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Foreign Technology Division
Wright-Patterson Air Force Base, Ohio

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Foreword

Various thin-walled three-dimensional structures combining high strength, rigidity and a light weight with the possibility of effective use of the inner volume are finding broad applications in modern aeronautical and rocket engineering, shipbuilding, and construction. The theoretical and experimental study of the operation of such structures in accordance with the requirements imposed by technical progress constitutes one of the important and complex areas of modern mechanics.

Thin-walled structures outlined by smooth surfaces are a traditional subject of study in shell theory. The development of this theory, which has a long history, has followed two directions. The first is represented by the so-called mathematical shell theory based on a minimum number of hypotheses and devoted to the substantiation of the fundamental equations, analysis of their accuracy and applicability, and qualitative study of comparatively simple problems. The second direction, applied shell theory, aims at the development of approximate methods of calculation, necessary for the creation of new technology. Major contributions to the development of both mathematical and applied shell theory have been made by Soviet scientists V.Z. Vlasov, A.L. Gol'denveyzr, A.I. Lur'ye, V.V. Novozhilov, Yu. N. Rabotnov, and others.

To date, shells of classical types have been thoroughly investigated: smooth cylindrical and conical shells of revolution and spherical shells. Studies dealing with these shells number in the thousands and are being steadily supplemented. For example, Nash's extensive bibliographies contain about 6,000 titles, most of which deal with shells of the indicated types.

Shells consisting of more complex, generally piecewise-smooth surfaces are in a different category. The general problem of design of such shells has not thus far been properly reflected in the literature, despite the fact that such shells, reinforced with longitudinal and transverse elements, despite the fact that such shells, reinforced with longitudinal and transverse elements, are widely employed in the construction of various types of wings, airplane fuselages, rocket bodies and ship hulls. When such "nonclassical" objects are considered, the direct integration of the differential equations describing the operation of shells involves practically insurmountable difficulties. In this connection, a very important role is played by the so-called technical theories, based on certain additional hypotheses concerning the comparative magnitude of the individual components of the stressed and strained state of the shell. Such assumptions, based on experimental data, permit the derivation of the equations of general shell theory and ultimately lead to results that can be used in practical applications.

The fundamental results of studies dealing with shells having complex, generally unsmooth surfaces pertain to the design of three-dimensional systems of the type of thin-walled bars. For such shells, which may have both a constant and a variable cross section, the concepts of principal axes of inertia and axis of rigidity characteristic of beams usually remain valid, but in contrast to beams, after deformation the cross sections of thin-walled bars do not remain planar and may also change their configuration. Unquestionably the greatest contribution to the theory of design of thin-walled bars has been made by Soviet scientists, chiefly by V.N. Belyayev, V.Z. Vlasov, and A.A. Umanskiy.

The thin-walled bar has proven a fruitful design model for three-dimensional systems of various applications. In aeronautical engineering, this model has been used to develop effective applied methods of design of fuselage and straight-wing structures of both constant and variable cross sections. Much credit for the development of these methods is due to A.I. Makarevskiy, G.G. Rostovtsev, A.M. Cheremukhin, A. Yu. Romashevskiy, L.I. Balabukh, V.F. Kiselev, S.N. Kan, Yu. G.

Odinokov, G. S. Yelenevskiy, A.F. Feofanov, V.V. Novitskiy, and others.

The development of high-speed aviation has resulted in the advent of more complex structure diagrams and external configurations. Oblique structures have appeared to which the traditional beam concept as well as individual concepts of thin walled bars are generally inapplicable. This refers primarily to wing structures. A distinctive feature of the operation of a swept wing, for example, is the fact that because of its obliqueness, any transverse load causes both bending and torsion of the wing, so that the concept of a rigidity axis loses its usual meaning, and the stressed and strained state of such a wing is changed considerably in comparison with a straight wing.

The operation of wings of low aspect ratio is even more complex in character, owing to oblique configurations in the plane, and to the fact that the longitudinal and transverse dimensions of the wing are of the same order. The beam concept is of little use in this case, even as a first approximation.

Oblique spatial systems also include certain structures of airplane fuselages, rocket bodies, and ship hulls, representing smooth and reinforced structures of conical and circular type, and also more complex configurations. With a sufficient degree of conicity, the stressed and strained state of such shells is also very complex in character.

The design of oblique spatial systems of the above-indicated type is one of the major problems of structural mechanics of thin-walled structures. The design of structures that meet present-day needs is unthinkable without the creation of adequately substantiated general methods of calculation combining the necessary accuracy with simplicity and permitting the study of a broad range of urgent problems from a unified point of view. However, despite the fact that individual problems have been dealt with in a number of studies, there is as yet no unified general approach to the design of oblique spatial systems.

In analyzing, for example, studies dealing with the design of swept and low-aspect wings, three basic trends can be distinguished.

The first trend, represented by earlier studies, is based mainly on the classical beam concept. The effect of obliqueness is taken into consideration approximately, through engineering analysis of the characteristics of wing structure diagrams of a given design, and through the use of Castigliano's variational principle in revealing static indeterminacy. This trend was developed by V.F. Kiselev, I.A. Sverdlov, A.F. Feofanov, and others. Without detracting from the advantages of this approach, which lie in the simplicity of the mathematical devices employed and in a clear representation of the physical picture of the structural interaction of the individual structural elements, we should note that the scope of application of such methods is limited mainly by certain typical static problems in the elastic formulation, whereas their application to calculations of complex external factors of dynamic and thermal character and also to studies of the lift effectiveness of wings is practically impossible. Another drawback of such methods is the necessity of considering wings of different configurations independently of one another, which creates an abundance of computational schemes and complicates the practical mastering of the methods.

The second trend includes studies based on the schematization of a thin-walled wing structure in the form of a structurally anisotropic oblique plate. Such a schematization is highly conventional, chiefly because the reception of an external load by a three-dimensional system and by a plate is fundamentally different. Whereas in a plate, an external transverse load is balanced by transverse forces, in a three-dimensional system this load is balanced off by flows of tangential forces acting along the contour of the cross section. Therefore, the possibilities of the use of a plate as a design model for a thin-walled-type wing are very limited. Such a model can be used only for the design of wings reinforced by a regular assembly of spars, whose walls receive the transverse load, as in the case of plates. A sufficient degree of accuracy can apparently be obtained in this case only when the wing is designed in the elastic formulation for the action of smooth aerodynamic loads, and also in determining the lowest frequencies and modes of natural vibrations, when the tangential stresses are comparatively small. The reliability of the results depends considerably on a successful choice of numerical values of the reduced elastic constants of the anisotropic plate. However, the design of a wing as an anisotropic plate for the action of large twisting moments and concentrated forces, when the tangential stresses may even exceed the normal ones, and the study of complex problems of thermoelasticity and life effectiveness on the basis of such a model appear to be altogether impossible.

The third trend includes methods based on a discrete computational model of an elastic body. According to this model, the structure is represented as a collection of a finite number of nodal points connected by a set of elementary rods with selected elastic properties. The stressed and strained state in the confines of a single rod is approximated in some manner, so that the study of a continuous system can be reduced to the determination of a finite number of unknown quantities in accordance with the adopted nodal partition of the structure. Mathematically, the problem reduces to systems of algebraic equations with a large number of unknowns, which can be obtained by the classical methods of structural mechanics of bar systems, i.e., the method of forces and the method of displacements. The discrete methods, covered by the general term "method of finite elements". appeared as a result of the advent of computers. This trend is primarily due to the works of J.H. Argyris* and S. Levy. Despite the conventional

*Modern Methods of Calculation of Complex Statically Indeterminate Systems. Collected Papers, translated from the English. Compiled and edited by Prof. A.P. Filin. Sudpromgiz, Leningrad, 1961.

character of schematization of the structure, discrete methods are of considerable interest. Their advantage lies in the possibility of calculating complex irregular structures on the basis of fairly universal matrix algorithms. However, like other numerical methods, discrete methods exclude the possibility of qualitative studies of the operation of structures, so necessary in designing. It should be noted at this point that the contrasting of discrete methods with analytical ones, which is occasionally encountered, is not justified. Both types of methods have their advantages and disadvantages, and th therefore, since they complement one another, should be developed to the same extent. Analytical methods, which describe the operation of a structure as a whole without excessive particularization, form the basis of qualitative studies. These methods should be employed in selecting the structure diagram and optimizing the main elements of the structure. At the stage of particularization of all the structural elements and checking of the calculations, when the structure is treated as a complex statically indeterminate system, an important role is played by discrete methods.

In the present book, the general subject of the study is a reinforced conical shell of arbitrary configuration, used as a universal computational model of an oblique thin-walled structure. Under appropriate assumptions regarding the configuration of the directrix of the conical surface and location of its apex, the book discusses, from a unified point of view, different types of wings (straight, swept, low-aspect, delta, etc.) and different types of airframes. The Lagrange variational principle is used to develop a general method of calculation of a reinforced conical shell of arbitrary configuration, amounting to the integration of a system of ordinary differential equations describing the operation of oblique thin-walled systems when the form of external actions is arbitrary. This method makes it possible to work out the solutions of the most diverse problems of structural mechanics of thin-walled structures. The solutions

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are constructed in the form of expansions whose first terms correspond to the hypothesis of plane sections, which is widely employed in practical engineering calculations. In this connection, the resolvent equations constitute a generalization of the equations of strength of materials. Keeping one to two additional terms of the expansion for such complex oblique three-dimensional systems as swept and delta wings, low-aspect wings, etc., one can derive solutions that are very effective as a whole and reduce them to compact general working formulas. When the number of expansion terms is large, the stressed and strained state of oblique systems can be studied in detail by means of a computer. The book discusses numerous problems pertaining to the analysis of wings and airframes for different types of external action of static, dynamic and thermal character. The solutions obtained are illustrated by a large number of numerical results, some of which are compared with experimental results. The appendices present a general method for deriving the solution of equations whose coefficients contain characteristics of the type of the delta function and its derivatives. Equations of this type describe objects combining continuous elements with discrete ones - bars, plates, or shells with masses and moments of inertia concentrated at points, on lines and individual surfaces, with reinforcing structures, supporting layers of zero thickness, etc. The proposed method makes it possible to study both "smooth" and "discrete-continuous" objects on the same methodological basis.

The book is based on the author's many years of research work. The results of experimental studies and certain theoretical conclusions of other authors mentioned in the book are accompanied by corresponding references. All the calculations were carried out in accordance with a single universal program written by Yu. S. Matyushev.

The authors express their gratitute to Engineer V.I. Narinskiy for his assistance in the preparation of this book.

Main Symbols

E, G, ν - Young's modulus, shear modulus, and Poisson's ratio;
$\Delta F_k(\bar{Z})$ - cross-sectional area of k-th element of longitudinal structure;
$h(\bar{Z}, S)$ - shell thickness;
l_0 - z-coordinate of cone apex;
l_s - length of generatrix S=const;
M, H - bending and twisting moments;
$B_n\ n_z,\ m_z$ - unit vectors of main trihedron oriented along the lines of principal curvatures;
$B_n,\ n_s,\ m_x$ - unit vectors of auxiliary trihedron ($m_5 \cdot m_z = \cos x$);
$P_i^n(\bar{Z})$ - generalized force;
p - surface load intensity vector;
p, ξ - circular coordinates in a shell of revolution with regular longitudinal structure: p - number of stringer, ξ - relative coordinate in the span between stringers;
Q - transverse force;
R_0 - base radius of right circular conical shell;
T, S - normal and tangential forces;
$t=\ln(1-\bar{Z})$ - independent variable in Euler-type equations;
$\bar{t}=\dfrac{t}{\sqrt{\gamma \sin \beta}}$ relative longitudinal coordinate;
$U(\bar{Z}, S)$ - elastic displacement vector;
$V_i(\bar{Z}),\ U_i(\bar{Z})$ - generalized displacements being sought;
x, y, z - Cartesian coordinates of point M of middle surface of shell in unstrained state: the plane xOz is perpendicular to the plane of the directrix, the origin lies in the plane of the directrix, and the Oz axis passes through the cone apex;
$x_0(S),\ y_0(S),\ z_0(S) = x_0(S)\ \mathrm{ctg} x_0$ - Cartesian coordinates of point M_0 of directrix;
$\bar{Z}$, S - curvilinear coordinates in the middle surface of shell; $\bar{Z}$ - relative coordinate, measured along the generatrix from the base to the apex of the cone, in fractions of its total length; S - arc of directrix of conical surface, constituting the parameter of the family of generatrices;
α - circular coordinate of right circular conical shell;
β - apex angle of right circular conical shell;
$\delta(x - x_0)$ - Dirac's delta function;
$e_t,\ \gamma_{t_1t_2},\ x_t,\ x_{t_1t_2}$ - strains of relative elongation, shear, bending, and torsion of middle surface;
$n(\bar{Z}),\ \theta(\bar{Z})$ - vectors of translational displacement and rotation of the contour $\bar{Z}$ = const;
$\theta(x - x_0)$ - Heaviside unit function;
x - ratio of skin area $2\pi R_0 h_0$ to total area of stringers ΔFn;
$\tilde{\sigma},\ \tilde{\tau}$ - normal and tangential stresses, referred to the magnitude

of the corresponding external force factor [in the case of moment, $(1/cm^3)$; in the case of force, $(1/cm^2)$; in the case of pressure, dimensionless];

Φ_{lsr}, Φ_{lsr} - special coordinate functions;

$\Phi(S)$ - coordinate vector function;

x_0 - angle between the coordinate plane zOy and the plane of the directrix;

$x(S)$ - angle between the coordinate lines $\overline{Z}$ = const, S = const;

$\psi_i(S)$, $\psi_i(S)$ - generalized deformation coordinates.

Part One

Conical Shell of Arbitrary Configuration

Chapter I. Fundamental Geometric, Static and Physical Relationships

1.1. Three-Dimensional-Oblique System of Curvilinear Coordinates.

First Quadratic Form

A conical shape of general type is considered. The contour of the directrix is completely arbitrary and can be both open and single- or multiclosed. The shell may be equipped with a reinforcing structure.

We will choose a system of Cartesian coordinates so that the Oy axis is in the plane of the directrix, and the Oz axis passes through the cone apex (Figure 1.1). Let the coordinates of the cone apex be

$$x=0,\ y=0,\ z=l_0. \tag{1.1}$$

and the equation of the directrix in parametric form becomes

$$x=x_0(S);\ y=y_0(S);\ z=x_0(S)\operatorname{ctg}\chi_0. \tag{1.2}$$

where S is some parameter measured along the contour of the directrix, and $x_0(S)$, $y_0(S)$ are specified functions determining the outlines of the directrix.

The location of point M of the middle surface of the shell will be fixed in a three-dimensional oblique coordinate grid consisting of the family of generatrices of the conical surface and family of lines of intersection of this surface by planes parallel to the plane of the directrix.

In the selected system of Cartesian coordinates, the equation of the generatrix passing through point M_0 lying on the directrix and having the coordinates $x = x_0(S)$, $y = y_0(S)$, $z = x_0(S)$ ctg χ_0 has the form

$$\frac{x}{x_0(S)} = \frac{y}{y_0(S)} = \frac{z - l_0}{x_0(S)\,\mathrm{ctg}\,\chi_0 - l_0}, \tag{1.3}$$

and the equation of an arbitrary plane parallel to the plane of the directrix is

$$z = l_0\bar{Z} + x\,\mathrm{ctg}\,\chi_0, \tag{1.4}$$

where $\bar{Z}$ is the distance between these planes, measured along the generatrix in fractions of its total length l_s.

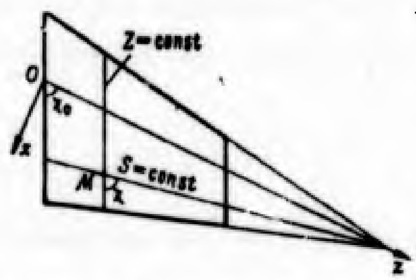

Figure 1.1. Conical shell of general type.

Solving Eqs. (1.3), (1.4) for x, y, z, we obtain the coordinates of point M:

$$\begin{aligned} x &= x_0(S)(1-\bar{Z}), \\ y &= y_0(S)(1-\bar{Z}), \\ z &= l_0\bar{Z} + x_0(S)\,\mathrm{ctg}\,\chi_0(1-\bar{Z}). \end{aligned} \tag{1.5}$$

Expressions (1.5) constitute parametric equations of a conical surface of arbitrary outline, and parameters $\bar{Z}$, S, the curvilinear coordinates of point M. It is evident that by setting S = const in (1.5), we will obtain the equations of the family of generatrices, and by setting $\bar{Z}$ = const, the equations of the family of lines of intersection of the middle surface by planes parallel to the plane of the directrix.

As the parameter S, we will take the magnitude of the arc of the directrix, measured from a certain fixed point. Then, taking (1.2) into consideration, we have

$$\frac{1}{\sin^2\chi_0} x_0'^2(S) + y_0'^2(S) = 1, \tag{1.6}$$

where

$$x_0' = \frac{dx_0}{dS}; \quad y_0' = \frac{dy_0}{dS}.$$

Let ds be an arbitrarily oriented small linear element of the middle surface, connecting the points with curvilinear coordinates $\bar{Z}$, S and $\bar{Z}$ + d$\bar{Z}$, S + dS. The square length of this element

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2. \tag{1.7}$$

It is evident that

$$dx = \frac{\partial x}{\partial \bar{Z}} d\bar{Z} + \frac{\partial x}{\partial S} dS,$$
$$dy = \frac{\partial y}{\partial \bar{Z}} d\bar{Z} + \frac{\partial y}{\partial S} dS,$$
$$dz = \frac{\partial z}{\partial \bar{Z}} d\bar{Z} + \frac{\partial z}{\partial S} dS. \tag{1.8}$$

Introducing (1.8) into (1.7) we find

$$(ds)^2 = A^2 (d\bar{Z})^2 + 2AB \cos\chi d\bar{Z} dS + B^2 (dS)^2. \tag{1.9}$$

where

$$A^2=\left(\frac{\partial x}{\partial \bar{Z}}\right)^2+\left(\frac{\partial y}{\partial \bar{Z}}\right)^2+\left(\frac{\partial z}{\partial \bar{Z}}\right)^2,$$
$$B^2=\left(\frac{\partial x}{\partial S}\right)^2+\left(\frac{\partial y}{\partial S}\right)^2+\left(\frac{\partial z}{\partial S}\right)^2,$$
$$AB\cos\chi=\frac{\partial x}{\partial \bar{Z}}\frac{\partial x}{\partial S}+\frac{\partial y}{\partial \bar{Z}}\frac{\partial y}{\partial S}+\frac{\partial z}{\partial \bar{Z}}\frac{\partial z}{\partial S}. \qquad (1.10)$$

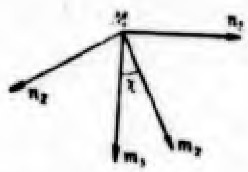

Figure 1.2. Tangential unit vectors of the main and auxiliary trihedra.

The first part of (1.9) is called the first quadratic form of the surface, and the quantities A^2, B^2, AB cos x are the coefficients of the first quadratic form. The first quadratic form determines the metric of the surface. Knowing the coefficients of the first quadratic form, one can compute the lengths of linear elements of the surface, and also the angles between them. Setting in (1.9) $dS = 0$ and $d\bar{Z} = 0$ successively, one can conclude that the coefficients A and B represent, respectively, the lengths of segments of coordinate lines S = const and $\bar{Z}$ = const corresponding to $d\bar{Z} = 1$ and $dS = 1$. It is also clear from geometrical considerations that x is the angle between the positive directions of the coordinate lines.

For an arbitrary conical surface, taking (1.5) into account, we find from (1.10):

$$A=l_s;\quad B=1-\bar{Z};\quad \cos\chi=-l_s'. \qquad (1.11)$$

where l_s is the length of the generatrix S = const:

$$l_1^2(S) = x_0^2(S) + y_0^2(S) + [l_0 - x_0(S)\,\mathrm{ctg}\,\chi_0]^2, \quad l_1' = \frac{dl_1}{dS}. \tag{1.12}$$

We introduce into consideration the tangential unit vectors m_z, m_s, n_z, n_s (Figure 1.2) and the unit vector of the normal n_n:

$$n_n = m_z \times n_z = n_s \times m_s. \tag{1.13}$$

It is easy to see that the tangential unit vectors are related as follows:

$$n_z = \frac{1}{\sin\chi} m_z - \frac{\cos\chi}{\sin\chi} m_s,$$
$$n_s = \frac{1}{\sin\chi} m_s - \frac{\cos\chi}{\sin\chi} m_z. \tag{1.14}$$

and the derivatives of the scalar vector functions of curvilinear coordinates $\bar{Z}$, S with respect to these unit vectors will be

$$\frac{\partial}{\partial m_z} = \frac{1}{A}\frac{\partial}{\partial \bar{Z}},$$
$$\frac{\partial}{\partial m_s} = \frac{1}{B}\frac{\partial}{\partial S},$$
$$\frac{\partial}{\partial n_z} = \frac{\partial}{\partial \bar{Z}}\frac{d\bar{Z}}{dn_z} + \frac{\partial}{\partial S}\frac{dS}{dn_z},$$
$$\frac{\partial}{\partial n_s} = \frac{\partial}{\partial \bar{Z}}\frac{d\bar{Z}}{dn_s} + \frac{\partial}{\partial S}\frac{dS}{dn_s}, \tag{1.15}$$

where according to (1.14)

$$\frac{d\bar{Z}}{dn_z} = \frac{1}{A}\frac{1}{\sin\chi}; \qquad \frac{d\bar{Z}}{dn_s} = -\frac{1}{A}\frac{\cos\chi}{\sin\chi};$$
$$\frac{dS}{dn_z} = -\frac{1}{B}\frac{\cos\chi}{\sin\chi}; \qquad \frac{dS}{dn_s} = \frac{1}{B}\frac{1}{\sin\chi}. \tag{1.16}$$

The direction cosines of unit vectors n_n, n_z, m_z, n_s, m_s in the basic system of Cartesian coordinates are:

	x	y	z
n_n	$\begin{vmatrix} m_{m_z} & n_{m_z} \\ m_{n_z} & n_{n_z} \end{vmatrix}$	$-\begin{vmatrix} l_{m_z} & n_{m_z} \\ l_{n_z} & n_{n_z} \end{vmatrix}$	$\begin{vmatrix} l_{m_z} & m_{m_z} \\ l_{n_z} & m_{n_z} \end{vmatrix}$
n_z	$\frac{dx}{dn_z}$	$\frac{dy}{dn_z}$	$\frac{dz}{dn_z}$
m_z	$\frac{dx}{dm_z}$	$\frac{dy}{dm_z}$	$\frac{dz}{dm_z}$
n_s	$\frac{dx}{dn_s}$	$\frac{dy}{dn_s}$	$\frac{dz}{dn_s}$
m_s	$\frac{dx}{dm_s}$	$\frac{dy}{dm_s}$	$\frac{dz}{dm_s}$

(1.17)

Expending (1.17) with the aid of (1.5), (1.11), (1.15), (1.16) we find

	x	y	z
n_n	$-\frac{y_0'}{\sin\chi}\frac{l_0}{l_s} + \frac{(x_0 y_0' - x_0' y_0)\operatorname{ctg}\chi_0}{l_s \sin\chi}$	$\frac{x_0'}{\sin\chi}\frac{l_0}{l_s}$	$\frac{x_0' y_0 - x_0 y_0'}{l_s \sin\chi}$
n_s	$\frac{x_0'}{\sin\chi} + \frac{\cos\chi}{\sin\chi}\frac{x_0}{l_s}$	$\frac{y_0'}{\sin\chi} + \frac{\cos\chi}{\sin\chi}\frac{y_0}{l_s}$	$\frac{-l_0\cos\chi + x_0' l_s \operatorname{ctg}\chi_0}{l_s \sin\chi} + \frac{x_0 \operatorname{ctg}\chi_0 \cos\chi}{l_s \sin\chi}$
m_s	$-\frac{x_0}{l_s}$	$-\frac{y_0}{l_s}$	$\frac{l_0}{l_s} - \frac{x_0 \operatorname{ctg}\chi_0}{l_s}$

(1.18)

The unit vectors n_z, m_z and n_s, m_s in combination with the unit vector n_n form two moving trihedra. The trihedron n_n, n_z, m_z, oriented along the lines of principal curvatures, will hereinafter be referred to as the main trihedron, and the trihedron n_n, n_s, m_s will be correspondingly referred to as the auxiliary trihedron.

Expanding any vector K in axes of the main and auxiliary trihedra,

	x	y	z
$\mathbf{n}_s$	$-x_0' \dfrac{\cos\chi}{\sin\chi} - \dfrac{x_0}{l_s \sin\chi}$	$-y_0' \dfrac{\cos\chi}{\sin\chi} - \dfrac{y_0}{l_s \sin\chi}$	$\dfrac{l_0 - x_0' \operatorname{ctg}\chi_0 l_s \cos\chi}{l_s \sin\chi} - \dfrac{x_0 \operatorname{ctg}\chi_0}{l_s \sin\chi}$
$\mathbf{m}_s$	x_0'	y_0'	$x_0' \operatorname{ctg}\chi_0$

(1.18)

we have

$$\mathbf{K} = K_{m_s}\mathbf{m}_s + K_{n_s}\mathbf{n}_s + K_{n_n}\mathbf{n}_n,$$
$$\mathbf{K} = K_{n_z}\mathbf{n}_z + K_{m_z}\mathbf{m}_z + K_{n_n}\mathbf{n}_n. \tag{1.19}$$

where the components of the vector are its scalar products by the corresponding unit vectors:

$$K_{m_s} = \mathbf{K}\cdot\mathbf{m}_s, \qquad K_{n_z} = \mathbf{K}\cdot\mathbf{n}_z,$$
$$K_{n_s} = \mathbf{K}\cdot\mathbf{n}_s, \qquad K_{m_z} = \mathbf{K}\cdot\mathbf{m}_z,$$
$$K_{n_n} = \mathbf{K}\cdot\mathbf{n}_n.$$

The tangential components of vector K are related by the obvious relations

$$K_{n_s} = K_{m_z}\sin\chi - K_{n_z}\cos\chi, \qquad K_{m_z} = K_{m_s}\cos\chi + K_{n_s}\sin\chi,$$
$$K_{m_s} = K_{m_z}\cos\chi + K_{n_z}\sin\chi, \qquad K_{n_z} = K_{m_s}\sin\chi - K_{n_s}\cos\chi. \tag{1.20}$$

1.2 Deformation of Conical Shell

1. Components of Tangential Deformation of the Middle Surface

Let $M(\bar{Z}, S)$ be the radius vector of point M of the middle surface before deformation, $M^1(\bar{Z}, S)$ the radius vector of point M after deformation, and $U(\bar{Z}, S)$, the vector of the elastic displacement of this point.

It is evident that

$$
\begin{aligned}
&\mathbf{M} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},\\
&\mathbf{M}^1 = x^1\mathbf{i} + y^1\mathbf{j} + z^1\mathbf{k},\\
&\mathbf{M}^1 = \mathbf{M} + \mathbf{U},
\end{aligned}
\tag{1.21}
$$

where i, j, k are unit vectors of the Cartesian coordinate system, and x, y, z and x^1, y^1, z^1 are the Cartesian coordinates of point M before and after deformation, respectively.

It follows from (1.21) that

$$
\begin{aligned}
&x^1 = \mathbf{M}^1\cdot\mathbf{i} = x + \mathbf{U}\cdot\mathbf{i},\\
&y^1 = \mathbf{M}^1\cdot\mathbf{j} = y + \mathbf{U}\cdot\mathbf{j},\\
&z^1 = \mathbf{M}^1\cdot\mathbf{k} = z + \mathbf{U}\cdot\mathbf{k}.
\end{aligned}
\tag{1.22}
$$

Expressions (1.22) represent parametric equations of the deformed surface of the shell. The linear element ds^1 of this surface is defined by the first quadratic form

$$(ds^1)^2 = A^{1^2}(d\bar{Z})^2 + 2A^1B^1\cos\chi^1 d\bar{Z}dS + B^{1^2}(dS)^2, \tag{1.23}$$

where, as in (1.10),

$$
\begin{aligned}
A^{1^2} &= \left(\frac{\partial x^1}{\partial \bar{Z}}\right)^2 + \left(\frac{\partial y^1}{\partial \bar{Z}}\right)^2 + \left(\frac{\partial z^1}{\partial \bar{Z}}\right)^2,\\
B^{1^2} &= \left(\frac{\partial x^1}{\partial S}\right)^2 + \left(\frac{\partial y^1}{\partial S}\right)^2 + \left(\frac{\partial z^1}{\partial S}\right)^2,\\
A^1B^1\cos\chi^1 &= \frac{\partial x^1}{\partial \bar{Z}}\frac{\partial x^1}{\partial S} + \frac{\partial y^1}{\partial \bar{Z}}\frac{\partial y^1}{\partial S} + \frac{\partial z^1}{\partial \bar{Z}}\frac{\partial z^1}{\partial S}.
\end{aligned}
\tag{1.24}
$$

From (1.22), taking (1.15) and (1.17) into account, we have

$$\frac{\partial x^1}{\partial Z}=A\left(\mathbf{m}_z+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\right)\cdot\mathbf{i}, \qquad \frac{\partial x^1}{\partial S}=B\left(\mathbf{m}_s+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\right)\cdot\mathbf{i},$$
$$\frac{\partial y^1}{\partial Z}=A\left(\mathbf{m}_z+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\right)\cdot\mathbf{j}, \qquad \frac{\partial y^1}{\partial S}=B\left(\mathbf{m}_s+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\right)\cdot\mathbf{j}, \qquad (1.25)$$
$$\frac{\partial z^1}{\partial Z}=A\left(\mathbf{m}_z+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\right)\cdot\mathbf{k}, \qquad \frac{\partial z^1}{\partial S}=B\left(\mathbf{m}_s+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\right)\cdot\mathbf{k}.$$

Introducing (1.25) into (1.24) and neglecting the squares of small quantities and their derivatives, we find

$$A^1=A\left(1+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\cdot\mathbf{m}_z\right),$$
$$B^1=B\left(1+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_s\right), \qquad (1.26)$$
$$A^1B^1\cos\chi^1=AB\left(\cos\chi+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\cdot\mathbf{m}_s+\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_z\right).$$

Let some curve given in parametric form

$$Z=Z(t), \quad S=S(t). \qquad (1.27)$$

pass through point M of the middle surface. We will represent the unit vector t, directed along a tangent to this curve, in the form of a linear combination of unit vectors m_z and m_s:

$$\mathbf{t}=a\mathbf{m}_z+b\mathbf{m}_s. \qquad (1.28)$$

Multiplying (1.28) scalarly by n_s and n_z successively, we obtain

$$\mathbf{n}_s\cdot\mathbf{t}=a\sin\chi,$$
$$\mathbf{n}_z\cdot\mathbf{t}=b\sin\chi, \qquad (1.29)$$

so that linear combination (1.28) reduces to the form

$$\mathbf{t}=\frac{1}{\sin\chi}\left[(\mathbf{t}\cdot\mathbf{n}_s)\mathbf{m}_z+(\mathbf{t}\cdot\mathbf{n}_z)\mathbf{m}_s\right]. \qquad (1.30)$$

Let ds_t be the differential of the arc of curve (1.27). From (1.9), we have

$$(ds_t)^2 = [A^2(\dot{\bar{Z}})^2 + 2AB\cos\chi\,\dot{\bar{Z}}\dot{S} + B^2(\dot{S})^2](dt)^2, \tag{1.31}$$

where

$$\dot{\bar{Z}} = \frac{d\bar{Z}(t)}{dt}, \qquad \dot{S} = \frac{dS(t)}{dt}.$$

The quantities $A\dot{\bar{Z}}dt = ds_{m_z}$ and $B\dot{S}dt = ds_{m_s}$ are differentials of coordinate lines S = const, $\bar{Z}$ = const corresponding to the differential ds_t. According to (1.30)

$$\frac{ds_{m_z}}{ds_t} = \frac{1}{\sin\chi}\,\mathbf{t}\cdot\mathbf{n}_s, \qquad \frac{ds_{m_s}}{ds_t} = \frac{1}{\sin\chi}\,\mathbf{t}\cdot\mathbf{n}_z. \tag{1.32}$$

The material segment ds_t after deformation will have the length ds_t^1. According to (1.23)

$$(ds_t^1)^2 = [A^{1^2}(\dot{\bar{Z}})^2 + 2A^1B^1\cos\chi^1\,\dot{\bar{Z}}\dot{S} + B^{1^2}(\dot{S})^2](dt)^2. \tag{1.33}$$

Let us expand (1.33) with the aid of (1.26). Neglecting squares of small quantities and their derivatives and considering (1.31), (1.32), we find

$$ds_t^1 = ds_t\left\{1 + \frac{1}{\sin^2\chi}\left[\left(\frac{\partial\mathbf{U}}{\partial m_z}\cdot\mathbf{m}_z\right)(\mathbf{t}\cdot\mathbf{n}_s)^2 + \left(\frac{\partial\mathbf{U}}{\partial m_s}\cdot\mathbf{m}_s\right)(\mathbf{t}\cdot\mathbf{n}_z)^2 + \right.\right.$$
$$\left.\left. + \left(\frac{\partial\mathbf{U}}{\partial m_z}\cdot\mathbf{m}_s + \frac{\partial\mathbf{U}}{\partial m_s}\cdot\mathbf{m}_z\right)(\mathbf{t}\cdot\mathbf{n}_s)(\mathbf{t}\cdot\mathbf{n}_z)\right]\right\}. \tag{1.34}$$

Expression (1.34) represents the length of the material segment of the middle surface after deformation, oriented in the direction of the arbitrary tangential unit vector t. The strain of this segment

$$\varepsilon_t^0 = \frac{ds_t^1 - ds_t}{ds_t}$$

on the basis of (1.34) will be

$$\varepsilon_t^0 = \frac{1}{\sin^2\chi}\left[\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_s\right)(\mathbf{t}\cdot\mathbf{n}_s)^2 + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_t}\cdot\mathbf{m}_t\right)(\mathbf{t}\cdot\mathbf{n}_t)^2 + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_t}\cdot\mathbf{m}_s + \frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_t\right)(\mathbf{t}\cdot\mathbf{n}_s)(\mathbf{t}\cdot\mathbf{n}_t)\right]. \tag{1.35}$$

We will reduce expression (1.35) to a more concise form permitting a simpler geometrical interpretation.

On the basis of (1.32)

$$\frac{\partial}{\partial \mathbf{t}} = \frac{1}{\sin\chi}\left[(\mathbf{t}\cdot\mathbf{n}_s)\frac{\partial}{\partial \mathbf{m}_s} + (\mathbf{t}\cdot\mathbf{n}_t)\frac{\partial}{\partial \mathbf{m}_t}\right], \tag{1.36}$$

where $\partial/\partial t$ is the derivative in the direction of the arbitrarily oriented unit vector t. Considering (1.36) and (1.30), from (1.35) it is easy to obtain

$$\varepsilon_t^0 = \frac{\partial \mathbf{U}}{\partial \mathbf{t}}\cdot\mathbf{t}. \tag{1.37}$$

Expression (1.37) is very clear. Figure 1.3 shows the elementary segment $ab = ds_t$ of the arc of the arbitrary curve on the middle surface of the shell and vectors U and $U + \partial U/\partial t(ds_t)$ of displacements of the ends of the segment. In this case, to within higher-order infinitesimals, vector $U + \partial U/\partial t(ds_t)$ also lies in the plane of the figure. It is obvious that the relative elongation of segment ab will be

$$\varepsilon_t^0 = \frac{a'b' - ab}{ab} = \frac{cd}{ab} = \frac{\partial \mathbf{U}}{\partial \mathbf{t}}\cdot\mathbf{t}.$$

On the basis of (1.37), the strain

$$\varepsilon_t^0 = \frac{\partial u_t}{\partial t} - U\cdot\frac{\partial t}{\partial t}. \tag{1.38}$$

where $u_t = U\cdot t$ is the projection of the elastic displacement vector U on the direction specified by unit vector t.

We will expand vector U in axes of the main trihedron. Denoting for the sake of brevity

$$m_s = t_1, \quad n_s = t_2, \quad a_s = t_3, \tag{1.39}$$

we have

$$U = \sum_{i=1}^{3} u_{t_i} t_i. \tag{1.40}$$

Using the adopted notation

$$U\cdot\frac{\partial t}{\partial t} = \sum_{i=1}^{3} u_{t_i} t_i\cdot\frac{\partial t}{\partial t} = \sum_{i=1}^{3} u_{t_i}\frac{\partial}{\partial t}(t\cdot t_i) - \sum_{i=1}^{3} u_{t_i} t\cdot\frac{\partial t_i}{\partial t}. \tag{1.41}$$

Considering that the unit vectors of the main trihedron according to (1.18) are independent of $\bar{Z}$, on the basis of (1.15) we have

$$\frac{\partial t_i}{\partial m_s} = 0, \qquad (i = 1, 2, 3). \tag{1.42}$$

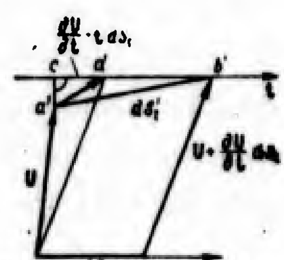

Figure 1.3. Concerning the determination of linear deformation of the middle surface.

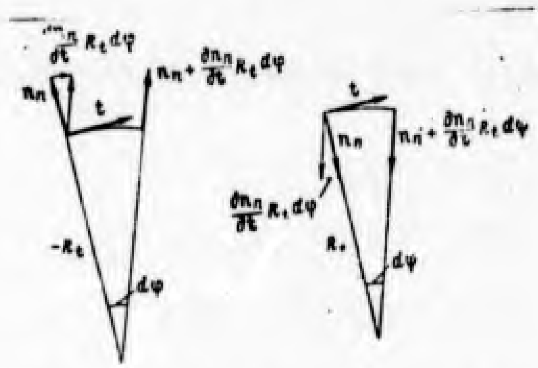

Figure 1.4. In reference to the determination of the radius of curvature of the conical surface.

Therefore, using (1.36), we obtain

$$\frac{\partial t_i}{\partial t_1} = \frac{1}{\sin\chi}(t_1\cdot n_s)\frac{\partial t_i}{\partial m_s} \qquad (i = 1, 2, 3). \tag{1.43}$$

We expand the derivatives $\partial t_i/\partial m_s$ in axes of the main trihedron:

$$\frac{\partial t_i}{\partial m_s} = \sum_{j=1}^{3}\left(t_j\cdot\frac{\partial t_i}{\partial m_s}\right)t_j. \tag{1.44}$$

Representing the unit vectors of the main trihedron in the form

$$t_i = l_{t_i}\mathbf{i} + m_{t_i}\mathbf{j} + n_{t_i}\mathbf{k},$$

we find

$$t_j\cdot\frac{\partial t_i}{\partial m_s} = l_{t_j}\frac{\partial l_{t_i}}{\partial m_s} + m_{t_j}\frac{\partial m_{t_i}}{\partial m_s} + n_{t_j}\frac{\partial n_{t_i}}{\partial m_s} \qquad (i, j = 1, 2, 3).. \tag{1.45}$$

Computing scalar products (1.45) with the aid of formulas (1.18) and the abvious relation

$$t_j\frac{\partial t_i}{\partial m_s} + t_i\frac{\partial t_j}{\partial m_s} = 0 \tag{1.46}$$

then introducing the result into (1.44), one can readily obtain

$$\begin{aligned}
\frac{\partial \mathbf{m}_z}{\partial \mathbf{m}_z} &= -\frac{\sin\chi}{AB}\,\mathbf{n}_z,\\
\frac{\partial \mathbf{n}_z}{\partial \mathbf{m}_z} &= \frac{\sin\chi}{AB}\,\mathbf{m}_z + \frac{1}{AB}\,\frac{l_0}{\sin^2\chi}\,(x_0' y_0'' - x_0'' y_0')\,\mathbf{n}_n,\\
\frac{\partial \mathbf{n}_n}{\partial \mathbf{m}_z} &= -\frac{1}{AB}\,\frac{l_0}{\sin^2\chi}\,(x_0' y_0'' - x_0'' y_0')\,\mathbf{n}_z.
\end{aligned} \qquad (1.47)$$

For an arbitrary normal section of the surface, the following relation is obvious (Figure 1.4):

$$\frac{\partial \mathbf{n}_n}{\partial t} = -\frac{1}{R_t}\,\mathbf{t}, \qquad (1.48)$$

where t is the unit vector of the intersection line, and R_t is the normal radius of its curvature, related to the principal radii of curvature of the surface R_1, R_2 by Euler's formula

$$\frac{1}{R_t} = \frac{\cos^2\alpha}{R_1} + \frac{\sin^2\alpha}{R_2}.$$

Here α is the angle between the arbitrary normal section and the section oriented along the line of principal curvature.

For an arbitrary conical surface, the lines of principal curvature are: the family of generatrices and the family of lines of intersection of this surface by spheres with the center at the cone apex, oriented along unit vectors m_z and n_z, respectively. Since for a conical as well as any linear surface, one of the radii of principal curvature is infinite, then

$$\frac{1}{R_t} = \frac{1}{R}\,(\mathbf{t}\cdot\mathbf{n}_z)^2. \qquad (1.49)$$

From (1.48) and (1.49), superposing the unit vector t on n_z and taking (1.36) and (1.42) into account, we obtain

$$\frac{\partial \mathbf{n}_n}{\partial \mathbf{m}_s} = -\frac{\sin\chi}{R}\mathbf{n}_s, \tag{1.50}$$

whence, using (1.47), we find

$$\frac{1}{R} = \frac{1}{AB}\frac{l_0}{\sin^3\chi}(x_0' y_0'' - x_0'' y_0'). \tag{1.51}$$

On the basis of expression (1.51) for the principal radius of curvature of an arbitrary conical surface, we reduce relations (1.47) to the form

$$\begin{aligned}
&\frac{\partial \mathbf{m}_s}{\partial \mathbf{m}_s} = -\frac{\sin\chi}{AB}\mathbf{n}_s,\\
&\frac{\partial \mathbf{n}_s}{\partial \mathbf{m}_s} = \frac{\sin\chi}{AB}\mathbf{m}_s + \frac{\sin\chi}{R}\mathbf{n}_n,\\
&\frac{\partial \mathbf{n}_n}{\partial \mathbf{m}_s} = -\frac{\sin\chi}{R}\mathbf{n}_s.
\end{aligned} \tag{1.52}$$

Expanding (1.41) with the aid of relations (1.43), (1.52), and introducing the result into (1.38), we finally obtain

$$\begin{aligned}
\varepsilon_t^c = \frac{\partial u_t}{\partial t} - u_{m_s}\left[\frac{\partial}{\partial t}(\mathbf{t}\cdot\mathbf{m}_s) + \frac{1}{AB}(\mathbf{t}\cdot\mathbf{n}_s)^2\right] -\\
- u_{n_s}\left[\frac{\partial}{\partial t}(\mathbf{t}\cdot\mathbf{n}_s) - \frac{1}{AB}(\mathbf{t}\cdot\mathbf{n}_s)(\mathbf{t}\cdot\mathbf{m}_s)\right] - u_{n_n}\frac{1}{R}(\mathbf{t}\cdot\mathbf{n}_s)^2.
\end{aligned} \tag{1.53}$$

In contrast to (1.35), expression (1.53) contains derivatives of scalar functions only. This expression is general in character. Superposing unit vector t on any given direction, one can readily obtain an expression for the corresponding linear strain. For the strain components in axes of the main and auxiliary trihedra, we find from (1.53)

$$\begin{aligned}
&\varepsilon_{m_s}^0 = \frac{\partial u_{m_s}}{\partial \mathbf{m}_s},\\
&\varepsilon_{m_s}^0 = \frac{\partial u_{m_s}}{\partial \mathbf{m}_s} - \frac{1 - \frac{1}{2}(l_s^2)^{\cdot}}{AB\sin\chi}u_{n_s} - \frac{\sin^2\chi}{R}u_{n_n},\\
&\varepsilon_{n_s}^0 = \frac{\partial u_{n_s}}{\partial \mathbf{n}_s} - \frac{1}{AB}u_{m_s} - \frac{1}{R}u_{n_n},\\
&\varepsilon_{n_s}^0 = \frac{\partial u_{n_s}}{\partial \mathbf{n}_0} - \operatorname{ctg}\chi\,\frac{1 - \frac{1}{2}(l_s^2)^{\cdot}}{AB\sin\chi}u_{m_s} - \frac{\cos^2\chi}{R}u_{n_\chi}.
\end{aligned} \tag{1.54}$$

Expanding (1.54) with the aid of (1.15), (1.16) and considering (1.11), we obtain expressions for the strain components in axes of the main and auxiliary trihedra in final form:

$$
\begin{aligned}
&\varepsilon^0_{m_z} = \frac{1}{l_s}\frac{\partial u_{m_z}}{\partial \bar{Z}},\\
&\varepsilon^0_{m_s} = \frac{1}{1-\bar{Z}}\frac{\partial u_{m_s}}{\partial S} - \frac{1-\frac{1}{2}(l_s^2)^{\cdot}}{(1-\bar{Z})\, l_s \sin\chi} u_{n_s} - \frac{\sin^2\chi}{R} u_{n_n},\\
&\varepsilon^0_{n_z} = -\frac{1}{l_s}\operatorname{ctg}\chi \frac{\partial u_{n_z}}{\partial \bar{Z}} + \frac{1}{(1-\bar{Z})\sin\chi}\frac{\partial u_{n_s}}{\partial S} - \frac{1}{(1-\bar{Z})\, l_s} u_{m_z} - \frac{1}{R} u_{n_n},\\
&\varepsilon^0_{n_s} = \frac{1}{l_s \sin\chi}\frac{\partial n_{n_s}}{\partial \bar{Z}} - \frac{1}{1-\bar{Z}}\operatorname{ctg}\chi \frac{\partial u_{n_s}}{\partial S} - \operatorname{ctg}\chi \frac{1-\frac{1}{2}(l_s^2)^{\cdot}}{(1-\bar{Z})\, l_s \sin\chi} u_{m_s} - \frac{\cos^2\chi}{R} u_{n_n}.
\end{aligned}
\tag{1.55}
$$

We will now consider the angle strains of the middle surface of the shell.

Let the following two curves pass through point M of the middle surface:

$$\bar{Z} = \bar{Z}_{\mathrm{I}}(t), \qquad S = S_{\mathrm{I}}(t),$$

and

$$\bar{Z} = \bar{Z}_{\mathrm{II}}(t), \qquad S = S_{\mathrm{II}}(t), \tag{1.56}$$

whose unit vectors are t_{I} and t_{II}. The cosine of the angle between these unit vectors is equal to their scalar product

$$\cos(t_{\mathrm{I}}, t_{\mathrm{II}}) = l_{t_{\mathrm{I}}} l_{t_{\mathrm{II}}} + m_{t_{\mathrm{I}}} m_{t_{\mathrm{II}}} + n_{t_{\mathrm{I}}} n_{t_{\mathrm{II}}}, \tag{1.57}$$

where the direction cosines are defined by the expressions

$$l = \frac{\partial x}{\partial t}, \quad m = \frac{\partial y}{\partial t}, \quad n = \frac{\partial z}{\partial t}. \tag{1.58}$$

The material lines coinciding with (1.56) will occupy a new position after deformation. Let $t^{\mathrm{I}}_{\mathrm{I}}$, $t^{\mathrm{I}}_{\mathrm{II}}$ be the unit vectors of these lines in the new position. Their direction cosines are given by the obvious expressions

$$l_l = \frac{1}{1+\varepsilon_l^0}\,\frac{\partial x^1}{\partial t},$$

$$m_l = \frac{1}{1+\varepsilon_l^0}\,\frac{\partial y^1}{\partial t},$$

$$n_l = \frac{1}{1+\varepsilon_l^0}\,\frac{\partial z^1}{\partial t}, \qquad (1.59)$$

where x^1, y^1, z^1 are the coordinates of point M after deformation, represented by expressions (1.22).

To within higher-order infinitesimals, the cosines of the angle between t_I^I and t_{II}^I, taking (1.22) and (1.57)-(1.59) into account, will be

$$\cos(t_I^I,\ t_{II}^I) = t_I^I \cdot t_{II}^I = (1-\varepsilon_{t_I}^0-\varepsilon_{t_{II}}^0)(t_I\cdot t_{II}) + t_I\cdot\frac{\partial U}{\partial t_{II}} + t_{II}\cdot\frac{\partial U}{\partial t_I}. \qquad (1.60)$$

In view of the smallness of the angle strain

$$\gamma^0_{t_I t_{II}} = (t_I,\ t_{II}) - (t_I^I,\ t_{II}^I)$$

we take

$$\cos(t_I^I,\ t_{II}^I) = \cos(t_I,\ t_{II}) + \gamma^0_{t_I t_{II}} \sin(t_I,\ t_{II}). \qquad (1.61)$$

Now, introducing (1.60) into (1.61), we find

$$\gamma^0_{t_I t_{II}} = \frac{1}{\sin(t_I,\ t_{II})}\left[t_I\cdot\frac{\partial U}{\partial t_{II}} + t_{II}\cdot\frac{\partial U}{\partial t_I}\right] - \mathrm{ctg}(t_I,\ t_{II})\,(\varepsilon_{t_I}^0+\varepsilon_{t_{II}}^0). \qquad (1.62)$$

Expression (1.62) represents the change of the angle between the arbitrary directions on the middle surface. For mutually perpendicular directions, we have

$$\gamma^0_{t_I t_{II}} = t_I\cdot\frac{\partial U}{\partial t_{II}} + t_{II}\cdot\frac{\partial U}{\partial t_I}. \qquad (1.63)$$

Vectorial expression (1.62) must be reduced to the coordinate form. For this purpose, we consider a scalar product of the type

$$\mathbf{a}\cdot\frac{\partial \mathbf{U}}{\partial \mathbf{b}}=\frac{\partial}{\partial \mathbf{b}}(\mathbf{U}\cdot\mathbf{a})-\mathbf{U}\cdot\frac{\partial \mathbf{a}}{\partial \mathbf{b}}. \tag{1.64}$$

Expanding vector U in axes of the main trihedron, we obtain

$$\mathbf{U}\cdot\frac{\partial \mathbf{a}}{\partial \mathbf{b}}=\sum_{i=1}^{3}u_{t_i}\left[\frac{\partial}{\partial \mathbf{b}}(\mathbf{a}\cdot\mathbf{t}_i)-\mathbf{a}\cdot\frac{\partial \mathbf{t}_i}{\partial \mathbf{b}}\right], \tag{1.65}$$

whence, on the basis of (1.43), for any tangential vectors a and b

$$\mathbf{U}\cdot\frac{\partial \mathbf{a}}{\partial \mathbf{b}}=\sum_{i=1}^{3}u_{t_i}\left[\frac{\partial}{\partial \mathbf{b}}(\mathbf{a}\cdot\mathbf{t}_i)-\frac{1}{\sin\chi}(\mathbf{b}\cdot\mathbf{n}_s)\left(\mathbf{a}\cdot\frac{\partial \mathbf{t}_i}{\partial \mathbf{m}_s}\right)\right]. \tag{1.66}$$

Now, using (1.64), (1.66) in a=t_I, b=t_{II} and on the other hand, its not difficult with aid of correlations (1.52) to reduce to

$$\begin{aligned}\gamma^0_{t_I t_{II}}=\frac{1}{\sin(\mathbf{t}_I,\mathbf{t}_{II})}\Bigg\{&\frac{\partial u_{t_I}}{\partial \mathbf{t}_{II}}+\frac{\partial u_{t_{II}}}{\partial \mathbf{t}_I}-u_{m_s}\left[\frac{\partial}{\partial \mathbf{t}_{II}}(\mathbf{t}_I\cdot\mathbf{m}_s)+\frac{\partial}{\partial \mathbf{t}_I}(\mathbf{t}_{II}\cdot\mathbf{m}_s)+\right.\\&\left.+\frac{2}{AB}(\mathbf{t}_I\cdot\mathbf{n}_s)(\mathbf{t}_{II}\cdot\mathbf{n}_s)\right]-u_{n_s}\left[\frac{\partial}{\partial \mathbf{t}_{II}}(\mathbf{t}_I\cdot\mathbf{n}_s)+\frac{\partial}{\partial \mathbf{t}_I}(\mathbf{t}_{II}\cdot\mathbf{n}_s)-\frac{1}{AB}((\mathbf{t}_{II}\cdot\mathbf{n}_s)(\mathbf{t}_I\cdot\mathbf{m}_s)+\right.\\&\left.+(\mathbf{t}_I\cdot\mathbf{n}_s)(\mathbf{t}_{II}\cdot\mathbf{m}_s))\right]-u_{n_n}\frac{2}{R}(\mathbf{t}_I\cdot\mathbf{n}_z)(\mathbf{t}_{II}\cdot\mathbf{n}_z)\Bigg\}-\operatorname{ctg}(\mathbf{t}_I,\mathbf{t}_{II})(\varepsilon^0_{t_I}+\varepsilon^0_{t_{II}}).\end{aligned} \tag{1.67}$$

In contrast to (1.62), expression (1.67) contains derivatives of scalar functions only. This expression is general in character. By superposing unit vectors t_I and t_{II} on any given directions, one can obtain expressions for the corresponding angle strain. For shearing strains in axes of the main and auxiliary trihedra, we find from (1.66)

$$\begin{aligned}\gamma^0_{n_s m_s}&=\frac{\partial u_{m_s}}{\partial \mathbf{n}_s}+\frac{\partial u_{n_s}}{\partial \mathbf{m}_s}+\frac{1}{AB}u_{n_s},\\ \gamma^0_{n_e m_e}&=\frac{\partial u_{m_s}}{\partial \mathbf{n}_e}+\frac{\partial u_{n_e}}{\partial \mathbf{m}_e}+\frac{1-\frac{1}{2}(l_s^2)}{AB\sin\chi}(u_{m_s}+\operatorname{ctg}\chi\, u_{n_s})+\frac{\sin 2\chi}{R}u_{n_n}.\end{aligned} \tag{1.68}$$

Expanding (1.68) with the aid of (1.15), (1.17) and considering (1.11), we obtain expressions for shearing strain in axes of the main and auxiliary trihedra in final form:

$$\gamma^0_{n_s m_s} = -\frac{\operatorname{ctg}\chi}{l_s}\,\frac{\partial u_{m_s}}{\partial \bar{Z}} + \frac{1}{1-\bar{Z}}\,\frac{1}{\sin\chi}\,\frac{\partial u_{m_s}}{\partial S} + \frac{1-\bar{Z}}{l_s}\,\frac{\partial}{\partial \bar{Z}}\left(\frac{u_{n_s}}{1-\bar{Z}}\right),$$
$$\gamma^0_{n_s m_s} = \frac{1}{l_s \sin\chi}\,\frac{\partial u_{m_s}}{\partial \bar{Z}} - \frac{1}{1-\bar{Z}}\operatorname{ctg}\chi\,\frac{\partial u_{m_s}}{\partial S} + \frac{1}{1-\bar{Z}}\,\frac{\partial u_{n_s}}{\partial S} +$$
$$+\frac{1-\frac{1}{2}(l_s^2)^{\cdot}}{l_s(1-\bar{Z})\sin\chi}\left(u_{m_s} + \operatorname{ctg}\chi\, u_{n_s}\right) + \frac{\sin 2\chi}{R}\,u_{n_n}. \tag{1.69}$$

2. Components of Bending Deformation of Middle Surface

Let $M^*(\bar{Z}, S, \gamma)$ be the radius vector of point M^* of an equidistant surface separated from the middle surface of the shell by distance γ measured along the normal n_n. Obviously,

$$M^* = M + \gamma n_n(M), \tag{1.70}$$

where $M(\bar{Z}, S)$ is the radius vector of point M of the middle surface.

We specify some curve $\bar{Z}=\bar{Z}(t)$, $S=S(t)$ on the middle surface. Its radius vector

$$r_t = M(\bar{Z}(t), S(t)). \tag{1.71}$$

Hence,

$$\frac{dr_t}{ds_t} = \frac{\partial M}{\partial t} = t. \tag{1.72}$$

where s_t and t are, respectively, the arc and unit vector of curve (1.71).

To curve (1.71) on the equidistant surface there corresponds the curve

$$r_t^* = M^*(\bar{Z}(t)\, S(t),\, \gamma).$$

On the basis of (1.70), (1.71), we have

$$\mathbf{r}_t^* = \mathbf{r}_t + \gamma \mathbf{n}_n(\mathbf{r}_t). \qquad (1.73)$$

Differentiating (1.73), we obtain

$$\frac{d\mathbf{r}_t^*}{ds_t} = \frac{\partial M^*}{\partial t} = \mathbf{t} + \gamma \frac{\partial \mathbf{n}_n}{\partial t}. \qquad (1.74)$$

The vector directed along the tangent to curve (1.73) is represented by expression (1.74). This vector is collinear with t only when vector $\partial n_n/\partial t$ is collinear with t or equal to zero. From (1.36), taking (1.42), (1.52) into consideration, we have

$$\frac{\partial \mathbf{n}_n}{\partial t} = -\frac{1}{R}(\mathbf{t}\cdot\mathbf{n}_s)\mathbf{n}_s. \qquad (1.75)$$

It follows that vector $\partial M^*/\partial t$ is collinear with t only when unit vector t is oriented along the line of principal curvature.

From (1.74), taking (1.75) into account, we have

$$\frac{\partial M^*}{\partial \mathbf{m}_s} = \mathbf{m}_s; \quad \frac{\partial M^*}{\partial \mathbf{n}_s} = \left(1 - \frac{\gamma}{R}\right)\mathbf{n}_s. \qquad (1.76)$$

Let M^{*1} be the radius vector of point M* after deformation. We have

$$M^{*1} = M^* + U^*(M^*), \qquad (1.77)$$

where $U^*(M^*) = U^*(\overline{Z}, S, \gamma)$ is the elastic displacement vector of point M*, which is a vector function of three variables: $\overline{Z}$, S, γ.

The theory of thin shells is based on the Kirchhoff-Love hypothesis, according to which a linear element of the shell, normal to the unstrained middle surface, is absolutely hard and in the course of deformation coincides with the normal to the strained middle surface. In accordance with this hypothesis,

$$M^{*1}(M, \gamma) = M^1(M) + \gamma n_n^1(M). \tag{1.78}$$

where M^1 and n_n^1 are the radius vector of point M and unit vector of the normal to the middle surface after deformation.

It is evident that the Kirchhoff-Love hypothesis permits one to reduce a three-dimensional problem to a two-dimensional one, since expression (1.78) contains the γ coordinate only in explicit form.

Equating the right-hand sides of (1.77) and (1.78) and taking (1.21) and (1.70) into account, we find

$$U^*(M^*) = U(M) + \gamma [n_n^1(M) - n_n(M)]. \tag{1.79}$$

Expression (1.79) relates the unknown vector function of three variables U^* with two unknown vector functions: U and n_n^1 of two variables, and determines the two-dimensional computational model of a thin shell.

We introduce the notation

$$\Delta n_n = n_n^1 - n_n. \tag{1.80}$$

From (1.78), taking (1.21) and (1.80) into account, we have

$$M^{*1} = M^* + U + \gamma \Delta n_n. \tag{1.81}$$

Expression (1.81) for a fixed value of γ represents the vector equation of the equidistant surface γ = const after deformation. The equation of this surface before deformation is represented by expression (1.70).

We will refer to the spatial material lines coinciding with curves

(1.73) before deformation as t lines. After deformation, the t lines lie on surface (1.81) and are determined by the equation

$$\mathbf{r}_t^{*1} \quad \mathbf{r}_t^*(\mathbf{r}_t)+\mathbf{U}(\mathbf{r}_t)+\gamma\Delta\mathbf{n}_n(\mathbf{r}_t). \tag{1.82}$$

Differentiating (1.82), we obtain

$$\frac{d\mathbf{r}_t^{*1}}{ds_t}=\frac{\partial\mathbf{M}^{*1}}{\partial t}=\frac{\partial\mathbf{M}^*}{\partial t}+\frac{\partial\mathbf{U}}{\partial t}+\gamma\frac{\partial\Delta\mathbf{n}_n}{\partial t}. \tag{1.83}$$

Expression (1.83) defines the vector directed along the tangent to the deformed t line.

Let ds_t^* be an elementary segment of arc of the t line before deformation. Considering (1.74),

$$ds_t^*=\left|\frac{\partial\mathbf{M}^*}{\partial t}\right|ds_t. \tag{1.84}$$

After deformation, the material segment ds_t^* will have a new length $(ds_t^*)^1$. Considering (1.83),

$$(ds_t^*)^1=\left|\frac{\partial\mathbf{M}^{*1}}{\partial t}\right|ds_t. \tag{1.85}$$

On the basis of (1.84), (1.85), the strain of an element of the t line

$$\varepsilon_t^*=\frac{(ds_t^*)^1-ds_t^*}{ds_t^*}$$

will be

$$\varepsilon_t^*=\frac{\left|\dfrac{\partial\mathbf{M}^{*1}}{\partial t}\right|}{\left|\dfrac{\partial\mathbf{M}^*}{\partial t}\right|}-1. \tag{1.86}$$

Multiplying expression (1.83) scalarly by itself and neglecting the squares and derivatives of small quantities, we find

$$\left|\frac{\partial \mathbf{M}^{*\prime}}{\partial t}\right| = \left|\frac{\partial \mathbf{M}^{*}}{\partial t}\right| + \frac{\frac{\partial \mathbf{M}^{*}}{\partial t}\cdot\left(\frac{\partial \mathbf{U}}{\partial t} + \gamma\frac{\partial \Delta \mathbf{n}_n}{\partial t}\right)}{\left|\frac{\partial \mathbf{M}^{*}}{\partial t}\right|}. \tag{1.87}$$

Introducing (1.87) into (1.86), we obtain

$$\varepsilon_t^{*} = \frac{\frac{\partial \mathbf{M}^{*}}{\partial t}\cdot\left(\frac{\partial \mathbf{U}}{\partial t} + \gamma\frac{\partial \Delta \mathbf{n}_n}{\partial t}\right)}{\left|\frac{\partial \mathbf{M}^{*}}{\partial t}\right|^2}. \tag{1.88}$$

Expression (1.88) is general in character. Superposing unit vector t on any direction in the middle surface, one can readily obtain expressions for the linear strain of the equidistant surface in the direction of vector (1.74). As was shown above, this vector is collinear with t if unit vector t is oriented along the line of principal curvature. In this case, from (1.88) and taking (1.76) into account, we find

$$\begin{aligned} \varepsilon_{m_t}^{*} &= \frac{\partial \mathbf{U}}{\partial \mathbf{m}_t}\cdot\mathbf{m}_t + \gamma\frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{m}_t}\cdot\mathbf{m}_t, \\ \varepsilon_{n_t}^{*} &= \frac{1}{1-\frac{\gamma}{R}}\left[\frac{\partial \mathbf{U}}{\partial \mathbf{n}_t}\cdot\mathbf{n}_t + \gamma\frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_t}\cdot\mathbf{n}_t\right]. \end{aligned} \tag{1.89}$$

We will now consider the angle strains in the equidistant surface. Let the following two curves pass through point M of the middle surface:

$$\mathbf{r}_{t_\mathrm{I}} = \mathbf{r}_{t_\mathrm{I}}(t), \quad \mathbf{r}_{t_\mathrm{II}} = \mathbf{r}_{t_\mathrm{II}}(t). \tag{1.90}$$

The following curves correspond to these curves on the equidistant surface γ = const:

$$\begin{aligned} \mathbf{r}_{t_\mathrm{I}}^{*} &= \mathbf{r}_{t_\mathrm{I}} + \gamma\mathbf{n}_n(\mathbf{r}_{t_\mathrm{I}}), \\ \mathbf{r}_{t_\mathrm{II}}^{*} &= \mathbf{r}_{t_\mathrm{II}} + \gamma\mathbf{n}_n(\mathbf{r}_{t_\mathrm{II}}). \end{aligned} \tag{1.91}$$

Differentiating (1.90), we have

$$\frac{d\mathbf{r}_{t_\mathrm{I}}}{ds_{t_\mathrm{I}}} = \frac{\partial \mathbf{M}}{\partial \mathbf{t}_\mathrm{I}} = \mathbf{t}_\mathrm{I}, \quad \frac{d\mathbf{r}_{t_\mathrm{II}}}{ds_{t_\mathrm{II}}} = \frac{\partial \mathbf{M}}{\partial \mathbf{t}_\mathrm{II}} = \mathbf{t}_\mathrm{II}. \tag{1.92}$$

where s_{t_I}, $s_{t_{II}}$ and t_I, t_{II} are the arcs and unit vectors of curves (1.91), respectively.

Differentiating (1.91) and taking (1.70), (1.92) into account, we find

$$
\begin{aligned}
&\frac{d\mathring{r}_{t_I}}{ds_{t_I}} = \frac{\partial M^\circ}{\partial t_I} = t_I + \gamma\frac{\partial n_n}{\partial t_I},\\
&\frac{d\mathring{r}_{t_{II}}}{ds_{t_{II}}} = \frac{\partial M^\circ}{\partial t_{II}} = t_{II} + \gamma\frac{\partial n_n}{\partial t_{II}}.
\end{aligned} \tag{1.93}
$$

Expression (1.93) represents the vectors directed along tangents to undeformed t lines. The corresponding unit vectors will be

$$
t_I^\circ = \frac{\dfrac{\partial M^\circ}{\partial t_I}}{\left|\dfrac{\partial M^\circ}{\partial t_I}\right|}, \quad t_{II}^\circ = \frac{\dfrac{\partial M^\circ}{\partial t_{II}}}{\left|\dfrac{\partial M^\circ}{\partial t_{II}}\right|}. \tag{1.94}
$$

After deformation, the t lines lie on surface (1.81). Differentiating (1.81), we have

$$
\begin{aligned}
&\frac{d\mathring{r}'_{t_I}}{ds_{t_I}} = \frac{\partial M^{\circ\prime}}{\partial t_I} = \frac{\partial M^\circ}{\partial t_I} + \frac{\partial U}{\partial t_I} + \gamma\frac{\partial \Delta n_n}{\partial t_I},\\
&\frac{d\mathring{r}'_{t_{II}}}{ds_{t_{II}}} = \frac{\partial M^{\circ\prime}}{\partial t_{II}} = \frac{\partial M^\circ}{\partial t_{II}} + \frac{\partial U}{\partial t_{II}} + \gamma\frac{\partial \Delta n_n}{\partial t_{II}}.
\end{aligned} \tag{1.95}
$$

Expressions (1.95) determine the vectors oriented along tangents to deformed t lines. The corresponding unit vectors

$$
t_I^{\circ\prime} = \frac{\dfrac{\partial M^{\circ\prime}}{\partial t_I}}{\left|\dfrac{\partial M^{\circ\prime}}{\partial t_I}\right|}, \quad t_{II}^{\circ\prime} = \frac{\dfrac{\partial M^{\circ\prime}}{\partial t_{II}}}{\left|\dfrac{\partial M^{\circ\prime}}{\partial t_{II}}\right|}. \tag{1.96}
$$

The cosine of the angles between the t lines before and after deformation are defined by scalar products of the corresponding unit vectors:

$$\cos(t_I^\circ, t_{II}^\circ) = \frac{\frac{\partial M^\circ}{\partial t_I} \cdot \frac{\partial M^\circ}{\partial t_{II}}}{\left|\frac{\partial M^\circ}{\partial t_I}\right| \left|\frac{\partial M^\circ}{\partial t_{II}}\right|}, \quad \cos(t_I^{\circ 1}, t_{II}^{\circ 1}) = \frac{\frac{\partial M^{\circ 1}}{\partial t_I} \cdot \frac{\partial M^{\circ 1}}{\partial t_{II}}}{\left|\frac{\partial M^{\circ 1}}{\partial t_I}\right| \left|\frac{\partial M^{\circ 1}}{\partial t_{II}}\right|}. \tag{1.97}$$

For small angle strains

$$\gamma^\circ_{t_I t_{II}} = (t_I^\circ, t_{II}^\circ) - (t_I^{\circ 1}, t_{II}^{\circ 1})$$

we will take

$$\sin \gamma^\circ_{t_I t_{II}} = \gamma^\circ_{t_I t_{II}}, \quad \cos \gamma^\circ_{t_I t_{II}} = 1. \tag{1.98}$$

Then

$$\gamma^\circ_{t_I t_{II}} = \frac{\cos(t_I^{\circ 1}, t_{II}^{\circ 1}) - \cos(t_I^\circ, t_{II}^\circ)}{\sin(t_I^\circ, t_{II}^\circ)}. \tag{1.99}$$

Now, expanding (1.97) with the aid of (1.93) and (1.95), and introducing the result into (1.99), we can readily find the expression for angle strain $\gamma^\circ_{t_I, t_{II}}$.

We superpose t_I on m_z, and t_{II} on n_z. We then have from (1.76)

$$t_I^\circ = m_z, \quad t_{II}^\circ = n_z. \tag{1.100}$$

On the basis of (1.100), it follows from (1.99) that

$$\gamma^\circ_{n_z m_z} = \cos(n_z^1, m_z^1), \tag{1.101}$$

where n_z^1, m_z^1 are the unit vectors of the n_z and m_z lines after deformation.

Let us expand expression (1.101). From (1.95), taking (1.76) into consideration, we have

$$\frac{\partial M^{*1}}{\partial m_z} = m_z + \frac{\partial U}{\partial m_z} + \gamma \frac{\partial \Delta n_n}{\partial m_z},$$
$$\frac{\partial M^{*1}}{\partial n_z} = \left(1 - \frac{\gamma}{R}\right) n_z + \frac{\partial U}{\partial n_z} + \gamma \frac{\partial \Delta n_n}{\partial n_z}. \tag{1.102}$$

Multiplying expressions (1.102) scalarly by themselves and neglecting the squares and derivatives of small quantities, we obtain

$$\left|\frac{\partial M^{*1}}{\partial m_z}\right| = 1 + \frac{\partial U}{\partial m_z} \cdot m_z + \gamma \frac{\partial \Delta n_n}{\partial m_z} \cdot m_z,$$
$$\left|\frac{\partial M^{*1}}{\partial n_z}\right| = 1 - \frac{\gamma}{R} + \frac{\partial U}{\partial n_z} \cdot n_z + \gamma \frac{\partial \Delta n_n}{\partial n_z} \cdot n_z. \tag{1.103}$$

Now, expanding (1.97) with the aid of (1.102) and (1.103), on the basis of (1.101) to within second-order small terms, we find

$$\gamma^{*}_{n_z m_z} = \frac{1}{1 - \frac{\gamma}{R}} \left[\frac{\partial U}{\partial n_z} \cdot m_z + \frac{\partial U}{\partial m_z} \cdot n_z + \right.$$
$$\left. + \gamma \left(\frac{\partial \Delta n_n}{\partial n_z} \cdot m_z + \frac{\partial \Delta n_n}{\partial m_z} \cdot n_z - \frac{1}{R} \frac{\partial U}{\partial m_z} \cdot n_z\right)\right]. \tag{1.104}$$

Thus, formulas (1.89) and (1.104) represent linear and angle strains of the equidistant surface in axes of the main trihedron. When $\gamma = 0$, these expressions coincide with the corresponding expressions for strain of the middle surface.

For the shells under consideration $\gamma/R \ll 1$, and therefore, setting

$$1 \pm \frac{\gamma}{R} \approx 1, \tag{1.105}$$

we will hereinafter proceed from the expressions

$$\varepsilon^{*}_{m_z} = \varepsilon^{0}_{m_z} + \gamma \frac{\partial \Delta n_n}{\partial m_z} \cdot m_z; \quad \varepsilon^{*}_{n_z} = \varepsilon^{0}_{n_z} + \gamma \frac{\partial \Delta n_n}{\partial n_z} \cdot n_z, \tag{1.106}$$
$$\gamma^{*}_{n_z m_z} = \gamma^{0}_{n_z m_z} + \gamma \left(\frac{\partial \Delta n_n}{\partial n_z} \cdot m_z + \frac{\partial \Delta n_n}{\partial m_z} \cdot n_z - \frac{1}{R} \frac{\partial U}{\partial m_z} \cdot n_z\right). \tag{1.107}$$

where $e^0_{m_z}$, e_{n_z}, $\gamma_{n_z m_z}$ are the strains of the middle surface, determined by expressions (1.55) and (1.69).

In order to represent (1.106) and (1.107) in coordinate form, we first express the auxiliary vector $\Delta n_n = n^1_n - n_n$ in terms of vector U of displacement of the middle surface.

Unit vector n^1_n will be represented in terms of the vector product of tangential unit vectors

$$n^1_n = \frac{m^1_z \times n^1_z}{\sin(m^1_z, n^1_z)}. \tag{1.108}$$

From (1.102) when $\gamma=0$, we have

$$m^1_z = \frac{m_z + \frac{\partial U}{\partial m_z}}{\left| m_z + \frac{\partial U}{\partial m_z} \right|}, \quad n^1_z = \frac{n_z + \frac{\partial U}{\partial n_z}}{\left| n_z + \frac{\partial U}{\partial n_z} \right|}. \tag{1.109}$$

For small strains

$$\sin(m^1_z, n^1_z) = \sin\left(\frac{\pi}{2} - \gamma_{n_z m_z}\right) \approx 1. \tag{1.110}$$

Introducing (1.109) into (1.108) and considering (1.110), to within the products of small quantities, it is easy to obtain

$$\Delta n_n = \frac{\partial U}{\partial m_z} \times n_z - \frac{\partial U}{\partial n_z} \times m_z - n_n \left(\frac{\partial U}{\partial m_z} \cdot m_z + \frac{\partial U}{\partial n_z} \cdot n_z \right). \tag{1.111}$$

Representing unit vectors n_z and m_z in the form

$$n_z = n_n \times m_z, \quad m_z = n_z \times n_n$$

and then using the rule of calculation of the double vector product

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b),$$

we find from (1.111)

$$\Delta \mathbf{n}_n = -\mathbf{m}_z\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\cdot \mathbf{n}_n\right) - \mathbf{n}_z\left(\frac{\partial \mathbf{U}}{\partial \mathbf{n}_z}\cdot \mathbf{n}_n\right). \tag{1.112}$$

Now, using formula (1.112), we can easily write strains (1.106), (1.107) in coordinate form.

We have:

$$\begin{gathered}\frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{m}_z}\cdot \mathbf{m}_z = \frac{\partial}{\partial \mathbf{m}_z}(\Delta \mathbf{n}_n \cdot \mathbf{m}_z) - \Delta \mathbf{n}_n \cdot \frac{\partial \mathbf{m}_z}{\partial \mathbf{m}_z},\\ \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_z}\cdot \mathbf{n}_z = \frac{\partial}{\partial \mathbf{n}_z}(\Delta \mathbf{n}_n \cdot \mathbf{n}_z) - \Delta \mathbf{n}_n \frac{\partial \mathbf{n}_z}{\partial \mathbf{n}_z},\\ \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_z}\cdot \mathbf{m}_z + \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{m}_z}\cdot \mathbf{n}_z = \frac{\partial}{\partial \mathbf{n}_z}(\Delta \mathbf{n}_n \cdot \mathbf{m}_z) +\\ + \frac{\partial}{\partial \mathbf{m}_z}(\Delta \mathbf{n}_n \cdot \mathbf{n}_z) - \Delta \mathbf{n}_n \cdot \frac{\partial \mathbf{m}_z}{\partial \mathbf{n}_z} - \Delta \mathbf{n}_n \cdot \frac{\partial \mathbf{n}_z}{\partial \mathbf{m}_z}.\end{gathered} \tag{1.113}$$

Using (1.42), (1.52) and (1.112), and also (1.15), (1.16), we obtain from (1.113)

$$\begin{gathered}\frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{m}_z}\cdot \mathbf{m}_z = -\frac{\partial^2}{\partial \mathbf{m}_z^2}(\mathbf{U}\cdot \mathbf{n}_n).\\ \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_z}\cdot \mathbf{n}_z = -\frac{\partial^2}{\partial \mathbf{n}_z^2}(\mathbf{U}\cdot \mathbf{n}_n) + \frac{\partial}{\partial \mathbf{n}_z}\left(\mathbf{U}\cdot \frac{\partial \mathbf{n}_n}{\partial \mathbf{n}_z}\right) + \frac{1}{AB}\,\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot \mathbf{n}_n).\end{gathered} \tag{1.114}$$

$$\begin{gathered}\frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_z}\cdot \mathbf{m}_z + \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{m}_z}\cdot \mathbf{n}_z = -\frac{\partial^2}{\partial \mathbf{n}_z \partial \mathbf{m}_z}(\mathbf{U}\cdot \mathbf{n}_n) - \frac{\partial^2}{\partial \mathbf{m}_z \partial \mathbf{n}_z}(\mathbf{U}\cdot \mathbf{n}_n) +\\ + \frac{\partial}{\partial \mathbf{m}_z}\left(\mathbf{U}\cdot \frac{\partial \mathbf{n}_n}{\partial \mathbf{n}_z}\right) - \frac{1}{AB}\left[\frac{\partial}{\partial \mathbf{n}_z}(\mathbf{U}\cdot \mathbf{n}_n) - \mathbf{U}\cdot \frac{\partial \mathbf{n}_n}{\partial \mathbf{n}_z}\right].\end{gathered} \tag{1.115}$$

The mixed derivatives in the direction of unit vectors of the main trihedron depend on the order of differentiation. Using (1.15), (1.16), one can establish the following relationship:

$$\frac{\partial^2}{\partial \mathbf{m}_z \partial \mathbf{n}_z} = \frac{\partial^2}{\partial \mathbf{n}_z \partial \mathbf{m}_z} + \frac{1}{AB}\cdot \frac{\partial}{\partial \mathbf{n}_z}. \tag{1.116}$$

Transforming (1.115) with the aid of (1.52) and (1.116), we readily obtain

$$\begin{gathered}\frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_z}\cdot \mathbf{n}_z = -\frac{\partial^2}{\partial \mathbf{n}_z^2}(\mathbf{U}\cdot \mathbf{n}_n) - \frac{\partial}{\partial \mathbf{n}_z}\left(\frac{1}{R}\,\mathbf{U}\cdot \mathbf{n}_z\right) + \frac{1}{AB}\,\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot \mathbf{n}_n),\\ \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{n}_z}\cdot \mathbf{m}_z + \frac{\partial \Delta \mathbf{n}_n}{\partial \mathbf{m}_z}\cdot \mathbf{n}_z = -2\left[\frac{\partial^2}{\partial \mathbf{m}_z \partial \mathbf{n}_z}(\mathbf{U}\cdot \mathbf{n}_n) + \frac{\partial}{\partial \mathbf{m}_z}\left(\frac{1}{R}\,\mathbf{U}\cdot \mathbf{n}_z\right)\right] +\\ + \frac{1}{R}\,\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot \mathbf{n}_z).\end{gathered}$$

Introducing these formulas and formula (1.114) into (1.106), (1.107) and considering (1.42), we obtain the scalar expressions:

$$
\begin{aligned}
&\varepsilon^{*}_{m_z} = \varepsilon^{0}_{m_z} - \gamma \frac{\partial^2 u_{n_n}}{\partial m_z^2},\\
&\varepsilon^{*}_{n_z} = \varepsilon^{0}_{n_z} - \gamma \left[\frac{\partial^2 u_{n_n}}{\partial n_z^2} + \frac{\partial}{\partial n_z}\left(\frac{u_{n_z}}{R}\right) - \frac{1}{AB}\frac{\partial u_{n_n}}{\partial m_z} \right],\\
&\gamma^{*}_{n_z m_z} = \gamma^{0}_{n_z m_z} - 2\gamma \left[\frac{\partial^2 u_{n_n}}{\partial m_z \partial n_z} + \frac{\partial}{\partial m_z}\left(\frac{u_{n_z}}{R}\right) \right].
\end{aligned}
\tag{1.117}
$$

Introducing the notation

$$
\begin{aligned}
&\frac{\partial^2 u_{n_n}}{\partial m_z^2} = -\varkappa_{m_z},\\
&\frac{\partial^2 u_{n_n}}{\partial n_z^2} + \frac{\partial}{\partial n_z}\left(\frac{u_{n_z}}{R}\right) - \frac{1}{AB}\frac{\partial u_{n_n}}{\partial m_z} = -\varkappa_{n_z},\\
&\frac{\partial^2 u_{n_n}}{\partial m_z \partial n_z} + \frac{\partial}{\partial m_z}\left(\frac{u_{n_z}}{R}\right) = -\varkappa_{n_z m_z},
\end{aligned}
\tag{1.118}
$$

we rewrite (1.117) in the form:

$$
\varepsilon^{*}_{m_z} = \varepsilon^{0}_{m_z} + \gamma\varkappa_{m_z};\quad \varepsilon^{*}_{n_z} = \varepsilon^{0}_{n_z} + \gamma\varkappa_{n_z},\quad \gamma^{*}_{n_z m_z} = \gamma^{0}_{n_z m_z} + 2\gamma\varkappa_{n_z m_z}.
\tag{1.119}
$$

The quantities $\varkappa_{m_z}$ and $\varkappa_{n_z}$ represent the components of bending strain of the middle surface; $\varkappa_{n_z m_z}$ determines the torsion of the middle surface. Expanding (1.118) with the aid of (1.15), (1.16) and considering (1.11), we write the expressions for $\varkappa_{m_z}$, $\varkappa_{n_z}$, $\varkappa_{n_z m_z}$ in the final form

$$
\begin{aligned}
&\varkappa_{m_z} = -\frac{1}{l_z^2}\frac{\partial^2 u_{n_n}}{\partial Z^2},\\
&\varkappa_{n_z} = -\left[\frac{\operatorname{ctg}\chi}{l_z}\frac{\partial}{\partial Z} - \frac{1}{(1-Z)\sin\chi}\frac{\partial}{\partial S}\right]\left[\frac{\operatorname{ctg}\chi}{l_z}\frac{\partial u_{n_n}}{\partial Z} - \frac{1}{(1-Z)\sin\chi}\frac{\partial u_{n_n}}{\partial S}\right] +\\
&\quad + \frac{1}{l_z^2(1-Z)}\frac{\partial u_{n_n}}{\partial Z} + \frac{\operatorname{ctg}\chi}{l_z R_0}\frac{\partial}{\partial Z}\left(\frac{u_{n_z}}{1-Z}\right) -\\
&\quad - \frac{1}{(1-Z)^2\sin\chi}\frac{\partial}{\partial S}\left(\frac{u_{n_z}}{R_0}\right),\\
&\varkappa_{n_z m_z} = \frac{\operatorname{ctg}\chi}{l_z^2}\frac{\partial^2 u_{n_n}}{\partial Z^2} - \frac{1}{l_z\sin\chi}\frac{\partial}{\partial Z}\left(\frac{1}{1-Z}\frac{\partial u_{n_n}}{\partial S}\right) - \frac{1}{l_z R_0}\frac{\partial}{\partial Z}\left(\frac{u_{n_z}}{1-Z}\right),
\end{aligned}
\tag{1.120}
$$

where

$$
R_0 = \frac{l_z}{l_0}\,\frac{\sin^3\chi}{x_0' y_0'' - x_0'' y_0'}
\tag{1.121}
$$

is the principal radius of curvature of the directrix of the middle surface.

