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DIFFERENTIABILITY OF OPTIMAL SEARCH PLANS

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MEMORANDUM

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From: H. R. Richardson

Subject: Differentiability of Optimal Search Plans

This memorandum investigates the differentiability of the optimal search plan m* for a stationary target. The principal result (Theorem 3) is that under suitable assumptions we may write (for almost all x)

$$m^*(T,x) = \int_0^T \mu(t,x)dt,$$

i.e., m* is absolutely continuous in the first variable. This is used in reference [a] to guarantee the existence of optimal search plans for a class of deterministically moving targets. These search plans are transformations of the functions μ.

Definitions and basic assumptions are presented in the first section followed in the second section by the investigation of differentiability. The last section provides illustrations.

Basic Definitions and Assumptions

This section provides the basic definitions and assumptions used throughout the memorandum. We shall use Λ to denote Lebesgue measure on E^N and χ_S to be the indicator function of a set $S \subset E^N$. For any real function θ of a real variable, let $q(\theta, \kappa, \Delta)$ be the difference quotient

$$q(\theta, \kappa, \Delta) = \frac{\theta(\kappa + \Delta) - \theta(\kappa)}{\Delta},$$

whenever $\kappa + \Delta$ and κ are within the domain of θ. The right derivative $D^+\theta$, when it exists, is given by

$$D^+\theta(\kappa) = \lim_{0 < \Delta \to 0} q(\theta, \kappa, \Delta).$$

The left derivative, when it exists, is given by

$$D^-\theta(\kappa) = \lim_{0 > \Delta \to 0} q(\theta, \kappa, \Delta).$$

The limits are permitted to be infinite. When $D^+\theta(\kappa) = D^-\theta(\kappa)$, θ has a derivative at κ which is usually denoted $\dot{\theta}(\kappa)$ or $D\theta(\kappa)$. These notations are also used for the one-sided derivatives at end points of intervals. In case θ is a real function of severa variables, then D_j^+, D_j^-, and D_j denote the partial derivatives with respect to the j^{th} argument. We use inv$[\theta]$ to denote the inverse of θ when it exists.

The local effectiveness function $b: [0, \infty) \to [0, 1]$ is assumed to have the following properties:

(1) b is strictly increasing with $b(0) = 0$ and $\lim_{z \to \infty} b(z) = 1$,

(2) $\dot{b}$ exists and is continuous and strictly decreasing on $[0, \infty)$. The right derivative $\dot{b}(0)$ satisfies $0 < \dot{b}(0) < \infty$, and

(3) $\ddot{b}$ exists and is continuous for all $z \epsilon [0,\infty)$ and the right derivative $\ddot{b}(0)$ satisfies $-\infty < \ddot{b}(0) < 0$.

A function $\Psi:(0,\infty) \to (0,\infty)$ is defined by

$$\Psi(u) = \begin{cases} \mathrm{inv}[\dot{b}](u) & \text{for } 0 < u \leq \dot{b}(0) \\ 0 & \text{for } u > \dot{b}(0). \end{cases}$$

In view of the conditions on the local effectiveness function b, Ψ is continuous and strictly decreasing with $\lim_{u \to 0} \Psi(u) = \infty$ and $\Psi(\dot{b}(0)) = 0$. The function Ψ is differentiable at every point except $\dot{b}(0)$. We have

$$\dot{\Psi}(u) = \begin{cases} \dfrac{1}{\ddot{b}(\Psi(u))} & \text{for } 0 < u < \dot{b}(0) \\ 0 & \text{for } u > \dot{b}(0). \end{cases}$$

Moreover,

$$D^{-}\Psi(\dot{b}(0)) = \frac{1}{\ddot{b}(0)} = \lim_{u \to \dot{b}(0)-} \dot{\Psi}(u)$$

and

$$D^{+}\Psi(\dot{b}(0)) = 0 = \lim_{u \to \dot{b}(0)+} \dot{\Psi}(u).$$

The derivative $\dot{\Psi}$ is continuous on $(0,\dot{b}(0))$ and

$$\lim_{u \to 0} \dot{\Psi}(u) = \infty.$$

An explicit bound on Ψ may be obtained by letting

$$\beta(u) = \sup\{|\dot{\Psi}(v)| : u \leq v < \dot{b}(0)\} \text{ for } 0 < u < \dot{b}(0).$$

The function β is finite since $\dot{\Psi}$ has a continuous extension on $[u, \dot{b}(0)]$ for $0 < u < \dot{b}(0)$.

