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OPTIMAL ALLOCATION OF RESOURCES IN SYSTEMS

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OPTIMAL ALLOCATION OF RESOURCES IN SYSTEMS

by

C. Derman, G.J. Lieberman, and S.M. Ross

In the design of a new system, or the maintenance of an old system, allocation of resources is of prime consideration. In allocating resources it is often beneficial to develop a solution that yields an optimal value of the system measure of desirability. In the context of the problems considered in this paper the resources to be allocated are components already produced (assembly problems) and money (allocation in the construction or repair of systems). The measure of desirability for system assembly will usually be maximizing the expected number of systems that perform satisfactorily and the measure in the allocation context will be maximizing the system reliability. Results are presented for these two types of general problems in both a sequential (when appropriate) and non-sequential context.

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1. Introduction

In the design of a new system, or the maintenance of an old system, allocation of resources is of prime consideration. In allocating resources it is often beneficial to develop a solution that yields an optimal value of the system measure of desirability. In the context of the problems considered in this paper the resources to be allocated are components already produced (assembly problems) and money (allocation in the construction or repair of systems). The measure of desirability for system assembly will usually be maximizing the expected number of systems that perform satisfactorily and the measure in the allocation context will be maximizing the system reliability.

2. General Allocation Problem

Let A denote a fixed amount of money to be used to build a single system consisting of n components. Define $P_i(x_i)$ as the probability that component i will work if x_i is allocated to its production. The problem is to choose $x_1, x_2, \ldots, x_n$ so as to maximize the probability that the system works, i.e.,

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maximize

$$R[P_1(x_1), P_2(x_2), \ldots, P_n(x_n)]$$

subject to

$$\sum x_i = A ,$$

where R is the probability that the system performs satisfactorily. A special case of this problem arises when the system can be represented by n independent modular subsystems connected in parallel and/or series; in such cases R has an identifiable simple form. This type of structure is often characteristic of a fault tree, where the fault tree diagram explicitly shows the decomposition via series and parallel modules. [Although in a fault tree a small number of components may appear in different modules resulting in a lack of independence.] There has been some work done on this problem [2], where algorithms have been developed, but essentially the solution is unknown. In order to get some insight into this general problem, a simpler version is considered by the authors in [5], [6], and [7]. This version assumes that $P_i(x) = P(x)$ for all components, and the system has a special structure, i.e., it is a k out of n system. However, another facet is added, namely in some of our models allocation decisions can be made sequentially.

3. Allocation of Money Resources

In [6], the authors consider the following problem. Suppose A denotes a fixed amount of money to build a single system consisting of n components, i.e., the n components are to be produced rather than taken or purchased from an existing stockpile. Define $P(x)$ as the probability that a component will work if amount x is allocated to its production (P is an increasing function and $P(0) = 0$). The non-sequential version of the problem is to choose $x_1, x_2, \ldots, x_n$ in order to maximize $R(P(x_1), \ldots, P(x_n))$, i.e., the probability that the system works. The sequential version assumes that the individual components are built sequentially in time and that knowledge as to whether or not a component functions is available to us before we have to allocate our investment in the next component. That is, x_1 is allocated to produce the first component. Using the information as to whether the first allocation produced a working or non working component, x_2 is then allocated to produce a 2nd component. We proceed in this manner, making no more than n allocations. The problem is to choose x_1, x_2, up to x_n, if necessary, sequentially to maximize the probability that the system will work.

If it is assumed that an n component system will work if at least k of the components function, then the following results are available:

(i) $k = 1$ (parallel system) - sequential or non-sequential version.

If $\log(1 - P(x))$ is convex, then the x's are chosen so that

$$x_1 = x_2 = \cdots = x_n = \frac{A}{n} .$$

If $\log(1 - P(x))$ is concave, then the x's are chosen so that

$$x_1 = A, \ x_2 = 0, \ \ldots , \ x_n = 0 .$$

It should be noted that the interpretation of the x's is that x_i dollars is to be invested in component i in the non-sequential case, and, in the sequential case x_i dollars is to be invested in the ith attempt if the first $i-1$ attempts to build a functioning component are unsuccessful. In this latter case, if equal investment is called for and the first attempt has been unsuccessful, then there is $A - \frac{A}{n} = \frac{A(n-1)}{n}$ dollars available to allocate to the remaining $n-1$ potential components, so that again, equal investment calls for investing A/n.

(ii) General k (note $k = n$ is series-system -- sequential or non-sequential version).

If $\log(1 - P(x))$ is (strictly) convex then if one wants to sequentially build k working components in at most n attempts, $n \geq k$, then it is (uniquely) optimal to allocate A/n at each stage when A is the total resource available. Thus, it also follows that the same allocation is optimal for the non-sequential model.