1.3. Elasticity Relations

As follows from (1.119), the Kirchhoff-Love hypothesis, which forms the basis of the theory of thin shells, makes it possible to express the components of tangential strain of the equidistant surface γ = const in terms of six components of strain of the middle surface:

$$\varepsilon^0_{m_s},\ \varepsilon^0_{n_s},\ \gamma^0_{n_s m_s}\ \text{и}\ \varkappa_{m_s},\ \varkappa_{n_s},\ \varkappa_{n_s m_s}.$$

Accordingly, the distribution of stresses over the shell thickness is determined by linear forces and moments normalized to the middle surface. For normal sections oriented along the lines of principal curvature, we have:

$$T_{m_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{m_s}\left(1-\frac{\gamma}{R_0}\right)d\gamma \qquad T_{n_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{n_s}\,d\gamma,$$

$$S_{m_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{n_s m_s}\left(1-\frac{\gamma}{R_0}\right)d\gamma \qquad S_{n_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{m_s n_s}\,d\gamma, \tag{1.122}$$

$$Q_{m_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{n_n m_s}\left(1-\frac{\gamma}{R_0}\right)d\gamma \qquad Q_{n_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{n_n n_s}\,d\gamma, \tag{1.123}$$

$$M_{m_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{m_s}\gamma\left(1-\frac{\gamma}{R_0}\right)d\gamma \qquad M_{n_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{n_s}\gamma\,d\gamma,$$

$$H_{m_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{n_s m_s}\gamma\left(1-\frac{\gamma}{R_0}\right)d\gamma \qquad H_{n_s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{m_s n_s}\gamma\,d\gamma, \tag{1.124}$$

where the components of the stresses are determined by the generalized Hooke's Law. For an isotropic three-dimensional medium, we have

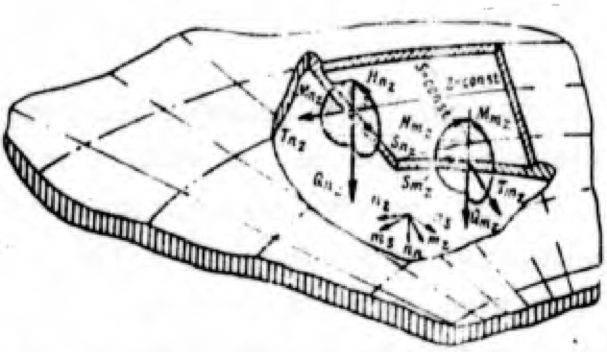

Figure 1.5. Linear forces and moments in the principal normal sections of the shell.

$$\varepsilon^{\circ}_{m_z} = \frac{1}{E}\left[\sigma_{m_z} - \nu(\sigma_{n_z} + \sigma_{n_n})\right], \quad (1.125)$$

$$\varepsilon^{\circ}_{n_z} = \frac{1}{E}\left[\sigma_{n_z} - \nu(\sigma_{n_n} + \sigma_{m_z})\right], \quad (1.126)$$

$$\varepsilon^{\circ}_{n_n} = \frac{1}{E}\left[\sigma_{n_n} - \nu(\sigma_{m_z} + \sigma_{n_z})\right], \quad (1.127)$$

$$\gamma^{\circ}_{n_z m_z} = \frac{1}{G}\tau_{n_z m_z}, \quad (1.128)$$

$$\gamma^{\circ}_{n_n m_z} = \frac{1}{G}\tau_{n_n m_z}, \quad (1.129)$$

$$\gamma^{\circ}_{n_n n_z} = \frac{1}{G}\tau_{n_n n_z}. \quad (1.130)$$

As we know, relations (1.127) and (1.129), (1.130) contradict the Kirchhoff-Love hypothesis. Indeed, according to this hypothesis, the elongation strains of a normal element and transverse shear strains are

$$\varepsilon^{\circ}_{n_n} = \gamma^{\circ}_{n_n m_z} = \gamma^{\circ}_{n_n n_z} = 0. \quad (1.131)$$

On the other hand, it follows from the conditions of equilibrium of the shell element that the corresponding tangential stresses $t_{n_n m_z}$, $t_{n_n n_z}$ are different from zero, and the normal σ_{n_n} are very small, so that the relation

$$\sigma_{n_n} - \nu(\sigma_{m_z} + \sigma_{n_z}) = 0,$$

formally resulting from (1.127) turns out to be incorrect. Contradictions of this type are characteristic of approximate theories based on a priori hypotheses. In the theory of thin shells, the Kirchhoff-Love hypothesis is such a hypothesis. However, this contradiction can be avoided by treating relations (1.131) not as a geometrical hypothesis, but as a consequence of the elastic properties of some fictitious transversally isotropic material identical to the real material of the shell in elastic properties in tangential directions, but differing from it in elastic properties in the direction of the normal. For materials of this type, the generalized Hooke's law is represented by relations of the type

$$\varepsilon^*_{m_z} = \frac{1}{E_1}(\sigma_{m_z} - \nu_1 \sigma_{n_z}) - \frac{\nu_2}{E_2}\sigma_{n_n}, \tag{1.132}$$

$$\varepsilon^*_{n_z} = \frac{1}{E_1}(\sigma_{n_z} - \nu_1 \sigma_{m_z}) - \frac{\nu_2}{E_2}\sigma_{n_n}, \tag{1.133}$$

$$\varepsilon^*_{n_n} = -\frac{\nu_2}{E_2}(\sigma_{m_z} + \sigma_{n_z}) + \frac{1}{E_2}\sigma_{n_n}, \tag{1.134}$$

$$\gamma^*_{n_z m_z} = \frac{1}{G_1}\tau_{n_z m_z}, \tag{1.135}$$

$$\gamma^*_{n_n m_z} = \frac{1}{G_2}\tau_{n_n m_z}, \tag{1.136}$$

$$\gamma^*_{n_n n_z} = \frac{1}{G_2}\tau_{n_n n_z}. \tag{1.137}$$

Now, if the fictitious material is assumed to be incompressible in the direction of the normal and absolutely hard with respect to transverse shear strain, it is necessary in (1.132)-(1.137) to set

$$E_2 = G_2 = \infty, \quad E_1 = E, \; G_1 = G, \; \nu_1 = \nu, \tag{1.138}$$

where E, G, v are elastic constants of the real material.

In this case, it follows from (1.134), (1.136), (1.137) that hypothetical equalities (1.131) are fulfilled exactly. Solving (1.132)-(1.137) for the components of stresses and taking (1.131), (1.138) into account, we have

$$\sigma_{m_z} = \frac{E}{1-\nu^2}(\varepsilon^*_{m_z} + \nu\varepsilon^*_{n_z}), \tag{1.139}$$

$$\sigma_{n_z} = \frac{E}{1-\nu^2}(\varepsilon^*_{n_z} + \nu\varepsilon^*_{m_z}). \tag{1.140}$$

$$\sigma_{n_n} = \infty \cdot 0 + \frac{\nu_2 E}{1-\nu}(\varepsilon_{m_z} + \varepsilon_{n_z}), \tag{1.141}$$

$$\tau_{n_z m_z} = G\gamma^0_{n_z m_z}; \tag{1.142}$$

$$\tau_{n_n m_z} = \infty \cdot 0, \tag{1.143}$$

$$\tau_{n_n n_z} = \infty \cdot 0. \tag{1.144}$$

Relations (1.141), (1.143), (1.144) now contain indeterminacies which are revealed on the basis of the equilibrium conditions. Thus, the components σ_{n_n}, $\tau_{n_n m_z}$, $\tau_{n_n n_z}$ are determined uniquely and do not contradict the elasticity relations.

Expanding relations (1.139), (1.140) and (1.142) with the aid of expressions (1.119), we obtain

$$\begin{aligned}
\sigma_{m_z} &= \frac{E}{1-\nu^2}\left[\varepsilon^0_{m_z} + \nu\varepsilon^0_{n_z} + \gamma(\varkappa_{m_z} + \nu\varkappa_{n_z}\right], \\
\sigma_{n_z} &= \frac{E}{1-\nu^2}\left[\varepsilon^0_{n_z} + \nu\varepsilon^0_{m_z} + \gamma(\varkappa_{n_z} + \nu\varkappa_{m_z})\right], \\
\tau_{n_z m_z} &= G(\gamma^0_{n_z m_z} + 2\gamma\varkappa_{n_z m_z}).
\end{aligned} \tag{1.145}$$

Assuming as above that $1-v/R \approx 1$ and considering the relation

$$G = \frac{E}{2(1+\nu)},$$

from (1.122), (1.124) with the aid of (1.145) we find

$$\begin{aligned}
T_{m_z} &= \frac{Eh}{1-\nu^2}(\varepsilon^0_{m_z} + \nu\varepsilon^0_{n_z}), \quad T_{n_z} = \frac{Eh}{1-\nu^2}(\varepsilon^0_{n_z} + \nu\varepsilon^0_{m_z}), \\
S_{m_z} &= S_{n_z} = S_{n_z m_z} = \frac{Eh}{2(1+\nu)}\gamma^0_{n_z m_z}, \\
M_{m_z} &= D(\varkappa_{m_z} + \nu\varkappa_{n_z}), \quad M_{n_z} = D(\varkappa_{n_z} + \nu\varkappa_{m_z}), \\
H_{m_z} &= H_{n_z} = H_{n_z m_z} = (1-\nu)D\varkappa_{n_z m_z},
\end{aligned} \tag{1.146}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{1.147}$$

Expressions (1.146) represent the elasticity relations for the components of linear forces and moments in areas of normal sections of the shell, oriented along the lines of principal curvatures of the middle surface. We will also be interested in the forces and moments in the areas of normal sections of the shell oriented perpendicular to the tangential unit vectors of the auxiliary trihedron. From the equilibrium conditions of elements of the middle surface (Figure 1.6) it is easy to obtain

$$
\begin{aligned}
&T_{n_s}=T_{m_z}\sin^2\chi+T_{n_z}\cos^2\chi-(S_{m_z}+S_{n_z})\sin\chi\cos\chi,\\
&T_{m_s}=T_{m_z}\cos^2\chi+T_{n_z}\sin^2\chi+(S_{m_z}+S_{n_z})\sin\chi\cos\chi,\\
&S_{n_s}=S_{m_s}=(T_{m_z}-T_{n_z})\sin\chi\cos\chi+S_{m_z}\sin^2\chi-S_{n_z}\cos^2\chi,\\
&M_{n_s}=M_{m_z}\sin^2\chi+M_{n_z}\cos^2\chi-H\sin 2\chi,\\
&M_{m_s}=M_{m_z}\cos^2\chi+M_{n_z}\sin^2\chi+H\sin 2\chi,\\
&H_{n_s}=H_{m_s}=(M_{m_z}-M_{n_z})\sin\chi\cos\chi-H\cos 2\chi,
\end{aligned}
\tag{1.148}
$$

and also

$$
\begin{aligned}
&Q_{m_s}=Q_{m_z}\cos\chi+Q_{n_z}\sin\chi,\\
&Q_{n_s}=Q_{m_z}\sin\chi-Q_{n_z}\cos\chi.
\end{aligned}
\tag{1.149}
$$

1.4. Differential Equations of Equilibrium

We will isolate an element of the shell by using the sections $\bar{Z}$ = const, S = const and $\bar{Z} + d\bar{Z}$ = const, S + dS = const (Figure 1.7). Let $\mathfrak{R}$ be the vector of the resultant and $\mathfrak{M}$, the vector of the moment with respect to point M of all the forces applied to the element. To within higher-order infinitesimals, vector $\mathfrak{R}$ may be represented in the form

$$
\begin{aligned}
\mathfrak{R}=\Big\{&\frac{\partial}{\partial S}[A(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})]+\\
&+\frac{\partial}{\partial \bar{Z}}[B(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})]+AB\sin\chi\,\mathbf{p}\Big\}\,d\bar{Z}\,dS,
\end{aligned}
\tag{1.150}
$$

where p is the vector of surface load intensity referred to the area of the middle surface, and T, S, Q are vectors of the linear forces applied to the contour of the isolated element:

$$
\begin{aligned}
&\mathbf{T}_{n_z}=T_{n_z}\mathbf{n}_z, \quad \mathbf{S}_{n_z}=S_{n_z}\mathbf{m}_z, \quad \mathbf{Q}_{n_z}=Q_{n_z}\mathbf{n}_n,\\
&\mathbf{T}_{n_s}=T_{n_s}\mathbf{n}_s, \quad \mathbf{S}_{n_s}=S_{n_s}\mathbf{m}_s, \quad \mathbf{Q}_{n_s}=Q_{n_s}\mathbf{n}_n.
\end{aligned}
\tag{1.151}
$$

We introduce (1.151) into (1.150). Using (1.52) and considering that the unit vectors of the main and auxiliary trihedra according to (1.18) are independent of $\overline{Z}$, we easily obtain

$$
\begin{gathered}
\mathfrak{R}=\Big\{\Big[\sin\chi T_{n_z}+\frac{\partial}{\partial S}(AS_{n_z})+\sin\chi\frac{\partial}{\partial\overline{Z}}(BT_{n_z})+\cos\chi\frac{\partial}{\partial\overline{Z}}(BS_{n_z})+\\
+AB\sin\chi\, p_{m_z}\Big]\mathbf{m}_z+\Big[\frac{\partial}{\partial S}(AT_{n_z})-\sin\chi S_{n_z}-\frac{\sin\chi}{R_0}AQ_{n_z}-\cos\chi\frac{\partial}{\partial\overline{Z}}(BT_{n_z})+\\
+\sin\chi\frac{\partial}{\partial\overline{Z}}(BS_{n_z})+AB\sin\chi\, p_{n_z}\Big]\mathbf{n}_z+\Big[\frac{\sin\chi}{R_0}AT_{n_z}+\frac{\partial}{\partial S}(AQ_{n_z})+\\
+\frac{\partial}{\partial\overline{Z}}(BQ_{n_z})+AB\sin\chi\, p_{n_n}\Big]\mathbf{n}_n\Big\}d\overline{Z}\,dS.
\end{gathered}
\tag{1.152}
$$

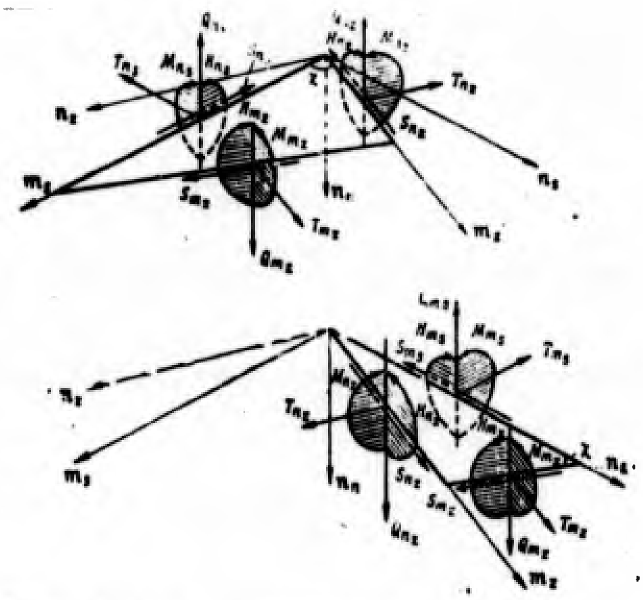

Figure 1.6. Concerning the determination of the components of internal forces and moments in axes of the auxiliary trihedron.

where p_{m_z}, p_{n_z}, p_{n_n} are the components of the vector of external surface load intensity in axes of the main trihedron.

To within higher-order infinitesimals, vector $\mathfrak{M}$ may be represented in the form

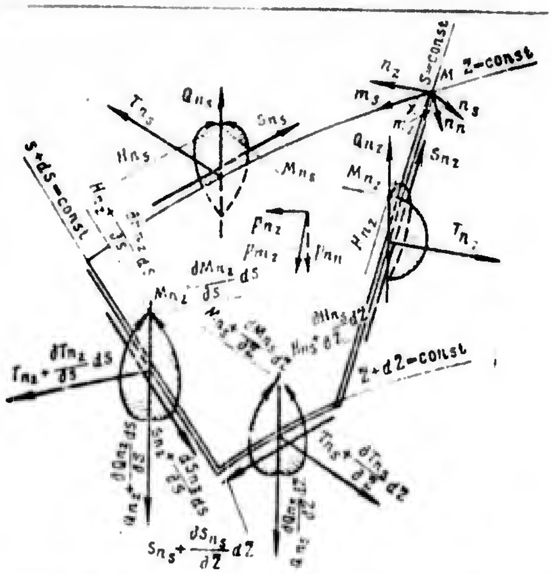

Figure 1.7. Concerning the differential equations of equilibrium.

$$\mathfrak{M}=\left\{\frac{\partial}{\partial S}[A(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})]+\frac{\partial}{\partial Z}[B(\mathbf{M}_{n_s}+\mathbf{H}_{n_s})]-\right.$$
$$\left.-AB[(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\times\mathbf{m}_s+(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\times\mathbf{m}_z]\right\}dZ\,dS, \tag{1.153}$$

where M and H are vectors of the linear bending and twisting moments applied to the contour of the element:

$$\begin{aligned}\mathbf{M}_{n_z}&=-M_{n_z}\mathbf{m}_z, & \mathbf{H}_{n_z}&=H_{n_z}\mathbf{n}_z,\\ \mathbf{M}_{n_s}&=M_{n_s}\mathbf{m}_s, & \mathbf{H}_{n_s}&=-H_{n_s}\mathbf{n}_s.\end{aligned} \tag{1.154}$$

We represent the unit vectors m_z and m_s in the form of vector products of unit vectors of the main trihedron. Using (1.14), we obtain

$$\begin{aligned}\mathbf{m}_z&=\mathbf{n}_z\times\mathbf{n}_n,\\ \mathbf{m}_s&=\mathbf{n}_n\times(\mathbf{m}_z\sin\chi-\mathbf{n}_z\cos\chi).\end{aligned} \tag{1.155}$$

Now, using the rule of calculation of the double vector product, taking (1.151) into account, we find

$$\begin{aligned}(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\times\mathbf{m}_z&=(-T_{n_s}\cos\chi+S_{n_s}\sin\chi)\mathbf{n}_n+\\ &+Q_{n_s}(\mathbf{n}_z\cos\chi-\mathbf{m}_z\sin\chi).\\ (\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\times\mathbf{m}_s&=(T_{n_z}\cos\chi-S_{n_z}\sin\chi)\mathbf{n}_n+Q_{n_z}\mathbf{n}_z.\end{aligned} \tag{1.156}$$

Introducing (1.156) into (1.153) and considering relations (1.52), we readily obtain

$$\mathfrak{R}=\Big\{\Big[-\frac{\partial}{\partial S}(AM_{n_s})+\sin\chi H_{n_s}+\cos\chi\frac{\partial}{\partial \bar Z}(BM_{n_s})-$$
$$-\sin\chi\frac{\partial}{\partial \bar Z}(BH_{n_s})+AB\sin\chi Q_{n_s}\Big]\mathfrak{m}_s+\Big[\sin\chi M_{n_s}+\frac{\partial}{\partial S}(AH_{n_s})+$$
$$+\sin\chi\frac{\partial}{\partial \bar Z}(BM_{n_s})+\cos\chi\frac{\partial}{\partial \bar Z}(BH_{n_s})-AB\cos\chi Q_{n_s}- \tag{1.157}$$
$$-ABQ_{\gamma_s}\Big]\mathfrak{n}_s+\Big[A\frac{\sin\chi}{R_0}H_{n_s}+AB\cos\chi(T_{n_z}-T_{n_s})-$$
$$-AB\sin\chi(S_{n_z}-S_{n_s})\Big]\mathfrak{n}_n\Big\}d\bar Z\,dS.$$

From the conditions of equilibrium of the separated element $\mathfrak{R}=0, \mathfrak{M}=0$ it follows that:

$$\sin\chi\frac{\partial}{\partial \bar Z}(BT_{n_s})+\cos\chi\frac{\partial}{\partial \bar Z}(BS_{n_s})+\frac{\partial}{\partial S}(AS_{n_s})+\sin\chi T_{n_z}+AB\sin\chi p_{m_s}=0, \tag{1.158}$$

$$-\cos\chi\frac{\partial}{\partial \bar Z}(BT_{n_s})+\sin\chi\frac{\partial}{\partial \bar Z}(BS_{n_s})+\frac{\partial}{\partial S}(AT_{n_z})-\sin\chi S_{n_s}-\frac{\sin\chi}{R_0}AQ_{n_s}+$$
$$+AB\sin\chi p_{n_z}=0, \tag{1.159}$$

$$\frac{\partial}{\partial \bar Z}(BQ_{n_s})+\frac{\partial}{\partial S}(AQ_{ns})+\frac{\sin\chi}{R_0}AT_{n_z}+AB\sin\chi p_{n_n}=0, \tag{1.160}$$

$$\cos\chi\frac{\partial}{\partial \bar Z}(BM_{n_s})-\sin\chi\frac{\partial}{\partial \bar Z}(BH_{n_s})-\frac{\partial}{\partial S}(AM_{n_s})+\sin\chi H_{n_z}+$$
$$+AB\sin\chi Q_{n_s}=0, \tag{1.161}$$

$$\sin\chi\frac{\partial}{\partial \bar Z}(BM_{n_s})+\cos\chi\frac{\partial}{\partial \bar Z}(BH_{n_s})+\frac{\partial}{\partial S}(AH_{n_s})+\sin\chi M_{n_s}-$$
$$-AB(\cos\chi Q_{n_s}+Q_{n_s})=0, \tag{1.162}$$

$$AB[\cos\chi(T_{n_z}-T_{n_s})-\sin\chi(S_{n_z}-S_{n_s})]+\frac{\sin\chi}{R_0}AH_{n_s}=0. \tag{1.163}$$

Equations (1.158)-(1.163) contain the components of linear forces and moments in axes of both the main and the auxiliary trihedron. We eliminate the components in axes of the auxiliary moving trihedron by using static relations (1.148), (1.149). We obtain

$$\sin\chi\frac{\partial}{\partial \bar Z}(BT_{m_s})-\cos\chi\frac{\partial}{\partial \bar Z}(BS_{n_s})+\frac{\partial}{\partial S}(AS_{n_s})+\sin\chi T_{n_z}+$$
$$+AB\sin\chi p_{m_s}=0, \tag{1.164}$$

$$-\cos\chi\frac{\partial}{\partial \bar Z}(BT_{n_s})+\frac{\partial}{\partial S}(AT_{n_s})+\sin\chi\frac{\partial}{\partial \bar Z}(BS_{m_s})-\sin\chi S_{n_s}-$$
$$-\frac{\sin\chi}{R_0}AQ_{n_s}+AB\sin\chi p_{n_s}=0, \tag{1.165}$$

$$\sin\chi \frac{\partial}{\partial Z}(BQ_{m_z}) - \cos\chi \frac{\partial}{\partial Z}(BQ_{n_z}) + \frac{\partial}{\partial S}(AQ_{n_s}) + \frac{\sin\chi}{R_0} AT_{n_z} + + AB \sin\chi p_{n_n} = 0, \tag{1.166}$$

$$\cos\chi \frac{\partial}{\partial Z}(BM_{n_z}) - \frac{\partial}{\partial S}(AM_{n_s}) - \sin\chi \frac{\partial}{\partial Z}(BH_{m_z}) + \sin\chi H_{n_z} + + AB \sin\chi Q_{n_z} = 0, \tag{1.167}$$

$$\sin\chi \frac{\partial}{\partial Z}(BM_{m_z}) - \cos\chi \frac{\partial}{\partial Z}(BH_{n_z}) + \frac{\partial}{\partial S}(AH_{n_s}) + \sin\chi M_{n_z} - - AB \sin\chi Q_{m_z} = 0, \tag{1.168}$$

$$S_{m_z} - S_{n_z} + \frac{1}{R} H_{n_z} = 0. \tag{1.169}$$

Thus, the equilibrium equations of the shell element are reduced to a system of five differential equations in partial derivatives (1.164)-(1.168) and one finite relation (1.169). Since in discussing the deformation of the shell and deriving the elasticity relations we used the approximate relation

$$1 \pm \frac{\gamma}{R} \approx 1, \tag{1.170}$$

it is easy to show that, with the same degree of accuracy,

$$S \pm \frac{1}{R} H \approx S \tag{1.171}$$

and hence, the last term of Eq. (1.169) should be omitted. Indeed,

$$S \pm \frac{1}{R} H = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau \, d\gamma \pm \frac{1}{R} \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau\gamma \, d\gamma = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau \left(1 \pm \frac{\gamma}{R}\right) d\gamma,$$

whence, in view of estimate (1.170), there results estimate (1.171).

In this case, Eq. (1.169) takes the form

$$S_{m_z} - S_{n_z} = 0$$

and is fulfilled identically, since according to the adopted elasticity relations (1.146), the law of parity

$$S_{m_z}=S_{n_z}=S_{n_z m_z},$$

applies.

Taking (1.15), (1.16) and (1.11) into consideration, one can readily obtain

$$\frac{\partial}{\partial n_z}(AB)=0 \tag{1.172}$$

then rewrite Eqs. (1.164)-(1.168) in the form

$$\begin{aligned}
&\frac{1}{B}\frac{\partial}{\partial m_z}(BT_{m_z})+\frac{\partial S_{n_z}}{\partial n_z}+\frac{1}{AB}T_{n_z}+p_{m_z}=0,\\
&\frac{\partial T_{n_z}}{\partial n_z}+\frac{1}{B}\frac{\partial}{\partial m_z}(BS_{m_z})-\frac{1}{AB}S_{n_z}-\frac{1}{R}Q_{n_z}+p_{n_z}=0,\\
&\frac{1}{B}\frac{\partial}{\partial m_z}(BQ_{m_z})+\frac{\partial Q_{n_z}}{\partial n_z}+\frac{1}{R}T_{n_z}+p_{n_n}=0,\\
&\frac{\partial M_{n_z}}{\partial n_z}+\frac{1}{B}\frac{\partial}{\partial m_z}(BH_{m_z})-\frac{1}{AB}H_{n_z}-Q_{n_z}=0,\\
&\frac{1}{B}\frac{\partial}{\partial m_z}(BM_{m_z})+\frac{\partial H_{n_z}}{\partial n_z}+\frac{1}{AB}M_{n_z}-Q_{m_z}=0.
\end{aligned} \tag{1.173}$$

It is easy to see that expressions (1.173) coincide in form with the equilibrium differential equations of an element whose contour is outlined by the lines of principal curvatures.

1.5. Strain Energy

Let us consider, in addition to the true deformed surface of the shell

$$\mathbf{M}'(\mathbf{M})=\mathbf{M}+\mathbf{U}(\mathbf{M})$$

some deformed surface adjacent to it

$$\tilde{M}^1(M) = M^1(M) + \delta M^1(M),$$

allowed by the geometric constraints imposed on the shell, and calculate the work δL done by external forces applied to the shell element (see figure 1.7) as the shell passes from equilibrium position M to position $\tilde{M}^1$.

Let t be an arbitrarily oriented tangential unit vector of the middle surface of the shell. The forces acting in the normal section of the shell perpendicular to t may be represented by vectors of their resultant $\mathfrak{N}_t$ and moment $\mathfrak{M}_t$, referred to a unit length of the contour of the middle surface section. By virtue of the Kirchhoff-Love hypothesis, the total work δL_t done by these forces is equal to the sum of the work of the resultant $\mathfrak{N}_t$ and net moment $\mathfrak{M}_t$. For a section of length ds, measured along the middle surface, we have

$$\delta L_t = (\mathfrak{N}_t \cdot \delta U + \mathfrak{M}_t \cdot \delta \Omega)\, \nu \cdot t\, dS. \tag{1.174}$$

Here

$$\delta U = \delta M^1$$

is the variation in the elastic displacement vector,

$$\delta \Omega = n_n^1 \times (n_n^1 + \delta n_n^1) = n_n^1 \times \delta n_n^1 \tag{1.175}$$

is the variation in the vector of the angle of rotation of the normal n_n to the middle surface,

v is the unit vector of the outer normal to the section, so that

$$\mathbf{v}\cdot\mathbf{t}=\pm 1.$$

For the shell element shown in Figure 1.7, using expression (1.174), to within higher-order infinitessimals, we find

$$\begin{aligned}\delta L=\Bigl(&\frac{\partial}{\partial S}\left\{A\left[(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cdot\delta\mathbf{U}+(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\cdot\delta\boldsymbol{\Omega}\right]\right\}+\\&+\frac{\partial}{\partial\bar{Z}}\left\{B\left[(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\cdot\delta\mathbf{U}+(\mathbf{M}_{n_s}+\mathbf{H}_{n_s})\cdot\delta\boldsymbol{\Omega}\right]\right\}+\\&+AB\sin\chi\,\mathbf{p}\cdot\delta\mathbf{U}\Bigr)\,d\bar{Z}\,dS.\end{aligned} \tag{1.176}$$

Carrying out the differentiation operation in (1.176), then changing the order of the variation and differentiation operations, we have

$$\begin{aligned}\delta L=\Bigl(&\Bigl\{\frac{\partial}{\partial S}\left[A(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\right]+\frac{\partial}{\partial\bar{Z}}\left[B(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\right]+\\&+AB\sin\chi\,\mathbf{p}\Bigr\}\delta\mathbf{U}+\Bigl\{\frac{\partial}{\partial S}\left[A(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\right]+\frac{\partial}{\partial\bar{Z}}\left[B(\mathbf{M}_{n_s}+\mathbf{H}_{n_s})\right]\Bigr\}\cdot\delta\boldsymbol{\Omega}+\\&+AB\Bigl[(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cdot\delta\frac{\partial\mathbf{U}}{\partial\mathbf{m}_s}+(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\cdot\delta\frac{\partial\mathbf{U}}{\partial\mathbf{m}_z}+\\&+(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\cdot\delta\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{m}_s}+(\mathbf{M}_{n_s}+\mathbf{H}_{n_s})\cdot\delta\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{m}_z}\Bigr]\Bigr)\,d\bar{Z}\,dS,\end{aligned} \tag{1.177}$$

whence, considering expressions (1.150) and (1.153), we obtain

$$\begin{aligned}\delta L=\mathfrak{R}\cdot\delta\mathbf{U}+\Bigl\{&\mathfrak{M}+AB\left[(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\times\mathbf{m}_s+\right.\\&\left.+(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\times\mathbf{m}_z\right]d\bar{Z}\,dS\Bigr\}\cdot\delta\boldsymbol{\Omega}+\\&+AB\Bigl[(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cdot\delta\frac{\partial\mathbf{U}}{\partial\mathbf{m}_s}+(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\cdot\delta\frac{\partial\mathbf{U}}{\partial\mathbf{m}_z}+\\&+(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\cdot\delta\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{m}_s}+(\mathbf{M}_{n_s}+\mathbf{H}_{n_s})\cdot\delta\frac{\partial\boldsymbol{\Omega}}{\partial\mathbf{m}_z}\Bigr]d\bar{Z}\,dS.\end{aligned} \tag{1.178}$$

However, $\mathfrak{R}$ and $\mathfrak{M}$ are the resultant and the net moment of forces external to the element under consideration and corresponding to the equilibrium state of the shell, and hence, in (1.178)

$$\mathfrak{R}=\mathfrak{M}=0. \tag{1.179}$$

Taking (1.179) into account and considering the admissibility of a cyclic permutation of the cofactors of the mixed product:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a},$$

we reduce (1.178) to the form

$$\begin{aligned}\delta L = AB\Big[&(\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s})\cdot\delta\left(\mathbf{m}_s\times\Omega+\frac{\partial\mathbf{U}}{\partial\mathbf{m}_s}\right)+\\ &+(\mathbf{T}_{n_t}+\mathbf{S}_{n_t}+\mathbf{Q}_{n_t})\cdot\delta\left(\mathbf{m}_t\times\Omega+\frac{\partial\mathbf{U}}{\partial\mathbf{m}_t}\right)+\\ &+(\mathbf{M}_{n_s}+\mathbf{H}_{n_s})\cdot\delta\frac{\partial\Omega}{\partial\mathbf{m}_s}+(\mathbf{M}_{n_t}+\mathbf{H}_{n_t})\cdot\delta\frac{\partial\Omega}{\partial\mathbf{m}_t}\Big]d\bar{Z}\,dS.\end{aligned} \tag{1.180}$$

Expression (1.180) contains variations in the vectors of displacement of the middle surface and rotation of the normal to it. These variations obviously are not independent.

Using formula (1.122) for the equilibrium surface of the shell $M^1(M)$ and the adjacent surface $\tilde{M}^1(M)$, we have

$$\mathbf{n}_n^1=\mathbf{n}_n-\mathbf{m}_s\left(\frac{\partial\mathbf{U}}{\partial\mathbf{m}_s}\cdot\mathbf{n}_n\right)-\mathbf{n}_s\left(\frac{\partial\mathbf{U}}{\partial\mathbf{n}_s}\cdot\mathbf{n}_n\right). \tag{1.181}$$

$$\mathbf{n}_n^1+\delta\mathbf{n}_n^1=\mathbf{n}_n-\mathbf{m}_s\left[\frac{\partial(\mathbf{U}+\delta\mathbf{U})}{\partial\mathbf{m}_s}\cdot\mathbf{n}_n\right]-\mathbf{n}_s\left[\frac{\partial(\mathbf{U}+\delta\mathbf{U})}{\partial\mathbf{n}_s}\cdot\mathbf{n}_n\right],$$

whence

$$\delta\mathbf{n}_n^1=-\mathbf{m}_s\left(\frac{\partial\delta\mathbf{U}}{\partial\mathbf{m}_s}\cdot\mathbf{n}_n\right)-\mathbf{n}_s\left(\frac{\partial\delta\mathbf{U}}{\partial\mathbf{n}_s}\cdot\mathbf{n}_n\right). \tag{1.182}$$

Introducing (1.181), (1.182) into (1.175), to within small terms of higher orders we find

$$\delta\Omega=\mathbf{m}_s\left(\frac{\partial\delta\mathbf{U}}{\partial\mathbf{n}_s}\cdot\mathbf{n}_n\right)-\mathbf{n}_s\left(\frac{\partial\delta\mathbf{U}}{\partial\mathbf{m}_s}\cdot\mathbf{n}_n\right)$$

or, changing the order of the variation and differentiation operations,

$$\delta\Omega = \delta\left[\mathbf{m}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{n}_s}\cdot\mathbf{n}_n\right) - \mathbf{n}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{n}_n\right)\right]. \tag{1.183}$$

Now, using (1.183) and considering (1.14)-(1.16), we easily obtain

$$\mathbf{m}_s \times \Omega = -\mathbf{n}_n\left(\mathbf{n}_n\cdot\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\right),$$
$$\mathbf{m}_s \times \Omega = -\mathbf{n}_n\left(\mathbf{n}_n\cdot\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\right),$$

whence, expanding the derivatives of vector U in axes of the main trihedron

$$\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} = \sum_{i=1}^{3}\mathbf{t}_i\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{t}_i\right), \quad \frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} = \sum_{i=1}^{3}\mathbf{t}_i\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{t}_i\right),$$

we find

$$\mathbf{m}_s\times\Omega + \frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} = \mathbf{m}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_s\right) + \mathbf{n}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{n}_s\right),$$
$$\mathbf{m}_s\times\Omega + \frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} = \mathbf{m}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_s\right) + \mathbf{n}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{n}_s\right). \tag{1.184}$$

Introducing (1.184) into (1.180), we obtain the final expression for work

$$\begin{aligned}\delta L = \iint\Big\{&(\mathbf{T}_{n_s} + \mathbf{S}_{n_s} + \mathbf{Q}_{n_s})\,\delta\left[\mathbf{m}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_s\right) + \mathbf{n}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{n}_s\right)\right] +\\ &+(\mathbf{T}_{n_s} + \mathbf{S}_{n_s} + \mathbf{Q}_{n_s})\,\delta\left[\mathbf{m}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{m}_s\right) + \mathbf{n}_s\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s}\cdot\mathbf{n}_s\right)\right] +\\ &+(\mathbf{M}_{n_s} + \mathbf{H}_{n_s})\,\delta\frac{\partial \Omega}{\partial \mathbf{m}_s} + (\mathbf{M}_{n_s} + \mathbf{H}_{n_s})\,\delta\frac{\partial \Omega}{\partial \mathbf{m}_s}\Big\}\,dZ\,dS.\end{aligned} \tag{1.185}$$

According to the origin of possible Lagrange displacements, when an elastic system passes from the equilibrium position to an arbitrary adjacent position allowed by the geometric constraints

$$\delta T - \delta U = 0.$$

where δT is the work done by external forces,
δU is the change in potential energy.

Using the origin of possible displacements toward the oblique-angled element shown in Figure 1.7, we have

$$\delta L - \int\limits_{\Sigma} \delta W \, d\Sigma = 0, \tag{1.186}$$

where

$$\Sigma = AB \sin \chi \, d\overline{Z} \, dS \tag{1.187}$$

is the area of the middle surface element,

δW is the change in potential energy referred to the area of the middle surface:

$$\int\limits_{\Sigma} \delta W \, d\Sigma = \delta U. \tag{1.188}$$

According to the theorem of the mean

$$\delta L = \delta W(M^*) \Sigma, \tag{1.189}$$

where M^* is some point of the middle surface within the confines of the element considered: $M^* = M(\overline{Z} + \xi d\overline{Z}, S + \eta dS)$, $0 \leq \xi, \eta \leq 1$.

Confining the oblique-angled element to the point $M = M(\overline{Z}, S)$ and passing in (1.189) to the limit, taking (1.185), (1.187) into account, we find

$$\begin{aligned} \delta W = \frac{1}{\sin \chi} \Big\{ & (\mathbf{T}_{n_z} + \mathbf{S}_{n_z} + \mathbf{Q}_{n_z}) \cdot \delta \left[\mathbf{m}_z \left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} \cdot \mathbf{m}_z \right) + \mathbf{n}_z \left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} \cdot \mathbf{n}_z \right) \right] + \\ & + (\mathbf{T}_{n_s} + \mathbf{S}_{n_s} + \mathbf{Q}_{n_s}) \cdot \delta \left[\mathbf{m}_s \left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z} \cdot \mathbf{m}_s \right) + \mathbf{n}_s \left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_s} \cdot \mathbf{n}_s \right) \right] + \\ & + (\mathbf{M}_{n_z} + \mathbf{H}_{n_z}) \cdot \delta \frac{\partial \Omega}{\partial \mathbf{m}_s} + (\mathbf{M}_{n_s} + \mathbf{H}_{n_s}) \cdot \delta \frac{\partial \Omega}{\partial \mathbf{m}_z} \Big\}. \end{aligned} \tag{1.190}$$

Expression (1.190) contains the components of linear forces and moments in axes of both the main and the auxiliary trihedron, but the components in axes of the latter can be eliminated by using the static relations resulting from the equilibrium condition of the middle-surface elements shown in Figure 1.6:

$$\begin{aligned} &\mathbf{T}_{n_s}+\mathbf{S}_{n_s}+\mathbf{Q}_{n_s}=(\mathbf{T}_{m_z}+\mathbf{S}_{m_z}+\mathbf{Q}_{m_z})\sin\chi-(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cos\chi, \\ &\mathbf{M}_{n_s}+\mathbf{H}_{n_s}=(\mathbf{M}_{m_z}+\mathbf{H}_{m_z})\sin\chi-(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\cos\chi, \end{aligned} \tag{1.191}$$

where the vectors

$$\begin{aligned} &\mathbf{T}_{m_z}=T_{m_z}\mathbf{m}_z;\quad \mathbf{S}_{m_z}=S_{m_z}\mathbf{n}_z;\quad \mathbf{Q}_{m_z}=Q_{m_z}\mathbf{n}_n; \\ &\mathbf{M}_{m_z}=M_{m_z}\mathbf{n}_z;\quad \mathbf{H}_{m_z}=-H_{m_z}\mathbf{m}_z, \end{aligned} \tag{1.192}$$

and the vectors T_{n_z}, S_{n_z}, Q_{n_z}, T_{n_s}, S_{n_s}, Q_{n_s} and M_{n_z}, H_{n_z}, M_{n_s}, S_{n_s} are represented by expressions (1.151), (1.154), respectively.

Introducing (1.191) into (1.190), with the aid of relation

$$\frac{\partial}{\partial \mathbf{m}_s}=\sin\chi\frac{\partial}{\partial \mathbf{n}_z}+\cos\chi\frac{\partial}{\partial \mathbf{m}_z}, \tag{1.193}$$

resulting from (1.14), one can readily obtain

$$\begin{aligned} \delta W=&(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cdot\delta\left[\mathbf{m}_z\left(\frac{\partial \mathbf{U}}{\partial \mathbf{n}_z}\cdot\mathbf{m}_z\right)+\mathbf{n}_z\left(\frac{\partial \mathbf{U}}{\partial \mathbf{n}_z}\cdot\mathbf{n}_z\right)\right]+ \\ &+(\mathbf{T}_{m_z}+\mathbf{S}_{m_z}+\mathbf{Q}_{m_z})\cdot\delta\left[\mathbf{m}_z\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\cdot\mathbf{m}_z\right)+\mathbf{n}_z\left(\frac{\partial \mathbf{U}}{\partial \mathbf{m}_z}\cdot\mathbf{n}_z\right)\right]+ \\ &+(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\cdot\delta\frac{\partial \mathbf{\Omega}}{\partial \mathbf{n}_z}+(\mathbf{M}_{m_z}+\mathbf{H}_{m_z})\cdot\delta\frac{\partial \mathbf{\Omega}}{\partial \mathbf{m}_z}. \end{aligned} \tag{1.194}$$

It is easy to see that expression (1.194) corresponds to the work of forces external to the element outlined by the lines of principal curvatures.

Using expression (1.183) and taking (1.15), (1.16), (1.42) and (1.52) into account, we find

$$\frac{\partial\Omega}{\partial n_z}=\left[\frac{\partial}{\partial n_z}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)-\frac{1}{AB}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right)\right]m_z-$$

$$-\left[\frac{\partial}{\partial n_z}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right)+\frac{1}{AB}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)\right]n_z-\frac{1}{R}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right)n_n, \tag{1.195}$$

$$\frac{\partial\Omega}{\partial m_z}=m_z\frac{\partial}{\partial m_z}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)-n_z\frac{\partial}{\partial m_z}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right).$$

Introducing into (1.194) the expanded expressions for the variations $\delta\ \partial\Omega/\partial_{n_z}\delta\ \partial\Omega/\partial m_z$ in accordance with (1.195) and considering representations (1.151), (1.154), (1.192), one can readily obtain, taking into account the parity of tangential forces and twisting moments (1.146),

$$\delta W=T_{m_z}\delta\left(\frac{\partial U}{\partial m_z}\cdot m_z\right)+S_{n_z m_z}\delta\left(\frac{\partial U}{\partial m_z}\cdot n_z+\frac{\partial U}{\partial n_z}\cdot m_z\right)+$$

$$+T_{n_z}\delta\left(\frac{\partial U}{\partial n_z}\cdot n_z\right)-M_{m_z}\delta\frac{\partial}{\partial m_z}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right)-$$

$$-H_{n_z m_z}\delta\left[\frac{\partial}{\partial m_z}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)+\frac{\partial}{\partial n_z}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right)+\frac{1}{AB}\frac{\partial U}{\partial n_z}\cdot n_n\right]- \tag{1.196}$$

$$-M_{n_z}\delta\left[\frac{\partial}{\partial n_z}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)-\frac{1}{AB}\frac{\partial U}{\partial m_z}\cdot n_n\right].$$

Transforming (1.196) with the aid of (1.52) and (1.116), we represent the variation δW in the form

$$\delta W=T_{m_z}\delta\left(\frac{\partial U}{\partial m_z}\cdot m_z\right)+\left(S_{n_z m_z}+\frac{1}{R}H_{n_z m_z}\right)\delta\frac{\partial U}{\partial m_z}\cdot n_z+S_{n_z m_z}\delta\frac{\partial U}{\partial n_z}\cdot m_z+$$

$$+T_{n_z}\delta\left(\frac{\partial U}{\partial n_z}\cdot n_z\right)-M_{m_z}\delta\frac{\partial}{\partial m_z}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right)-$$

$$-2H_{n_z m_z}\delta\left[\frac{\partial^2}{\partial m_z\partial n_z}(U\cdot n_n)+\frac{\partial}{\partial m_z}\left(\frac{U\cdot n_z}{R}\right)\right]- \tag{1.197}$$

$$-M_{n_z}\delta\left[\frac{\partial}{\partial n_z}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)-\frac{1}{AB}\frac{\partial U}{\partial m_z}\cdot n_n\right].$$

Using (1.42), (1.50), (1.93), we also reduce (1.118) to the form

$$\varkappa_{m_z}=-\frac{\partial}{\partial m_z}\left(\frac{\partial U}{\partial m_z}\cdot n_n\right),$$

$$\varkappa_{n_z m_z}=-\frac{\partial^2}{\partial m_z\partial n_z}(U\cdot n_n)-\frac{\partial}{\partial m_z}\left(\frac{U\cdot n_z}{R}\right), \tag{1.198}$$

$$\varkappa_{n_z}=-\frac{\partial}{\partial n_z}\left(\frac{\partial U}{\partial n_z}\cdot n_n\right)+\frac{1}{AB}\frac{\partial U}{\partial m_z}\cdot n_n.$$

Now, neglecting 1/R(H) in accordance with estimate (1.171) in comparison with the tangential force S, and considering expressions (1.37), (1.63) and (1.198), we finally obtain

$$\begin{gathered}\delta W = T_{m_z}\delta\varepsilon^0_{m_z} + S_{n_z m_z}\delta\gamma^0_{n_z m_z} + T_{n_z}\delta\varepsilon^0_{n_z} + \\ + M_{m_z}\delta\varkappa_{m_z} + 2H_{n_z m_z}\delta\varkappa_{n_z m_z} + M_{n_z}\delta\varkappa_{n_z}.\end{gathered} \tag{1.199}$$

Expression (1.199) represents the variation in potential energy per unit area of the middle surface of the shell. On the whole, according to (1.188), the variation in the potential energy of the shell

$$\begin{gathered}\delta U = \iint (T_{m_z}\delta\varepsilon^0_{m_z} + S_{n_z m_z}\delta\gamma^0_{n_z m_z} + T_{n_z}\delta\varepsilon^0_{n_z} + \\ + M_{m_z}\delta\varkappa_{m_z} + 2H_{n_z m_z}\delta\varkappa_{n_z m_z} + M_{n_z}\delta\varkappa_{n_z})\, AB\sin\chi\, d\overline{Z}\, dS.\end{gathered} \tag{1.200}$$

Chapter II. Resolvents of Two-Dimensional Problem

2.1. Static-Geometric Model. Fundamental Kinematic Unknowns

In conformity with the geometric, physical and static relations discussed in Chapter 1, the general calculation of a shell amounts to the determination of three components u_{m_z}, u_{n_z}, u_{n_n} of elastic displacement vector U, five components T_{m_z}, T_{n_z}, $S_{n_z m_z}$ and Q_{m_z}, Q_{n_z} of tangential and transverse forces, three components M_{m_z}, M_{n_z} and $H_{n_z m_z}$ of linear bending and twisting moments, three components $e^0_{m_z}$, $e^0_{n_z}$, $\gamma^0_{n_z m_z}$ of tangential strain of the middle surface of the shell, and three components x_{m_z}, x_{n_z} and $x_{n_z m_z}$ of its bending and twisting strain. Each of the 17 enumerated quantities is an unknown function of the variables $\bar{Z}$, S. To determine these functions, we have 17 relations: five differential equations of equilibrium:

$$
\begin{gathered}
\sin\chi\frac{\partial}{\partial\bar{Z}}(BT_{m_z})-\cos\chi\frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z})+\frac{\partial}{\partial S}(AS_{n_z m_z})+\\
+\sin\chi T_{n_z}+AB\sin\chi p_{m_z}=0,\\
-\cos\chi\frac{\partial}{\partial\bar{Z}}(BT_{n_z})+\frac{\partial}{\partial S}(AT_{n_z})+\sin\chi\frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z})-\\
-\sin\chi S_{n_z m_z}-\frac{\sin\chi}{R_0}AQ_{n_z}+AB\sin\chi p_{n_z}=0,\\
\sin\chi\frac{\partial}{\partial\bar{Z}}(BQ_{m_z})-\cos\chi\frac{\partial}{\partial\bar{Z}}(BQ_{n_z})+\frac{\partial}{\partial S}(AQ_{n_z})+\\
+\frac{\sin\chi}{R_0}AT_{n_z}+AB\sin\chi p_{n_n}=0,\\
\cos\chi\frac{\partial}{\partial\bar{Z}}(BM_{n_z})-\frac{\partial}{\partial S}(AM_{n_z})-\sin\chi\frac{\partial}{\partial\bar{Z}}(BH_{n_z m_z})+\\
+\sin\chi H_{n_z m_z}+AB\sin\chi Q_{n_z}=0,\\
\sin\chi\frac{\partial}{\partial\bar{Z}}(BM_{m_z})-\cos\chi\frac{\partial}{\partial\bar{Z}}(BH_{n_z m_z})+\frac{\partial}{\partial S}(AH_{n_z m_z})+\\
+\sin\chi M_{n_z}-AB\sin\chi Q_{m_z}=0,
\end{gathered}
\tag{2.1}
$$

six geometric relations:

$$
\begin{gathered}
e^0_{m_z}=\frac{1}{l_z}\frac{\partial u_{m_z}}{\partial\bar{Z}},\\
\gamma^0_{n_z m_z}=-\frac{\operatorname{ctg}\chi}{l_z}\frac{\partial u_{m_z}}{\partial\bar{Z}}+\frac{1}{1-\bar{Z}}\frac{1}{\sin\chi}\frac{\partial u_{m_z}}{\partial s}+\frac{1-\bar{Z}}{l_z}\frac{\partial}{\partial\bar{Z}}\left(\frac{u_{n_z}}{1-\bar{Z}}\right).
\end{gathered}
\tag{2.2}
$$

$$
\begin{aligned}
\varepsilon^0_{n_z} &= -\frac{1}{l_z}\operatorname{ctg}\chi\frac{\partial u_{n_z}}{\partial \bar{Z}} + \frac{1}{(1-\bar{Z})\sin\chi}\frac{\partial u_{n_z}}{\partial S} - \frac{1}{(1-\bar{Z})\,l_z}u_{m_z} - \frac{1}{R}u_{n_n},\\
\varkappa_{m_z} &= -\frac{1}{l_z^2}\frac{\partial^2 u_{n_n}}{\partial \bar{Z}^2},\\
\varkappa_{n_z m_z} &= \frac{\operatorname{ctg}\chi}{l_z^2}\frac{\partial^2 u_{n_n}}{\partial \bar{Z}^2} - \frac{1}{l_z\sin\chi}\frac{\partial}{\partial \bar{Z}}\left(\frac{1}{1-\bar{Z}}\frac{\partial u_{n_n}}{\partial S}\right) - \frac{1}{l_z R_0}\frac{\partial}{\partial \bar{Z}}\left(\frac{u_{n_z}}{1-\bar{Z}}\right),\\
\varkappa_{n_z} &= -\left[\frac{\operatorname{ctg}\chi}{l_z}\frac{\partial}{\partial \bar{Z}} - \frac{1}{(1-\bar{Z})\sin\chi}\frac{\partial}{\partial S}\right]\left[\frac{\operatorname{ctg}\chi}{l_z}\frac{\partial u_{n_n}}{\partial \bar{Z}} - \frac{1}{(1-\bar{Z})\sin\chi}\frac{\partial u_{n_n}}{\partial S}\right] +\\
&\quad + \frac{\operatorname{ctg}\chi}{l_z R_0}\frac{\partial}{\partial \bar{Z}}\left(\frac{u_{n_z}}{1-\bar{Z}}\right) - \frac{1}{(1-\bar{Z})^2\sin\chi}\frac{\partial}{\partial S}\left(\frac{u_{n_z}}{R_0}\right) + \frac{1}{l_z^2(1-\bar{Z})}\frac{\partial u_{n_n}}{\partial \bar{Z}}
\end{aligned}
\tag{2.2}
$$

and six elasticity relations:

$$
\begin{aligned}
T_{m_z} &= \frac{Eh}{1-\nu^2}(\varepsilon^0_{m_z} + \nu\varepsilon^0_{n_z}),\\
S_{n_z m_z} &= \frac{Eh}{2(1+\nu)}\gamma^0_{n_z m_z},\\
T_{n_z} &= \frac{Eh}{1-\nu^2}(\varepsilon^0_{n_z} + \nu\varepsilon^0_{m_z}),\\
M_{m_z} &= D(\varkappa_{m_z} + \nu\varkappa_{n_z}),\\
H_{n_z m_z} &= (1-\nu)D\varkappa_{n_z m_z},\\
M_{n_z} &= D(\varkappa_{n_z} + \nu\varkappa_{m_z}).
\end{aligned}
\tag{2.3}
$$

We will solve the problem in the displacements. For this purpose, it is necessary to reduce the system of 17 relations (2.1)-(2.3) to a system of three equilibrium equations relative to the components u_{m_z}, u_{n_z}, u_{n_n} of the elastic displacement vector, having eliminated all the force factors from (2.1).

The linear forces and moments T_{m_z}, T_{n_z}, $S_{n_z m_z}$, M_{m_z}, M_{n_z}, $H_{n_z m_z}$ are related to the components u_{m_z}, u_{n_z}, u_{n_n} by elasticity relations (2.3) and geometric relations (2.2). However, as already noted, the transverse forces Q_{m_z}, Q_{n_z} are purely static factors, in consequence of the Kirchhoff-Love hypothesis. From the fourth and fifth of equilibrium equations (2.1), we have

$$
\begin{aligned}
Q_{m_z} &= \frac{1}{AB}\left[\frac{\partial}{\partial \bar{Z}}(BM_{m_z}) - \operatorname{ctg}\chi\frac{\partial}{\partial \bar{Z}}(BH_{n_z m_z}) + \frac{1}{\sin\chi}\frac{\partial}{\partial S}(AH_{n_z m_z}) + M_{n_z}\right],\\
Q_{n_z} &= \frac{1}{AB}\left[-\operatorname{ctg}\chi\frac{\partial}{\partial \bar{Z}}(BM_{n_z}) + \frac{1}{\sin\chi}\frac{\partial}{\partial S}(AM_{n_z}) + \frac{\partial}{\partial \bar{Z}}(BH_{n_z m_z}) - H_{n_z m_z}\right].
\end{aligned}
\tag{2.4}
$$

Eliminating from the first three of equilibrium equations (2.1) the transverse forces Q_{m_z}, Q_{n_z} with the aid of relations (2.4), then, using relations (2.2) and (2.3), eliminating all the remaining force factors, we arrive at a system of differential resolvents in components u_{m_z}, u_{n_z}, u_{n_n} of the form

$$\begin{aligned} L_{11}(u_{m_z})+L_{12}(u_{n_z})+L_{13}(u_{n_n})&=p_{m_z},\\ L_{21}(u_{m_z})+L_{22}(u_{n_z})+L_{23}(u_{n_n})&=p_{n_z},\\ L_{31}(u_{m_z})+L_{32}(u_{n_z})+L_{33}(u_{n_n})&=p_{n_n}.\end{aligned} \tag{2.5}$$

Integration of these equations constitutes a very complex problem, solvable only for conical shells of revolution in the simplest cases. For shells of arbitrary configuration, however, the only known solution other than that of zero-moment theory is the partial solution, which is an extension of Saint-Venant's solution to prismatic bars.* However, a general solution taking into consideration the detailed character of application of the external load and, more importantly, the detailed fixing conditions of the shell apparently constitutes a practically insoluble problem.

All of the above pertains to smooth shells. In the presence of a reinforcing structure, however, the problem of integrating Eqs. (2.5) becomes even more complex, since in this case the components p_{m_z}, p_{n_z}, p_{n_n} of the load external to the shell not only are composed of the components of a given external load, but also include unknown forces of interaction of the reinforcing structure with the shell proper. For this reason, in the presence of a reinforcing structure, system (2.5) should be integrated jointly with the corresponding equations describing the work of the elements of the structure. Such a problem for shells of fairly complex configuration in the exact formulation is all the more insoluble as it makes it necessary to seek approximate solution methods. Two approaches are possible along these lines. The first consists in an approximate integration of systems of type (2.5) by numerical methods. The second approach is based on certain hypotheses regarding

*L.I. Balabukh. Bending and Torsion of Conical Shells. Trudy TsAGI, 1946, No. 577.

the character of deformation of the shell, which make it possible to reduce the problem to comparatively simple equations permitting integration in general form. Such an approach yields an analytical solution permitting the analysis of the operation of the shell from a quantitative as well as a qualitative point of view.

Any hypotheses regarding the character of deformation consist of certain geometric constraints imposed on the displacements of the shell, and determine its simplified computational model, which has a smaller number of degrees of freedom than the general model determined only by the Kirchhoff-Love hypothesis. However, when sufficiently valid hypotheses have been successfully chosen, the simplified computational model may permit not only a comparatively simple mathematical interpretation, but at the same time, a very satisfactory description of the properties of the real object.

In constructing a simplified geometric model, we will proceed from the representation of the elastic displacement vector in the form

$$U(\overline{Z}, S) = U^0(\overline{Z}, S) + U^1(\overline{Z}, S), \tag{2.6}$$

where vector function U^0 corresponds to an arbitrary spatial displacement of the contour $\overline{Z}$ = const as a solid, and vector function U^1 corresponds to certain additional displacements for which the configurations of the contour $\overline{Z}$ = const change. The first term on the right-hand side of (2.6), corresponding to a very rough but physically clear representation of the character of the operation of the reinforcing structure, determines the universal initial approximation, which in most cases reliably describes the picture as a whole. For straight, prismatic and slightly conical shells, this approximation determines the distribution of normal stresses in the cross section in accordance with the law of the plane and corresponds to the elementary solutions discussed in the strength of materials. For conical shells of arbitrary configuration, the displacement of the contour $\overline{Z}$ = const as a solid does not, generally speaking, signify a distribution of stresses in this section according to the law of the plane. We will say in this connection that U^0 defines the "generalized" law of plane sections, and the vector function

$U^1 = U - U^0$, which is a correction to the generalized law of plane sections, defines the warping of the contour $\bar{Z}$ = const.

As we know, the displacement of a body in space is determined by the displacement of an arbitrary pole rigidly bound to the body, and by the rotation of the body in relation to this pole. In vectorial form

$$\mathbf{U}^0(\mathbf{M}) = \boldsymbol{\eta} + \boldsymbol{\theta} \times (\mathbf{r} - \mathbf{r}_0),$$

where $U^0(M)$ is the displacement vector of point M of the solid; η is the displacement vector of the pole; θ is the rotation vector of the solid relative to the pole; r is the radius vector of point M; r_0 is the radius vector of the pole.

For the contour $\bar{Z}$ = const we have

$$\mathbf{U}^0(\bar{Z}, S) = \boldsymbol{\eta}(\bar{Z}) + \boldsymbol{\theta}(\bar{Z}) \times [\mathbf{r}(\bar{Z}, S) - \mathbf{r}_0(\bar{Z})], \tag{2.7}$$

where on the basis of (1.5)

$$\mathbf{r}(\bar{Z}, S) = x_0(S)(1-\bar{Z})\mathbf{i} + y_0(S)(1-\bar{Z})\mathbf{j} + [l_0\bar{Z} + x_0(S)\operatorname{ctg}\chi_0(1-\bar{Z})]\mathbf{k}. \tag{2.8}$$

Let us take as the pole of the contour $\bar{Z}$ = const the point of intersection of the plane of this contour with the Oz axis. Then

$$\mathbf{r}_0 = l_0\bar{Z}\,\mathbf{k}. \tag{2.9}$$

Introducing (2.8) and (2.9) into (2.7), we have

$$\mathbf{U}^0(\bar{Z}, S) = \boldsymbol{\eta}(\bar{Z}) + (1-\bar{Z})\boldsymbol{\theta}(\bar{Z}) \times [x_0(S)\mathbf{i} + y_0(S)\mathbf{j} + x_0(S)\operatorname{ctg}\chi_0\mathbf{k}]. \tag{2.10}$$

Expression (2.10) represents the vector function of the elastic displacement of the shell, corresponding to an arbitrary displacement of the contour $\bar{Z}$ = const as a solid. This function is completely defined by the vector functions η and θ, which are dependent only on the variable $\bar{Z}$, while its dependence on the variable S is explicit in character.

We will represent the vectors $\eta(\bar{Z})$, $\theta(\bar{Z})$ in the form

$$\begin{aligned} \eta(\bar{Z}) &= \eta_x(\bar{Z})\,\mathbf{i} + \eta_y(\bar{Z})\,\mathbf{j} + \eta_z(\bar{Z})\,\mathbf{k}, \\ \theta(\bar{Z}) &= \theta_x(\bar{Z})\,\mathbf{i} + \theta_y(\bar{Z})\,\mathbf{j} + \theta_z(\bar{Z})\,\mathbf{k}, \end{aligned} \tag{2.11}$$

where η_x, η_y, η_z are translational displacements of the contour $\bar{Z}$ = const along axes Ox, Oy, Oz; θ_x, θ_y, θ_z, are the angles of rotation of this contour relative to these axes.

Then, introducing (2.11) into (2.10), we obtain

$$\begin{aligned} \mathbf{U}^0(\bar{Z}, S) &= \eta_x(\bar{Z})\,\mathbf{i} + \eta_y(\bar{Z})\,\mathbf{j} + \eta_z(\bar{Z})\,\mathbf{k} + \\ &+ \{\theta_x(\bar{Z})[y_0(S)\mathbf{k} - x_0(S)\operatorname{ctg}\chi_0\,\mathbf{j}] + \theta_y(\bar{Z})[-x_0(S)\mathbf{k} + x_0(S)\operatorname{tg}\chi_0\,\mathbf{i}] + \\ &+ \theta_z(\bar{Z})[x_0(S)\mathbf{j} - y_0(S)\,\mathbf{i}]\}(1-\bar{Z}), \end{aligned} \tag{2.12}$$

i.e., the vector function $U^0(\bar{Z}, S)$ may be represented in the form

$$\mathbf{U}^0(\bar{Z}, S) = \sum_{i=1}^{6} V_i(\bar{Z})\varphi_i(S), \tag{2.13}$$

where the scalar functions

$$\begin{aligned} &V_1(\bar{Z}) = \eta_x(\bar{Z}), \quad V_2(\bar{Z}) = \eta_y(\bar{Z}), \quad V_3(\bar{Z}) = \eta_z(\bar{Z}), \\ &V_4(\bar{Z}) = (1-\bar{Z})\theta_x(\bar{Z}), \; V_5(\bar{Z}) = (1-\bar{Z})\theta_y(\bar{Z}), \; V_6(\bar{Z}) = (1-\bar{Z})\theta_z(\bar{Z}), \end{aligned} \tag{2.14}$$

and the vector functions are defined by the expressions

$$\begin{aligned}
&\varphi_1(S)=\mathrm{i}, \quad \varphi_2(S)=\mathrm{j}, \quad \varphi_3(S)=\mathrm{k},\\
&\varphi_4(S)=-x_0(S)\operatorname{ctg}\gamma_0\mathrm{j}+y_0(S)\mathrm{k},\\
&\varphi_5(S)=x_0(S)\operatorname{ctg}\gamma_0\mathrm{i}-x_0(S)\mathrm{k},\\
&\varphi_6(S)=-y_0(S)\mathrm{i}+x_0(S)\mathrm{j}.
\end{aligned} \tag{2.15}$$

Thus, the vector function $U^0(\overline{Z}, S)$, corresponding to an arbitrary displacement in space of the contour $\overline{Z}$ = const as a solid, is completely defined by six scalar functions of the variable $\overline{Z}$, which are the components of the translational displacement and rotation vectors of this contour.

Let t be an arbitrarily oriented unit vector. The displacement of point M in the direction of t, corresponding to the initial approximation U^0, will be

$$u_t^0=\mathrm{U}^0\cdot\mathrm{t} \tag{2.16}$$

or, using (2.13)

$$u_t^0(\overline{Z}, S)=\sum_{i=1}^{6} V_i(\overline{Z})\cdot\varphi_{it}(S), \tag{2.17}$$

where

$$\varphi_{it}=\varphi_i\cdot t. \tag{2.18}$$

Taking (2.15) into account, we find

$$\begin{aligned}
&\varphi_{1t}=l_t, \quad \varphi_{2t}=m_t, \quad \varphi_{3t}=n_t,\\
&\varphi_{4t}=-x_0(S)\operatorname{ctg}\chi_0 m_t+y_0(S)n_t,\\
&\varphi_{5t}=x_0(S)\operatorname{ctg}\gamma_0 l_t-x_0(S)n_t,\\
&\varphi_{6t}=-y_0(S)l_t+x_0(S)m_t.
\end{aligned} \tag{2.19}$$

By successively superposing t on the unit vectors of the main and auxiliary trihedra, one can readily obtain expanded expressions for the coordinate functions

ϕ_{im_z}, ϕ_{in_z}, ϕ_{im_s}, ϕ_{in_s}, ϕ_{in_n}. Since in accordance with formulas (1.20), the coordinate functions ϕ_{in_z}, ϕ_{in_s} can be represented in the form

$$\varphi_{in_z} = \frac{1}{\sin\chi}\varphi_{im_s} - \mathrm{ctg}\,\chi\,\varphi_{im_z},$$
$$\varphi_{in_s} = \frac{1}{\sin\chi}\varphi_{im_z} - \mathrm{ctg}\,\chi\,\varphi_{im_s}, \tag{2.20}$$

we will here give expanded expressions only for ϕ_{im_z}, ϕ_{im_s}, ϕ_{in_n}. Expanding (2.19) with the aid of (1.18), we obtain

i	V_i	φ_{im_z}	φ_{im_s}	φ_{in_n}
1	η_x	$-\frac{x_0}{l_s}$	x_0'	$-\frac{y_0'}{\sin\chi}\frac{l_0}{l_s} + \frac{x_0 y_0' - x_0' y_0}{\sin\chi\, l_s}\,\mathrm{ctg}\,\chi_0$
2	η_y	$-\frac{y_0}{l_s}$	y_0'	$\frac{x_0'}{\sin\chi}\frac{l_0}{l_s}$
3	η_z	$\frac{l_0}{l_s} - \frac{x_0}{l_s}\,\mathrm{ctg}\,\chi_0$	$x_0'\,\mathrm{ctg}\,\chi_0$	$\frac{x_0' y_0 - x_0 y_0'}{\sin\chi\, l_s}$
4	$(1-\overline{Z})\theta_x$	$y_0\frac{l_0}{l_s}$	$(x_0' y_0 - x_0 y_0')\,\mathrm{ctg}\,\chi_0$	$-\frac{x_0 x_0'}{\sin\chi}\frac{l_0}{l_s}\,\mathrm{ctg}\,\chi_0 + y_0\frac{x_0' y_0 - x_0 y_0'}{\sin\chi\, l_s}$
5	$(1-\overline{Z})\theta_y$	$-x_0\frac{l_0}{l_s}$	0	$-\frac{x_0 y_0'}{\sin\chi}\frac{l_0}{l_s}\,\mathrm{ctg}\,\chi_0 - x_0\frac{x_0' y_0 - x_0 y_0'}{\sin^2\chi_0 \sin\chi\, l_s}$
6	$(1-\overline{Z})\theta_z$	0	$x_0 y_0' - x_0' y_0$	$\frac{x_0 x_0' + y_0 y_0'}{\sin\chi}\frac{l_0}{l_s} + y_0\,\mathrm{ctg}\,\chi_0\frac{x_0' y_0 - x_0 y_0'}{\sin\chi\, l_s}$

Thus, in conformity with representations (2.6) and (2.13), the vector function of elastic displacement of the shell $U(\overline{Z}, S)$ is defined by six scalar functions $V_i(\overline{Z})$ of one variable and one vector function $U^1(\overline{Z}, S)$ of two variables.

Representation (2.16) in the absence of additional requirements for U^1 imposes no constraints on the general geometric model of the shell. In formulating a simplified geometric model, we will make certain assumptions only regarding the structure of the vector function $U^1(\bar{Z}, S)$, since the structure of the vector function $U^0(\bar{Z}, S)$ is completely defined by expression (2.13) and formulas (2.15). As the basic geometric hypothesis, we will take the assumption that in a conical shell, warping occurs only in the direction of the generatrices. Then the vector function $U^1(\bar{Z}, S)$ will be represented in the form

$$U^1(\bar{Z}, S) = \Omega(\bar{Z}, S)\,\mathbf{m}_1(S), \tag{2.22}$$

where $\Omega(\bar{Z}, S)$ is some scalar function to be determined.

The following considerations may be offered in support of the adopted hypothesis.

We are considering shells provided with a systematic structure of transverse diaphragmribs or frames. Such a structure plays the role of additional constraints preventing a change in the configuration of the contour of the cross section in its plane. Therefore, for example in analyzing cylindrical shells with a lateral structure for a load whose normal and transverse components are fairly smooth in character, the hypothesis of nondeformability of the cross section in its plane is very effective. Extending this hypothesis to conical shells of arbitrary configuration, in which the orientation of the diaphragms may, generally speaking, be different, one would have to assume the warping vector U^1 to be directed perpendicular to the plane of the diaphragms. A computational model of this kind will also be discussed below, but, as has been stated in several papers* and shown by studies made by the authors, the results of the analysis depend little on how the vector $U^1(\bar{Z}, S)$ is directed, whether parallel to the generatrices or perpendicular to the plane of the diaphragms. In particular, this leads to the well-known conclusion that for

*See for example L.I. Balabukh, Strength Analysis of Conical Torsion Boxes. Trudy TsAGI, 1947, No. 640.

most cases of loading, the stressed and strained state of wing-type shells depends little on the orientation of the ribs. In this connection, of decisive importance is the fact that our fundamental adopted hypothesis leads to simpler equations.

Here is one more proof in favor of the adopted hypothesis.

In solving the problem in the St.-Venant formulation*, it is shown that for a conical shell of arbitrary configuration, the displacements of the cross section, which is assumed to be the line of intersection of the middle surface with a sphere whose center is at the apex of the cone, consist of the displacements of the cross section as a solid and of its warpings in the direction of the generatrices. Therefore, the geometric model which we formulated can be regarded as the result of a natural transposition of certain properties of the solution in the St.-Venant formulation to the case in which the character of the application of the external load and the detailed fixing conditions are considered. This method is widely applied when the structure of the solution of a comparatively simple problem is used to construct an approximate solution of a more complex problem.

Thus, the analysis of a conical shell of arbitrary configuration on the basis of the formulated simplified geometric model amounts to the determination of two vector functions $\eta(\bar{Z})$ and $\theta(\bar{Z})$ of one independent variable and one scalar function $\Omega(\bar{Z}, S)$ of two variables. These functions in accordance with (2.6), (2.10), (2.22) completely define the deformed surface of the shell, and along with it, all the components of the stressed and strained state. We will therefore call them the fundamental kinematic unknowns. Let us also note that in (2.7), the pole r_0 can be chosen completely arbitrarily, and the vectors η and θ can be represented in the form of linear combinations of arbitrary nonorthogonal basis vectors. This permits one to raise the question of selection of the canonical unknowns determining the simplest structure of the resolvents, in contrast to the equations corresponding to the natural choice (2.9), (2.11) of the pole and basis vectors.

Let turn to the formulation of static hypotheses.

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*See footnote on p.

The interaction of the shell with the diaphragms is reduced to unknown contact forces applied to the shell in the planes of the diaphragms. We will proceed from the replacement of the discrete arrangement of the diaphragms by their continuous distribution. In this case, in the framework of the formulated geometric model, the contact forces, like the transverse forces Q_{m_z}, Q_{n_z}, constitute purely static factors. Introducing the contact forces together with the given external load into equilibrium differential equations (2.1), we obtain, as in (2.4), expressions establishing on the one hand, the relationship between the components of contact forces, and on the other hand, the components of internal forces in the shell proper as well as the components of the given external load. If we now eliminate the transverse forces and both components of contact forces from the equilibrium differential equations, we arrive at a single equilibrium equation in components of internal forces of the shell proper. Totthis equation must be added six equilibrium conditions of the finite portion of the shell cut off by the section parallel to the plane of the diaphragms. As we know, these equations are the first integrals of system (2.1), but hereinafter they will be obtained directly, this being easier to visualize. Further, using elasticity relations (2.3), geometric relations (2.2), and expressions (2.4), we obtain a complete system of equations in the fundamental kinematic unknowns η_x, η_y, η_z, θ_x, θ_y, θ_z and Ω. The first is a partial differential equation, and the others are integro-differential equations.

We will assume that the interaction of the shell with the diaphragms reduces essentially to tangential contact forces $q(\overline{Z}, S)$ applied to the shell in the direction of tangents to the contours of the diaphragms. Then, neglecting the other component of the contact forces, we represent the components $p_{m_z}(\overline{Z}, S)$ and $p_{n_z}(\overline{Z}, S)$ of the surface load acting on the shell in the form

$$p_{m_z} = p^0_{m_z} + q\cos\beta, \quad p_{n_z} = p^0_{n_z} + q\sin\beta, \tag{2.23}$$

where $p^0_{m_z}(\overline{Z}, S)$, $p^0_{n_z}(\overline{Z}, S)$ are the components of the given surface load;

$\beta(\bar{Z}, S)$ is the angle between the generatrix and the line of intersection of the middle surface by the plane of the diaphragms.

Introducing (2.23) into the second equation of system (2.1), we find

$$q = \frac{1}{AB\sin\beta}\left[\operatorname{ctg}\chi\frac{\partial}{\partial\bar{Z}}(BT_{n_z}) - \frac{1}{\sin\chi}\frac{\partial}{\partial S}(AT_{n_z}) - \frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z}) + S_{n_z m_z} + \frac{A}{R_0}Q_{n_z}\right] - \frac{p^0_{n_z}}{\sin\beta}. \tag{2.24}$$

In the adopted geometric model, a decisive role is played by tangential force factors, since comparative calculations show that the influence on the stressed and strained state of force factors connected with bending of the middle surface is usually unimportant. In turn, among the tangential force factors, the main ones are usually the normal forces T_{m_z} and tangential forces $S_{n_z m_z}$. Therefore, in the first two equations of system (2.1), one can neglect the transverse forces Q_{n_z} and normal tangential forces T_{n_z}. Then, (2.23) being taken into account, these equations become

$$\begin{gathered}\sin\chi\frac{\partial}{\partial\bar{Z}}(BT_{m_z}) - \cos\chi\frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z}) + \frac{\partial}{\partial S}(AS_{n_z m_z}) + \\ + AB\sin\chi(p^0_{m_z} + q\cos\beta) = 0, \\ \frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z}) - S_{n_z m_z} + AB(p^0_{n_z} + q\sin\beta) = 0,\end{gathered} \tag{2.25}$$

and the forces q, represented by expression (2.24), will be

$$q = \frac{1}{\sin\beta}\left\{\frac{1}{AB}\left[S_{n_z m_z} - \frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z})\right] - p^0_{n_z}\right\}. \tag{2.26}$$

Eliminating the forces q with the aid of (2.26) from the first equation of (2.25), we obtain

$$\begin{gathered}\sin\chi\frac{\partial}{\partial\bar{Z}}(BT_{m_z}) - (\operatorname{ctg}\beta\sin\chi + \cos\chi)\frac{\partial}{\partial\bar{Z}}(BS_{n_z m_z}) + \\ + \frac{\partial}{\partial S}(AS_{n_z m_z}) + \operatorname{ctg}\beta\sin\chi S_{n_z m_z} = AB\sin\chi(\operatorname{ctg}\beta p^0_{n_z} - p^0_{m_z}).\end{gathered} \tag{2.27}$$

Expression (2.27) represents the fundamental differential equation of equilibrium for our adopted static-geometric model of a reinforced conical shell of arbitrary configuration. This equation contains two unknown forces T_{m_z} and $S_{n_z m_z}$, indicating that the adopted model is statically indeterminate.

Equation (2.27) has variable coefficients dependent on both $\bar{Z}$ and S, and therefore its integration involves major difficulties. If the diaphragms are oriented parallel to the plane of the generatrix, then

$$\beta(\bar{Z}, S) = \chi(S), \tag{2.28}$$

as a result of which Eq. (2.27) becomes somewhat simplified, assuming the form

$$\sin\chi \frac{\partial}{\partial \bar{Z}}(BT_{m_z}) - 2\cos\chi \frac{\partial}{\partial \bar{Z}}(BS_{n_z m_z}) + A\frac{\partial S_{n_z m_z}}{\partial S} = -ABp^0_{n_z}, \tag{2.29}$$

where

$$p^0_{n_z} = p^0_{m_z}\sin\chi - p^0_{h_z}\cos\chi. \tag{2.30}$$

As before, the integration of this equation constitutes a very complex problem. A simpler equation is

$$\sin\chi \frac{\partial}{\partial \bar{Z}}(BT_{m_z}) - \cos\chi \frac{\partial}{\partial \bar{Z}}(BS_{n_z m_z}) + \frac{\partial}{\partial S}(AS_{n_z m_z}) = -AB\sin\chi p^0_{m_z}, \tag{2.31}$$

which formally results from the first equation of system (2.1) if in the latter one neglects the forces T_{n_z} and component q cos β of the reactive load exerted on the shell by the diaphragms in comparison with the forces T_{m_z} and $S_{n_z m_z}$. The following arguments can be adduced to substantiate Eq. (2.31). First, the lateral structure is usually very weak, so that the forces q are small. Secondly, practice shows that the stressed state of the shell proper changes comparatively little with changing orientation of the diaphragm, so that angle β may be considered close to π/2. In this case, Eq. (2.31) follows from (2.27).

Adding the equilibrium conditions of the end portion of the shell as a solid to Eq. (2.31) and changing to the fundamental kinematic unknowns in the system obtained, after integrating this system we will have the complete solution of the problem: the displacements are determined according to (2.6), (2.13), (2.14), (2.22), the strains according to (2.2), and the internal forces and moments, according to (2.3), (2.4), (2.24). Moreover, the first of elasticity relations (2.3) can be replaced by the simpler relation

$$T_{m_z} = Eh\varepsilon^0_{m_z}, \tag{2.32}$$

since, as already mentioned, $T_{n_z} \ll T_{m_z}$. Let us note that the tangential forces T_{n_z} and contact forces q, as well as, in particular, the transverse forces Q_{m_z}, Q_{n_z} and moment M_{m_z}, $H_{n_z m_z}$, M_{n_z}, discussed in the framework of the adopted computational model as secondary force factors, are obviously determined very roughly. However, having gotten an idea of their orders of magnitude, we are afforded an opportunity to evaluate the validity of the adopted hypotheses.

Neglecting the force factors related to bending of the middle surface, we thus assume that the stressed state of the shell, as in zero-moment theory, remains unchanged along its thickness. It should be noted that, nevertheless, the solution based on our adopted static-geometric model differs qualitatively from the solution according to zero-moment theory. As we know, according to this theory, the problem of determination of internal forces T_{m_z}, $S_{n_z m_z}$, T_{n_z} is statically indeterminate, since we have three equations of equilibrium in these forces. Consequently, it is easy to show that zero-moment theory does not yield damped solutions characteristic of the systems under consideration. Indeed, when $Q_{n_z} = Q_{m_z} = 0$ and in the absence of an external load, it is easy to obtain the following from the first three equations of system (2.1).

$$\begin{gathered}
S_{n_z m_z}(\bar{Z}, S) = \frac{1}{(1-\bar{Z})^2} S_{n_z m_z}(0, S), \\
T_{m_z}(\bar{Z}, S) = \frac{1}{1-\bar{Z}} \left[T_{m_z}(0, S) + \frac{2}{1-\bar{Z}} \operatorname{ctg}\chi S_{n_z m_z}(0, S) - \right. \\
\left. - \frac{A}{\sin\chi(1-\bar{Z})} \frac{\partial}{\partial S} S_{n_z m_z}(0, S) \right], \\
T_{n_z}(\bar{Z}, S) = 0,
\end{gathered}$$

whence, considering relations (1.148), we conclude that in a conical shell of arbitrary configuration, an undamaged system of internal forces corresponds to any system of boundary forces. This stands in direct contradiction to the forces of the systems under consideration, in which, as we know, the stressed state due to a self-balanced boundary load has a rapidly damping character.

The model allowing for the presence of lateral stiffeners does not have this disadvantage. In the framework of our computat ional model, the first three equilibrium equations, in addition to the components of tangential forces, contain the unknown forces of interaction of the lateral assembly with the shell. The problem proves to be statically indeterminate, so that damped solutions prove possible.