By the mean value theorem of differential calculus,

$$\Psi(\dot{b}(0)) - \Psi(v) = \dot{\Psi}(\tau)[\dot{b}(0)-v] \text{ for } v < \tau < \dot{b}(0),$$

and since $\Psi(\dot{b}(0)) = 0$,

$$|\Psi(v)| = |\dot{\Psi}(\tau)| \; [\dot{b}(0)-v] \leq \beta(u)[\dot{b}(0)-v] \text{ for } u \leq v \leq \dot{b}(0).$$

The target location probability density function $f: E^N \to [0, \infty)$ is assumed to be essentially bounded by $\Gamma < \infty$, i.e., $\Lambda(\{x : f(x) > \Gamma\}) = 0$ and for $\gamma < \Gamma$ $\Lambda(\{x : f(x) > \gamma\}) > 0$. We denote by κ_0 the product $\kappa_0 = \dot{b}(0)\Gamma < \infty$.

For $0 < \kappa \leq \kappa_0$, we define $S(\kappa)$ by

$$S(\kappa) = \left\{ x : f(x) \geq \frac{\kappa}{\dot{b}(0)} \right\}$$

and define $\theta : (0, \kappa_0] \to (0, \infty)$ by

$$\theta(\kappa) = \int_{S(\kappa)} \Psi\left(\frac{\kappa}{f(x)}\right) dx.$$

Note that $0 \leq \Lambda(S(\kappa)) < \infty$, since $\int_{E^N} f(x) dx = 1$.

It is not difficult to show that under the assumptions on b and f, θ is continuous and strictly decreasing with $\lim_{\kappa \to 0} \theta(\kappa) = \infty$ and $\theta(\kappa_0) = 0$.

It can be shown (see, for example, reference [b]) that under assumptions (1) and (2) for b, the search plan $m^* : [0, \infty) \times E^N \to [0, \infty)$ which maximizes detection probability is given by

$$m^*(t,x) = \begin{cases} \Psi(\frac{\lambda(t)}{f(x)}) & \text{for } f(x) \neq 0, \text{ and} \\ 0 & \text{for } f(x) = 0. \end{cases}$$

The function $\lambda:[0,\infty) \to (0,\kappa_0]$ is defined by $\lambda = \text{inv}[\theta] \circ C$ where $C:[0,\infty) \to [0,\infty)$ is some differentiable (finite derivative) strictly increasing function with $C(0) = 0$. It follows immediately that λ is continuous and strictly decreasing; $\lambda(0) = \kappa_0$ and $\lim_{t\to\infty} \lambda(t) = 0$. For each fixed $x \in E^N$, $m^*(\cdot,x)$ is continuous since it is the composition of continuous functions.

Differentiability of m^*

Differentiability of the optimal search plan m^* is investigated in this section. The principal result given as Theorem 3 is that under the assumptions of the first section, $m^*(\cdot,x)$ is absolutely continuous for almost all $x \in E^N$.

Lemma 1. The function $\theta:(0,\kappa_0] \to [0,\infty)$ has finite and non-zero left and right derivatives on the open interval $(0,\kappa_0)$. The left derivative $D^-\theta(\kappa_0)$ is finite but may be zero. Moreover θ is differentiable at $\kappa \in (0,\kappa_0)$ if and only if $\Lambda(\{x:f(x) = \kappa/\dot{b}(0)\}) = 0$.

Proof. In order to establish the existence of the left derivative, assume $\Delta < 0$ and $\kappa \in (0,\kappa_0]$. Then in view of the definition of θ

$$q(\theta,\kappa,\Delta) = \int_{S(\kappa)} q(\Psi, \frac{\kappa}{f(x)}, \frac{\Delta}{f(x)}) \frac{1}{f(x)} dx - \int_{S(\kappa+\Delta)-S(\kappa)} \frac{1}{\Delta} \Psi(\frac{\kappa+\Delta}{f(x)}) dx.$$

