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ADDITIONAL TWO-DIMENSIONAL WAKE AND JET-LIKE FLOWS

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A number of authors have described wake and jet-like similarity solutions to the steady, two-dimensional, laminar, incompressible boundary layer equations. These solutions are shown to be a subset of a larger class of solutions including those with algebraic velocity profile functions near the outer edge of the boundary layer. In addition to describing the relationship between solutions, the limiting cases are described in which the pressure gradient parameter becomes infinite and where the wall and edge velocities are equal.

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I. INTRODUCTION

A number of authors[1-4] have described wake and jet-like similarity solutions to the steady laminar, incompressible boundary layer equations. The purpose of this note is to extend these solutions into several new areas.

II. EQUATIONS

Solutions are sought to the two-dimensional incompressible boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad , \tag{2}$$

with the general boundary conditions

$$y = 0: u = u_w(x), \ v = 0,$$
$$y \rightarrow \infty: u \rightarrow u_e(x).$$

A convenient form of the similarity equations may be obtained by assuming that

$$G'(\eta) = (u - u_e) \ / \ (u_w - u_e), \ \eta = y/h(x), \tag{3}$$

so that (1) and (2) become

$$\left.\begin{aligned} G''' + A \left[(G + B\eta) \ G'' - \beta \ (G'^2 + 2 \ BG')\right] = 0 \\ G(0) = G'(\infty) = 0, \ G'(0) = 1, \end{aligned}\right\} \tag{4}$$

where

$$A = \pm 1$$
$$B = u_e/(u_w - u_e)$$

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1. Stewartson, K., "Further Solutions of the Falkner-Skan Equation," Proc. Cambridge Phil. Soc. 50, pp. 454-465 (1954).

2. Kennedy, E.D., "Wake-Like Solutions of the Laminar Boundary Layer Equations," AIAA Journal 2, pp. 225-231 (1964).

3. Steiger, M.H. and Chen, K., "Further Similarity Solutions of Two-Dimensional Wakes and Jets," AIAA Journal 3, pp. 520-528 (1965).

4. Schlichting, H., "Laminare Strahlausbreitung," S. angew. Math. Mech. 13, pp. 260-263 (1933).

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$$u_e \sim u_w \sim x^{\beta/(2-\beta)}$$

$$h(x) = \sqrt{(2-\beta)\ AB\nu\ x/u_e}$$

Wake and jet-like solutions to (4) have the added restriction that $G''(0) = 0$. The present work is intended to consider those values of A, B, and β for which such solutions can be found.

The parameter B essentially defines the wall to edge velocity ratio, and β is the usual velocity gradient parameter. The constant A is a more complicated parameter because it takes on the values of ± 1 depending on both B and β. If the solutions are to be real then h must be real and $(2-\beta)\ ABx/u_e \geq 0$. Furthermore, by considering the asymptotic behavior of (4) at large η, one may show that the outer boundary condition is approached through an exponential decrease (decay) of G' for the case when $AB > 0$ and $\beta \leq 2$. There are two other cases: first for $\beta > 2$, $AB > 0$ the outer boundary condition is again approached with exponential decay of G' but this is a so called "backward boundary layer"[5] in which $x/u_e < 0$. The second case is where $AB < 0$ with $\beta < -1/2$ and $x/u_e < 0$ and thus this is also a backward boundary layer but with asymptotic algebraic decay of G'. The latter case is not the usual kind of boundary layer although they have been discussed elsewhere[5,6] in some detail.

III. DISCUSSION

Stewartson's[1] original work and the more recent paper by Kennedy[2] concerning the wake-like flows in the region $B < 0$ is shown in Figure 1. Note that $B = -1$ and $\beta = -.1988$ specifies the Falkner-Skan separation profile, and at $B = -1$ and $\beta = 0$ the solution corresponds to Chapman's free shear layer solution[2]. Steiger and Chen[3] have extended these results to $B > 0$ which are jet-like solutions (u_e may be considered positive without loss of generality; thus $B \geq 0$ corresponds to $u_e \leq u_w$

5. *Goldstein, S., "On Backward Boundary Layers and Flow in Converging Passages," Journal of Fluid Mechanics 21, Part 1, pp. 33-45 (1965).*