2.2. General System of Equilibrium Integro-Differential Equations of a Conical Shell of Arbitrary Configuration

As already noted, to equilibrium equation (2.34) it is necessary to add the equilibrium conditions of the end portion of the shell. Obviously, the simplest will be the equilibrium conditions of the end portion cut-off by the section $\bar{Z}$=const. Moreover, if the transverse diaphragms are not parallel to the plane of the directrix, the equilibrium equations of the cut-off portion of the shell will include, in addition to the forces in the shell proper, the forces in the diaphragms, which, however, are comparatively small and cannot be taken into consideration under the equilibrium conditions of the cut-off portion.

Let $\mathfrak{R}$ be the vector of the resultant of internal forces in the section $\bar{Z}$ = const, and $\mathfrak{M}$, the vector of the moment of these forces with respect to the point of intersection of the plane $\bar{Z}$ = const with the Oz axis. We have

$$\mathfrak{R}=\oint(\mathbf{T}_{n_s}+\mathbf{S}_{m_s n_s}+\mathbf{Q}_{n_s})B\,dS,$$
$$\mathfrak{M}=\oint[-(\mathbf{T}_{n_s}+\mathbf{S}_{m_s n_s}+\mathbf{Q}_{n_s})\times(\mathbf{r}-\mathbf{r}_0)+\mathbf{M}_{n_s}+\mathbf{H}_{n_s}]\,B\,dS, \tag{2.33}$$

where the vectors of the forces T_{n_s}, $S_{n_s m_s}$, Q_{n_s} and moments M_{n_s}, H_{n_s} are represented by expressions (1.151), (1.154), and radius vector r of a point on the contour

$\bar{Z}$ = const, and radius vector r_0 of the point of intersection of the plane of this contour with the Oz axis are represented by expressions (2.8), (2.9). The integral in (2.33) is extended to the entire contour of the directrix.

Expressions (2.33) pertain to a section, the outer normal to which makes an acute angle with the positive direction of the Oz axis. Therefore, the equilibrium conditions of the cut-off portion have the form

$$-\mathfrak{R}+\mathfrak{R}^{*}=0, \quad -\mathfrak{M}+\mathfrak{M}^{*}=0, \tag{ .34}$$

where $\mathfrak{R}^{*}$, $\mathfrak{M}^{*}$ are vectors of the resultant and net moment of the external forces applied to the cut-off portion of the shell:

$$\begin{gathered}
\mathfrak{R}^{*}=\mathfrak{R}_1^0+\int_{\bar{Z}}^{\bar{Z}_1}\oint\left[p^0_{m_z}(\xi, S)\,\mathbf{m}_z(S)+p^0_{n_z}(\xi, S)\,\mathbf{n}_z(S)+p^0_{n_n}(\xi, S)\,\mathbf{n}_n(S)\right]\times \\
\times AB\sin\gamma\, dS\, d\xi, \\
\mathfrak{M}^{*}=\mathfrak{M}_1^0-\mathfrak{R}_1^0\times\left[\mathbf{r}_0(\bar{Z}_1)-\mathbf{r}_0(\bar{Z})\right]- \\
-\int_{\bar{Z}}^{\bar{Z}_1}\oint\left[p^0_{m_z}(\xi, S)\,\mathbf{m}_z(S)+p^0_{n_z}(\xi, S)\,\mathbf{n}_z(S)+p^0_{n_n}(\xi, S)\,\mathbf{n}_n(S)\right]\times \\
\times\left[\mathbf{r}(\xi, S)-\mathbf{r}_0(\bar{Z})\right]AB\sin\chi\, dS\, d\xi.
\end{gathered} \tag{2.35}$$

Here $\mathfrak{R}_1^0$, $\mathfrak{M}_1^0$ are vectors of the resultant and neg moment of the external forces applied to the end $\bar{Z} = \bar{Z}_1$.

Expressions (2.33) contain the components of linear forces and moments in axes of the auxiliary moving trihedron. Switching to the components in axes of the main trihedron with the aid of (1.191), we have

$$\begin{gathered}
\mathfrak{R}=\oint\left[(\mathbf{T}_{m_z}+\mathbf{S}_{m_z}+\mathbf{Q}_{m_z})\sin\chi-(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cos\chi\right]B\,dS, \\
\mathfrak{M}=\oint\{-\left[(\mathbf{T}_{m_z}+\mathbf{S}_{m_z}+\mathbf{Q}_{m_z})\sin\chi-(\mathbf{T}_{n_z}+\mathbf{S}_{n_z}+\mathbf{Q}_{n_z})\cos\chi\right]\times(\mathbf{r}-\mathbf{r}_0)+ \\
+(\mathbf{M}_{m_z}+\mathbf{H}_{m_z})\sin\chi-(\mathbf{M}_{n_z}+\mathbf{H}_{n_z})\cos\chi\}\,B\,dS.
\end{gathered} \tag{2.36}$$

whence, considering (1.151), (1.154) and (1.192), we obtain

$$\begin{gathered}
\mathfrak{R}=\oint[(T_{m_z}\sin\chi-S_{n_z}\cos\chi)\mathbf{m}_z+(S_{m_z}\sin\chi-T_{n_z}\cos\chi)\mathbf{n}_z+\\
+(Q_{m_z}\sin\chi-Q_{n_z}\cos\chi)\mathbf{n}_n]\,BdS,\\
\mathfrak{M}=\oint\{-[(T_{m_z}\sin\chi-S_{n_z}\cos\chi)\mathbf{m}_z+(S_{m_z}\sin\chi-T_{n_z}\cos\chi)\mathbf{n}_z+\\
+(Q_{m_z}\sin\chi-Q_{n_z}\cos\chi)\mathbf{n}_n]\times(\mathbf{r}-\mathbf{r}_0)+(M_{n_z}\cos\chi-H_{m_z}\sin\chi)\mathbf{m}_z+\\
+(M_{m_z}\sin\chi-H_{n_z}\cos\chi)\mathbf{n}_z\}\,BdS.
\end{gathered} \tag{2.37}$$

Considering (2.8), (2.9) and expressions (2.19), one can readily obtain the following relations for an arbitrary unit vector t:

$$\mathbf{t}=\varphi_{1t}\mathbf{i}+\varphi_{2t}\mathbf{j}+\varphi_{3t}\mathbf{k}, \tag{2.38}$$

$$\begin{gathered}
\mathbf{t}\times(\mathbf{r}-\mathbf{r}_0)=-(1-\bar{Z})(\varphi_{4t}\mathbf{i}+\varphi_{5t}\mathbf{j}+\varphi_{6t}\mathbf{k}),\\
\mathbf{t}\times[\mathbf{r}(\xi,S)-\mathbf{r}_0(\bar{Z})]=\mathbf{t}\times[\mathbf{r}(\xi,S)-\mathbf{r}_0(\xi)]+\mathbf{t}\times[\mathbf{r}_0(\xi)-\mathbf{r}_0(\bar{Z})]=\\
=-(1-\xi)(\varphi_{4t}\mathbf{i}+\varphi_{5t}\mathbf{j}+\varphi_{6t}\mathbf{k})+(\xi-\bar{Z})l_0(\varphi_{2t}\mathbf{i}-\varphi_{1t}\mathbf{j}).
\end{gathered} \tag{2.39}$$

We will expand expressions (2.35), (2.37) with the aid of relations (2.38), (2.39). Superposing t on the unit vectors of the main trihedron, we obtain from (2.35)

$$\begin{gathered}
\mathfrak{R}''=Q_x(\bar{Z})\mathbf{i}+Q_y(\bar{Z})\mathbf{j}+N_z(\bar{Z})\mathbf{k},\\
\mathfrak{M}''=M_x(\bar{Z})\mathbf{i}+M_y(\bar{Z})\mathbf{j}+M_z(\bar{Z})\mathbf{k},
\end{gathered} \tag{2.40}$$

there

$$\begin{gathered}
Q_x(\bar{Z})=Q_{1x}+\int_{\bar{Z}}^{\bar{Z}_1}R_1(\xi)\,d\xi,\\
Q_y(\bar{Z})=Q_{1y}+\int_{\bar{Z}}^{\bar{Z}_1}R_2(\xi)\,d\xi,\\
N_z(\bar{Z})=N_{1z}+\int_{\bar{Z}}^{\bar{Z}_1}R_3(\xi)\,d\xi,
\end{gathered} \tag{2.41}$$

$$\begin{gathered}
M_x(\bar{Z})=M_{1x}+\int_{\bar{Z}}^{\bar{Z}_1}[R_4(\xi)-l_0Q_y(\xi)]\,d\xi,\\
M_y(\bar{Z})=M_{1y}+\int_{\bar{Z}}^{\bar{Z}_1}[R_5(\xi)+l_0Q_x(\xi)]\,d\xi,\\
M_z(\bar{Z})=M_{1z}+\int_{\bar{Z}}^{\bar{Z}_1}R_6(\xi)\,d\xi,
\end{gathered} \tag{2.42}$$

where

$$Q_{1x}=\mathfrak{R}_1^0\cdot\mathbf{i},\quad Q_{1y}=\mathfrak{R}_1^0\cdot\mathbf{j},\quad N_{1z}=\mathfrak{R}_1^0\cdot\mathbf{k},$$
$$M_{1x}=\mathfrak{M}_1^0\cdot\mathbf{i},\quad M_{1y}=\mathfrak{M}_1^0\cdot\mathbf{j},\quad M_{1z}=\mathfrak{M}_1^0\cdot\mathbf{k},\tag{2.43}$$

$$R_l=\lambda_l(1-\bar{Z})\oint[p^0_{m_z}\varphi_{lm_z}+p^0_{n_z}\varphi_{ln_z}+p^0_{n_n}\varphi_{ln_n}]l_z\sin\chi\,dS\tag{2.44}$$
$$(l=1,\,2,\ldots,\,6),$$

$$\lambda_l=1\qquad(l=1,\,2,\,3),$$
$$\lambda_l=1-\bar{Z}\quad(l=4,\,5,\,6).\tag{2.45}$$

From (2.37) we also have

$$\mathfrak{R}=(1-\bar{Z})\Big[\mathbf{i}\oint(T\varphi_{1m_z}+S\varphi_{1n_z}+Q\varphi_{1n_n})dS+$$
$$+\mathbf{j}\oint(T\varphi_{2m_z}+S\varphi_{2n_z}+Q\varphi_{2n_n})\,dS+\mathbf{k}\oint(T\varphi_{3m_z}+S\varphi_{3n_z}+Q\varphi_{3n_n})dS\Big],$$
$$\mathfrak{M}=(1-\bar{Z})\Big\{\mathbf{i}\Big[(1-\bar{Z})\oint(T\varphi_{4m_z}+S\varphi_{4n_z}+Q\varphi_{4n_n})dS+\tag{2.46}$$
$$+\oint(H\varphi_{1m_z}+M\varphi_{1n_z})dS\Big]+\mathbf{j}\Big[(1-\bar{Z})\oint(T\varphi_{5m_z}+S\varphi_{5n_z}+Q\varphi_{5n_n})\,dS+$$
$$+\oint(H\varphi_{2m_z}+M\varphi_{2n_z})dS\Big]+\mathbf{k}\Big[(1-\bar{Z})\oint(T\varphi_{6m_z}+S\varphi_{6n_z}+Q\varphi_{6n_n})dS+$$
$$+\oint(H\varphi_{3m_z}+M\varphi_{3n_z})dS\Big]\Big\},$$

where

$$T=T_{m_z}\sin\chi-S_{n_z}\cos\chi,$$
$$S=S_{m_z}\sin\chi-T_{n_z}\cos\chi,$$
$$Q=Q_{m_z}\sin\chi-Q_{n_z}\cos\chi,$$
$$M=M_{m_z}\sin\chi-H_{n_z}\cos\chi,$$
$$H=M_{n_z}\cos\chi-H_{m_z}\sin\chi.\tag{2.47}$$

Now, using (2.40) and (2.46), from the equilibrium conditions of the cut-off portion of the shell (2.34) we find

$$(1-\bar{Z})\oint(T\varphi_{1m_z}+S\varphi_{1n_z}+Q\varphi_{1n_n})dS=Q_x(\bar{Z}),$$
$$(1-\bar{Z})\oint(T\varphi_{2m_z}+S\varphi_{2n_z}+Q\varphi_{2n_n})\,dS=Q_y(\bar{Z}),$$
$$(1-\bar{Z})\oint(T\varphi_{3m_z}+S\varphi_{3n_z}+Q\varphi_{3n_n})dS=N_z(\bar{Z}),$$
$$(1-\bar{Z})^2\oint(T\varphi_{4m_z}+S\varphi_{4n_z}+Q\varphi_{4n_n})dS+$$
$$+(1-\bar{Z})\oint(H\varphi_{1m_z}+M\varphi_{1n_z})dS=M_x(\bar{Z}),\tag{2.48}$$
$$(1-\bar{Z})^2\oint(T\varphi_{5m_z}+S\varphi_{5n_z}+Q\varphi_{5n_n})dS+$$
$$+(1-\bar{Z})\oint(H\varphi_{2m_z}+M\varphi_{2n_z})dS=M_y(\bar{Z}),$$
$$(1-\bar{Z})^2\oint(T\varphi_{6m_z}+S\varphi_{6n_z}+Q\varphi_{6n_n})dS+$$
$$+(1-\bar{Z})\oint(H\varphi_{3m_z}+M\varphi_{3n_z})dS=M_z(\bar{Z}).$$

Expressions (2.48), which contain all the components of internal forces, are exact. Considering the static hypotheses formulated above, we have approximately

$$\oint [(T_{m_z} \sin \chi - S_{n_z} \cos \chi) \varphi_{im_z} + S_{m_z} \sin \chi \varphi_{in_z}] dS = \frac{1}{(1-\bar{Z})\lambda_i} P_i \qquad (i = 1, 2, \ldots, 6), \tag{2.49}$$

where

$$\begin{gathered} P_1 = Q_x(\bar{Z}), \quad P_2 = Q_y(\bar{Z}), \quad P_3 = N_z(\bar{Z}), \\ P_4 = M_x(\bar{Z}), \quad P_5 = M_y(\bar{Z}), \quad P_6 = M_z(\bar{Z}). \end{gathered} \tag{2.50}$$

Differential equation (2.39) and six integral equations (2.49) in the solution of the problem in displacements form a complete system of equations in the unknown functions θ_z, Ω, which in the framework of our adopted static-geometric model determine the deformed surface of the shell, and along with it, all the components of internal forces.

We first write Eqs. (2.31) and (2.49) in the operator form.

In accordance with expressions (1.54), (1.68), (2.3), (2.32), the forces T_{m_z} and $S_{n_z m_z}$ may be represented in the form

$$\begin{gathered} T_{m_z} = Eh \frac{\partial u_{m_z}}{\partial m_z}, \\ S_{n_z m_z} = \frac{Eh}{2(1+\nu)} \left[\frac{\partial u_{m_z}}{\partial n_z} + B \frac{\partial}{\partial m_z} \left(\frac{u_{n_z}}{B} \right) \right]. \end{gathered} \tag{2.51}$$

Under the assumptions we made, Eq. (2.31) coincides with the first equation of system (1.173) if one sets $T_{n_z} = 0$. Using representation (2.51), we have

$$\frac{2(1+\nu)}{B} \frac{\partial}{\partial m_z} \left(B \frac{\partial u_{m_z}}{\partial m_z} \right) + \frac{\partial^2 u_{m_z}}{\partial n_z^2} + \frac{\partial}{\partial n_z} \left[B \frac{\partial}{\partial m_z} \left(\frac{u_{n_z}}{B} \right) \right] + \frac{2(1+\nu)}{Eh} p^0_{m_z} = 0. \tag{2.52}$$

Introducing (2.51) into Eqs. (2.49), we also obtain

$$\begin{gathered} 2(1+\nu) \oint \frac{\partial u_{m_z}}{\partial m_z} \varphi_{im_z} \sin \chi h dS + \oint \left[\frac{\partial u_{m_z}}{\partial n_z} + B \frac{\partial}{\partial m_z} \left(\frac{u_{n_z}}{B} \right) \right] \varphi_i \sin \chi h dS = \\ = \frac{2(1+\nu)}{EB\lambda_i} P_i \qquad (i = 1, 2, \ldots, 6), \end{gathered} \tag{2.53}$$

where the functions $\psi_i(S)$ will be

$$\psi_i = \varphi_{in_z} - \mathrm{ctg}\,\chi\,\varphi_{im_z}. \tag{2.54}$$

or, taking (2.20) into account

$$\psi_i = \frac{1}{\sin\chi}\chi_{im_z} - 2\,\mathrm{ctg}\,\chi\,\varphi_{im_z}. \tag{2.55}$$

In accordance with expressions (2.6), (2.13) and (2.22), the displacements in the direction of the arbitrarily oriented unit vector t may be represented in the form

$$u_t(\bar{Z}, S) = \sum_{i=1}^{6} V_i(\bar{Z})\varphi_{it}(S) + \Omega(\bar{Z}, S)\,\mathbf{t}\cdot\mathbf{m}_z, \tag{2.56}$$

where the functions $\phi_{it}(S)$ are determined by expressions (2.19), and the functions $V_i(\bar{Z})$ are related to the desired components of displacement of the contour $\bar{Z}$ = const as a solid by relations (2.14).

From (2.56) we have

$$u_{m_z} = \sum_{i=1}^{6} V_i\varphi_{im_z} + \Omega, \tag{2.57}$$

$$u_{n_z} = \sum_{i=1}^{6} V_i\varphi_{in_z}. \tag{2.58}$$

Introducing the displacements u_{n_z} represented by expression (2.58) into Eqs. (2.52), (2.53), we obtain

$$\frac{2(1+\nu)}{B}\frac{\partial}{\partial \mathbf{m}_z}\left(B\frac{\partial u_{m_z}}{\partial \mathbf{m}_z}\right) + \frac{\partial^2 u_{m_z}}{\partial \mathbf{n}_z^2} + \sum_{i=1}^{6}\frac{\partial}{\partial \mathbf{n}_z}\left[\frac{B}{A}\left(\frac{V_i}{B}\right)'\varphi_{in_z}\right] = -\frac{2(1+\nu)}{Eh}p^0_{m_z}, \tag{2.59}$$

$$2(1+\nu)\oint \frac{\partial u_{m_z}}{\partial m_z}\varphi_{jm_z}\sin\chi h dS+\oint \frac{\partial u_{m_z}}{\partial n_z}\psi_j \sin\chi h dS+$$
$$+B\sum_{i=1}^{6}\left(\frac{V_i}{B}\right)'\oint \varphi_{in_z}\psi_j \frac{\sin\chi}{A} h dS=\frac{2(1+\nu)}{EB_j}P_j$$
$$(j=1,\ 2,\ \ldots,\ 6). \qquad (2.60)$$

Expressions (2.59), (2.60) constitute a complete system of integro-differential equations in the unknown function u_{m_z}, dependent on two variables, and six functions V_i of one variable. These functions are related to the basic kinematic unknowns by expressions (2.14) and (2.57).

As was noted above, the main problem is to integrate system (2.59), (2.60). Having determined $V_i(\bar{Z})$ (i=1, 2, ..., 6) and $u_{m_z}(\bar{Z}, S)$, and having further determined $\Omega(\bar{Z}, S)$ from relation (2.57), one can find the displacements and stresses at any point of the shell by using general expressions (2.56) and (2.2), (2.3).

2.3. Integro-Differential Resolvents

It is apparent that integro-differential equations (2.60) form a system of linear algebraic equations in the derivatives (V_i/B) (i=1, 2, ..., 6). Solving this system, we obtain from Cramer's formulas

$$B\left(\frac{V_i}{B}\right)'=\frac{1}{D}\sum_{j=1}^{6}A_{ji}\left[\frac{2(1+\nu)}{EB_j}P_j-2(1+\nu)\oint \frac{\partial u_{m_z}}{\partial m_z}\varphi_{jm_z}\sin\chi h dS-\right.$$
$$\left.-\oint \frac{\partial u_{m_z}}{\partial n_z}\psi_j \sin\chi h dS\right] \qquad (2.61)$$
$$(i=1,\ 2,\ \ldots,\ 6).$$

Here

D is the determinant of the matrix $||a_{ji}||$, where

$$a_{ji}=\oint \varphi_{in_z}\psi_j \frac{\sin\chi}{A} h dS; \qquad (2.62)$$

A_{ji} are the adjoints of the elements of determinant D.

Now, using expression (2.61), we eliminate from Eq. (2.59) the unknown functions of one variable $V_i(\bar{Z})$. We obtain

$$\frac{2(1+\nu)}{B}\frac{\partial}{\partial m_z}\left(B\frac{\partial u_{m_z}}{\partial m_z}\right)+\frac{\partial^2 u_{m_z}}{\partial n_z^2}-\\ -\frac{1}{D}\frac{\partial}{\partial n_z}\left\{\frac{1}{A}\sum_{i=1}^{6}\varphi_{in_z}\oint\left[a_i(S)\frac{\partial u_{m_z}}{\partial m_z}+b_i(S)\frac{\partial u_{m_z}}{\partial n_z}\right]\sin\chi h\,dS\right\}=L(\mathbf{p}), \tag{2.63}$$

where

$$a_i(S)=2(1+\nu)\sum_{j=1}^{6}A_{ji}\varphi_{jm_z},\\ b_i(S)=\sum_{j=1}^{6}A_{ji}\psi_j, \tag{2.64}$$

$$L(\mathbf{p})=-\frac{2(1+\nu)}{E}\left[\frac{p^0_{m_z}}{h}+\frac{1}{D}\frac{1}{AB}\frac{\partial}{\partial n_z}\sum_{i=1}^{6}\varphi_{in_z}\sum_{j=1}^{6}A_{ji}\frac{1}{\lambda_j}P_j\right].$$

Expression (2.63) is the integro-differential resolvent of a conical shell of arbitrary configuration in the displacement component $u_{m_z}(\bar{Z}, S)$. In expanded form, (1.11), (1.15), (1.16) being taken into account, this equation takes the form

$$\frac{2(1+\nu)}{AB}\frac{\partial}{\partial\bar{Z}}\left(\frac{B}{A}\frac{\partial u_{m_z}}{\partial\bar{Z}}\right)+\left(-\frac{1}{A}\operatorname{ctg}\chi\frac{\partial}{\partial\bar{Z}}+\frac{1}{B\sin\chi}\frac{\partial}{\partial S}\right)\times\\ \times\left(-\frac{1}{A}\operatorname{ctg}\chi\frac{\partial u_{m_z}}{\partial\bar{Z}}+\frac{1}{B\sin\chi}\frac{\partial u_{m_z}}{\partial S}\right)-\frac{1}{D}\left(-\frac{1}{A}\operatorname{ctg}\chi\frac{\partial}{\partial\bar{Z}}+\frac{1}{B\sin\chi}\frac{\partial}{\partial S}\right)\times\\ \times\left\{\frac{1}{A}\sum_{i=1}^{6}\varphi_{in_z}\oint\left[\frac{a_i}{A}\frac{\partial u_{m_z}}{\partial\bar{Z}}+b_i\left(-\frac{1}{A}\operatorname{ctg}\chi\frac{\partial u_{m_z}}{\partial\bar{Z}}+\frac{1}{B\sin\chi}\frac{\partial u_{m_z}}{\partial S}\right)\right]\sin\chi h\,dS\right\}=\\ =-\frac{2(1+\nu)}{E}\left\{\frac{p^0_{m_z}}{h}+\frac{1}{D}\frac{1}{AB}\sum_{i=1}^{6}\left[\left(-\frac{1}{A}\operatorname{ctg}\chi\frac{\partial}{\partial\bar{Z}}+\frac{1}{B\sin\chi}\frac{\partial}{\partial S}\right)\times\right.\right.\\ \left.\left.\times\,\varphi_{in_z}\sum_{j=1}^{6}A_{ji}\frac{1}{\lambda_j}P_j\right]\right\}. \tag{2.65}$$

Equation (2.65) has variable coefficients dependent on the $\bar{Z}$ coordinate as well as the S coordinate. For this reason, the integration of this equation constitutes a very complex problem solvable exactly only in special cases, the most important of which are right conical shells of revolution discussed in Part Three. Generally, however, this problem has to be solved approximately for conical shells of arbitrary configuration.

Integro-differential equation (2.65) can be reduced to a simpler form corresponding to the canonical form of partial differential equations. For this purpose, we represent (2.65) in the form

$$a_{11}\frac{\partial^2 u_{m_s}}{\partial \bar{Z}^2}+2a_{12}\frac{\partial^2 u_{m_s}}{\partial \bar{Z}\,\partial S}+a_{22}\frac{\partial^2 u_{m_s}}{\partial S^2}+F\left(\bar{Z},\ S,\ \frac{\partial u_{m_s}}{\partial \bar{Z}},\ \frac{\partial u_{m_s}}{\partial S}\right)+T(u_{m_s})=L(p), \tag{2.66}$$

where T is the integro-differential operator entering into Eq. (2.65).

From (2.65), we have

$$\begin{aligned} a_{11}&=\frac{1}{A^2}\left[2(1+\nu)+\mathrm{ctg}^2\chi\right],\\ a_{12}&=-\frac{1}{AB}\,\frac{\cos\chi}{\sin^2\chi},\\ a_{22}&=\frac{1}{B^2\sin^2\chi}. \end{aligned} \tag{2.67}$$

In view of (2.67), the equation of characteristics takes the form

$$\frac{d\bar{Z}}{dS}=\frac{B}{A}\cos\chi\pm i\sin\chi\sqrt{2(1+\nu)}, \tag{2.68}$$

whence, taking (1.11) into account, we find

$$\ln(1-\bar{Z})l_s\pm i\sqrt{2(1+\nu)}\int\frac{\sin\chi}{l_s}\,dS=\ln C. \tag{2.69}$$

Expressions (2.69) represent the complex characteristics of Eq. (2.66). Changing to the real variables $\alpha = \alpha(\bar{Z},\ S)$, $\beta = \beta(\bar{Z},\ S)$, we have

$$\alpha=(1-\bar{Z})\,l_s;\quad \beta=\sqrt{2(1+\nu)}\int\frac{\sin\chi}{l_s}\,dS. \tag{2.70}$$

It is easy to see that the coordinate grid corresponding to variables (2.70) coincides with the lines of principal curvatures of an arbitrary conical surface. The family of lines α = const consists of the lines of intersection of the arbitrary conical surface by spheres of radius $R = (1 - \overline{Z})l_s$ with the center at the apex of the cone. The lines β = const obviously determine the family of generatrices. The quantity $\beta/\sqrt{2(1+\nu)} = \int \sin\chi/l_s(dS)$ represents the angle between the current and the fixed generatrix, measured along the conical surface.

In the new variables, Eq (2.65) takes the form

$$\alpha\frac{\partial}{\partial\alpha}\left(\alpha\frac{\partial u_{m_z}}{\partial\alpha}\right)+\frac{\partial^2 u_{m_z}}{\partial\beta^2}-$$
$$-\frac{1}{D}\sum_{i=1}^{6}\left(\frac{\varphi_{in_z}}{A}\right)'\oint\left[-\frac{a_i}{\sqrt{2(1+\nu)}}\alpha\frac{\partial u_{m_z}}{\partial\alpha}+b_i\frac{\partial u_{m_z}}{\partial\beta}\right]hl_s d\beta=L(\mathbf{p}), \tag{2.71}$$

where

$$L(\mathbf{p})=-\frac{1}{E}\left[\frac{p_{m_z}}{h}\alpha^2+\frac{1}{D}\sqrt{2(1+\nu)}\sum_{i=1}^{6}\sum_{j=1}^{6}A_{ji}\frac{\partial}{\partial\beta}\left(\frac{\varphi_{in_z}}{\lambda_j}P_j\right)\right]. \tag{2.72}$$

Equation (2.71) is considerably simpler than Eq. (2.65); as before, it has coefficients dependent on both variables, but by making the substitution of variable

$$\alpha=e^t, \tag{2.73}$$

we arrive at an equation with variable coefficients dependent only one one variable, β:

$$\frac{\partial^2 u_{m_z}}{\partial t^2}+\frac{\partial^2 u_{m_z}}{\partial\beta^2}-\frac{1}{D}\sum_{i=1}^{6}\left(\frac{\varphi_{in_z}}{l_s}\right)'\oint\left[-\frac{a_i}{\sqrt{2(1+\nu)}}\frac{\partial u_{m_z}}{\partial t}+b_i\frac{\partial u_{m_z}}{\partial\beta}\right]hl_s d\beta=L(\mathbf{p}). \tag{2.74}$$

As already noted, Eq. (2.74) is integrated exactly when applied to right conical shells of revolution. In this case, the derivative $\partial u_{m_z}/\partial_t$ is absent from the integro-differential operator, and at the same time, the shell edges coincide with the coordinate lines t = const. This makes it possible to apply the method of separation of variables to Eq. (2.74). The corresponding homogeneous problem has an infinite spectrum of orthogonal eigenfunctions. For conical shells of arbitrary configuration, the problem is complicated not only by the fact that the derivative $\partial u_{m_z}/\partial_t$ is present in the integro-differential operator of Eq. (2.74), but also by the fact that the boundary conditions must be set at edges that do not coincide with the coordinate lines. The latter fact causes the greatest difficulties characterizing the calculation of oblique systems.

Equations (2.74) can also be extended to the case of cylindrical shells by means of a limiting transition. In this case, our adopted static-geometric model defines a shell with a cross-sectional contour nondeformable in its plane. For right cylindrical shells, the solution on the basis of Eq. (2.74) in the absence of a longitudinal structure essentially coincides with the solution obtained by R.A. Adadurov.**

2.4. Canonical Kinematic Unknowns

For the basic kinematic unknowns selected above, the structure of Eqs. (2.34) expressing the equilibrium conditions of the cut-off portion of the shell turns out to be very complex. In expanded form, these equations, written for a conical shell of arbitrary configuration, form a coupled system of six integro-differential equations containing, in addition to the basic kinematic unknowns, also their first derivatives. The present section will discuss the question of the choice of the canonical kinematic unknowns which define the simplest possible structure of this system.* We will not neglect the tangential forces T_{n_z}, since in shells of marked conicity these forces are appreciable in certain regions.

On the basis of the adopted geometrical hypotheses, the elastic displacement vector may be represented in the form

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*G.G. Onanov, Canonical Kinematic Unknowns of the Generalized Law of Plane Sections. In: Strength and Stability of Thin-Walled Aeronautical Structures. Trudy MAI, No. 180. Moscow, Mashinostroyeniye, 1970.

**R.A. Adadurov. Stresses and Strains in a Cylindrical Shell with Rigid Cross Sections. DAN, 62, No. 2, 1948.

$$\mathbf{U}(\bar{Z}, S) = \boldsymbol{\eta}(\bar{Z}) + \boldsymbol{\theta}(\bar{Z}) \times [\mathbf{r}(\bar{Z}, S) - \mathbf{r}_0(\bar{Z})] + \Omega(\bar{Z}, S)\, \mathbf{m}_z(S). \tag{2.75}$$

Taking (1.54), (1.68), (2.3) into consideration, we can represent the components of tangential forces T_{m_z}, $S_{n_z m_z}$, T_{n_z} in vector notation:

$$\begin{aligned}
T_{m_z} &= \frac{Eh}{1-\nu^2}\left\{\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot\mathbf{m}_z) + \nu\left[\frac{\partial}{\partial \mathbf{n}_z}(\mathbf{U}\cdot\mathbf{n}_z) - \frac{1}{AB}(\mathbf{U}\cdot\mathbf{m}_z) - \frac{1}{R}(\mathbf{U}\cdot\mathbf{n}_R)\right]\right\},\\
S_{n_z m_z} &= \frac{Eh}{2(1+\nu)}\left[\frac{\partial}{\partial \mathbf{n}_z}(\mathbf{U}\cdot\mathbf{m}_z) + B\frac{\partial}{\partial \mathbf{m}_z}\left(\frac{\mathbf{U}\cdot\mathbf{n}_z}{B}\right)\right],\\
T_{n_z} &= \frac{Eh}{1-\nu^2}\left[\frac{\partial}{\partial \mathbf{n}_z}(\mathbf{U}\cdot\mathbf{n}_z) - \frac{1}{AB}(\mathbf{U}\cdot\mathbf{m}_z) - \frac{1}{R}(\mathbf{U}\cdot\mathbf{n}_R) + \nu\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot\mathbf{m}_z)\right].
\end{aligned} \tag{2.76}$$

For the derivatives entering into (2.76), using representation (2.75) and relations (1.52), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot\mathbf{m}_z) &= \left[\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{m}_z} + \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{m}_z}\times\boldsymbol{\rho} + \boldsymbol{\theta}\times\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{m}_z}\right]\cdot\mathbf{m}_z + \frac{\partial \Omega}{\partial \mathbf{m}_z},\\
\frac{\partial}{\partial \mathbf{n}_z}(\mathbf{U}\cdot\mathbf{m}_z) &= \left[\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{n}_z} + \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{n}_z}\times\boldsymbol{\rho} + \boldsymbol{\theta}\times\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{n}_z}\right]\cdot\mathbf{m}_z - \frac{1}{AB}(\boldsymbol{\eta}+\boldsymbol{\theta}\times\boldsymbol{\rho})\cdot\mathbf{n}_z + \frac{\partial \Omega}{\partial \mathbf{n}_z},\\
\frac{\partial}{\partial \mathbf{m}_z}(\mathbf{U}\cdot\mathbf{n}_z) &= \left[\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{m}_z} + \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{m}_z}\times\boldsymbol{\rho} + \boldsymbol{\theta}\times\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{m}_z}\right]\cdot\mathbf{n}_z,\\
\frac{\partial}{\partial \mathbf{n}_z}(\mathbf{U}\cdot\mathbf{n}_z) &= \left[\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{n}_z} + \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{n}_z}\times\boldsymbol{\rho} + \boldsymbol{\theta}\times\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{n}_z}\right]\cdot\mathbf{n}_z +\\
&\quad + (\boldsymbol{\eta}+\boldsymbol{\theta}\times\boldsymbol{\rho})\cdot\left(\frac{1}{AB}\mathbf{m}_z + \frac{1}{R}\mathbf{n}_R\right),
\end{aligned} \tag{2.77}$$

where

$$\boldsymbol{\rho}(\bar{Z}, S) = \mathbf{r}(\bar{Z}, S) - \mathbf{r}_0(\bar{Z}) \tag{2.78}$$

is the radius vector of the point $M(\bar{Z}, S)$ relative to some pole $M_0(\bar{Z})$ rigidly bound to the section $\bar{Z}$ = const.

Expressions (2.77), corresponding to an arbitrarily chosen pole $M_0(\bar{Z})$, are general in character. These expressions, and hence, expressions (2.76) for the components of tangential forces contain both the functions $\eta(\bar{Z})$, $\theta(\bar{Z})$ themselves, and their derivatives. However, by selecting the pole $M_0(\bar{Z})$ in a certain manner, expressions (2.76) can be simplified, as already stated in the formulation of a simplified geometric model of the shell.

In Section 2.1, the point of intersection of the plane of this contour with the Oz axis was chosen as the pole of the contour $\bar{Z}$ = const, as is done in the analysis of beams. This fact ultimately leads to a complex structure of the equilibrium equations of the cut-off portion of the shell, since, as before, the expressions for the forces contain both the kinematic unknowns and their derivatives.

We will now place the pole, corresponding to displacements of the section $\bar{Z}$ = const as a solid, at some fixed point common to all the sections. In this case, the radius vector $\mathbf{r}_0$ is constant. Differentiating relative radius vector (2.78), we have

$$\frac{\partial \boldsymbol{\rho}}{\partial m_z} = \mathbf{m}_z; \quad \frac{\partial \boldsymbol{\rho}}{\partial n_z} = \mathbf{n}_z. \tag{2.79}$$

In view of (2.79), expressions (2.77) take the form

$$\begin{aligned}
&\frac{\partial}{\partial m_z}(\mathbf{U}\cdot\mathbf{m}_z) = \left(\frac{\partial \boldsymbol{\eta}}{\partial m_z} + \frac{\partial \boldsymbol{\theta}}{\partial m_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{m}_z + \frac{\partial \Omega}{\partial m_z},\\
&\frac{\partial}{\partial n_z}(\mathbf{U}\cdot\mathbf{m}_z) = \left(\frac{\partial \boldsymbol{\eta}}{\partial n_z} + \frac{\partial \boldsymbol{\theta}}{\partial n_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{m}_z - \boldsymbol{\theta}\cdot\mathbf{n}_n - \frac{1}{AB}(\boldsymbol{\eta}+\boldsymbol{\theta}\times\boldsymbol{\rho})\cdot\mathbf{n}_z + \frac{\partial \Omega}{\partial n_z},\\
&\frac{\partial}{\partial m_z}(\mathbf{U}\cdot\mathbf{n}_z) = \left(\frac{\partial \boldsymbol{\eta}}{\partial m_z} + \frac{\partial \boldsymbol{\theta}}{\partial m_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{n}_z + \boldsymbol{\theta}\cdot\mathbf{n}_n,\\
&\frac{\partial}{\partial n_z}(\mathbf{U}\cdot\mathbf{n}_z) = \left(\frac{\partial \boldsymbol{\eta}}{\partial n_z} + \frac{\partial \boldsymbol{\theta}}{\partial n_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{n}_z + (\boldsymbol{\eta}+\boldsymbol{\theta}\times\boldsymbol{\rho})\left(\frac{1}{AB}\mathbf{m}_z + \frac{1}{R}\mathbf{n}_n\right).
\end{aligned} \tag{2.80}$$

Introducing (2.75) and (2.80) into expressions (2.76), we find

$$\begin{aligned}
&T_{m_z} = \frac{Eh}{1-\nu^2}\left[\left(\frac{\partial \boldsymbol{\eta}}{\partial m_z} + \frac{\partial \boldsymbol{\theta}}{\partial m_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{m}_z + \nu\left(\frac{\partial \boldsymbol{\eta}}{\partial n_z} + \frac{\partial \boldsymbol{\theta}}{\partial n_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{n}_z + \frac{\partial \Omega}{\partial m_z}\right],\\
&S_{n_z m_z} = \frac{Eh}{2(1+\nu)}\left[\left(\frac{\partial \boldsymbol{\eta}}{\partial n_z} + \frac{\partial \boldsymbol{\theta}}{\partial n_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{m}_z + \left(\frac{\partial \boldsymbol{\eta}}{\partial m_z} + \frac{\partial \boldsymbol{\theta}}{\partial m_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{n}_z + \frac{\partial \Omega}{\partial n_z}\right],\\
&T_{n_z} = \frac{Eh}{1-\nu^2}\left[\left(\frac{\partial \boldsymbol{\eta}}{\partial n_z} + \frac{\partial \boldsymbol{\theta}}{\partial n_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{n}_z + \nu\left(\frac{\partial \boldsymbol{\eta}}{\partial m_z} + \frac{\partial \boldsymbol{\theta}}{\partial m_z}\times\boldsymbol{\rho}\right)\cdot\mathbf{m}_z + \nu\frac{\partial \Omega}{\partial m_z}\right].
\end{aligned} \tag{2.81}$$

It is apparent that expressions (2.81) corresponding to an arbitrary position of the poles but common to all the sections, contain only the derivatives of vector functions $\eta(\bar{Z})$, $\theta(\bar{Z})$, which ultimately, simplifies the reslvents considerably.

On the basis of (1.15), (1.16), we have

$$\frac{\partial \eta}{\partial n_z} = -\mathrm{ctg}\,\chi \frac{\partial \eta}{\partial m_z}; \quad \frac{\partial \theta}{\partial n_z} = -\mathrm{ctg}\,\chi \frac{\partial \theta}{\partial m_z}. \tag{2.82}$$

Introducing (2.82) into (2.81), we obtain

$$\begin{aligned} T_{m_z} &= \frac{Eh}{1-\nu^2}\left[\left(\frac{\partial \eta}{\partial m_z} + \frac{\partial \theta}{\partial m_z} \times \varrho\right)\cdot(\mathbf{m}_z - \nu\,\mathrm{ctg}\,\chi\,\mathbf{n}_z) + \frac{\partial \Omega}{\partial m_z}\right], \\ S_{n_z m_z} &= \frac{Eh}{2(1+\nu)}\left[\left(\frac{\partial \eta}{\partial m_z} + \frac{\partial \theta}{\partial m_z} \times \varrho\right)\cdot(-\mathrm{ctg}\,\chi\,\mathbf{m}_z + \mathbf{n}_z) + \frac{\partial \Omega}{\partial n_z}\right], \\ T_{n_z} &= \frac{Eh}{1-\nu^2}\left[\left(\frac{\partial \eta}{\partial m_z} + \frac{\partial \theta}{\partial m_z} \times \varrho\right)\cdot(-\mathrm{ctg}\,\chi\,\mathbf{n}_z + \nu\mathbf{m}_z) + \nu\frac{\partial \Omega}{\partial m_z}\right]. \end{aligned} \tag{2.83}$$

We will now place the pole of each of the sections $\overline{Z}$ = const at the cone apex. In this case, the radius vector of a point belonging to the middle surface of the conical shell, with respect to the apex of the shell and (1.11), being taken into account, will be

$$\varrho(Z, S) = -AB\mathbf{m}_z. \tag{2.84}$$

Introducing (2.84) into (2.83), we finally obtain

$$\begin{aligned} T_{m_z} &= \frac{Eh}{1-\nu^2}\left[\frac{\partial \eta}{\partial m_z}\cdot(\mathbf{m}_z - \nu\,\mathrm{ctg}\,\chi\,\mathbf{n}_z) + \nu AB\,\mathrm{ctg}\,\chi \frac{\partial \theta}{\partial m_z}\cdot\mathbf{n}_z + \frac{\partial \Omega}{\partial m_z}\right], \\ S_{n_z m_z} &= \frac{Eh}{2(1+\nu)}\left[\frac{\partial \eta}{\partial m_z}\cdot(-\mathrm{ctg}\,\chi\,\mathbf{m}_z + \mathbf{n}_z) - AB\frac{\partial \theta}{\partial m_z}\cdot\mathbf{n}_z + \frac{\partial \Omega}{\partial n_z}\right], \\ T_{n_z} &= \frac{Eh}{1-\nu^2}\left[\frac{\partial \eta}{\partial m_z}\cdot(-\mathrm{ctg}\,\chi\,\mathbf{n}_z + \nu\mathbf{m}_z) + AB\,\mathrm{ctg}\,\chi \frac{\partial \theta}{\partial m_z}\cdot\mathbf{n}_n + \nu\frac{\partial \Omega}{\partial m_z}\right]. \end{aligned} \tag{2.85}$$

Expressions (2.85), contain the derivatives of two unknown vector functions $\eta(\overline{Z})$, $\theta(\overline{Z})$ and one scalar function $\Omega(\overline{Z}, S)$. In order to switch to expressions containing only scalar unknowns, it is necessary to establish the correspondence between vector functions $\eta(\overline{Z})$, $\theta(\overline{Z})$ and their affine coordinates.

Let e_1, e_2, e_3 be some basis triple of noncoplanar vectors. An arbitrary vector a may be represented as a linear combination of basis vectors:

$$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3. \tag{2.86}$$

where a^1, a^2, a^3 are the affine coordinates of vector a.

Using the notation adopted in tensor calculus, we write expression (2.86) in the contracted form

$$\mathbf{a} = a^{\alpha}\mathbf{e}_{\alpha}, \tag{2.87}$$

where the repeated index (superscript and subscript) denoted by a Greek letter signifies summation over this index from 1 to 3.

Together with the basis e_1, e_2, e_3, it is necessary to introduce the reciprocal basis defined by the relations

$$\mathbf{e}^1 = k(\mathbf{e}_2 \times \mathbf{e}_3), \quad \mathbf{e}^2 = k(\mathbf{e}_3 \times \mathbf{e}_1), \quad \mathbf{e}^3 = k(\mathbf{e}_1 \times \mathbf{e}_2), \tag{2.88}$$

where

$$k = \frac{1}{\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3}.$$

Obviously, the vectors of the main and reciprocal bases satisfy the relations

$$\mathbf{e}^{\beta} \cdot \mathbf{e}_{\alpha} = \begin{cases} 0, & \alpha \neq \beta, \\ 1, & \alpha = \beta, \end{cases} \quad (\alpha, \beta = 1, 2, 3). \tag{2.89}$$

Multiplying (2.87) scalarly by the vector of the reciprocal basis e^{β} and taking (2.89) into account, we obtain

$$a^{\beta} = \mathbf{a} \cdot \mathbf{e}^{\beta} \quad (\beta = 1, 2, 3). \tag{2.90}$$

Vector a can also be represented in the form of a linear combination of vectors of the reciprocal basis:

$$\mathbf{a} = a_1\mathbf{e}^1 + a_2\mathbf{e}^2 + a_3\mathbf{e}^3 = a_{\alpha}\mathbf{e}^{\alpha}, \tag{2.91}$$

where the affine coordinates a_{α} in correspondence with (2.89) will be

$$a_\alpha = \mathbf{a}\cdot\mathbf{e}_\alpha \quad (\alpha = 1, 2, 3). \tag{2.92}$$

The affine coordinates a^α and a_α are said to be contravariant and covariant coordinates of vector $\mathbf{a}$, respectively.

Introducing into consideration two different bases e_α and $\hat{e}_\alpha$, we represent the vector functions $\eta(\bar{Z})$ and $\theta(\bar{Z})$ in the form

$$\eta(\bar{Z}) = \eta^\alpha(\bar{Z})\mathbf{e}_\alpha, \quad \theta(\bar{Z}) = \hat{\theta}^\alpha(\bar{Z})\mathbf{e}_\alpha, \tag{2.93}$$

where $\eta^\alpha(\bar{Z})$, $\hat{\theta}^\alpha(\bar{Z})$ are scalar unknowns.

Introducing (2.93) into (2.75) and considering (2.84), we have

$$\mathbf{U}(\bar{Z}, S) = \eta^\alpha(\bar{Z})\mathbf{e}_\alpha - AB\theta^\alpha(\bar{Z})\hat{\mathbf{e}}_\alpha \times \mathbf{m}_z(S) + \Omega(\bar{Z}, S)\mathbf{m}_z(S). \tag{2.94}$$

Setting

$$\eta^\alpha(\bar{Z}) = V_1^\alpha(\bar{Z}), \quad (1-\bar{Z})\theta^\alpha(\bar{Z}) = V_2^\alpha(\bar{Z}), \tag{2.95}$$

we write

$$V^\alpha(\bar{Z}, S) = V_1^\alpha(\bar{Z})\mathbf{q}_{1\alpha} + V_2^\alpha(\bar{Z})\mathbf{q}_{2\alpha}, \tag{2.96}$$

where

$$\mathbf{q}_{1\alpha} = \mathbf{e}_\alpha, \quad \mathbf{q}_{2\alpha} = -A\hat{\mathbf{e}}_\alpha \times \mathbf{m}_z(S). \tag{2.97}$$

Comparing expressions (2.96) and (2.13), we see that formulas (2.97) constitute an extension of formulas (2.15) to the case of arbitrary (in the general case, oblique-angled) bases e_α, $\hat{e}_\alpha$, the only difference being that formulas (2.15) correspond to poles with coordinates x=0, y=0, $z = i_0(\bar{Z})$, and formulas (2.97), to poles fixed at the point x=0, y=0, $z = 1_0$.

Introducing (2.93) into (2.85), we also have

$$T_{m_z} = \frac{Eh}{1-\nu^2}\left[\frac{\partial \eta^\circ}{\partial m_z}\mathbf{e}_0\cdot(\mathbf{m}_z - \nu \operatorname{ctg}\chi\,\mathbf{n}_z) + \nu AB \operatorname{ctg}\chi \frac{\partial \hat{\vartheta}^\circ}{\partial m_z}\hat{\mathbf{e}}_0\cdot\mathbf{n}_n + \frac{\partial \mathbf{\Omega}}{\partial m_z}\right],$$
$$S_{n_z m_z} = \frac{Eh}{2(1+\nu)}\left[\frac{\partial \eta^\circ}{\partial m_z}\mathbf{e}_0\cdot(-\operatorname{ctg}\chi\,\mathbf{m}_z + \mathbf{n}_z) - AB\frac{\partial \hat{\vartheta}^\circ}{\partial m_z}\hat{\mathbf{e}}_0\cdot\mathbf{n}_n + \frac{\partial \mathbf{\Omega}}{\partial n_z}\right], \qquad (2.98)$$
$$T_{n_z} = \frac{Eh}{1-\nu^2}\left[\frac{\partial \eta^\circ}{\partial m_z}\mathbf{e}_0\cdot(-\operatorname{ctg}\chi\,\mathbf{n}_z + \nu\mathbf{m}_z) + AB \operatorname{ctg}\chi \frac{\partial \hat{\vartheta}^\circ}{\partial m_z}\hat{\mathbf{e}}_0\cdot\mathbf{n}_n + \nu\frac{\partial \mathbf{\Omega}}{\partial m_z}\right].$$

We now write the equilibrium equations of the cut-off portion of the shell. These equations assume the simplest form if the vector of the moment of forces external to the cut-off portion is calculated with respect to the apex of the cone. Expanding expressions (2.37) for the resultant and the net moment of internal forces acting in the section $\overline{Z}$ = const, with the aid of expressions (2.98) and (2.84) and neglecting the transverse forces Q_{m_z}, Q_{n_z} and moments M_{m_z}, $H_{n_z m_z}$, M_{n_z}, we obtain

$$\mathfrak{R}(\overline{Z}) = \frac{\partial \eta^\circ(\overline{Z})}{\partial \overline{Z}}(1-\overline{Z})\mathbf{\Psi}_{R\eta 0} + \frac{\partial \hat{\vartheta}^\circ(\overline{Z})}{\partial \overline{Z}}(1-\overline{Z})^2\mathbf{\Psi}_{R\theta 0} + (1-\overline{Z})\mathbf{R}_0(\overline{Z}),$$
$$\mathfrak{M}(\overline{Z}) = \frac{\partial \eta^\circ(\overline{Z})}{\partial \overline{Z}}(1-\overline{Z})^2\mathbf{\Psi}_{M\eta 0} + \frac{\partial \hat{\vartheta}^\circ(\overline{Z})}{\partial \overline{Z}}(1-\overline{Z})^3\mathbf{\Psi}_{M\theta 0} + (1-\overline{Z})^2\mathbf{M}_0(\overline{Z}), \qquad (2.99)$$

where the fixed vectors

$$\mathbf{\Psi}_{R\eta 0} = \oint\left\{\mathbf{e}_0\cdot\mathbf{m}_z\left[\frac{1}{\sin\chi}(1+\sin^2\chi - \nu\cos^2\chi)\mathbf{m}_z - (1+\nu)\cos\chi\,\mathbf{n}_z\right] - \right.$$
$$\left. - \mathbf{e}_0\cdot\mathbf{n}_z\left[(1+\nu)\cos\chi\,\mathbf{m}_z - \frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\mathbf{n}_z\right]\right\}\times$$
$$\times\frac{Eh}{2(1-\nu^2)}\frac{1}{A}\,dS,$$
$$\mathbf{\Psi}_{R\theta 0} = \oint\hat{\mathbf{e}}_0\cdot\mathbf{n}_n\left[(1+\nu)\cos\chi\,\mathbf{m}_z - \frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\mathbf{n}_z\right]\times$$
$$\times\frac{Eh}{2(1-\nu^2)}\,dS, \qquad (2.100)$$
$$\mathbf{\Psi}_{M\eta 0} = \oint\left[\mathbf{e}_0\cdot\mathbf{m}_z(1+\nu)\cos\chi - \mathbf{e}_0\cdot\mathbf{n}_z\frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\right]\mathbf{n}_n\times$$
$$\times\frac{Eh}{2(1-\nu^2)}\,dS,$$
$$\mathbf{\Psi}_{M\theta 0} = \oint\hat{\mathbf{e}}_0\cdot\mathbf{n}_n\frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\mathbf{n}_n\frac{Eh}{2(1-\nu^2)}A\,dS.$$

and the vector functions

$$\mathbf{R}_{\theta}=\oint\left\{\frac{\partial\Omega}{\partial \mathbf{m}_{s}}\left[\frac{1}{\sin\chi}(1+\sin^{2}\chi-\nu\cos^{2}\chi)\mathbf{m}_{s}-(1+\nu)\cos\chi\,\mathbf{n}_{s}\right]-\right.$$
$$\left.-\frac{\partial\Omega}{\partial \mathbf{m}_{s}}(\cos\chi\,\mathbf{m}_{s}-\sin\chi\,\mathbf{n}_{s})\frac{1-\nu}{\sin\chi}\right\}\frac{Eh}{2(1-\nu^{2})}dS,$$
$$\mathbf{M}_{\theta}=\oint\left[\frac{\partial\Omega}{\partial \mathbf{m}_{s}}(1+\nu)\cos\chi-\frac{\partial\Omega}{\partial \mathbf{m}_{s}}(1-\nu)\right]\mathbf{n}_{s}\frac{Eh}{2(1-\nu^{2})}A\,dS. \qquad (2.101)$$

Introducing (2.101) into equilibrium equations (2.34) of cut-off portion, we have

$$\frac{\partial\eta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})\boldsymbol{\Psi}_{R\eta_{\alpha}}+\frac{\partial\vartheta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})^{2}\boldsymbol{\Psi}_{R\vartheta_{\alpha}}=\mathfrak{R}^{0}-(1-\bar{Z})\mathbf{R}_{\theta},$$
$$\frac{\partial\eta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})\boldsymbol{\Psi}_{M\eta_{\alpha}}+\frac{\partial\vartheta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})^{2}\boldsymbol{\Psi}_{M\vartheta_{\alpha}}=\frac{1}{1-\bar{Z}}\mathfrak{M}^{0}-(1-\bar{Z})\mathbf{M}_{\theta}. \qquad (2.102)$$

Vector equations (2.102) contain six scalar unknowns

$$\frac{\partial\eta^{\alpha}(\bar{Z})}{\partial\bar{Z}},\quad\frac{\partial\vartheta^{\alpha}(\bar{Z})}{\partial\bar{Z}}\quad(\alpha=1,2,3), \qquad (2.103)$$

entering linearly into these equations. In this connection, the six scalar equations corresponding to Eqs. (2.102) will also be linear with respect to unknowns (2.103).

Introducing the two (in the general case, different) bases $\mathbf{e}'_{\beta}$, $\hat{\mathbf{e}}'_{\beta}$, we have

$$\boldsymbol{\Psi}_{R\eta_{\alpha}}=\Psi'^{\beta}_{R\eta_{\alpha}}\mathbf{e}'_{\beta},\quad\boldsymbol{\Psi}_{R\vartheta_{\alpha}}=\Psi'^{\beta}_{R\vartheta_{\alpha}}\mathbf{e}'_{\beta},\quad\mathbf{R}_{\theta}=R'^{\beta}_{\theta}\mathbf{e}'_{\beta},\quad\mathfrak{R}^{0}=\mathfrak{R}^{0'\beta}\mathbf{e}'_{\beta}, \qquad (2.104)$$
$$\boldsymbol{\Psi}_{M\eta_{\alpha}}=\Psi'^{\beta}_{M\eta_{\alpha}}\hat{\mathbf{e}}'_{\beta},\quad\boldsymbol{\Psi}_{M\vartheta_{\alpha}}=\hat{\Psi}'^{\beta}_{M\vartheta_{\alpha}}\mathbf{e}'_{\beta},\quad\mathbf{M}_{\theta}=\hat{M}'^{\beta}_{\theta}\hat{\mathbf{e}}'_{\beta},\quad\mathfrak{M}^{0}=\hat{\mathfrak{M}}^{0'\beta}\hat{\mathbf{e}}'_{\beta}. \qquad (2.105)$$

Introducing relations (2.104), (2.105) into Eqs. (2.102) and equating the coefficients of the basis vectors $\mathbf{e}'_{\beta}$ and $\hat{\mathbf{e}}'_{\beta}$ in the left and right members of these equations, we obtain

$$\frac{\partial\eta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})\Psi'^{\beta}_{R\eta_{\alpha}}+\frac{\partial\vartheta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})^{2}\Psi'^{\beta}_{R\vartheta_{\alpha}}=\mathfrak{R}^{0'\beta}-(1-\bar{Z})R'^{\beta}_{\theta},$$
$$\frac{\partial\eta^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})\Psi'^{\beta}_{M\eta_{\alpha}}+\frac{\partial\hat{\vartheta}^{\alpha}}{\partial\bar{Z}}(1-\bar{Z})^{2}\Psi'^{\beta}_{M\vartheta_{\alpha}}=\frac{1}{1-\bar{Z}}\hat{\mathfrak{M}}^{0'\beta}-(1-\bar{Z})M'^{\beta}_{\theta}$$
$$(\beta=1,2,3). \qquad (2.106)$$

Thus, by selecting as the kinematic unknowns the components of translational displacement of the contour $\overline{Z}$ = const together with the apex of the cone, and the components of rotation of this contour about the apex, we can always reduce the integro-differential equilibrium equations of the cut-off portion of the shell to a system of six linear algebraic equations in derivatives of these components.

Solving Eqs. (2.106) for the derivative $\partial\eta^\alpha/\partial\overline{Z}$, $\partial\hat{\theta}^\alpha/\partial\overline{Z}$, then integrating the relations obtained, we arrive at relations of the type

$$\eta^\bullet(\overline{Z}) = \eta^\bullet(0) + L_1(p, \Omega), \quad \hat{\theta}^\bullet(\overline{Z}) = \hat{\theta}^\bullet(0) + L_2(p, \Omega), \tag{2.107}$$

which establish a definite relationship between the vector function $\eta(\overline{Z})$, $\theta(\overline{Z})$, on the one hand, and the external load and warping $\Omega(\overline{Z}, S)$, on the other. We will therefore apply the term "canonical kinematic unknowns" to the components of translational displacement of the contour $\overline{Z}$ = const together with the apex of the shell and to the components of rotation of this contour about the apex.

Equations (2.106) corresponding to arbitrarily chosen bases e_α, e_α, e_α, e_α, are general in character. It is natural to ask, how should these bases be chosen so that Eqs. (2.106) assume the simplest structure possible? For example, we will require that each of the equations of system (2.106) contain not more than one of the components of translational displacement and rotation. (It is easy to show that the equilibrium equations of the cut-off portion of a right circular conical shell have such a structure). It is necessary for this purpose that the basis e'_α be reciprocal to both the basis formed by vectors $\Psi_{Rr_1\alpha}$, and that formed by vectors $\Psi_{R\theta\alpha}$, and that the basis e_α be reciprocal to the bases formed by vectors $\Psi_{Rr_1\alpha}$, and $\Psi_{M\theta\alpha}$. It is easy to see that these requirements are equivalent to the conditions of collinearity of vectors $\Psi_{Rr_1\alpha}$ and $\Psi_{R\theta\alpha}$ and correspondingly, $\Psi_{Mr_1\alpha}$, $\Psi_{M\theta\alpha}$.

Indeed, let b_α be a basis reciprocal to certain bases c^α and d^α. Then,

where

$$\mathbf{c}^1 = k_b(\mathbf{b}_2 \times \mathbf{b}_3), \quad \mathbf{c}^2 = k_b(\mathbf{b}_3 \times \mathbf{b}_1), \quad \mathbf{c}^3 = k_b(\mathbf{b}_1 \times \mathbf{b}_2), \tag{2.108}$$

$$k_b = \frac{1}{\mathbf{b}_1 \times \mathbf{b}_2 \cdot \mathbf{b}_3}$$

and

$$\mathbf{b}_1 = k_d(\mathbf{d}^2 \times \mathbf{d}^3), \quad \mathbf{b}_2 = k_d(\mathbf{d}^3 \times \mathbf{d}^1), \quad \mathbf{b}_3 = k_d(\mathbf{d}^1 \times \mathbf{d}^2),$$

$$k_d = \frac{1}{\mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{d}_3}. \tag{2.109}$$

Introducing (2.109) into (2.108), we obtain

$$\begin{aligned} \mathbf{c}^1 &= k_b k_d^2 (\mathbf{d}^3 \times \mathbf{d}^1) \times (\mathbf{d}^1 \times \mathbf{d}^2), \\ \mathbf{c}^2 &= k_b k_d^2 (\mathbf{d}^1 \times \mathbf{d}^2) \times (\mathbf{d}^2 \times \mathbf{d}^3), \\ \mathbf{c}^3 &= k_b k_d^2 (\mathbf{d}^2 \times \mathbf{d}^3) \times (\mathbf{d}^3 \times \mathbf{d}^1), \end{aligned} \tag{2.110}$$

whence it follows that the corresponding vectors of bases c^α and d^α are collinear.

Thus, in order for Eqs. (2.106) to have the desired structure, it is necessary that the vectors of bases $\Psi_{R\eta\alpha}$, be collinear with the corresponding vectors of bases $\Psi_{R\theta\alpha}$, and the vectors of bases $\Psi_{M\eta\alpha}$, with the corresponding vectors of bases $\Psi_{M\theta\alpha}$.

The collinearity condition of certain vectors a and b may be formulated in two ways:

$$\mathbf{a} \times \mathbf{b} = 0$$

or

$$\mathbf{a} + k\mathbf{b} = 0,$$

where k is some scalar different from zero. The second form of the notation is more convenient in this case. We have

$$\begin{aligned} k_R \boldsymbol{\Psi}_{R\eta\alpha} + \boldsymbol{\Psi}_{R\theta\alpha} &= 0, \\ k_M \boldsymbol{\Psi}_{M\eta\alpha} + \boldsymbol{\Psi}_{M\theta\alpha} &= 0, \end{aligned} \tag{2.111}$$

where k_R, k_M are some scalars, as yet undetermined.

Vectors Ψ in accordance with expressions (2.100) depend on the choice of bases e_α and $\hat{e}_\alpha$, which should be chosen so that vectors Ψ meet the collinearity conditions (2.111).

Let the basis t^β define some Cartesian coordinate system

$$t^1 = \mathbf{i}, \quad t^2 = \mathbf{j}, \quad t^3 = \mathbf{k}. \tag{2.112}$$

We expand the basis vectors e_α and $\hat{e}_\alpha$ in the basis t^β. We have

$$\mathbf{e}_\alpha = e_{\alpha\beta} \cdot t^\beta, \quad \hat{\mathbf{e}}_\alpha = \hat{e}_{\alpha\beta} t^\beta, \tag{2.113}$$

where

$$e_{\alpha\beta} = \mathbf{e}_\alpha \cdot t^\beta, \quad \hat{e}_{\alpha\beta} = \hat{\mathbf{e}}_\alpha \cdot t^\beta. \tag{2.114}$$

Introducing (2.113) into (2.100), we represent the vectors Ψ in the form

$$\Psi_{R\gamma\alpha} = e_{\alpha\beta} \Lambda^\beta_{R\gamma}, \quad \Psi_{R\theta\alpha} = \hat{e}_{\alpha\beta} \Lambda^\beta_{R\theta},$$
$$\Psi_{M\gamma\alpha} = e_{\alpha\beta} \Lambda^\beta_{M\gamma}, \quad \Psi_{M\theta\alpha} = \hat{e}_{\alpha\beta} \Lambda^\beta_{M\theta},$$

where

$$\Lambda^\beta_{R\gamma} = \oint \Big\{ t^\beta \cdot \mathbf{m}_z \Big[\frac{1}{\sin\chi} (1 + \sin^2\chi - \nu\cos^2\chi)\mathbf{m}_z - (1+\nu)\cos\chi\, \mathbf{n}_z \Big] - \tag{2.115}$$
$$- t^\beta \cdot \mathbf{n}_z \Big[(1+\nu)\cos\chi\, \mathbf{m}_z - \frac{1}{\sin\chi} (1 + \cos^2\chi - \nu\sin^2\chi)\mathbf{n}_z \Big] \Big\} \times$$
$$\times \frac{Eh}{2(1-\nu^2)} \frac{1}{A} dS,$$

$$\Lambda^\beta_{R\theta} = \oint t^\beta \cdot \mathbf{n}_z \Big[(1+\nu)\cos\chi\, \mathbf{m}_z - \frac{1}{\sin\chi}(1 + \cos^2\chi - \nu\sin^2\chi)\mathbf{n}_z \Big] \times \frac{Eh}{2(1-\nu^2)} dS,$$

$$\Lambda^\beta_{M\gamma} = \oint \Big[t^\beta \cdot \mathbf{m}_z (1+\nu)\cos\chi - t^\beta \cdot \mathbf{n}_z \frac{1}{\sin\chi}(1 + \cos^2\chi - \nu\sin^2\chi) \Big] \mathbf{n}_z \times \frac{Eh}{2(1-\nu^2)} dS, \tag{2.116}$$

$$\Lambda^\beta_{M\theta} = \oint t^\beta \cdot \mathbf{n}_z \frac{1}{\sin\chi}(1 + \cos^2\chi - \nu\sin^2\chi)\mathbf{n}_z \frac{Eh}{2(1-\nu^2)} A\, dS.$$

Now, (1.115) being taken into account, collinearity conditions (1.111) take the form

$$k_R e_{\alpha\beta} \Lambda^\beta_{R\gamma} + \hat{e}_{\alpha\beta} \Lambda^\beta_{R\theta} = 0,$$
$$k_M e_{\alpha\beta} \Lambda^\beta_{M\gamma} + \hat{e}_{\alpha\beta} \Lambda^\beta_{M\theta} = 0. \tag{2.117}$$

To the two vector relations (1.117), there correspond six scalar ones. Expanding (2.117) in the base t^β, we have

$$
\begin{gathered}
k_R e_{\alpha\beta}\Delta^{\beta}_{R\eta\gamma} + \hat{e}_{\alpha\beta}\Delta^{\beta}_{R\theta\gamma} = 0, \\
k_M e_{\alpha\beta}\Delta^{\beta}_{M\eta\gamma} + \hat{e}_{\alpha\beta}\Delta^{\beta}_{M\theta\gamma} = 0 \\
(\gamma = 1, 2, 3),
\end{gathered}
\tag{2.118}
$$

where, by (1.116),

$$
\begin{aligned}
\Delta^{\beta}_{R\eta\gamma} = \oint \Big\{ & \mathbf{t}^{\beta}\cdot\mathbf{m}_s \Big[\frac{1}{\sin\chi}(1+\sin^2\chi-\nu\cos^2\chi)\,\mathbf{t}_\gamma\cdot\mathbf{m}_s - \\
& -(1+\nu)\cos\chi\,\mathbf{t}_\gamma\cdot\mathbf{n}_s\Big] - \mathbf{t}^{\beta}\cdot\mathbf{n}_s\Big[(1+\nu)\cos\chi\,\mathbf{t}_\gamma\mathbf{m}_s - \\
& - \frac{1}{\sin\chi}(1+\cos^2\chi-\nu\sin^2\chi)\,\mathbf{t}_\gamma\cdot\mathbf{n}_s\Big]\Big\}\frac{Eh}{2(1-\nu^2)}\frac{1}{A}\,dS, \\
\Delta^{\beta}_{R\theta\gamma} = \oint & \mathbf{t}^{\beta}\cdot\mathbf{n}_s\Big[(1+\nu)\cos\chi\,\mathbf{t}_\gamma\cdot\mathbf{m}_s - \frac{1}{\sin\chi}(1+\cos^2\chi - \\
& -\nu\sin^2\chi)\,\mathbf{t}_\gamma\cdot\mathbf{n}_s\Big]\frac{Eh}{2(1-\nu^2)}\,dS, \\
\Delta^{\beta}_{M\eta\gamma} = \oint & \Big[\mathbf{t}^{\beta}\cdot\mathbf{m}_s(1+\nu)\cos\chi - \mathbf{t}^{\beta}\cdot\mathbf{n}_s\frac{1}{\sin\chi}(1+\cos^2\chi-\nu\sin^2\chi)\Big]\mathbf{t}_\gamma\cdot\mathbf{n}_s \times \\
& \times\frac{Eh}{2(1-\nu^2)}\,dS, \\
\Delta^{\beta}_{M\theta\gamma} = \oint & \mathbf{t}^{\beta}\cdot\mathbf{n}_s\frac{1}{\sin\chi}(1+\cos^2\chi-\nu\sin^2\chi)\,\mathbf{t}_\gamma\cdot\mathbf{n}_s\frac{Eh}{2(1-\nu^2)}A\,dS.
\end{aligned}
\tag{2.119}
$$

Expressions (2.118) represent a homogeneous system of six linear algebraic equations in the unknowns $e_{\alpha\beta}$, $\hat{e}_{\alpha\beta}$. The latter are the components of the desired basis vectors e_α, $\hat{e}_\alpha$ in the basis t^β. As follows from expressions (2.119), system (2.118) has a symmetric matrix that can be represented in the form

$$
A = \begin{pmatrix} k_R^2\,(\Delta^{\beta}_{R\eta\gamma}) & (\Delta^{\beta}_{R\theta\gamma}) \\ k_M^2\,(\Delta^{\beta}_{M\eta\gamma}) & (\Delta^{\beta}_{M\theta\gamma}) \end{pmatrix},
\tag{2.120}
$$

where matrices $(\Delta^{\beta}_{R\eta t})$ and $(\Delta^{\beta}_{M\theta t})$ are symmetric, and matrix $(\Delta^{\beta}_{M\eta t})$ is the transposed matrix $(\Delta^{\beta}_{R\theta t})$. To matrix A is equivalent the matrix

$$B=\begin{pmatrix} \lambda^{\beta} & (\Lambda^{\beta}_{R\eta t}) & (\Lambda^{\beta}_{R\theta t}) \\ & (\Lambda^{\beta}_{M\eta t}) & (\Lambda^{\beta}_{M\theta t}) \end{pmatrix}, \qquad (2.121)$$

where

$$\lambda=\frac{k_R}{k_M}. \qquad (2.122)$$

Since we are interested only in the nontrivial solutions of system (2.118), the determinant of matrix A, or, what amounts to the same thing, the determinant of matrix B should be equated to zero. The condition

$$\det B=0 \qquad (2.123)$$

is a cubic equation in the parameter λ. If all the roots of this equation are real, the problem we have formulated has a solution. System (2.118) for each of the values of parameter λ obtained has a nontrivial solution containing two arbitrary factors. To determine these factors, it is necessary each time to add to system (2.118) two additional conditions imposing limitations on the length of basis vectors e_α, $\hat{e}_\alpha$. Thus, for example, for unit vectors e_α, $\hat{e}_\alpha$, the following conditions must be fulfilled:

$$\begin{aligned}(e_{\alpha 1})^2+(e_{\alpha 2})^2+(e_{\alpha 3})^2&=1,\\ (\hat{e}_{\alpha 1})^2+(\hat{e}_{\alpha 2})^2+(\hat{e}_{\alpha 3})^2&=1.\end{aligned} \qquad (2.124)$$

Thus, to each of the roots $\lambda = \lambda_\alpha$ of Eqs. (2.123), there corresponds a pail of basis vectors e_α and $\hat{e}_\alpha$. If all three roots λ_1, λ_2 are real, the bases being sought exist. Since one of the roots of Eq. (2.123) is always real, it is always possible to obtain two equilibrium equations of the cut-off portion of the shell of the desired structure, i.e., equations containing only one of the components of translational displacement and rotation.

2.5. Generalized Law of Plane Section. Self-Balanced State of a Shell.

The general system of equilibrium equations of the cut-off portion of a conical shell of arbitrary configuration is defined by expressions (2.106), constituting a linear system of algebraic equations in derivatives of canonical kinematic unknowns:

$$AX = Y^0 - Y_\Omega. \tag{2.125}$$

Here

A is a square matrix:

$$A = \begin{pmatrix} (\Psi'^{\beta}_{R\eta\alpha}) & (\Psi'^{\beta}_{R\theta\alpha}) \\ (\hat{\Psi}'^{\beta}_{M\eta\alpha}) & (\hat{\Psi}'^{\beta}_{M\theta\alpha}) \end{pmatrix}, \tag{2.126}$$

where $(\Psi'^{\beta}_{R\eta\alpha})$, $(\Psi'^{\beta}_{R\theta\alpha})$, $(\hat{\Psi}'^{\beta}_{M\eta\alpha})$, $(\hat{\Psi}'^{\beta}_{M\theta\alpha})$ are blocks constituting third-order square matrices;

$$X = \begin{pmatrix} (1-\bar{Z})\left[\dfrac{\partial \eta^{\beta}}{\partial \bar{Z}}\right] \\ (1-\bar{Z})^{\beta}\left[\dfrac{\partial \hat{\theta}^{\beta}}{\partial \bar{Z}}\right] \end{pmatrix}, \tag{2.127}$$

where $\left[\dfrac{\partial \eta^{\beta}}{\partial \bar{Z}}\right]$, $\left[\dfrac{\partial \hat{\theta}^{\beta}}{\partial \bar{Z}}\right]$ are blocks constituting third-order column matrices;

Y^0, Y_Ω are the column matrices:

$$Y^0 = \begin{pmatrix} [\mathfrak{N}^{0\beta}] \\ \dfrac{1}{1-\bar{Z}}[\hat{\mathfrak{M}}^{0\beta}] \end{pmatrix}, \quad Y_\Omega = \begin{pmatrix} (1-\bar{Z})\,[R_\Omega^{\beta}] \\ (1-\bar{Z})\,[\hat{M}_\Omega^{\beta}] \end{pmatrix}, \tag{2.128}$$

where $[\mathfrak{N}^{0\beta}]$, $[\hat{\mathfrak{M}}^{0\beta}]$, $[R_\Omega^{\beta}]$, $[\hat{M}_\Omega^{\beta}]$ are blocks constituting third-order column matrices.

According to (2.104), (2.105), the block elements of matrices A, Y^0, Y_Ω are determined by the expressions

$$\Psi'^{\beta}_{R\gamma\alpha} = \oint \Big\{ \mathbf{e}_\alpha \cdot \mathbf{m}_s \Big[\frac{1}{\sin\chi}(1+\sin^2\chi - \nu\cos^2\chi)\mathbf{e}'^{\beta} \cdot \mathbf{m}_s -$$
$$-(1+\nu)\cos\chi\,\mathbf{e}'^{\beta} \cdot \mathbf{n}_s \Big] - \mathbf{e}_\alpha \cdot \mathbf{n}_s \Big[(1+\nu)\cos\chi\,\mathbf{e}'^{\beta} \cdot \mathbf{m}_s -$$
$$- \frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\mathbf{e}'^{\beta} \cdot \mathbf{n}_s \Big] \Big\} \frac{Eh}{2(1-\nu^2)} \frac{1}{A}\, dS,$$
$$\Psi'^{\beta}_{R\alpha} = \oint \hat{\mathbf{e}}_\alpha \cdot \mathbf{n}_s \Big[(1+\nu)\cos\chi\,\mathbf{e}'^{\beta} \cdot \mathbf{m}_s - \frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\mathbf{e}'^{\beta}\mathbf{n}_s \Big] \times$$
$$\times \frac{Eh}{2(1-\nu^2)}\, dS,$$
$$\hat{\Psi}'^{\beta}_{M\gamma\alpha} = \oint \Big[\mathbf{e}_\alpha \cdot \mathbf{m}_s (1+\nu)\cos\chi - \mathbf{e}_\alpha \cdot \mathbf{n}_s \frac{1}{\sin\chi}(1+\cos^2\chi -$$
$$- \nu\sin^2\chi) \Big] \hat{\mathbf{e}}'^{\beta} \cdot \mathbf{n}_s \frac{Eh}{2(1-\nu^2)}\, dS,$$
$$\hat{\Psi}'^{\beta}_{M\alpha} = \oint \hat{\mathbf{e}}_\alpha \cdot \mathbf{n}_s \frac{1}{\sin\chi}(1+\cos^2\chi - \nu\sin^2\chi)\hat{\mathbf{e}}'^{\beta} \cdot \mathbf{n}_s \frac{Eh}{2(1-\nu^2)} A\, dS \qquad (2.129)$$
$$(\alpha, \beta = 1, 2, 3).$$

$$\mathfrak{R}^{0\cdot\beta} = \mathbf{e}'^{\beta} \cdot \mathfrak{R}^0,$$
$$\hat{\mathfrak{M}}^{0\cdot\beta} = \hat{\mathbf{e}}'^{\beta} \cdot \mathfrak{M}^0, \qquad (2.130)$$
$$R'^{\beta}_{\theta} = \oint \Big\{ \frac{\partial\Omega}{\partial \mathbf{m}_s} \Big[\frac{1}{\sin\chi}(1+\sin^2\chi - \nu\cos^2\chi)\mathbf{e}'^{\beta} \cdot \mathbf{m}_s -$$
$$-(1+\nu)\cos\chi\,\mathbf{e}'^{\beta} \cdot \mathbf{n}_s \Big] - \frac{\partial\Omega}{\partial \mathbf{m}_s}(\cos\chi\,\mathbf{e}'^{\beta} \cdot \mathbf{m}_s - \sin\chi\,\mathbf{e}'^{\beta} \cdot \mathbf{n}_s)\frac{1-\nu}{\sin\chi} \Big\} \times$$
$$\times \frac{Eh}{2(1-\nu^2)}\, dS, \qquad (2.131)$$
$$\hat{M}'^{\beta}_{\theta} = \oint \Big[\frac{\partial\Omega}{\partial \mathbf{m}_s}(1+\nu)\cos\chi - \frac{\partial\Omega}{\partial \mathbf{m}_s}(1-\nu) \Big] \hat{\mathbf{e}}'^{\beta} \cdot \mathbf{n}_s \frac{Eh}{2(1-\nu^2)} A\, dS.$$

The solution of system (2.106) may be written in the forn

$$\mathbf{X} = \mathbf{X}^0 + \mathbf{X}_\theta, \qquad (2.132)$$

where

$$\mathbf{X}^0 = \mathbf{A}^{-1}\mathbf{Y}^0, \quad \mathbf{X}_\theta = -\mathbf{A}^{-1}\mathbf{Y}_\theta. \qquad (2.133)$$

On the basis of Cramer's formulas

$$\eta^{*0}(\bar{Z}) = \eta^*(0) + \frac{1}{D}\Big[A^*_{R\eta\beta} \int_0^{\bar{Z}} \frac{1}{1-\xi}\mathfrak{R}^{0\cdot\beta}(\xi)\, d\xi +$$
$$+ A^*_{M\eta\beta} \int_0^{\bar{Z}} \frac{1}{(1-\xi)^2}\hat{\mathfrak{M}}^{0\cdot\beta}(\xi)\, d\xi \Big],$$
$$\hat{\theta}^{*0}(\bar{Z}) = \hat{\theta}^*(0) + \frac{1}{D}\Big[A^*_{R\theta\beta} \int_0^{\bar{Z}} \frac{1}{(1-\xi)^2}\mathfrak{R}^{0\cdot\beta}(\xi)\, d\xi + \qquad (2.134)$$
$$+ A^*_{M\theta\beta} \int_0^{\bar{Z}} \frac{1}{(1-\xi)^3}\hat{\mathfrak{M}}^{0\cdot\beta}(\xi)\, d\xi \Big].$$
$$\eta^*_\theta(\bar{Z}) = -\frac{1}{D}\Big[A^*_{R\eta\beta} \int_0^{\bar{Z}} R'^{\beta}_{\theta}(\xi)\, d\xi + A^*_{M\eta\beta} \int_0^{\bar{Z}} \hat{M}'^{\beta}_{\theta}(\xi)\, d\xi \Big],$$
$$\hat{\theta}^*_\theta(\bar{Z}) = -\frac{1}{D}\Big[A^*_{R\theta\beta} \int_0^{\bar{Z}} \frac{1}{1-\xi}R'^{\beta}_{\theta}(\xi)\, d\xi + A^*_{M\theta\beta} \int_0^{\bar{Z}} \frac{1}{1-\xi}\hat{M}'^{\beta}_{\theta}(\xi)\, d\xi \Big]. \qquad (2.135)$$

Here $\eta^{*0}, \tilde{\theta}^{*0}$ correspond to the elements of column matrix X^0;

$\eta^{*}_{\Omega}, \tilde{\theta}^{*}_{\Omega}$ correspond to the elements of column matrix X_{Ω};

D is the determinant of matrix A;

$A^{\bullet}_{R\eta\varphi}, A^{\bullet}_{M\eta\varphi}$,

$A^{\bullet}_{R\theta\beta}, A^{\bullet}_{M\theta\beta}$ are the adjoints of elements of corresponding blocks of matrix A.

According to representation (2.132), the vector function $U^0(\overline{Z}, S)$ corresponding to the generalized law of plane sections may be represented as a sum of two components:

$$U^0(\overline{Z}, S) = U^{00}(\overline{Z}, S) + U^0_{\Omega}(\overline{Z}, S). \tag{2.136}$$

The first component is statically determinate, since according to expressions (2.134), it is completely determined by a given external load. The second component according to (2.128), (2.131), (2.135) is related to warping and therefore statically indeterminate. In this connection, we will call $U^{00}(\overline{Z}, S)$ the generalized beam solution, while $U^0_{\Omega}(\overline{Z}, S)$, in accordance with representation (2.6), will be referred to $U^1(\overline{Z}, S)$, the combined vector function $U^0_{\Omega} + U^1$ being called generalized warping.