Without loss of generality, assume that for some fixed $\epsilon > 0$, we have

$$|\Delta| < \epsilon \leq \frac{\kappa}{2}.$$

Since Γ is the essential bound of f, for almost all $x \epsilon S(\kappa+\Delta)$,

$$\dot{b}(0) \geq \frac{\kappa+\Delta}{f(x)} \geq \frac{\kappa-\epsilon}{\Gamma} > 0,$$

and

$$\left|\Psi\left(\frac{\kappa+\Delta}{f(x)}\right)\right| \leq \beta\left(\frac{\kappa-\epsilon}{\Gamma}\right) \left(\dot{b}(0) - \frac{\kappa+\Delta}{f(x)}\right).$$

Since

$$S(\kappa+\Delta)-S(\kappa) = \left\{x: \frac{\kappa+\Delta}{\dot{b}(0)} \leq f(x) < \frac{\kappa}{\dot{b}(0)}\right\},$$

it is easily shown that

$$\dot{b}(0) - \frac{\kappa+\Delta}{f(x)} \leq \dot{b}(0) \left(\frac{-\Delta}{\kappa}\right) \text{ for } x \epsilon [S(\kappa+\Delta)-S(\kappa)].$$

Thus for almost all $x \epsilon S(\kappa+\Delta)$

$$\left|\frac{1}{\Delta} \Psi\left(\frac{\kappa+\Delta}{f(x)}\right)\right| \leq \beta\left(\frac{\kappa-\epsilon}{\Gamma}\right) \frac{\dot{b}(0)}{\kappa}.$$

Since $S(\kappa+\Delta_1)-S(\kappa) \subset S(\kappa+\Delta_2)-S(\kappa)$ for $\Delta_1 > \Delta_2$ and

$$\bigcap_{\Delta<0} S(\kappa+\Delta) - S(\kappa) = 0.$$

We have

$$\lim_{0>\Delta\to 0} \int_{S(\kappa+\Delta)-S(\kappa)} \frac{1}{\Delta} \Psi\left(\frac{\kappa+\Delta}{f(x)}\right) dx = 0,$$

since the integrands are uniformly bounded.

For $\kappa \in (0, \kappa_0]$ and for all $x \in S(\kappa)$

$$\lim_{0 > \Delta \to 0} q(\Psi, \frac{\kappa}{f(x)}, \frac{\Delta}{f(x)}) \frac{1}{f(x)} = D^-\Psi(\frac{\kappa}{f(x)}) \frac{1}{f(x)}.$$

The convergence is essentially bounded since for almost all $x \in S(\kappa)$

$$\left| q(\Psi, \frac{\kappa}{f(x)}, \frac{\Delta}{f(x)}) \frac{1}{f(x)} \right| \leq \beta(\frac{\kappa - \epsilon}{\Gamma}) [\frac{\dot{b}(0)}{\kappa}]^2.$$

Thus the bounded convergence theorem insures that for $0 < \kappa \leq \kappa_0$

$$D^-\theta(\kappa) = \int_{S(\kappa)} D^-\Psi(\frac{\kappa}{f(x)}) \frac{1}{f(x)} dx < \infty,$$

since $\Lambda(S(\kappa)) < \infty$. If $\kappa < \kappa_0$, then $D^-\Psi(\kappa/f(x))/f(x) < 0$ on the set $\{x : f(x) > \kappa/\dot{b}(0)\}$ which has non-zero measure (otherwise Γ_0 would not be the essential bound). Thus for $0 < \kappa < \kappa_0$, $D^-\theta(\kappa) \neq 0$.

Similarly, the existence of the right derivative of θ on $(0, \kappa_0)$ is established by assuming $\Delta > 0$ and writing

$$q(\theta, \kappa, \Delta) = \int_{S(\kappa+\Delta)} q(\Psi, \frac{\kappa}{f(x)}, \frac{\Delta}{f(x)}) \frac{1}{f(x)} dx - \int_{S(\kappa)-S(\kappa+\Delta)} \frac{1}{\Delta} \Psi(\frac{\kappa}{f(x)}) dx.$$

For almost all $x \in S(\kappa) - S(\kappa+\Delta)$ we have

$$\left| \frac{1}{\Delta} \Psi(\frac{\kappa}{f(x)}) \right| \leq \beta(\frac{\kappa}{\Gamma}) \dot{b}(0) \frac{1}{\kappa+\Delta}.$$