6. *Brown, S.N. and Stewartson, K., "Similarity Solutions of Boundary-Layer Equations with Algebraic Decay," Journal of Fluid Mechanics 23, pp. 673-687 (1965).*

7. *Danberg, J.E. and Fansler, K.S., "Similarity Solutions of the Boundary Layer Equations for Flow Over a Moving Wall," BRL Report No. 1714, U.S. Army Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland (1973).*

and these flows may be designated as jets). They did not point out, however, that the limiting case of $u_e = 0$ (i.e., $B = 0$) corresponds to Schlichting's[4] two-dimensional jet into a quiescent fluid. Steiger and Chen also reported the counterflow jet solutions where $-1/2 \leq B \geq -1/3$ and $\beta \geq 1$ but no solutions were found in the $0 < \beta < 1$ range. In a recent investigation[7], no solutions with $G''(0) = 0$ were found in the $0 < \beta < 1$ range which confirms Steiger and Chen's observation.

Limit $\beta \to \pm \infty$

Steiger and Chen[3] reported that when $\beta = \infty$, $u_w/u_e = -2$ (i.e., $B = -1/3$), based on numerical integration of the similarity equations. A closed form solution can be obtained if (4) is treated as a singular perturbation problem where the limiting form of the equation is obtained through the use of the following transformations:

$$\bar{G} = \sqrt{|\beta|}\; G \quad \text{and} \quad \bar{\eta} = \sqrt{|\beta|}\; \eta \quad .$$

Dividing the resulting equation by β and taking the limit for large β gives

$$\bar{G}''' - A\sigma\,(\bar{G}'^2 + 2B\bar{G}') = 0$$

where $\sigma = \beta/|\beta|$. This equation can be integrated twice in closed form using the boundary conditions that as $\bar{\eta} \to \infty$, $\bar{G}' \to 0$, $\bar{G}'' \to 0$ and $\bar{\eta} = 0$, $\bar{G}' = 1$ to give (note that $AB\sigma = |B|$):

$$\bar{G}' = 3B\left\{ \tanh^2\left[\sqrt{|B|/2}\;\bar{\eta} + \tanh^{-1}\sqrt{1/3B + 1}\right] - 1\right\} \qquad (5)$$

where $B \leq -1/3$ and $\bar{G}' > 0$. Equation (5) is a generalization of the converging channel flow problem[8] ($B = -1$) to include the moving wall situation. The $B = -1/3$ limit agrees with the results of Steiger and Chen and has the following velocity distribution

$$\bar{G}' = 1 - \tanh^2\,(\eta/\sqrt{6}).$$

Algebraic Decay Solutions

The limiting cases at $B = -1/3$, $\beta = -\infty$ and $B = 0$, $\beta = -1$ are connected by a set of solutions involving asymptotic algebraic decay of G' with $\bar{\eta}$. The asymptotic form of G' at large η with $AB < 0$ and $\beta < 0$ can be written as (following techniques discussed in reference 9):

$$G'(\eta) \sim [\sqrt{|AB|}\;(\eta - \delta^*/B)]^{2\beta} \quad .$$

Although no wake-like solutions exist for $\beta > -1$, the displacement thickness becomes infinite as $\beta \to -1/2$ for solutions with nonzero $G''(0)$. In general the algebraic-decay solutions are more difficult to compute since any error in the solutions grows exponentially with η for large values of η.

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8. *Pohlhausen, D., "Zur naherungsweisen Integration der Differentlialgleichung der Grenzschicht," ZAMM* 1, *pp. 252 (1921).*

9. *Laminar Boundary Layers, L. Rosenhead (Ed.), Oxford University Press, Oxford (1963).*

Solutions were obtained at $\beta = -1.1$, -1.5 and -2.0 over a range of B and were used to determine the value of B for $G''(0) = 0$ by interpolation; the results are shown in Figure 1.