(iii) Special case of $P(x) = x$ -- sequential case.

Since $\log(1-x)$ is a concave function, the results presented under (i) hold for $k = 1$, i.e., $x_1 = A$, $x_2 = 0$, ... , $x_n = 0$. Exact results can also be obtained for the case of $k = 2$. The optimal policy π^* can be described as follows. When the present amount available is y and at most n additional components can be built, then

a) if only one additional working component is needed, π^* allocates $\min(y,1)$, and

b) if two additional working components are needed, π^* allocates

$$\begin{cases} y-1 & \text{if } y \geq \frac{n}{n-1} \\ \frac{y}{n} & \text{if } y \leq \frac{n}{n-1} \end{cases}$$

For the general case (any k), the authors conjecture that the optimal policy π^{**} is such that when the present amount available is y and if k additional working components are needed with at most n stages to go, then π^{**} calls for allocating

$$\begin{cases} \frac{y}{n} & \text{if } y \leq \frac{n}{n-1}(k-1) \\ y-(k-1) & \text{if } y \geq \frac{n}{n-1}(k-1)\ . \end{cases}$$

(iv) Special case of $P(x) = x$ -- non-sequential case.

The non-sequential case with general k was considered by the authors in [5]. It was shown that the optimal allocation $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is such that all of the non-zero elements of $\underline{x}^*$ are equal. It is not clear how many non-zero elements are present in an optimal allocation, although some indications are available for A near k or zero; for A near k the number of non-zero elements is small while for A near zero the number is large.

4. A Stochastic Sequential Allocation Model

The following model is considered in [7], and is described in terms of an investment problem, although several other interpretations are available for this model. We have D units available for investment. During each of N time periods an opportunity to invest will occur with probability p. As soon as an opportunity presents itself we must decide how much of our available resources to invest. If we invest x, then we obtain an expected profit $P(x)$, where P is a nondecreasing continuous function. The amount x then becomes unavailable for future investment. The problem is to decide how much to invest at each opportunity so as to maximize the total expected profit. When $P(x)$ is convex, it is easily shown that the optimal policy is to invest everything when an opportunity presents itself. When $P(x)$ is a concave function, it is only possible to describe the structure of the optimal policy. In particular, if $V(n,A)$ denotes the supremal expected additional profit attainable when there are n time periods to go, A dollars available, and an investment opportunity is at hand, and $x_n(A)$ is the optimal amount to invest at this time, then

(i) $V(n,A)$ is a concave function of A,

(ii) $x_n(A)$ is a nondecreasing function of A, and

(iii) $x_n(A)$ is a nonincreasing function of n.

This structure can be used to simplify the necessary computations, but does not yield a closed form expression for the optimal value to invest.

One special case for which the optimal policy can be completely specified is when $P(x) = \log x$.[1] In this case, it turns out that when there are n time periods to go it is optimal to allocate a fixed proportion of the remaining resources available. Specifically, if there are n time periods to go and A dollars available then

$$x_n(A) = \frac{A}{1 + (n-1)p} .$$

In [8] Klinger and Brown considered a model for allocating unreliable units to a random number of demands. When a demand occurs a certain number of units must be allocated and if i units are allocated then the demand will be successfully met with probability $1-q^i$. They assumed that demands occur at random (Poisson) time points and their objective was to maximize the probability that all demands are successfully met in a fixed time range. Letting $x(t,i)$ denote the optimal number of units to allocate to a demand occurring when there is time t remaining and the inventory stockpile consists of

[1]We have also attained computational results for other specific forms of $P(x)$.

i units Klinger and Brown conjectured that $x(t,i)$ is nondecreasing in i, and then proved that if this is so $x(t,i)$ is also nonincreasing in t. It was later directly proven in [10] that $x(t,i)$ is non-increasing in t. Whether or not it is also nondecreasing in i remains an open question.

Our model is quite similar to the above with the exception that we have a more general return function and have chosen to work in discrete time periods as opposed to continuous time. By interpreting investment opportunities as demands and by interpreting $P(x)$ as the probability of meeting the demand when we allocate x units then it follows that our problem is of a similar nature as the above but with the criterion of maximizing the expected number of demands met rather than the probability that all demands are met.

The stochastic sequential allocation model can be interpreted as a target assignment problem. Suppose that there are D units of ammunition available, and for each of N days the target may be under attack. During each of the N days enemy planes will attack with probability p. As soon as planes appear, we must decide how much of our ammunition to expend.

Another application is concerned with allocation of research effort. A proposal is received and sent out for review. From past history the fraction of those receiving favorable reviews are p. (p may be thought of as the probability of the referee recommending funding.) However, the review comes in as recommending approval or rejection. If the review is positive how much should be allocated to each proposal. We have a total of D dollars available. If x is allocated, then $P(x)$ is the return on the investment. We have n proposals to be sent for review and decisions must be made sequentially.