Obviously, the components of tangential forces T_{m_z}, $S_{n_z m_z}$, T_{n_z} can also be represented in the form of a sum of two components, one of which corresponds to the generalized beam solution, and the other, to generalized warping:

$$T_{m_z} = T^0_{m_z} + T_{m_z\Omega}; \quad T_{n_z} = T^0_{n_z} + T_{n_z\Omega}, \quad S_{n_z m_z} = S^0_{n_z m_z} + S_{n_z m_z\Omega}, \tag{2.137}$$

where according to (2.98), (2.134), (2.135)

$$\begin{aligned}
T^0_{m_z} &= \frac{Eh}{1-\nu^2}\,\frac{1}{D}\left\{\frac{1}{A}(m_{zs} - \nu\,\mathrm{ctg}\,\chi\, n_{zs})\left[\frac{1}{B}A^{\bullet}_{R\eta\beta}\mathfrak{M}^{0\beta} + \right.\right.\\
&\left.\left. + \frac{1}{B^2}A^{\bullet}_{M\eta\beta}\mathfrak{M}^{0\beta}\right] + \nu\,\mathrm{ctg}\,\chi\,\tilde{n}_{ns}\left[\frac{1}{B}A^{\bullet}_{R\eta\beta}\mathfrak{M}^{0\beta} + \frac{1}{B^2}A^{\bullet}_{M\eta\beta}\mathfrak{M}^{0\beta}\right]\right\},\\
S^0_{n_z m_z} &= \frac{Eh}{2(1+\nu)}\,\frac{1}{D}\left\{\frac{1}{A}(-\mathrm{ctg}\,\chi\, m_{zs} + n_{zs})\left[\frac{1}{B}A^{\bullet}_{R\eta\beta}\mathfrak{M}^{0\beta} + \right.\right.\\
&\left.\left. + \frac{1}{B^2}A^{\bullet}_{M\eta\beta}\mathfrak{M}^{0\beta}\right] - \tilde{n}_{ns}\left[\frac{1}{B}A^{\bullet}_{R\eta\beta}\mathfrak{M}^{0\beta} + \frac{1}{B^2}A^{\bullet}_{M\eta\beta}\mathfrak{M}^{0\beta}\right]\right\},\\
T^0_{n_z} &= \frac{Eh}{1-\nu^2}\,\frac{1}{D}\left\{\frac{1}{A}(-\mathrm{ctg}\,\chi\, n_{zs} + \nu m_{zs})\left[\frac{1}{B}A^{\bullet}_{R\eta\beta}\mathfrak{M}^{0\beta} + \right.\right.\\
&\left.\left. + \frac{1}{B^2}A^{\bullet}_{M\eta\beta}\mathfrak{M}^{0\beta}\right] + \mathrm{ctg}\,\chi\,\tilde{n}_{ns}\left[\frac{1}{B}A^{\bullet}_{R\eta\beta}\mathfrak{M}^{0\beta} + \frac{1}{B^2}A^{\bullet}_{M\eta\beta}\mathfrak{M}^{0\beta}\right]\right\},\\
T_{m_z\Omega} &= -\frac{Eh}{1-\nu^2}\left\{\frac{1}{D}\left[\frac{1}{A}(m_{zs} - \nu\,\mathrm{ctg}\,\chi\, n_{zs})(A^{\bullet}_{R\eta\beta}R^{\beta}_{y} + A^{\bullet}_{M\eta\beta}M^{\beta}_{\theta}) + \right.\right.\\
&\left.\left. + \nu\,\mathrm{ctg}\,\chi\,\tilde{n}_{ns}(A^{\bullet}_{R\eta\beta}R^{\beta}_{y} + A^{\bullet}_{M\eta\beta}M^{\beta}_{\theta})\right] - \frac{\partial\Omega}{\partial m_z}\right\}.
\end{aligned} \tag{2.138}$$

$$S_{n_z m_z \Omega} = \frac{Eh}{2(1+\nu)} \left\{ \frac{1}{D} \left[-\frac{1}{A} (-\operatorname{ctg} \chi m_{z1} + n_{z2})(A^{*}_{R\alpha\beta} R^{\beta}_{\Omega} + \right. \right.$$
$$\left. \left. + A^{*}_{M\alpha\beta} M^{\beta}_{\Omega}) + n_{z0}(A^{*}_{R\alpha\beta} R^{\beta}_{\Omega} + A^{*}_{M\alpha\beta} M^{\beta}_{\Omega}) \right] + \frac{\partial\Omega}{\partial n_z} \right\},$$
$$T_{n_z\Omega} = -\frac{Eh}{1-\nu^2} \left\{ \frac{1}{D} \left[\frac{1}{A} (-\operatorname{ctg} \chi n_{z1} + \nu m_{z1})(A^{*}_{R\alpha\beta} R^{\beta}_{\Omega} + \right. \right.$$
$$\left. \left. + A^{*}_{M\alpha\beta} M^{\beta}_{\Omega}) + \operatorname{ctg} \chi n_{z1}(A^{*}_{R\alpha\beta} R^{\beta}_{\Omega} + A^{*}_{M\alpha\beta} M^{\beta}_{\Omega}) \right] + \nu \frac{\partial\Omega}{\partial m_z} \right\}, \tag{2.139}$$
$$m_{z1} = \mathbf{e}_1 \cdot \mathbf{m}_z, \quad n_{z1} = \mathbf{e}_1 \cdot \mathbf{n}_z, \quad n_{z1} = \mathbf{e}_1 \cdot \mathbf{n}_z.$$

Using the expressions obtained, one can readily write down the integro-differential resolvent in warping $\Omega(\overline{Z}, S)$. In Section 2.3, in deriving the integro-differential resolvent, we proceeded from equilibrium differential equation (2.31), which coincided with the first equation of system (1.173) if the component of tangential force T_{n_z} is neglected in that equation. However, as was noted above, the forces T_{n_z} in certain regions of shells of marked conicity may be appreciable. Therefore, here as in Section 2.4, we will consider the forces T_{n_z}, proceeding in the derivation of the integro-differential equation from the first equation of system (1.173).

Considering representation (2.137), we write the first equation of system (2.173) in the form

$$A \frac{\partial}{\partial m_z}(BT_{m_z\Omega}) + \frac{\partial}{\partial n_z}(ABS_{n_z m_z\Omega}) + T_{n_z\Omega} + L(\mathbf{p}) = 0, \tag{2.140}$$

where

$$L(\mathbf{p}) = ABp_{m_z} + A\frac{\partial}{\partial m_z}(BT^{0}_{m_z}) + \frac{\partial}{\partial n_z}(ABS^{0}_{n_z m_z}) + T^{0}_{n_z}. \tag{2.141}$$

Now, using the elasticity relations, it is necessary to eliminate the forces $T_{m_z\Omega}$, $S_{n_z m_z\Omega}$, $T_{n_z\Omega}$ from (2.140). We thus obtain an equation relating the warping $\Omega(\overline{Z}, S)$ with the canonical kinematic unknowns.

Setting $\eta^{\alpha} = \eta^{\alpha}_{\Omega}$, $\hat{\theta}^{\alpha} = \hat{\theta}^{\alpha}_{\Omega}$ in (2.98), we obtain

$$A\frac{\partial}{\partial\mathfrak{m}_s}\left(B\frac{\partial\Omega}{\partial\mathfrak{m}_s}\right)+\nu\frac{\partial\Omega}{\partial\mathfrak{m}_s}+\frac{1-\nu}{2}AB\frac{\partial^2\Omega}{\partial\mathfrak{n}_s^2}+k_{1s}(S)A\frac{\partial}{\partial\mathfrak{m}_s}\left(B\frac{\partial\eta_\Omega^0}{\partial\mathfrak{m}_s}\right)+$$

$$+k_{2s}(S)\frac{\partial\eta_\Omega^0}{\partial\mathfrak{m}_s}+k_{3s}(S)A^0\frac{\partial}{\partial\mathfrak{m}_s}\left(B^0\frac{\partial\hat{\theta}_\Omega^0}{\partial\mathfrak{m}_s}\right)+ \tag{2.142}$$

$$+k_{4s}(S)AB\frac{\partial\hat{\theta}_\Omega^0}{\partial\mathfrak{m}_s}+\frac{1-\nu^2}{Eh}L(\mathrm{p})=0,$$

где

where

$$k_{1s}(S)=m_{ss}\left(1+\frac{1-\nu}{2}\operatorname{ctg}^2\chi\right)-n_{ss}\frac{1+\nu}{2}\operatorname{ctg}\chi,$$

$$k_{2s}(S)=\frac{1-\nu}{2}\frac{1}{\sin\chi}\frac{\partial}{\partial S}\left[A(-\operatorname{ctg}\chi m_{ss}+n_{ss})\right]-\operatorname{ctg}\chi n_{ss}+\nu m_{ss},$$

$$k_{3s}(S)=\hat{n}_{ms}\frac{1+\nu}{2}\operatorname{ctg}\chi,$$

$$k_{4s}(S)=\hat{n}_{ms}\operatorname{ctg}\chi-\frac{1-\nu}{2}\frac{1}{A\sin\chi}\frac{\partial}{\partial S}(A^2\hat{n}_{ms}). \tag{2.143}$$

Eliminating the canonical kinematic unknowns from (2.142) with the aid of relations (2.135), we can readily obtain

$$A\frac{\partial}{\partial\mathfrak{m}_s}\left(B\frac{\partial\Omega}{\partial\mathfrak{m}_s}\right)+\nu\frac{\partial\Omega}{\partial\mathfrak{m}_s}+\frac{1-\nu}{2}AB\frac{\partial^2\Omega}{\partial\mathfrak{n}_s^2}-\frac{1}{D}\Big\{\Big[k_{1s}(S)A^0_{R\eta\beta}+$$

$$+Ak_{3s}(S)A^0_{M\theta\beta}\Big]\frac{\partial}{\partial\mathfrak{m}_s}(BR_\Omega^{'\beta})+\left[\frac{1}{A}k_{2s}(S)A^0_{R\eta\beta}+k_{4s}(S)A^0_{R\theta\beta}\right]R_\Omega^{'\beta}+ \tag{2.144}$$

$$+\left[k_{1s}(S)A^0_{M\eta\beta}+Ak_{3s}(S)A^0_{M\theta\beta}\right]\frac{\partial}{\partial\mathfrak{m}_s}(B\hat{M}_\Omega^{'\beta})+\Big[\frac{1}{A}k_{2s}(S)A^0_{M\eta\beta}+$$

$$+k_{4s}(S)A^0_{M\theta\beta}\Big]\hat{M}_\Omega^{'\beta}\Big\}+\frac{1-\nu^2}{Eh}L(\mathrm{p})=0.$$

whence, using (2.131), we finally obtain

$$\frac{1}{A}\frac{\partial}{\partial\bar{Z}}\left(B\frac{\partial\Omega}{\partial\bar{Z}}\right)+\frac{\nu}{A}\frac{\partial\Omega}{\partial\bar{Z}}+\frac{1-\nu}{2AB}\left(-\operatorname{ctg}\chi B\frac{\partial}{\partial\bar{Z}}+\frac{A}{\sin\chi}\frac{\partial}{\partial S}\right)\left(-\operatorname{ctg}\chi B\frac{\partial\Omega}{\partial\bar{Z}}+\right.$$

$$\left.+\frac{A}{\sin\chi}\frac{\partial\Omega}{\partial S}\right)-\frac{1}{D}\oint\Big[f_1(S,\xi)\frac{\partial}{\partial\bar{Z}}\left(B\frac{\partial\Omega}{\partial\bar{Z}}\right)-f_2(S,\xi)\frac{\partial^2\Omega}{\partial\bar{Z}\,\partial S}+ \tag{2.145}$$

$$+f_3(S,\xi)\frac{\partial\Omega}{\partial\bar{Z}}-f_4(S,\xi)\frac{\partial\Omega}{\partial S}\Big]\frac{Eh}{2(1-\nu^2)}\,d\xi+\frac{1-\nu^2}{Eh}L(\mathrm{p})=0.$$

Here

$$f_1(S,\xi)=\varphi_{1\beta}(S)\psi_1^\beta(\xi)+\varphi_{3\beta}(S)\psi_3^\beta(\xi),$$

$$f_2(S,\xi)=\varphi_{1\beta}(S)\psi_2^\beta(\xi)+\varphi_{3\beta}(S)\psi_4^\beta(\xi),$$

$$f_3(S,\xi)=\varphi_{2\beta}(S)\psi_1^\beta(\xi)+\varphi_{4\beta}(S)\psi_3^\beta(\xi),$$

$$f_4(S,\xi)=\varphi_{2\beta}(S)\psi_2^\beta(\xi)+\varphi_{4\beta}(S)\psi_4^\beta(\xi). \tag{2.146}$$

where

$$
\begin{aligned}
\varphi_{1\beta}(S) &= \frac{1}{A} k_{1\alpha}(S) A^{*}_{R\eta\beta} + k_{3\alpha}(S) A^{*}_{R\theta\beta},\\
\varphi_{2\beta}(S) &= \frac{1}{A} k_{2\alpha}(S) A^{*}_{R\eta\beta} + k_{4\alpha}(S) A^{*}_{R\theta\beta},\\
\varphi_{3\beta}(S) &= \frac{1}{A} k_{1\alpha}(S) A^{*}_{M\eta\beta} + k_{3\alpha}(S) A^{*}_{M\theta\beta},\\
\varphi_{4\beta}(S) &= \frac{1}{A} k_{2\alpha}(S) A^{*}_{M\eta\beta} + k_{4\alpha}(S) A^{*}_{M\theta\beta},
\end{aligned}
\tag{2.147}
$$

$$
\begin{aligned}
\psi_1^{\beta}(\xi) &= \frac{1}{A}\left[\frac{A}{\sin\chi}(1+\sin^2\chi-\nu\cos^2\chi)\, m_z'^{\beta}-(1+\nu)\cos\chi\, n_z'^{\beta}\right],\\
\psi_2^{\beta}(\xi) &= \frac{1-\nu}{\sin\chi}(\cos\chi\, m_z'^{\beta}-\sin\chi\, n_z'^{\beta}),\\
\psi_3^{\beta}(\xi) &= (1+\nu)\cos\chi\, \hat{n}_n^{\beta},\\
\psi_4^{\beta}(\xi) &= A(1-\nu)\, \hat{n}_n^{\beta}.
\end{aligned}
\tag{2.148}
$$

Expression (2.145) represents the solving integro-differential equation in warpings $\Omega(\bar{Z}, S)$. This equation has variable coefficients dependent on both $\bar{Z}$ and S. Substitution of variables $1 - \bar{Z} = e^t$ reduces Eq. (2.145) to an equation with variable coefficients dependent only on the S coordinate:

$$
\begin{aligned}
&\frac{1}{A}\frac{\partial^2\Omega}{\partial t^2}-\frac{\nu}{A}\frac{\partial\Omega}{\partial t}+\frac{1-\nu}{2A}\left(\operatorname{ctg}\chi\frac{\partial}{\partial t}+\frac{A}{\sin\chi}\frac{\partial}{\partial S}\right)\left(\operatorname{ctg}\chi\frac{\partial\Omega}{\partial t}+\frac{A}{\sin\chi}\frac{\partial\Omega}{\partial S}\right)-\\
&-\frac{1}{D}\oint\left[f_1(S,\xi)\frac{\partial^2\Omega}{\partial t^2}+f_2(S,\xi)\frac{\partial^2\Omega}{\partial t\,\partial S}-f_3(S,\xi)\frac{\partial\Omega}{\partial t}-\right.\\
&\left.-f_4(S,\xi)\frac{\partial\Omega}{\partial S}\right]\frac{Eh}{2(1-\nu^2)}d\xi+\frac{1-\nu^2}{Eh}BL(p)=0.
\end{aligned}
\tag{2.149}
$$

Like Eq. (2.65), discussed in Section 2.3, Eq. (2.149) is integrated exactly only in special cases, the most important of which is the case of right conical shells of revolution. In the general case, Eq. (2.149) can be integrated approximately. After determining the warping $\Omega(\bar{Z}, S)$ in some manner, one can, using expressions (2.135) and (2.131), find the components $\eta^{\alpha}_{\Omega}(\bar{Z})$ and $\hat{\theta}^{\alpha}_{\Omega}(\bar{Z})$ of translational displacement and rotation of the contour $\bar{Z}$ = const about the cone apex, components which along with $\Omega(\bar{Z}, S)$ determine the generalized warping of the shell. The corresponding components of internal tangential forces determining the self-balanced state of the shell are represented by expressions (2.139).

2.6. Refinement of Tangential Forces

In accordance with the adopted computational model, the contour of the section $\bar{Z}$ = const has an unlimited number of degrees of freedom with respect to the displacements u_{m_z} and only six degrees of freedom with respect to u_{n_z} and u_{n_n}. In this connection, the strain e_{m_z} is determined most exactly, while the shear strain $\gamma_{n_z m_z}$ and particularly the strain e_{n_z} are determined less exactly. Since the magnitude of the strain e_{n_z} is usually insignificant, the error in its determination is also unimportant. However, for the shear strain $\gamma_{n_z m_z}$, and hence, for tangential forces $S_{n_z m_z}$, a refined solution obtained from the equilibrium conditions may be of interest. We will proceed from the first equation of system (1.173), again omitting in the latter, as above, the reactive flow q from the ribs. Expanding this equation by considering (1.11), (1.15), (1.16), we obtain

$$-\frac{1}{l_s}\operatorname{ctg}\chi\frac{\partial S_{n_z m_z}}{\partial \bar{Z}}+\frac{1}{(1-\bar{Z})\sin\chi}\frac{\partial S_{n_z m_z}}{\partial S}=F(\bar{Z}, S), \tag{2.150}$$

where

$$F(\bar{Z}, S)=-\frac{1}{(1-\bar{Z})l_s}\left\{\frac{\partial}{\partial \bar{Z}}\left[(1-\bar{Z})T_{m_z}\right]+T_{n_z}\right\}-p^0_{m_z}. \tag{2.151}$$

Expression (2.150) will be regarded as a first-order linear partial differential equation, assuming the function $F(\bar{Z}, S)$ to be known. The following characteristic system corresponds to this equation:

$$-l_s\operatorname{tg}\chi\, d\bar{Z}=(1-\bar{Z})\sin\chi\, dS=\frac{dS_{n_z m_z}}{F(\bar{Z}, S)}. \tag{2.152}$$

We will take the S coordinate as the independent variable. Then the equations of characteristics $\bar{Z}=\bar{Z}(S)$, $S_{n_z m_z}=S_{n_z m_z}(S)$ will be

$$\frac{d\bar{Z}}{dS}=-\frac{\cos\chi}{l_s}(1-\bar{Z}), \tag{2.153}$$

$$\frac{dS_{n_z m_z}}{dS}=[1-\bar{Z}(S)]\sin\chi\, F[\bar{Z}(S), S]. \tag{2.154}$$

Equation (2.153) is integrated by separating the variables. Considering (1.11), we find

$$(1-\overline{Z})l_z=C_1, \tag{2.155}$$

whence

$$\overline{Z}(S)=1-\frac{C_1}{l_z}. \tag{2.156}$$

Introducing (2.156) into (2.154), we have

$$\frac{dS_{n_z m_z}}{dS}=\frac{C_1}{l_z}\sin\chi F\left[1-\frac{C_1}{l_z}, S\right], \tag{2.157}$$

whence

$$S_{n_z m_z}(S)=C_1\int F\left[1-\frac{C_1}{l_z}, S\right]\frac{\sin\chi}{l_z}\,dS+C_2. \tag{2.158}$$

Expressions (2.156) and (2.158) represent families of characteristics containing arbitrary constants C_1 and C_2. We will write the equation of the characteristics passing through a fixed point $S=S_0$, $\overline{Z}=\overline{Z}_0$, $S_{n_z m_z}=(S_{n_z m_z})_0$. We have

$$\begin{gathered}\overline{Z}(S)=1-(1-\overline{Z}_0)\frac{l_{z0}}{l_z},\\ S_{n_z m_z}(S)=(S_{n_z m_z})_0+(1-\overline{Z})l_z\int_{S_0}^{S}F\left[1-(1-\overline{Z}_0)\frac{l_{z0}}{l_\xi}, \xi\right]\frac{\sin\chi(\xi)}{l_\xi}\,d\xi.\end{gathered} \tag{2.159}$$

Let the desired integral surface of Eq. (2.150) pass through the curve

$$S=S_0,\quad S_{n_z m_z}=S^0_{n_z m_z}(\overline{Z}). \tag{2.160}$$

Then, assuming the fixed point to lie on curve (2.160), we obtain

$$(S_{n_z m_z})_0=S^0_{n_z m_z}(\overline{Z}_0). \tag{2.161}$$

Taking (2.161) into account, we can write equation of characteristics (2.159) in the form

$$(1-\bar{Z})l_s = (1-\bar{Z}_0)l_{s0},$$

$$S_{n_z m_z} = S^0_{n_z m_z}(\bar{Z}_0) + (1-\bar{Z})l_s \int_{S_0}^{S} F\left[1-(1-\bar{Z}_0)\frac{l_{s0}}{l_\xi}, \xi\right] \frac{\sin\chi(\xi)}{l_\xi} d\xi. \tag{2.162}$$

Expressions (2.162) determine the parametric representation of the desired integral surface. Eliminating $\bar{Z}_0$, we find

$$\begin{gathered} S_{n_z m_z}(\bar{Z}, S) = S^0_{n_z m_z}\left(1-\frac{l_s}{l_{s0}}(1-\bar{Z})\right) + \\ +(1-\bar{Z})l_s \int_{S_0}^{S} F\left(1-\frac{l_s}{l_\xi}(1-\bar{Z}), \xi\right) \frac{\sin\chi(\xi)}{l_\xi} d\xi. \end{gathered} \tag{2.163}$$

Expanding (2.163) with the aid of (2.151), we finally obtain

$$\begin{gathered} S_{n_z m_z}(\bar{Z}, S) = S^0_{n_z m_z}\left(1-\frac{l_s}{l_{s0}}(1-\bar{Z})\right) - \\ -\int_{S_0}^{S} \frac{\partial}{\partial\alpha}\left[(1-\alpha)T_{m_z}(\alpha, \xi)\right]_{\alpha=1-\frac{l_s}{l_\xi}(1-\bar{Z})} \frac{\sin\chi(\xi)}{l_\xi} d\xi - \\ -\int_{S_0}^{S} T_{n_z}\left(1-\frac{l_s}{l_\xi}(1-\bar{Z}), \xi\right) \frac{\sin\chi(\xi)}{l_\xi} d\xi - \\ -(1-\bar{Z})l_s \int_{S_0}^{S} p^0_{m_z}\left(1-\frac{l_s}{l_\xi}(1-\bar{Z}), \xi\right) \frac{\sin\chi(\xi)}{l_\xi} d\xi. \end{gathered} \tag{2.164}$$

Expression (2.164) serves to determine the tangential forces $S_{n_z m_z}$ in terms of the forces T_{m_z}, T_{n_z}; the latter are determined from the elasticity relations. The function $S^0_{n_z m_z}$ entering into (2.164) represents the distribution of tangential forces $S_{n_z m_z}$ in the section $S=S_0$. This function for shells of open profile is determined from the conditions of loading of longitudinal edges. For singly closed shells, this function is determined from the equilibrium condition of the cut-off portion. Finally, for multiclosed shells, the strain compatibility equations of the contours must be added to the equilibrium condition of the cut-off portion.

Expression (2.164) admits of a clear geometric interpretation. It is easy to see that the expression constitutes a curvilinear integral along the section of arc of the line of principal curvature connecting an arbitrary point with coordinates $\bar{Z}$, S and a point lying on the generatrix $S=S_0$. Such an interpretation obviously

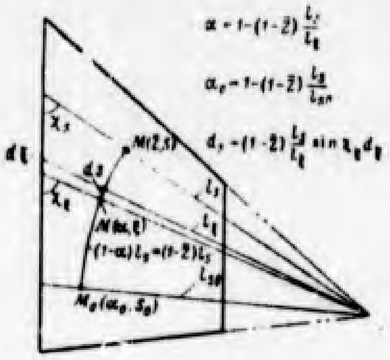

Figure 2.1. In reference to the refinement of tangential forces.

results from the first equation of system (1.173), which is a contracted version of Eq. (2.150). In expression (2.163), the quantity $(1-\overline{Z})l_s(\sin x(\xi)/l\xi)d\xi$ is the differential of the arc of the line of principal curvature passing through point M with coordinate $\overline{Z}$, S; the quantities $1-l_x/l_\xi(1-\overline{Z})$ and ξ are the coordinates of the point of intersection of the generatrix l_ξ with the line of principal curvature passing through point M; finally, the quantity $1-l_s/l_{s0}(1-\overline{Z})$ is the coordinate of point M_0 (Figure 2.1).

Expression (2.164) does not take into account the actual location of the transverse diaphragms. A more accurate result can be obtained on the basis of Eq. (2.27), written for an arbitrarily oriented transverse assembly. This equation is integrated in the manner described above for Eq. (2.150).

Chapter III

Reduction of a Two-Dimensional to a One-Dimensional Problem

It was shown in the previous chapter that the problem of integration of complex systems of partial differential equations describing the stressed and strained state of a conical shell of arbitrary configuration can be reduced to the integration of a single integro-differential resolvent. Sections 2.3, 2.5 derived an integro-differential resolvent in displacement $u_{m_z}(\overline{Z}, S)$ in the direction of the generatrices and in warping $\Omega(\overline{Z}, S)$, pertaining to the self-balanced state of a shell. These equations are very complex and, as already noted, can generally be integrated only approximately. The integration can be carried out by direct application of numerical methods to these equations. In a more convenient approach, however, the approximate solution is based on a variational treatment of the problem. Using the origin of possible Lagrange displacements, we arrive in this case at a resolvent system of ordinary differential equations, omitting the integro-differential equations discussed above. Let us note that while remaining within the framework of the hypotheses adopted earlier, we will obtain a solution equivalent to the approximate solution of the integro-differential equations of Chapter II. However, by partly abandoning the adopted assumptions, we can also obtain an approximate solution for more complex computational models that do not permit a sufficiently clear treatment in differential form.

3.1. Resolvent System of Ordinary Differential Equations of a Conical Shell of Arbitrary Configuration*

As before, we will represent the elastic displacement vector $U(\overline{Z}, S)$ in the form

$$\mathbf{U}(\overline{Z}, S) = \mathbf{U}^0(\overline{Z}, S) + \mathbf{U}^1(\overline{Z}, S), \tag{3.1}$$

where U^0 corresponds to displacements of the contour $\overline{Z}$ = const as a solid, and U^1

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*G.G. Onanov. Computation of Wing Type Shells by V.Z. Vlasov's method. Candidate's dissertation, MAI, 1963.

determines the displacements for which the relative positions of the points of this contour change.

For the vector function $U^0(\overline{Z}, S)$ we have

$$U^0(\overline{Z}, S)=\eta(\overline{Z})+\theta(\overline{Z})\times[r(\overline{Z}, S)-r_0(\overline{Z})], \tag{3.2}$$

where $\eta(\overline{Z})$ and $\theta(\overline{Z})$ are vectors of translational displacement of the contour $\overline{Z}$ = const together with some pole M_0 and rotation of this contour about M_0; r_0 is the radius vector of the pole.

According to representation (3.2), the vector function $U^0(\overline{Z}, S)$ may be written in the form

$$U^0(\overline{Z}, S)=\sum_{i=1}^{6}V_i(\overline{Z})\varphi_i(S). \tag{3.3}$$

Here $V_i(\overline{Z})$ are unknown scalar functions proportional to the components of vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ in certain bases chosen in advance. The vector functions $\phi_i(S)$ are known and are determined by the choice of these bases.

If the chosen pole M_0 of the contour $\overline{Z}$ = const is the point of intersection of the plane of this contour with the Oz axis, and vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ are expanded in a basis coinciding with the Cartesian system of coordinate xyz, the corresponding functions $\phi_i(S)$ are represented by expressions (2.15). Here the functions $V_i(\overline{Z})$ are related to the components of vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ by relations (2.14). If however, the pole of the contour $\overline{Z}$ = const coincides with the apex of the conical shell, and vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ are expanded in arbitrary and different bases, the corresponding functions $\phi_i(S)$ are represented by general expressions (2.97), and functions $V_i(\overline{Z})$ are related to the components of vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ by relations (2.95). As the pole of the contour $\overline{Z}$ = const, let us choose, for example, the apex of the conical surface, and vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ and expanded in single axes parallel to the axes of the Cartesian coordinate system. In this case, on the basis of (2.97) one can readily obtain

$$\varphi_1 = \mathbf{i}, \quad \varphi_2 = \mathbf{j}, \quad \varphi_3 = \mathbf{k},$$

$$\varphi_4 = l_s(-l_{n_n}\mathbf{n}_s + l_{n_s}\mathbf{n}_n),$$

$$\varphi_5 = l_s(-m_{n_n}\mathbf{n}_s + m_{n_s}\mathbf{n}_n), \tag{3.4}$$

$$\varphi_6 = l_s(-n_{n_n}\mathbf{n}_s + n_{n_s}\mathbf{n}_n). \tag{3.5}$$

Using relations (3.4), (3.5) and (1.18), we have for the components ϕ_{im_z}, ϕ_{im_s}, ϕ_{in_n}

i	U_i	φ_{im_z}	φ_{im_s}	φ_{in_n}
1	η_x	$-\frac{x_0}{l_s}$	x_0'	$-\frac{y_0'}{\sin\chi}\frac{l_0}{l_s} + \frac{x_0 y_0' - x_0' y_0}{\sin\chi\, l_s}\operatorname{ctg}\chi_0$
2	η_y	$-\frac{y_0}{l_s}$	y_0'	$\frac{x_0'}{\sin\chi}\frac{l_0}{l_s}$
3	η_z	$\frac{l_0}{l_s} - \frac{x_0}{l_s}\operatorname{ctg}\chi_0$	$x_0'\operatorname{ctg}\chi_0$	$\frac{x_0' y_0 - x_0 y_0'}{\sin\chi\, l_s}$
4	θ_x	0	$l_0 y_0' + (x_0' y_0 - x_0 y_0')\operatorname{ctg}\chi_0$	$\frac{l_s}{\sin\chi}x_0' + x_0\operatorname{ctg}\chi$
5	θ_y	0	$-l_0 x_0'$	$\frac{l_s}{\sin\chi}y_0' +$ [illegible] $\operatorname{ctg}\chi$
6	θ_z	0	$x_0 y_0' - x_0' y_0$	$\frac{l_s}{\sin\chi}x_0'\operatorname{ctg}\chi_0 - (l_0 - x_0\operatorname{ctg}\chi_0)\operatorname{ctg}\chi$

(3.6)

The components ϕ_{in_z} and ϕ_{in_s} are determined in terms of $U^0(\overline{Z}, S)$ on the basis of relations (2.20).

By analogy with (3.3), we represent the vector function $U^1(\overline{Z}, S)$ in the form of the expansion

$$\mathbf{U}^1(\overline{Z}, S) = \sum_k w_k(\overline{Z})\varphi_k^1(S), \tag{3.7}$$

where $w_k(\overline{Z})$ are unknown scalar functions and $\phi_k^1(S)$ is some specified system of coordinate vector functions.

If according to (2.22) we assume that the vector U^1 is directed along the generatrices, the coordinate functions $\phi_k^1(S)$ may be represented in the form

$$\varphi_k^I(S) = m_r \varphi_k^I(S), \tag{3.8}$$

where $\phi_k^I(S)$ is some system of scalar functions.

Introducing (3.8) into (3.7), we have

$$U^I(\bar{Z}, S) = m_r \sum_k \omega_k(\bar{Z}) \varphi_k^I(S). \tag{3.9}$$

If $\{\phi_k^I\}$ is a complete system of functions, expression (3.9) represents the expansion of the vector function $U^I(\bar{Z}, S)$ in an infinite series. If however the system of functions ϕ_k^I is incomplete, expansion (3.9) defines the conical shell as a continuous system with a finite number of degrees of freedom with respect to the S coordinate. This means that a limited number of possible forms of spatial displacement, which in combination approximate the actual displacement of the contour, $\bar{Z}$ = const are imposed on that contour.

Introducing (3.3) and (3.7) into (3.1), we finally represent the vector function of total displacement $U(\bar{Z}, S)$ in the form

$$U(\bar{Z}, S) = \sum_{i=1}^{6+n} U_i(\bar{Z}) \lambda_i(\bar{Z}) \varphi_i(S). \tag{3.10}$$

Here, the first six terms pertain to the vector function $U^0(\bar{Z}, S)$;

U_1, U_2, U_3 are the components of the translational displacement vector η;
U_4, U_5, U_6 are the components of the rotation vector θ;

$$\lambda_i(\bar{Z}) = \begin{cases} 1 & (i = 1, 2, 3); \\ 1 - \bar{Z} & (i = 4, 5, 6). \end{cases} \tag{3.11}$$

The remaining terms of expansion (3.10) correspond to the vector function $U^I(\bar{Z}, S)$;

$\lambda_i(\bar{Z})|_{i=6+k}=\lambda_k(\bar{Z})$ is an arbitrarily chosen function;

$$U_i(\bar{Z})\lambda_i(\bar{Z})|_{i=6+k}=\omega_k(\bar{Z}),$$
$$\varphi_i(S)|_{i=6+k}=\varphi_k^1(S). \qquad (3.12)$$

n is the number of degrees of freedom of the contour $\bar{Z}$ = const with respect to warping $U^i(\bar{Z}, S)$.

The functions $U^i(\bar{Z})$ in expansion (3.10) are the principal scalar unknowns. After V.Z. Vlasov, these functions will be referred to as generalized displacements. The products of known functions $\lambda_i(\bar{Z})\phi_i(S)$ corresponding to them will be referred to as generalized coordinates.

Let t be an arbitrarily oriented unit vector. The displacement in the direction of t in accordance with (3.10) will be

$$u_t(\bar{Z}, S)=\sum_{i=1}^{6+n} V_i(\bar{Z})\varphi_{it}(S), \qquad (3.13)$$

where

$$\varphi_{it}=\varphi_i t \qquad (3.14)$$

and for brevity, we set

$$V_i(\bar{Z})=U_i(\bar{Z})\ \lambda_i(\bar{Z}). \qquad (3.15)$$

Superposing t on the unit vectors of the main moving trihodron, we expand geometric relations (2.2) with the aid of (3.13). The tangential strains of the middle surface can be represented in the form

$$\varepsilon^0_{m_s}=\sum_{i=1}^{6+n}\left[V'_i\frac{1}{l_s}\varphi_{im_s}+\frac{1}{1-\bar{Z}}V_i\bar{\varphi}_{im_s}\right],$$
$$\gamma^0_{n_s m_s}=\sum_{i=1}^{6+n}\left[V'_i\frac{1}{l_s}\varphi_{in_s m_s}+\frac{1}{1-\bar{Z}}V_i\bar{\varphi}_{in_s m_s}\right], \qquad (3.16)$$
$$\varepsilon^0_{n_s}=\sum_{i=1}^{6+n}\left[V'_i\frac{1}{l_s}\varphi_{in_s}+\frac{1}{1-\bar{Z}}V_i\bar{\varphi}_{in_s}\right].$$

where

$$\psi_{im_s}=\varphi_{im_s}; \quad \psi_{in_s}=-\operatorname{ctg}\chi\varphi_{in_s}; \quad \psi_{in_sm_s}=-\operatorname{ctg}\chi\varphi_{im_s}+\varphi_{in_s}; \tag{3.17}$$

$$\bar{\psi}_{im_s}=0; \quad \bar{\psi}_{in_s}=\frac{1}{\sin\chi}\varphi_{in_s}-\frac{1}{l_s}\varphi_{im_s}-\frac{1}{R_0}\varphi_{in_n};$$
$$\bar{\psi}_{in_sm_s}=\frac{1}{\sin\chi}\varphi'_{im_s}+\frac{1}{l_s}\varphi_{in_s}. \tag{3.18}$$

The bending strains of the middle surface will be represented in the form

$$\varkappa_{m_s}=\sum_{i=1}^{6+n}\left[V''_i\frac{1}{l_s^2}\vartheta_{im_s}+\frac{1}{1-Z}V'_i\frac{1}{l_s}\bar{\vartheta}_{im_s}+\frac{1}{(1-Z)^2}V_i\bar{\bar{\vartheta}}_{im_s}\right],$$
$$\varkappa_{n_sm_s}=\sum_{i=1}^{6+n}\left[V''_i\frac{1}{l_s^2}\vartheta_{in_sm_s}+\frac{1}{1-Z}V'_i\frac{1}{l_s}\bar{\vartheta}_{in_sm_s}+\frac{1}{(1-Z)^2}V_i\bar{\bar{\vartheta}}_{in_sm_s}\right], \tag{3.19}$$
$$\varkappa_{n_s}=\sum_{i=1}^{6+n}\left[V''_i\frac{1}{l_s^2}\vartheta_{in_s}+\frac{1}{1-Z}V'_i\frac{1}{l_s}\bar{\vartheta}_{in_s}+\frac{1}{(1-Z)^2}V_i\bar{\bar{\vartheta}}_{in_s}\right],$$

where

$$\vartheta_{im_s}=-\varphi_{in_n}, \quad \vartheta_{in_s}=-\operatorname{ctg}^2\chi\varphi_{in_n}, \quad \vartheta_{in_sm_s}=\operatorname{ctg}\chi\varphi_{in_n}, \tag{3.20}$$

$$\bar{\vartheta}_{im_s}=0, \quad \bar{\vartheta}_{in_sm_s}=-\frac{1}{\sin\chi}\varphi'_{in_n}-\frac{1}{R_0}\varphi_{in_s},$$
$$\bar{\vartheta}_{in_s}=\frac{\cos\chi}{\sin^2\chi}\varphi'_{in_n}+\frac{l_s}{\sin\chi}\left(\frac{\operatorname{ctg}\chi}{l_s}\varphi_{in_n}\right)'-\frac{1}{R_0}\operatorname{ctg}\chi\varphi_{in_s}+\frac{1}{l_s}\varphi_{in_n}, \tag{3.21}$$

$$\bar{\bar{\vartheta}}_{im_s}=0, \quad \bar{\bar{\vartheta}}_{in_sm_s}=-\frac{1}{l_s\sin\chi}\varphi'_{in_n}-\frac{1}{l_sR_0}\varphi_{in_s},$$
$$\bar{\bar{\vartheta}}_{in_s}=\frac{1}{l_s}\frac{\cos\chi}{\sin^2\chi}\varphi'_{in_n}-\frac{1}{\sin\chi}\left(\frac{1}{\sin\chi}\varphi'_{in_n}\right)'+\frac{\operatorname{ctg}\chi}{l_sR_0}\varphi_{in_s}-\frac{1}{\sin\chi}\left(\frac{1}{R_0}\varphi_{in_s}\right)'. \tag{3.22}$$

Expanding elasticity relations (2.3) with the aid of expressions (3.16)-(3.19), we have

$$T_{m_s}=\frac{Eh}{1-\nu^2}\sum_{i=1}^{6+n}\left[V'_i\frac{1}{l_s}(\psi_{im_s}+\nu\psi_{in_s})+\frac{1}{1-Z}V_i(\bar{\psi}_{im_s}+\nu\bar{\psi}_{in_s})\right],$$
$$S_{n_sm_s}=\frac{Eh}{2(1+\nu)}\sum_{i=1}^{6+n}\left[V'_i\frac{1}{l_s}\psi_{in_sm_s}+\frac{1}{1-Z}V_i\bar{\psi}_{in_sm_s}\right], \tag{3.23}$$
$$T_{n_s}=\frac{Eh}{1-\nu^2}\sum_{i=1}^{6+n}\left[V'_i\frac{1}{l_s}(\psi_{in_s}+\nu\psi_{im_s})+\frac{1}{1-Z}V_i(\bar{\psi}_{in_s}+\nu\bar{\psi}_{im_s})\right].$$

and

$$
\begin{aligned}
M_{m_s} &= D\sum_{i=1}^{6+n}\Big[V_i''\frac{1}{l_s^2}(\theta_{im_s}+\nu\theta_{in_s})+\frac{1}{1-\bar{Z}}V_i'\frac{1}{l_s}(\bar{\theta}_{im_s}+\nu\bar{\theta}_{in_s})+\\
&+\frac{1}{(1-\bar{Z})^2}V_i\left(\bar{\bar{\theta}}_{im_s}+\nu\bar{\bar{\theta}}_{in_s}\right)\Big],\\
H_{n_s m_s} &= (1-\nu)D\sum_{i=1}^{6+n}\Big[V_i''\frac{1}{l_s^2}\theta_{in_s m_s}+\frac{1}{1-\bar{Z}}V_i'\frac{1}{l_s}\bar{\theta}_{in_s m_s}+\\
&+\frac{1}{(1-\bar{Z})^2}V_i\bar{\bar{\theta}}_{in_s m_s}\Big],\\
M_{n_s} &= D\sum_{i=1}^{6+n}\Big[V_i''\frac{1}{l_s^2}(\theta_{in_s}+\nu\theta_{im_s})+\frac{1}{1-\bar{Z}}V_i'\frac{1}{l_s}(\bar{\theta}_{in_s}+\nu\bar{\theta}_{im_s})+\\
&+\frac{1}{(1-\bar{Z})^2}V_i\left(\bar{\bar{\theta}}_{in_s}+\nu\bar{\bar{\theta}}_{im_s}\right)\Big].
\end{aligned}
\tag{3.24}
$$

For the elements of the longitudinal structure, oriented along the generatrices, we have

$$
\begin{aligned}
N_k &= E\Delta F_k\sum_{i=1}^{6+n}V_i'\frac{1}{l_s}\varphi_{im_s},\\
M_k &= E\Delta I_k\sum_{i=1}^{6+n}V_i''\frac{1}{l_s^2}\theta_{im_s},\\
H_k &= \frac{E\Delta I_{tk}}{2(1+\nu)}\sum_{i=1}^{6+n}\Big[V_i''\frac{1}{l_s^2}\theta_{in_s m_s}+\frac{1}{1-\bar{Z}}V_i'\frac{1}{l_s}\bar{\theta}_{in_s m_s}+\\
&+\frac{1}{(1-\bar{Z})^2}V_i\bar{\bar{\theta}}_{in_s m_s}\Big],
\end{aligned}
\tag{3.25}
$$

where N_k, M_k, H_k are the axial force and the bending and twisting moments in the reinforcing element with number k;

I_k, I_{tk} are the moments of inertia.

Expressions (3.13) and(3.16)-(3.24) completely define the stressed and strained state of a conical shell of arbitrary configuration to within the functions $V_i(\bar{Z})$, related to the unknown generalized displacements $U_i(\bar{Z})$ by relations (3.15).

Since the relationship between the strains and functions V_i, which is represented by expressions (3.16)-(3.19), is established on the basis of geometrical relations (2.2), the unknown generalized displacements U_i should be determined from the equilibrium conditions of the shell, which are conveniently formulated on the basis of the origin of possible Lagrangian displacements.

Let the shell be at equilibrium under the action of a surface load and of some system of forces applied to the ends $\bar{Z} = 0$ and $\bar{Z} = \bar{Z}_1$. According to the origin of possible Lagrangian displacements, when the strained surface of the shell passes from the equilibrium position to any adjacent position allowed by the geometric constraints imposed on the shell,

$$\delta T - \delta U = 0, \tag{3.26}$$

where δT is the work done by the external forces;

δU is the potential energy change.

For the work of external forces we have the obvious formula

$$\begin{aligned} \delta T = \int_0^{\bar{Z}_1} \oint \mathbf{p} \cdot \delta \mathbf{U}\, AB \sin\gamma \, dS\, d\bar{Z} + \oint \bar{\mathbf{q}}^0 \cdot \delta \mathbf{U}\, B\, dS \Big|_{\bar{Z}=0} + \\ + \oint \mathbf{q}^1 \cdot \delta \mathbf{U}\, B\, dS \Big|_{\bar{Z}=\bar{Z}_1}. \end{aligned} \tag{3.27}$$

Here

$p(\bar{Z}, S)$ is the surface load vector referred to a unit area of the middle surface;

$\bar{q}^0(S)$, $\bar{q}^1(S)$ are vectors of the external load applied to the ends $\bar{Z} = 0$, $\bar{Z} = \bar{Z}_1$ and referred to a unit length of the contour.

In accordance with (3.10), we have

$$\delta \mathbf{U}(\bar{Z}, S) = \sum_{i=1}^{6+n} \delta U_i(\bar{Z}) \lambda_i(\bar{Z}) \boldsymbol{\varphi}_i(S). \tag{3.28}$$

Introducing (3.28) into (3.27), we find

$$\delta T = \int_0^{\bar{Z}_1} \sum_{i=1}^{6+n} R_i \delta U_i \, d\bar{Z} + \sum_{i=1}^{6+n} \bar{P}_i \delta U_i \Big|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1}. \tag{3.29}$$

where

$$R_i=\lambda_i B\oint \mathbf{p}\cdot\boldsymbol{\varphi}_i A\sin\chi\, dS,$$

$$\bar{P}_i(0)=-\oint \bar{\mathbf{q}}^0\cdot\boldsymbol{\varphi}_i\, dS, \tag{3.30}$$

$$\bar{P}_i(\bar{Z}_1)=\lambda_i(\bar{Z}_1)(1-\bar{Z}_1)\oint \mathbf{q}^1\cdot\boldsymbol{\varphi}_i\, dS. \tag{3.31}$$

The potential energy change is given by

$$\delta U=\int_0^{\bar{Z}_1}\oint \delta W AB\sin\chi\, dS\, d\bar{Z}. \tag{3.32}$$

We expand (3.32) with the aid of general expression (1.199). Keeping only the tangential forces, we have

$$\delta U=\int_0^{\bar{Z}_1}\oint (T_{m_2}\delta\varepsilon^0_{m_2}+S_{n_2m_2}\delta\gamma^0_{n_2m_2}+T_{n_2}\delta\varepsilon^0_{n_2})\, AB\sin\chi\, dS\, d\bar{Z}. \tag{3.33}$$

In accordance with expressions (3.16)

$$\begin{aligned}
\delta\varepsilon^0_{m_2} &= \sum_{i=1}^{6+n}\left[\delta V_i'\frac{1}{l_2}\psi_{im_2}+\frac{1}{1-\bar{Z}}\delta V_i\bar{\psi}_{im_2}\right],\\
\delta\gamma^0_{n_2m_2} &= \sum_{i=1}^{6+n}\left[\delta V_i'\frac{1}{l_2}\psi_{in_2m_2}+\frac{1}{1-\bar{Z}}\delta V_i\bar{\psi}_{in_2m_2}\right],\\
\delta\varepsilon^0_{n_2} &= \sum_{i=1}^{6+n}\left[\delta V_i'\frac{1}{l_2}\psi_{in_2}+\frac{1}{1-\bar{Z}}\delta V_i\bar{\psi}_{in_2}\right].
\end{aligned} \tag{3.34}$$

Introducing (3.34) into (3.33) and taking (3.15) into account, we find

$$\delta U=\int_0^{\bar{Z}_1}\sum_{i=1}^{6+n}\left[P_i\frac{(\delta U_i\lambda_i)'}{\lambda_i}+Q_i\delta U_i\right]d\bar{Z}, \tag{3.35}$$

where

$$P_i=\lambda_i B\oint (T_{m_2}\psi_{im_2}+S_{n_2m_2}\psi_{in_2m_2}+T_{n_2}\psi_{in_2})\sin\chi\, dS, \tag{3.36}$$

$$Q_i=\lambda_i\oint (T_{m_2}\bar{\psi}_{im_2}+S_{n_2m_2}\bar{\psi}_{in_2m_2}+T_{n_2}\bar{\psi}_{in_2})A\sin\chi\, dS. \tag{3.37}$$

Integrating the first term under the summation sign by parts in (3.35), we finally obtain

$$\delta U = \int_0^{\bar{Z}_1} \sum_{i=1}^{6+n} \left[\left(\frac{P_i}{\lambda_i}\right)' \lambda_i + Q_i \right] \delta U_i \, d\bar{Z} + \sum_{i=1}^{6+n} P_i \delta U_i \Bigg|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1}. \tag{3.38}$$

Expanding variational Eq. (3.26) with the aid of expressions (3.29) and (3.38), we obtain

$$\int_0^{\bar{Z}_1} \sum_{i=1}^{6+n} \left[\left(\frac{P_i}{\lambda_i}\right)' \lambda_i - Q_i + R_i \right] \delta U_i \, d\bar{Z} + \sum_{i=1}^{6+n} (\bar{P}_i - P_i) \delta U_i \Bigg|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1} = 0. \tag{3.39}$$

In the interval $(0, \bar{Z}_1)$, the variations of generalized displacements δU_i are arbitrary and independent. Therefore, on the basis of the well-known Lagrange lemma, the following equations follow from (3.39):

$$\left(\frac{P_j}{\lambda_j}\right)' \lambda_j - Q_j + R_j = 0 \quad (j = 1, 2, \ldots, 6+n) \tag{3.40}$$

and the natural boundary conditions are

$$(\bar{P}_j - P_j) \delta U_j \Bigg|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1} = 0 \quad (j = 1, 2, \ldots 6+n). \tag{3.41}$$

We will write Eqs. (3.40) in expanded form.

Using (3.23), we can readily obtain

$$\frac{1}{G} P_j(\bar{Z}) = \lambda_j \left[(1-\bar{Z}) \sum_{i=1}^{6+n} a_{ji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i \right]. \tag{3.42}$$

$$\frac{1}{G} Q_j(\bar{Z}) = \lambda_j \left[\sum_{i=1}^{6+n} b_{ij} V_i' + \frac{1}{1-\bar{Z}} \sum_{i=1}^{6+n} c_{ji} V_i \right]. \tag{3.43}$$

Here

$$a_{ji}=\frac{2}{1-\nu}\oint\left[\psi_{jm_2}\psi_{im_2}+\nu\left(\psi_{jm_2}\psi_{in_2}+\psi_{jn_2}\psi_{im_2}\right)+\psi_{jn_2}\psi_{in_2}\right]\frac{\sin\chi}{l_2}\,hdS+$$
$$+\oint\psi_{jn_2m_2}\psi_{in_2m_2}\frac{\sin\chi}{l_2}\,hdS,$$
$$b_{ji}=\frac{2}{1-\nu}\oint\left[\psi_{jm_2}\bar{\psi}_{im_2}+\nu\left(\psi_{jm_2}\bar{\psi}_{in_2}+\psi_{jn_2}\bar{\psi}_{im_2}\right)+\psi_{jn_2}\bar{\psi}_{in_2}\right]\sin\chi\,hdS+ \tag{3.44}$$
$$+\oint\psi_{jn_2m_2}\bar{\psi}_{in_2m_2}\sin\chi\,hdS,$$
$$c_{ji}=\frac{2}{1-\nu}\oint\left[\bar{\psi}_{jm_2}\bar{\psi}_{im_2}+\nu\left(\bar{\psi}_{jm_2}\bar{\psi}_{in_2}+\bar{\psi}_{jn_2}\bar{\psi}_{im_2}\right)+\bar{\psi}_{jn_2}\bar{\psi}_{in_2}\right]l_2\sin\chi\,hdS+$$
$$+\oint\bar{\psi}_{jn_2m_2}\bar{\psi}_{in_2m_2}\,l_2\sin\chi\,hdS.$$

Introducing expressions (3.42), (3.43) into Eqs. (3.40), we have

$$\sum_{i=1}^{6+n}\left\{\left[(1-\bar{Z})a_{ji}V_i'+b_{ji}V_i\right]'-b_{ij}V_i'-\frac{1}{1-\bar{Z}}c_{ji}V_i\right\}=-\frac{1}{\lambda_j G}R_j \quad (j=1,2,\ldots,6+n). \tag{3.45}$$

Expressions (3.45) represent a resolvent system of ordinary differential equations in 6+n unknowns $V_i(\bar{Z})$, related to the desired generalized displacements as follows:

$$V_i(\bar{Z})=\lambda_i(\bar{Z})U_i(\bar{Z}) \quad (i=1,2,\ldots,6+n).$$

From expressions (3.44) it is easy to see that the matrices $\|a_{ji}\|$ and $\|c_{ji}\|$ are symmetric about the main diagonal, this being in complete accord with the Betti reciprocity theory.

As can be readily ascertained by using (3.30), Eqs. (3.45) do not contain the factors $\lambda_j(\bar{Z})$ in explicit form. Hence, when (3.13) and (3.16) are taken into account, it follows that when j>6, the factor $\lambda_i(\bar{Z})$ can be chosen arbitrarily. Therefore, hereinafter we will take

$$\lambda_j(\bar{Z})=1 \quad (j=7,8,\ldots,6+n). \tag{3.46}$$

Expanding the right-hand sides of (3.45) with the aid of (3.30), we obtain

$$\frac{1}{\nu_j t_i} R_j = \frac{1}{t_i}(1-\bar{Z}) \oint [p^0_{m_1}(\bar{Z}, S)\varphi_{jm_1}(S) + p^0_{n_1}(\bar{Z}, S)\varphi_{jn_1}(S) + + p^0_{n_n}(\bar{Z}, S)\varphi_{jn_n}(S)]\, l_s \sin\gamma\, dS \quad (j = 1, 2, \ldots, 6+n). \tag{3.47}$$

The general solution of system (3.45) contains 2(6+n) arbitrary constraints; this fully corresponds to the number of degrees of freedom of the end sections of the shell $\bar{Z} = 0$ and $\bar{Z} = \bar{Z}_l$. These constraints are subject to determination from natural boundary conditions (3.41).

Let the displacements be given at one of the shell ends. Then, the values of all the generalized displacements U_i in this section are known. By setting

$$U_i = U_i^0 \quad (i = 1, 2, \ldots, 6+n), \tag{3.48}$$

for the corresponding value of the $\bar{Z}$ coordinate, where U_i^0 are known values of generalized displacements, we will satisfy the natural boundary conditions for this section, since in the latter all variations $\delta U_i = 0$ because of (3.41).

Let the forces be given in one of the end sections. Then conditions (3.41) for this section for arbitrary and independent variations δU_i will be satisfied only when we set for the corresponding value of the $\bar{Z}$ coordinate

$$P_j = \bar{P}_j \quad (j = 1, 2, \ldots, 6+n), \tag{3.49}$$

where the quantities $\bar{P}_j$, pertaining to the end sections of the shell, are represented by expressions (3.31).

It is easy to see that the products of P_j and $\bar{P}_j$ by the generalized displacement δU_j entering into (3.41) have the dimension of work. Accordingly, P_j and $\bar{P}_j$ will be referred to as generalized forces. Thus, the static boundary conditions according to

(3.49) and (3.31), (3.36) are formulated in the form of equality of the unknown generalized forces to the given forces.

The case of mixed boundary conditions, when the displacements are given on some portions of the end section of the shell, and the forces are given on other portions is also possible. In this case, the variations δU_i for the corresponding value of the $\bar{Z}$ coordinate are not completely independent, substantially complicating the determination of arbitrary constants. This case will be examined separately.

Coefficients (3.44) of differential resolvents (3.45) are written for a shell without a reinforcing structure. In the presence of the latter, in variational Eq. (3.26), it is also necessary to consider the potential energy change of the elements of this structure. As shown by comparative calculations and experimental studies, within the scope of applicability of the adopted static-geometric model, only the consideration of the potential energy of longitudinal elements oriented in the direction of the generatrices is essential, while the potential energy of transverse diaphragms can be neglected. In this case, the potential energy variation of the entire system

$$\delta U_{\Sigma} = \delta U + \delta U_F. \tag{3.50}$$

Here

δU is the variation in the potential energy of the shell proper, represented by expression (3.35);

δU_F is the variation in the potential energy of the elements of the longitudinal structure:

$$\delta U_F = \sum_k \int_0^{\bar{Z}_1} \delta W_{F_k} l_k \, d\bar{Z}, \tag{3.51}$$

where

$$\delta W_{F_k} = N_k \delta\varepsilon^0_{m_k}, \tag{3.52}$$

are variations in the potential energy of a unit length of the kth longitudinal element.

Introducing (3.52) into (3.51) and taking (3.34) and (3.18) into account, we obtain

$$\delta U_F=\int_0^{\bar{Z}_1}\sum_{i=1}^{6+n}P_{F_i}\frac{(\delta U_i\lambda_i)'}{\lambda_i}\,d\bar{Z}, \tag{3.53}$$

where

$$P_{F_i}=\lambda_i\sum_k N_k\psi_{im_k}(S_k). \tag{3.54}$$

From (3.50) and considering (3.35) and (3.54), we now have

$$\delta U_\Sigma=\int_0^{\bar{Z}_1}\sum_{i=1}^{6+n}\left[P_{\Sigma i}\frac{(\delta U_i\cdot\lambda_i)'}{\lambda_i}+Q_i\delta U_i\right]d\bar{Z}, \tag{3.55}$$

where

$$P_{\Sigma i}=P_i+P_{F_i}. \tag{3.56}$$

Substituting (3.55) into variational Eq. (3.26), as before, we obtain

$$\left(\frac{P_{\Sigma j}}{\lambda_j}\right)'\lambda_j-Q_j+R_j=0\quad (j=1,\ 2,\dots,\ 6+n), \tag{3.57}$$

$$(\bar{P}_{\Sigma j}-P_{\Sigma j})\delta U_j\Big|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1}=0\quad (j=1,\ 2,\dots,\ 6+n). \tag{3.58}$$

Here

$$\bar{P}_{\Sigma j}=\bar{P}_j+\bar{P}_{F_j}, \tag{3.59}$$

where $\bar{P}_{Fj}$ corresponds to the work done by external forces $\bar{N}_k^0$, $\bar{N}_k^1$ applied at sections $\bar{Z}=0$ and $\bar{Z}=\bar{Z}_1$ respectively to the elements of the longitudinal structure:

$$\begin{aligned}\bar{P}_{F_j}(0)&=-\sum_k\bar{N}_k^0\cdot\varphi_j(S_k),\\ \bar{P}_{F_j}(\bar{Z}_1)&=\lambda_j(\bar{Z}_1)\sum_k\bar{N}_k^1\cdot\varphi_j(S_k).\end{aligned} \tag{3.60}$$

Let us expand expression (3.57). Using (3.25), we obtain

$$\frac{1}{G} P_{Fj}(\bar{Z}) = \lambda_j (1-\bar{Z}) \sum_{i=1}^{6+n} a_{Fji} V_i', \tag{3.61}$$

where

$$a_{Fji} = \frac{2(1+\nu)}{1-\bar{Z}} \sum_k \gamma_{jm_k} \gamma_{im_k} \frac{\Delta F_k}{l_{sk}}. \tag{3.62}$$

Introducing expressions (3.42) and (3.61) into (3.56), we have

$$\frac{1}{G} P_{\Sigma j}(\bar{Z}) = \lambda_j \left[(1-\bar{Z}) \sum_{i=1}^{6+n} a_{\Sigma ji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i \right], \tag{3.63}$$

where

$$a_{\Sigma ji} = a_{ji} + a_{Fji}. \tag{3.64}$$

Now, using (3.63), we obtain from (3.57)

$$\sum_{i=1}^{6+n} \left\{ \left[(1-\bar{Z}) a_{\Sigma ji} V_i' + b_{ji} V_i \right]' - b_{ij} V_i' - \frac{1}{1-\bar{Z}} c_{ji} V_i \right\} = -\frac{1}{\lambda_j G} R_j \quad (j=1, 2, \ldots, 6+n). \tag{3.65}$$

Thus, the resolvent systems of ordinary differential equations for a reinforced and nonreinforced shell are structurally identical. Consideration of the reinforcing structure affects only the matrix of coefficients $||a_{\Sigma ji}|| = ||a_{ji}|| + ||a_{Fji}||$.

Equations (3.65) have variable coefficients. Making the substitution of variables

$$1-\bar{Z} = e^t, \tag{3.66}$$

we obtain

$$\sum_{i=1}^{6+n} (a_{\Sigma ji} V_i' + b_{ji} V_i)' - \sum_{i=1}^{6+n} b_{ij} V_i' - \sum_{i=1}^{6+n} c_{ji} V_i = -\frac{e^t}{\lambda_j G} R_j \quad (j=1, 2, \ldots, 6+n), \tag{3.67}$$

where

$$(\ )' = \frac{d}{dt} (\ ).$$

Now, turning to expressions (3.44), (3.62), (3.64), we can readily observe that the coefficients of system (3.67) are constant is the shell thickness depends only on the S coordinate:

$$h(Z, S) = h(S),$$

and the cross-sectional areas of the elements of the longitudinal structure depend linearly on the $\overline{Z}$ coordinate:

$$\Delta F_k = \Delta F_k^0 (1 - \overline{Z}).$$

In this case, system (3.67) assumes the form

$$\sum_{i=1}^{6+n} a_{zji} V_i'' + \sum_{i=1}^{6+n} (b_{ij} - b_{ji}) V_i' - \sum_{i=1}^{6+n} c_{ji} V_i = -\frac{e^t}{\lambda_j G} R_j \tag{3.68}$$

$$(j = 1, 2, \ldots, 6+n).$$

Switching to the variable t according to (3.66), from (3.23) and (3.42), (3.61) and considering (3.56), we obtain

$$\begin{aligned} T_{m_s}(t, S) &= \frac{Eh}{1-\nu^2} e^{-t} \sum_{i=1}^{6+n} \left[-V_i' \frac{1}{l_s} (\psi_{im_s} + \nu \psi_{in_s}) + V_i (\dot{\psi}_{im_s} + \nu \dot{\psi}_{in_s}) \right], \\ S_{n_s m_s}(t, S) &= \frac{Eh}{2(1+\nu)} e^{-t} \sum_{i=1}^{6+n} \left[-V_i' \frac{1}{l_s} \psi_{in_s m_s} + V_i \dot{\psi}_{in_s m_s} \right], \\ T_{n_s}(t, S) &= \frac{Eh}{1-\nu^2} e^{-t} \sum_{i=1}^{6+n} \left[-V_i' \frac{1}{l_s} (\psi_{in_s} + \nu \psi_{im_s}) + V_i (\dot{\psi}_{in_s} + \nu \dot{\psi}_{im_s}) \right]. \end{aligned} \tag{3.69}$$

$$\frac{1}{G} P_{zj}(t) = \lambda_j(t) \left[-\sum_{i=1}^{6+n} a_{zji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i \right]. \tag{3.70}$$

3.2. Normal Form of Differential Resolvents. First Integrals

A resolvent system of ordinary differential equations is conveniently represented in normal form if one introduces the new unknowns

$$P_j^* = \frac{1}{\lambda_j} P_{3j} \quad (j=1, 2, \ldots, 6+n). \tag{3.71}$$

Expanding Eqs. (3.57) with the aid of (3.43), adding to the expressions obtained relations (3.63) with j=1,2, ..., 6+n, and taking (3.71) into consideration, we obtain

$$\begin{gathered} \frac{1}{G} P_i^{*\prime} - \sum_{i=1}^{6+n} b_{ij} V_i' - \frac{1}{1-\overline{Z}} \sum_{i=1}^{6+n} c_{ji} V_i = -\frac{1}{\lambda_j G} R_j, \\ (1-\overline{Z}) \sum_{i=1}^{6+n} a_{3ji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i - \frac{1}{G} P_j^* = 0 \\ (j=1, 2, \ldots, 6+n). \end{gathered} \tag{3.72}$$

Expressions (3.72) represent in mixed form a system of 2(6+n) first-order differential resolvents in 6+n unknown functions $V_i(\overline{Z})$ and the same number of unknown functions $P_i^*(\overline{Z})$. It is evident that with the aid of a linear transformation, system (3.72) can be represented in the normal Cauchy form.

System (3.72) has variable coefficients. However, if the shell thickness depends only on S, and the sections of the elements of the longitudinal structure change linearly with the length, then, on passing in (3.72) to the new variable t in accordance with (3.66), we arrive at a system with constant coefficients:

$$\begin{gathered} \frac{1}{G} P_j^{*\prime} - \sum_{i=1}^{6+n} b_{ij} V_i' + \sum_{i=1}^{6+n} c_{ji} V_i = \frac{e^t}{\lambda_j G} R_j, \\ \sum_{i=1}^{6+n} a_{3ji} V_i' - \sum_{i=1}^{6+n} b_{ji} V_i + \frac{1}{G} P_j^* = 0 \\ (j=1, 2, \ldots, 6+n). \end{gathered} \tag{3.73}$$

System (3.72) allows a reduction of the order form 2(6+n) to 6+2n with the aid of the first six integrals, which can be found for a shell of arbitrary configuration.

Let us turn to expressions (3.16). Assuming that the shell undergoes translational displacement as a solid in an arbitrary direction, we have

$$V_i^0(\overline{Z}) = \begin{cases} \text{const} & (i=1, 2, 3), \\ 0 & (i>3). \end{cases} \tag{3.74}$$

Since there are no strains in the shell when it undergoes displacement, we have from (3.16), taking (3.74) into consideration,

$$\sum_{i=1}^{4} V_i \psi_{im_s} - \sum_{i=1}^{4} V_i \psi_{in_s m_s} - \sum_{i=1}^{4} V_i \psi_{in_s} = 0, \tag{3.75}$$

whence because of the arbitrariness of V_i, we find

$$\bar{\psi}_{im_s} = \bar{\psi}_{in_s m_s} = \bar{\psi}_{in_s} = 0 \quad (i = 1,\ 2,\ 3). \tag{3.76}$$

Let the shell now arbitrarily rotate as a solid about the apex. We will first consider the case in which the point of intersection of the plane of the contour $\bar{Z}$ = const with the Oz axis is chosen as the pole of rotation of this contour, and the corresponding vectors $\eta(\bar{Z})$, $\theta(\bar{Z})$ are expanded in the Cartesian system of coordinates xyz. In this case, taking (2.14) into account, we have

$$V_4(\bar{Z}) = (1-\bar{Z})\theta_x; \quad V_5(\bar{Z}) = (1-\bar{Z})\theta_y; \quad V_6(\bar{Z}) = (1-\bar{Z})\theta_z, \tag{3.77}$$

where θ_x, θ_y, θ_z are arbitrary constants.

The functions $V_1(\bar{Z})$, $V_2(\bar{Z})$, $V_3(\bar{Z})$ represent, according to (2.14), the components of the displacement vector of the point $M_0(0,\ 0,\ l_0\bar{Z})$. Setting in (2.7) $\eta=0$, $\theta=\theta_x i + \theta_y j+\theta_z k$ and also $r_0=l_0 k$ and $r=l_0\bar{Z}k$, we find

$$V_1(\bar{Z}) = -\theta_y l_0(1-\bar{Z}); \quad V_2(\bar{Z}) = \theta_x l_0(1-\bar{Z}); \quad V_3(\bar{Z}) = 0. \tag{3.78}$$

Since strains and warping of the section are absent when the shell moves as a solid, by introducing (3.77) and (3.78) into (3.16) and taking (3.76) into account, we obtain relations of the type

$$\theta_x\left(-\frac{l_0}{l_s}\psi_{2\tau_k} - \frac{1}{l_s}\psi_{4\tau_k} + \bar{\psi}_{1\tau_k}\right) + \theta_y\left(\frac{l_0}{l_s}\psi_{1\tau_k} - \frac{1}{l_s}\psi_{5\tau_k} + \bar{\psi}_{2\tau_k}\right) + \\ + \theta_z\left(-\frac{1}{l_s}\psi_{6\tau_k} + \bar{\psi}_{3\tau_k}\right) = 0 \quad (k = 1,\ 2,\ 3). \tag{3.79}$$

where for brevity of notation, γ_1, γ_2, γ_3 denote the indices m_z, $n_z m_z$, n_z respectively.

From (3.79), taking the independence of θ_x, θ_y, θ_z into consideration, we have the following relations:

$$\begin{aligned}
\bar{\psi}_{4\gamma_k} &= \frac{1}{l_s}\left(l_0\dot{\psi}_{2\gamma_k} + \dot{\psi}_{4\gamma_k}\right),\\
\bar{\psi}_{5\gamma_k} &= \frac{1}{l_s}\left(-l_0\dot{\psi}_{1\gamma_k} + \dot{\psi}_{5\gamma_k}\right),\\
\bar{\psi}_{6\gamma_k} &= \frac{1}{l_s}\dot{\psi}_{6\gamma_k} \qquad (k=1,\ 2,\ 3).
\end{aligned} \tag{3.80}$$

According to formula (3.18), for any i

$$\bar{\psi}_{im_z}=0. \tag{3.81}$$

Taking (3.81) into consideration, we have from (3.80)

$$\dot{\psi}_{4m_z} = -l_0\dot{\psi}_{2m_z};\quad \dot{\psi}_{5m_z} = l_0\dot{\psi}_{1m_z};\quad \dot{\psi}_{6m_z}=0. \tag{3.82}$$

Considering (3.76) and relations (3.80), (3.82) from expressions (3.36) (3.37), (3.54), (3.56) we can readily obtain

$$Q_1=Q_2=Q_3=0, \tag{3.83}$$

$$Q_4=l_0P_2^{\circ}+P_4^{\circ};\quad Q_5=-l_0P_1^{\circ}+P_5^{\circ};\quad Q_6=P_6^{\circ}. \tag{3.84}$$

Relations (3.84) make it possible to find the first six integrals of the resolvent system of ordinary differential equations (3.72).

Let us turn to Eqs. (3.57). Considering (3.71) and relations (3.83), (3.84), we find

$$\begin{aligned}
&P_j^{\circ\prime} = -R_j \qquad (j=1,\ 2,\ 3),\\
&\left[(1-\bar{Z})P_4^{\circ}\right]' - l_0P_2^{\circ} = -R_4,\\
&\left[(1-\bar{Z})P_5^{\circ}\right]' + l_0P_1^{\circ} = -R_5,
\end{aligned} \tag{3.85}$$

$$\left[(1-\bar{Z})P_6^{\circ}\right]' = -R_6. \tag{3.86}$$

The system of six Eqs. (3.85), (3.86) is easily integrated. From Eqs. (3.85), we have

$$P_j^* = \overline{P}_j^* \qquad (j = 1, 2, 3), \tag{3.87}$$

where

$$\overline{P}_j^*(\overline{Z}) = P_j^*(\overline{Z}_1) - \int_{\overline{Z}_1}^{\overline{Z}} R_j(\xi)\, d\xi. \tag{3.88}$$

Eliminating P_1^* and P_2^* from Eqs. (3.86) with the aid of expressions (3.87), we also obtain

$$P_j^* = \overline{P}_j^* \qquad (j = 4, 5, 6). \tag{3.89}$$

Here

$$\begin{aligned}
\overline{P}_4^*(\overline{Z}) &= \frac{1}{1-\overline{Z}} \left\{ (1-\overline{Z}_1) P_4^*(\overline{Z}_1) - \int_{\overline{Z}_1}^{\overline{Z}} [R_4(\xi) - l_0 \overline{P}_2^*(\xi)]\, d\xi \right\}, \\
\overline{P}_5^*(\overline{Z}) &= \frac{1}{1-\overline{Z}} \left\{ (1-\overline{Z}_1) P_5^*(\overline{Z}_1) - \int_{\overline{Z}_1}^{\overline{Z}} [R_5(\xi) + l_0 \overline{P}_1^*(\xi)]\, d\xi \right\}, \\
\overline{P}_6^*(\overline{Z}) &= \frac{1}{1-\overline{Z}} \left\{ (1-\overline{Z}_1) P_6^*(\overline{Z}_1) - \int_{\overline{Z}_1}^{\overline{Z}} R_6(\xi)\, d\xi \right\}.
\end{aligned} \tag{3.90}$$

Expressions (3.87), (3.89), which determine the six unknown functions P_j^* $(j=1,2,\ldots 6)$ to within the arbitrary constants $\overline{P}_j^*(\overline{Z}_1)$, represent the first six integrals of the resolvent system of ordinary differential Eqs. (3.72). This system contains 2(6+n) equations. The first six of them, equivalent to Eqs. (3.85), (3.86), can now be discarded and replaced by expressions (3.87), (3.86). The order of system (3.72) will thereby be reduced by six units.

The first integrals of (3.87), (3.89) also make it possible to reduce the order of the resolvent system of differential equations (3.65) by six units. This system contains 6+n equations representing the expanded notation of Eqs. (3.57). For this reason, the first six equations of system (3.65) can be replaced by the first integrals of (3.87), (3.89), P_j^* having been eliminated from the latter. Using (3.63), and taking (3.71) into account, we obtain

$$(1-\overline{Z})\sum_{i=1}^{6+n} a_{\varepsilon ji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i = \frac{1}{G}\overline{P}_j^{\circ} \qquad (j=1, 2, \ldots, 6) \tag{3.91}$$

or, replacing the variable $\overline{Z}$ according to (3.66),

$$-\sum_{i=1}^{6+n} a_{\varepsilon ji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i = \frac{1}{G}\overline{P}_j^{\circ} \qquad (j=1, 2, \ldots, 6). \tag{3.92}$$

Adding Eqs. (3.91) to Eqs. (3.65) when $j>6$, we obtain a complete system of differential resolvents of order $6+2n$.

We have discussed the case in which the point of intersection of the plane of the contour $\overline{Z}$ = const with the Oz axis is taken as the pole of rotation of this contour, and the corresponding vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ are expanded in the Cartesian system of coordinates xyz. We will now consider the case in which the apex of the conical surface is taken as the pole common to all sections of $\overline{Z}$ = const, and the corresponding vectors $\eta(\overline{Z})$, $\theta(\overline{Z})$ are expanded in arbitrary bases e_α and $\hat{e}_\alpha$;

$$\eta(\overline{Z}) = \eta^{\alpha}(\overline{Z})\, e_{\alpha}, \quad \theta(\overline{Z}) = \hat{\theta}^{\alpha}(\overline{Z})\, \hat{e}_{\alpha},$$

where we apply the term of canonical kinematic unknowns to the components η^α, θ^α

In this case, for an arbitrary displacement of the shell as a solid, all the components of vectors η and θ are independent of the $\overline{Z}$ coordinate. Therefore, taking relations (2.95) into consideration, we have

$$U_l(\overline{Z}) = \frac{V_l(\overline{Z})}{\lambda_l(\overline{Z})} = \begin{cases} \eta^l = \text{const} & (l=1, 2, 3), \\ \hat{\theta}^{l-3} = \text{const} & (l=4, 5, 6). \end{cases} \tag{3.93}$$

Since strains and warping of the section are absent for any displacement of the shell as a solid, from (3.16) and taking (3.93) into account, we have

$$\eta^{\circ}\dot{\psi}_{am_z}=\eta^{\circ}\dot{\psi}_{an_z m_z}=\eta^{\circ}\dot{\psi}_{an_z}=0 \tag{3.94}$$

and

$$\bar{\theta}^{\circ}\left(\dot{\bar{\psi}}_{am_z}-\frac{1}{l_s}\psi_{am_z}\right)=\bar{\theta}^{\circ}\left(\dot{\bar{\psi}}_{an_z m_z}-\frac{1}{l_s}\psi_{an_z m_z}\right)=\bar{\theta}^{\circ}\left(\dot{\bar{\psi}}_{an_z}-\frac{1}{l_s}\psi_{an_z}\right)=0, \tag{3.95}$$

whence in view of the arbitrariness of the displacement of the shell, the following important relations result:

$$\dot{\bar{\psi}}_{im_z}=\dot{\bar{\psi}}_{in_z m_z}=\dot{\bar{\psi}}_{in_z}=0 \qquad (i=1,\ 2,\ 3), \tag{3.96}$$

$$\dot{\bar{\psi}}_{im_z}=\frac{1}{l_s}\psi_{im_z};\quad \dot{\bar{\psi}}_{in_z m_z}=\frac{1}{l_s}\psi_{in_z m_z};\quad \dot{\bar{\psi}}_{in_z}=\frac{1}{l_s}\psi_{in_z} \qquad (i=4,\ 5,\ 6). \tag{3.97}$$

Considering (3.96) and relations (3.97), we can readily obtain from (3.36), (3.37), (3.54), (3.56)

$$\begin{aligned} &Q_i=0 \qquad (i=1,\ 2,\ 3),\\ &Q_i=P_i^{\circ} \qquad (i=4,\ 5,\ 6). \end{aligned} \tag{3.98}$$

Now, using (3.98), we can readily find the first six integrals of system (3.72).

Let us turn to Eqs. (3.57). Considering (3.11), (3.71) and (3.98), we have

$$(\lambda_j P_j^{\circ})'=-R_j \qquad (j=1,\ 2,\ldots,\ 6), \tag{3.99}$$

whence

$$P_j^{\circ}=\bar{P}_j^{\circ} \qquad (j=1,\ 2,\ldots,\ 6), \tag{3.100}$$

where

$$\bar{P}_j^{\circ}(\bar{z})=\frac{1}{\lambda_j(\bar{z})}\left[\lambda_j(\bar{z}_1)P_j^{\circ}(\bar{z}_1)-\int_{\bar{z}_1}^{\bar{z}}R_j(\xi)d\xi\right]. \tag{3.101}$$

Expressions (3.100) represent the first six integrals of system (3.72) for the case under consideration, which make it possible to reduce the order of both the system of 6+n resolvents (3.65) and the system of 2(6+n) resolvents (3.72) by six units.

Let us elucidate the meaning of the first integrals which we obtained for the system of differential resolvents of a conical shell of arbitrary configuration.

We will expand in arbitrary bases e_α and $\hat{e}_\alpha$ the vectors of the resultant and net moment of the internal forces acting in the section $\bar{Z}$ = const. Omitting the forces Q_{m_z}, Q_{n_z} and moments M_{m_z}, $H_{n_z m_z}$, M_{n_z} in expressions (2.37), we get

$$\begin{aligned} \mathfrak{R} = B\mathbf{e}_\alpha \oint [T_{m_z} \mathbf{m}_z \cdot \mathbf{e}^\alpha + S_{n_z m_z}(-\operatorname{ctg}\chi\, \mathbf{m}_z + \mathbf{n}_z)\cdot \mathbf{e}^\alpha - \\ - T_{n_z} \operatorname{ctg}\chi\, \mathbf{n}_z \cdot \mathbf{e}^\alpha] \sin\chi\, dS, \\ \mathfrak{M} = -B\hat{\mathbf{e}}_\alpha \oint [T_{m_z} \mathbf{m}_z \times (\mathbf{r}-\mathbf{r}_0)\cdot \hat{\mathbf{e}}^\alpha + S_{n_z m_z}(-\operatorname{ctg}\chi\, \mathbf{m}_z + \mathbf{n}_z)\times \\ \times (\mathbf{r}-\mathbf{r}_0)\cdot \hat{\mathbf{e}}^\alpha - T_{n_z} \operatorname{ctg}\chi\, \mathbf{n}_z \times (\mathbf{r}-\mathbf{r}_0)\cdot \hat{\mathbf{e}}^\alpha] \sin\chi\, dS. \end{aligned} \tag{3.102}$$

Expressions (3.102) pertain to the shell proper. For the resultant and net moment of the internal forces acting in the elements of the longitudinal structure in the section $\bar{Z}$ = const, we also have

$$\begin{aligned} \mathfrak{R}_p = \mathbf{e}_\alpha \sum_k N_k \mathbf{m}_z \cdot \mathbf{e}^\alpha, \\ \mathfrak{M}_p = -\hat{\mathbf{e}}_\alpha \sum_k N_k \mathbf{m}_z \times (\mathbf{r}-\mathbf{r}_0)\cdot \hat{\mathbf{e}}^\alpha. \end{aligned} \tag{3.103}$$

Let a point with radius vector r_0 be chosen as the pole of rotation of the contour $\bar{Z}$ = const as a solid. Then, developing the vector $\eta(\bar{Z})$ of translational displacement of the contour $\bar{Z}$ = const together with the pole in the basis e^α, and vector $\theta(\bar{Z})$ of rotation of the contour $\bar{Z}$ = const relative to this pole in the basis $\hat{e}^\alpha$, we represent the vector function $U^0(\bar{Z}, S)$ in the form

$$\mathbf{U}^0(\bar{Z}, S) = \eta_\alpha(\bar{Z})\mathbf{e}^\alpha + \hat{\theta}_\alpha(\bar{Z})\hat{\mathbf{e}}^\alpha \times (\mathbf{r}-\mathbf{r}_0). \tag{3.104}$$

or, introducing the notation

$$\begin{aligned} \eta_\alpha = U_i, \quad \mathbf{e}^\alpha = \tilde{\varphi}_i \qquad (i=\alpha=1, 2, 3), \\ \hat{\theta}_\alpha = U_i, \quad \hat{\mathbf{e}}^\alpha \times (\mathbf{r}-\mathbf{r}_0) = \tilde{\varphi}_i \quad (i=\alpha+3=4, 5, 6), \end{aligned} \tag{3.105}$$

in the form

$$\mathbf{U}^0(\bar{Z}, S) = \sum_{i=1}^{6} U_i(\bar{Z})\tilde{\varphi}_i(\bar{Z}, S). \tag{3.106}$$

where in accordance with the terminology adopted above, the unknown functions U_i will be referred to as, generalized displacements, and the known functions $\tilde{\phi}_i$ as generalized coordinates.

For the displacement $u_t^0 = U^0 \cdot t$ in the direction of an arbitrarily oriented unit vector t, we have from (3.106)

$$u_t^0(\bar{Z}, S) = \sum_{i=1}^{6} U_i(\bar{Z}) \tilde{\varphi}_{it}(\bar{Z}, S), \tag{3.107}$$

where

$$\tilde{\varphi}_{it} = \tilde{\varphi}_i \cdot t. \tag{3.108}$$

Expression (3.106) is general in character. If the pole of rotation of the contour $\bar{Z}$ = const as a solid coincides with the apex of a conical shell or with the point of intersection of the plane of this contour with the Oz axis, then, as was shown above, the generalized coordinates $\tilde{\phi}(\bar{Z}, S)$ may be represented in the form

$$\tilde{\varphi}_i(\bar{Z}, S) = \lambda_i(\bar{Z}) \varphi_i(S), \tag{3.109}$$

where

$$\lambda_i(\bar{Z}) = \begin{cases} 1 & (i = 1, 2, 3), \\ 1 - \bar{Z} & (i = 4, 5, 6). \end{cases} \tag{3.110}$$

In the remaining cases, the dependence of $\tilde{\phi}$ on the coordinate $\bar{Z}$ is more complex in character.