Making use of the fact that $\Phi(\dot{b}(0)) = 0$, it follows that

$$\left| \int_{S(\kappa)-S(\kappa+\Delta)} \frac{1}{\Delta} \Psi(\frac{\kappa}{f(x)}) dx \right| \leq \Lambda\{x : \frac{\kappa}{\dot{b}(0)} < f(x) < \frac{\kappa+\Delta}{\dot{b}(0)}\} \beta(\frac{\kappa}{\Gamma}) \dot{b}(0) \frac{1}{\kappa+\Delta}$$

which vanishes as Δ approaches 0. For $\chi_{S(\kappa+\Delta)}$, the indicator function of the set

$S_{\kappa+\Delta}$, it is easy to show that for $x \in S(\kappa)$ and $0 < \kappa < \kappa_0$

$$\lim_{0<\Delta\to 0} \chi_{S(\kappa+\Delta)}(x)\, q(\Psi, \frac{\kappa}{f(x)}, \frac{\Delta}{f(x)}) \frac{1}{f(x)} = D^+\Psi(\frac{\kappa}{f(x)}) \frac{1}{f(x)}.$$

It is also not difficult to show that for $x \in S(\kappa)$ and $0 < \kappa < \kappa_0$

$$\left| q(\Psi, \frac{\kappa}{f(x)}, \frac{\Delta}{f(x)}) \frac{1}{f(x)} \right| \leq \beta\, (\frac{\kappa}{\Gamma}) [\frac{\dot{b}(0)}{\kappa}]^2.$$

Since $\Lambda(S(\kappa)) < \infty$, it follows that for $0 < \kappa < \kappa_0$

$$D^+\theta(\kappa) = \int_{S(\kappa)} D^+\Psi(\frac{\kappa}{f(x)}) \frac{1}{f(x)} dx.$$

Finally, for $0 < \kappa < \kappa_0$

$$D^-\theta(\kappa) - D^+\theta(\kappa) = \int_{S(\kappa)} [D^-\Psi(\frac{\kappa}{f(x)}) - D^+\Psi(\frac{\kappa}{f(x)})] \frac{1}{f(x)} dx$$

$$= \int_{\{x:f(x)=\frac{\kappa}{\dot{b}(0)}\}} D^-\Psi(\dot{b}(0)) \frac{1}{f(x)} dx = \frac{\dot{b}(0)}{\kappa} D^-\Psi(\dot{b}(0)) \Lambda(\{x:f(x) = \frac{\kappa}{\dot{b}(0)}\}),$$

since $D^+\Psi(\kappa/f(x)) = D^-\Psi(\kappa/f(x))$ for all x such that $\kappa/f(x) < \dot{b}(0)$.

Lemma 2. The function $\lambda : [0, \infty) \to (0, \kappa_0]$ has finite right and left derivatives on $(0, \infty)$. The right derivative exists at zero but may not be finite. The function λ has a derivative at $t \in (0, \infty)$ if and only if

$$\Lambda(\{x : f(x) = \frac{\lambda(t)}{\dot{b}(0)}\}) = 0.$$

Proof. By definition

$$\lambda(T) = \mathrm{inv}[\theta](C(T)) \qquad \text{for } 0 \leq T < \infty.$$

Therefore, we may write

$$D^{\pm} \lambda(T) = \frac{\dot{C}(T)}{D^{\mp} \theta(\lambda(T))} \qquad \text{for } 0 < T < \infty,$$

whenever the derivative in the right-hand side of the equation exists. For $T = 0$, we have

$$D^{+} \lambda(0) = \frac{\dot{C}(0)}{D^{-} \theta(\kappa_0)}.$$

Noting that $\lambda(T) < \kappa_0$ for $T > 0$, the conclusions follow from Lemma 1.

Theorem 1. The optimal search plan $m^* : [0, \infty) \times E^N \to [0, \infty)$ has the property that for all $x \in E^N$, $m^*(\cdot, x)$ has finite right and left derivatives on $(0, \infty)$. Moreover, for all $x \in E^N$, $m^*(\cdot, x)$ is differentiable for any $t \in (0, \infty)$ for which

$$\Lambda(\{x : f(x) = \frac{\lambda(t)}{\dot{b}(0)}\}) = 0.$$

Proof. Assume without loss of generality that $f(x) > 0$. Since

$$m^*(t, x) = \Psi(\frac{\lambda(t)}{f(x)}),$$

we may write

$$D^{\pm} m^*(t,x) = D^{\mp} \Psi\left(\frac{\lambda(t)}{f(x)}\right) D^{\pm} \lambda(t) \frac{1}{f(x)}$$

for all $t \in (0, \infty)$ for which the derivatives on the right-hand side exist and are finite. For all $t \in (0, \infty)$, $\lambda(t) \neq 0$ and, therefore, $D^{-}\Psi(\lambda(t)/f(x))$ and $D^{+}\Psi(\lambda(t)/f(x))$ exist and are finite. The theorem then follows from Lemma 2.