This stochastic sequential allocation model can be thought of as a variation of the sequential stochastic assignment problem treated by the authors [3]. There it is assumed that there are n men available to perform n jobs. The jobs arrive in sequential order with each job being categorized before a man is assigned to it. It is assumed that the category θ_j of the jth job is determined by a probability measure over all possible categories and that $\{\theta_j\}$ $(j = 1, \ldots, n)$ are independent with the same probability measure. The ith man has a value x_i $(0 \leq x_i \leq 1,\ i = 1, \ldots, n)$ associated with him. If the ith man is assigned to the jth job the (expected) return is a known function $P(x_i, \theta_j)$. After a man is assigned to a job, he is unavailable for future assignments. The objective is to assign the workers sequentially to maximize total expected return. In the stochastic sequential allocation model the possible categories are two in number. The first category, which occurs with probability $1-p$, corresponds to $P(x, \theta_1) \equiv 0$ (no occurrence of an opportunity); the second, which occurs with probability p, to $P(x, \theta_2) = P(x)$ (occurrence of an opportunity). The n men each having a value x_i $(i = 1, \ldots, n)$ is equivalent to a predetermined division of the total resources $\sum_{i=1}^{n} x_i = D$ and the problem is simply that of assigning the predetermined values. The allocation problem requires instead of a sequential assignment of values a sequential division of the resources. Beyond occurring or not occurring the present allocation model does not permit a more refined weighting of the return function since $P(x)$ is assumed to be the same for each occurrence. However, the authors are currently investigating an extension where $P(x)$ is replaced by $P(x, \theta)$ with θ random and taking on many possible values.

5. Assembly Problems

The previous discussion was concerned with the optimal allocation of money. Suppose now that the resources consist of a stockpile of components, and these components are to be arranged in some fashion into a set of working systems. This problem was considered by the authors in [4] and [5]. In particular we assume that a single system has m different types of components. Associated with each component is a numerical value. Let $\{b^i\}$, $i = 1, 2, \ldots, m$, denote this set of numerical values of the m components. Let $R(b^1, b^2, \ldots, b^m)$ denote the probability that the system will perform satisfactorily, i.e., $R(b^1, b^2, \ldots, b^m)$ is the reliability of the system. For example, suppose $Y_1, Y_2, \ldots, Y_m$ is a random vector representing the component pressure under operating conditions such that the system performs satisfactorily if $(Y_1, Y_2, \ldots, Y_m \leq b^1, b^2, \ldots, b^m)$. Then the reliability is just $P\{Y_1, Y_2, \ldots, Y_m \leq b^1, b^2, \ldots, b^m\}$, allowing for the possibility that the Y's may be dependent. Alternatively, let b^i denote the probability that the ith component will work, component performances are independent, and all components must work then the reliability is just $R = b^1 \cdot b^2 \cdots b^m$. Now suppose that there are n units of each component with corresponding $b_1^i, b_2^i, \ldots, b_n^i$ for every i. The problem considered is to arrange the nm units into n systems, to maximize the expected number of systems that perform satisfactorily, i.e., maximize $E(N)$, where N is the number of systems that work. Of course this criterion is equivalent to maximizing the sum of the n reliabilities.

The following results were obtained by the authors in [4]:

If R is a distribution function (such as arises in the aforementioned examples), and if $b_1^i \leq b_2^i \leq \cdots \leq b_n^i$ for $i = 1,2,\ldots,m$, then the n systems represented by the partitions $(b_1^1, b_1^2, \ldots, b_1^m)$, $\ldots, (b_n^1, b_n^2, \ldots, b_n^m)$ is the optimal arrangement, i.e., put the "worst" together, the second "worst" together, ... , and finally, the "best" together. Furthermore, if $R(b^1, \ldots, b^m) \geq \frac{1}{2}$ for every permutation of the units, then this same arrangement also minimizes the variance of the number of systems that perform satisfactorily. Finally, if

$$R(b^1, b^2, \ldots, b^m) = b^1 b^2 \cdots b^m ,$$

where

$$b^i = P\{i\text{th component works}\} ,$$

then this same arrangement maximizes

$$P\{N \geq r\} ,$$

for each r.