Limit $u_w = u_e$

The fact that the wake-like solutions ($B < -1$) or the jet-like solutions ($B > 0$) have a limiting point at $B = \pm\infty$ and $\beta = -1/2$ has been recognized by other investigators. At that point $u_w = u_e$ and thus $u \equiv u_e$ throughout the boundary layer so long as G' is finite. As shown here, other limiting solutions can also be found. Although the velocity distributions are degenerate, the limiting case is considered as a singular perturbation problem where the singularity is removed by the transformations:

$$G(\eta) = F(\eta_1)/\sqrt{|AB|} \quad \text{and} \quad \eta = \eta_1/\sqrt{|AB|} \ ,$$

one finds that the equation for $F(\eta_1)$ is identical to the large η_1 approximation, i.e.,

$$F''' + \eta_1 F'' - 2\beta F' = 0, \quad F'(\eta_1) \equiv \frac{dF}{d\eta_1} \ . \qquad (6)$$

A series solution of the following form can be assumed:

$$F' = K\,\eta_1^{\gamma} \exp(m\eta_1^2) \sum_{n=1}^{\infty} p_n/\eta_1^n \ .$$

The result, which has been derived in reference 7, has special properties at certain values of β given by

$$\beta_n = -(2n - 1)/2, \ n = 1, 2, 3, \ldots$$

The series then reduces to a finite number of terms and can be made to satisfy the inner as well as outer boundary conditions. Some of these solutions are:

n	β_n	$F'(\eta_1)$
1	-1/2	$\exp(-\eta_1^2/2)$
2	-3/2	$(1 - \eta_1^2)\exp(-\eta_1^2/2)$
3	-5/2	$(1 - 2\eta_1^2 + \eta_1^4/3)\exp(-\eta_1^2/2)$
.	.	.
.	.	.

The n = 2 case corresponds to the limiting case for a Libby-Liu[10] family

10. *Libby, P.A. and Liu, T.M., "Further Solutions of the Falkner-Skan Equations," AIAA Journal 5, pp. 1040-1042 (1967).*

of solutions with boundary layer velocity overshoot. The $G''(0) = 0$ solutions for this family are indicated as a dashed line in Figure 1 because the $B = -1$ point is the only other value currently available. Higher n's correspond to the larger negative β families whose existence was noted by Libby-Liu[10]. Solutions to (6) may be obtained for any β by numerical integration; however, $G''(0)$ is infinite for all β (except at β_n) because $G''(0) = \sqrt{|AB|}\, F''(0)$. These results for the limiting case may be used to determine the large B approximations in a perturbation series approach similar to that used by Mirels[11] for $\beta = 0$; one has

$$G(\eta,B) = (1/\sqrt{|B|})\ F\ (\sqrt{|B|}\ \eta) + 0\ (1/\ |B|\ ).$$

11. *Mirels, H., "Laminar Boundary Layer Behind Shock Advancing into Stationary Fluid," NACA TN 3401, March 1955.*