Theorem 2. If $\Lambda(\{x: f(x) = \Gamma\}) > 0$, then $D_1^{+} m^*(0,x)$ exists and is finite for all $x \in E^N$. If $\Lambda(\{x: f(x) = \Gamma\}) = 0$, then $D_1^{+} m^*(0,x)$ exists and is finite for all x for which $f(x) \neq \Gamma$.

Proof. If $D^{-}\Psi(\lambda(0)/f(x))$ and $D^{+}\lambda(0)$ exist and are finite, then $D_1^{+} m^*(0,x)$ exists and is finite. In this case

$$D_1^{+} m^*(0,x) = D^{-} \Psi\left(\frac{\lambda(0)}{f(x)}\right) D^{+}\lambda(0) \frac{1}{f(x)},$$

where

$$D^{-} \Psi\left(\frac{\lambda(0)}{f(x)}\right) = \begin{cases} \dfrac{1}{\ddot{b}(0)} & \text{for } f(x) = \Gamma \\ 0 & \text{otherwise,} \end{cases}$$

and

$$D^{+}\lambda(0) = \frac{\dot{C}(0)}{D^{-}\theta(\kappa_0)}.$$

Now

$$D^{-}\theta(\kappa_0) = \int_{\{x:f(x) = \Gamma\}} D^{-}\Psi(\dot{b}(0)) \frac{1}{f(x)} dx$$

$$= \frac{1}{\ddot{b}(0)\,\Gamma} \Lambda(\{x:f(x) = \Gamma\}).$$

Therefore, the first statement of the theorem is established.

The second statement is proved by observing that for $x \in E^N$, if $f(x) \neq \Gamma$, then there exists a $\kappa_1 < \kappa_0$ such that $f(x) < \kappa_1/\dot{b}(0)$. Also there exists $t_1 > 0$ such that $\kappa_1 \leq \lambda(t) < \kappa_0$ for $0 \leq t \leq t_1$. This means that for $0 \leq t \leq t_1$,

$$\frac{\lambda(t)}{f(x)} > \frac{\lambda(t)\,\dot{b}(0)}{\kappa_1} > \dot{b}(0),$$

and, therefore,

$$m^*(t,x) = 0 \qquad \text{for } 0 \leq t \leq t_1.$$

Thus $D^+ m^*(0,x) = 0$, and this completes the proof.

<u>Theorem 3</u>. For almost all $x \in E^N$, the functions $m^*(\cdot,x)$ are absolutely continuous and the function values may be written

$$m^*(T,x) = \int_0^T D_1^+(t,x)\,dt \qquad \text{for } 0 \leq T < \infty.$$

<u>Proof</u>. For all $x \in E^N$, $m^*(\cdot,x)$ is continuous and non-decreasing on $[0,T]$. Thus $m^*(\cdot,x)$ is of bounded variation on $[0,T]$. By Theorems 1 and 2, $D_1^+ m^*(\cdot,x)$ exists and is finite on $[0,T]$ for almost all x. Then by Lebesgue's version of the fundamental theorem of integral calculus (page 596 of reference [c])

$$m^*(T,x) = \int_0^T D_1^+ m^*(t,x)\,dt$$

except on the exceptional set of x having measure zero. This concludes the proof.

Illustrations

As an illustration of the above results, let b be the exponential effectiveness function $b(z) = 1 - e^{-z}$ for $z \epsilon [0, \infty)$. Thus $\dot{b}(z) = e^{-z}$ and $\Psi(u) = -\ln(u)$. Note that $\dot{b}(0) = 1$. Assume that $C(t) = t$.