Considering (3.105) and (3.108), the vectors of the resultant and net moment of internal forces in the section $\bar{Z}$ = const, represented by expressions (3.102) and (3.103), may be written in the form

$$\mathfrak{R} = B \sum_{i=1}^{3} \mathbf{e}_i \oint [T_{m_z}\tilde{\varphi}_{im_z} + S_{n_z m_z}(-\operatorname{ctg}\chi\tilde{\varphi}_{im_z} + \tilde{\varphi}_{in_z}) -$$
$$- T_{n_z} \operatorname{ctg}\chi\tilde{\varphi}_{in_z}] \sin\chi\, dS,$$
$$\mathfrak{M} = B \sum_{i=4}^{6} \mathbf{e}_{i-3} \oint [T_{m_z}\tilde{\varphi}_{im_z} + S_{n_z m_z}(-\operatorname{ctg}\chi\tilde{\varphi}_{im_z} + \tilde{\varphi}_{in_z}) - \tag{3.111}$$
$$- T_{n_z} \operatorname{ctg}\chi\tilde{\varphi}_{in_z}] \sin\chi\, dS,$$
$$\mathfrak{R}_F = \sum_{i=1}^{3} \mathbf{e}_i \sum_k N_k \tilde{\varphi}_{im_z},$$
$$\mathfrak{M}_F = \sum_{i=4}^{6} \mathbf{e}_{i-3} \sum_k N_k \tilde{\varphi}_{im_z}. \tag{3.112}$$

Comparing expressions (3.111) with (3.36) and (3.112) with (3.54), and taking (3.17), (3.108), (3.109), into consideration, we see that the generalized forces P_i and P_{Fi} (i=1, 2,3) are the components of the resultant, and generalized forces P_i, P_{Fi} (i=4,5,6) are the components of the net moment of the external forces applied to the cut-off portion of the shell between the current section $\bar{Z}$ = const and the end section $\bar{Z} = \bar{Z}_l$. Let

$$\mathbf{p}(\bar{Z}, S) = p^0_{m_z}(\bar{Z}, S)\mathbf{m}_z(S) + p^0_{n_z}(\bar{Z}, S)\mathbf{n}_z(S) + p^0_{n_n}(\bar{Z}, S)\mathbf{n}_n(S)$$

be the vector of the external surface load referred to the area of the middle surface. Then expression (2.35) can be rewritten in the form

$$\mathfrak{R}^*(\bar{Z}) = \mathfrak{R}^0_l + \int_{\bar{Z}}^{\bar{Z}_l} \oint \mathbf{p}(\xi, S) AB \sin\chi\, dS\, d\xi, \tag{3.113}$$
$$\mathfrak{M}^*(\bar{Z}) = \mathfrak{M}^0_l - \mathfrak{R}^0_l \times [\mathbf{r}_l - \mathbf{r}_0(\bar{Z})] -$$
$$- \int_{\bar{Z}}^{\bar{Z}_l} \oint \mathbf{p}(\xi, S) \times [\mathbf{r}(\xi, S) - \mathbf{r}_0(\bar{Z})] A(S) B(\xi) \sin\chi(S)\, dS\, d\xi, \tag{3.114}$$

where $r_l = r_0(\bar{Z}_l)$ is the radius vector of the point of application of vectors $\mathfrak{R}^0_l$, $\mathfrak{M}^0_l$, which are, respectively, the resultant and the net moment of the external forces applied to the end $\bar{Z} = \bar{Z}_l$.

Differentiating (3.114), we have

$$\frac{d\mathfrak{M}^{e}}{d\bar{Z}}=\mathfrak{M}_1^0\times\frac{d\mathbf{r}_0}{d\bar{Z}}+\oint\mathbf{p}(\bar{Z},S)\times[\mathbf{r}(\bar{Z},S)-\mathbf{r}_0(\bar{Z})]\,AB\sin\chi\,dS-$$

$$-\frac{d\mathbf{r}_0}{d\bar{Z}}\times\int_{\bar{Z}}^{\bar{Z}_1}\oint\mathbf{p}(\xi,S)\,AB\sin\chi\,dS\,d\xi, \tag{3.115}$$

or, using (3.113),

$$\frac{d\mathfrak{M}^{e}}{d\bar{Z}}=\mathfrak{R}^{e}\times\frac{d\mathbf{r}_0}{d\bar{Z}}+\oint\mathbf{p}(\bar{Z},S)\times[\mathbf{r}(\bar{Z},S)-\mathbf{r}_0(\bar{Z})]\,AB\sin\chi\,dS. \tag{3.116}$$

Integrating (3.116), we obtain

$$\mathfrak{M}^{e}(\bar{Z})=\mathfrak{M}_1^0+\int_{\bar{Z}_1}^{\bar{Z}}\mathfrak{R}^{e}\times\frac{d\mathbf{r}_0}{d\bar{Z}}\,d\xi+\int_{\bar{Z}_1}^{\bar{Z}}\oint\mathbf{p}(\xi,S)\times[\mathbf{r}(\xi,S)-\mathbf{r}_0(\xi)]\,AB\sin\chi\,dS\,d\xi. \tag{3.117}$$

We will now expand the vectors $\mathfrak{R}^{e}$ and $\mathfrak{M}^{e}$ in bases $\mathbf{e}_\alpha$ and $\hat{\mathbf{e}}_\alpha$. We have

$$\mathfrak{R}^{e}=(\mathfrak{R}_1^0\cdot\mathbf{e}^{\alpha})\,\mathbf{e}_\alpha-\mathbf{e}_\alpha\int_{\bar{Z}_1}^{\bar{Z}}\int\mathbf{p}\cdot\mathbf{e}^{\alpha}\,AB\sin\chi\,dS\,d\xi, \tag{3.118}$$

$$\mathfrak{M}^{e}=(\mathfrak{M}_1^0\cdot\hat{\mathbf{e}}^{\alpha})\,\hat{\mathbf{e}}_\alpha+\hat{\mathbf{e}}_\alpha\int_{\bar{Z}_1}^{\bar{Z}}\mathfrak{R}^{e}\times\frac{d\mathbf{r}_0}{d\bar{Z}}\cdot\hat{\mathbf{e}}^{\alpha}\,d\xi+$$

$$+\hat{\mathbf{e}}_\alpha\int_{\bar{Z}_1}^{\bar{Z}}\oint\mathbf{p}\times(\mathbf{r}-\mathbf{r}_0)\cdot\hat{\mathbf{e}}^{\alpha}AB\sin\chi\,dS\,d\xi. \tag{3.119}$$

Taking (3.105) into account, we can finally represent the vectors $\mathfrak{R}^{e}$ and $\mathfrak{M}^{e}$ in the form

$$\mathfrak{R}^{e}=(\mathfrak{R}_1^0\cdot\mathbf{e}^{\alpha})\,\mathbf{e}_\alpha-\sum_{i=1}^{3}\mathbf{e}_i\int_{\bar{Z}_1}^{\bar{Z}}\oint\mathbf{p}\,\varphi_i\,AB\sin\chi\,dS\,d\xi, \tag{3.120}$$

$$\mathfrak{M}^{e}=(\mathfrak{M}_1^0\cdot\hat{\mathbf{e}}^{\alpha})\,\hat{\mathbf{e}}_\alpha+\hat{\mathbf{e}}_\alpha\int_{\bar{Z}_1}^{\bar{Z}}\mathfrak{R}^{e}\cdot\left(\frac{d\mathbf{r}_0}{d\bar{Z}}\times\hat{\mathbf{e}}^{\alpha}\right)d\xi-$$

$$-\sum_{i=4}^{6}\hat{\mathbf{e}}_{i-3}\int_{\bar{Z}_1}^{\bar{Z}}\oint\mathbf{p}\cdot\varphi_i\,AB\sin\chi\,dS\,d\xi. \tag{3.121}$$

Expression (3.121) includes the derivative $dr_0/d\bar{Z}$ of the radius vector of the pole of rotation of the contour $\bar{Z}$ = const as a solid. If this pole is chosen common to all the sections of $\bar{Z}$ = const, then obviously

$$\frac{d\mathbf{r}_0}{d\bar{Z}}=0. \tag{3.122}$$

If however the point of intersection of the plane of contour $\bar{Z}$ = const with the Oz axis is chosen as this pole, then

$$\mathbf{r}_0(\bar{Z})=l_0\bar{Z}\mathbf{k},$$

whence

$$\frac{d\mathbf{r}_0}{d\bar{Z}}=l_0\mathbf{k}. \tag{3.123}$$

Now, using expressions (3.120), (3.121) and considering (3.122), (3.123), we can readily ascertain that the right-hand sides of Eq. (3.87), (3.89) and (3.100) constitute the components, referred to the factor $\lambda_j(\bar{Z})$, of the vectors of the resultant and net moment of the external forces applied to the cut-off portion of the shell. As follows from the above, in view of (3.59), (3.73), the left-hand sides of these equations constitute the components of the resultant and of the net moment of all internal forces in the section $\bar{Z}$ = const, also referred to the factor $\lambda_j(\bar{Z})$. It follows that the first integrals of (3.87), (3.89) and (3.100) of the system of differential resolvents of a conical shell of arbitrary configuration represent the conditions of equilibrium of the cut-off portion of the shell, and hence, the first six equations of system (3.57) corresponding to them express the equilibrium conditions of an elementary strip defined by sections $\bar{Z}$ = const and $\bar{Z} + d\bar{Z}$ = const. For example, Eqs. (3.85), (3.86), equivalent to the first six equations of system (3.57), as follows from the above, can be written in the form

$$\begin{aligned}&Q_x'=-R_x; &&Q_y'=-R_y; &&N_z'=-R_z,\\ &M_x'-l_0Q_y=-R_{M_x}; &&M_y'+l_0Q_x=-R_{M_y}; &&M_z'=-R_{M_z},\end{aligned} \tag{3.124}$$

where $Q_x=P_1$, $Q_y=P_2$, $N_z=P_3$ are the components of the resultant of internal forces in the section $\overline{Z}$ = const, and $M_x=P_4$, $M_y=P_5$, $M_z=P_6$ are the components of the vector moment of the same forces with respect to the axes passing through point $M_0(0, 0, l_0\overline{Z})$ parallel to the axes of the Cartesian coordinate system.

Expressions (3.124) represent an extension of the known Zhuravskiy formulas to the case of oblique beam. Therefore, we will hereinafter refer to the generalized forces P_1, P_2 as transverse forces, to the generalized force P_3 as axial force, and to the generalized forces P_4, P_5, and P_6 as bending and twisting moments, respectively.

3.3. Canonical System of Resolvents

If a single point coinciding with the apex of a conical shell is chosen as the pole of rotation of the contour $\overline{Z}$ = const as a solid, the corresponding system of solving differential equations will be called canonical.

We will represent the matrices $||a_{\Sigma jl}||$, $||b_{jl}||$, $||c_{jl}||$ of the coefficients of the differential resolvent in the form of block matrices:

$$A=\left(\begin{array}{c|c|c} A_{\eta\eta} & A_{\eta\theta} & A_{\eta\Omega} \\ \hline A_{\theta\eta} & A_{\theta\theta} & A_{\theta\Omega} \\ \hline A_{\Omega\eta} & A_{\Omega\theta} & A_{\Omega\Omega} \end{array}\right), \qquad B=\left(\begin{array}{c|c|c} B_{\eta\eta} & B_{\eta\theta} & B_{\eta\Omega} \\ \hline B_{\theta\eta} & B_{\theta\theta} & B_{\theta\Omega} \\ \hline B_{\Omega\eta} & B_{\Omega\theta} & B_{\Omega\Omega} \end{array}\right), \tag{3.125}$$

$$C=\left(\begin{array}{c|c|c} C_{\eta\eta} & C_{\eta\theta} & C_{\eta\Omega} \\ \hline C_{\theta\eta} & C_{\theta\theta} & C_{\theta\Omega} \\ \hline C_{\Omega\eta} & C_{\Omega\theta} & C_{\Omega\Omega} \end{array}\right).$$

where blocks of the form

$M_{\eta\eta}$, $M_{\eta\theta}$, $M_{\theta\eta}$, $M_{\theta\theta}$ are square matrices of third order;
$M_{\eta\Omega}$, $M_{\theta\Omega}$ are [3 x n] matrices;
$M_{\Omega\eta}$, $M_{\Omega\theta}$ are [n x 3] matrices;
$M_{\Omega\Omega}$ are square matrices of nth order.

The coefficients a_{jl}, b_{jl}, c_{jl} are represented by expressions (3.44), (3.62), (3.64). Since according to (3.18)

$$\dot{\psi}_i = 0 \quad (i = 1, 2, \ldots, 6 + n), \tag{3.126}$$

then, also considering (3.96) and relations (3.97), for matrix blocks (3.125) one can readily establish that

$$B_{\xi\eta} = B_{\theta\eta} = B_{\zeta\eta} = 0, \tag{3.127}$$

$$C_{\xi\xi} = C_{\eta\theta} = C_{\theta\xi} = C_{\xi\zeta} = C_{\zeta\xi} = 0, \tag{3.128}$$

$$B_{\eta\theta} = A_{\xi\theta}; \quad B_{\theta\xi} = A_{\theta\theta}; \quad B_{\zeta\theta} = A_{\zeta\eta}, \tag{3.129}$$

$$C_{\theta\theta} = A_{\xi\theta}; \quad C_{\xi\zeta} = B_{\xi\zeta}. \tag{3.130}$$

We will also need the matrix of coefficient B_{lj}, which can evidently be obtained by transforming matrix B:

$$B' = \left(\begin{array}{c|c|c} B'_{\xi\xi} & B'_{\theta\xi} & B'_{\zeta\xi} \\ \hline B'_{\xi\theta} & B'_{\theta\theta} & B'_{\zeta\theta} \\ \hline B'_{\xi\zeta} & B'_{\theta\zeta} & B'_{\zeta\zeta} \end{array}\right), \tag{3.131}$$

where [ ]' denotes the transposition of the corresponding matrix.

On the basis of (1.127), we conclude that

$$B'_{\eta\eta} = B'_{\theta\eta} = B'_{\zeta\eta} = 0. \tag{3.132}$$

Using expression (3.44), (3.62), (3.64), we can readily observe that matrices $||a_{jl}||$ and $||c_{jl}||$ are symmetric. Hence

$$A'_{\theta\theta} = A_{\theta\theta}, \quad A'_{\xi\eta} = A_{\eta\xi}, \quad A'_{\zeta\eta} = A_{\eta\zeta}, \quad C'_{\xi\zeta} = C_{\zeta\xi}. \tag{3.133}$$

Considering (3.129), (3.130) and (1.133), we also have

$$\mathbf{B}'_{\eta\theta}=\mathbf{A}_{\theta\eta},\quad \mathbf{B}'_{\theta\theta}=\mathbf{A}_{\theta\theta},\quad \mathbf{B}'_{\Omega\theta}=\mathbf{A}_{\theta\Omega}. \tag{3.134}$$

On the basis of (3.127)-(3.130) and (3.132)-(3.134), matrices B, B' and C can be represented in the form:

$$\mathbf{B}=\left(\begin{array}{c|c|c} 0 & \mathbf{A}_{\eta\theta} & \mathbf{B}_{\eta\Omega} \\ \hline 0 & \mathbf{A}_{\theta\theta} & \mathbf{B}_{\theta\Omega} \\ \hline 0 & \mathbf{A}_{\Omega\theta} & \mathbf{B}_{\Omega\Omega} \end{array}\right),\qquad \mathbf{B}'=\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \mathbf{A}_{\theta\eta} & \mathbf{A}_{\theta\theta} & \mathbf{A}_{\theta\Omega} \\ \hline \mathbf{B}'_{\eta\Omega} & \mathbf{B}'_{\theta\Omega} & \mathbf{B}'_{\Omega\Omega} \end{array}\right)$$

$$\mathbf{C}=\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \mathbf{A}_{\theta\theta} & \mathbf{B}_{\theta\Omega} \\ \hline 0 & \mathbf{B}'_{\theta\Omega} & \mathbf{C}_{\Omega\Omega} \end{array}\right). \tag{3.135}$$

We will represent the generalized forces $P_{\Sigma I}(\overline{Z})$ determined by expressions (3.63) in the form of a block column matrix

$$\mathbf{P}=\left(\begin{array}{c} \mathbf{P}_{\eta} \\ \hline \mathbf{P}_{\theta} \\ \hline \mathbf{P}_{\Omega} \end{array}\right). \tag{3.136}$$

where P_{η}, P_{θ} are column matrices, each composed of three elements; P_{Ω} is a column matrix of n elements.

In accordance with (3.42), we have

$$\frac{1}{G}\Lambda^{-1}\mathbf{P}=(1-\overline{Z})\,\mathbf{A}\frac{d}{d\overline{Z}}\mathbf{V}+\mathbf{BV}. \tag{3.137}$$

Here

Λ is a diagonal matrix. In block form

$$\Lambda = \begin{pmatrix} \Lambda_\eta & 0 & 0 \\ 0 & (1-\bar{Z})\Lambda_\theta & 0 \\ 0 & 0 & \Lambda_\Omega \end{pmatrix}. \tag{3.138}$$

where Λ_η Λ_θ are unit matrices of third order;

Λ_Ω is a unit matrix of nth order;

V is a block column matrix:

$$V = \begin{pmatrix} V_\eta \\ V_\theta \\ V_\Omega \end{pmatrix}. \tag{3.139}$$

where V_η, V_θ are column matrices of three elements each;

V_Ω is a column matrix of n elements.

According to (3.15)

$$V = \Lambda U. \tag{3.140}$$

Here U is a block column matrix:

$$U = \begin{pmatrix} \eta \\ \theta \\ \Omega \end{pmatrix}. \tag{3.141}$$

where η, θ are common matrices of three elements each:

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}; \quad \theta = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \end{pmatrix}; \tag{3.142}$$

Ω is a column matrix of n elements. According to (3.12) and (3.46)

$$\Omega=\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}. \tag{3.143}$$

Introducing representation (3.140) into expression (3.137) and carrying out the differentiation operation, we reduce (3.137) to the form

$$\frac{1}{G}\,\mathrm{A}^{-1}\mathrm{P}=(1-\bar{Z})\,\mathrm{A}\Lambda\,\frac{d}{d\bar{Z}}\,\mathrm{U}+\left[(1-\bar{Z})\,\mathrm{A}\,\frac{d}{d\bar{Z}}\,\Lambda+\mathrm{B}\Lambda\right]\mathrm{U}. \tag{3.144}$$

Considering (3.125), (3.135) and (3.138), and also the fact that matrices Λ_η, Λ_θ, Λ_Ω are unit matrices, we can readily obtain

$$\mathrm{A}\,\frac{d}{d\bar{Z}}\,\Lambda=\left(\begin{array}{c:c:c} 0 & -\mathrm{A}_{\eta\theta} & 0 \\ \hdashline 0 & -\mathrm{A}_{\theta\theta} & 0 \\ \hdashline 0 & -\mathrm{A}_{\Omega\theta} & 0 \end{array}\right), \tag{3.145}$$

$$\mathrm{B}\Lambda=(1-\bar{Z})\left(\begin{array}{c:c:c} 0 & \mathrm{A}_{\eta\theta} & \dfrac{1}{1-\bar{Z}}\,\mathrm{B}_{\eta\Omega} \\ \hdashline 0 & \mathrm{A}_{\theta\theta} & \dfrac{1}{1-\bar{Z}}\,\mathrm{B}_{\theta\Omega} \\ \hdashline 0 & \mathrm{A}_{\Omega\theta} & \dfrac{1}{1-\bar{Z}}\,\mathrm{B}_{\Omega\Omega} \end{array}\right), \tag{3.146}$$

whence we find

$$(1-\bar{Z})\,\mathrm{A}\,\frac{d}{d\bar{Z}}\,\Lambda+\mathrm{B}\Lambda=\left(\begin{array}{c:c:c} 0 & 0 & \mathrm{B}_{\eta\Omega} \\ \hdashline 0 & 0 & \mathrm{B}_{\theta\Omega} \\ \hdashline 0 & 0 & \mathrm{B}_{\Omega\Omega} \end{array}\right)=\mathrm{F}. \tag{3.147}$$

Considering (3.147), from (3.144) we have

$$\frac{1}{G} A^{-1}P = (1-\bar{Z})\, A\Lambda \frac{d}{d\bar{Z}} U + FU. \tag{3.148}$$

Let us introduce into consideration the block column matrix

$$Q = \begin{pmatrix} Q_\eta \\ Q_\theta \\ Q_\Omega \end{pmatrix}, \tag{3.149}$$

where Q_η, Q_θ are column matrices of three elements each;

Q_Ω is a column matrix of n elements.

Matrix elements Q_η, Q_θ, Q_Ω are determined by expression (3.43). Considering (3.125), (3.131), (3.138), (3.139) and (3.149), from (3.43) we have

$$\frac{1}{G} A^{-1}Q = B' \frac{d}{d\bar{Z}} V + CV. \tag{3.150}$$

Using (3.140), we reduce expression (3.150) to the form

$$\frac{1}{G} A^{-1}Q = B'\Lambda \frac{d}{d\bar{Z}} U + \left[B' \frac{d}{d\bar{Z}} \Lambda + \frac{1}{1-\bar{Z}} C\Lambda\right] U. \tag{3.151}$$

Considering (3.135) and (3.138), we can readily obtain

$$B' \frac{d}{d\bar{Z}} \Lambda = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & -A_{\theta\theta} & 0 \\ \hline 0 & -B_{\Omega\theta}' & 0 \end{array}\right). \tag{3.152}$$

$$C\Lambda=(1-\bar{Z})\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & A_{\theta\theta} & \frac{1}{1-\bar{Z}}B_{\theta\Omega} \\ \hline 0 & B'_{\Omega\theta} & \frac{1}{1-\bar{Z}}C_{\Omega\Omega} \end{array}\right). \tag{3.153}$$

whence

$$(1-\bar{Z})B'\frac{d}{d\bar{Z}}\Lambda+C\Lambda=\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & B_{\theta\Omega} \\ \hline 0 & 0 & C_{\Omega\Omega} \end{array}\right)=G. \tag{3.154}$$

Considering (3.154), from (3.151) we have

$$\frac{1}{a}\Lambda^{-1}Q=B'\Lambda\frac{d}{d\bar{Z}}U+\frac{1}{1-\bar{Z}}GU. \tag{3.155}$$

Let us now turn to Eqs. (3.57). In matrix notation, these equations may be represented in the form

$$\frac{d}{d\bar{Z}}(\Lambda^{-1}P)-\Lambda^{-1}Q=-\Lambda^{-1}R. \tag{3.156}$$

Here R is a column matrix whose elements are determined by expression (3.47). In block form

$$R=\begin{pmatrix} R_\eta \\ R_\theta \\ R_\Omega \end{pmatrix}. \tag{3.157}$$

where R_η, R_θ are column matrices of three elements each;
R_Ω is a column matrix of n elements.

Expanding Eqs. (3.156) with the aid of (3.148) and (3.155), we have

$$\frac{d}{d\bar{Z}}(1-\bar{Z})A\Lambda\frac{d}{d\bar{Z}}U+(F-B'\Lambda)\frac{d}{d\bar{Z}}U+\left(\frac{d}{d\bar{Z}}F-\frac{1}{1-\bar{Z}}G\right)U=-\frac{1}{G}\Lambda^{-1}R. \qquad (3.158)$$

Expression (3.158) represents a matrix differential resolvent in column matrix U of generalized displacements.

We will write the matrix differential resolvents in mixed form. Considering (3.148), (3.155), (3.156) we have

$$\frac{1}{G}\Lambda^{-1}P=(1-\bar{Z})A\Lambda\frac{d}{d\bar{Z}}U+FU, \qquad (3.159)$$

$$\frac{1}{G}\frac{d}{d\bar{Z}}(\Lambda^{-1}P)-B'\Lambda\frac{d}{d\bar{Z}}U-\frac{1}{1-\bar{Z}}GU=-\frac{1}{G}\Lambda^{-1}R. \qquad (3.160)$$

To the system of block equations (3.159), (3.160), there corresponds a system of six matrix equations. Considering (3.125), (3.135), (3.138), (3.147), (3.154), and (3.136), (3.141), (3.147), we have

$$\begin{aligned}
&\frac{1}{G}P_\eta=(1-\bar{Z})\left[A_{\eta\eta}\frac{d}{d\bar{Z}}\eta+(1-\bar{Z})A_{\eta\theta}\frac{d}{d\bar{Z}}\theta+A_{\eta\Omega}\frac{d}{d\bar{Z}}\Omega\right]+B_{\eta\Omega}\Omega,\\
&\frac{1}{G}\frac{1}{1-\bar{Z}}P_\theta=(1-\bar{Z})\left[A_{\theta\eta}\frac{d}{d\bar{Z}}\eta+(1-\bar{Z})A_{\theta\theta}\frac{d}{d\bar{Z}}\theta+\right.\\
&\qquad\left.+A_{\theta\Omega}\frac{d}{d\bar{Z}}\Omega\right]+B_{\theta\Omega}\Omega,\\
&\frac{1}{G}P_\Omega=(1-\bar{Z})\left[A_{\Omega\eta}\frac{d}{d\bar{Z}}\eta+(1-\bar{Z})A_{\Omega\theta}\frac{d}{d\bar{Z}}\theta+\right.\\
&\qquad\left.+A_{\Omega\Omega}\frac{d}{d\bar{Z}}\Omega\right]+B_{\Omega\Omega}\Omega.
\end{aligned} \qquad (3.161)$$

$$\begin{aligned}
&\frac{1}{G}\frac{d}{d\bar{Z}}P_\eta=-\frac{1}{G}R_\eta,\\
&\frac{1}{G}\frac{d}{d\bar{Z}}\left(\frac{1}{1-\bar{Z}}P_\theta\right)-A_{\theta\eta}\frac{d}{d\bar{Z}}\eta-(1-\bar{Z})A_{\theta\theta}\frac{d}{d\bar{Z}}\theta-A_{\theta\Omega}\frac{d}{d\bar{Z}}\Omega-\\
&\qquad-\frac{1}{1-\bar{Z}}B_{\theta\Omega}\Omega=-\frac{1}{G}\frac{1}{1-\bar{Z}}R_\theta,\\
&\frac{1}{G}\frac{d}{d\bar{Z}}P_\Omega-B'_{\eta\Omega}\frac{d}{d\bar{Z}}\eta-(1-\bar{Z})B'_{\theta\Omega}\frac{d}{d\bar{Z}}\theta-B'_{\Omega\Omega}\frac{d}{d\bar{Z}}\Omega-\\
&\qquad-\frac{1}{1-\bar{Z}}C_{\Omega\Omega}\Omega=-\frac{1}{G}R_\Omega.
\end{aligned} \qquad (3.162)$$

Integrating the first equation of system (3.162), we have

$$\mathbf{P}_{\eta}=\overline{\mathbf{P}}_{\eta}, \tag{3.163}$$

where

$$\overline{\mathbf{P}}_{\eta}=\overline{\mathbf{P}}_{\eta}(\overline{Z}_1)-\int_{\overline{Z}_1}^{\overline{Z}}\mathbf{R}_{\eta}(\xi)\,d\xi. \tag{3.164}$$

Multiplying the second equation of system (3.162) by $1-\overline{Z}$ and subtracting the result from the second equation of system (3.161), we obtain

$$\frac{d}{d\overline{Z}}\mathbf{P}_{\theta}=-\mathbf{R}_{\theta}, \tag{3.165}$$

whence

$$\mathbf{P}_{\theta}=\overline{\mathbf{P}}_{\theta}, \tag{3.166}$$

where

$$\overline{\mathbf{P}}_{\theta}=\overline{\mathbf{P}}_{\theta}(\overline{Z}_1)-\int_{\overline{Z}_1}^{\overline{Z}}\mathbf{R}_{\theta}(\xi)\,d\xi. \tag{3.167}$$

Expressions (3.163), (3.164) and (3.166), (3.167) represent the first six integrals of the system of matrix equations (3.161), (3.162). These first integrals express the equilibrium conditions of the cut-off portion of the shell. The corresponding results were obtained in scalar form in the preceding section.

We will now represent the system of Eqs. (3.159), (3.160) in the normal Cauchy form. Multiplying Eq. (3.159) on the left by the product of matrices $\Lambda^{-1}A^{-1}$ reciprocal to Λ and A, we obtain

$$(1-\overline{Z})\frac{d}{d\overline{Z}}\mathbf{U}=\frac{1}{G}\,\Lambda^{-1}\mathbf{A}^{-1}\mathbf{A}^{-1}\mathbf{P}-\Lambda^{-1}\mathbf{A}^{-1}\mathbf{F}\mathbf{U}. \tag{3.168}$$

Introducing (3.168) into (3.160), we also have

$$\frac{1}{G}(1-\overline{Z})\frac{d}{d\overline{Z}}(\Lambda^{-1}\mathbf{P})=\frac{1}{G}\,\mathbf{B}'\Lambda^{-1}\mathbf{A}^{-1}\mathbf{P}+(\mathbf{G}-\mathbf{B}'\Lambda^{-1}\mathbf{F})\mathbf{U}-\frac{1}{G}(1-\overline{Z})\Lambda^{-1}\mathbf{R}. \tag{3.169}$$

Expressions (3.168), (3.169) represent a mixed form of the differential rosolvents in the normal Cauchy form. A system of six matrix equations corresponds to these equations.

Representing matrix A^{-1} in the form

$$A^{-1}=\begin{pmatrix} \tilde{A}_{\eta\eta} & \tilde{A}_{\eta\varphi} & \tilde{A}_{\eta\theta} \\ \tilde{A}_{\varphi\eta} & \tilde{A}_{\varphi\varphi} & \tilde{A}_{\varphi\theta} \\ \tilde{A}_{\theta\eta} & \tilde{A}_{\theta\varphi} & \tilde{A}_{\theta\theta} \end{pmatrix}, \tag{3.170}$$

and considering (3.135), (3.138), (3.147), (3.154) we can readily obtain

$$A^{-1}F=\begin{pmatrix} 0 & 0 & D_\eta \\ 0 & 0 & D_\varphi \\ 0 & 0 & D_\theta \end{pmatrix}, \quad B'A^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ H_{\varphi\eta} & H_{\varphi\varphi} & H_{\varphi\theta} \\ D'_\eta & D'_\varphi & D'_\theta \end{pmatrix}, \quad G-B'A^{-1}F=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & L_\varphi \\ 0 & 0 & L_\theta \end{pmatrix}, \tag{3.171}$$

where

$$\tag{3.172}$$

$$\begin{aligned} D_\eta &= \tilde{A}_{\eta\eta}B_{\eta\theta} + \tilde{A}_{\eta\varphi}B_{\varphi\theta} + \tilde{A}_{\eta\theta}B_{\theta\theta}, \\ D_\varphi &= \tilde{A}_{\varphi\eta}B_{\eta\theta} + \tilde{A}_{\varphi\varphi}B_{\varphi\theta} + \tilde{A}_{\varphi\theta}B_{\theta\theta}, \\ D_\theta &= \tilde{A}_{\theta\eta}B_{\eta\theta} + \tilde{A}_{\theta\varphi}B_{\varphi\theta} + \tilde{A}_{\theta\theta}B_{\theta\theta}. \end{aligned} \tag{3.173}$$

$$\begin{aligned} H_{\varphi\eta} &= A_{\varphi\eta}\tilde{A}_{\eta\eta} + A_{\varphi\varphi}\tilde{A}_{\varphi\eta} + A_{\varphi\theta}\tilde{A}_{\theta\eta}, \\ H_{\varphi\varphi} &= A_{\varphi\eta}\tilde{A}_{\eta\varphi} + A_{\varphi\varphi}\tilde{A}_{\varphi\varphi} + A_{\varphi\theta}\tilde{A}_{\theta\varphi}, \\ H_{\varphi\theta} &= A_{\varphi\eta}\tilde{A}_{\eta\theta} + A_{\varphi\varphi}\tilde{A}_{\varphi\theta} + A_{\varphi\theta}\tilde{A}_{\theta\theta}. \\ L_\varphi &= B_{\varphi\theta} - H_{\varphi\eta}B_{\eta\theta} - H_{\varphi\varphi}B_{\varphi\theta} - H_{\varphi\theta}B_{\theta\theta}, \\ L_\theta &= C_{\theta\theta} - D'_\eta B_{\eta\theta} - D'_\varphi B_{\varphi\theta} - D'_\theta B_{\theta\theta}. \end{aligned} \tag{3.174}$$

Using (3.138), (3.170), (3.171), and (3.136), (3.141), (3.157), from (3.168), (3.169) we have

$$
\begin{aligned}
&(1-\bar{Z})\frac{d}{d\bar{Z}}\eta=\frac{1}{G}\left(\tilde{A}_{\eta\eta}P_\eta+\frac{1}{1-\bar{Z}}\tilde{A}_{\eta\theta}P_\theta+\tilde{A}_{\eta\Omega}P_\Omega\right)-D_\eta\Omega,\\
&(1-\bar{Z})\frac{d}{d\bar{Z}}\theta=\frac{1}{G}\frac{1}{1-\bar{Z}}\left(\tilde{A}_{\theta\eta}P_\eta+\frac{1}{1-\bar{Z}}\tilde{A}_{\theta\theta}P_\theta+\tilde{A}_{\theta\Omega}P_\Omega\right)-\frac{1}{1-\bar{Z}}D_\theta\Omega,\\
&(1-\bar{Z})\frac{d}{d\bar{Z}}\Omega=\frac{1}{G}\left(\tilde{A}_{\Omega\eta}P_\eta+\frac{1}{1-\bar{Z}}\tilde{A}_{\Omega\theta}P_\theta+\tilde{A}_{\Omega\Omega}P_\Omega\right)-D_\Omega\Omega,\\
&\frac{1}{G}(1-\bar{Z})\frac{d}{d\bar{Z}}P_\eta=-\frac{1}{G}(1-\bar{Z})R_\eta,\\
&\frac{1}{G}(1-\bar{Z})\frac{d}{d\bar{Z}}\left(\frac{1}{1-\bar{Z}}P_\theta\right)=\frac{1}{G}\left(H_{\theta\eta}P_\eta+\frac{1}{1-\bar{Z}}H_{\theta\theta}P_\theta+H_{\theta\Omega}P_\Omega\right)+\\
&\qquad\qquad+L_\theta\Omega-\frac{1}{G}R_\theta,\\
&\frac{1}{G}(1-\bar{Z})\frac{d}{d\bar{Z}}P_\Omega=\frac{1}{G}\left(D'_\eta P_\eta+\frac{1}{1-\bar{Z}}D'_\theta P_\theta+D'_\Omega P_\Omega\right)+\\
&\qquad\qquad+L_\Omega\Omega-\frac{1}{G}(1-\bar{Z})R_\Omega.
\end{aligned}
\qquad (3.176)
$$

Expressions (3.175), (3.176) form a complete system of six equations in six unknowns: η, θ, Ω and P_η, P_θ, P_Ω.

As was shown above, the first two integrals of this system are represented by relations (3.163), (3.166). By adding (3.163), (3.166) to system (3.175), (3.176), this permits one to exclude from consideration the first two equations of (3.176). In this connection, the mixed form of the resolvent system of ordinary differential equations in the normal Cauchy form can be finally written in the form

$$
\begin{aligned}
&P_\eta=\bar{P}_\eta,\\
&P_\theta=\bar{P}_\theta,
\end{aligned}
\qquad (3.177)
$$

$$
\begin{aligned}
&(1-\bar{Z})\frac{d}{d\bar{Z}}\eta=\frac{1}{G}\left(\tilde{A}_{\eta\eta}\bar{P}_\eta+\frac{1}{1-\bar{Z}}\tilde{A}_{\eta\theta}\bar{P}_\theta+\tilde{A}_{\eta\Omega}P_\Omega\right)-D_\eta\Omega,\\
&(1-\bar{Z})\frac{d}{d\bar{Z}}\theta=\frac{1}{G}\frac{1}{1-\bar{Z}}\left(\tilde{A}_{\theta\eta}\bar{P}_\eta+\frac{1}{1-\bar{Z}}\tilde{A}_{\theta\theta}\bar{P}_\theta+\tilde{A}_{\theta\Omega}P_\Omega\right)-\frac{1}{1-\bar{Z}}D_\theta\Omega,\\
&(1-\bar{Z})\frac{d}{d\bar{Z}}\Omega+D_\Omega\Omega-\frac{1}{G}\tilde{A}_{\Omega\Omega}P_\Omega=\frac{1}{G}\left(\tilde{A}_{\Omega\eta}\bar{P}_\eta+\frac{1}{1-\bar{Z}}\tilde{A}_{\Omega\theta}\bar{P}_\theta\right).
\end{aligned}
\qquad (3.178)
$$

$$
\begin{aligned}
&\frac{1}{G}(1-\bar{Z})\frac{d}{d\bar{Z}}P_\Omega-L_\Omega\Omega-\frac{1}{G}D'_\Omega P_\Omega=\frac{1}{G}\left(D'_\eta\bar{P}_\eta+\frac{1}{1-\bar{Z}}D'_\theta\bar{P}_\theta\right)-\\
&\qquad\qquad-\frac{1}{G}(1-\bar{Z})R_\Omega.
\end{aligned}
\qquad (3.179)
$$

Expressions (3.177), (3.179) constitute a canonical system of resolvents. This system was partially decomposed. Equations (3.179) constitute an independent system with respect to warping Ω and the corresponding generalized force P_Ω. Integrating Eqs.

(3.179), then introducing the result into the right-hand members of Eqs. (3.178), we obtain two independent equations in vectors η and θ. Integrating these equations, we have

$$\begin{aligned}
\eta(\bar{Z}) &= \eta(0)+\frac{1}{G}\int_0^{\bar{Z}}\left(\tilde{A}_{\eta\eta}\bar{P}_\eta+\frac{1}{1-\xi}\tilde{A}_{\eta\theta}\bar{P}_\theta+\tilde{A}_{\eta\Omega}P_\Omega\right)\frac{d\xi}{1-\xi}-\int_0^{\bar{Z}}D_\eta\Omega\frac{d\xi}{1-\xi}, \\
\theta(\bar{Z}) &= \theta(0)+\frac{1}{G}\int_0^{\bar{Z}}\left(\tilde{A}_{\theta\eta}\bar{P}_\eta+\frac{1}{1-\xi}\tilde{A}_{\theta\theta}\bar{P}_\theta+\tilde{A}_{\theta\Omega}P_\Omega\right)\frac{d\xi}{(1-\xi)^2}-\int_0^{\bar{Z}}D_\theta\Omega\frac{d\xi}{(1-\xi)^2}.
\end{aligned} \tag{3.180}$$

Thus, the canonical kinematic unknowns, which are the vector components of translational displacement of the contour $\bar{Z}$ = const together with the cone apex, and of rotation of this contour about the apex, make it possible to decrease the total order of the system of resolvents by 12 units. Equations (3.179) in Ω and P_Ω in expanded form are of order 2n. The remaining unknowns are represented explicitly: P_η and P_θ by relations (3.177), and η and θ by relations (3.180) for previously determined Ω and P_Ω.

Equations (3.179) have variable coefficients. However, if the shell thickness is constant, and the cross-sectional area of the elements of the longitudinal structure depend linearly on the $\bar{Z}$ coordinate, they will be Euler type equations. Changing in this case to variable t according to (3.66), we arrive at equations with constant coefficients:

$$\begin{aligned}
&-\frac{d}{dt}\Omega+D_\Omega\Omega-\frac{1}{G}\tilde{A}_{\Omega\Omega}P_\Omega=\frac{1}{G}(\tilde{A}_{\Omega\eta}\bar{P}_\eta+e^{-t}\tilde{A}_{\Omega\theta}\bar{P}_\theta), \\
&-\frac{1}{G}\frac{d}{dt}P_\Omega-L_\Omega\Omega-\frac{1}{G}D_\Omega'P_\Omega=\frac{1}{G}(D_\eta'\bar{P}_\eta+e^{-t}D_\theta'\bar{P}_\theta)-\frac{1}{G}e^tR_\Omega.
\end{aligned} \tag{3.181}$$

In accordance with (3.180), vectors $\eta(\bar{Z})$, $\theta(\bar{Z})$ may be represented in the form

$$\eta=\eta^0+\eta_\Omega,\quad \theta=\theta^0+\theta_\Omega. \tag{3.182}$$

where
$$\eta^{0}=\eta(0)+\frac{1}{G}\int\limits_{0}^{Z}\left(\tilde{A}_{\eta\eta}\bar{P}_{\eta}+\frac{1}{1-\xi}\tilde{A}_{\eta\theta}\bar{P}_{\theta}\right)\frac{d\xi}{1-\xi},$$

$$\theta^{0}=\theta(0)+\frac{1}{G}\int\limits_{0}^{Z}\left(\tilde{A}_{\theta\eta}\bar{P}_{\eta}+\frac{1}{1-\xi}\tilde{A}_{\theta\theta}\bar{P}_{\theta}\right)\frac{d\xi}{(1-\xi)^{2}}. \tag{3.183}$$

$$\eta_{\Omega}=\frac{1}{G}\int\limits_{0}^{Z}\tilde{A}_{\eta\Omega}P_{\Omega}\frac{d\xi}{1-\xi}-\int\limits_{0}^{Z}D_{\eta}\Omega\frac{d\xi}{1-\xi},$$

$$\theta_{\Omega}=\frac{1}{G}\int\limits_{0}^{Z}\tilde{A}_{\theta\Omega}P_{\Omega}\frac{d\xi}{(1-\xi)^{2}}-\int\limits_{0}^{Z}D_{\theta}\Omega\frac{d\xi}{(1-\xi)^{2}}. \tag{3.184}$$

Expressions (3.183) constitute the statically determinate part of the solution, which according to the terminology adopted in Section 2.5 corresponds to the generalized law of plane sections. The additional vectors η_{Ω}, θ_{Ω} represented by expressions (3.184) as well as warping Ω and generalized force P_{Ω}, pertain to the statically indeterminate part of the solution. In accordance with the terminology adopted in Section 2.5, η_{Ω}, θ_{Ω}, Ω and P_{Ω} determine the generalized warping and corresponding self-balanced state of the shell.

The system of two matrix resolvents (3.179) can be easily reduced to a single second-order matrix equation in warping Ω or in generalized force P^{Ω}.

Multiplying the left-hand side of the first equation of (3.179) by matrix $\tilde{A}^{-1}_{\Omega\Omega}$ which the reciprocal of matrix $\tilde{A}_{\Omega\Omega}$, we find

$$\frac{1}{G}P_{\Omega}=(1-Z)\tilde{A}^{-1}_{\Omega\Omega}\frac{d}{d\bar{Z}}\Omega+\tilde{A}^{-1}_{\Omega\Omega}D_{\Omega}\Omega-\frac{1}{G}\tilde{A}^{-1}_{\Omega\Omega}\left(\tilde{A}_{\Omega\eta}\bar{P}_{\eta}+\frac{1}{1-\bar{Z}}\tilde{A}_{\Omega\theta}\bar{P}_{\theta}\right). \tag{3.185}$$

Introducing (3.185) into the second equation of (3.179), we obtain

$$\begin{aligned}&(1-Z)\frac{d}{d\bar{Z}}\left[(1-Z)\tilde{A}^{-1}_{\Omega\Omega}\frac{d}{d\bar{Z}}\Omega+\tilde{A}^{-1}_{\Omega\Omega}D_{\Omega}\Omega\right]-(1-Z)D'_{\Omega}\tilde{A}^{-1}_{\Omega\Omega}\frac{d}{d\bar{Z}}\Omega-\\&-\left[D'_{\Omega}\tilde{A}^{-1}_{\Omega\Omega}D_{\Omega}+L_{\Omega}\right]\Omega=\frac{1}{G}\left\{(1-Z)\frac{d}{d\bar{Z}}\left[\tilde{A}^{-1}_{\Omega\Omega}\left(\tilde{A}_{\Omega\eta}\bar{P}_{\eta}+\frac{1}{1-\bar{Z}}\tilde{A}_{\Omega\theta}\bar{P}_{\theta}\right)\right]+\right.\\&\left.(D_{\eta}-D'_{\Omega}\tilde{A}^{-1}_{\Omega\Omega}\tilde{A}_{\Omega\eta})\bar{P}_{\eta}+\frac{1}{1-\bar{Z}}(D'_{\theta}-D'_{\Omega}\tilde{A}^{-1}_{\Omega\Omega}\tilde{A}_{\Omega\theta})\bar{P}_{\theta}-(1-Z)R_{\Omega}\right\}.\end{aligned} \tag{3.186}$$

Expression (3.186) represents an ordinary differential matrix equation in warping Ω. This equation is obviously equivalent to the partial integro-differential resolvent (2.145). Thus, the problem of integrating (2.145) reduces to the integration of a system of ordinary differential equations corresponding to matrix equation (3.186). It is easy to see that this system reduces to a system with constant coefficients if the shell thickness and longitudinal structure satisfy the above-indicated conditions. Carrying out the substitution of variables according to (3.66), we have

$$\begin{gathered}
\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}\frac{d^2}{dt^2}\Omega+\left(\mathbf{D}'_{\Omega}\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}-\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}\mathbf{D}_{\Omega}\right)\frac{d}{dt}\Omega-\left(\mathbf{D}'_{\Omega}\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}\mathbf{D}_{\Omega}+\mathbf{L}_{\Omega}\right)\Omega=\\
=-\frac{1}{G}\Big\{-\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}\Big[\tilde{\mathbf{A}}_{\Omega\eta}\frac{d}{dt}\bar{\mathbf{P}}_{\eta}+\tilde{\mathbf{A}}_{\Omega\theta}\frac{d}{dt}\left(e^{-t}\bar{\mathbf{P}}_{\theta}\right)\Big]+\left(\mathbf{D}'_{\eta}-\mathbf{D}'_{\Omega}\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}\tilde{\mathbf{A}}_{\Omega\eta}\right)\bar{\mathbf{P}}_{\eta}+\\
+e^{-t}\left(\mathbf{D}'_{\theta}-\mathbf{D}'_{\Omega}\tilde{\mathbf{A}}^{-1}_{\Omega\Omega}\tilde{\mathbf{A}}_{\Omega\theta}\right)\bar{\mathbf{P}}_{\theta}-e^{t}\mathbf{R}_{\Omega}\Big\}.
\end{gathered}\tag{3.187}$$

In Eq. (3.187), matrix $\tilde{A}^{-1}_{\Omega\Omega}$ is obviously symmetric, whereas matrix $D'_{\Omega}\tilde{A}^{-1}_{\Omega\Omega}-\tilde{A}^{-1}_{\Omega\Omega}D_{\Omega}$ is inversely symmetric. Using expressions (3.172), (3.174), we can also readily ascertain that matrix $D'_{\Omega}\tilde{A}^{-1}_{\Omega\Omega}D_{\Omega}+L_{\Omega}$ is symmetric. These properties of the matrices result from the Betti reciprocity theorem.

It follows from Eq. (3.187), in view of (3.172), that if the vector Ω is represented by only one component, i.e., is a scalar function, the corresponding resolvent always has the form

$$\frac{d^2\omega}{dt^2}-k\omega=F(t).\tag{3.188}$$

3.4. Differential Resolvents of a Conical Shell Allowing for Bending Strain of the Middle Surface

As above, we will proceed from a representation of the elastic displacement vector $U(\bar{Z}, S)$ in the form

$$\mathbf{U}(Z, S)=\mathbf{U}^0(Z, S)+\mathbf{U}^1(Z, S),\tag{3.189}$$

where vector U^0, corresponding to the displacement of the contour $\bar{Z}$ = const as a solid, can be written in the form

$$\mathbf{U}^{*}(\bar{Z}, S)=\sum_{l=1}^{6} V_{l}(\bar{Z})\,\boldsymbol{\varphi}_{l}(S), \tag{3.190}$$

and vector U^{I}, determining the displacements for which the relative arrangement of the points of the contour $\bar{Z}$ = const changes, can be represented in the form of the expansion

$$\mathbf{U}^{1}(\bar{Z}, S)=\sum_{k} \omega_{k}(\bar{Z})\,\boldsymbol{\varphi}_{k}^{1}(S), \tag{3.191}$$

where $\{\phi_{k}^{I}\}$ is a given system of coordinate vector functions.

Considering the bending strains of the middle surface of the shell, we will have to abandon the basic geometric hypothesis according to which vector U^{I} is directed along the generatrices. In this case, expanding vector function U^{I} in a certain basis

$$\mathbf{U}^{1}(\bar{Z}, S)=U^{1\alpha}(\bar{Z}, S)\,\mathbf{e}_{\alpha}(\bar{Z}, S), \tag{3.192}$$

we have to represent the components $U^{I\alpha}(\bar{Z}, S)$ in the form of expansions

$$U^{1\alpha}(\bar{Z}, S)=\sum_{j} \omega_{j}^{\alpha}(\bar{Z})\,\varphi_{j}^{1\alpha}(S), \quad \alpha=1, 2, 3, \tag{3.193}$$

where $\{\phi_{j}^{I\alpha}\}$, α = 1, 2, 3 are three given systems of scalar cooroinate functions.

Introducing (3.193) into (3.192), we have

$$\mathbf{U}^{1}(\bar{Z}, S)=\sum_{j} \left[\omega_{j}^{1}(\bar{Z})\,\boldsymbol{\varphi}_{j1}^{1}(S)+\omega_{j}^{2}(\bar{Z})\,\boldsymbol{\varphi}_{j2}^{1}(S)+\omega_{j}^{3}(\bar{Z})\,\boldsymbol{\varphi}_{j3}^{1}(S)\right], \quad (3.194)$$

(3.194)

where

$$\boldsymbol{\varphi}_{j1}^{1}=\varphi_{j}^{11}\,\mathbf{e}_{1}, \quad \boldsymbol{\varphi}_{j2}^{1}=\varphi_{j}^{12}\,\mathbf{e}_{2}, \quad \boldsymbol{\varphi}_{j3}^{1}=\varphi_{j}^{13}\,\mathbf{e}_{3}. \tag{3.195}$$

Numbering the terms of expansion (3.194) successively, we will hereinafter use the representation of vector function U^{1} in the form (3.191). For this purpose, we introduce the notation

$$\omega_j^{\alpha}=\omega_k, \quad \varphi_{j\alpha}^{1}=\varphi_k^{1}, \tag{3.196}$$

where

$$k=3(j-1)+\alpha.$$

Now, introducing (3.190) and (3.191) into (3.189), we can finally represent vector function $U(\overline{Z}, S)$ in the form

$$U(\bar{Z}, S)=\sum_{i=1}^{6+N} V_i(\bar{Z})\varphi_i(S). \tag{3.197}$$

Here N is the total number of degrees of freedom of the contour $\overline{Z}$ = const with respect to the displacements, accompanied by a change in the relative arrangement of the points of this contour:

$$N=n_1+n_2+n_3, \tag{3.198}$$

where n_{α}(α = 1, 2, 3) is the number of degrees of freedom in relation to the displacements in the direction of e^{α}. If n_2=n_3=0, and the unit vectors e^{α} are superposed on the unit vectors of the main moving trihedron, then e^{1}=m_z, and we arrive at the fundamental static-geometric model discussed above. If however n_2 or n_3 is different from zero, we have a more complete computational model permitting a refinement of the results obtained on the basis of the fundamental geometric hypothesis.

Expanding geometric relations (2.2) with the aid of expansions (3.197), we can readily obtain the corresponding expansions for components of tangential and flexural strain. These expansions formally coincide with expressions (3.16), (3.19) if in these expressions the upper limit 6+n of the sum is replaced according to (3.198) by the limit 6+N=6+n_1+n_2+n_3. Similarly, the expansions for tangential forces and moments coincide with expansions (3.23)-(3.25), if in the latter n is replaced by index N.

As above, the differential resolvents in unknown functions V_i can be easily obtained by using the Lagrange variational principle.

The work done by external forces can be written in the case at hand in the form

$$\delta T=\int_0^{\bar{Z}_1}\oint \mathbf{p}\cdot\delta\mathbf{U}\,AB\sin\chi\, dS\,d\bar{Z}+\oint(\bar{\mathbf{q}}^0\delta\mathbf{U}+\bar{\mathbf{m}}^0\delta\boldsymbol{\Omega})\,BdS\big|_{\bar{Z}=0}+ \\ +\oint(\bar{\mathbf{q}}^1\delta\mathbf{U}+\bar{\mathbf{m}}^1\delta\boldsymbol{\Omega})\,BdS\big|_{\bar{Z}=\bar{Z}_1}. \tag{3.199}$$

Here $p(\bar{Z}, S)$ is the vector of the surface load referred to a unit area of the middle surface;

$\bar{q}^0(S)$, $\bar{q}^1(S)$ are vectors of the resultants of the external load applied to the ends $\bar{Z}=0$, $\bar{Z}=Z_1$ and referred to a unit length of the contour;

$\bar{m}^0(S)$, $\bar{m}^1(S)$ are vectors of net moments of the external load applied to the ends $\bar{Z}=0$, $\bar{Z}=\bar{Z}_1$ and referred to a unit length of the contour;

Ω is the vector of the angle of rotation of the normal to the middle surface of the shell.

Considering (3.15), we represent expansion (3.197) in the form

$$\mathbf{U}(\bar{Z}, S)=\sum_{i=1}^{6+N}U_i(\bar{Z})\lambda_i(\bar{Z})\boldsymbol{\varphi}_i(s), \tag{3.200}$$

where

$$\lambda_i(\bar{Z})=\begin{cases}1 & (i\neq 4,5,6),\\ 1-\bar{Z} & (i=4,5,6).\end{cases} \tag{3.201}$$

In accordance with (3.200)

$$\delta\mathbf{U}(\bar{Z}, S)=\sum_{i=1}^{6+N}\delta U_i(\bar{Z})\lambda_i(\bar{Z})\boldsymbol{\varphi}_i(S). \tag{3.202}$$

The variation $d\Omega$ is represented by expression (1.183). Using (3.202) and formulas (1.11), (1.14), (1.15), (1.16), we can easily obtain

$$\delta\Omega(\bar{Z}, S) = \frac{1}{\sin\chi}\sum_{i=1}^{6+N}\Big\{-[\delta U_i(\bar{Z})\lambda_i(\bar{Z})]'\frac{1}{l_s}(\varphi_i\cdot n_n)m_s +$$
$$+\frac{1}{1-\bar{Z}}\delta U_i(\bar{Z})\lambda_i(\bar{Z})(\varphi_i'\cdot n_n)m_s\Big\}. \tag{3.203}$$

Introducing (3.202) and (3.203) into (3.199), we find

$$\delta T = \int_0^{\bar{Z}_1}\sum_{i=1}^{6+N} R_i\delta U_i d\bar{Z} + \sum_{i=1}^{6+N}\Big[(\bar{P}_i+\Delta\bar{P}_i)\delta U_i + \frac{\bar{M}_i}{\lambda_i}(\delta U_i\lambda_i)'\Big]_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1}.$$

where

$$R_i = \lambda_i B\oint \mathbf{p}\cdot\varphi_i A\sin\chi dS, \tag{3.204}$$

$$\bar{P}_i(0) = -\oint \bar{q}^0\cdot\varphi_i dS, \tag{3.205}$$
$$\bar{P}_i(\bar{Z}_1) = \lambda_i(\bar{Z}_1)(1-\bar{Z}_1)\oint \bar{q}^1\cdot\varphi_i dS. \tag{3.206}$$
$$\Delta\bar{P}_i(0) = -\oint \bar{m}^0\cdot m_s(\varphi_i'\cdot n_n)\frac{dS}{\sin\chi},$$
$$\Delta\bar{P}_i(\bar{Z}_1) = \lambda_i(\bar{Z}_1)\oint \bar{m}^1\cdot m_s(\varphi_i'\cdot n_n)\frac{dS}{\sin\chi}, \tag{3.207}$$
$$\bar{M}_i(0) = \oint \bar{m}^0\cdot m_s(\varphi_i\cdot n_n)\frac{1}{l_s}\frac{dS}{\sin\chi},$$
$$\bar{M}_i(\bar{Z}_1) = -\lambda_i(\bar{Z}_1)(1-\bar{Z}_1)\oint \bar{m}^1\cdot m_s(\varphi_i\cdot n_n)\frac{1}{l_s}\frac{dS}{\sin\chi}. \tag{3.208}$$

The potential energy variation allowing for bending strain is given by expression (1.200)

$$\delta U = \int_0^{\bar{Z}_1}\oint(T_{m_z}\delta\varepsilon^0_{m_z} + S_{n_z m_z}\delta\gamma^0_{n_z m_z} + T_{n_z}\delta\varepsilon^0_{n_z} + M_{m_z}\delta\varkappa_{m_z} +$$
$$+2H_{n_z m_z}\delta\varkappa_{n_z m_z} + M_{n_z}\delta\varkappa_{n_z})AB\sin\chi d\bar{Z}dS. \tag{3.209}$$

The variations $\delta\varepsilon^0_{m_z}$, $\delta\gamma^0_{n_z m_z}$, $\delta\varepsilon^0_{n_z}$ of the components of tangential strain of the middle surface are represented by expressions (3.34). The variations $\delta\varkappa_{m_z}$, $\delta\varkappa_{n_z m_z}$, $\delta\varkappa_{n_z}$ in accordance with (3.19) will be determined by expressions of the form

$$\delta\varkappa = \sum_{i=1}^{6+N}\Big[(\delta U_i\lambda_i)''\frac{1}{l_s^2}\vartheta_i + \frac{1}{1-\bar{Z}}(\delta U_i\lambda_i)'\frac{1}{l_s}\bar{\vartheta}_i + \frac{1}{(1-\bar{Z})^2}\delta U_i\lambda_i\bar{\bar{\vartheta}}_i\Big]_{r}. \tag{3.210}$$

Expanding (3.209) with the aid of (3.210), we obtain

$$\delta U=\int_0^{\bar Z_1}\sum_{i=1}^{6+N}\left[M_i\frac{(\delta U_i\lambda_i)''}{\lambda_i}+P_i\frac{(\delta U_i\lambda_i)'}{\lambda_i}+Q_i\delta U_i\right]d\bar Z, \tag{3.211}$$

where

$$M_i=\lambda_i B\oint\left(M_{m_z}\vartheta_{im_z}+2H_{n_z m_z}\vartheta_{in_z m_z}+M_{n_z}\vartheta_{in_z}\right)\frac{\sin\gamma}{A}dS, \tag{3.212}$$

$$P_i=\lambda_i\Big[B\oint\left(T_{m_z}\psi_{im_z}+S_{n_z m_z}\psi_{in_z m_z}+T_{n_z}\psi_{in_z}\right)\sin\gamma dS+ \\ +\oint\left(M_{m_z}\bar\vartheta_{im_z}+2H_{n_z m_z}\bar\vartheta_{in_z m_z}+M_{n_z}\bar\vartheta_{in_z}\right)\sin\gamma dS\Big], \tag{3.213}$$

$$Q_i=\lambda_i\Big[\oint\left(T_{m_z}\bar\psi_{im_z}+S_{n_z m_z}\bar\psi_{in_z m_z}+T_{n_z}\bar\psi_{in_z}\right)A\sin\gamma dS+ \\ +\frac{1}{B}\oint\left(M_{m_z}\bar\vartheta_{im_z}+2H_{n_z m_z}\vartheta_{in_z m_z}+M_{n_z}\bar\vartheta_{in_z}\right)A\sin\gamma dS\Big]. \tag{3.214}$$

Integrating by parts, we finally reduce the variation δU represented by expression (3.211) to the form

$$\delta U=\int_0^{\bar Z_1}\sum_{i=1}^{6+N}\left\{\left[\left(\frac{M_i}{\lambda_i}\right)'-\frac{P_i}{\lambda_i}\right]'\lambda_i+Q_i\right\}\delta U_i d\bar Z+ \\ +\sum_{i=1}^{6+N}\left\{\left[\frac{P_i}{\lambda_i}-\left(\frac{M_i}{\lambda_i}\right)'\right]\delta U_i\lambda_i+\frac{M_i}{\lambda_i}(\delta U_i\lambda_i)'\right\}\Big|_{\bar Z=0}^{\bar Z=\bar Z_1}. \tag{3.215}$$

Expanding variational Eq. (3.26) with the aid of (3.204) and (3.215), we have

$$\int_0^{\bar Z_1}\sum_{i=1}^{6+N}\left\{\left[\frac{P_i}{\lambda_i}-\left(\frac{M_i}{\lambda_i}\right)'\right]'\lambda_i-Q_i+R_i\right\}\delta U_i d\bar Z+ \\ +\sum_{i=1}^{6+N}\left\{\left[\bar P_i+\Delta\bar P_i-P_i+\lambda_i\left(\frac{M_i}{\lambda_i}\right)'\right]\delta U_i+\frac{1}{\lambda_i}(\bar M_i-M_i)(\delta U_i\lambda_i)'\right\}\Big|_{\bar Z=0}^{\bar Z=\bar Z_1}=0. \tag{3.216}$$

Variations of generalized displacements δU_i over the interval $(0, \bar Z_1)$ are arbitrary and independent. Consequently, from variational Eq. (3.126) there follows the system of equations

$$\left[\frac{P_j}{\lambda_j}-\left(\frac{M_j}{\lambda_j}\right)'\right]'\lambda_j-Q_j+R_j=0 \qquad (j=1,2,\ldots,6+N) \tag{3.217}$$

and the natural boundary conditions

$$\left[\bar{P}_j+\Delta\bar{P}_j-P_j+\lambda_j\left(\frac{M_j}{\lambda_j}\right)'\right]\mathfrak{W}_j+\frac{1}{\lambda_j}(\bar{M}_j-M_j)(\mathfrak{W}_j\lambda_j)'\Big|_{\bar{z}=0}^{\bar{z}=\bar{z}_1}=0$$

$$(j=1,2,\ldots,6+N). \tag{3.218}$$

We will write Eqs. (3.127) in expanded form. Using (3.23), (3.24), one can easily obtain

$$M_j(\bar{Z})=D\lambda_j\left[(1-\bar{Z})\sum_{l=1}^{6+N}d_{jl}V_l''+\sum_{l=1}^{6+N}e_{jl}V_l'+\frac{1}{1-\bar{Z}}\sum_{l=1}^{6+N}f_{jl}V_l\right], \tag{3.219}$$

$$P_j(\bar{Z})=\lambda_j\left\{D\sum_{l=1}^{6+N}e_{lj}V_l''+\sum_{l=1}^{6+N}\left[G(1-\bar{Z})a_{jl}+\frac{D}{1-\bar{Z}}g_{jl}\right]V_l'+\right.$$

$$\left.+\sum_{l=1}^{6+N}\left[Gb_{jl}+\frac{D}{(1-\bar{Z})^2}h_{jl}\right]V_l\right\}, \tag{3.220}$$

$$Q_j(\bar{Z})=\lambda_j\left\{\frac{D}{1-\bar{Z}}\sum_{l=1}^{6+N}f_{lj}V_l''+\sum_{l=1}^{6+N}\left[Gb_{lj}+\frac{D}{(1-\bar{Z})^2}h_{lj}\right]V_l'+\right.$$

$$\left.+\frac{1}{1-\bar{Z}}\sum_{l=1}^{6+N}\left[Gc_{jl}+\frac{D}{(1-\bar{Z})^2}k_{jl}\right]V_l\right\}. \tag{3.221}$$

Here the coefficients a_{jl}, b_{jl}, c_{jl} are represented by expressions (3.44). Using the symbolic notation

$$(X_jY_l)=X_{jm_s}Y_{lm_s}+\nu(X_{jm_s}Y_{ln_s}+X_{jn_s}Y_{lm_s})+X_{jn_s}Y_{ln_s}+$$

$$+2(1-\nu)X_{jn_sm_s}Y_{ln_sm_s}. \tag{3.222}$$

we can represent the remaining coefficients in the form

$$\begin{aligned}
d_{jl}&=\oint(\vartheta_j\vartheta_l)\frac{\sin\chi}{l_s^3}\,dS,\\
e_{jl}&=\oint(\vartheta_j\bar{\vartheta}_l)\frac{\sin\chi}{l_s^2}\,dS,\\
f_{jl}&=\oint(\vartheta_j\bar{\bar{\vartheta}}_l)\frac{\sin\chi}{l_s}\,dS,\\
g_{jl}&=\oint(\bar{\vartheta}_j\,\bar{\vartheta}_l)\frac{\sin\chi}{l_s}\,dS,\\
h_{jl}&=\oint(\bar{\vartheta}_j\,\bar{\bar{\vartheta}}_l)\sin\chi\,dS,\\
k_{jl}&=\oint(\bar{\bar{\vartheta}}_j\,\bar{\bar{\vartheta}}_l)\,l_s\sin\chi\,dS.
\end{aligned} \tag{3.223}$$

Introducing expressions (3.219)-(3.221) into Eqs. (3.217), we have

$$\sum_{i=1}^{6+N}\Big\{-\Big[\gamma(1-\bar{Z})d_{ji}V_i''+\gamma e_{ji}V_i'+\gamma\frac{1}{1-\bar{Z}}f_{ji}V_i\Big]'+\gamma e_{ij}\tilde{V}_i'+$$
$$+\Big[(1-\bar{Z})a_{ji}+\gamma\frac{1}{1-\bar{Z}}g_{ji}\Big]V_i'+\Big[b_{ji}+\gamma\frac{1}{(1-\bar{Z})^2}h_{ji}\Big]V_i\Big\}'-$$
$$-\sum_{i=1}^{6+N}\Big\{\gamma\frac{1}{1-\bar{Z}}f_{ij}V_i''+\Big[b_{ij}+\gamma\frac{1}{(1-\bar{Z})^2}h_{ij}\Big]V_i'+\tag{3.224}$$
$$+\frac{1}{1-\bar{Z}}\Big[c_{ji}+\gamma\frac{1}{(1-\bar{Z})^2}k_{ji}\Big]V_i\Big\}=-\frac{1}{\lambda_j G}R_j\quad (j=1,2,\ldots,6+N),$$

where

$$\gamma=\frac{D}{G}.\tag{3.225}$$

Expressions (3.224) constitute a general resolvent system of ordinary differential equations of a conical shell of arbitrary configuration. Comparing systems (3.224) and (3.45), we can conclude that consideration of bending strain of the middle surface raises the order of each of the equations of the resolvent system by two units. It is also easy to see that Eqs. (3.224) do not reduce to Euler type equations, as is the case of Eqs. (3.45). Therefore, numerical integration methods are basic for system (3.224). However, this does not preclude the possibility of analytical solutions based on asymptotic methods. Then the solution of system (3.224) is usefully represented in the form

$$\mathbf{V}=\mathbf{V}^0+\Delta\mathbf{V},\tag{3.226}$$

where vector V^0 is the solution of system (3.45). Introducing representation (3.226) into Eqs. (3.224) enables us to estimate the error of the solution $V=V^0$.

For system (3.224), it is easy to find the first six integrals, as was done in Section 3.2. These integrals, which constitute the equilibrium conditions of the cut-off portion of the shell, make it possible to lower the order of system (3.224) by six units, but this is unimportant in a numerical solution of this system.

Coefficients (3.223) of system of differential resolvents (3.224) correspond to a smooth shell without a reinforcing structure. If the shell is provided with a longitudinal reinforcing structure, the corresponding coefficients of system (3.224) are easily obtained by using (3.25), as was done in Section 3.4.

3.5. Differential Resolvents of a Conical Shell with a Nondeformable Cross-Sectional Contour

The adopted fundamental static-geometric model according to which the contour $\bar{Z}$ = const is sought in the direction of the generatrices has a very universal character and yields satisfactory results for shells of the most diverse configurations for any arrangement of the reinforcing transverse diaphragms. However, if the shell is reinforced with a regular structure of frequent and sufficiently rigid diaphragms, it may be useful to adopt the hypothesis of nondeformability of the diaphragms in their plane. This leads to a computational model in which the warpings take place in a direction perpendicular to the plane of the diaphragms.

We will now consider shells provided with a regular set of diaphragms oriented parallel to the plane $\bar{Z}$ = const. We will neglect the strain energy of the bending of the middle surface of the shell.

Representing the elastic displacement vector $U(\bar{Z}, S)$ in the form

$$\mathbf{U}(\bar{Z}, S)=\mathbf{U}^0(\bar{Z}, S)+\mathbf{U}^1(\bar{Z}, S), \tag{3.227}$$

where U^0 corresponds to the displacement of the contour $\bar{Z}$ = const as a solid, and $U^1(\bar{Z}, S)$ defines the warping of this contour, we obtain

$$\mathbf{U}^0(\bar{Z}, S)=\sum_{i=1}^{6} V_i(\bar{Z})\varphi_i(S), \tag{3.228}$$

$$\mathbf{U}^1(\bar{Z}, S)=\mu\sum_{k} w_k(\bar{Z})\varphi_k^1(S), \tag{3.229}$$

where $\{\phi_k^1\}$ is a given system of coordinate functions;

μ is a unit vector normal to the plane of the diaphragm.

For a set of diaphragms oriented parallel to the plane $\overline{Z}$ = const,

$$\mu = -\cos\chi_0 \mathbf{i} + \sin\chi_0 \mathbf{k}. \tag{3.230}$$

Introducing expansions (3.228), (3.229) into expression (3.227) for vector U, we will represent (3.227) in the form

$$\mathbf{U}(\overline{Z}, S) = \sum_{i=1}^{6+n} U_i(\overline{Z}) \lambda_i(\overline{Z}) \varphi_i(S). \tag{3.231}$$

Here as before,

U_i (i = 1, 2, ..., 6) are the components of vectors of translational displacement of the contour $\overline{Z}$ = const together with a certain pole and of the rotation of this contour about the pole in arbitrary bases e_α and $\hat{e}_\alpha$;

$$\lambda_i = \begin{cases} 1, & i = 1, 2, 3; \\ 1 - \overline{Z}, & i = 4, 5, 6; \end{cases}$$

$\phi_i(S)$ (i = 1, 2, ..., 6) are vector functions dependent on the choice of the pole and bases e_α, $\hat{e}_\alpha$;

$$\varphi_i(S)|_{i=6+k} = \mu \varphi_k^1(S);$$

where λ_i are arbitrarily chosen functions;

$$U_i \lambda_i|_{i=6+k} = w_k \quad (k = 1, 2, \dots, n), \tag{3.232}$$

n is the number of degrees of freedom of the contour $\overline{Z}$ = const in relation to warping.

Setting as before $U_i\lambda_i = V_i$, we can further write

$$\mathbf{U}(\bar{Z}, S) = \sum_{i=1}^{6+n} V_i(\bar{Z}) \boldsymbol{\varphi}_i(S) \tag{3.233}$$

$$\mathbf{U} \cdot \mathbf{t} = u_t = \sum_{i=1}^{6+n} V_i(\bar{Z}) \varphi_{it} \tag{3.234}$$

where u_t is the displacement along the direction of the arbitrarily oriented unit vector t;

$$\varphi_{it} = \boldsymbol{\varphi}_i \cdot \mathbf{t}. \tag{3.235}$$

All of the subsequent treatment will be given for an arbitrary pole and arbitrary bases e_α and $\hat{e}_\alpha$. Of greatest interest here are two cases, when the pole coincides with the cone apex and when the point of intersection of the plane of the contour $\bar{Z}$ = const with the Oz axis is chosen as the pole. In the first of these cases, vector functions $\phi_i(S)$ for arbitrary bases e_α and $\phi_i(S)$ are determined by expressions (2.97).

Let us consider the second case. For displacement of the contour $\bar{Z}$ = const as a solid, taking (2.8) into account, we have from (2.10)

$$\mathbf{U}^0(\bar{Z}, S) = \boldsymbol{\eta}(\bar{Z}) + (1 - \bar{Z}) \boldsymbol{\theta}(\bar{Z}) \times \mathbf{r}(0, S), \tag{3.236}$$

where

$$\mathbf{r}(0, S) = x_0(S)\mathbf{i} + y_0(S)\mathbf{j} + x_0(S)\operatorname{ctg}\gamma_0 \mathbf{k} \tag{3.237}$$

is the radius vector of the point M(0, S) belonging to the directrix of the conical surface.

Expanding vectors η and θ in bases e_α and $\hat{e}_\alpha$ respectively, we obtain

$$\mathbf{U}^0(\bar{Z}, S) = \eta^\alpha(\bar{Z}) \mathbf{e}_\alpha + (1 - \bar{Z}) \theta^\alpha \hat{\mathbf{e}}_\alpha \times \mathbf{r}(0, S). \tag{3.238}$$

Comparing (3.238) and (3.228), we have

$$V_i(\bar{Z}) = \begin{cases} \psi(\bar{Z}) & (i=\alpha=1,2,3), \\ (1-\bar{Z})\psi & (i=\alpha+3=4,5,6), \end{cases} \tag{3.239}$$

$$\boldsymbol{\varphi}_i = \begin{cases} \mathbf{e}_\alpha & (i=\alpha=1,2,3), \\ \hat{\mathbf{e}}_\alpha \times \mathbf{r}(0,S) & (i=\alpha+3=4,5,6). \end{cases} \tag{3.240}$$

Like expressions (2.97), expressions (2.40) are general in character and permit the determination of the coordinate vector functions ϕ_i, corresponding to the displacement of the contour $\bar{Z}$ = const as a solid for arbitrarily chosen bases e_α and $\hat{e}_\alpha$. If these bases coincide with the bases of the fundamental system of Cartesian coordinates, the coordinate functions ϕ_i (i=1, 2, ..., 6) and their components are determined by expressions (2.15), (2.21) for a pole superimposed on the point of intersection of the plane of the contour $\bar{Z}$ = const with the Oz axis, and by expressions (3.4)-(3.6) for a pole coinciding with the apex of the conical surface.

We will introduce into consideration an auxiliary unit basis so that one of its vectors is directed along the Oy axis, and the other, perpendicular to the plane of the directrix, i.e., in the case at hand, perpendicular to the plane of the diaphragms. Superposing vectors e_α and $\hat{e}_\alpha$, on the basis thus chosen, we have

$$\begin{aligned} &\mathbf{e}_1 = \hat{\mathbf{e}}_1 = \sin\chi_0 \mathbf{i} + \cos\chi_0 \mathbf{k}, \\ &\mathbf{e}_2 = \hat{\mathbf{e}}_2 = \mathbf{j}, \\ &\mathbf{e}_3 = \hat{\mathbf{e}}_3 = \boldsymbol{\mu} = -\cos\chi_0 \mathbf{i} + \sin\chi_0 \mathbf{k}. \end{aligned} \tag{3.241}$$

Developing relations (3.240) with the aid of (3.237) and (3.241), we obtain

$$\begin{aligned} &\boldsymbol{\varphi}_1 = \sin\chi_0 \mathbf{i} + \cos\chi_0 \mathbf{k}, \\ &\boldsymbol{\varphi}_2 = \mathbf{j}, \\ &\boldsymbol{\varphi}_3 = -\cos\chi_0 \mathbf{i} + \sin\chi_0 \mathbf{k}, \\ &\boldsymbol{\varphi}_4 = (\sin\chi_0 \mathbf{k} - \cos\chi_0 \mathbf{i}) y_0, \\ &\boldsymbol{\varphi}_5 = (-\mathbf{k} + \operatorname{ctg}\chi_0 \mathbf{i}) x_0, \\ &\boldsymbol{\varphi}_6 = \frac{x_0}{\sin\chi_0} \mathbf{j} - (\cos\chi_0 \mathbf{k} + \sin\chi_0 \mathbf{i}) y_0. \end{aligned} \tag{3.242}$$

Relations (3.242) represent the coordinate vector functions. Considering (3.242), the scalar coordinate functions corresponding to displacements in the direction of the arbitrarily oriented unit vector t will be

$$
\begin{aligned}
\varphi_{1t} &= \sin\chi_0 l_t + \cos\chi_0 n_t, \\
\varphi_{2t} &= m_t, \\
\varphi_{3t} &= -\cos\chi_0 l_t + \sin\chi_0 n_t, \\
\varphi_{4t} &= (\sin\chi_0 n_t - \cos\chi_0 l_t) y_0, \\
\varphi_{5t} &= (-n_t + \operatorname{ctg}\chi_0 l_t) x_0, \\
\varphi_{6t} &= \frac{x_0}{\sin\chi_0} m_t - (\cos\chi_0 n_t + \sin\chi_0 l_t) y_0
\end{aligned}
\qquad (3.243)
$$

Using relations (3.243) for the components $\phi_{in_s}(S)$, $\phi_{im_s}(S)$, $\phi_{in_n}(S)$ of coordinate vector functions $\phi_i(S)(i=1, 2, \ldots, 6)$ and considering (1.18), we can easily obtain

i	V_i	φ_{in_s}	φ_{im_s}	φ_{in_n}
1	η_1	$\dfrac{l_0\cos\chi_0 - x_0' l_s \cos\chi - x_0}{l_s \sin\chi\sin\chi_0}$	$\dfrac{x_0'}{\sin\chi_0}$	$-\dfrac{\sin\chi_0}{\sin\chi} y_0' \dfrac{l_0}{l_s}$
2	η_2	$-y_0' \operatorname{ctg}\chi - \dfrac{y_0}{l_s\sin\chi}$	y_0'	$\dfrac{x_0'}{\sin\chi} \dfrac{l_0}{l_s}$
3	η_3	$\dfrac{\sin\chi_0}{\sin\chi} \dfrac{l_0}{l_s}$	0	$\dfrac{\cos\chi_0}{\sin\chi} y_0' \dfrac{l_0}{l_s} + \dfrac{x_0' y_0 - x_0 y_0'}{l_s \sin\chi\sin\chi_0}$
4	$(1-\bar{Z})\theta_1$	$\dfrac{\sin\chi_0}{\sin\chi} y_0 \dfrac{l_0}{l_s}$	0	$\dfrac{\cos\chi_0}{\sin\chi} y_0 y_0' \dfrac{l_0}{l_s} + y_0 \dfrac{x_0' y_0 - x_0 y_0'}{l_s\sin\chi\sin\chi_0}$
5	$(1-\bar{Z})\theta_2$	$-\dfrac{x_0}{\sin\chi} \dfrac{l_0}{l_s}$	0	$-\operatorname{ctg}\chi_0 \dfrac{x_0 y_0'}{\sin\chi} \dfrac{l_0}{l_s} + x_0 \dfrac{x_0 y_0' - x_0' y_0}{l_s \sin\chi \sin^2\chi_0}$
6	$(1-\bar{Z})\theta_3$	$\dfrac{\operatorname{ctg}\chi}{\sin\chi_0}(x_0' y_0 - x_0 y_0') - \dfrac{\cos\chi_0}{\sin\chi} y_0 \dfrac{l_0}{l_s}$	$\dfrac{x_0 y_0' - x_0' y_0}{\sin\chi_0}$	$\dfrac{\sin\chi_0}{\sin\chi} \times \left(\dfrac{x_0 x_0'}{\sin^2\chi_0} + y_0 y_0'\right) \dfrac{l_0}{l_s}$

(3.244)

Relations (3.244) correspond to the case where the pole coincides with the point of intersection of the plane of the contour $\bar{Z}$ = const with the Oz axis.

Let us now consider the case in which the apex of the conical surface is chosen as the pole. Expanding expressions (2.97) with the aid of (3.241), we can readily obtain

$$\begin{aligned}
\varphi_1 &= \sin\chi_0\,\mathbf{i} + \cos\chi_0\,\mathbf{k},\\
\varphi_2 &= \mathbf{j},\\
\varphi_3 &= -\cos\chi_0\,\mathbf{i} + \sin\chi_0\,\mathbf{k},\\
\varphi_4 &= l_s\left[-\mathbf{n}_s(\sin\chi_0\, l_{n_n} + \cos\chi_0\, n_{n_n}) + \mathbf{n}_n(\sin\chi_0\, l_{n_s} + \cos\chi_0\, n_{n_s})\right],\\
\varphi_5 &= l_s(-\mathbf{n}_s m_{n_n} + \mathbf{n}_n m_{n_s}),\\
\varphi_6 &= l_s\left[\mathbf{n}_s(\cos\chi_0\, l_{n_n} - \sin\chi_0\, n_{n_n}) + \mathbf{n}_n(-\cos\chi_0\, l_{n_s} + \sin\chi_0\, n_{n_s})\right].
\end{aligned} \qquad (3.245)$$

whence

i	V_i	φ_{in_s}	φ_{im_s}	φ_{in_n}
1	φ_1	$\dfrac{l_0\cos\chi_0 - x_0' l_s\cos\chi - x_0}{l_s\sin\chi\sin\chi_0}$	$\dfrac{x_0'}{\sin\chi_0}$	$-\dfrac{\sin\chi_0}{\sin\chi}\,y_0'\,\dfrac{l_0}{l_s}$
2	φ_2	$-y_0'\operatorname{ctg}\chi - \dfrac{y_0}{l_s\sin\chi}$	y_0'	$\dfrac{x_0'}{\sin\chi}\,\dfrac{l_0}{l_s}$
3	φ_3	$\dfrac{\sin\chi_0}{\sin\chi}\,\dfrac{l_0}{l_s}$	0	$\dfrac{\cos\chi_0}{\sin\chi}\,y_0'\,\dfrac{l_0}{l_s} + \dfrac{x_0' y_0 - x_0 y_0'}{l_s\sin\chi\sin\chi_0}$
4	$(1-\bar{Z})\theta_1$	$-\operatorname{ctg}\chi\sin\chi_0\, l_0 y_0'$	$\sin\chi_0 l_0 y_0'$	$-\dfrac{\cos\chi_0}{\operatorname{tg}\chi}\,\dfrac{l_0}{l_s} + \dfrac{x_0'}{\sin\chi\sin\chi_0} + \dfrac{x_0\cos\chi}{l_s\sin\chi\sin\chi_0}$
5	$(1-\bar{Z})\theta_2$	$\operatorname{ctg}\chi\, l_0 x_0'$	$-l_0 x_0'$	$\dfrac{1}{\sin\chi}(x_0' y_0 - x_0 y_0')$
6	$(1-\bar{Z})\theta_3$	$\dfrac{x_0' y_0 - x_0 y_0'}{\operatorname{tg}\chi\sin\chi_0} + l_0\cos\chi_0\, y_0'\operatorname{ctg}\chi$	$\dfrac{x_0 y_0' - x_0' y_0}{\sin\chi_0} - l_0\cos\chi_0\, y_0'$	$\dfrac{x_0' y_0 - x_0 y_0'}{\sin\chi\sin\chi_0} - l_0\cos\chi_0\,\dfrac{y_0'}{\sin\chi}$

(3.246)

Expressions (3.244), (3.246) represent scalar coordinate functions corresponding to displacements in the direction of the axes of the auxiliary moving trihedron during displacement of the contour $\bar{Z}$ = const as a solid. The coordinate vector functions corresponding to warpings of this contour are represented by expressions (3.232). For the corresponding scalar coordinate functions, taking (3.230) into account, we have

$$\varphi_{lt}(S)|_{l=6+k} = (-\cos\chi_0 l_t + \sin\chi_0 n_t)\varphi_k^1(S), \tag{3.247}$$

where t is an arbitrarily oriented unit vector.