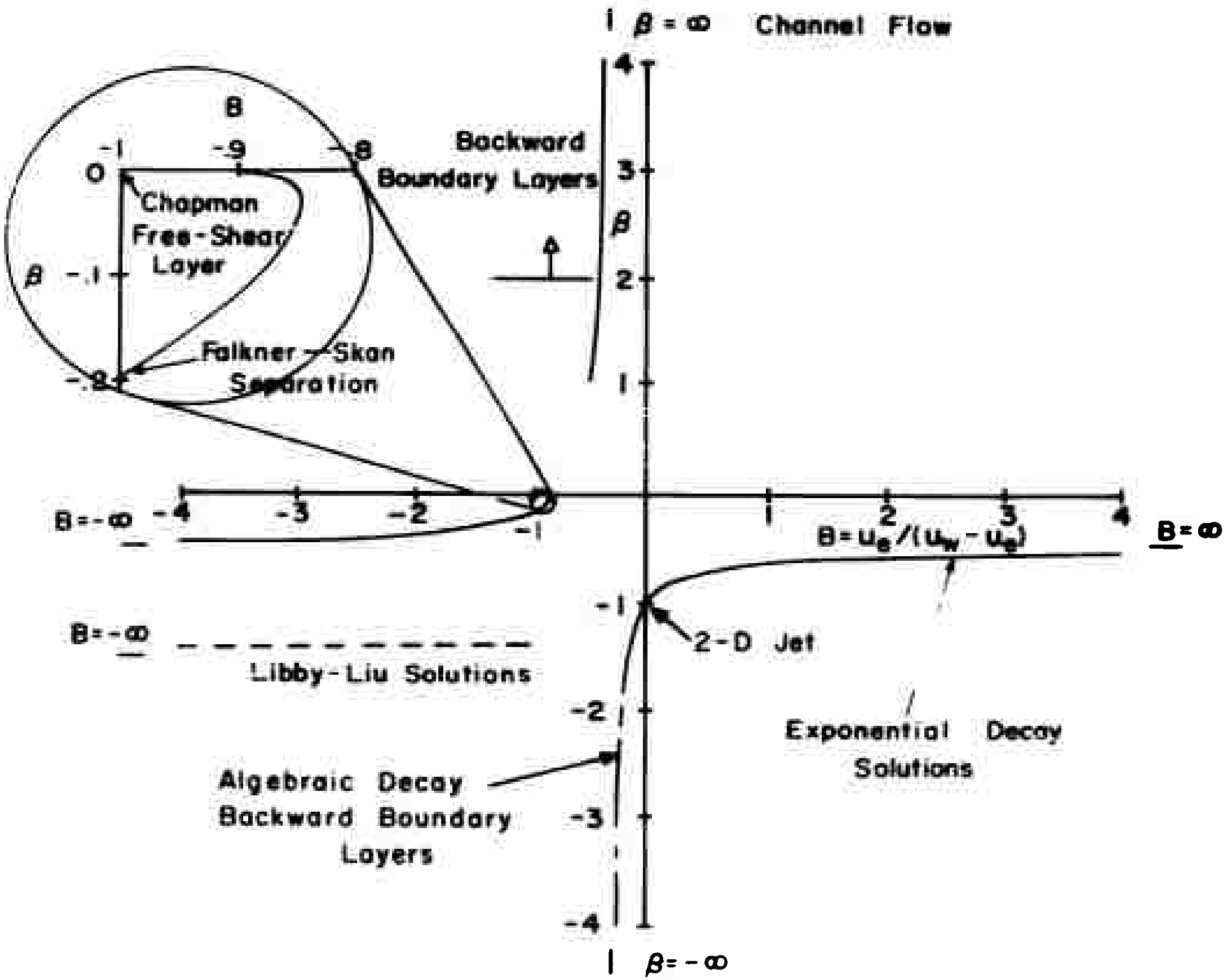

Figure 1. B-β Map of Wake and Jet-Like Solutions

REFERENCES

1. Stewartson, K., "Further Solutions of the Falkner-Skan Equation," Proc. Cambridge Phil. Soc. 50, pp. 454-465 (1954).

2. Kennedy, E.D., "Wake-Like Solutions of the Laminar Boundary Layer Equations," AIAA Journal 2, pp. 225-231 (1964).

3. Steiger, M. H. and Chen, K., "Further Similarity Solutions of Two-Dimensional Wakes and Jets," AIAA Journal 3, pp. 520-528 (1965).

4. Schlichting, H., "Laminare Strahlausbreitung," Z. angew. Math. Mech. 13, pp. 260-263 (1933).

5. Goldstein, S., "On Backward Boundary Layers and Flow in Converging Passages," Journal of Fluid Mechanics 21, Part 1, pp. 33-45 (1965).

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7. Danberg, J. E. and Fansler, K. S., "Similarity Solutions of the Boundary Layer Equations for Flow Over a Moving Wall," BRL Report 1714, U.S. Army Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland (1973).

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9. Laminar Boundary Layers, L. Rosenhead (Ed.), Oxford University Press, Oxford, p. 247 (1963).

10. Libby, P. A. and Liu, T. M., "Further Solutions of the Falkner-Skan Equations," AIAA Journal 5, pp. 1040-1042 (1967).

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LIST OF SYMBOLS

A	constant = ± 1
B	wall and edge velocity parameter, $u_e / (u_w - u_e)$
F	transformed streamfunction, $\sqrt{\lvert AB \rvert}\ G$
G	nondimensional similarity streamfunction
h	function of x in similarity variable, $\eta = y/h(x)$
K	constant
m	constant
n	index
p_n	constants
u	local velocity in x direction
u_e	boundary layer edge velocity
u_w	wall velocity
v	local velocity in y direction
x	coordinate along the surface
y	coordinate normal to the surface
β	pressure gradient parameter $u_e \sim u_e \sim x^{\beta/(2-\beta)}$
δ^*	displacement thickness
γ	constant
ν	kinematic viscosity
η	similarity variable, $\eta = y/h(x)$
η_1	transformed similarity variable, $\eta_1 = \sqrt{\lvert AB \rvert}\ \eta$