First consider a two cell example. For $p_1 > p_2$ and disjoint R_1, $R_2 \subset E^2$ such that $\Lambda(R_1) = \Lambda(R_2) = 1$, let

$$f(x) = \begin{cases} p_1 & \text{for } x \epsilon R_1, \\ p_2 & \text{for } x \epsilon R_2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\Gamma = \kappa_0 = p_1$ and

$$S(\kappa) = \{x : f(x) \geq \kappa\} = \begin{cases} R_1 \cup R_2 & \text{for } 0 < \kappa \leq p_2 \\ R_1 & \text{for } p_2 < \kappa \leq p_1. \end{cases}$$

Since

$$\theta(\kappa) = \int_{S(\kappa)} \Psi\left(\frac{\kappa}{f(x)}\right) dx,$$

one has

$$\theta(\kappa) = \begin{cases} \ln \dfrac{p_1 p_2}{\kappa^2} & \text{for } 0 < \kappa \leq p_2 \\ \ln \dfrac{p_1}{\kappa} & \text{for } p_2 < \kappa \leq p_1. \end{cases}$$

Thus

$$\lambda(t) = \begin{cases} p_1 e^{-t} & \text{for } 0 \leq t < \ln(p_1/p_2) \\ \sqrt{p_1 p_2}\, e^{-t/2} & \text{for } t \geq \ln(p_1/p_2). \end{cases}$$

Finally for $x \epsilon R_1$,

$$m^*(t,x) = \begin{cases} t & \text{for } 0 \leq t < \ln(p_1/p_2) \\ \frac{t}{2} + \ln(p_1/p_2) & \text{for } t \geq \ln(p_1/p_2) \end{cases}$$

and for $x \epsilon R_2$,

$$m^*(t,x) = \begin{cases} 0 & \text{for } t < \ln(p_1/p_2) \\ \frac{t}{2} + \ln(p_1/p_2) & \text{for } t \geq \ln(p_1/p_2). \end{cases}$$

Note that in accordance with Theorem 1, $m^*(\cdot, x)$ has a finite derivative for all $t \epsilon (0, \infty)$ for which

$$\Lambda(\{x: f(x) = \lambda(t)\}) = 0.$$

This condition fails to hold only when $t = \ln(p_1/p_2)$, in which case

$$\Lambda(\{x: f(x) = \lambda(t)\}) = \Lambda\{x: f(x) = p_2\} = 1.$$

In addition, $\Lambda(\{x: f(x) = \Gamma = p_1\}) = 1$, and, therefore, by Theorem 2, $D_1^+ m^*(0,x)$ exists and is finite for all $x \epsilon E^2$.

The next illustration considers a situation where $D_1^+ m^*(0,x)$ is not finite for all x. Let f be a bivariate normal distribution, i.e., for $x = (x_1, x_2) \epsilon E^2$ and $r(x_1, x_2) = ([x_1/\sigma_1]^2 + [x_2/\sigma_2]^2)^{\frac{1}{2}}$,

$$f(x) = \frac{1}{2\pi\sigma_1\sigma_2} \exp(-\tfrac{1}{2} r(x,y)^2).$$

Since $\Lambda(\{x : f(x) = \lambda(t)\}) = 0$ for all $t \geq 0$, $m^*(\cdot, x)$ must be differentiable for all $x \in E^N$ when $t > 0$ according to Theorem 2.

It is well known that if $C(t) = t$, then for $t \geq 0$,

$$m^*(t, x_1, x_2) = \begin{cases} K\sqrt{t} - \frac{1}{2}r(x_1, x_2)^2 & \text{for } r(x_1, x_2)^2 \leq 2K\sqrt{t} \\ 0 & \text{otherwise,} \end{cases}$$

where $K = (\pi \sigma_1 \sigma_2)$. For all $t > 0$ and $(x_1, x_2) \in E^2$,

$$D_1 m^*(t, x_1, x_2) = \frac{K}{\sqrt{t}}.$$

In accordance with Theorem 3, $D_1^+ m^*(0, x_1, x_2) < \infty$, for $(x_1, x_2) \neq (0, 0)$. Only at the point $(0, 0)$ (a set of measure zero) does $D_1^+ m^*$ fail to be finite.

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References

[a] Optimal Search for Deterministically Moving Target, Daniel H. Wagner, Associates Memorandum to Office of Naval Research, by H. R. Richardson, dated February 19, 1971, Unclassified.

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[c] The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Volume One, E. W. Hobson, Dover Publications, Inc., 1957.