Superposing t on the unit vectors of the auxiliary moving trihedron and taking (1.18) into consideration, we obtain

$$\begin{gathered}\varphi_{ln_s}(S) = \frac{\sin\chi_0}{\sin\chi}\frac{l_0}{l_s}\varphi_k'(S),\\ \varphi_{lm_s}(S) = 0,\\ \varphi_{ln_n}(S) = \left(\frac{\cos\chi_0}{\sin\chi}y_0'\frac{l_0}{l_s} + \frac{x_0' y_0 - x_0 y_0'}{l_s\sin\chi\sin\chi_0}\right)\varphi_k'(S),\\ l = 6+k.\end{gathered} \tag{3.248}$$

Using expansion (3.233), we can readily obtain the corresponding expansions for the components of the stressed and strained state. We will switch in the case at hand to components in axes of the auxiliary moving trihedron n_s, m_s, n_n, since for shells with a known deformable contour $\bar{Z}$ = const

$$\varepsilon_{m_s}^0 = 0, \tag{3.249}$$

which simplifies the elasticity relations.

The geometric relations in axes of the auxiliary moving trihedron are represented by expressions (1.55), (1.69). Considering (3.249), we have

$$\frac{1}{1-\bar{Z}}\frac{\partial u_{m_s}}{\partial S} - \frac{1-\frac{1}{2}(l_s^2)'}{(1-\bar{Z})l_s\sin\chi}u_{n_s} - \frac{\sin^2\chi}{R}u_{n_n} = 0. \tag{3.250}$$

Transforming the expression for shear strain $\gamma_{n_s m_s}^0$ with the aid of (3.250), we can readily obtain

$$\gamma_{n_s m_s}^0 = \frac{1}{l_s\sin\chi}\frac{\partial u_{m_s}}{\partial\bar{Z}} + \frac{1}{1-\bar{Z}}\frac{\partial u_{n_s}}{\partial S} + \frac{1-\frac{1}{2}(l_s^2)'}{(1-\bar{Z})l_s\sin\chi}u_{m_s} + \frac{\sin\chi\cos\chi}{R}u_{n_n}. \tag{3.251}$$

Now, using expansion (3.234), we obtain

$$\varepsilon^0_{n_s}=\sum_{i=1}^{6+n}\left[V_i'\frac{1}{l_s}\varphi_{in_s}+\frac{1}{1-\bar{z}}V_i\bar{\varphi}_{in_s}\right],$$
$$\gamma^0_{n_s m_s}=\sum_{i=1}^{6+n}\left[V_i'\frac{1}{l_s}\varphi_{in_s m_s}+\frac{1}{1-\bar{z}}V_i\bar{\varphi}_{in_s m_s}\right]. \qquad (3.252)$$

where

$$\psi_{in_s}=\frac{1}{\sin\chi}\varphi_{in_s},\quad \varphi_{in_s m_s}=\frac{1}{\sin\chi}\varphi_{im_s}, \qquad (3.253)$$

$$\bar{\varphi}_{in_s}=-\operatorname{ctg}\chi\varphi_{in_s}'-\operatorname{ctg}\chi\frac{1-\frac{1}{2}(l_s^2)'}{l_s\sin\chi}\varphi_{im_s}-\frac{\cos^2\chi}{R_0}\varphi_{in_n},$$
$$\bar{\varphi}_{in_s m_s}=\varphi_{in_s}'+\frac{1-\frac{1}{2}(l_s^2)''}{l_s\sin\chi}\varphi_{im_s}+\frac{\sin\chi\cos\chi}{R_0}\varphi_{in_n}. \qquad (3.254)$$

In the case at hand, the elasticity relations, allowing for (3.249), will be

$$T_{n_s}=\frac{Eh}{1-\nu^2}\varepsilon^0_{n_s};\quad S_{n_s m_s}=\frac{Eh}{2(1+\nu)}\gamma^0_{n_s m_s}. \qquad (3.255)$$

Expanding (3.255) with the aid of (3.252), we have

$$T_{n_s}=\frac{Eh}{1-\nu^2}\sum_{i=1}^{6+n}\left[V_i'\frac{1}{l_s}\varphi_{in_s}+\frac{1}{1-\bar{z}}V_i\bar{\varphi}_{in_s}\right],$$
$$S_{n_s m_s}=\frac{Eh}{2(1+\nu)}\sum_{i=1}^{6+n}\left[V_i'\frac{1}{l_s}\psi_{in_s m_s}+\frac{1}{1-\bar{z}}V_i\bar{\varphi}_{in_s m_s}\right]. \qquad (3.256)$$

We will obtain the differential resolvents in the unknown functions V_i as above by using the Lagrange principle.

The work done by external forces δT is represented by expression (3.27). The variation δW of potential strain energy referred to the area of the middle surface, in components along the axis of the auxiliary moving trihedron without consideration of bending strain, can be easily shown to be

$$\delta W=T_{n_s}\delta\varepsilon^0_{n_s}+S_{n_s m_s}\delta\gamma^0_{n_s m_s}+T_{m_s}\delta\varepsilon^0_{m_s}. \qquad (3.257)$$

For shells with a nondeformable contour $\bar{Z}$ = const, in view of conditions (3.249)

$$\delta\varepsilon^{0}_{m_s}=0. \tag{3.258}$$

In view of (3.258), the variation of potential strain energy of the shell will be

$$\delta U=\int_0^{\bar{Z}_1}\oint(T_{n_s}\delta\varepsilon^0_{n_s}+S_{n_s m_s}\delta\gamma^0_{n_s m_s})\,AB\sin\chi\,dS\,d\bar{Z}. \tag{3.259}$$

For elements of the reinforcing structure, oriented in the direction of the generatrices, we have

$$\delta U_F=\int_0^{\bar{Z}_1}\sum_k N_k\delta\varepsilon^0_{m_s k}\,l_{sk}\,d\bar{Z}. \tag{3.260}$$

Expanding (3.259) and (3.260) with the aid of expansions (3.252) and (3.34) and considering (3.15), we obtain

$$\delta U+\delta U_F=\int_0^{\bar{Z}_1}\sum_{i=1}^{6+n}\left[P_{\Sigma i}\frac{(\delta U_i\lambda_i)'}{\lambda_i}+Q_i\delta U_i\right]d\bar{Z}, \tag{3.61}$$

where

$$P_{\Sigma i}=P_i+P_{Fi},$$

$$P_i=\lambda_i B\oint(T_{n_s}\varphi_{in_s}+S_{n_s m_s}\psi_{in_s m_s})\sin\chi\,dS, \tag{3.262}$$

$$P_{Fi}=\lambda_i\sum_k N_k\varphi_{im_s}(S_k), \tag{3.263}$$

$$Q_i=\lambda_i\oint(T_{n_s}\bar{\varphi}_{in_s}+S_{n_s m_s}\bar{\psi}_{in_s m_s})\,A\sin\chi\,dS. \tag{3.264}$$

(3.265)

Now, using (3.27) and (3.261), from Lagrange's variational equation (3.26) we can readily obtain

$$\int_0^{\bar{Z}_1}\sum_{i=1}^{6+n}\left[\left(\frac{P_{\Sigma i}}{\lambda_i}\right)'\lambda_i-Q_i+R_i\right]\delta U_i\,d\bar{Z}+\sum_{i=1}^{6+n}[\bar{P}_i-P_{\Sigma i}]\,\delta U_i\Big|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1}=0, \tag{3.266}$$

whence, because of the arbitrariness and independence of variations δU_i over the interval $(0, \bar{Z}_i)$ there follow the equations

$$\left(\frac{P_{zj}}{\lambda_j}\right)' \lambda_j - Q_j + R_j = 0 \quad (j = 1, 2, \ldots, 6+n) \tag{3.267}$$

and natural boundary conditions

$$(\bar{P}_j - P_{zj})\delta U_j \Big|_{Z=0}^{Z=\bar{Z}_i} = 0 \quad (j = 1, 2, \ldots, 6+n). \tag{3.268}$$

We will write Eqs. (3.267) in expanded form. Using (3.61) and (3.256), we get

$$\frac{1}{G} P_{zj}(\bar{Z}) = \lambda_j \left[(1-\bar{Z}) \sum_{i=1}^{6+n} a_{zji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i\right], \tag{3.269}$$

$$\frac{1}{G} Q_j(\bar{Z}) = \lambda_j \left[\sum_{i=1}^{6+n} b_{ij} V_i' + \frac{1}{1-\bar{Z}} \sum_{i=1}^{6+n} c_{ji} V_i\right], \tag{3.270}$$

where

$$\begin{aligned} a_{zji} &= \oint\left(\frac{2}{1-\nu}\psi_{jn_s}\psi_{in_s} + \psi_{jn_s m_s}\psi_{in_s m_s}\right)\frac{\sin\gamma}{l_s} h\, dS + \\ &+ \frac{2(1-\nu)}{1-\bar{Z}} \sum_k \psi_{jm_k}\psi_{im_k} \frac{\Delta F_k}{l_{s0}}, \\ b_{ji} &= \oint\left(\frac{2}{1-\nu}\psi_{jn_s}\bar{\psi}_{in_s} + \psi_{jn_s m_s}\bar{\psi}_{in_s m_s}\right)\sin\gamma h\, dS, \\ c_{ji} &= \oint\left(\frac{2}{1-\nu}\bar{\psi}_{jn_s}\bar{\psi}_{in_s} + \bar{\psi}_{jn_s m_s}\bar{\psi}_{in_s m_s}\right) l_s \sin\gamma h\, dS. \end{aligned} \tag{3.271}$$

Introducing expressions (3.269), (3.270) into Eqs. (3.267), we obtain a resolvent system of ordinary differential equations of the same form as (3.45):

$$\sum_{i=1}^{6+n}\left\{[(1-\bar{Z})a_{ji}V_i' + b_{ji}V_i]' - b_{ij}V_i' - \frac{1}{1-\bar{Z}}c_{ji}V_i\right\} = -\frac{1}{\lambda_j G}R_j \quad (3.272)$$

$$(j = 1, 2, \ldots, 6+n). \tag{3.272}$$

In the general case, Eqs. (3.272) have variable coefficients, but substitution of variables according to (3.66) reduces these equations to equations with constant coefficients under the same conditions as Eqs. (3.65).

Let us write the system of resolvents in mixed form. Introducing the additional unknowns

$$\frac{1}{\lambda_j} P_{sj} = P_j^{*} \quad (j = 1, 2, \ldots, 6+n), \tag{3.273}$$

we obtain

$$\begin{gathered} \frac{1}{a} P_j^{*\prime} - \sum_{i=1}^{6+n} b_{ij} V_i' - \frac{1}{1-\bar{z}} \sum_{i=1}^{6+n} c_{ji} V_i = -\frac{1}{\lambda_j a} R_j, \\ (1-\bar{z}) \sum_{i=1}^{6+n} a_{sji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i - P_j^{*} = 0 \\ (j = 1, 2, \ldots, 6+n). \end{gathered} \tag{3.274}$$

In the most general case, system (3.274) allows a reduction of the order by six units with the aid of the first six integrals, which, as in Section 3.2, express the equilibrium conditions of the cut-off portion of the shell.

Let us turn to geometric relations (1.55), (1.69). It is easy to see that any of the components of tangential strain in axes of both the main and the auxiliary trihedron may be represented in the form of an expression of the type

$$\sum_{i=1}^{6+n} \left[V_i' \frac{1}{l_s} \phi_{i\gamma_k} + \frac{1}{1-\bar{z}} V_i \bar{\phi}_{i\gamma_k} \right], \tag{3.275}$$

where γ_k is the index of the component in question.

Such expansions correspond to both linear and angular strains. For example, for linear strain $e^0_{m_z}$ in (3.275), instead of $\phi_{i\gamma_k}$, $\bar{\phi}_{i\gamma_k}$ one should write ϕ_{im_z}, $\bar{\phi}_{im_z}$, and for shear strain $\gamma^0_{n_s m_s}$, one should write $\phi_{in_s m_s}$, $\bar{\phi}_{in_s m_s}$.

Assuming that the shell undergoes arbitrary displacement in space as a solid, and taking (3.275) into account, we have

$$\sum_{i=1}^{6} \left[V_i' \frac{1}{l_s} \phi_{i\gamma_k} + \frac{1}{1-\bar{z}} V_i \bar{\phi}_{i\gamma_k} \right] = 0. \tag{3.276}$$

where the six functions $V_i(\bar{Z})$ can obviously always be found to within six constants.

Vector U(M) of displacement of an arbitrary point M of the solid may be represented in the form

$$\mathbf{U}(M)=\eta^0+\theta^0\times[\mathbf{r}(M)-\mathbf{r}_0], \tag{3.277}$$

where r_0 is the radius vector of some fixed point M_0; η^0 is the displacement vector of point M_0; θ^0 is the vector of rotation of the solid about M_0.

For translational displacement of the solid, $\theta^0=0$, and from (3.277) it follows that

$$\mathbf{U}(M)=\eta^0=\text{const}. \tag{3.278}$$

For rotation of the solid about point M_0, vector $\eta^0=0$, and it follows from (3.277) that

$$\mathbf{U}(M)=\theta^0\times[\mathbf{r}(M)-\mathbf{r}_0]. \tag{3.279}$$

We now superpose point M with the pole of contour $\bar{Z}$=const of the shell, coinciding with the point of intersection of the plane of this contour with the Oz axis. In this case

$$\mathbf{r}(M)=\bar{Z}l_0\mathbf{k} \tag{3.280}$$

and

$$\mathbf{U}(M)=\eta(\bar{Z}). \tag{3.281}$$

For convenience, superposing point M_0 on the cone apex, we have

$$\mathbf{r}_0=l_n\mathbf{k}. \tag{3.282}$$

Then, considering (3.280), (3.281) we find from (3.279)

$$\eta(\bar{Z}) = l_0(1 - \bar{Z})\,\mathbf{k}\times\theta^0. \tag{3.283}$$

We will now assume that the shell executes translational motion as a solid. In this case

$$V_i(\bar{Z}) = \begin{cases} \text{const} & (i = 1, 2, 3), \\ 0 & (i = 4, 5, 6), \end{cases} \tag{3.284}$$

and from (3.276), in view of the arbitrariness of V_1, V_2, V_3, we find

$$\bar{\psi}_{in_3} = \bar{\psi}_{in_3 m_3} = 0 \qquad (i = 1, 2, 3). \tag{3.285}$$

Now, let the shell rotate as a solid about the cone apex. In this case, in view of (3.283), we have

$$\begin{aligned} &\eta(\bar{Z}) = l_0(1 - \bar{Z})\,\mathbf{k}\times\theta^0, \\ &\theta(\bar{Z}) = \theta^0 = \text{const}. \end{aligned} \tag{3.286}$$

Expanding vectors (3.286) in bases e_α and $\hat{e}_\alpha$, respectively, we have

$$\eta^\alpha \mathbf{e}_\alpha = l_0(1 - \bar{Z})\,\hat{\theta}^\beta\,\mathbf{k}\times\hat{\mathbf{e}}_\beta, \tag{3.287}$$

whence

$$\eta^\alpha = l_0(1 - \bar{Z})\,\hat{\theta}^\beta\,\hat{\mathbf{e}}_\beta\times\mathbf{e}^\alpha\cdot\mathbf{k} \qquad (\alpha = 1, 2, 3). \tag{3.288}$$

Expression (3.287) makes it possible to establish the following relations which constitute an extension of relations (3.78) to the case of arbitrary bases e_α and $\hat{e}_\alpha$:

$$V_j = l_0 \sum_{i=1}^{3} V_{i+3}\,\hat{\mathbf{e}}_i\times\mathbf{e}^j\cdot\mathbf{k} \quad (j = 1, 2, 3). \tag{3.289}$$

Introducing (3.289) into relations (3.276) and considering that in view of (3.239)

$$\frac{V_{i+3}}{1-\bar{Z}} = -V'_{i+3} = \hat{\theta}^{0i} \qquad (i = 1, 2, 3), \tag{3.290}$$

taking (3.285) into account, we can readily obtain relations of the type

$$\sum_{i=1}^{3} \hat{\theta}^{0i} \left[\bar{\psi}_{i+3,1_0} - \frac{1}{l_s} \psi_{i+3,1_0} - \frac{l_0}{l_s} \sum_{j=1}^{3} \psi_{j,1_0} \hat{\mathbf{e}}_i \times \mathbf{e}^j \cdot \mathbf{k} \right] = 0, \tag{3.291}$$

whence in view of the arbitrariness of vector θ^0, we have

$$\bar{\psi}_{i+3,1_0} = \frac{1}{l_s} \left(\psi_{i+3,1_0} + l_0 \sum_{j=1}^{3} \psi_{j,1_0} \hat{\mathbf{e}}_i \times \mathbf{e}^j \cdot \mathbf{k} \right). \tag{3.292}$$

From relations (3.292), taking (3.81) into account, we have

$$\psi_{i+3,m_z} = -l_0 \sum_{j=1}^{3} \psi_{j,m_z} \hat{\mathbf{e}}_i \times \mathbf{e}^j \cdot \mathbf{k} \qquad (i = 1, 2, 3). \tag{3.293}$$

Now, using (3.285) and relations (3.292), (3.293), from expressions (3.263)-(3.265) we can readily obtain

$$Q_i = 0 \quad (i = 1, 2, 3), \tag{3.294}$$

$$Q_i = P_i^{\cdot} + l_0 \sum_{j=1}^{3} P_j^{\cdot} \hat{\mathbf{e}}_i \times \mathbf{e}^j \cdot \mathbf{k} \quad (i = 4, 5, 6). \tag{3.295}$$

Relations (3.294), (3.295) make it possible to find the first six integrals of system (3.274). Let us turn to Eqs. (3.267). Considering (3.273) and relations (3.294), (3.295), we find

$$P_j^{\cdot\cdot} = -R_j \quad (j = 1, 2, 3), \tag{3.296}$$

$$[(1-\bar{Z}) P_j^{\cdot}]^{\cdot} - l_0 \sum_{j=1}^{3} P_j^{\cdot} \hat{\mathbf{e}}_i \times \mathbf{e}^j \cdot \mathbf{k} = -R_j \, (j = 4, 5, 6). \tag{3.297}$$

Relations (3.296), (3.297) form a system of six differential equations in unknowns P_j^*, constituting a generalization of system (3.85), (3.86). From Eqs. (3.296) we find

$$P_j^* = \bar{P}_j^* \quad (j=1, 2, 3), \tag{3.298}$$

where

$$\bar{P}_j^*(\bar{Z}) = P_j^*(\bar{Z}_1) - \int_{\bar{Z}_1}^{\bar{Z}} R_j(\xi)\, d\xi. \tag{3.299}$$

Eliminating the unknowns P_j^*(j=1, 2, 3) from Eqs. (3.297), with the aid of (3.298) we also obtain

$$P_j^* = \bar{P}_j^* \quad (j=4, 5, 6). \tag{3.300}$$

Here

$$\bar{P}_j^*(\bar{Z}) = \frac{1}{1-\bar{Z}}\left\{(1-\bar{Z}_1)P_j^*(\bar{Z}_1) - \int_{\bar{Z}_1}^{\bar{Z}} R_j(\xi)\, d\xi + \right.$$

$$\left. + l_0 \sum_{i=1}^{3} \hat{e}_i \times e^j \cdot \mathbf{k} \int_{\bar{Z}_1}^{\bar{Z}} \bar{P}_i^*(\xi)\, d\xi \right\}. \tag{3.301}$$

Expressions (3.298), (3.300) represent the first six integrals of system (3.274), which permit one to reduce the order of this system by six units, so that the first six equations of system (3.274) should be discarded. The first integrals of (3.298), (3.300) also permit one to reduce by six units the order of resolvent system (3.272). Eliminating P_j^* from (3.186), (3.189) with the aid of relations (3.269), we obtain

$$(1-\bar{Z}) \sum_{i=1}^{6+n} a_{\xi ji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i = \frac{1}{G} \bar{P}_j^* \quad (j=1, 2, \ldots, 6). \tag{3.302}$$

Adding Eqs. (3.302) to equations (3.272) for j=7, 8, ..., n we obtain a complete system of differential resolvents of order 6+2n.

Let us now consider the case in which the cone apex is chosen as the only pole for all the sections $\bar{Z}$= const, and the corresponding vectors $\eta(\bar{Z})$, $\theta(\bar{Z})$ are expanded in arbitrary bases e_α, $\hat{e}_\alpha$. In this case, from the condition of absence of strains during arbitrary displacement of the shell as a solid, we can readily obtain

$$\bar{\psi}_{i\gamma_k} = 0 \quad (i = 1, 2, 3), \tag{3.303}$$

$$\bar{\psi}_{i\gamma_k} = \frac{1}{l_i}\psi_{i\gamma_k} \quad (i = 4, 5, 6). \tag{3.304}$$

In expressions (3.303), (3.304), as in (3.275), γ_k denotes the index of any of the strain components. Considering (3.303), (3.304), we find from (3.263)-(3.265)

$$\begin{aligned} &Q_i = 0 \quad (i = 1, 2, 3), \\ &Q_i = P_i^{*} \quad (i = 4, 5, 6). \end{aligned} \tag{3.305}$$

Taking (3.305) into account, from Eqs. (3.267) we obtain

$$(r_j P_j^{*})' = -R_j \quad (j = 1, 2, \ldots, 6), \tag{3.306}$$

whence

$$P_j^{*} = \bar{P}_j^{*} \quad (j = 1, 2, \ldots, 6), \tag{3.307}$$

where

$$\bar{P}_j^{*} = \frac{1}{r_j(\bar{Z})}\left[r_j(\bar{Z}_1)P_j^{*}(\bar{Z}_1) - \int_{\bar{Z}_1}^{\bar{Z}} R_j(\xi)\,d\xi\right]. \tag{3.308}$$

Expressions (3.307) represent the first six integrals of system (3.274) for the case under consideration, which permit one to reduce by six units the order of the system of differential resolvents represented in any form.

Like the first integrals obtained in Section 3.2, the first integrals of (3.298), (3.300) and (3.307) constitute the equilibrium conditions written for the cut-off portion of the shell. The generalized forces P_j (j=1, 2, 3) are the components of the resultant;the generalized forces P_j (j=4, 5, 6) are the components of the net moment of the internal forces in the section $\bar{Z}$=const with respect to the chosen pole. As was done in Section 3.2, this can be readily ascertained by using expressions (2.33). Accordingly, $\bar{P}_j$ (j=1, 2, ..., 6) constitute the components of the resultant and net moment of all the external forces applied to the cut-off portion of the shell.

It is easy to see that if the cone apex is chosen as the pole of the contour

$\overline{Z}$=const, the overall resolvent system of ordinary differential equations of a conical shell of arbitrary configuration with a deformable contour (3.272) can be reduced to canonical form, as was done in Section 3.3. To this end, it is sufficient to make sure that the matrices of the coefficients of system (3.272) satisfy relations (3.127)-(3.130); this can be easily checked by using expressions (3.271) as well as (3.126), (3.303), (3.304).

3.6. Coordinate Functions

Thus far, we have ignored the question of selection of coordinate functions $\phi_k^I(S)$, which in combination approximate the functions

$$U^I(\overline{Z}, S) = \sum_k w_k(\overline{Z})\varphi_k^I(S),$$

for which the relative arrangement of the points of the contour $\overline{Z}$=const changes.

If the problem is solved on the basis of the fundamental static-geometric model or on the basis of the hypothesis of deformability of the contour $\overline{Z}$=const, the direction of vector function U^I turns out to be defined, and in accordance with (3.8) or (3.232), the system of scalar functions $\phi_k^I(S)$ is subject to selection. If however the problem is solved in the general formulation without additional geometric constraints imposed on the displacements $U^I(\overline{Z}, S)$, then the three systems of scalar functions $\phi_j^{I1}(S)$, $\phi_j^{I2}(S)$, $\phi_j^{I3}(S)$ are subject to selection in accordance with (3.194).

It is evident that the following general requirements must be placed on vector functions $\phi_k^I(S)$;

a) continuity;

b) single-valuedness on the single- or multiclosed contour $\overline{Z}$=const;

c) linear independence with respect to both each other and the functions $\phi_i(S)$ (i=1, 2, ..., 6), corresponding to displacement of the contour $\overline{Z}$ = const as a solid.

In selecting the system of scalar functions $\phi_k^1(S)$ which satisfy the above general requirements, one can proceed from two concepts. The first is based on the idea of approximation and is applicable in cases where, for shells of a given type, the nature of warping of the contour $\overline{Z}$ = const can be established in advance on the basis of an analysis of experimental data or on the basis of heuristic considerations. In this case, $\{\phi_k^1\}$ consists of several functions approximating the principal forms of warping. As will be shown below, such an approach is very effective in many cases. Moreover, since the number of unknowns is small, the solution of the resolvent system of differential equations can be found in analytical form. This is conveniently done by using the canonical form of the resolvents. In this case, when approximating the warping with one function, we arrive at one second-order differential equation, and when approximating the warping with n functions, at a system of n second-order equations each.

The chief advantage of the approximation concept is that the analytical solutions obtained on its basis permit both a quantitative and a qualitative analysis of the problems under consideration. Among the disadvantages inherent in this concept one must include first of all the necessity of preliminary information on the operation of the structure being studied, and also the fact that the quality of the solution completely depends on how successfully the approximating functions are chosen. The second concept, based on a series expansion of warping, is free of these disadvantages. Moreover, if $\{\varphi_k^1\}$ is a complete system of functions on the contour $\overline{Z}$ = const, we obtain a solution that is exact in the range of the adopted computational scheme.

For an arbitrarily outlined single- or multiclosed contour $\overline{Z}$ = const, one can indicate various methods of constructing complete systems of functions. For example, as $\{\phi_k\}$ one can adopt a system of eigenfunctions of a problem with eigenvalues, i.e., functions which on the contour $\overline{Z}$ = const satisfy a homogeneous differential or integro-differential equation of the form

$$M[\varphi^1(S)]+\lambda N[\varphi^1(S)]=0.$$

the conditions of single-valuedness on the contour, and certain additional conditions imposed on the derivatives. Such a method is employed later in Part Three in the construction of special coordinate functions for circular conical shells reinforced with a regular stringer structure. On the cross-sectional contour of a stringer shell, these functions possess the same properties as the trigonometric functions six $k\alpha$, cos $k\alpha$ on the contour of a smooth circular shell, and constitute their generalization.

Such a method of selection of coordinate functions is characterized by the fact that for shells with different outlines of the contour $\overline{Z}$ = const and even for shells with the same outline but with different reinforcements, different systems of coordinate functions will be obtained. These systems are very convenient, since, by analyzing the corresponding generating equations, one can establish certain common properties of the eigenfunctions that permit a simplification of the resolvent systems of ordinary differential equations. However, for an arbitrarily outlined contour, finding the eigenfunctions constitutes a very cumbersome problem solvable only by numerical methods. For this reason, such an approach is not always justified.

The indicated difficulties can be avoided by using a certain system of functions known in advance, known to be complete for both a single- and a multiclosed contour of any outline. For example, proceeding from the possibility of representing the functions of two variables Z(x, y) in the form of the power series

$$Z(x,y) \sim \sum_{i,j=0}^{\infty} a_{i,j} x^i y^j, \tag{3.309}$$

where $x_0(S)$, $y_0(S)$ are the coordinates of point M_0 belonging to the directrix of the cone.

It is easy to see that coordinate functions (3.309) are linearly independent of one another, continuous, and single-valued on a completely arbitrary single- or multiclosed contour $\overline{Z}$=const. At the same time, as we know, the functions $x^i y^j$ (i, j=0, 1, 2, ... ∞) in any bounded region of the xOy plane form a complete system. Hence there clearly follows the completeness of the coordinate functions (3.309) on an arbitrary single- or multiclosed plane contour. Therefore, as the system of coordinate functions $\phi_k^{l}(S)$, we can take system (3.309), first eliminating from it those functions that are linearly dependent with the coordinate functions corresponding to the displacement of the contour $\overline{Z}$=const as a solid. Among such functions can obviously be included only three functions of system (3.309): 1, $x_0(S)$, $y_0(S)$, depending on the outlines of the shell. The system of coordinate functions thus obtained is infinite, but in practical calculations, if the system of differential resolvents does not decompose completely, one can confine oneself to a finite number of coordinate functions which corresponds to the limited number of degrees of freedom of the contour $\overline{Z}$=const of the shell. The accuracy of the results naturally increases with increasing number of retained coordinate functions. However, raising the accuracy above a certain limit in this manner usually causes certain difficulties, since the numerical realization of boundary value problems reducing to differential equations of high orders usually requires a high accuracy of the computations that goes beyond the scope of the limited accuracy of the computer. A system of local coordinate functions is more suitable in this respect.

As the desired generalized displacements $w_k(\overline{Z})$ we will choose warping displacements of n points (nodes) of the contour $\overline{Z}$=const. Then the corresponding scalar coordinate functions $\phi_k^{l(n)}(S)$ should obviously satisfy the conditions

$$\phi_k^{l(n)}(S)=\begin{cases} 1, & S=S_k, \\ 0, & S=S_l,\ l\neq k, \end{cases} \tag{3.310}$$

where S_l, l=1, 2, ..., k, ..., n are the coordinates of the chosen nodes.

Assuming that in the spaces between any neighboring points the functions $\phi_k^{l(n)}(S)$ change linearly, in view of conditions (3.310) we will completely determine the functions $\phi_k^{l(n)}(S)$ on the contour $\overline{Z}$= const.

The coordinate functions selected in this manner will obviously be linearly independent from one another, continuous, and single-valued on the contour $\overline{Z}$=const. In view of the fact that these functions are different from zero only on portions adjacent to the point $S=S_k$, we will call them local coordinate functions. It is also evident that in the class of piecewise-linear continuous functions corresponding to the isolation of n nodes on the contour, the system $\{\phi_k^{I(n)}(S)\}$ (k=1, 2, ..., n) is complete.

Thus, the number of nodes n is generally the same as the number of degrees of freedom of the contour $\overline{Z}$=const with respect to warping displacements. However, one must make sure at the same time that the local coordinate functions be linearly independent with respect to the coordinate functions corresponding to the displacement of the contour $\overline{Z}$=const as a solid. If such a linear dependence does take place, it is necessary to omit a suitable number of any local coordinate functions, but this number will obviously be no higher than three.

Computational practice shows that numerical integration on a computer of systems of differential resolvents corresponding to local coordinate functions is possible for a greater number n of degrees of freedom than in the case of polynomial coordinate functions (3.309). This is due to the fact that the blocks $A_{\Omega\Omega}$, $B_{\Omega\Omega}$, $C_{\Omega\Omega}$ of matrices A, B, C (3.125) of the coefficients of differential resolvents constitute Jacobi matrices.

Everything that has been said thus far about the selection of local coordinate functions pertained to the case where the direction of the warping vector U^I proves to be defined, so that only one system of coordinate functions $\phi_k^I(S)$ is subject to selection. In solving the problem in the general formulation, as already noted, it is necessary to choose three systems of coordinate functions. In that case, when bending strains of the middle surface are taken into account, piecewise-linear local coordinate functions are inapplicable. However, each of the three systems of coordinate functions $\{\phi_j^{I1}(S)\}$, $\{\phi_j^{I2}(S)\}$, $\{\phi_j^{I3}(S)\}$ can be given in the form of polynomial system (3.309).

CHAPTER IV.

NATURAL BOUNDARY CONDITIONS

The general solutions of differential resolvents obtained in the preceding chapter on the basis of the Lagrange variational principle contain arbitrary constants that must be determined by satisfying the natural boundary conditions, which also result from the variational principle. For 6 + N degrees of freedom of the contour $\bar{Z}$ = const, the natural boundary conditions reduce to 6 + N equations at each of the ends of the shell. The relationship between natural boundary conditions, and "physical" ones is discussed below. We will consider : the static boundary conditions, when a system of external forces at the end of the shell is given; kinematic boundary conditions, when the displacements at the end are given, and finally mixed discrete-type conditions, when the end of the shell, loaded by a given external load, is fixed at a finite number of points. Boundary conditions of the latter type will be called "special".

For brevity, we will confine our discussion to natural boundary conditions obtained without considering the bending strain of the middle surface of the shell.

4.1. STATIC BOUNDARY CONDITIONS

Let a system of external forces be given on any of the ends of the shell $\bar{Z} = \bar{Z}_T$ $(\bar{Z}_0, \bar{Z}_1)$ * Then the variations of the generalized displacements $\delta U_j(\bar{Z}_T)$ $(j = 1, 2, \ldots, 6+n)$ — are arbitrary and independent. In this case, it follows from natural boundary conditions (3.58) and (3.268 that

$$p_{zj}(\bar{Z}_T) = \bar{p}_{zj}(\bar{Z}_T) \quad (j = 1, 2, \ldots, 6+n). \tag{4.1}$$

Relations (4.1), which express the equality of the generalized forces at the end $\bar{Z} = \bar{Z}_t$ to the given forces represent a system of 6 + N equations in the integration constants. Expanding (4.1) with the aid of (3.63) or (3.269), we obtain

$$G\lambda_j \left[(1-\bar{Z}) \sum_{i=1}^{6+n} a_{zji} V_i' + \sum_{i=1}^{6+n} b_{ji} V_i \right] \Bigg|_{\bar{Z}=\bar{Z}_T} = \bar{p}_{zj}(\bar{Z}_T)$$
$$(j = 1, 2, \ldots, 6+n). \tag{4.2}$$

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*Translator's note : subscript T refers to "end".

Let us eludidate the meaning of relations (..1). We turn to relations (1.191) expressing the equilibrium conditions of an element of the middle surface. We multiply scalarly the right and left-hand sides of the first of these relations by the vector function φ_i (S). Omitting the transverse forces and taking (1.151), (1.192) into account we find

$$T_{m_s}\varphi_{jm_s}\sin\chi + S_{n_s m_s}(\varphi_{jn_s}\sin\chi - \varphi_{jm_s}\cos\chi) - T_{n_s}\varphi_{jn_s}\cos\chi = \\ = T_{n_s}\varphi_{jn_s} + S_{n_s m_s}\varphi_{jm_s} \quad (j = 1, 2, \ldots, 6+n). \tag{4.3}$$

Using expressions (4.3), we can readily ascertain that the generalized forces $P\Sigma_j$ for both the fundamental static-geometric model and shells with an undeformable contour $\bar{Z}$ = const. may be represented in the form

$$P_{\Sigma j} = \nu_j \left[(1-\bar{Z}) \int (T_{n_s}\varphi_{jn_s} + S_{n_s m_s}\varphi_{jm_s})\, dS \cdot \sum_k N_k \varphi_{jm_s} \right]. \tag{4.4}$$

As follows from (3.313), expression (4.4) applies if one neglects the force factors related to the bending of the middle surface, and in the general case, when no limitations are imposed on the warping U^1.

In accordance with expressions (4.4), the generalized force $P\Sigma_j$ is the sum of the work done by the internal forces in the section $\bar{Z}$ = const of the shell on geometrically possible displacements determined by the conditions.

$$U_i = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases} \tag{4.5}$$

It follows that the generalized forces $P\Sigma_j$ (j = 1,2,3) are the components of the resultant, and $P\Sigma_j$ (j = 4,5,6) are the components of the net moment in the section $\bar{Z}$ = const of the shell with respect to the pole of rotation of this section.

Let q be the vector of internal forces in the section $\bar{Z}$ = const of the shell proper; N_k be the vector of the force in the kth element of the reinforcing structure. We have

$$\mathbf{q} = T_{n_s}\mathbf{n}_s + S_{n_s m_s}\mathbf{m}_s; \quad \mathbf{N}_k = N_k \mathbf{m}_k. \tag{4.6}$$

In view of (4.6), the generalized forces $P_j\Sigma$ (4.4) may be represented in the form

$$P_{\Sigma j}=\lambda_j\left[(1-\overline{Z})\oint \mathbf{q}\cdot\boldsymbol{\varphi}_j dS+\sum_k \mathbf{N}_k\cdot\boldsymbol{\varphi}_j\right]. \tag{4.7}$$

Now, taking (3.31), (3.60) and (4.6) into account, we can finally represent boundary conditions (4.1) in the form

$$(1-\overline{Z}_\tau)\oint(\mathbf{q}-\overline{\mathbf{q}}')\cdot\boldsymbol{\varphi}_j dS+\sum_k(\mathbf{N}_k-\overline{\mathbf{N}}_k')\cdot\boldsymbol{\varphi}_j=0$$

$$(j=1,2,\ldots,6+n). \tag{4,8}$$

Thus, as follows from (4.8), boundary conditions (4.1) express the orthogonality conditions on the contour $\overline{Z}=\overline{Z}_t$ of the difference between the external and internal forces on this contour with respect to the system of coordinate functions ϕ_j.

4.2 KINEMATIC BOUNDARY CONDITIONS

Let the displacements $\overline{Z}=\overline{Z}_t$ $(\overline{Z}_0, \overline{Z}_1)$ be given on any of the ends of the shell U_t (S). The following conditions should then be fulfilled on this end.

$$U(\overline{Z}_t, S)-\overline{U}_t(S)=0, \tag{4.9}$$

or in expanded form

$$\sum_{i=1}^{6+N} U_i(\overline{Z}_\tau)\lambda_i(\overline{Z}_\tau)\varphi_i(S)-\overline{U}_\tau(S)=0. \tag{4.10}$$

Expression (4.10) is an equation of constraint in values of the desired generalized displacements $U_j(\overline{Z})$ $(j=1, 2, \ldots\ldots, 6+N)$ when $\overline{Z}=\overline{Z}_t$. In this expression, summation in accordance with (3.198) is carried out up to $i=6+n_1+n_2+n_3$. This means that for arbitrary displacements U_t (S), no limitations must be imposed on the warping of the contour $\overline{Z}=$ const. However, if the displacements U_t (S) satisfy the geometric onstraints corresponding to the fundamental static-geometric model or model with a nondeformable contour $\overline{Z}=$ const, then in (4.10) the summation is carried out up to $i=6+n$.

On the basis of the concept of expansion as a series in coordinate functions, the equation of constraint can be satisfiec by orthogonalizing difference (4.9), by analogy w th the preceding section, to each of the coordinate functions ϕi on the contour $\overline{Z}=\overline{Z}_t$, including the elements of the longitudinal structure as well :

$$\oint [\mathbf{U}(\bar{Z}_\tau, S) - \bar{\mathbf{U}}_\tau(S)] \cdot \boldsymbol{\varphi}_j(S) h (1-\bar{Z}_\tau) dS + \sum_k [\mathbf{U}(\bar{Z}_\tau, S_k) - \bar{\mathbf{U}}_\tau(S_k)] \cdot \mathbf{m}_z (\boldsymbol{\varphi}_j \cdot \mathbf{m}_z) \Big|_{S=S_k} \frac{\Delta F_k}{\sin \gamma_k} = 0 \quad (j = 1, 2, \ldots, 6+N), \tag{4.11}$$

or in expanded form considering (4.10)

$$\sum_{i=1}^{6+N} U_i(\bar{Z}_\tau) \lambda_i(\bar{Z}_\tau) \left[(1-\bar{Z}_\tau) \oint \boldsymbol{\varphi}_i \boldsymbol{\varphi}_j h\, dS + \sum_k \varphi_{im_z} \varphi_{jm_z} \Big|_{S=S_k} \frac{\Delta F_k}{\sin \gamma_k} \right] - (1-\bar{Z}_\tau) \oint \bar{\mathbf{U}}_\tau \boldsymbol{\varphi}_j h dS - \sum_k \bar{\mathbf{U}}_\tau \cdot \mathbf{m}_z \varphi_{jm_z} \Big|_{S=S_k} \frac{\Delta F_k}{\sin \gamma_k} = 0 \quad (4.12)$$

$$(j = 1, 2, \ldots, 6+N). \tag{4.12}$$

Conditions (4.12) can also be obtained on the basis of the variational principle by using the method of Lagrangian multipliers.

Let us first assume that the longitudinal reinforcing structure is absent. In order to satisfy equation of constraint (4.9), we will proceed not from variational eq. (3.26), but from the equation.

$$\delta T - \delta U + \delta \oint \boldsymbol{\Lambda}_\tau(S) [\mathbf{U}(\bar{Z}_\tau, S) - \bar{\mathbf{U}}_\tau(S)] (1-\bar{Z}_\tau) h dS = 0. \tag{4.13}$$

Where Λ_τ (S) is an as yet undetermined vector function.

Considering (4.10) we have from Eq. (4.13)

$$\delta T - \delta U + \sum_{i=1}^{6+N} \delta U_i(\bar{Z}_\tau) \lambda_i(\bar{Z}_\tau)(1-\bar{Z}_\tau) \oint \boldsymbol{\Lambda}_\tau(S) \cdot \boldsymbol{\varphi}_i(s) h dS + \oint \delta \boldsymbol{\Lambda}_\tau(s) [\mathbf{U}(\bar{Z}_\tau, S) - \bar{\mathbf{U}}_\tau(S)] (1-\bar{Z}_\tau) h dS = 0. \tag{4.14}$$

In Eq. (4.14) the variations of generalized displacements δU_i are arbitrary and independent both within the interval $(0, Z_1)$ and on the end of thes shell $Z = Z_t$.

We will seek Λ_t (S) in the form of an expansion in coordinate vector functions ϕ_j :

$$\boldsymbol{\Lambda}_\tau(S) = \sum_{i=1}^{6+N} X_i \boldsymbol{\varphi}_i(S), \tag{4.15}$$

where X_i are unknown coefficients.

It follows form (4.15) that

$$\delta \boldsymbol{\Lambda}_\tau(S) = \sum_{i=1}^{6+N} \delta X_i \boldsymbol{\varphi}_i(S). \tag{4.16}$$

Expanding variational Eq. (4.14) with the aid of general expressions (3.204), (3.215) and considering (4.16), we can readily obtain

$$\int_0^{Z_t}\sum_{j=1}^{6+N}\left[\left(\frac{P_j}{\phi_j}\right)'\phi_j-Q_j+R_j\right]\delta U_j\,dZ-\sum_{j=1}^{6+N}\left[(\bar P_j-P_j)\,\delta U_j\right]\Big|_{Z=0}^{Z=Z_t}+$$
$$+\delta U_j(Z_t)\,\phi_j(Z_t)(1-Z_t)\int X_j(S)\,q_j(S)\,h\,dS+$$
$$+\delta X_j(1-Z_t)\int\left[U(Z_t,S)-\bar U_t(S)\right]q_j(S)\,h\,dS=0. \qquad (4.17)$$

In Eq. (4.17), all the variations δU_i and δX_i are arbitrary and independent. By virtue of the arbitrariness of δU_i on $(0, Z_1)$, we have from (4.17).

$$\left(\frac{P_j}{\phi_j}\right)'\phi_j-Q_j+R_j=0$$
$$(j=1,2,\ldots,6+N). \qquad (4.18)$$

When the coordinate functions ϕ_j are appropriately chose, Eqs. (4.18) coincide with any of the variants of the resolvents discussed above.

On the end $Z = Z_t$, the variations $\delta U_i(Z_t)$ must also be considered arbitrary and independent. Since the displacements are given on this end, all the generalized forces $\bar P_j(Z_t)$, corresponding to a given external load must be taken equal to zero :

$$\bar P_j(Z_t)=0\ (j=1,2,\ldots,6+N) \qquad (4.19)$$

Considering (4.19), we have from Eq. (4.17)

$$\phi_j(Z_t)(1-Z_t)\int X_j(S)\,q_j(S)\,h\,dS=P_j(Z_t)$$
$$(j=1,2,\ldots,6+n). \qquad (4.20)$$

On the other shell end, as before, we obtain

$$(\bar P_j-P_j)\,\delta U_j=0\quad (j=1,2,\ldots,6+N). \qquad (4.21)$$

In ddition to relations (4,20), (4.21) in view of the arbitrariness of the variation δX_i, we obtain from Eq. (4.17)

$$(1-Z_t)\int\left[U(Z_t,S)-\bar U_t(S)\right]q_j(S)\,h\,dS=0$$
$$(j=1,2,\ldots,6+N). \qquad (4.22)$$

Relations (4.20), (4.22) represent the set of natural boundary conditions on the end $\bar{Z} = \bar{Z}_t$. It is easy to see that relations (4.22) coincide with orthogonality conditions (4.11) if in the latter one omits the terms corresponding to the elements of the longitudinal structure. Relations (4.20) establish the relationship between the vector function $\Lambda_r(S)$ (Lagrangian multiplier) and generalized forces P_j in the section $\bar{Z} = \bar{Z}_t$. Introducing into (4.20) the generalized forces represented by vectorial expression (4.7), and omitting in this expression the terms pertaining to the elements of the longitudinal structure, we readily obtain

$$\lambda_t(\bar{Z}_t)(1-\bar{Z}_t)\oint\left[\Lambda_t(S)-\frac{1}{h}\,q(\bar{Z}_t,\,S)\right]\cdot\varphi_j(S)\,h\,dS=0$$

$$(j=1,\,2,\ldots,\,6+N). \tag{4.23}$$

Whence it follows that in the sense of the expansion as a series in coordinate functions ϕ_j

$$\Lambda_t(S)=\frac{1}{h}\,q(\bar{Z}_t,\,S), \tag{4.24}$$

i.e., the Lagrangian multiplier $\Lambda_t(S)$ is the vector function of reactive stresses on the end $\bar{Z} = \bar{Z}_t$. This makes it possible to interpret the last term of variational Eq. (4.13) :

$$\delta\oint\Lambda_t(S)\,[\mathrm{U}(\bar{Z}_t,\,S)-\bar{\mathrm{U}}_t(S)]\,(1-\bar{Z}_t)\,h\,dS \tag{4.25}$$

as the work of reactive forces to the fixing of geometric constraints (4.9)

In presenting the variational treatment of rothogonality conditions (4.12) for simplicity we omitted the terms corresponding to the longitudinal reinforcing structure. For a shell with a structure, the third term of the variational equation must obviously be written in the form

$$\delta\left\{\oint\Lambda_t(S)\,[\mathrm{U}(\bar{Z}_t,\,S)-\bar{\mathrm{U}}_t(S)]\,(1-\bar{Z}_t)\,h\,dS+\right.$$
$$\left.+\sum_k(\Lambda_t\cdot\mathrm{m}_t)\,\mathrm{m}_t\Big|_{S=S_k}\cdot[\mathrm{U}(\bar{Z}_t,\,S_k)-\bar{\mathrm{U}}_t(S_k)]\,\frac{\Delta F_k}{\sin\gamma_k}\right\}. \tag{4.26}$$

where the sum over the elements of the longitudinal structure represents the work done by the concentrated forces applied to them in maintaining geometric constraints (4.9) in the direction of the generatrices. Using (4.26) and reasoning as above, we can readily arrive at orthogonality conditions (4.11).

4.3 SPECIAL BOUNDARY CONDITIONS

Let either of the ends of the shell $\bar{Z} = \bar{Z}_t$ $(\bar{Z}_0, \bar{Z}_1)$ fastened at p points against displacements

$$u_t = U \cdot t, \tag{4.27}$$

where t = t (S) - is a unit vector specifying a certain given direction at each of the points. A given external load is applied at the remaining points of this end. In this case, the boundary conditions at the end $\bar{Z} = \bar{Z}_t$ should be classified as mixed-type boundary conditions.

The conditions

$$U(\bar{Z}_t, S_m) \cdot t(S_m) = 0 \quad (m = 1, 2, \ldots, p) \tag{4.28}$$

represent p equations of constraint in 6 + N values of the desired generalized displacements U_j $(\bar{Z})$ (j = 1,2,.....6 + N) when $\bar{Z} = \bar{Z}_t$. In this connection, the Lagrange variational equation should be written in the form

$$\delta T - \delta U + \delta \sum_{m=1}^{p} \Lambda_{t,m} \, U(\bar{Z}_t, S_m) = 0. \tag{4.29}$$

where p vectors $\Lambda_t m$ represent the Lagrangian multipliers.

In accordance with the above, the vector $\Lambda_t m$ represents the reaction of the mth constraint. Therefore

$$\Lambda_{t,m} = R_m t. \tag{4.30}$$

where R_m is the magnitude of the reaction at the mth point. Consequently, in variational equation (4.29), the third term

$$\delta \sum_{m=1}^{p} \Lambda_{t,m} \cdot U(\bar{Z}_t, S_m)$$

represents the work of the reactions of the supports, expended in maintaining geometric constraints (4.28).

Let us consider the case which is most important in practice, i. e. , when the end $\bar{Z} = \bar{Z}_t$ is discretely fastened in the direction of the generatrices. In this case, $t = m_z$. Let us note that if the direction of t does not coincide with m_z, all the arguments remain in force. We can also examine the case in which the section $\bar{Z} = Z_t$ is completely fastened at p points. In this case, the Lagrangian multipliers (reactions of supports) in contrast to (4.30) are unknown in both magnitude and direction, but the nature of the reasoning remains unchanged.

Expanding variational equation (4.29) with the aid of general expressions (3.204), (3.215) and considering (4.30), we obtain

$$\int_0^{Z_t} \sum_{i=1}^{6+N} \left[\left(\frac{P_i}{\lambda_i} \right)' \lambda_i - Q_i + R_i \right] \delta U_i \, d\bar{Z} + \sum_{i=1}^{6+N} \Big\{ (\bar{P}_i - P_i) \delta U_i \Big|_{\bar{Z}=0}^{\bar{Z}=Z_t} + \\ + \delta U_i (\bar{Z}_t) \lambda_i (\bar{Z}_t) \sum_{m=1}^{p} R_m \varphi_{im_z} (S_m) \Big\} + \sum_{m=1}^{p} \delta R_m \sum_{i=1}^{6+N} U_i (\bar{Z}_t) \lambda_i (\bar{Z}_t) \varphi_{im_z} (S_m) = 0. \tag{4.31}$$

In Eq. (4.31), all the variations δU_i and δR_m are arbitrary and independent In view of the arbitrariness of δU_i on $(0, \bar{Z}_1)$ from (4.31) there follows equation (4.18), which, as was noted above, coincides with any of the variants of the resolvents when the coordinate functions are appropriately selected.

On the discretely fastened end $\bar{Z} = Z_t$ the variations $\delta U_i (Z_t)$ (i = 1, 2, ...6 should also be considered arbitrary and independent. Hence, when $\bar{Z} = Z_t$ we have

$$\bar{P}_j (Z_t) + \Delta P_j (\bar{Z}_t) = P_j (\bar{Z}_t) \quad (j = 1, 2, \ldots, 6+N), \tag{4.32}$$

where

$$\Delta P_j (\bar{Z}_t) = \lambda_j (\bar{Z}_t) \sum_{m=1}^{p} R_m \varphi_{jm_z} (S_m). \tag{4.33}$$

For the other end of the shell as in all the preceding cases, we obtain

$$(\bar{P}_j - P_j) \delta U_j = 0 \quad (j = 1, 2, \ldots, 6+N). \tag{4.34}$$

In addition to relations (4.32), (4.34) as a result of the arbitrariness and independence of p variations δR_m we find from Eq. (4.31)

$$\sum_{i=1}^{6+N} U_i(\bar{Z}_t)\, y_i(\bar{Z}_t)\, \varphi_{i,m_l}(S_l) = 0 \qquad (l = 1, 2, \ldots, p). \tag{4.35}$$

Relations (4.32), (4.35) represent a set of natural boundary conditions formulated for a discretely fastened end $\bar{Z} = \bar{Z}_t$. Relations (4.32) correspond to the static boundary conditions on this end. Here the quantities $\bar{F}_j(\bar{Z}_t)$ correspond to a given external load on the end $\bar{Z} = \bar{Z}_t$, and the quantities $\Delta P_j(\bar{Z}_t)$ defined by expression (4.33) clearly constitute additional generalized forces due to the desired reactions of the supports. Relations (4.35) obviously constitute the conditions of discrete fastening of the shell.

Thus, the boundary conditions corresponding to discrete fastening of a shell end are realized as follows. After discarding, the discrete supports and replacing their action by unknown reactions, it is necessary to satisfy the static boundary conditions (4.28) on this end. Then, all the 2 (6 + N) integration constants will be expressed in terms of p desired reactions of the supports. These reactions must then be found from the end conditions at p points of (4.35).

5.5. Elasticity Relations

Let t=t(M) be the temperature change at point M of a shell, measured from some initial state characterized by a uniform distribution of temperature, which is the same for the shell itself and for the ambient medium. In the general case, the temperature change will depend on three coordinates:

$$t=t(Z,\ S,\ \gamma). \tag{5.1}$$

We visualize an elementary parallelepiped cut out of the shell, bounded by two surfaces equidistant from the middle surface and by normal sections oriented along the lines of principal curvatures of the middle surface. According to the Duhamel-Neumann hypothesis, the strain of the isolated element consists of the elastic strain caused by the action of stressed state components applied to the surface of the element, and of thermal expansion.

For an isotropic material, in view of (1.125)-(1.130), we have

$$\begin{aligned}
\varepsilon^{*}_{m_z} &= \frac{1}{E}\left[\sigma_{m_z}-\nu\left(\sigma_{n_z}+\sigma_{n_n}\right)\right]+\alpha t,\\
\varepsilon^{*}_{n_z} &= \frac{1}{E}\left[\sigma_{n_z}-\nu\left(\sigma_{n_n}+\sigma_{m_z}\right)\right]+\alpha t,\\
\varepsilon^{*}_{n_n} &= \frac{1}{E}\left[\sigma_{n_n}-\nu\left(\sigma_{m_z}+\sigma_{n_z}\right)\right]+\alpha t,\\
\gamma^{*}_{n_z m_z} &= \frac{1}{G}\tau_{n_z m_z};\quad \gamma^{*}_{n_n m_z}=\frac{1}{G}\tau_{n_n m_z};\quad \gamma^{*}_{n_n n_z}=\frac{1}{G}\tau_{n_n n_z},
\end{aligned} \tag{5.2}$$

where α is the linear expansion coefficient.

Applying the Kirchhoff-Love hypothesis to the shell

$$\varepsilon^{*}_{n_n}=\gamma^{*}_{n_n m_z}=\gamma^{*}_{n_n n_z}=0, \tag{5.3}$$

as before, we will use the representation of a fictitious transversally isotropic material identical to the real material in physical properties in the tangential directions, but differing from it in the properties in the direction of the normal. Then, considering (1.132)-(1.137), we have

$$
\begin{aligned}
\varepsilon^{\bullet}_{m_z} &= \frac{1}{E_1}\left(\sigma_{m_z} - \nu_1\sigma_{n_z}\right) - \frac{\nu_2}{E_2}\sigma_{n_n} + \alpha_1 t, \\
\varepsilon^{\bullet}_{n_z} &= \frac{1}{E_1}\left(\sigma_{n_z} - \nu_1\sigma_{m_z}\right) - \frac{\nu_2}{E_2}\sigma_{n_n} + \alpha_1 t, \\
\varepsilon^{\bullet}_{n_n} &= -\frac{\nu_2}{E_2}\left(\sigma_{m_z} + \sigma_{n_z}\right) + \frac{1}{E_2}\sigma_{n_n} + \alpha_2 t, \\
\gamma^{\bullet}_{n_z m_z} &= \frac{1}{G_1}\tau_{n_z m_z}; \quad \gamma^{\bullet}_{n_n m_z} = \frac{1}{G_2}\tau_{n_n m_z}; \quad \gamma^{\bullet}_{n_n n_z} = \frac{1}{G_2}\tau_{n_n n_z}.
\end{aligned}
\tag{5.4}
$$

Now, assuming the fictitious material of the shell to be absolutely rigid with respect to transverse shear strain and nondeformable in the direction of the normal, it is necessary to set in relations (5.4):

$$
\begin{aligned}
&E_2 = G_2 = \infty, \\
&\alpha_2 = 0, \\
&E_1 = E, \; G_1 = G, \; \nu_1 = \nu, \\
&\alpha_1 = \alpha,
\end{aligned}
\tag{5.5}
$$

where E, G, ν, α are physical constants of the real material.

In this case, from the first, second and fourth of relations (5.4), we can readily obtain

$$
\begin{aligned}
\sigma_{m_z} &= \frac{E}{1-\nu^2}\left(\varepsilon^{\bullet}_{m_z} + \nu\varepsilon^{\bullet}_{n_z}\right) - \frac{E}{1-\nu}\alpha t, \\
\sigma_{n_z} &= \frac{E}{1-\nu^2}\left(\varepsilon^{\bullet}_{n_z} + \nu\varepsilon^{\bullet}_{m_z}\right) - \frac{E}{1-\nu}\alpha t, \\
\tau_{n_z m_z} &= G\gamma^{\bullet}_{n_z m_z}.
\end{aligned}
\tag{5.6}
$$

The remaining elasticity relations will, as before, contain indeterminacies which can be revealed by the equilibrium conditions.

Thus, the physical relations for a shell allowing for a temperature field differ from analogous relations without consideration of the temperature field in the additional terms -E/1-v in the expressions for normal stresses. Accordingly, in problems involving a temperature field, we will add the symbol t to the components of stress: $\sigma^t_{m_z}$, $\sigma^t_{n_z}$, $\tau^t_{m_z n_z}$. We will also keep responding to physical relations without consideration of temperature. In this connection, expanding relations (5.6) with the aid of expressions (1.119), we write

$$\sigma_{m_z}^t = \sigma_{m_z} - \sigma_t; \quad \sigma_{n_z}^t = \sigma_{n_z} - \sigma_t; \quad \tau_{n_z m_z}^t = \tau_{n_z m_z}, \tag{5.7}$$

where

$$\sigma_t = \frac{E}{1-\nu}\alpha t, \tag{5.8}$$

and σ_{m_z}, σ_{n_z}, $\tau_{m_z n_z}$ are determined by expressions (1.145).

Adopting approximate relations (1.105), from (1.122), (1.124) with the aid of (5.7) we find

$$\begin{gathered} T_{m_z}^t = T_{m_z} - T_t, \quad T_{n_z}^t = T_{n_z} - T_t, \\ S_{m_z}^t = S_{n_z}^t = S_{n_z m_z}, \\ M_{m_z}^t = M_{m_z} - M_t, \quad M_{n_z}^t = M_{n_z} - M_t, \\ H_{m_z}^t = H_{n_z}^t = H_{n_z m_z}, \end{gathered} \tag{5.9}$$

where

$$T_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu}\alpha t\, d\gamma; \qquad M_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu}\alpha t \gamma\, d\gamma, \tag{5.10}$$

the terms T_{m_z}, T_{n_z}, $S_{n_z m_z}$, M_{m_z}, M_{n_z}, $H_{n_z m_z}$, formally corresponding to physical relations without temperature, are determined by expressions (1.146).

In expressions (5.10) it is assumed that the elastic modulus E and coefficient of linear expansion α may in the general case depend on temperature and hence, in an arbitrary temperature field, on the coordinate γ. If the shell is heated uniformly along the thickness, then $t=t(\bar{Z}, S)$ and from (5.10) we obtain

$$T_t = \frac{Eh}{1-\nu}\alpha t; \quad M_t = 0. \tag{5.11}$$

For longitudinal structure elements oriented in the direction of the generatrices, we have

$$N_k^l = N_k - N_{kl}; \quad M_k^l = M_k - M_{kl}; \quad H_k^l = H_k, \tag{5.12}$$

where

$$N_{kl} = \int_{\Delta F_k} E\alpha t dF; \qquad M_{kl} = \int_{\Delta F_k} E\alpha t y dF, \tag{5.13}$$

and N_k, M_k, H_k are determined by relations (3.25).

In (5.13), the quadratures extend to the area ΔF_k of the cross section of the kth reinforcing element; y is the coordinate of the moving point of the cross section of the reinforcing element relative to the middle surface of the shell.

5.2. Resolvents of the Problem*

On the basis of the concept of expansion as a series in coordinate vector functions $\phi_i(S)$, we will represent the elastic displacement vector $U(\overline{Z}, S)$ in the form

$$U(\overline{Z}, S) = \sum_{i=1}^{6+N} U_i(\overline{Z}) \lambda_i(\overline{Z}) \varphi_i(S). \tag{5.14}$$

In expansion (5.14), as has always been the case thus far, the first six terms correspond to the specimens of the contour $\overline{Z}$ = const as a solid. The remaining terms of expansion (5.14) determine the warping of the contour $\overline{Z}$ = const. As already noted, for a suitable selection of the coordinate functions $\phi_i(S)$, expansion (5.14) may correspond to the fundamental static-geometric model, to the model with the contour $\overline{Z}$ = const nondeformable in its plane, and also to the general case, in which no limitations are

*For analogous equations for straight prismatic caissons, see I.F. Obraztsov, Variational Methods of Calculation of Thin-Walled Aeronautical Structures. Moscow, Mashinostroyeniye, 1966.

imposed on the warping of the contour $\bar{Z}$ = const.

Since geometric relations (1.55), (1.69), (1.120) are not related to the character of the external action on the shell, and the elastic displacement vector is represented by expansion (5.14), whose form is the same as that of the analogous expansions in the preceding cases, general expansions (3.16)-(3.22) and (3.252)-(3.254), which determine the tangential and flexural strains of the middle surface of the shell, retain their form. Consequently, it is easy to see that the resolvents resulting from the principle of possible Lagrange displacements can, as in the preceding cases, be represented by expressions of the form

$$\left(\frac{P^t_{\Sigma j}}{\lambda_j}\right)' - \frac{Q^t_j}{\lambda_j} + \frac{R_j}{\lambda_j} = 0 \qquad (j = 1, 2, \ldots, 6+N), \tag{5.15}$$

the general expressions for $P^t_{\Sigma j}$, Q^t_j, R_j also retaining their form, with replacement of the components of internal forces without the index t by the corresponding components with this index.

For the sake of specificity, let us turn to expressions (3.30), (3.36), (3.37), (3.54), pertaining to the fundamental static-geometric model. Let the external load on the shell be absent. Then, in view of (5.9)

$$P^t_{\Sigma j} = P_{\Sigma j} - P_{\Sigma jt}; \qquad Q^t_j = Q_j - Q_{jt}; \qquad R_j = 0. \tag{5.16}$$

Here $P_{\Sigma j}$ and Q_j are determined by expressions (3.36), (3.37), (3.54), (3.56) where T_{m_z}, $S_{n_z m_z}$, T_{n_z}, N_k are determined by the expressions

$$\begin{aligned} P_{\Sigma jt} &= \lambda_j \left[(1-\bar{Z}) \oint (\varphi_{jm_z} - \varphi_{jn_z}) T_t \sin\gamma \, dS + \sum_k N_{kt} \varphi_{jm_z}(S_k) \right], \\ Q_{jt} &= \lambda_j \oint (\varphi_{jm_z} - \varphi_{jn_z}) T_t l_s \sin\gamma \, dS. \end{aligned} \tag{5.17}$$

In view of (5.16), resolvents (5.15) can be represented in the form

$$\left(\frac{P_{\Sigma j}}{\lambda_j}\right)' - \frac{Q_j}{\lambda_j} + R_{jt} = 0 \qquad (j = 1, 2, \ldots, 6+N), \tag{5.18}$$

where

$$R_{jt} = Q_{jt} - \lambda_j \left(\frac{P_{\Sigma jt}}{\lambda_j}\right)'. \tag{5.19}$$

Now, considering that expressions (3.43), (3.63) relating $P_{\Sigma j}$ and Q_j to the generalized displacements U_j are valid for problems with and without consideration of a temperature field, we can readily conclude that the differential resolvents for the case of a temperature field differ from the resolvents without consideration of a temperature field in the right-hand sides only. Calculation for the action of the temperature field yields

$$\sum_{i=1}^{6+N}\left\{\left[(1-\bar{Z})\, a_{\Sigma ji} V_i' + b_{ji} V_i\right]' - b_{ij} V_i' - \frac{1}{1-\bar{Z}} c_{ji} V_i\right\} = -\frac{1}{\lambda_j G} R_{jt} \qquad (j = 1, 2, \ldots, 6+N), \tag{5.20}$$

where R_{jt} are represented by (5.19).

As can be readily seen, the natural boundary conditions in the case under consideration have a form similar to (3.58):

$$\left(\bar{P}_{\Sigma j} - P^t_{\Sigma j}\right) \delta U_j \Big|_{Z=0}^{Z=Z_1} = 0 \qquad (j = 1, 2, \ldots, 6+N), \tag{5.21}$$

where $\bar{P}_{\Sigma j}(0)$, $\bar{P}_{\Sigma j}(\bar{Z}_1)$ are determined by expressions (3.31), (3.59) and (3.60).

If the external load on the shell is absent, $\bar{P}_{\Sigma j}(0) = \bar{P}_{\Sigma j}(\bar{Z}_1) = 0$, and in view of (5.16), natural boundary conditions (5.21) assume the following form:

$$(P_{zj} - P_{tjt})\delta U_j \Big|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_T} = 0 \qquad (j=1,\, 2, \ldots,\, 6+N). \tag{5.22}$$

For the free end of the shell $\bar{Z}=\bar{Z}_T$ it follows from (5.22) that

$$P_{tj}(\bar{Z}_T) = P_{tjt}(\bar{Z}_T) \qquad (j=1,\, 2, \ldots,\, 6+N). \tag{5.23}$$

If the displacements on the end of the shell are specified, the boundary conditions are formulated in the same way as in the problem without the temperature field.

The matrices of the coefficients of system (5.20) obviously possess the same general properties as the corresponding matrices for the problem without the temperature field. Consequently, the resolvent system of ordinary differential equations of the temperature problem can be reduced to the canonical form by switching to canonical kinematic unknowns, as was shown in Section 3.3. The order of the resolvent system is thus reduced by 12 units.

System (5.20) also permits a decrease of the order by six units when the first six equations of the system are replaced by the equilibrium conditions of the cut-off portion of the shell:

$$P_{zj}^{t}(\bar{Z}) = \bar{P}_j(\bar{Z}) \qquad (j=1,\, 2, \ldots,\, 6). \tag{5.24}$$

If there is no external load on the shell, in view of (5.16), we have from (5.24)

$$P_{tj}(\bar{Z}) = P_{zjt}(\bar{Z}) \qquad (j=1,\, 2, \ldots,\, 6) \tag{5.25}$$

or in expanded form in view of (3.42),

$$(1-\bar{Z}) \sum_{i=1}^{6+N} a_{tji} V_i' + \sum_{i=1}^{6+N} b_{ji} V_i = \frac{1}{\lambda_j G} P_{jt} \qquad (j=1,\, 2, \ldots,\, 6). \tag{5.26}$$

It is also obvious that if the thickness h of the shell proper depends only on the S coordinate, and the areas ΔF_k of elements of the longitudinal structure change in accordance with the law $\Delta F_k(\bar{Z}) = \Delta F_k^0(1-\bar{Z})$, the resolvents of the temperature problem reduce to equations with constant coefficients, as do the resolvents which do not consider the temperature field.

We will now turn to the model based on the hypothesis of nondeformability of the contour $\bar{Z}$ = const. For brevity, we will assume that the shell heats up uniformly along the thickness. Obviously, in the presence of a temperature field, the hypothesis of nondeformability of the contour must be modified, by taking

$$\varepsilon^0_{m_s} - \alpha t = 0, \tag{5.27}$$

which means that the elastic strains are equal to zero, whereas the total strain should be sufficient from zero and equal to the temperature expansion:

$$\varepsilon^0_{m_s} = \alpha t. \tag{5.28}$$

Therefore, strictly speaking, a term corresponding to the thermal expansion of the contour $\bar{Z}$ = const in its plane should be introduced into expansion (2.231). However, strain (5.28) is more conveniently introduced directly into Hooke's law without affecting expansion (2.231), i.e., considering that in the presence of a temperature field, the warping displacements are, as before, perpendicular to the plane of the contour $\bar{Z}$ = const. The error thus introduced is practically imperceptible, while most of the fundamental formulas preserve their form, this being highly important.

Following this approach, one can readily obtain

$$T'_{n_s} = \frac{Eh}{1-\nu^2}(\varepsilon^0_{n_s} - \alpha t); \qquad S'_{n_s m_s} = \frac{Eh}{2(1+\nu)} \gamma^0_{n_s m_s}, \tag{5.29}$$

whence, transforming the shear strain as was done in Section 3.5, we find

$$T'_{n_s} = T_{n_s} - T_t; \qquad S'_{n_s m_s} = S_{n_s m_s} - \operatorname{ctg} \gamma S_t, \tag{5.30}$$

where

$$T_t = \frac{Eh}{1 - \nu^2} \alpha t; \qquad S_t = \frac{Eh}{2(1 + \nu)} \alpha t, \tag{5.31}$$

and the terms T_{n_s}, $S_{n_s m_s}$, which formally correspond to the physical relations without temperature, are determined by expressions (3.256).

It is easy to see that, since the temperature t is specified, the strain variations $\delta\epsilon^0_{n_s}$, $\delta\gamma^0_{n_s m_s}$ will be determined by the same expressions as before without consideration of the temperature field, and the strain variation $e^0_{m_s}$ will as before be equal to zero. As a result, the resolvents resulting from the Lagrange principle can, as in all previous cases, be represented by expressions of the form (5.15). At the same time, the general expressions for $P^t_{\Sigma j}$, Q^t_j, R_j also retain their form when the forces T_{n_s}, $S_{n_s m_s}$ without index t are replaced by the corresponding forces with this index.

In the absence of an external load on the shell, the following relations result from expressions (3.30), (3.262)-(3.265) in view of (5.12) and (5.30), (5.31):

$$P'_{\Sigma j} = P_{\Sigma j} - P_{\Sigma jt}; \quad Q'_j = Q_j - Q_{jt}; \quad R_j = 0, \tag{5.32}$$

where $P_{\Sigma j}$ and Q_j are determined, as before, by expressions (3.262), (3.265) without index t on the components of the internal forces, and

$$\begin{gathered} P_{\Sigma jt} = e_j \left[(1 - Z) \int \left(\frac{2}{1 - \nu} \varphi_{j n_s} + \varphi_{j n_s m_s} \right) S_t \sin \gamma \, dS + \sum_k N_{kt} \varphi_{j m_s}(S_k) \right], \\ Q_{jt} = e_j \int \left(\frac{2}{1 - \nu} \bar{\varphi}_{j n_s} + \bar{\varphi}_{j n_s m_s} \right) S_t l_s \sin \gamma \, dS. \end{gathered} \tag{5.33}$$

It follows from (5.32) that in the case under consideration, the resolvents can also be represented in the form (5.18), (5.19). It follows that the system of differential resolvents of the temperature problem for the model with a nondeformable contour $\overline{Z}$ = const differs from the corresponding system without consideration of the temperature field in the right-hand sides only, which are determined by expressions (5.19), where $P_{\Sigma jt}$, Q_{jt} should be calculated from formulas (5.33). The boundary conditions in the case at hand are formulated in the same way as in the preceding case.

As was done above, it is easy to obtain the differential resolvents of the temperature problem by considering the bending strains of the middle surface of the shell. These equations will obviously differ from the corresponding equations without consideration of temperature in the right-hand sides only, which as above are determined by the expression

$$-\frac{1}{\lambda_j G} R_{jt} \quad (j=1,\ 2, \ldots,\ 6+N). \tag{5.34}$$

In the case under consideration

$$R_{jt}=Q_j+\lambda_j\left[\left(\frac{M_{jt}}{\lambda_j}\right)' - \frac{P_{jt}}{\lambda_j}\right], \tag{5.35}$$

where

$$\begin{aligned}
M_{jt} &= \lambda_j(1-\overline{Z})\oint(\vartheta_{jm_s}+\vartheta_{jn_s})M_t\frac{\sin\chi}{l_s}dS,\\
P_{jt} &= \lambda_j\Big[(1-\overline{Z})\oint(\dot\psi_{jm_s}+\dot\psi_{jn_s})T_t\sin\chi dS+\\
&\quad+\oint(\bar\vartheta_{jm_s}+\bar\vartheta_{jn_s})M_t\sin\chi dS\Big].\\
Q_{jt} &= \lambda_j\Big[\oint(\dot\psi_{jm_s}+\dot\psi_{jn_s})T_t l_s\sin\chi dS+\\
&\quad+\frac{1}{1-\overline{Z}}\oint(\bar\vartheta_{jm_s}+\bar\vartheta_{jn_s})M_t l_s\sin\chi dS\Big].
\end{aligned} \tag{5.36}$$

The natural boundary conditions allowing for bending strain of the middle surface of the shell are represented by expressions (3.218). In the case at hand, in the absence of an external load on the shell, we have

$$\left[P_j - \lambda_j \left(\frac{M_j}{\lambda_j}\right)' - P_{jl} + \lambda_j \left(\frac{M_{jl}}{\lambda_j}\right)'\right] \delta U_j +$$
$$+ \frac{1}{\lambda_j} (M_j - M_{jl}) (\delta U_j \rho_j)' \Big|_{z=0}^{z=z_l} = 0 \qquad (j = 1, 2, \ldots, 6 + N_p) \tag{5.37}$$

Chapter VI. Oblique Cylindrical Shells

The present chapter presents resolvent systems of ordinary differential equations for cylindrical shells of different types.* Emphasis is placed on oblique cylindrical shells. Shells with a set of transverse diaphragms oriented perpendicular to the generatrices or parallel to the oblique end are discussed. In addition, a straight cylindrical shell with an arbitrarily outlined cross-sectional contour is also discussed. The systems of differential resolvents for shells of these types constitute special cases of resolvents of a conical shell of arbitrary configuration, obtained in Chapter III. The treatment is therefore in the nature of a survey. Only the principal results are given, along with references to the more general relations corresponding to conical shells of arbitrary configuration; most intermediate operations are omitted.

6.1. Summary of Fundamental Relations

In discussing straight and oblique cylindrical shells as special modifications of conical shells of arbitrary configuration, it is necessary in the general relations for conical shells to carry out the limiting transition when $l_0 \to \infty$ without changing the configuration of the directrix or the origin of the Cartesian coordinates. Thus, we arrive at an oblique cylindrical shell with generatrices parallel to the Oz axis, which makes an angle x_0 with the plane of the directrix (Figure 6.1).

Since when $l_0 \to \infty$ the length l_s of all the generatrices of the conical shell also tends to infinity, in order to achieve the indicated passage to the limit, it is necessary to change, from the relative oblique-angled coordinate $\overline{Z}=\overline{Z}(M)$ constituting the relative distance from the point M of the middle surface of the shell to the plane of the directrix, measured along the generatrix in fractions of its total length, to an

*See footnote on page .

absolute coordinate. For this purpose, we introduce a new variable

$$Z = l_0 \bar{Z}, \tag{6.1}$$

which is the Cartesian coordinate of the point of intersection of the plane of the contour $\bar{Z}$=const with the Oz axis. It is obvious that as a result of the limiting transition to a cylinder, the new oblique-angled coordinate Z=Z(M) will be the absolute distance from the point of the middle surface of the cylindrical shell to the plane of the directrix, measured along the generatrix passing through point M.

We will summarize the fundamental relations in coordinates Z, S for an oblique cylindrical shell.

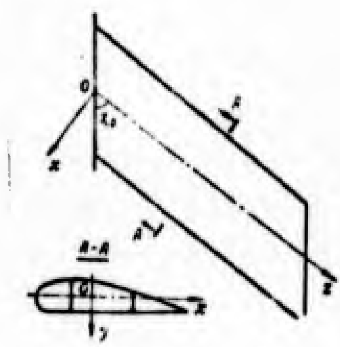


Figure 6.1. Oblique cylindrical shell.

The parametric equations of an oblique cylindrical surface arising from (1.5) as a result of the passage to the limit have the form

$$\begin{gathered} x = x(S);\quad y = y(S);\\ z = Z + x(S)\operatorname{ctg}\chi_0. \end{gathered} \tag{6.2}$$

The coefficients of the first quadratic form of an oblique cylindrical surface of arbitrary configuration referred to curvilinear coordinates Z, S, (6.2) will be, in view of (1.6),

$$A^2=\left(\frac{\partial x}{\partial Z}\right)^2+\left(\frac{\partial y}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2=1,$$
$$B^2=\left(\frac{\partial x}{\partial S}\right)^2+\left(\frac{\partial y}{\partial S}\right)^2+\left(\frac{\partial z}{\partial S}\right)^2=1,$$
$$\cos\chi=\frac{1}{AB}\left(\frac{\partial x}{\partial Z}\frac{\partial x}{\partial S}+\frac{\partial y}{\partial Z}\frac{\partial y}{\partial S}+\frac{\partial z}{\partial Z}\frac{\partial z}{\partial S}\right)=x'\operatorname{ctg}\chi_0. \qquad (6.3)$$

The direction cosines of unit vectors n_n, n_z, n_ε, m_z, m_s in the fundamental system of Cartesian coordinates are determined by the expressions arising from (1.18) as a result of the passage to the limit:

	x	y	z
n_n	$-\frac{1}{\sin\chi}y'$	$\frac{1}{\sin\chi}x'$	0
n_z	$\frac{1}{\sin\chi}x'$	$\frac{1}{\sin\chi}y'$	0
m_z	0	0	1
n_s	$-\operatorname{ctg}\chi\, x'$	$-\operatorname{ctg}\chi\, y'$	$\sin\chi$
m_s	x'	y'	$\cos\chi$

(6.4)

The components of tangential strain of the middle surface of the oblique cylindrical shell in axes of the main and auxiliary trihedra are determined by expressions arising from (1.55), (1.65) as a result of the passage to the limit:

$$\varepsilon^0_{m_z}=\frac{\partial u_{m_z}}{\partial Z},$$
$$\varepsilon^0_{n_z}=-\operatorname{ctg}\chi\frac{\partial u_{n_z}}{\partial Z}+\frac{1}{\sin\chi}\frac{\partial u_{n_z}}{\partial S}-\frac{1}{R}u_{n_n}, \qquad (6.5)$$
$$\gamma^0_{n_z m_z}=-\operatorname{ctg}\chi\frac{\partial u_{m_z}}{\partial Z}+\frac{1}{\sin\chi}\frac{\partial u_{m_z}}{\partial S}+\frac{\partial u_{n_z}}{\partial Z},$$

$$\varepsilon^0_{m_s}=\frac{\partial u_{m_s}}{\partial S}-\frac{\sin^2\chi}{R}u_{n_n},$$
$$\varepsilon^0_{n_s}=\frac{1}{\sin\chi}\frac{\partial u_{n_s}}{\partial Z}-\operatorname{ctg}\chi\frac{\partial u_{n_s}}{\partial S}-\frac{\cos^2\chi}{R}u_{n_n}, \qquad (6.6)$$
$$\gamma^0_{n_s m_s}=\frac{1}{\sin\chi}\frac{\partial u_{m_s}}{\partial Z}-\operatorname{ctg}\chi\frac{\partial u_{m_s}}{\partial S}+\frac{\partial u_{n_s}}{\partial S}+\frac{\sin 2\chi\, u_{n_n}}{R},$$

where in accordance with (1.121)

$$\frac{1}{R}=\frac{1}{\sin\chi}(x'y''-x''y'). \tag{6.7}$$

If the point of intersection of the plane of the contour $\overline{Z}$=const with the Oz axis is chosen as the pole, and the vectors of translational displacement and rotation of this contour are expanded in axes of the main Cartesian system, then on the basis of (2.21), the components of the coordinate vector functions $\phi_i(S)$ corresponding to displacements of the contour $\overline{Z}$ = const of an oblique cylindrical shell of arbitrary configuration as a solid will have the form

i	U_i	φ_{im_z}	φ_{im_s}	φ_{in_n}
1	η_x	0	x'	$-\frac{1}{\sin\chi}y'$
2	η_y	0	y'	$\frac{1}{\sin\chi}x'$
3	η_z	1	$x'\operatorname{ctg}\chi_0$	0
4	θ_x	y	$(x'y-xy')\operatorname{ctg}\chi_0$	$-\frac{\operatorname{ctg}\chi_0}{\sin\chi}xx'$
5	θ_y	$-x$	0	$-\frac{\operatorname{ctg}\chi_0}{\sin\chi}xy'$
6	θ_z	0	$xy'-x'y$	$-\frac{1}{\sin\chi}(xx'+yy')$

(6.8)

If however the vectors of translational displacement and rotation of the contour $\overline{Z}$ = const are expanded in the auxiliary basis (3.241), then in view of (3.244), we can readily obtain

i	U_i	φ_{in_z}	φ_{im_z}	φ_{in_n}
1	η_1	$\frac{\cos^2\gamma_0 - \cos^2\chi}{\sin\chi\cos\chi_0}$	$\frac{x'}{\sin\chi_0}$	$-\frac{\sin\chi_0}{\sin\chi}y'$
2	η_2	$-\operatorname{ctg}\chi y'$	y'	$\frac{1}{\sin\chi}x'$
3	η_3	$\frac{\sin\chi_0}{\sin\chi}$	0	$\frac{\cos\chi_0}{\sin\chi}y'$
4	θ_1	$\frac{\sin\chi_0}{\sin\chi}y$	0	$\frac{\cos\chi_0}{\sin\chi}yy'$
5	θ_2	$-\frac{1}{\sin\chi}x$	0	$-\frac{\cos\chi_0}{\sin\chi}\frac{x}{\sin\chi_0}y'$
6	θ_3	$\frac{x'y - xy'}{\sin\chi_0\operatorname{tg}\chi} - \frac{\cos\chi_0}{\sin\chi}y$	$\frac{1}{\sin\chi_0}(xy' - x'y)$	$\frac{\sin\chi_0}{\sin\chi}\left(\frac{xx'}{\sin^2\chi_0} + yy'\right)$

(6.9)

For coordinate functions corresponding to warpings of the contour Z = const, in view of (3.248), in the case of a contour nondeformable in its plane, we have

$$\varphi_{in_z}(S)|_{i=6+k} = \frac{\sin\chi_0}{\sin\chi}\varphi_k^1(S),$$
$$\varphi_{im_z}(S)|_{i=7,8,\ldots,6+n} = 0,$$
$$\varphi_{in_n}(S)|_{i=6+k} = \frac{\cos\chi_0}{\sin\chi}y'\varphi_k^1(S). \tag{6.10}$$

For the fundamental model $\phi(6+k)m_z=\phi_k^1$, $\phi(6+k)n_z=\phi(6+k)n_n=0$.