The previous material presents some results for general systems, and is specifically applicable to the case of series systems having independent components. What can be said for the case of assembling parallel systems having independent components? In this situation a system will be said to perform satisfactorily if at least one of its m components perform satisfactorily and again the problem is to arrange the nm units with n systems to maximize the expected number of systems that work, $E(N)$. In this case,

$$R(b^1, b^2, \ldots , b^m) = 1 - \prod_{i=1}^{m} (1-b^i) = 1 - \prod_{i=1}^{m} a^i ,$$

where

$$a^i = P\{i\text{th component fails}\} ,$$

so that

$$E(N) = \sum_{j=1}^{n} (1 - \prod_{i=1}^{m} a_j^i) = n - \sum_{j=1}^{n} \prod_{i=1}^{m} a_j^i .$$

This formulation requires that each (parallel) system contain exactly m components, and such a requirement may degrade the performance measure in that $E(N)$ may be larger if we allow for the possibility that some systems contain less than m units while others contain more. This more general parallel problem is treated by the authors in [5]. In particular, a set of t units is to be partitioned into n disjoint parallel systems.

After completion of a partition the number of units contained in the jth system $(j = 1, 2, \ldots, n)$ is denoted by m_j, with the added restriction that $\sum_{j=1}^{n} m_j = t$.[1] For a given partition, the reliability of system j, R_j, is given by

$$R_j = 1 - \prod_{\substack{\text{all } i \\ \text{units in} \\ \text{system } j}} a_j^i ,$$

so that

$$E(N) = \sum_{j=1}^{n} R_j = n - \sum_{j=1}^{n} \prod_{\substack{\text{all } i \\ \text{units in} \\ \text{system } j}} a_j^i .$$

The solution to this problem, i.e., the arrangement that maximizes $E(N)$, attempts to make the reliabilities of each system as equal as possible. Indeed, it is shown that if a partition exists that makes the reliabilities equal, it is optimal. Unfortunately, such an arrangement may not exist. Hence, bounds are presented so that the maximum expected number of systems that perform satisfactorily will be within these bounds; the bounds being a function of an arbitrarily chosen partition. Finally, an improvement algorithm is given. Essentially, this algorithm looks for pairwise interchanges of units which make two systems have "more equal" reliabilities. Incidently, all the results obtained for this problem carry over to the original problem where each system is required to contain exactly m components.

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[1] A partition will allow for one or more systems to contain no units so long as $\sum_{j=1}^{n} m_j = t$. The reliability of a system containing no units is taken to be zero.

These results on assembly of systems have other applications. A version of the target assignment problem can be related to the general parallel system assembly formulation. Manne [9] treats essentially the following target assignment problem. There are t weapons to be assigned against n targets. Let p_{ij} be the probability that the ith weapon will destroy the jth target if it alone is assigned to it. The objective is to minimize the expected number of surviving targets. If x_{ij} denotes the probability that the ith weapon is assigned to the jth target, then the x_{ij} are sought that minimize

$$\sum_{j=1}^{n} \prod_{i=1}^{t} (1 - p_{ij} x_{ij}) ,$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 , \qquad i = 1, 2, \ldots , t ,$$

and

$$x_{ij} \geq 0 .$$

Manne points out that this is a nonlinear problem and an exact solution is not known. However, by making some simplifying assumptions he presents two approximate solutions (one due to himself and one due to G. B. Dantzig).

The analogous concepts in the assembly model version would assume that x_{ij} is zero or one. The ith weapon corresponds to the ith unit. The jth target corresponds to the jth system. Whereas p_{ij} depends on both the weapon and target, the probability of a unit working in the assembly context is assumed to be independent of which system it is placed in and hence is denoted by p_i. This would imply that the ith weapon has the same probability of destroying each target. Under this assumption (which is less stringent than those proposed by Manne) the system assembly results are relevant.

An independent and earlier discussion of the assembly problem with other application can be found in Abe [1]. He uses somewhat different techniques, particularly in the parallel case. In the reliability context he always assumes independence of components, and his version of the parallel system problem requires each system to contain m units. However, for this case he obtains some sufficient conditions for optimality weaker than equal reliabilities. He also points out that the assembly model can be used in search and assignment contexts.

6. Conclusions

A large number of results have been presented, and one could question their relevance for solving the general allocation problem posed in Section 2. Can they be used to aid in the design of a new system or in the maintenance of an old system? Obtaining an explicit solution to the general allocation problem requires intimate knowledge of cost functions and system performance. Similarly, this information also appears to be necessary for obtaining explicit solutions to the "simplified" models considered in this paper -- with one important difference -- namely, most solutions lead to qualitative results. For example, in assembly problems for series systems, one should put the best components into one system, and the worst in another; whereas in assembly of parallel systems, one should tend to equalize the reliabilities in each system. In allocating resources to k out of n systems, with certain cost functions convex, then equal allocations to each component is the optimal way to proceed. Admittedly, the "optimal solution" to the general allocation problem is still open, but the results presented in this paper are useful in enhancing "engineering intuition" for the purpose of getting "good" answers to a most difficult problem.

Finally, the models presented, usually in a reliability context, are quite broad so that they are useful in other areas, e.g., the assembly of parallel systems and the stochastic sequential allocation model are related to target assignment problems.

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