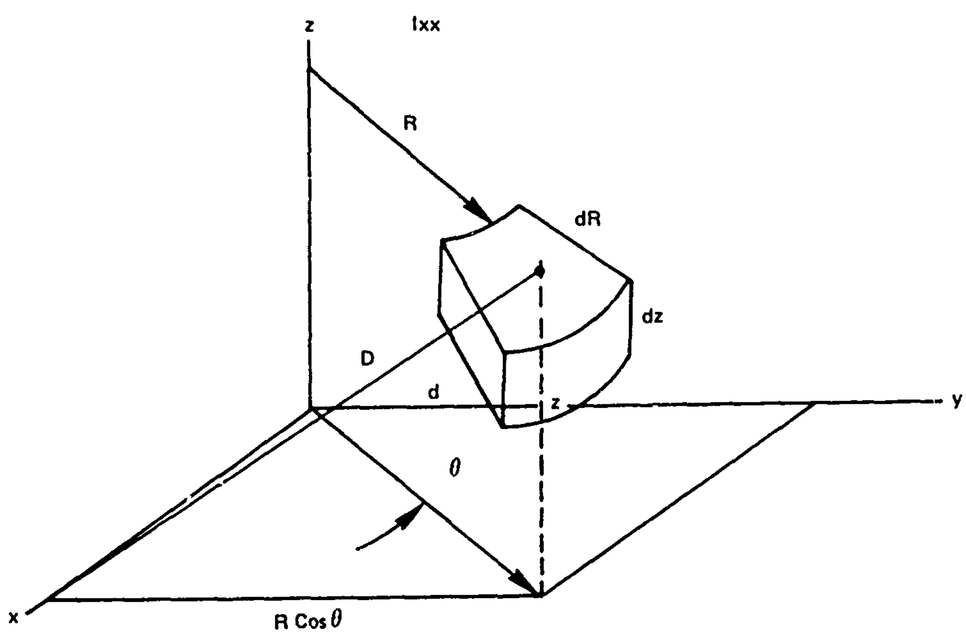
# Appendix A

## MOMENT OF INERTIA EQUATIONS FOR BODIES THE SHAPE OF THE DISCOS PACKAGES

To determine if the moments of inertia calculated on the bifilar pendulum were reasonable, the equations for the moment of inertia of the three principal axis of the Discos package were determined as follows:



The radius to the element of mass is

$$D = [z^2 + r^2 \cos^2 \theta]^{\frac{1}{2}} . \qquad \text{(A-1)}$$

The element of mass is

$$dm = \rho r \, d\theta dr dz . \qquad \text{(A-2)}$$

Therefore, the M of Ixx is

$$Ixx = \rho \int\int\int D^2 r \, d\theta dr dz , \qquad \text{(A-3)}$$

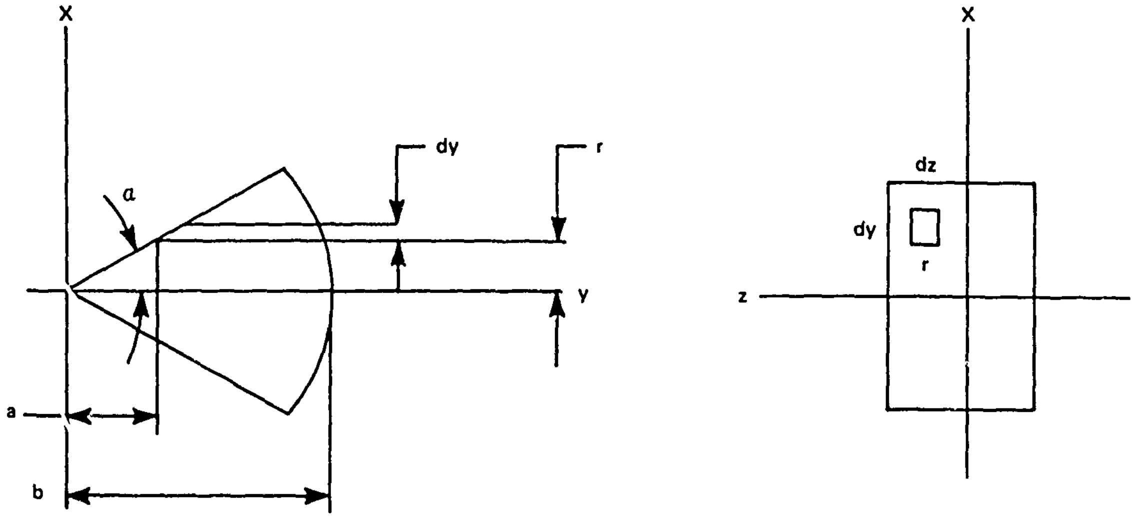
$$Ixx = \rho \int_{-\frac{H}{2}}^{\frac{H}{2}} \int_{R_1}^{R_2} \int_{-\alpha}^{\alpha} (z^2 + r^2 \cos^2\theta) \, r^2 d\theta dr dz , \qquad \text{(A-4)}$$