Passing to the limit when $l_0\to\infty$, we have

$$\lambda_i = 1 \quad (i=1,2,\ldots,6+n), \tag{6.11}$$

so that the expansion for the elastic displacement vector will have the form

$$\mathbf{U}(Z,S) = \sum_{i=1}^{6+n} U_i(Z)\varphi_i(S). \tag{6.12}$$

On the basis of (3.16), (3.252), the components of tangential strain of the middle surface of an oblique cylindrical shell of arbitrary configuration in variables Z, S will be written in the form

$$\varepsilon^0_{m_z} = \sum_{i=1}^{6+n} (U_i' \psi_{im_z} + U_i \bar{\psi}_{im_z}),$$ (6.13)

$$\gamma^0_{n_z m_z} = \sum_{i=1}^{6+n} (U_i' \psi_{in_z m_z} + U_i \bar{\psi}_{in_z m_z}),$$

$$\varepsilon^0_{n_z} = \sum_{i=1}^{6+n} (U_i' \psi_{in_z} + U_i \bar{\psi}_{in_z})$$

and

$$\varepsilon^0_{n_s} = \sum_{i=1}^{6+n} (U_i' \psi_{in_s} + U_i \bar{\psi}_{in_s}),$$

$$\gamma^0_{n_s m_s} = \sum_{i=1}^{6+n} (U_i' \psi_{in_s m_s} + U_i \bar{\psi}_{in_s m_s}),$$ (6.14)

where in view of (3.17), (3.18) and (3.254)

$$\psi_{im_z} = \varphi_{im_z}, \quad \psi_{in_z m_z} = -\operatorname{ctg}\chi \varphi_{im_z} + \varphi_{in_z}, \quad \psi_{in_z} = -\operatorname{ctg}\chi \varphi_{in_z},$$ (6.15)

$$\bar{\psi}_{im_z} = 0, \quad \bar{\psi}_{in_z m_z} = \frac{1}{\sin\chi} \varphi'_{im_z}, \quad \bar{\psi}_{in_z} = \frac{1}{\sin\chi} \varphi'_{in_z} - \frac{1}{R} \varphi_{in_n}$$ (6.16)

$$\psi_{in_s} = \frac{1}{\sin\chi} \varphi_{in_s}, \quad \psi_{in_s m_s} = \frac{1}{\sin\chi} \varphi_{im_s}.$$ (6.17)

$$\bar{\psi}_{in_s} = -\operatorname{ctg}\chi \varphi'_{in_s} - \frac{\cos^2\chi}{R} \varphi_{in_n}, \quad \bar{\psi}_{in_s m_s} = \varphi_{in_s} + \frac{\sin\chi\cos\chi}{R} \varphi_{in_n}.$$ (6.18)

The expansions for components of tangential forces (3.23), (3.256) for an oblique cylindrical shell take the form

$$T_{m_z} = \frac{Eh}{1-\nu^2} \sum_{i=1}^{6+n} [U_i' (\psi_{im_z} + \nu\psi_{in_z}) + U_i (\bar{\psi}_{im_z} + \nu\bar{\psi}_{in_z})],$$

$$S_{n_s m_s} = \frac{Eh}{2(1+\nu)} \sum_{i=1}^{6+n} [U_i' \psi_{in_s m_s} + U_i \bar{\psi}_{in_s m_s}],$$ (6.19)

$$T_{\alpha_s} = \frac{Eh}{1-\nu^2} \sum_{i=1}^{6+n} \left[U_i' (\psi_{i\alpha_s} + \nu \psi_{im_s}) + U_i (\bar{\psi}_{i\alpha_s} + \nu \bar{\psi}_{im_s}) \right],$$

$$T_{\alpha_s} = \frac{Eh}{1-\nu^2} \sum_{i=1}^{6+n} (U_i' \psi_{i\alpha_s} + U_i \bar{\psi}_{i\alpha_s}),$$

$$S_{\alpha_s m_s} = \frac{Eh}{2(1+\nu)} \sum_{i=1}^{6+n} (U_i' \psi_{i\alpha_s m_s} + U_i \bar{\psi}_{i\alpha_s m_s}). \tag{6.20}$$

For elements of longitudinal structure we also obtain

$$N_k = E \Delta F_k \sum_{i=1}^{6+n} U_i' \psi_{im_s}. \tag{6.21}$$

6.2. Differential Resolvents

The differential resolvents of a conical shell of arbitrary configuration for both the fundamental static-geometric model and the model with contour $\bar{Z}$ = const nondeformable in its plane have the form (3.45). Changing in Eqs. (3.45) to the variable Z according to (6.1), we represent them in the form

$$\sum_{i=1}^{6+n} \left\{ \left[\left(1 - \frac{Z}{l_0}\right) \tilde{a}_{\Sigma ji} V_i' + \tilde{b}_{ji} V_i \right]' - \tilde{b}_{ij} V_i' - \frac{1}{1 - \frac{Z}{l_0}} \tilde{c}_{ji} V_i \right\} = \frac{1}{\lambda_j G} \tilde{R}_j \tag{6.22}$$
$$(j = 1, 2, \ldots, 6+n),$$

where the derivatives

$$V_i' = \frac{dV_i}{dZ},$$

and the coefficients $\tilde{\tilde{a}}_{\Sigma ji}$, $\tilde{\tilde{b}}_{ji}$, $\tilde{\tilde{b}}_{ij}$, $\tilde{\tilde{c}}_{ji}$ and load terms R_j will be

$$\tilde{a}_{\Sigma ji} = l_0 a_{\Sigma ji}; \quad \tilde{b}_{ji} = b_{ji}; \quad \tilde{b}_{ij} = b_{ij}; \quad \tilde{c}_{ji} = \frac{1}{l_0} c_{ji}. \tag{6.23}$$

$$\tilde{R}_j = \frac{1}{l_0} R_j. \tag{6.24}$$

We can now easily pass to the limit when $l_0 \to \infty$. Equations (6.22) take the form

$$\sum_{i=1}^{6+n} \left[(\tilde{a}_{zji} U_i' + \tilde{b}_{ji} U_i)' - \tilde{b}_{ij} U_i' - \tilde{c}_{ji} U_i \right] = -\frac{1}{G} \tilde{R}_j \quad (j = 1, 2, \ldots, 6+n). \tag{6.25}$$

For the fundamental static-geometric model, in view of (3.44), (3.64), (3.65), coefficients (6.23) take the form

$$\begin{aligned}
\tilde{a}_{zji} &= \frac{2}{1-\nu} \oint \left[\psi_{jm_z} \psi_{im_z} + \nu (\psi_{jn_z} \psi_{im_z} + \psi_{jm_z} \psi_{in_z}) + \psi_{jn_z} \psi_{in_z} \right] \sin \chi h dS + \\
&\quad + \oint \psi_{jn_z m_z} \psi_{in_z m_z} \sin \chi h dS + 2(1+\nu) \sum_k \psi_{jm_z} \psi_{im_z} \Delta F_k, \\
\tilde{b}_{ji} &= \frac{2}{1-\nu} \oint \left[\psi_{jm_z} \bar{\psi}_{im_z} + \nu (\psi_{jn_z} \bar{\psi}_{im_z} + \psi_{jm_z} \bar{\psi}_{in_z}) + \psi_{jn_z} \bar{\psi}_{in_z} \right] \sin \chi h dS + \\
&\quad + \oint \psi_{jn_z m_z} \bar{\psi}_{in_z m_z} \sin \chi h dS, \\
\tilde{c}_{ji} &= \frac{2}{1-\nu} \oint \left[\bar{\psi}_{jm_z} \bar{\psi}_{im_z} + \nu (\bar{\psi}_{jn_z} \bar{\psi}_{im_z} + \bar{\psi}_{jm_z} \bar{\psi}_{in_z}) + \bar{\psi}_{jn_z} \bar{\psi}_{in_z} \right] \sin \chi h dS + \\
&\quad + \oint \bar{\psi}_{jn_z m_z} \bar{\psi}_{in_z m_z} \sin \chi h dS.
\end{aligned} \tag{6.26}$$

For shells with contour Z = const nondeformable in its plane, in view of (3.271), we have

$$\begin{aligned}
\tilde{a}_{zji} &= \oint \left[\frac{2}{1-\nu} \psi_{jn_s} \psi_{in_s} + \psi_{jn_s m_s} \psi_{in_s m_s} \right] \sin \chi h dS + 2(1+\nu) \sum_k \psi_{jm_s} \psi_{im_s} \Delta F_k, \\
\tilde{b}_{ji} &= \oint \left[\frac{2}{1-\nu} \psi_{jn_s} \bar{\psi}_{in_s} + \psi_{jn_s m_s} \bar{\psi}_{in_s m_s} \right] \sin \chi h dS, \\
\tilde{c}_{ji} &= \oint \left[\frac{2}{1-\nu} \bar{\psi}_{jn_s} \bar{\psi}_{in_s} + \bar{\psi}_{jn_s m_s} \bar{\psi}_{in_s m_s} \right] \sin \chi h dS.
\end{aligned} \tag{6.27}$$

The right-hand sides of Eqs. (6.25), in view of (3.47), are determined by the expressions

$$\frac{1}{G} \tilde{R}_j = \frac{1}{G} \oint \left[p^0_{m_z}(Z, S) \varphi_{jm_z}(S) + p^0_{n_s}(Z, S) \varphi_{jn_s}(S) + p^0_{n_n}(Z, S) \varphi_{jn_n}(S) \right] \sin \chi dS \quad (j = 1, 2, \ldots, 6+n). \tag{6.28}$$

In view of (3.31), the generalized forces corresponding to the external load given on the shell ends will be

$$\bar{P}_l(0) = -\oint \bar{q}^0 \cdot \varphi_l dS; \quad \bar{P}_l(z_1) = \oint \bar{q}^1 \cdot \varphi_l dS. \tag{6.29}$$

If the shell is acted on by a given temperature field t=t(Z, S, γ), and the external surface load is absent, we can readily obtain for the fundamental static-geometric model, in view of (5.17), (5.19),

$$\bar{R}_{jt} = \oint \left(\dot{\psi}_{jm_z} + \dot{\psi}_{jn_z}\right) T_t \sin\chi\, dS, \tag{6.30}$$

where T_t is determined by expression (5.10).

For shells with contour z=const nondeformable in its plane, in view of (5.19), (5.33), we have

$$\bar{R}_{jt} = \oint \left(\frac{2}{1-\nu} \dot{\psi}_{jn_z} + \dot{\psi}_{jn_z m_z}\right) S_t \sin\chi\, dS, \tag{6.31}$$

where S_t is determined according to (5.31).

The order of the system of differential resolvents (6.25) can be lowered by six units by using the first integrals of this system, written in mixed form. Considering (3.71), (3.72) and (6.23), we have

$$\begin{gathered} \frac{1}{G} P_j' - \sum_{l=1}^{6+N} (\tilde{b}_{jl} U_l' + \tilde{c}_{jl} U_l) = -\frac{1}{G} \bar{R}_j, \\ \sum_{l=1}^{6+n} (\tilde{a}_{2jl} U_l' + \tilde{b}_{jl} U_l) - \frac{1}{G} P_j = 0 \\ (j = 1, 2, \ldots, 6+n). \end{gathered} \tag{6.32}$$

The first integrals of system (6.32), which express the equilibrium conditions of the cut-off portion of the oblique cylindrical shell, will be

$$P_j = \bar{P}_j \quad (j = 1, 2, \ldots, 6), \tag{6.33}$$

where $\bar{P}_j$, the components of resultant $\mathfrak{R}^0$ and net moment $\mathfrak{M}^0$ of the external load, applied to the cut-off portion of the shell, are determined by general expressions (3.120), (3.121).

Let the chosen pole of rotation of the contour z=const as a solid be the point of intersection of the plane of this contour with the Oz axis. Then, for a cylindrical shell, taking (3.123) and (3.30), (3.109), (6.24) into consideration, we can readily obtain from (3.120), (3.121)

$$\begin{aligned} \mathfrak{R}^0 &= (\mathfrak{R}_1^0 \cdot \mathbf{e}^0)\mathbf{e}_0 - \sum_{i=1}^{3} \mathbf{e}_i \int_{Z_1}^{Z} R_i \, d\xi, \\ \mathfrak{M}^0 &= (\mathfrak{M}_1^0 \cdot \hat{\mathbf{e}}^0)\hat{\mathbf{e}}_0 + \hat{\mathbf{e}}_0 \int_{Z_1}^{Z} \mathfrak{R}^0 (\mathbf{k} \times \hat{\mathbf{e}}^0)\, d\xi - \sum_{i=4}^{6} \hat{\mathbf{e}}_{i-3} \int_{Z_1}^{Z} R_i \, d\xi. \end{aligned} \tag{6.34}$$

Superposing the bases e_α and $\hat{e}_\alpha$ on the basis of the main Cartesian coordinate system, we obtain from (6.34)

$$\begin{aligned} \bar{P}_j(Z) &= P_j(Z_1) - \int_{Z_1}^{Z} \bar{R}_j(\xi)\, d\xi \quad (j = 1, 2, 3), \\ \bar{P}_4(Z) &= P_4(Z_1) - \int_{Z_1}^{Z} [R_4(\xi) - \bar{P}_2(\xi)]\, d\xi, \\ \bar{P}_5(Z) &= P_5(Z_1) - \int_{Z_1}^{Z} [R_5(\xi) + \bar{P}_1(\xi)]\, d\xi, \\ \bar{P}_6(Z) &= P_6(Z_1) - \int_{Z_1}^{Z} \bar{R}_6(\xi)\, d\xi. \end{aligned} \tag{6.35}$$

If however we superpose the bases e_α and $\hat{e}_\alpha$ on the auxiliary bases (3.241), we find from (6.34)

$$
\begin{aligned}
\overline{P}_j(Z) &= P_j(Z_1) - \int_{Z_1}^{Z} \overline{R}_j(\xi)\,d\xi \quad (j=1,2,3),\\
\overline{P}_4(Z) &= P_4(Z_1) - \int_{Z_1}^{Z} \left[\overline{R}_4(\xi) - \sin\chi_0 \overline{P}_3(\xi)\right] d\xi,\\
\overline{P}_5(Z) &= P_5(Z_1) - \int_{Z_1}^{Z} \left[\overline{R}_5(\xi) + \sin\chi_0 \overline{P}_1(\xi) - \cos\chi_0 \overline{P}_3(\xi)\right] d\xi,\\
\overline{P}_6(Z) &= P_6(Z_1) - \int_{Z_1}^{Z} \left[\overline{R}_6(\xi) + \cos\chi_0 \overline{P}_2(\xi)\right] d\xi.
\end{aligned}
\tag{6.36}
$$

Adding first integrals (6.33) to system (6.32), we must eliminate the first six equations from (6.32).

In expanded form, in view of (6.63), first integrals (6.33) have the form

$$
\sum_{i=1}^{6+n} \left(\tilde{a}_{3ji} U_i' + \tilde{b}_{ji} U_i\right) = \frac{1}{G} \overline{P}_j \quad (j=1,2,\ldots,6).
\tag{6.37}
$$

Adding relations (6.37) to system (6.25), we should also eliminate the first six equations from (6.25).

6.3. Canonical System of Resolvents

As was shown in Section 3.3, the general system of differential resolvents of a conical shell of arbitrary configuration makes it possible to lower the order by 12 units if one adopts as the desired generalized displacements $U_i(\overline{Z})$ (i=1, 2, ..., 6) the canonical kinematic unknowns which are the components of the translational displacement vector of the contour $\overline{Z}$=const together with the cone apex and the components of the vector of rotation of this contour as a solid about the apex. In this case, the system of resolvents reduces to an independent system of order 2n with respect to warpings and six relations solved for the derivatives of generalized displacements U_i (i=1, 2, ..., 6). On passing to a cylindrical shell, the cone apex goes to infinity, and the indicated kinematic unknowns lose meaning. However, as can be readily proven, the system of differential resolvents of an oblique cylindrical shell of arbitrary configuration also reduces to the canonical form if some point common to all the section Z = const is chosen as the pole. In this connection, the canonical kinematic unknowns for cylindrical shells are the components of the vectors of translational displacement and rotation of the contour Z=const with respect to any fixed point common to all the sections.

Let r_0 be the radius vector of the common pole. We represent the vector function $U^0(Z, S)$, corresponding to the displacements of the contour Z=const as a solid, in the form

$$\mathbf{U}^0(Z, S) = \boldsymbol{\eta}(Z) + \boldsymbol{\theta}(Z) \times [\mathbf{r}(Z, S) - \mathbf{r}_0], \tag{6.38}$$

where r(Z, S) is the radius vector of the moving point of the middle surface of an oblique cylindrical shell of arbitrary configuration:

$$\mathbf{r}(Z, S) = x(S)\mathbf{i} + y(S)\mathbf{j} + [Z + x(S)\operatorname{ctg}\chi_0]\mathbf{k}. \tag{6.39}$$

Expanding vectors η and θ in arbitrary bases e_α and $\hat{e}_\alpha$, respectively, we have

$$\mathbf{U}^0(Z, S) = \eta^\alpha(Z)\mathbf{e}_\alpha + \hat{\theta}^\alpha(Z)\hat{\mathbf{e}}_\alpha \times [\mathbf{r}(Z, S) - \mathbf{r}_0]. \tag{6.40}$$

As the generalized displacements $U_i(Z)$ (i=1, 2, ..., 6) we take the components of vectors η and θ

$$U_i(Z) = \begin{cases} \eta^\alpha(Z) & (i = \alpha = 1, 2, 3), \\ \hat{\theta}^\alpha(Z) & (i = \alpha + 3 = 4, 5, 6). \end{cases} \tag{6.41}$$

On the basis of (6.39)-(6.41), the displacement vector $U^0(Z, S)$ will be represented as the expansion

$$U^{*}(Z,S)=\sum_{i=1}^{6} U_i(Z)[\varphi_i(S)+v_i Z], \tag{6.42}$$

where

$$\varphi_i(S)=\begin{cases} e_\alpha & (i=\alpha=1,2,3), \\ \hat{e}_\alpha \times [r(0,S)-r_0] & (i=\alpha+3=4,5,6). \end{cases} \tag{6.43}$$

$$v_i=\begin{cases} 0 & (i=1,2,3), \\ \hat{e}_\alpha \times k & (i=\alpha+3=4,5,6). \end{cases} \tag{6.44}$$

It follows from (6.42)-(6.44) that for a single pole for all the sections Z=const, the elastic displacement vector of the cylindrical shell cannot be represented in the form (6.12). For this reason, we will proceed in the present section from the representation

$$U(Z,S)=\sum_{i=1}^{6+n} U_i(Z)[\varphi_i(S)+v_i Z]. \tag{6.45}$$

Here the vector functions $\phi_i(S)$ for i=1, 2, ..., 6 are determined by expression (6.43), and for $i > 6$, are subjected to preliminary selection; v_i are constant vectors. In accordance with (6.44), we take

$$v_i=\begin{cases} \hat{e}_{i-3} \times k & (i=4,5,6), \\ 0 & (i \neq 4,5,6). \end{cases} \tag{6.46}$$

For displacements in the direction of an arbitrarily oriented unit vector t, we have

$$u_t = U\cdot t=\sum_{i=1}^{6+n} U_i[\varphi_{it}+v_{it} Z]. \tag{6.47}$$

where

$$\varphi_{it}=\varphi_i\cdot t, \quad v_{it}=v_i\cdot t. \tag{6.48}$$

The components of tangential strain of the middle surface, which are determined by expressions (6.5), (6.6), will be, in view of (6.47):

$$\varepsilon^0_{m_z} = \sum_{i=1}^{6+n} (U'_i \hat{\varphi}_{im_z} + U_i \hat{\hat{\varphi}}_{im_z}),$$

$$\gamma^0_{n_z m_z} = \sum_{i=1}^{6+n} (U'_i \hat{\varphi}_{in_z m_z} + U_i \hat{\hat{\varphi}}_{in_z m_z}), \qquad (6.49)$$

$$\varepsilon^0_{n_z} = \sum_{i=1}^{6+n} (U'_i \hat{\varphi}_{in_z} + U_i \hat{\hat{\varphi}}_{in_z})$$

and

$$\varepsilon^0_{n_s} = \sum_{i=1}^{6+n} (U'_i \hat{\varphi}_{in_s} + U_i \hat{\hat{\varphi}}_{in_s}),$$

$$\gamma^0_{n_s m_s} = \sum_{i=1}^{6+n} (U'_i \hat{\varphi}_{in_s m_s} + U_i \hat{\hat{\varphi}}_{in_s m_s}).$$

(6.50)

Here the functions $\hat{\varphi}_i$ and $\hat{\hat{\varphi}}_i$ are determined by

$$\hat{\varphi}_i(Z, S) = \varphi_i(S) + \mu_i(S) Z,$$
$$\hat{\hat{\varphi}}_i(Z, S) = \bar{\varphi}_i(S) + \mu_i(S) + \bar{\mu}_i(S) Z, \qquad (6.51)$$

where the function $\varphi_i(S), \bar{\varphi}_i(S)$ are represented by (6.15)-(6.18), and the functions $\mu_i(S), \bar{\mu}_i(S)$ will be

$$\mu_{im_z} = \nu_{im_z};\ \mu_{in_z} = -\operatorname{ctg}\chi \nu_{in_z};$$

$$\mu_{in_z m_z} = -\operatorname{ctg}\chi \nu_{im_z} + \nu_{in_z}; \qquad (6.52)$$

$$\bar{\mu}_{im_z} = 0;\ \bar{\mu}_{in_z m_z} = 0;\ \bar{\mu}_{in_z} = -\frac{1}{R}\nu_{in_n} + \frac{1}{\sin\chi}\nu'_{in_z}, \qquad (6.53)$$

$$\mu_{in_s} = \frac{1}{\sin\chi}\nu_{in_s};\ \mu_{in_s m_s} = \frac{1}{\sin\chi}\nu_{im_s},$$

$$\bar{\mu}_{in_s} = -\frac{\cos^2\chi}{R}\nu_{in_n} - \operatorname{ctg}\chi \nu'_{in_s}, \qquad (6.54)$$

$$\bar{\mu}_{in_s m_s} = \frac{\sin\chi\cos\chi}{R}\nu_{in_n} + \nu'_{in_s}. \qquad (6.55)$$

The expansions for the components of tangential forces (6.19), (6.20) in view of (6.49), (6.50), take the form

$$T_{m_z} = \frac{Eh}{1-\nu^2}\sum_{i=1}^{6+n}\left[U'_i\left(\hat{\varphi}_{im_z} + \nu\hat{\varphi}_{in_z}\right) + U_i\left(\hat{\hat{\varphi}}_{im_z} + \nu\hat{\hat{\varphi}}_{in_z}\right)\right],$$

$$S_{n_z m_z} = \frac{Eh}{2(1+\nu)}\sum_{i=1}^{6+n}\left(U'_i\hat{\varphi}_{in_z m_z} + U_i\hat{\hat{\varphi}}_{in_z m_z}\right), \qquad (6.56)$$

$$T_{n_z} = \frac{Eh}{1-\nu^2}\sum_{i=1}^{6+n}\left[U'_i\left(\hat{\varphi}_{in_z} + \nu\hat{\varphi}_{im_z}\right) + U_i\left(\hat{\hat{\varphi}}_{in_z} + \nu\hat{\hat{\varphi}}_{im_z}\right)\right]$$

and

$$T_{n_s} = \frac{Eh}{1-\nu^2} \sum_{i=1}^{6+n} \left(U_i' \hat{\varphi}_{in_s} + U_i \hat{\bar{\psi}}_{in_s} \right),$$

$$S_{n_s m_s} = \frac{Eh}{2(1+\nu)} \sum_{i=1}^{6+n} \left(U_i' \hat{\psi}_{in_s m_s} + U_i \hat{\bar{\psi}}_{in_s m_s} \right). \quad (6.57)$$

For elements of the longitudinal structure, in view of (6.53), we also have

$$N_k = E \Delta F_k \sum_{i=1}^{6+n} \left(U_i' \hat{\varphi}_{im_s} + U_i \hat{\bar{\psi}}_{im_s} \right). \quad (6.58)$$

We now turn to variational Eq. (3.26). Considering (6.45), the work done by external forces and represented by general expression (3.27) is

$$\delta T = \int_0^{Z_1} \sum_{i=1}^{6+n} \hat{\bar{R}}_i \delta U_i dZ + \sum_{i=1}^{6+n} \hat{\bar{P}}_i \delta U_i \Big|_{Z=0}^{Z=Z_1}. \quad (6.59)$$

Here

$$\hat{\bar{R}}_i = \bar{R}_i + \bar{R}_i^\circ, \quad (6.60)$$

$$\hat{\bar{P}}_i(0) = \bar{P}_i(0) + \bar{P}_i^\circ(0); \quad \hat{\bar{P}}_i(Z_1) = \bar{P}_i(Z_1) + \bar{P}_i^\circ(Z_1), \quad (6.61)$$

where

$$\bar{R}_i^\circ = Z \nu_i \oint p \sin \chi \, dS, \quad (6.62)$$

$$\bar{P}_i^\circ(0) = -Z \nu_i \oint \bar{q}^0 dS; \quad \bar{P}_i^\circ(Z_1) = Z \nu_i \oint \bar{q}^1 dS, \quad (6.63)$$

and $\bar{R}_i$, $\bar{P}_i(0)$, $\bar{P}_i(Z_1)$ are determined by expressions (6.28), (6.29).

In view of (6.49), the potential energy variation represented by general expression (3.33)

$$\delta U = \int_0^{Z_1} \sum_{i=1}^{6+n} \left[\hat{P}_{zi} \delta U_i' + \hat{Q}_i \delta U_i \right] dZ, \quad (6.64)$$

where

$$\hat{P}_{zi}=\oint\left(T_{m_z}\hat{\psi}_{in_z}+S_{n_zm_z}\hat{\psi}_{in_zm_z}+T_{n_z}\hat{\psi}_{in_z}\right)\sin\chi\,dS+\sum_k N_k\hat{\psi}_{im_z}(S_k). \tag{6.65}$$

$$\hat{Q}_i=\oint\left(T_{m_z}\hat{\hat{\psi}}_{im_z}+S_{n_zm_z}\hat{\hat{\psi}}_{in_zm_z}+T_{n_z}\hat{\hat{\psi}}_{in_z}\right)\sin\chi\,dS+\sum_k N_k\hat{\hat{\psi}}_{im_z}(S_k). \tag{6.66}$$

Expanding variational Eq. (3.26) with the aid of expression (6.59) and (6.64), we obtain the equations

$$\hat{P}'_{zj}-\hat{Q}_j+\hat{R}_j=0\ (j=1,2,\ldots,6+n) \tag{6.67}$$

and natural boundary conditions

$$(\hat{P}_j-\hat{P}_{zj})\delta U_j\Big|_{Z=0}^{Z=Z_1}=0\ (j=1,2,\ldots,6+n). \tag{6.68}$$

We write Eqs. (6.67) in expanded form. Using (6.56), we can obtain

$$\frac{1}{G}\hat{P}_{zj}=\sum_{l=1}^{6+n}\left(\hat{\hat{a}}_{zjl}U'_l+\hat{\hat{b}}_{zjl}U_l\right). \tag{6.69}$$

$$\frac{1}{G}\hat{Q}_j=\sum_{l=1}^{6+n}\left(\hat{\hat{b}}_{zlj}U'_l+\hat{\hat{c}}_{zjl}U_l\right). \tag{6.70}$$

where the coefficients $\hat{\hat{a}}_{zjl},\ \hat{\hat{b}}_{zjl},\ \hat{\hat{b}}_{zlj},\ \hat{\hat{c}}_{zjl}$ are determined by the expressions

$$\begin{aligned}
\hat{\hat{a}}_{zjl}&=\frac{2}{1-\nu}\oint\left[\hat{\psi}_{jm_z}\hat{\psi}_{lm_z}+\nu\left(\hat{\psi}_{jn_z}\hat{\psi}_{lm_z}+\hat{\psi}_{jm_z}\hat{\psi}_{ln_z}\right)+\hat{\psi}_{jn_z}\hat{\psi}_{ln_z}\right]\sin\chi\,hdS+\\
&\quad+\oint\hat{\psi}_{jn_zm_z}\hat{\psi}_{ln_zm_z}\sin\chi\,hdS+2(1+\nu)\sum_k\hat{\psi}_{jm_z}\hat{\psi}_{lm_z}\Delta F_k,\\
\hat{\hat{b}}_{zjl}&=\frac{2}{1-\nu}\oint\left[\hat{\psi}_{jm_z}\hat{\hat{\psi}}_{lm_z}+\nu\left(\hat{\psi}_{jn_z}\hat{\hat{\psi}}_{lm_z}+\hat{\psi}_{jm_z}\hat{\hat{\psi}}_{ln_z}\right)+\hat{\psi}_{jn_z}\hat{\hat{\psi}}_{ln_z}\right]\sin\chi\,hdS+\\
&\quad+\oint\hat{\psi}_{jn_zm_z}\hat{\hat{\psi}}_{ln_zm_z}\sin\chi\,hdS+2(1+\nu)\sum_k\hat{\psi}_{jm_z}\hat{\hat{\psi}}_{lm_z}\Delta F_k,\\
\hat{\hat{c}}_{zjl}&=\frac{2}{1-\nu}\oint\left[\hat{\hat{\psi}}_{jm_z}\hat{\hat{\psi}}_{lm_z}+\nu\left(\hat{\hat{\psi}}_{jn_z}\hat{\hat{\psi}}_{lm_z}+\hat{\hat{\psi}}_{jm_z}\hat{\hat{\psi}}_{ln_z}\right)+\hat{\hat{\psi}}_{jn_z}\hat{\hat{\psi}}_{ln_z}\right]\sin\chi\,hdS+\\
&\quad+\oint\hat{\hat{\psi}}_{jn_zm_z}\hat{\hat{\psi}}_{ln_zm_z}\sin\chi\,hdS+2(1+\nu)\sum_k\hat{\hat{\psi}}_{jm_z}\hat{\hat{\psi}}_{lm_z}\Delta F_k.
\end{aligned} \tag{6.71}$$

Introducing (6.69) and (6.70) into (6.67), we obtain a system of ordinary differential equations of the form

$$\sum_{i=1}^{6+n}\left[\left(\hat{\bar{a}}_{zji}U'_i+\hat{\bar{b}}_{zji}U_i\right)'-\hat{\bar{b}}_{zij}U'_i-\hat{\bar{c}}_{zji}U_i\right]=\frac{1}{G}\,\hat{\bar{R}}_j \qquad (j=1,2,\ldots,6+n). \tag{6.72}$$

We will now show that Eqs. (6.72) can be reduced to the canonical form, as was done in Section 3.3.

We turn to expression (6.42). Assuming that the shell moves arbitrarily as a solid, in view of (6.41) we have

$$U_i(Z)=\begin{cases}\eta^i=\text{const} & (i=1,2,3),\\ \hat{\theta}^{i-3}=\text{const} & (i=4,5,6),\\ 0 & (i>6).\end{cases} \tag{6.73}$$

Considering (6.73), in view of the arbitrariness of the components η^α and $\hat{\theta}^\alpha$, we obtain the following important relations from expressions (6.49):

$$\hat{\bar{\psi}}_{im_z}=\hat{\bar{\psi}}_{in_zm_z}=\hat{\bar{\psi}}_{in_z}=0 \quad (i=1,2,\ldots,6). \tag{6.74}$$

Relations (6.74) express an identical equality to zero of the functions of two variables. In accordance with (6.51), have from (6.74)

$$\bar{\mu}_{im_z}=\bar{\mu}_{in_zm_z}=\bar{\mu}_{in_z}=0 \quad (i=1,2,\ldots,6) \tag{6.75}$$

and

$$\bar{\psi}_{im_z}+\mu_{im_z}=0,\ \bar{\psi}_{in_zm_z}+\mu_{in_zm_z}=0,\ \bar{\psi}_{in_z}+\mu_{in_z}=0 \quad (i=1,2,\ldots,6). \tag{6.76}$$

In view of (6.74), it follows from (6.66) that

$$\bar{\bar{Q}}_i=0 \quad (i=1,2,\ldots,6). \tag{6.77}$$

On the basis of (6.77), we have from (6.67)

$$\hat{\bar{P}}'_j = -\hat{\bar{R}}_j \quad (j=1,2,\dots,6), \tag{6.78}$$

whence

$$\hat{P}_j = \hat{\bar{P}}_j \quad (j=1,2,\dots,6), \tag{6.79}$$

where

$$\hat{\bar{P}}_j(Z) = \hat{P}_j(Z_1) - \int_{Z_1}^{Z} \hat{\bar{R}}_j(\xi)\,d\xi. \tag{6.80}$$

Relations (6.79) represent the equilibrium conditions of the cut-off portion of the shell. Here $\hat{P}_j$ are the components in bases e_α and $\hat{e}_\alpha$ of vectors of the resultant and net moment with respect to the common pole of internal forces in the section z = const.

We represent the matrices of coefficients $\hat{a}_{\Sigma ji}$, $\hat{b}_{\Sigma ji}$, $\hat{b}_{\Sigma ij}$, $\hat{c}_{\Sigma ji}$ in the form of block matrices, as in (3.125), (3.131). As can be seen from (6.71), in view of (6.74), these matrices take the form

$$\hat{\tilde{\mathbf{A}}} = \begin{pmatrix} \hat{\tilde{\mathbf{A}}}_{\eta\eta} & \hat{\tilde{\mathbf{A}}}_{\eta\varphi} & \hat{\tilde{\mathbf{A}}}_{\eta\theta} \\ \hat{\tilde{\mathbf{A}}}_{\varphi\eta} & \hat{\tilde{\mathbf{A}}}_{\varphi\varphi} & \hat{\tilde{\mathbf{A}}}_{\varphi\theta} \\ \hat{\tilde{\mathbf{A}}}_{\theta\eta} & \hat{\tilde{\mathbf{A}}}_{\theta\varphi} & \hat{\tilde{\mathbf{A}}}_{\theta\theta} \end{pmatrix}, \quad \hat{\tilde{\mathbf{B}}} = \begin{pmatrix} 0 & 0 & \hat{\tilde{\mathbf{B}}}_{\eta\theta} \\ 0 & 0 & \hat{\tilde{\mathbf{B}}}_{\varphi\theta} \\ 0 & 0 & \hat{\tilde{\mathbf{B}}}_{\theta\theta} \end{pmatrix},$$

$$\hat{\tilde{\mathbf{B}}}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hat{\tilde{\mathbf{B}}}'_{\eta\theta} & \hat{\tilde{\mathbf{B}}}'_{\varphi\theta} & \hat{\tilde{\mathbf{B}}}'_{\theta\theta} \end{pmatrix}, \quad \hat{\tilde{\mathbf{C}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{\tilde{\mathbf{C}}}_{\theta\theta} \end{pmatrix}. \tag{6.81}$$

We conclude from (6.81) that expressions (6.69), (6.70), which determine $\hat{P}_j$ and $\hat{Q}_j$ and hence, all Eqs. (6.72), do not contain the generalized displacements (6.41) corresponding to displacements of the contour Z=const as a solid. Consequently, first integral (6.79) can be regarded as a system of six algebraic equations in the derivatives of six generalized displacements U_i (i=1, 2, ..., 6). Therefore, solving the first integrals for U_i' (i=1, 2, ..., 6), then eliminating these derivatives with the aid of the expressions obtained from Eqs. (6.72) for j=7, 8, ..., 6+n, we arrive at a canonical representation of the resolvents that is analogous to (3.180), (3.186). This pertains not only to oblique cylindrical shells, whose equations are obtained on the basis of the fundamental static-geometric model, but equally to oblique cylindrical shells with the contour Z=const nondeformable in its plane. The latter statement is based on the fact that the resolvents of both models clearly turn out, to be completely identical if the components in the axes of the main moving trihedron are adopted as the components of the stressed and strained state of a shell with a nondeformable contour (as in the case of the fundamental static-geometric model). In this case, generally speaking, only the coordinate vectors $\phi_i(S)$ $(i > 6)$ are distinguished, directed in one case along the generatrices, and in the other, perpendicular to the plane of the diaphragms. The remaining coordinate vectors and hence the corresponding coefficients of the resolvents pertaining to displacements of the contour Z=const as a solid obviously have the same common properties, which in both cases provide for the transition to the canonical form.

As the single pole for all the sections Z=const, we will select the origin of the fundamental system of Cartesian coordinates. Then, in (6.38) we must set $r_0=0$. In this case, as can be readily seen by comparing (6.43) with (3.240), the coordinate functions $\phi_i(S)$ contained in expansion (6.42) turn out to be the same as in the case of a moving pole coinciding with the point of intersection of the plane of the contour Z=const with the Oz axis. Expanding the vector η of translational displacement and vector θ of rotation of the contour Z=const as a solid about the origin of the fundamental system of Cartesian coordinates in the basis of this system, from (6.45) we have

$$\mathbf{U}(Z,S)=\sum_{i=1}^{6+n} U_i(Z)[\boldsymbol{\varphi}_i(S)+\mathbf{v}_i Z], \qquad (6.82)$$

where in view of (6.39), (6.43), (6.44), (6.46)

$$\begin{aligned} &\varphi_1 = \mathbf{i},\ \varphi_2 = \mathbf{j},\ \varphi_3 = \mathbf{k}, \\ &\varphi_4 = y\mathbf{k} - x \operatorname{ctg} \chi_0 \mathbf{j}, \\ &\varphi_5 = -x\mathbf{k} + x \operatorname{ctg} \chi_0 \mathbf{i}, \\ &\varphi_6 = x\mathbf{j} - y\mathbf{i}. \end{aligned} \tag{6.83}$$

$$\begin{gathered} \mathbf{v}_i = 0\ (i = 1, 2, 3), \\ \mathbf{v}_4 = -\mathbf{j};\ \mathbf{v}_5 = \mathbf{i};\ \mathbf{v}_6 = 0. \end{gathered} \tag{6.84}$$

For the coordinate functions corresponding to the generalized displacement $U_i (i>6)$, we have

$$\mathbf{v}_i = 0\ (i = 7, 8, \ldots, 6+n), \tag{6.85}$$

$\phi_i = \phi_k^1 k$ is the fundamental model,

$\phi_i = \phi_k^1 (\sin \chi_0 k - \cos \chi_0 i)$ is the model with the nondeformable contour

$$(i = 6+k;\ k = 1, 2, \ldots, n). \tag{6.86}$$

Using expressions (6.52), (6.84), (6.85) and (6.4), we have

$$\begin{gathered} \mu_{i m_z} = 0 \quad (i = 1, 2, \ldots, 6+n), \\ \mu_{i n_z m_z} = 0 \quad (i \neq 4, 5), \\ \mu_{4 n_z m_z} = -\frac{1}{\sin \chi} y';\quad \mu_{5 n_z m_z} = \frac{1}{\sin \chi} x', \\ \mu_{i n_z} = 0\ (i \neq 4, 5), \\ \mu_{4 n_z} = \frac{\cos \chi}{\sin^2 \chi} y';\quad \mu_{5 n_z} = -\frac{\cos \chi}{\sin^2 \chi} x'. \end{gathered} \tag{6.87}$$

Expanding (6.51) with the aid of (6.15), (6.16), (6.83), (6.86), (6.87) and considering (6.74), we obtain

i	U_i	$\hat{\psi}_{i m_z}$	$\hat{\psi}_{i n_z m_z}$	$\hat{\psi}_{i n_z}$
1	η_x	0	$\frac{1}{\sin \chi} x'$	$-\frac{\cos \chi}{\sin^2 \chi} x'$
2	η_y	0	$\frac{1}{\sin \chi} y'$	$-\frac{\cos \chi}{\sin^2 \chi} y'$
3	η_z	1	$-\operatorname{ctg} \chi$	0
4	θ_x	y	$-y \operatorname{ctg} \chi - (x \operatorname{ctg} \chi_0 + Z) \frac{y'}{\sin \chi}$	$(x \operatorname{ctg} \chi_0 + Z) \frac{\cos \chi}{\sin^2 \chi} y'$

(6.88)

5	θ_y	$-x$	$x \operatorname{ctg} \chi + (x \operatorname{ctg} \chi_0 + Z) \frac{x'}{\sin \chi}$	$-(x \operatorname{ctg} \chi_0 + Z) \frac{\cos \chi}{\sin^2 \chi} x'$
6	θ_z	0	$\frac{1}{\sin \chi} (xy' - x'y)$	$\frac{\cos \chi}{\sin^2 \chi} (x'y - xy')$
ω_k		φ_{im_z}	$-\operatorname{ctg} \chi \varphi_{im_z} + \varphi_{in_z}$	$-\operatorname{ctg} \chi \varphi_{in_z}$

In expression (6.88), the last row of the table corresponds to the generalized displacements which determine the displacements for which the mutual arrangement of the points of the contour Z=const changes. For the fundamental static-geometric model and the model with a nondeformable contour, the expanded expressions ψ_i (i>6) are easily obtained for any chosen system of functions ϕ_k^1 by using expressions (6.86). However, if no assumptions are made regarding the coordinate functions ϕ_{im_z}, ϕ_{in_z} (i > 6), expression (6.88) will correspond to the general case where the deformation of the cross section in its plane is not considered. Then the general expressions for the coefficients of the differential resolvents for i, j>6 change. However, it can be seen that this change does not affect the results presented below. In this connection, all further conclusions pertain equally to the model with both the deformable and the nondeformable contour.

Taking (6.88) and (6.16), (6.74) into consideration, strain components (6.49) take the form

$$
\begin{aligned}
\varepsilon^0_{m_z} &= \eta'_z + \theta'_x y - \theta'_y x + \sum_{i=7}^{6+n} U'_i \varphi_{im_z}, \\
\gamma^0_{n_z m_z} &= \eta'_x \frac{x'}{\sin \chi} + \eta'_y \frac{y'}{\sin \chi} - \eta'_z \operatorname{ctg} \chi - \theta'_x \left[y \operatorname{ctg} \chi + \frac{y'}{\sin \chi} (x \operatorname{ctg} \chi_0 + Z) \right] + \\
&+ \theta'_y \left[x \operatorname{ctg} \chi + \frac{x'}{\sin \chi} (x \operatorname{ctg} \chi_0 + Z) \right] + \theta'_z \frac{xy' - x'y}{\sin \chi} + \\
&+ \sum_{i=7}^{6+n} \left[U'_i (-\operatorname{ctg} \chi \varphi_{im_z} + \varphi_{in_z}) - U_i \frac{1}{\sin \chi} \varphi'_{im_z} \right], \\
\varepsilon^0_{n_z} &= -\eta'_x \frac{\cos \chi}{\sin^2 \chi} x' - \eta'_y \frac{\cos \chi}{\sin^2 \chi} y' + \theta'_x (x \operatorname{ctg} \chi_0 + Z) \frac{\cos \chi}{\sin^2 \chi} y' - \\
&- \theta'_y (x \operatorname{ctg} \chi_0 + Z) \frac{\cos \chi}{\sin^2 \chi} x' + \theta'_z \frac{\cos \chi}{\sin^2 \chi} (x'y - xy') + \\
&+ \sum_{i=7}^{6+n} \left[-U'_i \operatorname{ctg} \chi \varphi_{in_z} + U_i \left(\frac{1}{\sin \chi} \varphi'_{in_z} - \frac{1}{R} \varphi_{in_n} \right) \right].
\end{aligned}
\tag{6.89}
$$

We will write the system of differential resolvents in canonical kinematic unknowns. Replacing the first six equations by the first integrals (6.79), in system (6.72), in view of (6.69) and (6.81), we obtain

$$\sum_{i=1}^{6} \hat{\hat{a}}_{zji} U_i' + \sum_{i=7}^{6+n} \left[\hat{\hat{a}}_{zji} U_i' + \hat{\hat{b}}_{zji} U_i\right] = \frac{1}{G} \hat{\hat{P}}_j \ (j=1, 2, \ldots, 6), \tag{6.90}$$

$$\sum_{i=1}^{6} \left[\hat{\hat{a}}_{zji} U_i' + \hat{\hat{b}}_{zji} U_i\right]' + \sum_{i=7}^{6+n} \left[\left(\hat{\hat{a}}_{zji} U_i' + \hat{\hat{b}}_{zji} U_i\right)' - \hat{\hat{b}}_{zij} U_i' - \hat{\hat{c}}_{zji} U_i\right] =$$
$$= \frac{1}{G} \hat{\hat{R}}_j \ (j=7, 8, \ldots, 6+n). \tag{6.91}$$

Now, solving Eqs. (6.90) for the derivatives U_i' $(i=1, 2, \ldots, 6)$, then eliminating these derivatives from Eqs. (6.91), we arrive at an independent system of n differential equations in generalized displacements U_i $(i=7, 8, \ldots, 6+n)$, corresponding to the change in the spatial configuration of the contour Z=const. We will show that these equations for shells, the thickness and sections of whose reinforcing elements are independent of the Z coordinate, will have constant coefficients despite the fact that Eqs. (6.90), (6.91) have variable coefficients, in accordance with (6.71), (6.88).

Let us represent Eqs. (6.90) and (6.91) in matrix form. Considering (6.81), we have

$$\hat{\hat{\mathbf{A}}}_{\eta\eta} \frac{d}{dZ} \eta + \hat{\hat{\mathbf{A}}}_{\eta\theta} \frac{d}{dZ} \theta + \hat{\hat{\mathbf{A}}}_{\eta\Omega} \frac{d}{dZ} \Omega + \hat{\hat{\mathbf{B}}}_{\eta\Omega} \Omega = \frac{1}{G} \hat{\hat{\mathbf{P}}}_{\eta}, \tag{6.92}$$

$$\hat{\hat{\mathbf{A}}}_{\theta\eta} \frac{d}{dZ} \eta + \hat{\hat{\mathbf{A}}}_{\theta\theta} \frac{d}{dZ} \theta + \hat{\hat{\mathbf{A}}}_{\theta\Omega} \frac{d}{dZ} \Omega + \hat{\hat{\mathbf{B}}}_{\theta\Omega} \Omega = \frac{1}{G} \hat{\hat{\mathbf{P}}}_{\theta}, \tag{6.93}$$

$$\frac{d}{dZ}\left[\hat{\hat{\mathbf{A}}}_{\Omega\eta} \frac{d}{dZ} \eta + \hat{\hat{\mathbf{A}}}_{\Omega\theta} \frac{d}{dZ} \theta\right] - \hat{\hat{\mathbf{B}}}_{\eta\Omega}' \frac{d}{dZ} \eta - \hat{\hat{\mathbf{B}}}_{\theta\Omega}' \frac{d}{dZ} \theta +$$
$$+ \hat{\hat{\mathbf{A}}}_{\Omega\Omega} \frac{d^2}{dZ^2} \Omega + (\hat{\hat{\mathbf{B}}}_{\Omega\Omega} - \hat{\hat{\mathbf{B}}}_{\Omega\Omega}') \frac{d}{dZ} \Omega - \hat{\hat{\mathbf{C}}}_{\Omega\Omega} \Omega = \frac{1}{G} \hat{\hat{\mathbf{R}}}_{\Omega}. \tag{6.94}$$

On the basis of (6.44), (6.46), (6.51)-(6.53), (6.71), it can be readily established that the elements of matrices $\hat{\hat{\mathbf{A}}}_{\eta\eta}$, $\hat{\hat{\mathbf{A}}}_{\eta\Omega}$, $\hat{\hat{\mathbf{A}}}_{\Omega\eta}$, $\hat{\hat{\mathbf{A}}}_{\Omega\Omega}$, $\hat{\hat{\mathbf{B}}}_{\eta\Omega}$, $\hat{\hat{\mathbf{B}}}_{\eta\Omega}'$, $\hat{\hat{\mathbf{B}}}_{\Omega\Omega}$, $\hat{\hat{\mathbf{B}}}_{\Omega\Omega}'$, $\hat{\hat{\mathbf{C}}}_{\Omega\Omega}$ are numbers, whereas the elements of the remaining matrices generally depend on the Z coordinate.

We will carry out two successive elementary transformations on Eqs. (6.92) by multiplying it on the left by the product ZE_2E_1, where

$$\mathbf{E}_1=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.95}$$

Combining the result with Eq. (6.93), we obtain

$$(Z\mathbf{E}_3\hat{\hat{\mathbf{A}}}_{\eta\eta}+\hat{\hat{\mathbf{A}}}_{\theta\eta})\frac{d}{dZ}\eta+(Z\mathbf{E}_3\hat{\hat{\mathbf{A}}}_{\eta\theta}+\hat{\hat{\mathbf{A}}}_{\theta\theta})\frac{d}{dZ}\theta+(Z\mathbf{E}_3\hat{\hat{\mathbf{A}}}_{\eta\Omega}+\hat{\hat{\mathbf{A}}}_{\theta\Omega})\frac{d}{dZ}\Omega+$$
$$+(Z\mathbf{E}_3\hat{\hat{\mathbf{B}}}_{\eta\Omega}+\hat{\hat{\mathbf{B}}}_{\theta\Omega})\Omega=\frac{1}{G}(Z\mathbf{E}_3\hat{\hat{\mathbf{P}}}_{\eta}+\hat{\hat{\mathbf{P}}}_{\theta}), \tag{6.96}$$

where

$$\mathbf{E}_3=\mathbf{E}_2\mathbf{E}_1=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.97}$$

It is easy to establish the static meaning of the transformations performed. Matrix E_1 corresponds to the mutual transposition of the first two equations of system (6.90). Equations (6.92) and (6.93), respectively, represent the equilibrium conditions in the form of equality to zero of the resultant and net moment relative to the origin of all the forces external to the cut-off portion of the shell. In this connection, it is evident that Eq. (6.96) expresses the conditiosn of equality to zero of the net moment of all the external forces applied to the cut-off portion of the shell with respect to the point of intersection of the plane of the contour Z=const with the Oz axis.

We introduce into consideration the column matrix

$$\boldsymbol{\Xi}=\begin{pmatrix} \Xi_x \\ \Xi_y \\ \Xi_z \end{pmatrix}, \tag{6.98}$$

where

$$\Xi_x=\eta'_x+Z\theta'_y;\quad \Xi_y=\eta'_y-Z\theta'_x;\quad \Xi_z=\eta'_z. \tag{6.99}$$

In view of (6.97), we have

$$\frac{d}{dZ}\eta=\boldsymbol{\Xi}-Z\mathbf{E}_3\frac{d}{dZ}\theta. \tag{6.100}$$

Using (6.100), we eliminate vector η from Eqs. (6.92), (6.96), and (6.94). We obtain

$$\hat{\tilde{A}}_{\eta\eta}\Xi + (\hat{\tilde{A}}_{\eta\theta} - Z\hat{\tilde{A}}_{\eta\zeta}E_3)\frac{d}{dZ}\Theta + \hat{\tilde{A}}_{\eta\vartheta}\frac{d}{dZ}\Omega + \hat{B}_{\eta\vartheta}\Omega = \frac{1}{G}\hat{P}_{\eta}, \quad (6.101)$$

$$(ZE_3\hat{\tilde{A}}_{\eta\eta} + \hat{\tilde{A}}_{\theta\eta})\Xi + [\hat{A}_{\theta\theta} + Z(E_3\hat{A}_{\eta\theta} - \hat{A}_{\theta\zeta}E_3) - Z^2E_3\hat{\tilde{A}}_{\eta\eta}E_3]\frac{d}{dZ}\Theta +$$

$$+(ZE_3\hat{\tilde{A}}_{\eta\vartheta} + \hat{\tilde{A}}_{\theta\vartheta})\frac{d}{dZ}\Omega + (ZE_3\hat{B}_{\eta\vartheta} + \hat{B}_{\theta\vartheta})\Omega = \frac{1}{G}(ZE_3P_{\eta} + \hat{P}_{\theta}). \quad (6.102) \qquad (6.101)$$

$$\hat{\tilde{A}}_{\vartheta\eta}\frac{d}{dZ}\Xi - \hat{B}'_{\eta\vartheta}\Xi + \frac{d}{dZ}\left[(\hat{A}_{\vartheta\theta} - ZA_{\vartheta\zeta}E_3)\frac{d}{dZ}\Theta\right] - (\hat{B}'_{\theta\vartheta} - Z\hat{B}'_{\eta\vartheta}E_3)\frac{d}{dZ}\Theta + \qquad (6.102)$$

$$+\hat{A}_{\vartheta\vartheta}\frac{d^2}{dZ^2}\Omega + (\hat{B}_{\vartheta\vartheta} - \hat{B}'_{\vartheta\vartheta})\frac{d}{dZ}\Omega - \hat{C}_{\vartheta\vartheta}\Omega = \frac{1}{G}\hat{R}_{\vartheta}.$$

$$(6.103)$$

Analyzing the matrices in Eqs. (6.101)-(6.103) we can easily ascertain with the aid of relations (6.46), (6.51)-(6.53), (6.71), (6.88) that the coefficients of Eqs. (6.101)-(6.103) are independent of the Z coordinate. On this basis, we have

$$\begin{gathered}
\hat{\tilde{A}}_{\eta\theta}(Z) - Z\hat{\tilde{A}}_{\eta\zeta}E_3 = \hat{\tilde{A}}_{\eta\theta}(0) = \tilde{A}_{\eta\theta},\\
ZE_3\hat{\tilde{A}}_{\eta\eta} + \hat{\tilde{A}}_{\theta\eta}(Z) = \hat{\tilde{A}}_{\theta\eta}(0) = \tilde{A}_{\theta\eta},\\
\hat{\tilde{A}}_{\theta\theta}(Z) + Z[E_3\hat{\tilde{A}}_{\eta\theta}(Z) - \hat{\tilde{A}}_{\theta\zeta}(Z)E_3] - Z^2E_3\hat{\tilde{A}}_{\eta\eta}E_3 = \hat{\tilde{A}}_{\theta\theta}(0) = \tilde{A}_{\theta\theta},\\
ZE_3\hat{\tilde{A}}_{\eta\vartheta} + \hat{A}_{\theta\vartheta}(Z) = \hat{\tilde{A}}_{\theta\vartheta}(0) = \tilde{A}_{\theta\vartheta},\\
ZE_3\hat{B}_{\eta\vartheta} + \hat{B}_{\theta\vartheta}(Z) = \hat{B}_{\theta\vartheta}(0) = \tilde{B}_{\theta\vartheta},\\
\hat{\tilde{A}}_{\vartheta\theta}(Z) - Z\hat{A}_{\vartheta\zeta}E_3 = \hat{\tilde{A}}_{\vartheta\theta}(0) = \tilde{A}_{\vartheta\theta},\\
\hat{B}'_{\theta\vartheta}(Z) - Z\hat{B}'_{\eta\vartheta}E_3 = \hat{B}'_{\theta\vartheta}(0) = \tilde{B}_{\theta\vartheta},
\end{gathered} \qquad (6.104)$$

and

$$\begin{gathered}
\hat{\tilde{A}}_{\eta\eta} = \tilde{A}_{\eta\eta},\quad \hat{\tilde{A}}_{\eta\vartheta} = \tilde{A}_{\eta\vartheta},\quad \hat{\tilde{A}}_{\vartheta\eta} = \tilde{A}_{\vartheta\eta},\quad \hat{\tilde{A}}_{\vartheta\vartheta} = \tilde{A}_{\vartheta\vartheta},\\
\hat{B}_{\eta\vartheta} = \tilde{B}_{\eta\vartheta},\quad \hat{B}'_{\eta\vartheta} = \tilde{B}'_{\eta\vartheta},\quad \hat{B}_{\vartheta\vartheta} = \tilde{B}_{\vartheta\vartheta},\quad \hat{B}'_{\vartheta\vartheta} = \tilde{B}'_{\vartheta\vartheta},\\
\hat{\hat{C}}_{\vartheta\vartheta} = \tilde{C}_{\vartheta\vartheta},
\end{gathered} \qquad (6.105)$$

where the elements of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are determined by expression (6.26), corresponding to the case where the point of intersection of the plane of the contour Z=const with the Oz axis is chosen as the pole.

In view of (6.104), (6.105), Eqs. (6.101)-(6.103) take the form

$$\tilde{A}_{\eta\eta}\Xi + \tilde{A}'_{\eta\theta}\frac{d}{dZ}\Theta + \tilde{A}_{\eta\vartheta}\frac{d}{dZ}\Omega + \tilde{B}_{\eta\vartheta}\Omega = \frac{1}{G}\tilde{P}_{\eta}, \qquad (6.106)$$

$$\tilde{A}_{\theta\eta}\Xi + \tilde{A}_{\theta\theta}\frac{d}{dZ}\Theta + \tilde{A}_{\theta\vartheta}\frac{d}{dZ}\Omega + \tilde{B}_{\theta\vartheta}\Omega = \frac{1}{G}\tilde{P}_{\theta}, \qquad (6.107)$$

$$\begin{gathered}
\tilde{A}_{\vartheta\eta}\frac{d}{dZ}\Xi - \tilde{B}'_{\eta\vartheta}\Xi + \tilde{A}_{\vartheta\theta}\frac{d^2}{dZ^2}\Theta - \tilde{B}'_{\theta\vartheta}\frac{d}{dZ}\Theta + \tilde{A}_{\vartheta\vartheta}\frac{d^2}{dZ^2}\Omega +\\
+(\tilde{B}_{\vartheta\vartheta} - \tilde{B}'_{\vartheta\vartheta})\frac{d}{dZ}\Omega - \tilde{C}_{\vartheta\vartheta}\Omega = \frac{1}{G}R_{\vartheta}.
\end{gathered} \qquad (6.108)$$

where the right-hand members $1/G(\bar{P}_\eta)$, $1/G(\bar{P}_\theta)$, $1/G(\tilde{R}_\Omega)$ also correspond to the case where the point of intersection of the plane of the contour Z=const with the Oz axis is chosen as the pole, and are correspondingly determined by expressions (6.35), (6.28).

System (6.106)-(6.108) can be easily reduced to the canonical form. We represent Eqs. (6.106), (6.107) in the form

$$\tilde{A}_0\begin{pmatrix}\Sigma \\ \frac{d}{dZ}\theta\end{pmatrix}+\begin{pmatrix}\tilde{A}_{\eta\Omega} \\ \tilde{A}_{\theta\Omega}\end{pmatrix}\frac{d}{dZ}\Omega+\begin{pmatrix}\tilde{B}_{\eta\Omega} \\ \tilde{B}_{\theta\Omega}\end{pmatrix}\Omega=\frac{1}{G}\begin{pmatrix}\bar{P}_\eta \\ \bar{P}_\theta\end{pmatrix}, \tag{6.109}$$

where

$$\tilde{A}_0=\begin{pmatrix}\tilde{A}_{\eta\eta} & \tilde{A}_{\eta\theta} \\ \tilde{A}_{\theta\eta} & \tilde{A}_{\theta\theta}\end{pmatrix}. \tag{6.110}$$

Introducing the matrix

$$\tilde{A}_0^{-1}=\begin{pmatrix}\tilde{\tilde{A}}_{\eta\eta} & \tilde{\tilde{A}}_{\eta\theta} \\ \tilde{\tilde{A}}_{\theta\eta} & \tilde{\tilde{A}}_{\theta\theta}\end{pmatrix}, \tag{6.111}$$

the reciprocal of $\tilde{A}_0$, we have from Eq. (6.109)

$$\Sigma=-H_\Sigma^1\frac{d}{dZ}\Omega-H_\Sigma^0\Omega+\frac{1}{G}H_\Sigma^P, \tag{6.112}$$

$$\frac{d}{dZ}\theta=-H_\theta^1\frac{d}{dZ}\Omega-H_\theta^0\Omega+\frac{1}{G}H_\theta^P, \tag{6.113}$$

where

$$\begin{aligned}
H_\Sigma^1&=\tilde{\tilde{A}}_{\eta\eta}\tilde{A}_{\eta\Omega}+\tilde{\tilde{A}}_{\eta\theta}\tilde{A}_{\theta\Omega},\\
H_\theta^1&=\tilde{\tilde{A}}_{\theta\eta}\tilde{A}_{\eta\Omega}+\tilde{\tilde{A}}_{\theta\theta}\tilde{A}_{\theta\Omega},\\
H_\Sigma^0&=\tilde{\tilde{A}}_{\eta\eta}\tilde{B}_{\eta\Omega}+\tilde{\tilde{A}}_{\eta\theta}\tilde{B}_{\theta\Omega},\\
H_\theta^0&=\tilde{\tilde{A}}_{\theta\eta}\tilde{B}_{\eta\Omega}+\tilde{\tilde{A}}_{\theta\theta}\tilde{B}_{\theta\Omega},\\
H_\Sigma^P&=\tilde{\tilde{A}}_{\eta\eta}\bar{P}_\eta+\tilde{\tilde{A}}_{\eta\theta}\bar{P}_\theta,\\
H_\theta^P&=\tilde{\tilde{A}}_{\theta\eta}\bar{P}_\eta+\tilde{\tilde{A}}_{\theta\theta}\bar{P}_\theta.
\end{aligned} \tag{6.114}$$

Eliminating the unknowns Σ and $d/dZ(\theta)$ from Eq. (6.108) with the aid of (6.112), (6.113), we obtain

$$(\tilde{A}_{\Omega\Omega} - \tilde{A}_{\Omega\eta} H_\eta^1 - \tilde{A}_{\Omega\theta} H_\theta^1) \frac{d^2}{dZ^2} \Omega + (\tilde{B}_{\Omega\Omega} - \tilde{B}'_{\Omega\Omega} - \tilde{A}_{\Omega\eta} H_\eta^0 +$$

$$+ \tilde{B}'_{\eta\Omega} H_\eta^1 - \tilde{A}_{\Omega\theta} H_\theta^0 + \tilde{B}'_{\theta\Omega} H_\theta^1) \frac{d}{dZ} \Omega - (\tilde{C}_{\Omega\Omega} - \tilde{B}'_{\eta\Omega} H_\eta^0 - \tilde{B}'_{\theta\Omega} H_\theta^0) \Omega =$$

$$= \frac{1}{G} \left(\tilde{R}_\Omega - \tilde{A}_{\Omega\eta} \frac{d}{dZ} H_\eta^p - \tilde{A}_{\Omega\theta} \frac{d}{dZ} H_\theta^p + \tilde{B}'_{\eta\Omega} H_\eta^p + \tilde{B}'_{\theta\Omega} H_\theta^p \right). \quad (6.115)$$

Expression (6.115) is an independent matrix differential equation in warping Ω with constant coefficients. Considering (6.114), one can readily ascertain that the matrix coefficients on $d^2/dZ^2(\Omega)$ and Ω are symmetric, whereas the matrix coefficient on $d/dZ(\Omega)$ is inversely symmetric. These properties of the matrices result from the Betti reciprocity theorem. In view of these properties, it follows that if vector Ω is represented by only one component, i.e., is a scalar function, the corresponding resolvent always has the form

$$\frac{d^2\omega}{dZ^2} - k\omega = F(Z). \quad (6.116)$$

Having determined Ω, we can also easily find the components of the law of plane sections by integrating relations (6.112), (6.113) and taking (6.100) into account. We have

$$\eta = \eta^0 + \eta_\Omega, \quad \theta = \theta^0 + \theta_\Omega. \quad (6.117)$$

where

$$\eta^0 = \eta(0) + \frac{1}{G} \int_0^Z [H_\eta^p(\xi) - \xi E_3 H_\theta^p(\xi)] \, d\xi,$$

$$\theta^0 = \theta(0) + \frac{1}{G} \int_0^Z H_\theta^p(\xi) \, d\xi. \quad (6.118)$$

$$\eta_\Omega = - \int_0^Z \left[(H_\eta^1 - \xi E_3 H_\theta^1) \frac{d}{d\xi} \Omega + (H_\eta^0 - \xi E_3 H_\theta^0) \Omega \right] d\xi,$$

$$\theta_\Omega = - \int_0^Z \left(H_\theta^1 \frac{d}{d\xi} \Omega + H_\theta^0 \Omega \right) d\xi. \quad (6.119)$$

6.4. Right Cylindrical Shells of Arbitrary Configuration

Treating right cylindrical shells as a special modification of oblique cylindrical shells, it is necessary to set

$$\chi_0 = \frac{\pi}{2}. \qquad (6.120)$$

in the general relations obtained in the present chapter.

In view of (6.120), parametric Eqs. (6.2) assume the form

$$x = x(S); \quad y = y(S); \quad Z = Z, \qquad (6.121)$$

and the coefficients of the first quadratic form (6.3)

$$A = 1, \quad B = 1, \quad \cos\chi = 0, \qquad (6.122)$$

whence it follows that the coordinate grid in this case will be orthogonal, so that the contour Z=const coincides with the cross section of the shell.

The main and auxiliary trihedra merge, as do the basis of the fundamental system of Cartesian coordinates and the auxiliary basis (3.241). The direction cosines of the unit vectors oriented in the direction of the normal to the cylindrical surface and along the tangents to the orthogonal coordinate lines will be, in view of (6.4), (6.122):

	x	y	z
n_n	$-y'$	x'	0
n_s	x'	y'	0
m_s	0	0	1

(6.123)

Strain components (6.5) take the form

$$\varepsilon^0_{m_s} = \frac{\partial u_{m_s}}{\partial Z}; \quad \varepsilon^0_{n_s} = \frac{\partial u_{n_s}}{\partial S} - \frac{1}{R} u_{n_n},$$
$$\gamma^0_{n_s m_s} = \frac{\partial u_{m_s}}{\partial S} + \frac{\partial u_{n_s}}{\partial Z}. \qquad (6.124)$$

where in accordance (6.7)

$$\frac{1}{R}=x'y''-x''y' \qquad (6.125)$$

If the point of intersection of the plane of the cross section with the Oz axis is chosen as the pole, and the vectors of translational displacement and rotation of the cross section are expanded in axes of the fundamental Cartesian coordinate system, the components $\phi_{im_z}(S)$, $\phi_{in_z}(S)$, $\phi_{in_n}(S)$ of coordinate vector functions $\phi_i(S)$, corresponding to displacements of the cross section as a solid, will be, on the basis of (6.8):

i	U_i	φ_{im_z}	φ_{in_z}	φ_{in_n}
1	η_x	0	x'	$-y'$
2	η_y	0	y'	x'
3	η_z	1	0	0
4	θ_x	y	0	0
5	θ_y	$-x$	0	0
6	θ_z	0	$xy'-x'y$	$xx'+yy'$

(6.126)

Comparison of formulas (6.8) and (6.126) shows that in a cylindrical shell, the longitudinal and transverse displacements separate, whereas in an oblique cylindrical shell and particularly in a conical one, arbitrary displacements of the contour Z=const as a solid are associated with longitudinal as well as transverse displacements simultaneously.

For coordinate functions corresponding to warpings of the cross section, on the basis of (6.10), we have

$$\begin{gathered}\varphi_{im_z}(S)|_{i=6+k}=\varphi_k'(S),\\ \varphi_{in_z}(S)=0;\quad \varphi_{in_n}(S)=0\\ (i=7,8,\ldots,6+n).\end{gathered} \qquad (6.127)$$

Expansion in coordinate vector functions for the elastic displacement vector, as before, has the form

$$\mathbf{U}(Z,S)=\sum_{i=1}^{6+n}U_i(Z)\varphi_i(S). \qquad (6.128)$$

In the case at hand, in view of (6.15), (6.16), components (6.13) of tangential strain of the middle surface

$$
\begin{aligned}
&\varepsilon^0_{m_z} = \sum_{i=1}^{6+n} U'_i\varphi_{im_z}; \quad \varepsilon^0_{n_z}=0;\\
&\gamma^0_{n_z m_z} = \sum_{i=1}^{6+n} (U'_i\varphi_{in_z} + U_i\varphi'_{im_z}).
\end{aligned}
\tag{6.129}
$$

so that the expansions for components of tangential forces (6.19), (6.20) assume the form

$$
\begin{aligned}
&T_{m_z} = \frac{Eh}{1-\nu^2}\sum_{i=1}^{6+n} U'_i\varphi_{im_z},\\
&S_{n_z m_z} = \frac{Eh}{2(1+\nu)}\sum_{i=1}^{6+n} (U'_i\varphi_{in_z} + U_i\varphi'_{im_z}).
\end{aligned}
\tag{6.130}
$$

For elements of the longitudinal structure, we also have

$$
N_k = E\Delta F_k \sum_{i=1}^{6+n} U'_i\varphi_{im_z}.
\tag{6.131}
$$

As in the preceding case, the resolvent system of ordinary differential equations for the case of right cylindrical shells has the form

$$
\sum_{i=1}^{6+n} [(\tilde{a}_{\Sigma ji}U'_i + \tilde{b}_{ji}U_i)' - \tilde{b}_{ij}U'_i - \tilde{c}_{ji}U_i] = \frac{1}{G}\tilde{R}_j \; (j=1,2,\ldots,6+n).
\tag{6.132}
$$

The coefficients $\tilde{a}_{\Sigma ji}$, $\tilde{b}_{ji}$, $\tilde{c}_{ji}$ and load terms $\tilde{R}_j$ are determined by

$$
\begin{aligned}
&\tilde{a}_{\Sigma ji} = \oint\left(\frac{2}{1-\nu}\varphi_{jm_z}\varphi_{im_z} + \varphi_{jn_z}\varphi_{in_z}\right)hdS + 2(1+\nu)\sum_k \varphi_{jm_z}\varphi_{im_z}\Delta F_k,\\
&\tilde{b}_{ji} = \oint \varphi_{jn_z}\varphi'_{im_z}hdS,\\
&\tilde{c}_{ji} = \oint \varphi'_{jm_z}\varphi'_{im_z}hdS.
\end{aligned}
\tag{6.133}
$$

It can be shown that, to within the notation, Eqs. (6.132) coincide with the known equations of Vlasov*, if the contour of the cross section

*V.Z. Vlasov, Thin-Walled Three-Dimensional Systems, Ch. III. Moscow, Gosstroyizdat, 1958.

of the prismatic system is considered to be nondeformable in its plane.

The system of resolvents allowing for deformation of the cross section in its plane is identical to system (6.132), except for the fact that the coefficients $\tilde{c}_{ji}$ for j, i>6 will generally have the form

$$\tilde{c}_{ji} = \oint \varphi'_{jm_z} \varphi'_{im_z} h dS + \frac{1}{G} \oint \frac{M_{jn_z} M_{in_z}}{EI} dS, \tag{6.134}$$

where M_{jn_z}, M_{in_z} are the linear bending moments M_{n_z} corresponding to the states

$$U_k = \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases}; \quad U_k = \begin{cases} 1 & (k = i) \\ 0 & (k \neq i) \end{cases} \quad (k = 7, 8, \ldots, 6 + n). \tag{6.135}$$

Let us note that since the longitudinal and transverse displacements separate in a right cylindrical shell, in any of coefficients (6.134), one of the integrals will always be zero.

The system of differential resolvents of a right cylindrical shell of arbitrary configuration allowing for deformation of the cross section in its plane can obviously be easily obtained as a special case from system (3.224) of resolvents of a conical shell of arbitrary configuration allowing for bending strain of the middle surface if the terms corresponding to bending moments M_{m_z} and twisting moments $H_{n_z m_z}$ are neglected in coefficients (3.223).

If the shell is provided with reinforcing diaphragms, and the deformation of the contour Z=const is taken into consideration, it is necessary in that case to take into consideration the work of the diaphragms as well. Then the system of resolvents for shells of any configurations will have the same form as before. The expressions for the coefficients of the resolvents also remain unchanged, with the exception of the coefficients c_{ji}. These coefficients acquire additional terms which can be different from zero only when i, j>6. The corresponding expressions will be obtained in Part Two.

Obviously, Eqs. (6.132) are easily reduced to the canonical form by using the results of the preceding section. All the final expressions are simplified if the major axes of inertia of the cross section of the shell are taken as the Ox, Oy axes.

Let us turn to Eqs. (6.106)-(6.108). The matrix elements of the coefficients of these equations are determined by expressions (6.133), where the coordinate functions ϕ_{im_z}, ϕ_{in_z} (i=1, 2, ..., 6) are represented by expressions (6.126). For the major axes of inertia of a cross section of the shell we have

$$
\begin{aligned}
&\frac{1}{1-\nu^2}\oint xhdS+\sum_k x_k\Delta F_k=0,\\
&\frac{1}{1-\nu^2}\oint yhdS+\sum_k y_k\Delta F_k=0,\\
&\frac{1}{1-\nu^2}\oint xyhdS+\sum_k x_k y_k\Delta F_k=0.
\end{aligned}
\tag{6.136}
$$

In view of (6.136), matrices $\bar{A}_{\eta\eta}$, $\bar{A}_{\eta\theta}$, $\bar{A}_{\theta\eta}$, $\bar{A}_{\theta\theta}$ will have the form

$$
A_{\eta\eta}=\begin{pmatrix} F^*_{xx} & F^*_{xy} & 0\\ F^*_{yx} & F^*_{yy} & 0\\ 0 & 0 & \frac{2}{1-\nu}F\end{pmatrix},\quad A_{\eta\theta}=\begin{pmatrix} 0 & 0 & S^*_x\\ 0 & 0 & S^*_y\\ 0 & 0 & 0\end{pmatrix},
$$

$$
A_{\theta\eta}=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ S^*_x & S^*_y & 0\end{pmatrix},\quad A_{\theta\theta}=\begin{pmatrix} \frac{2}{1-\nu}I_x & 0 & 0\\ 0 & \frac{2}{1-\nu}I_y & 0\\ 0 & 0 & I^*_z\end{pmatrix}.
\tag{6.137}
$$

Here

$$
\begin{aligned}
&F=\oint hdS+(1-\nu^2)\sum_k\Delta F_k,\\
&I_x=\oint y^2hdS+(1-\nu^2)\sum_k y_k^2\Delta F_k,\\
&I_y=\oint x^2hdS+(1-\nu^2)\sum_k x_k^2\Delta F_k.
\end{aligned}
\tag{6.138}
$$

$$F^{*}_{xx}=\oint x'^{2}h\,dS,$$

$$F^{*}_{xy}=F^{*}_{yx}=\oint x'y'h\,dS,$$

$$F^{*}_{yy}=\oint y'^{2}h\,dS, \tag{6.139}$$

$$S^{*}_{x}=\oint x'(xy'-x'y)h\,dS,$$

$$S^{*}_{y}=\oint y'(xy'-x'y)h\,dS, \tag{6.140}$$

$$I^{*}_{z}=\oint (xy'-x'y)^{2}h\,dS. \tag{6.141}$$

Let us write Eqs. (6.106)-(6.108) in expanded form. Using (6.137), we obtain

$$F^{*}_{xx}\Xi_x+F^{*}_{xy}\Xi_y+S^{*}_{x}\theta'_z+\sum_{i=7}^{6+n}(\tilde{a}_{1i}U'_i+\tilde{b}_{1i}U_i)=\frac{1}{G}Q_x,$$

$$F^{*}_{yx}\Xi_x+F^{*}_{yy}\Xi_y+S^{*}_{y}\theta'_z+\sum_{i=7}^{6+n}(\tilde{a}_{2i}U'_i+\tilde{b}_{2i}U_i)=\frac{1}{G}Q_y, \tag{6.142}$$

$$\frac{2}{1-\nu}F\eta'_z+\sum_{i=7}^{6+n}(\tilde{a}_{3i}U'_i+\tilde{b}_{3i}U_i)=\frac{1}{G}N_z,$$

$$\frac{2}{1-\nu}I_x\theta'_x+\sum_{i=7}^{6+n}(\tilde{a}_{4i}U'_i+\tilde{b}_{4i}U_i)=\frac{1}{G}M_x, \tag{6.143}$$

$$\frac{2}{1-\nu}I_y\theta'_y+\sum_{i=7}^{6+n}(\tilde{a}_{5i}U'_i+\tilde{b}_{5i}U_i)=\frac{1}{G}M_y, \tag{6.144}$$

$$S^{*}_{x}\Xi_x+S^{*}_{y}\Xi_y+I^{*}_{z}\theta'_z+\sum_{i=7}^{6+n}(\tilde{a}_{6i}U'_i+\tilde{b}_{6i}U_i)=\frac{1}{G}M_z.$$

. . . .

$$\tilde{a}_{j1}\Xi'_x+\tilde{a}_{j2}\Xi'_y+\tilde{a}_{j3}\eta''_z-\tilde{b}_{1j}\Xi_x-\tilde{b}_{2j}\Xi_y-\tilde{b}_{3j}\eta'_z+\tilde{a}_{j4}\theta''_x+\tilde{a}_{j5}\theta''_y+\tilde{a}_{j6}\theta''_z-\;-\tilde{b}_{4j}\theta'_x-\tilde{b}_{5j}\theta'_y-\tilde{b}_{6j}\theta'_z+\sum_{i=7}^{6+n}[\tilde{a}_{ji}U''_i+(\tilde{b}_{ji}-\tilde{b}_{ij})U'_i-\tilde{c}_{ji}U_i]=\frac{1}{G}\tilde{R}_j \tag{6.145}$$

$$(j=7,\,8,\ldots,\,6+n).$$

Hereinafter, instead of the two symbols a_{ji} and $a_{\Sigma ji}$, we will mostly use the single symbol a_{ji}, keeping in mind that the longitudinal structure is taken into account.

Solving Eqs. (6.142), (6.144) for Σ_x, Σ_y, θ'_z, we find

$$\Xi_x=\frac{1}{k}\Bigg\{\frac{1}{G}\left[k_{yy}\left(Q_x-\frac{S^{*}_{x}}{I^{*}_{z}}M_z\right)-k_{xy}\left(Q_y-\frac{S^{*}_{y}}{I^{*}_{z}}M_z\right)\right]-$$

$$-k_{yy}\sum_{i=7}^{6+n}\left[\left(\tilde{a}_{1i}-\frac{S^{*}_{x}}{I^{*}_{z}}\tilde{a}_{6i}\right)U'_i+\left(\tilde{b}_{1i}-\frac{S^{*}_{x}}{I^{*}_{z}}\tilde{b}_{6i}\right)U_i\right]+ \tag{6.146}$$

$$+k_{xy}\sum_{i=7}^{6+n}\left[\left(\tilde{a}_{2i}-\frac{S^{*}_{y}}{I^{*}_{z}}\tilde{a}_{6i}\right)U'_i+\left(\tilde{b}_{2i}-\frac{S^{*}_{y}}{I^{*}_{z}}\tilde{b}_{6i}\right)U_i\right]\Bigg\}.$$

$$\xi'_y = \frac{1}{k}\left\{\frac{1}{G}\left[k_{xx}\left(Q_y - \frac{S^*_y}{I^*_z} M_z\right) - k_{xy}\left(Q_x - \frac{S^*_x}{I^*_z} M_z\right)\right] - \right.$$

$$- k_{xx}\sum_{i=7}^{6+n}\left[\left(\tilde{a}_{2i} - \frac{S^*_y}{I^*_z}\tilde{a}_{6i}\right)U'_i + \left(\tilde{b}_{2i} - \frac{S^*_y}{I^*_z}\tilde{b}_{6i}\right)U_i\right] +$$

$$\left. + k_{xy}\sum_{i=7}^{6+n}\left[\left(\tilde{a}_{1i} - \frac{S^*_x}{I^*_z}\tilde{a}_{6i}\right)U'_i + \left(\tilde{b}_{1i} - \frac{S^*_x}{I^*_z}\tilde{b}_{6i}\right)U_i\right]\right\}. \qquad (6.147)$$

$$\theta'_z = \frac{1}{k}\left\{\frac{1}{G}\left[\left(k_{xy}\frac{S^*_y}{I^*_z} - k_{yy}\frac{S^*_x}{I^*_z}\right)\left(Q_x - \frac{S^*_x}{I^*_z} M_z\right) + \right.\right.$$

$$\left. + \left(k_{xy}\frac{S^*_x}{I^*_z} + k_{xx}\frac{S^*_y}{I^*_z}\right)\left(Q_y - \frac{S^*_y}{I^*_z} M_z\right)\right] +$$

$$+ \left(k_{yy}\frac{S^*_x}{I^*_z} - k_{xy}\frac{S^*_y}{I^*_z}\right)\sum_{i=7}^{6+n}\left[\left(\tilde{a}_{1i} - \frac{S^*_x}{I^*_z}\tilde{a}_{6i}\right)U'_i + \left(\tilde{b}_{1i} - \frac{S^*_x}{I^*_z}\tilde{b}_{6i}\right)U_i\right] +$$

$$\left. + \left(k_{xx}\frac{S^*_y}{I^*_z} - k_{xy}\frac{S^*_x}{I^*_z}\right)\sum_{i=7}^{6+n}\left[\left(\tilde{a}_{2i} - \frac{S^*_y}{I^*_z}\tilde{a}_{6i}\right)U'_i + \left(\tilde{b}_{2i} - \frac{S^*_y}{I_z}\tilde{b}_{6i}\right)U_i\right]\right\} + \qquad (6.148)$$

$$+ \frac{1}{GI^*_z} M_z - \frac{1}{I^*_z}\sum_{i=7}^{6+n}\left(\tilde{a}_{6i}U'_i + \tilde{b}_{6i}U_i\right).$$

Here

$$k = k_{xx}k_{yy} - k^2_{xy}, \qquad (6.149)$$

where

$$k_{xx} = F_{xx} - \frac{S^{*2}_x}{I^*_z}; \quad k_{xy} = F_{xy} - \frac{S^*_x S^*_y}{I^*_z}; \quad k_{yy} = F_{yy} - \frac{S^{*2}_y}{I^*_z}. \qquad (6.150)$$

From Eqs. (6.143) we also find

$$\eta'_z = \frac{1-\nu}{2F}\left[\frac{1}{G} N_z - \sum_{i=7}^{6+n}\left(\tilde{a}_{3i}U'_i + \tilde{b}_{3i}U_i\right)\right],$$

$$\theta'_x = \frac{1-\nu}{2I_x}\left[\frac{1}{G} M_x - \sum_{i=7}^{6+n}\left(\tilde{a}_{4i}U'_i + \tilde{b}_{4i}U_i\right)\right], \qquad (6.151)$$

$$\theta'_y = \frac{1-\nu}{2I_y}\left[\frac{1}{G} M_y - \sum_{i=7}^{6+n}\left(\tilde{a}_{5i}U'_i + \tilde{b}_{5i}U_i\right)\right].$$

Now, with the aid of expressions (6.146)-(6.148), and (6.151), we can readily eliminate from Eq. (6.145) the displacements corresponding to the law of plane sections, thus obtaining an independent system of differential equations in warpings U_i ($i=7, 8, \ldots, 6+n$) with constant coefficients.