and, finally,

$$Ixx = \frac{H\rho\alpha [R_2^4 - R_1^4]}{4} + \frac{H\rho \sin^2\alpha [R_2^4 - R_1^4]}{8} \qquad \text{(A-5)}$$

$$+ \frac{H^3 \alpha\rho [R_2^2 - R_1^2]}{12} .$$

Iyy




The element of mass is

$$dm = dy\, dz(b-a)\,\rho \ . \qquad \text{(A-6)}$$

The radius to the element is

$$r = [y^2 + z^2]^{\frac{1}{2}} \ . \qquad \text{(A-7)}$$

Therefore, the M of Iyy is

$$Iyy = \int\int r^2 dm \qquad \text{(A-8)}$$

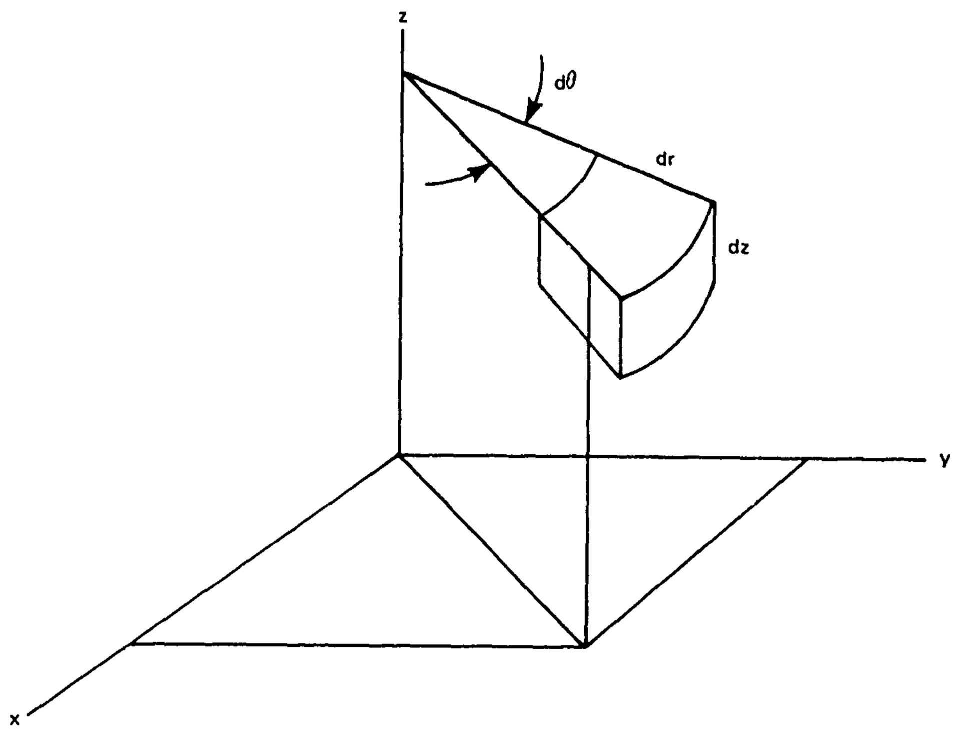
$$Iyy = \rho\int\int (b-a)\ (y^2 + z^2)\ dydz \ . \qquad \text{(A-9)}$$

And, finally,

$$Iyy = \frac{h\rho R^4\alpha}{4} - \frac{2h\rho R^4 \sin^4\alpha}{16} - \frac{h\rho R^4 \sin^4\alpha}{2\tan\alpha} + \frac{\rho R^2 h^3 \alpha}{12} \tag{A-10}$$

$$+ \frac{\rho R^2 h^3 \sin^2\alpha}{24} - \frac{\rho R^2 h^3 \sin^2\alpha}{12 \tan\alpha}$$

Izz




The element of mass is

$$dm = rd\theta drdz\rho \ . \tag{A-11}$$

The radius of the element is R; therefore Izz is

$$Izz = \rho \int_{-\frac{H}{2}}^{\frac{H}{2}} \int_{R_1}^{R_2} \int_{-\alpha}^{\alpha} R^3 d\theta dr dz \ . \qquad (A-12)$$

And, finally,

$$Izz = \rho\, 2\alpha \left[ \frac{R_2^4}{4} - \frac{R_1^4}{4} \right] H \ . \qquad (A-13)$$

The above equations were programmed, and the mass properties of homogeneous packages of the same weight as the actual Discos packages were determined.

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