Integrating this system, and introducing the result into the right hand sides of relations (6.146)-(6.148) and (6.151), in view of (6.99), we can readily find all the components of the law of plane sections: η_x, η_y, η_z, θ_x, θ_y, θ_z.

Like Eqs. (6.145), relations (6.146)-(6.148), (6.151) are most general in character, and, as already noted, apply not only to shells with a cross-sectional contour nondeformable in its plane, but also when the deformation of the contour is taken into account, in the presence of transverse elastic diaphragms, etc., since $\tilde{a}_{ji}$, $\tilde{b}_{ji}$, $\tilde{b}_{ij}$, $\tilde{c}_{ji}$ can be any coefficients. These relations can be substantially simplified via a linear transformation of the unknowns η_x, η_y, θ_z by taking

$$\eta_x^{*}=\eta_x^{**}+b\theta_z^{**};\quad \eta_y^{*}=\eta_y^{**}-a\theta_z^{**};\quad \theta_z=\theta_z^{**}. \tag{6.152}$$

Here

$$\eta_x^{*}=\eta_x+Z\theta_y;\quad \eta_y^{*}=\eta_y-Z\theta_x \tag{6.153}$$

are the displacements corresponding to a pole placed at point O* with coordinates (o,o,Z);
η_x^{**}, η_y^{**}, θ_z^{**} are the displacements and twisting angle corresponding to a pole placed at point O** with coordinates (a, b, Z).

In accordance with formulas (6.126), the coordinate functions

$$\begin{gathered}\varphi_{\eta_x}^{**}=(x-a)'=x';\quad \varphi_{\eta_y}^{**}=(y-b)'=y',\\ \varphi_{\theta_z}^{**}=(x-a)(y-b)'-(x-a)'(y-b)=(x-a)y'-x'(y-b).\end{gathered} \tag{6.154}$$

We will select point O** so that the following orthogonality conditions are fulfilled:

$$\oint\varphi_{\theta_z}^{**}\varphi_{\eta_x}^{**}h\,dS=\oint\varphi_{\theta_z}^{**}\varphi_{\eta_y}^{**}h\,dS=0. \tag{6.155}$$

In this case, using (6.154), we can readily obtain

$$a=\frac{S_y^{*}F_{xx}^{*}-S_x^{*}F_{xy}^{*}}{F_{xx}^{*}F_{yy}^{*}-F_{xy}^{*2}},\qquad b=\frac{S_y^{*}F_{xy}^{*}-S_x^{*}F_{yy}^{*}}{F_{xx}^{*}F_{yy}^{*}-F_{xy}^{*2}}. \tag{6.156}$$

Taking (6.99), (6.152), (6.153) into account, we can easily establish the following relations:

$$\Xi_x=\Xi_x^{**}+b\theta_z^{**\prime},\qquad \Xi_y=\Xi_y^{**}-a\theta_z^{**\prime}, \tag{6.157}$$

where Σ_x^{**}, Σ_y^{**} are determined by expressions (6.99) provided that for displacements in the direction of axes Ox, Oy, the pole is placed at the points with coordinates (a, b, 0).

Now, carrying out the substitution of variables in Eqs. (6.142), (6.144), in view of (6.156), we can readily obtain

$$\begin{aligned}
&F_{xx}^{*}\Xi_x^{**}+F_{xy}^{*}\Xi_y^{**}+\sum_{i=7}^{6+n}(\tilde{a}_{1i}U_i'+\tilde{b}_{1i}U_i)=\frac{1}{G}Q_x,\\
&F_{yx}^{*}\Xi_x^{**}+F_{yy}^{*}\Xi_y^{**}+\sum_{i=7}^{6+n}(\tilde{a}_{2i}U_i'+\tilde{b}_{2i}U_i)=\frac{1}{G}Q_y,\\
&S_x^{*}\Xi_x^{**}+S_y^{*}\Xi_y^{**}+(I_z^{*}+bS_x^{*}-aS_y^{*})\theta_z'+\sum_{i=7}^{6+n}(\tilde{a}_{6i}U_i'+\tilde{b}_{6i}U_i)=\frac{1}{G}M_z,
\end{aligned} \tag{6.158}$$

where

$$\Xi_x^{**}=\eta_x^{**\prime}-\theta_y,\qquad \Xi_y^{**}=\eta_y^{**\prime}+\theta_x. \tag{6.159}$$

It now becomes obvious that point O** (a, b, Z) is the flexural center of the cross section. The first two equations of system (6.158) represent the equilibrium conditions of the cut-off portion of the shell with respect to the displacements in the direction of the Ox, Oy axes. The third equation represents the equilibrium condition with respect to rotation about the Oz axis. Having eliminated the unknowns Σ_x^{**}, Σ_y^{**} from the third equation of system (6.158) with the aid of the first two equations, we arrive at the equilibrium condition of the cut-off portion of the shell with respect to rotation relative to the line of flexural centers.

CHAPTER VII

SLIGHTLY OBLIQUE AND SLIGHTLY CONICAL SHELLS

The present chapter deals with certain typical modifications of conical shells, for which the general resolvents can be substantially simplified on the basis of characteristic approximate relations.

We will introduce into consideration the parameter H,i.e., the height of a conical shell, representing the distance between the apex of the conical surface and the plane of the directrix :

$$H = l_0 \sin \chi_0 \qquad (7.1)$$

Chosing the origin of the fundamental system of Cartesian coordinates so that the Oz axis is perpendicular to the plane of the directrix, we have from (1.12)

$$\left(\frac{l_s}{H}\right)^2 = 1 + \left(\frac{x}{H}\right)^2 + \left(\frac{y}{H}\right)^2. \qquad (7.2)$$

Using (7.2), from 1.11) we also find

$$\cos \chi = -x' \frac{x}{l_s} - y' \frac{y}{l_s}, \qquad (7.3)$$

whence

$$\max |\cos \chi| < \frac{|x| + |y|}{l_s}. \qquad (7.4)$$

We will consider shells, the length of whose generatrices changes so little that in calculating the coefficients of the resolvents it can be assumed that

$$l_s = \text{const}. \qquad (7.5)$$

Among such shells, it is useful to distinguish two main types. The first includes shells characterized by the approximate relation

$$\frac{l_s}{H} \approx 1. \qquad (7.6)$$

In this case, on the basis of (7.2)

$$\left(\frac{x}{H}\right)^2+\left(\frac{y}{H}\right)^2\ll 1, \tag{7.7}$$

So that the following approximate relation results from (7.4) :

$$\chi\approx\frac{\pi}{2}. \tag{7.8}$$

Shells satisfying relations (7.6) will be refferred to as slightly oblique shells.

The second type of shells is characterized by the approximate relation.

$$\frac{l_s}{H}\approx \mathrm{const} \neq 1. \tag{7.9}$$

Two cases are possible here. The first is characterized by the fact that angle X is close to $\pi/2$. Included here are shells of marked conicity, close to right circular ones. The second case is characterized by the fact that the coordinate angle X can take values substantially different from $\pi/2$. Included here are shells close to oblique cylindrical ones. Such shells will be referred to as oblique slightly conical ones.

7.1 SLIGHTLY CONICAL OBLIQUE SHELLS

For slightly conical oblique shells, on the basis of (7.9), we assume that

$$\frac{Is}{H} = k \tag{7.10}$$

where k is some constant different form unity.

In view of (7.10), the general expressions for the coordinate funct.ons, corresponding to the displacement of the contour Z = const as a solid, are somewhat simplified. We will confine ourselves here to the case in which the point of intersection of the Oz axis with the plane Z = const is chosen as the pole, and vectors η and θ are expanded in an orthogonal basis coinciding with the basis of the fundamental system of Cartesian coordinates in the case of the fundamental static-geometric model, and with the basis defined by relations (3.241) in the case of a model with the contour Z = const nondeformable in its plane.

For the fundamental static geometric model, in view of (7.10), we have from (2.21):

i	V_i	φ_{im_x}	φ_{im_y}	φ_{im_z}
1	η_x	$-\frac{1}{k}\frac{x_0}{H}$	x_0'	$-\frac{1}{k\sin\chi_0}\frac{y_0'}{\sin\chi}+\frac{1}{k}\operatorname{ctg}\chi_0\frac{x_0y_0'-x_0'y_0}{H\sin\chi}$
2	η_y	$-\frac{1}{k}\frac{y_0}{H}$	y_0'	$\frac{1}{k\sin\chi_0}\frac{x_0'}{\sin\chi}$
3	η_z	$\frac{1}{k\sin\chi_0}\left(1-\cos\chi_0\frac{x_0}{H}\right)$	$x_0'\operatorname{ctg}\chi_0$	$\frac{1}{k}\frac{x_0'y_0-x_0y_0'}{H\sin\chi}$
4	$\left(1-\frac{Z}{l_0}\right)\theta_x$	$\frac{1}{k\sin\chi_0}y_0$	$(x_0'y_0-x_0y_0')\operatorname{ctg}\chi_0$	$\frac{1}{k\sin\chi_0}\operatorname{ctg}\chi_0\frac{x_0x_0'}{\sin\chi}+\frac{y_0}{H}\frac{x_0'y_0-x_0y_0'}{k\sin\chi}$
5	$\left(1-\frac{Z}{l_0}\right)\theta_y$	$-\frac{1}{k\sin\chi_0}x_0$	0	$\frac{1}{k\sin\chi_0}\operatorname{ctg}\chi_0\frac{x_0y_0'}{\sin\chi}-\frac{x_0}{H}\frac{x_0'y_0-x_0y_0'}{k\sin^2\chi_0\sin\chi}$
6	$\left(1-\frac{Z}{l_0}\right)\theta_z$	0	$x_0y_0'-x_0'y_0$	$\frac{1}{k\sin\chi_0}\frac{x_0x_0'+y_0y_0'}{\sin\chi}+\frac{y_0}{H}\frac{x_0'y_0-x_0y_0'}{k\operatorname{tg}\chi_0\sin\chi}$

(7.11)

For the model with the nondeformable contour, in view of (7.10), we have from (3.244) :

i	V_i	φ_{in_s}	φ_{in_t}	φ_{in_n}
1	η_1	$\dfrac{\frac{1}{k}\,\mathrm{ctg}\,\chi_0 - x_0' \cos\chi}{\sin\chi \sin\chi_0} - \dfrac{x_0}{Hk \sin\chi_0 \sin\chi}$	$\dfrac{x_0'}{\sin\chi_0}$	$-\dfrac{1}{k\sin\chi}\, y_0'$
2	η_2	$-y_0' \,\mathrm{ctg}\,\chi - \dfrac{1}{k\sin\chi}\dfrac{y_0}{H}$	y_0'	$\dfrac{1}{k\sin\chi_0 \sin\chi}\, x_0'$
3	η_3	$\dfrac{1}{k\sin\chi}$	0	$\dfrac{1}{k}\dfrac{\mathrm{ctg}\,\chi_0}{\sin\chi}\, y_0' + \dfrac{x_0' y_0 - x_0 y_0'}{Hk \sin\chi_0 \sin\chi}$
4	$\left(1-\dfrac{Z}{l_0}\right)\theta_1$	$\dfrac{1}{k\sin\chi}\, y_0$	0	$\dfrac{1}{k}\dfrac{\mathrm{ctg}\,\chi_0}{\sin\chi}\, y_0 y_0' + \dfrac{y_0}{H}\dfrac{x_0' y_0 - x_0 y_0'}{k\sin\chi_0 \sin\chi}$
5	$\left(1-\dfrac{Z}{l_0}\right)\theta_2$	$-\dfrac{1}{k\sin\chi_0 \sin\chi}\, x_0$	0	$\dfrac{\cos\chi_0}{k\sin^2\chi_0 \sin\chi}\, x_0 y_0' + \dfrac{x_0}{H}\dfrac{x_0 y_0' - x_0' y_0}{k\sin^2\chi_0 \sin\chi}$
6	$\left(1-\dfrac{Z}{l_0}\right)\theta_3$	$\dfrac{\mathrm{ctg}\,\chi}{\sin\chi_0}(x_0' y_0 - x_0 y_0') - \dfrac{\mathrm{ctg}\,\chi_0}{k\sin\chi}\, y_0$	$\dfrac{x_0 y_0' - x_0' y_0}{\sin\chi_0}$	$\dfrac{1}{k\sin\chi}\left(\dfrac{x_0 x_0'}{\sin^2\chi_0} + y_0 y_0'\right)$

(7.12)

In the resolvent system of ordinary differential equations of conical shells of arbitrary configuration (3.65), the relative $\bar{Z}$ coordinate was chosen as the dependent variable. However, for slightly conical oblique shells, the absolute coordinate $Z = l_0 \bar{Z}$. is more convenient to use. In this case, we have

$$\sum_{i=1}^{6+n}\left[\left(1-\frac{Z}{l_0}\right)\tilde{a}_{z\,ji} V_i' + \tilde{b}_{ji} V_i\right] = \frac{1}{\lambda_j G}\bar{P}_j \quad (j=1,2,\ldots,6), \quad (7.13)$$

$$\sum_{i=1}^{6+n}\left\{\left[\left(1-\frac{Z}{l_0}\right)\tilde{a}_{z\,ji} V_i' + \tilde{b}_{ji} V_i\right]' - \tilde{b}_{ij} V_i' - \frac{1}{1-\frac{Z}{l_0}}\tilde{c}_{ji} V_i\right\} = \frac{1}{G} R_j$$

$$(j = 7,8, \ldots, 6+n), \quad (7.14)$$

where for the fundamental static-geometric model, the coefficients are determined by the expressions

$$\begin{aligned}
\tilde{a}_{z\,ji} &= \frac{1}{k\sin\gamma_0}\oint\Big\{\frac{2}{1-\nu}\big[\dot\psi_{jm_z}\dot\psi_{im_z}+\nu(\dot\psi_{jn_z}\dot\psi_{im_z}+\dot\psi_{jm_z}\dot\psi_{in_z})+\\
&+\dot\psi_{jn_z}\dot\psi_{in_z}\big]+\dot\psi_{jn_z m_z}\dot\psi_{in_z m_z}\Big\}\sin\gamma h\,dS+\\
&+\frac{2(1+\nu)}{k\sin\gamma_0}\frac{1}{1-\frac{Z}{l_0}}\sum_k \psi_{jm_z}\psi_{im_z}\Delta F_k.\\
\tilde{b}_{ji} &= \oint\Big\{\frac{2}{1-\nu}\big[\dot\psi_{jm_z}\bar{\dot\psi}_{im_z}+\nu(\dot\psi_{jn_z}\bar{\dot\psi}_{im_z}+\dot\psi_{jm_z}\bar{\dot\psi}_{in_z})+\dot\psi_{jn_z}\bar{\dot\psi}_{in_z}\big]+\\
&+\dot\psi_{jn_z m_z}\bar{\dot\psi}_{in_z m_z}\Big\}\sin\gamma h\,dS,\\
\tilde{c}_{ji} &= k\sin\gamma_0\oint\Big\{\frac{2}{1-\nu}\big[\bar{\dot\psi}_{jm_z}\bar{\dot\psi}_{im_z}+\nu(\bar{\dot\psi}_{jn_z}\bar{\dot\psi}_{im_z}+\bar{\dot\psi}_{jm_z}\bar{\dot\psi}_{in_z})+\\
&+\bar{\dot\psi}_{jn_z}\bar{\dot\psi}_{in_z}\big]+\bar{\dot\psi}_{jn_z m_z}\bar{\dot\psi}_{in_z m_z}\Big\}\sin\gamma h\,dS,
\end{aligned}\tag{7.15}$$

and for the model with the contour $Z=$ const nondeformable in its plane, by the expressions :

$$\begin{aligned}
\tilde{a}_{z\,ji} &= \frac{1}{k\sin\gamma_0}\Big[\oint\Big(\frac{2}{1+\nu}\dot\psi_{jn_z}\dot\psi_{in_z}+\dot\psi_{jn_z m_z}\dot\psi_{in_z m_z}\Big)\sin\gamma h\,dS+\\
&+\frac{2(1+\nu)}{1-\frac{Z}{l_0}}\sum_k \psi_{jm_z}\psi_{im_z}\Delta F_k\Big].\\
\tilde{b}_{ji} &= \oint\Big(\frac{2}{1-\nu}\dot\psi_{jn_z}\bar{\dot\psi}_{in_z}+\dot\psi_{jn_z m_z}\bar{\dot\psi}_{in_z m_z}\Big)\sin\gamma h\,dS,\\
\tilde{c}_{ji} &= k\sin\gamma_0\oint\Big(\frac{2}{1-\nu}\bar{\dot\psi}_{jn_z}\bar{\dot\psi}_{in_z}+\bar{\dot\psi}_{jn_z m_z}\bar{\dot\psi}_{in_z m_z}\Big)\sin\gamma h\,dS.
\end{aligned}\tag{7.16}$$

The right-hand sides of Eqs. (7.13) are determined by expressions (3.88), (3.90) for the fundamental static-geometric model and by expressions (3.299), (3.301) for the model with the nondeformable contour.

The right-hand sides of Eqs.(7.14) are determined by the expression

$$\frac{1}{\lambda f\Omega}\bar{R}_j=\frac{k\sin\chi_0}{\Omega}\left(1-\frac{Z}{l_0}\right)\int_0^s\left(p^0_{m_s}\bar{\varphi}_{jm_s}+p_{n_s}\bar{\varphi}_{jn_s}+p^1_{n_s}\bar{\varphi}_{jn_s}\right)\sin\gamma\,dS$$

$$(j = 7, 8, \ldots 6 + n) \qquad (7.17)$$

Resolvent Eqs. (7.13), (7.14), and expressions (7.15), (7.16) and (7.17) for their coefficients and right-hand sides also remain valid when the first six desired generalized displacements chosen are the canonical kinematic unknowns, which are the components of the displacement Z = const as a solid about a single pole coinciding with the apex of the conical surface. In this case, matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{B}}', \tilde{\mathbf{C}}$ of the coefficients of the resolvents, in view of (3.125), (3.135), (6.23) take the form

$$\tilde{\mathbf{A}}=\begin{pmatrix}\tilde{\mathbf{A}}_{\gamma\gamma} & \tilde{\mathbf{A}}_{\gamma\varphi} & \tilde{\mathbf{A}}_{\gamma\theta}\\ \tilde{\mathbf{A}}_{\varphi\gamma} & \tilde{\mathbf{A}}_{\varphi\varphi} & \tilde{\mathbf{A}}_{\varphi\theta}\\ \tilde{\mathbf{A}}_{\theta\gamma} & \tilde{\mathbf{A}}_{\theta\varphi} & \tilde{\mathbf{A}}_{\theta\theta}\end{pmatrix},\quad \tilde{\mathbf{B}}=\begin{pmatrix}0 & \frac{1}{l_0}\tilde{\mathbf{A}}_{\gamma\varphi} & \tilde{\mathbf{B}}_{\gamma\theta}\\ 0 & \frac{1}{l_0}\tilde{\mathbf{A}}_{\varphi\varphi} & \tilde{\mathbf{B}}_{\varphi\theta}\\ 0 & \frac{1}{l_0}\tilde{\mathbf{A}}_{\theta\varphi} & \tilde{\mathbf{B}}_{\theta\theta}\end{pmatrix},$$

$$\tilde{\mathbf{B}}'=\begin{pmatrix}0 & 0 & 0\\ \frac{1}{l_0}\tilde{\mathbf{A}}_{\varphi\gamma} & \frac{1}{l_0}\tilde{\mathbf{A}}_{\varphi\varphi} & \frac{1}{l_0}\tilde{\mathbf{A}}_{\varphi\theta}\\ \tilde{\mathbf{B}}'_{\theta\gamma} & \tilde{\mathbf{B}}'_{\theta\varphi} & \tilde{\mathbf{B}}'_{\theta\theta}\end{pmatrix},\quad \tilde{\mathbf{C}}=\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{l_0^2}\tilde{\mathbf{A}}_{\varphi\varphi} & \frac{1}{l_0}\tilde{\mathbf{B}}_{\varphi\theta}\\ 0 & \frac{1}{l_0}\tilde{\mathbf{B}}'_{\theta\varphi} & \tilde{\mathbf{C}}_{\theta\theta}\end{pmatrix}.$$

$$(7.18)$$

The coordinate functions corresponding to the canonical kinematic unknowns are represented in the general case by expressions (3.6). These expressions can be simplified somewhat by using (7.10).

As a result of the change to canonical kinematic unknowns, in view of (7.18) resolvents (7.13), (7.14) assume the form

$$\left(1-\frac{Z}{l_0}\right)\sum_{i=1}^{3}\tilde{a}_{2\,ji}U'_i+\left(1-\frac{Z}{l_0}\right)^2\sum_{i=4}^{6}\tilde{a}_{2\,ji}U'_i+$$

$$+\sum_{i=7}^{6+n}\left[\left(1-\frac{Z}{l_0}\right)\tilde{a}_{2\,ji}U'_i+\tilde{b}_{ji}U_i\right]=\frac{1}{\lambda f\Omega}\bar{P}_j$$

$$(j=1,2,\ldots,6),$$

$$\sum_{i=1}^{3}\left\{\tilde{a}_{2\,ji}\left[\left(1-\frac{Z}{l_0}\right)U'_i\right]'-\tilde{b}_{ij}U'_i\right\}+\sum_{i=4}^{6}\left\{\tilde{a}_{2\,ji}\left[\left(1-\frac{Z}{l_0}\right)^2U'_i\right]'-\right.$$

$$\left.-\tilde{b}_{ij}\left(1-\frac{Z}{l_0}\right)U'_i\right\}+\sum_{i=7}^{6+n}\left\{\left[\left(1-\frac{Z}{l_0}\right)\tilde{a}_{2\,ji}U'_i+\tilde{b}_{ji}U_i\right]'-\right. \qquad (7.19)$$

$$\Big\} - \bar{b}_i U_i - \frac{1}{1-\frac{Z}{l_0}} \bar{c}_i U'_i \Big\} - \frac{1}{a} R_j$$

$$(j = 7, 8, \ldots, 6+n). \qquad (7.20)$$

Solving Eqs. (7.19), as a system of algebraic equations for the unknowns $\left(\frac{u_l}{Z} - 1\right)$ U'_i (i = 1, 2, 3,) and $\left(1 - \frac{Z}{l_0}\right)^2$ U'_i (i = 4, 5, 6) then eliminating them from Eqs. (7.20), we arrive at a system of differential equations in the generalized displacements U_i (i = 7,8 . . . ., 6 + n), corresponding to the warpings. These equations reduce to equations with constant coefficients when the thickness h of the shell is independent of the Z coordinate, and the cross-sectional area of the elements of the longitudinal structure changes in accordance with the linear law

$$\Delta F_k (Z) = \Delta F_k (0) \left\{ 1 - \frac{Z}{T_0} \right\}$$

Let us recall that in Eqs. (7.19), $\overline{P_j}$ (j = 1, 2, 3) constitute the components of an equivalent external load applied to the cut-off portion of the shell, and $\overline{P}_j$ (j =4,5,6) are the components of the moment of external forces with respect to the apex of the conical surface.

7.2. SLIGHTLY OBLIQUE SHELLS

For slightly oblique shells, in the relations represented in the preceding section, it is necessary to assume that

$$\frac{l_s}{H} = k = 1; \qquad X = \frac{n}{2}$$

$$(7.21)$$

Moreover, taking (7.7) into consideration, we will neglect the quantities $\frac{x_0}{H}$, $\frac{y_0}{H}$ in comparison with unity in the expressions for the coordinate functions, This assumption has practically no effect on the magnitude of the coefficients of the resolvents if the origin of the fundamental system of Cartesian coordinates lies within the contour of the directrix. It is usually desirable to select the origin at the center of gravity of the contour of the directrix. Then the Oz axis coincides with the line of the centers of gravity of the section Z=const. If this line is perpendicular to the plane of the section Z=const, we are dealing with a slightly conical shell in the sense in which this is usually understood in the literature. The concept of major axes of inertia is preserved in such shells, and the equations corresponding to bending are separated form the equations corresponding to torsion. In the general case, however, in slightly oblique shells, the line of centroids is inclined on the plane of the section Z =const, and bending is not separated from torsion.

For slightly oblique shells, the difference between the fundamental static-geometric model and the model with a nondeformable contour Z=const practically vanishes. We therefore give preference to the former model as the simpler one.

For coordinate functions corresponding to displacement of the contour Z=const as a solid, on the basis of (7.11) we obtain

i	V_i	[illegible]	[illegible]	[illegible]
1	v_x	$-\frac{x_0}{H}$	x_0'	$-\frac{1}{\sin\chi_0} y_0'$
2	v_y	$-\frac{y_0}{H}$	y_0'	$\frac{1}{\sin\chi_0} x_0'$
3	v_z	$\frac{1}{\sin\chi_0}$	$x_0' \operatorname{ctg}\chi_0$	$\frac{1}{H}(x_0'y_0 - x_0y_0')$
4	$\left(1-\frac{Z}{l_0}\right)\theta_x$	$\frac{1}{\sin\chi_0} y_0$	$(x_0'y_0 - x_0y_0')\operatorname{ctg}\chi_0$	$\frac{\cos\chi_0}{\sin^2\chi_0} x_0x_0'$
5	$\left(1-\frac{Z}{l_0}\right)\theta_y$	$-\frac{1}{\sin\chi_0} x_0$	0	$\frac{\cos\chi_0}{\sin^2\chi_0} x_0y_0'$
6	$\left(1-\frac{Z}{l_0}\right)\theta_z$	0	$x_0y_0' - x_0'y_0$	$\frac{1}{\sin\chi_0}(x_0x_0' + y_0y_0')$

(7.22)

The differential resolvents in the case under consideration coincide in form with Eqs. (7.13), (7.14). Coefficients (7.15) of the resolvents take the form

$$\tilde{a}_{jl} = \frac{1}{\sin\gamma_0}\left[\oint\left(\frac{2}{1-\nu}\varphi'_{jm_z}\varphi'_{lm_z} + \varphi_{jn_z}\varphi_{ln_z}\right)h\,dS + \right.$$
$$\left. + \frac{2(1+\nu)}{1-\frac{Z}{l_0}}\sum_k \varphi_{jm_z}\varphi_{lm_z}\Delta F_k\right].$$
$$\tilde{b}_{jl} = \oint \varphi_{jn_z}\left(\varphi'_{lm_z} + \frac{1}{H}\varphi_{ln_z}\right)h\,dS.$$
$$\tilde{c}_{jl} = \sin\gamma_0\oint\left(\varphi'_{jm_z} + \frac{1}{H}\varphi_{jn_z}\right)\left(\varphi'_{lm_z} + \frac{1}{H}\varphi_{ln_z}\right)h\,dS.$$

(7.23)

The right-hand sides of Eqs. (7.13) are determined by expressions (3.88), (3.90). The right-hand sides of Eqs. (7.14) are determined by the expression

$$\frac{1}{\lambda_j G}\tilde{R}_j = \frac{\sin\gamma_0}{G}\left(1-\frac{Z}{l_0}\right)\oint\left(p^0_{m_z}\varphi_{jm_z} + p^0_{n_z}\varphi_{jn_z} + p^0_{n_n}\varphi_{jn_n}\right)dS.$$

(7.24)

Part Two. Analysis of Wing-Type Shells

Chapter VIII. Analysis of a Wing Neglecting the Elasticity of Ribs

This chapter deals with the analysis of wing-type shells of various configurations. Shells of straight and swept wing types as well as shells of low-aspect wing type are discussed. The calculations are based on resolvents obtained in Part One for the fundamental static-geometric model. In addition, equations obtained on the basis of a model with the contour Z=const nondeformable in its plane are discussed. This model corresponds to the case where the ribs are oriented parallel to the flow, while the fundamental static-geometric model, which assumes that the warping takes place along the generatrices, effectively corresponds to the case in which the ribs are perpendicular to the generatrices. Comparative calculations show that the influence of the rib orientation on the stressed state of the wing is usually insignificant.

Given below are computational results illustrating the range of wing configurations fitting the fundamental static-geometric model. The calculations, performed on a computer, starting with straight wings and ending with low-aspect wings, were performed in accordance with a single universal program written for a conical shell of arbitrary configuration on the assumption that the thickness h of the shell and cross-sectional area ΔF_k of the elements of the longitudinal structure are variable, and it is assumed in the general case that the shell thickness changes along two coordinates: $h=h(Z, S)$. Coordinate functions $\phi_k^1(S)$ of two types were analyzed: power functions (3.309) and piecewise-linear local functions (3.310).

In addition, for shells of all the types considered, simplified analytical solutions are given, obtained by approximating the warping with only one function. It was assumed that the shell thickness h was independent of the Z coordinate, which made it possible to change from equations with variable coefficients to equations with constant ones. On the basis of an analysis of experimental data on shells of swept and low-aspect wing types, a single approximating function is proposed which gives satisfactory results for shells of these types in the majority of cases.

8.1. Straight-Wing Type Shells

1. Shells of Constant Cross Section

The resolvents and other relations necessary for the analysis of a right cylindrical shell of constant cross section (Figure 8.1) were obtained in Section 6.4 from general expressions for a conical shell of arbitrary configuration.*

We will use numerical examples to analyze the character of the convergence of expansions corresponding to warping of the cross section. We will consider expansions in powerful functions as well as piecewise-linear local coordinate functions.

We superpose the Oz axis on the neutral axis of the cross section. Since for the moments of inertia with respect to the principal central axes of the cross section of any wing, the following relation is typical:

$$I_{min} \ll I_{max}, \tag{8.1}$$

then, as the neutral axis for any loads typical of wings, we can take with a high degree of accuracy the principal central axis with respect to which the moment of inertia $I=I_{min}$.

The preliminary position of the neutral axis, having thus been established, it is permissible to assume that the displacements corresponding to the warping change linearly with the height, since the height of the cross section is considerably smaller than the chord. In this connection, instead of (3.309), one should select as the coordinate power functions

$$\varphi_k^1(S) = x^k(S)\,y(S) \qquad (k = 1, 2, \ldots, n), \tag{8.2}$$

thus reducing the total number of generalized displacements retained. The assumption

*See also I.F. Obraztsov, Methods of Strength Analysis of Wing-Type Caisson Structures. Moscow, Oborongiz, 1960.

of linearity of warping along the height of the cross section permits one to reduce the number of terms retained in the case of piecewise-linear local coordinate func tions as well. Choosing paired points with the same abscissa x_k as the nodes on the upper and lower portions of the cross-sectional contour of the wing, by using the relation

$$\frac{u_{m_z}(x_k, y_k^{u})}{u_{m_z}(x_k, y_k^{l})} = \frac{y_k^{u}}{y_k^{l}}, \qquad (8.3)$$

where y_k^u, y_k^l are the ordinates of the upper and lower kth node, we can cut the number of generalized displacements corresponding to warping by exactly one-half.

The number of generalized displacements determining the displacements of the cross section as a solid is equal to six, which corresponds to six equilibrium conditions of the cut-off portion of the shell. However, in some cases, the number of generalized displacements U_i retained can be reduced. For example, for sections symmetric about the Ox axis, in bending and torsion of the wing, of the six generalized displacements $U_i (i=1, 2, \ldots, 6)$, only three, η_y, θ_x, θ_z, can be retained. Moreover, the equilibrium equations corresponding to the generalized displacements η_z, η_x, θ_y will be fulfilled identically.

Given below are results of comparative bending and torsion calculations for right caissons of rectangular cross section for the two variants of geometric dimensions shown in Figure 8.2.

Figure 8.3 shows diagrams of the coordinate functions ϕ_{im_z}, ϕ_{in_z}, corresponding to the displacement of the section as a solid in bending and torsion.

The diagrams of the coordinate power functions ϕ_k^l, corresponding to warping of the cross section, are shown in Figure 8.4. Power functions whose diagrams are symmetric about the Oy axis pertain to bending; functions with diagrams inversely symmetric about the Oy axis correspond to torsion.

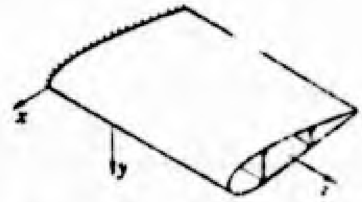
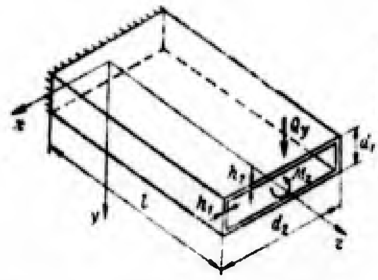


Figure 8.1. Straight cylindrical shell.



	l	d_1	d_2	h_1	h_2
			см		
I	120	4	30	0,1	0,1
II	36	4	30	0,1	0,1

Figure 8.2. Straight caisson of constant rectangular cross section.

Diagrams of piecewise-linear local coordinate functions ϕ_k^1 are given in Figure 8.5. These functions pertain equally to torsion and bending. In the latter case, in view of the obvious linear relation

$$\sum_{k=1}^{n} \varphi_k^1(S) = \gamma_{x_m z}, \tag{8.4}$$

any one of n local coordinate functions corresponding to the decomposition of the contour along its width into n pairs of nodes should be discarded.

Figures 8.6-8.8 show the results of analysis of caissons I and II for different numbers n of retained coordinate functions $\phi_k^1(S)$. The following notation is used:

n^* - index of the last of the power coordinate functions ϕ_k^1; in torsion, n^* is odd: $xy, x^3y, \ldots, x^{n^*}y$);

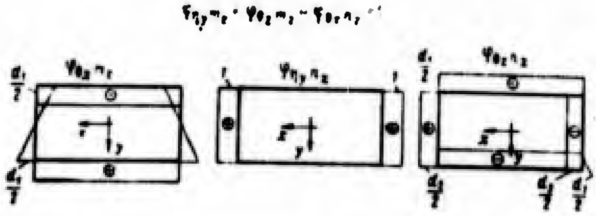


Figure 8.3. Coordinate functions corresponding to the law of plane sections.

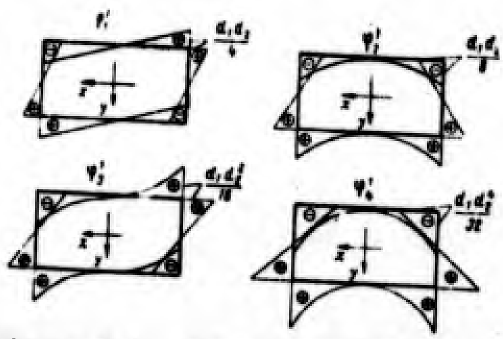


Figure 8.4. Power coordinate functions.

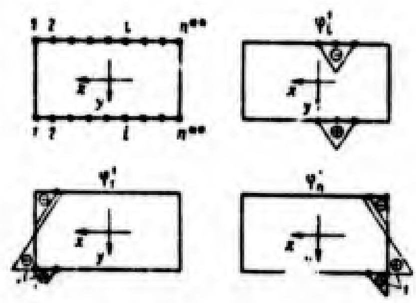


Figure 8.5. Local coordinate functions.

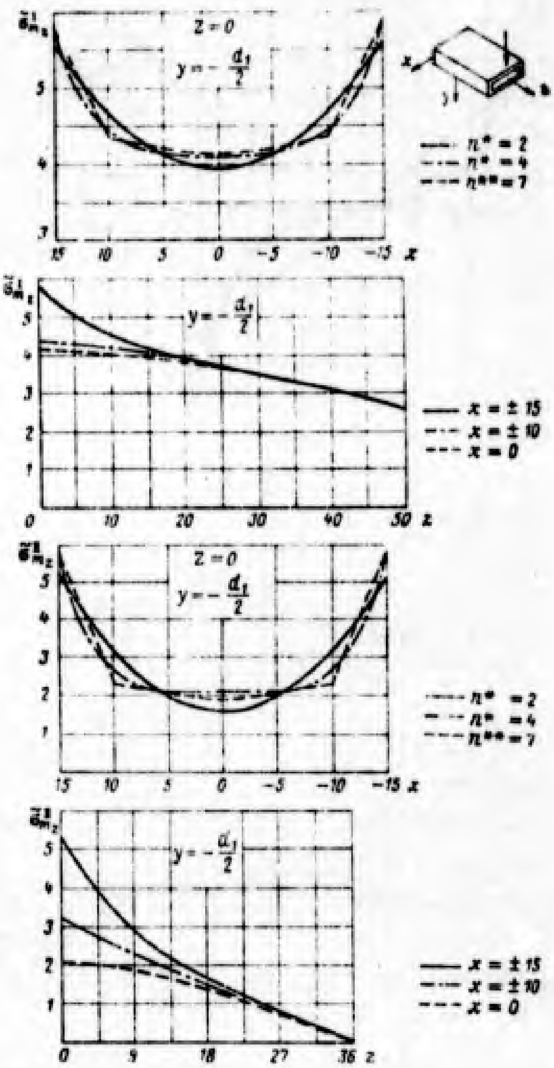


Figure 8.6. Distribution of normal stresses in bending of caissons I and II.

n^{**} - is the number of retained local coordinate functions ϕ_k^1 when the cross section is divided along its width into $n^{**}+1$ pairs of nodes in bending, and n^{**} pairs of nodes in torsion.

As an illustration, Figures 8.10 and 8.11 show the results of calculations of a shell of straight wing type I (Figure 8.9) acted on by an air load.

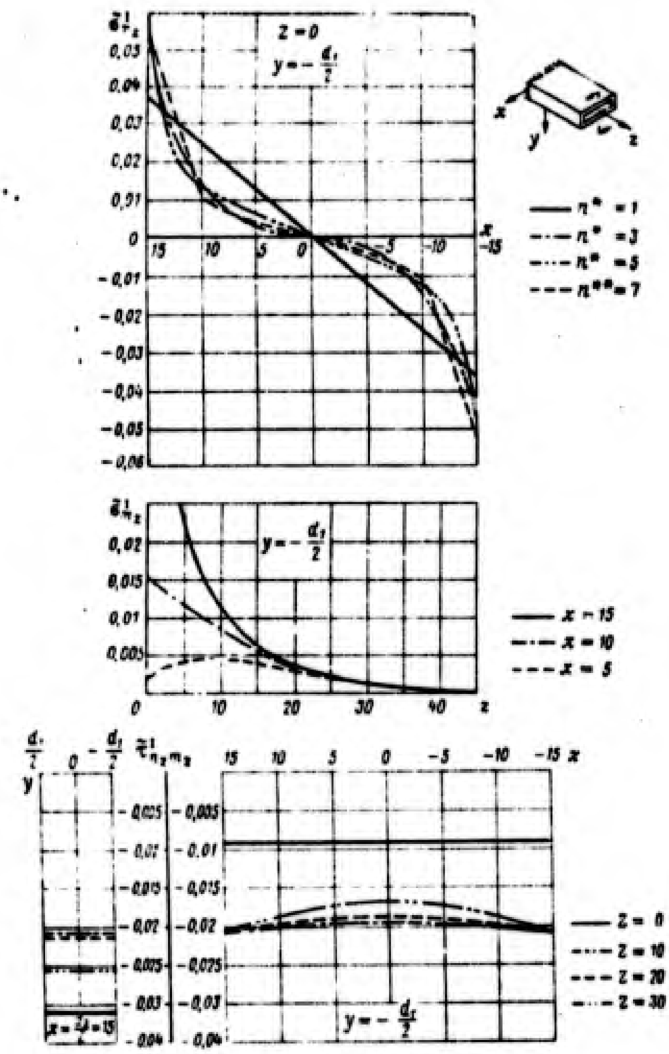


Figure 8.7. Distribution of normal and tangential stresses in torsion of caisson I.

2. Shells of Variable Cross Section

In the general case, when the line of centroids of the cross sections is not perpendicular to the edge, we are dealing with slightly oblique shells. The resolvents and other relations necessary for analyzing such shells were obtained from general expressions for a conical shell of arbitrary configuration in Section 7.2. If the line of centroids is perpendicular to the edge, then, as already noted, we are dealing with a slightly conical shell in the generally accepted sense of this term.

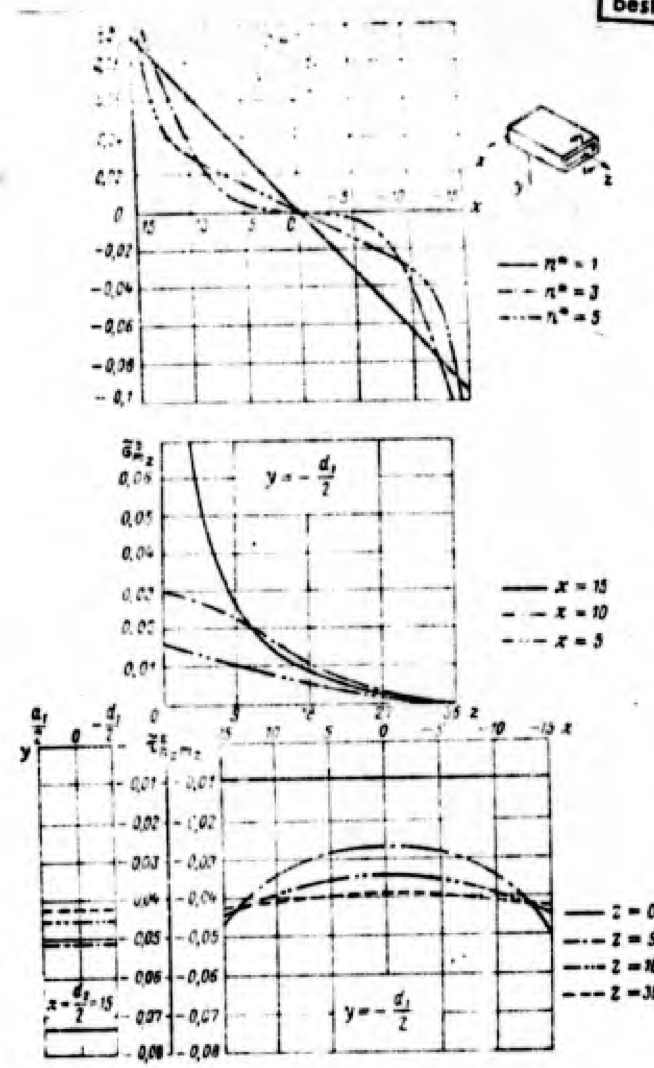
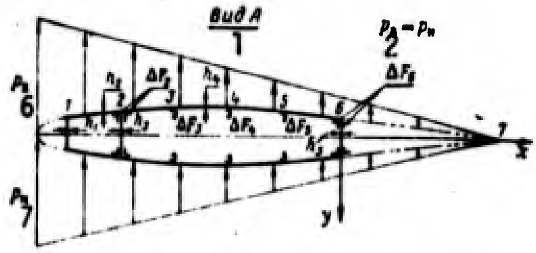


Figure 8.8. Distribution of normal and tangential stresses in torsion of caisson II.

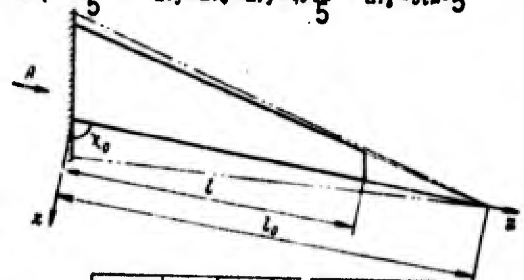
We will use numerical examples to analyze the convergence of the expansions corresponding to warping of the cross section. Let us again consider expansions in power as well as local piecewise-linear coordinate functions as in Subsection 1.

Given below are results of comparative bending and torsion analyses of caissons of variable rectangular cross section for the two variants of geometric dimensions shown in Figure 8.12. These caissons should obviously be regarded as slightly conical straight ones. For such boxes, superposing the oz axis on the lines of centroids of the cross sections, we have



	1	2	3	4	5	6	7
x_0 в см 3	-100	-80	-60	-40	-20	0	60
y_0 в см	-5,5	-8,4	-10,0	-10,3	-9,3	-7,0	0

$h_1 = h_2 = 0,1$ см 4 $h_3 = h_5 = 0,15$ см 4 $h_4 = 0,2$ см 4
$\Delta F_2 = 5$ см² 5 $\Delta F_3 = \Delta F_4 = \Delta F_5 = 1,5$ см² 5 $\Delta F_6 = 3$ см² 5



	I	II	III	IV	V
l_0 в см 3	∞	720	720	∞	720
l в см	400	400	400	400	400
χ_0 3	π/2	π/2	35°	π/4	π/4

Figure 8.9. Wings with symmetric curve contour. Key: 1) View A, 2) $p_u = p_1$, 3) in cm, 4) cm, 5) cm^2, 6) p_u, 7) p_1.

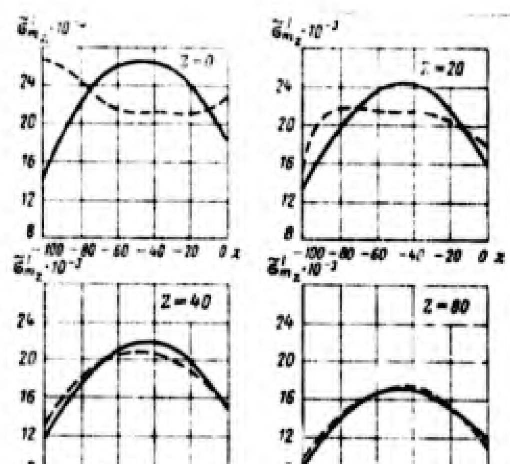


Figure 8.10. Normal stresses in a straight wing with a curved contour. Key: 1) law of plane sections.

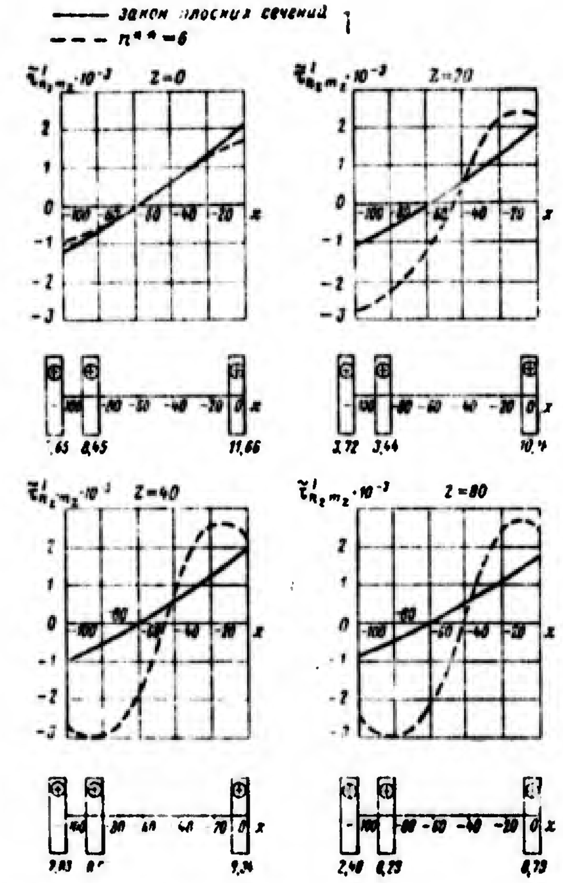
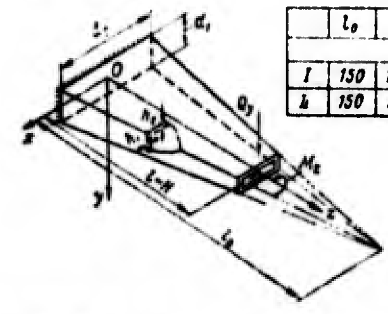


Figure 8.11. Tangential stresses in the skin and walls of a straight wing with a curved contour. Key: 1) law of plane sections.



	l_0	l	d_1	d_2	h_1	h_2
	см					
I	150	120	8	30	0,1	0,1
II	150	36	4	30	0,1	0,1

Figure 8.12. Slightly conical straight caisson of rectangular cross section.

$$\gamma_0 = \frac{\pi}{2}, \tag{8.5}$$

as a consequence of which coordinate functions (7.22), corresponding to the displacement of the cross section as a solid, are considerably simplified. Superposing

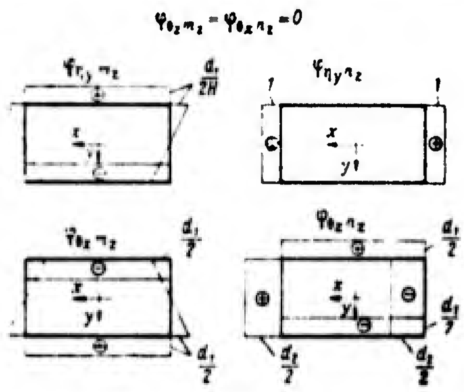


Figure 8.13. Coordinate functions corresponding to the law of plane sections.

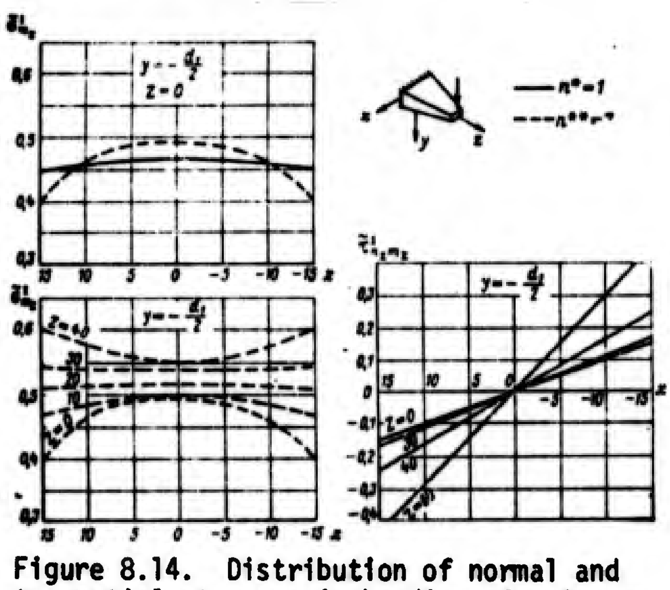


Figure 8.14. Distribution of normal and tangential stresses in bending of caisson I.

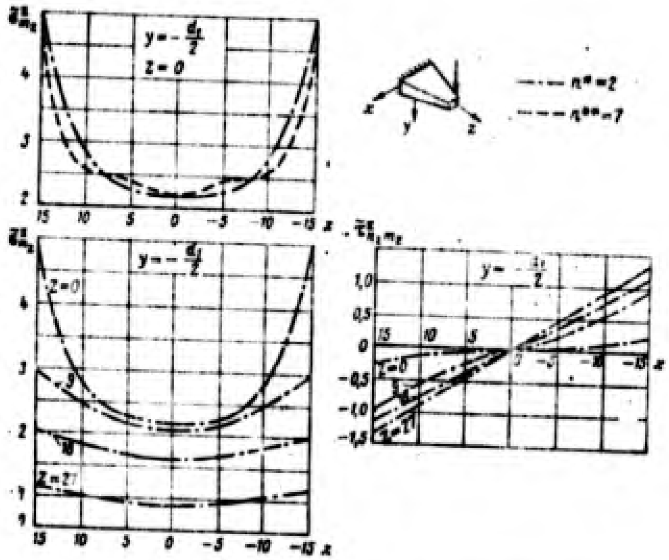


Figure 8.15. Distribution of normal and tangential stresses in bending of caisson II.

the Ox axis on the neutral side of the cross section, in the case at hand, as in the preceding case, we should retain the two generalized displacements η_y, θ_x in bending and one θ_z in torsion. The diagrams of the corresponding coordinate functions are shown in Figure 8.13. The diagrams of the coordinate functions $\phi_k^1(S)$ corresponding to warpings of the contour of the cross section coincide with the analogous diagrams for a shell of constant cross section pictured in Figures 8.4 nad 8.5.

Figures 8.14-8.17 show the results of calculations for bending and torsion of caissons I and II for different numbers n of retained coordinate functions $\phi_k^1(S)$.

Figures 8.18-8.21 illustrate the results of analysis for the action of an air load on slightly conical and slightly oblique shells of straight wing types II and III (see Figure 8.9).

For the shells discussed in Subsections 1 and 2, we will give simplified analytical solutions by approximating the warping of the cross section by a single function in the case of bending as well as in the case of torsion.

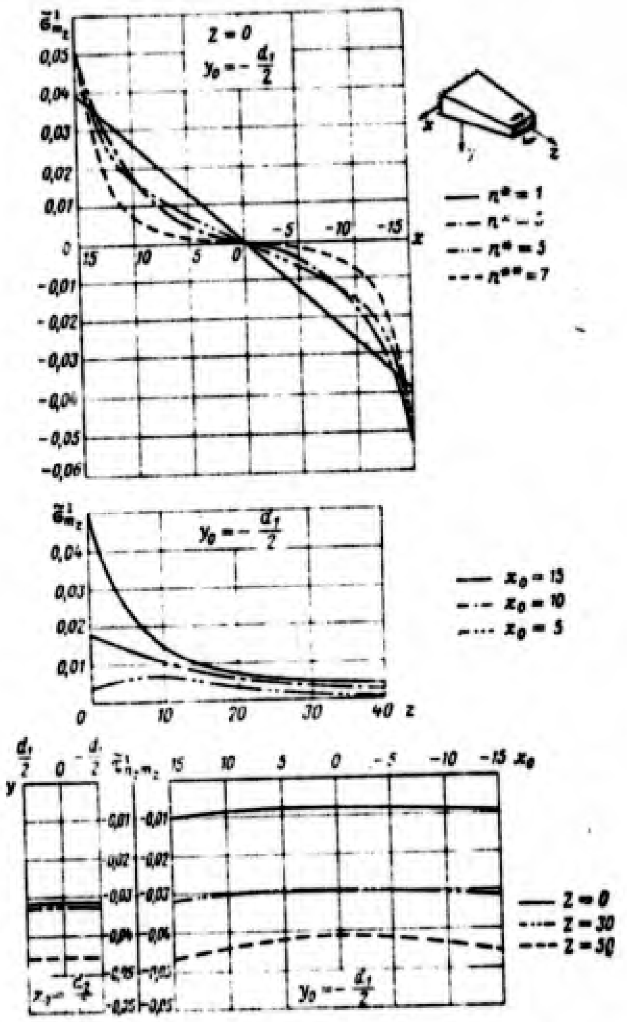


Figure 8.16. Distribution of normal and tangential stresses in torsion of caisson I.

3. Simplified Analysis of Shells of Constant Cross Section

We superimpose the Ox, Oy axes on the principal central axes of the cross section. Assuming that the external load on the wing is represented by normal pressure distributed in accordance with an arbitrary law, we will retain the generalized displacements η_y, θ_x, θ_z corresponding to bending in the yOz plane and to torsion, and also the two generalized displacements w_1, w_2, which determine the corresponding warpings. As the coordinate functions $\phi_1^1(S)$, $\phi_2^1(S)$, equally suited for shells with

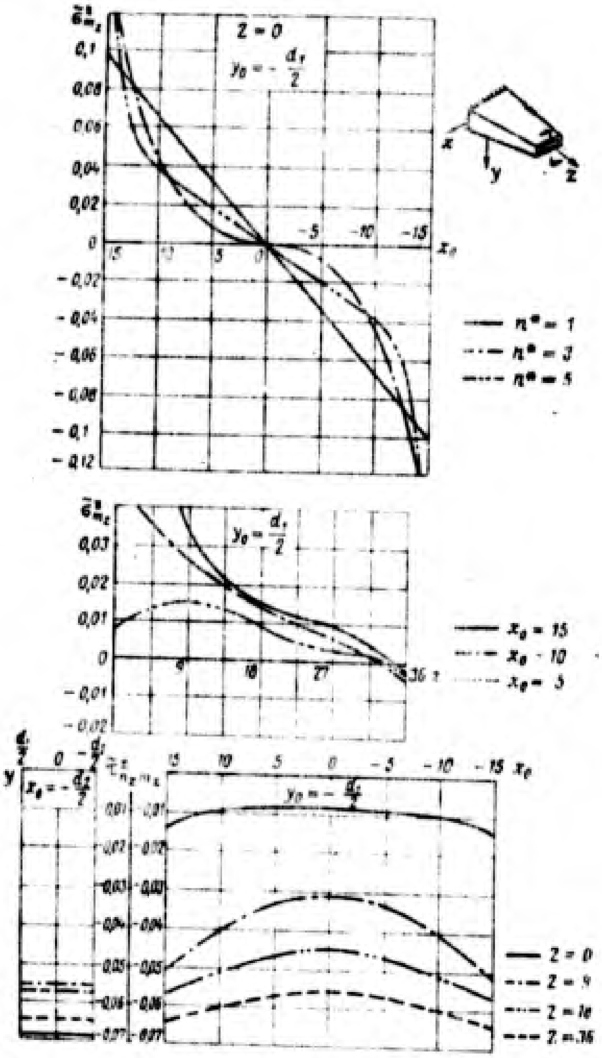


Figure 8.17. Distribution of normal and tangential stresses in torsion of caisson II.

different outlines of the cross-sectional profile, it is natural to take the first even function and odd function with respect to the x coordinate of the system of power coordinate functions (8.2):

$$\varphi_1(S) = x S [illegible] (S) \tag{8.6}$$

$$\bar{\varphi}_1(S) = x^2 S [illegible] (S) \tag{8.7}$$

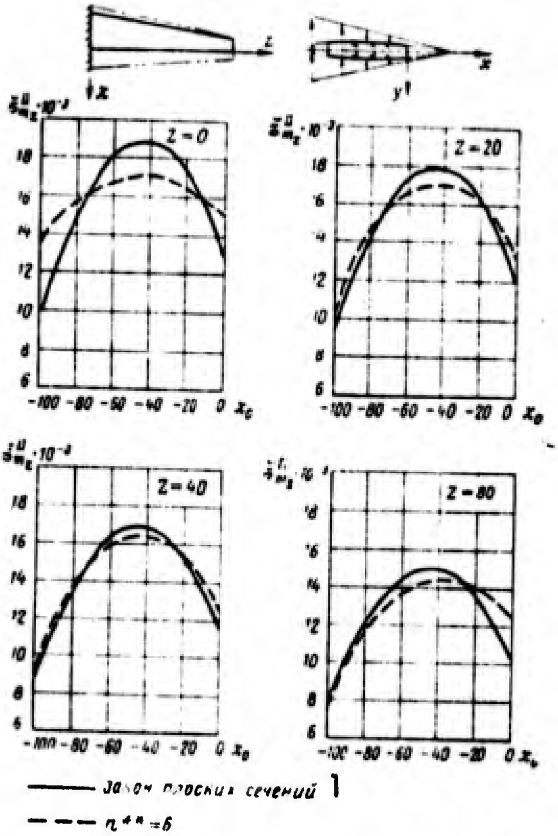


Figure 8.18. Normal stresses in a slightly conical straight wing with a curved contour. Key: 1) law of plane sections.

These functions together with the coordinate function $\phi_{\theta_x m_z}(S)=y(S)$ determine the longitudinal displacements of the cross section. However, it is more convenient to choose the coordinate functions ϕ_1^1 and ϕ_1^2 so that they are orthogonal to one another and to all the coordinate functions ϕ_{im_z} corresponding to displacements of the cross section as a solid. For this purpose, considering (6.126), we take

$$\varphi_2^1 = xy + k_{2x}x + k_{2y}y + k_2, \tag{8.8}$$

$$\varphi_1^1 = x^2y + k_{1x}x + k_{1y}y + k_1 + k_{12}\varphi_2^1. \tag{8.9}$$

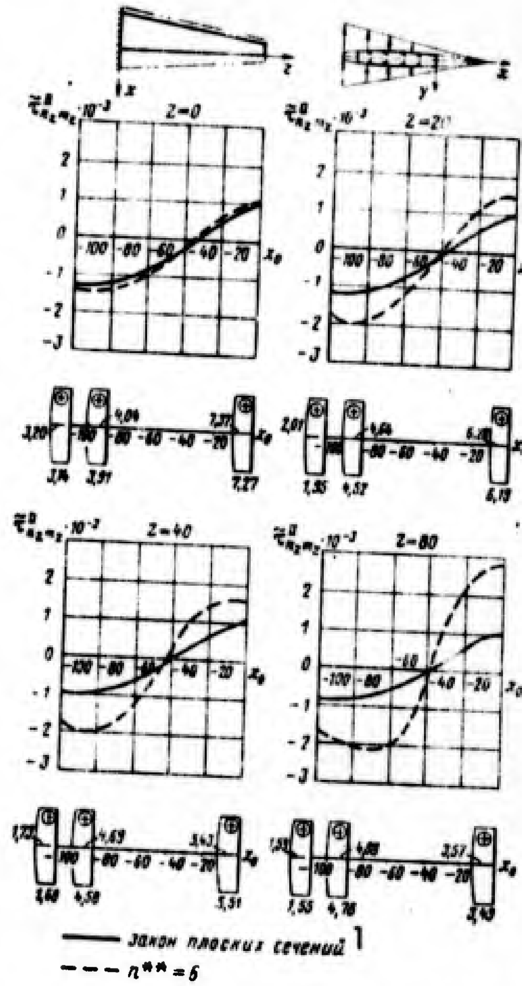


Figure 8.19. Tangential stresses in the skin and webs of a slightly conical straight wing with a curved contour.

where, from orthogonality conditions

$$\oint x\varphi_k^1 h\,dS = \oint y\varphi_k^1 h\,dS = \oint \varphi_k^1 h\,dS = 0 \quad (k = 1, 2),$$
$$\oint \varphi_1^1\varphi_2^1 h\,dS = 0, \tag{8.10}$$

the orthogonalization coefficients are determined by the formulas

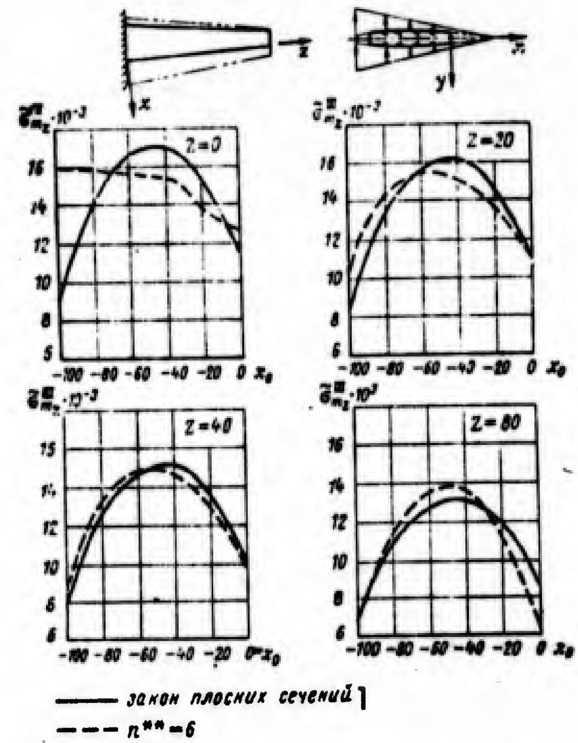


Figure 8.20. Normal stresses in a slightly oblique wing with a curved contour. Key: 1) law of plane sections.

$$
\begin{aligned}
k_{1x} &= -\frac{\oint x^3 y h\, dS}{\oint x^2 h\, dS}, & k_{2x} &= -\frac{\oint x^2 y h\, dS}{\oint x^2 h\, dS},\\
k_{1y} &= -\frac{\oint x^2 y^2 h\, dS}{\oint y^2 h\, dS}, & k_{2y} &= -\frac{\oint x y^2 h\, dS}{\oint y^2 h\, dS},\\
k_{1} &= -\frac{\oint x^2 y h\, dS}{\oint h\, dS}, & k_{2} &= -\frac{\oint x y h\, dS}{\oint h\, dS},
\end{aligned}
\qquad (8.11)
$$

$$
k_{12} = -\frac{\oint \psi_2^1 x^2 y h\, dS}{\oint (\psi_2^1) h\, dS}.
$$

Formulas (8.11) are applicable to a single- and multiclosed cross section of arbitrary configuration. If the shell is reinforced with a longitudinal structure, the quadratures in (8.11) should be understood in the sense of Stieltjes integrals, assuming in accordance with (6.133) that the finite measures $(1-\nu^2)\Delta F_k$ are concentrated at the points $S=S_k$ of the cross sectional contour.

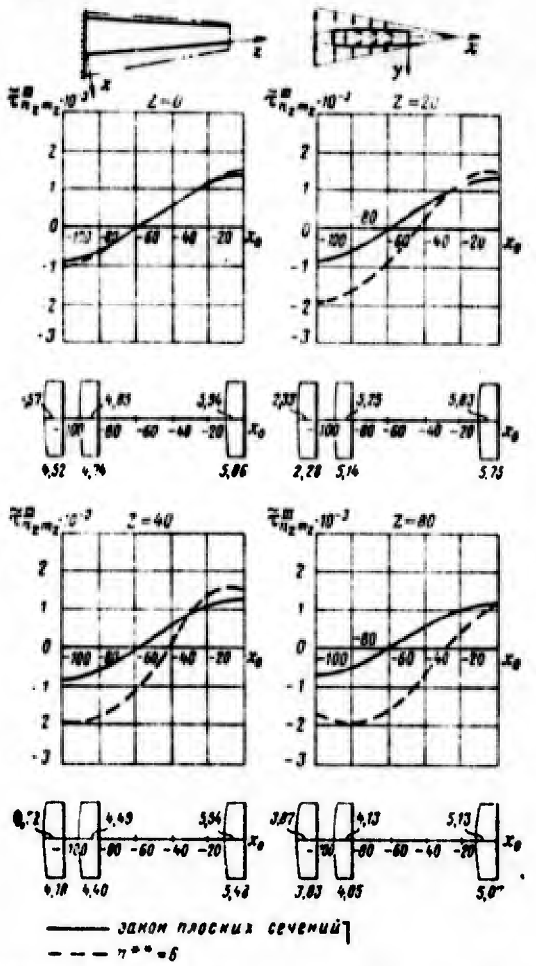


Figure 8.21. Tangential stresses in the skin and walls of a slightly oblique wing with a curved contour.

Resolvents in the desired generalized displacements η_y, θ_x, θ_z, w_1, w_2 are readily obtained from general canonical relations (6.145)-(6.151). For wing type shells, these relations are substantially simplified if it is assumed in accordance with (8.1) that for any external load typical for a wing

$$\Xi_z = \eta_z + Z\theta'_y \approx 0. \tag{8.12}$$

On the basis of (8.12), it follows from expression (6.146) that

$$k_{xy} \approx 0; \quad S'_z \approx 0; \quad l \approx 0. \tag{8.13}$$

In view of (8.13), we have from (6.149), (6.150)

$$K \approx k_{xx} k_{yy}; \quad k_{xx} \approx F_{xx}. \tag{8.14}$$

Taking (8.14) into account, we also find from (6.146), (6.147)

$$\frac{1}{k_{yy}} \gg \frac{1}{k_{xx}}. \tag{8.15}$$

It is easy to see that relations (8.13)-(8.15) become exact for wings with a profile symmetric about the Ox axis.

Considering relations (8.13), (8.14), we have from (6.147), (6.148)

$$\Xi_y = \frac{1}{k_{yy}} \left[\frac{Q_y}{G} - \frac{S_y^*}{I_z^*} \frac{M_z}{G} - \sum_{i=7}^{8} \left(\tilde{b}_{2i} - \frac{S_y^*}{I_z^*} \tilde{b}_{6i} \right) \omega_{i-6} \right], \tag{8.16}$$

$$\theta_z' = \frac{1}{I_z^*} \left[\frac{M_z}{G} + S_y^* \Xi_y - \sum_{i=7}^{8} \tilde{b}_{6i} \omega_{i-6} \right]. \tag{8.17}$$

From (6.151), in view of orthogonality conditions (8.10), we also find

$$\theta_x' = \frac{1-\nu}{2GI_x} M_x. \tag{8.18}$$

Expressions (8.16)-(8.18) represent the equilibrium conditions of the cut-off portion of the shell as a solid. On the basis of orthogonality conditions (8.10) and relation (8.12), equilibrium conditions (6.145) corresponding to warping of the section of the shell take the form

$$-\tilde{b}_{2j} \Xi_y - \tilde{b}_{6j} \theta_z' + \tilde{a}_{jj} \omega_{j-6}' - \sum_{i=7}^{8} \tilde{c}_{ji} \omega_{i-6} = 0 \quad (j = 7, 8). \tag{8.19}$$

Eliminating from Eqs. (8.19) the components of the law of plane sections Ξ_y and θ_z with the aid of (8.16), (8.17), we can readily obtain

$$\tilde{a}_{jj}\omega^{\bullet}_{j-6} - \sum_{i=7}^{8}\left[\tilde{c}_{ji} + \frac{1}{k_{yy}}\tilde{b}_{2j}\left(\frac{S_y^{\bullet}}{I_x^{\bullet}}\tilde{b}_{6i} - \tilde{b}_{2i}\right) - \frac{1}{I_x^{\bullet}}\tilde{b}_{6j}\tilde{b}_{6i}\right]\omega_{i-6} =$$

$$= \tilde{b}_{2j}\frac{Q_y}{Gk_{yy}} + \left(\tilde{b}_{6j} - \frac{S_y^{\bullet}}{k_{yy}}\tilde{b}_{2j}\right)\frac{M_z}{GI_x^{\bullet}} \qquad (j = 7, 8). \tag{8.20}$$

Expressions (8.20) represent a system of two simultaneous equations in flexural warping w_1 and torsional warping w_2. This system decomposes only when the cross section is symmetric about both the Ox axis and the Oy axis. We write Eqs. (8.20) in the form

$$\omega^{\bullet}_{j-6} - \sum_{i=7}^{8} c^{\bullet}_{ji}\omega_{i-6} = R^{\bullet}_{j-6} \qquad (j = 7, 8), \tag{8.21}$$

where

$$c^{\bullet}_{ji} = \frac{1}{\tilde{a}_{jj}}\left[\tilde{c}_{ji} + \frac{1}{k_{yy}}\tilde{b}_{2j}\left(\frac{S_y^{\bullet}}{I_x^{\bullet}}\tilde{b}_{6i} - \tilde{b}_{2i}\right) - \frac{1}{I_x^{\bullet}}\tilde{b}_{6j}\tilde{b}_{6i}\right],$$

$$R^{\bullet}_{j-6} = \frac{1}{\tilde{a}_{jj}}\left[\tilde{b}_{2j}\frac{Q_y}{Gk_{yy}} + \left(\tilde{b}_{6j} - \frac{S_y^{\bullet}}{k_{yy}}\tilde{b}_{2j}\right)\frac{M_z}{GI_x^{\bullet}}\right]. \tag{8.22}$$

The characteristic equation of system (8.21) has the form

$$r^4 - (c^{\bullet}_{77} + c^{\bullet}_{88})r^2 - c^{\bullet}_{78}c^{\bullet}_{87} + c^{\bullet}_{77}c^{\bullet}_{88} = 0, \tag{8.23}$$

where

$$r_{1,2,3,4} = \pm\sqrt{\frac{1}{2}\left[c^{\bullet}_{77} + c^{\bullet}_{88} \pm \sqrt{(c^{\bullet}_{77} - c^{\bullet}_{88})^2 + 4c^{\bullet}_{78}c^{\bullet}_{87}}\right]}. \tag{8.24}$$

The general solution of the homogeneous system corresponding to (8.21) is readily seen to have the form

$$\omega_1^0(Z) = C_1e^{k_1Z} + C_2e^{-k_1Z} + C_3e^{k_2Z} + C_4e^{-k_2Z},$$

$$\omega_2^0(Z) = \frac{c^{\bullet}_{87}}{k_1^2 - c^{\bullet}_{88}}(C_1e^{k_1Z} + C_2e^{-k_1Z}) + \frac{c^{\bullet}_{87}}{k_2^2 - c^{\bullet}_{88}}(C_3e^{k_2Z} + C_4e^{-k_2Z}), \tag{8.25}$$

where

$$k_1 = \sqrt{\frac{1}{2}\left[c^{\bullet}_{77} + c^{\bullet}_{88} + \sqrt{(c^{\bullet}_{77} - c^{\bullet}_{88})^2 + 4c^{\bullet}_{78}c^{\bullet}_{87}}\right]},$$

$$k_2 = \sqrt{\frac{1}{2}\left[c^{\bullet}_{77} + c^{\bullet}_{88} - \sqrt{(c^{\bullet}_{77} - c^{\bullet}_{88})^2 + 4c^{\bullet}_{78}c^{\bullet}_{87}}\right]}. \tag{8.26}$$

A particular solution of system (8.21), the method of variation of arbitrary constants being used, will be represented in the form

$$\omega_1^*(Z)=\frac{1}{p_2-p_1}\left\{\frac{1}{k_1}\int_{Z_0}^{Z}[p_2R_1^*(\xi)-R_2^*(\xi)]\,\mathrm{sh}\,k_1(Z-\xi)\,d\xi+\right.$$

$$\left.+\frac{1}{k_2}\int_{Z_0}^{Z}[R_2^*(\xi)-p_1R_1^*(\xi)]\,\mathrm{sh}\,k_2(Z-\xi)\,d\xi\right\}, \tag{8.27}$$

$$\omega_2^*(Z)=\frac{1}{p_2-p_1}\left\{\frac{p_1}{k_1}\int_{Z_0}^{Z}[p_2R_1^*(\xi)-R_2^*(\xi)]\,\mathrm{sh}\,k_1(Z-\xi)\,d\xi+\right.$$

where

$$\left.+\frac{p_2}{k_2}\int_{Z_0}^{Z}[R_2^*(\xi)-p_1R_1^*(\xi)]\,\mathrm{sh}\,k_2(Z-\xi)\,d\xi\right\},$$

$$p_1=\frac{c_{47}^*}{k_1^2-c_{44}^*},\quad p_2=\frac{c_{47}^*}{k_2^2-c_{44}^*}. \tag{2.28}$$

Introducing the general solution of system (8.21) in the form

$$\omega_1=\omega_1^0+\omega_1^*,\quad \omega_2=\omega_2^0+\omega_2^*,$$

into the right-hand sides of (8.16) and (8.17), and also using expressions (8.18) and (6.99), one can easily find all the components of the law of plane sections. At the same time, it should be remembered that in the expression

$$\Xi_y=\eta_y'-Z\vartheta_x \tag{8.29}$$

the function $\eta_y(Z)$ constitutes the displacement, in the direction of the axis, of a point rigidly bound to the cross section Z=const and coinciding up to the deformation with the origin of the fundamental system of Cartesian coordinates.

Let $\eta_y^*(Z)$ be the vertical displacement of the point of intersection of the cross sectional plane with the Oz axis. It is easy to see that

$$\eta_y^*=\eta_y-Z\vartheta_x. \tag{8.30}$$

In view of (8.30), we obtain from (8.29)

$$\Xi_y = \eta_y^{*\prime} + \vartheta_x. \tag{8.31}$$

Using (8.31), instead of the function $\eta_y(Z)$ we find the function $\eta_y^*(Z)$, which constitutes the ordinary deflection of the wing axis.

We have considered a simplified solution for a cross section of general type. It should be noted that this solution will be more rigorous for shells with a cross section symmetric about the Ox axis. In this case, the equilibrium conditions of the cut-off portion

$$\sum X = \sum Z = \sum \mathfrak{M}_y = 0 \tag{8.32}$$

are fulfilled identically if the external load is such that

$$N_z = Q_x = M_y = 0. \tag{8.33}$$

Strictly speaking, however, in the general case, equilibrium conditions (8.32) are not fulfilled. However, for the cross sections of wings and at typical loads for the latter, this imbalance is slight and has practically no effect on the results.

Let us now consider the case in which the cross section is symmetric about both the Ox axis and the Oy axis. It is easy to see that as a result of symmetry we have

$$\begin{aligned} &S_y^* = 0, \\ &\bar{b}_{3i}|_{i=4} = \bar{b}_{3j}|_{j=4} = 0, \\ &\bar{b}_{6i}|_{i=7} = \bar{b}_{6j}|_{j=7} = 0, \\ &\bar{c}_{78} = \bar{c}_{87} = 0. \end{aligned} \tag{8.34}$$

In the case under consideration, the approximating functions represented by general expressions (8.8), (8.9), (8.11) are

$$\varphi_1^1(S)=x^2(S)\,y(S)+k_{1y}y(S),$$
$$\varphi_2^1(S)=x(S)\,y(S). \tag{8.35}$$

In view of (8.34), system (8.20) takes the form

$$\begin{aligned}&\tilde{a}_{77}\omega_1''-\left(\tilde{c}_{77}-\frac{1}{F_{yy}^{*}}\tilde{b}_{27}^{2}\right)\omega_1=\tilde{b}_{27}\frac{Q_y}{GF_{yy}^{*}},\\&\tilde{a}_{88}\omega_2''-\left(\tilde{c}_{88}-\frac{1}{I_z^{*}}\tilde{b}_{68}^{2}\right)\omega_2=\tilde{b}_{68}\frac{M_z}{GI_z^{*}}.\end{aligned} \tag{8.36}$$

Hence

$$\begin{aligned}&\omega_1(Z)=C_1e^{k_1Z}+C_2e^{-k_1Z}+\frac{\tilde{b}_{27}}{\tilde{a}_{77}k_1GF_{yy}^{*}}\int_{Z_0}^{Z}Q_y(\xi)\,\mathrm{sh}\,k_1(Z-\xi)\,d\xi,\\&\omega_2(Z)=C_3e^{k_2Z}+C_4e^{-k_2Z}+\frac{\tilde{b}_{68}}{\tilde{a}_{88}k_2GI_z^{*}}\int_{Z_0}^{Z}M_z(\xi)\,\mathrm{sh}\,k_2(Z-\xi)\,d\xi,\end{aligned} \tag{8.37}$$

where

$$\begin{aligned}&k_1=\sqrt{\frac{1}{\tilde{a}_{77}}\left(\tilde{c}_{77}-\frac{1}{F_{yy}^{*}}\tilde{b}_{27}^{2}\right)},\\&k_2=\sqrt{\frac{1}{\tilde{a}_{88}}\left(\tilde{c}_{88}-\frac{1}{I_z^{*}}\tilde{b}_{68}^{2}\right)}.\end{aligned} \tag{8.38}$$

In view of (8.34), relations (8.16), (8.17) become

$$\begin{aligned}&\Xi_y=\frac{1}{F_{yy}^{*}}\left(\frac{Q_y}{G}-\tilde{b}_{27}\omega_1\right),\\&\theta_z'=\frac{1}{I_z^{*}}\left(\frac{M_z}{G}-\tilde{b}_{68}\omega_2\right).\end{aligned} \tag{8.39}$$

Integrating relations (8.18), (8.39) and taking (8.31) into consideration, we find

$$\begin{aligned}&\theta_x(Z)=\theta_x(0)+\frac{(1-\nu)}{2EI_x}\int_0^{Z}M_x(\xi)\,d\xi,\\&\eta_y^{*}(Z)=\eta_y^{*}(0)-\theta_x(0)\,Z+\frac{1}{G}\int_0^{Z}\left[\frac{1}{F_{yy}^{*}}Q_y(\zeta)-\right.\\&\qquad\left.-\frac{1-\nu}{2I_x}\int_0^{\zeta}M_x(\xi)\,d\xi\right]d\zeta-\tilde{b}_{27}\int_0^{Z}\omega_1(\xi)\,d\xi,\\&\theta_z(Z)=\theta_z(0)+\frac{1}{GI_z^{*}}\int_0^{Z}M_z(\xi)\,d\xi-\frac{\tilde{b}_{68}}{I_z^{*}}\int_0^{Z}\omega_2(\xi)\,d\xi.\end{aligned} \tag{8.40}$$

It follows from expressions (8.37) and (8.40) that for shells with a cross section symmetric about the Ox and Oy axes, bending becomes separate from torsion.

Let us consider a shell with a rectangular cross section (see Figure 8.2).

Computing the quadratures, in view of (8.35), we can readily obtain

$$
\begin{aligned}
&I_x = \frac{d_2 d_1^2}{2} h_2 + \frac{d_1^3}{6} h_1, \\
&F^*_{ww} = 2 d_1 h_1, \\
&I^*_s = \frac{d_1 d_2}{2} (d_2 h_1 + d_1 h_2).
\end{aligned}
\tag{8.41}
$$

$$
\begin{aligned}
&k_{1w} = -\frac{1}{I_s} \frac{d_1 d_2^2}{24} (d_1 h_1 + d_2 h_2), \\
&\tilde{a}_{77} = \frac{2}{1-\nu} \frac{d_1^2 d_2^4}{32} \left(\frac{d_1}{3} h_1 + \frac{d_2}{5} h_2 \right) - \frac{1-\nu}{2} \frac{\tilde{a}_{s8}^2}{I_s}, \\
&\tilde{a}_{88} = \frac{2}{1-\nu} \frac{d_1^2 d_2^2}{24} (d_1 h_1 + d_2 h_2), \\
&\tilde{b}_{77} = d_1 h_1 \left(\frac{d_2^2}{2} + 2k_{1w} \right), \\
&\tilde{b}_{88} = \frac{d_1 d_2}{2} (d_2 h_1 - d_1 h_2), \\
&\tilde{c}_{77} = \frac{d_1^2 d_2^3}{6} h_2 + 2d_1 \left(\frac{d_2^2}{4} + k_{1w} \right)^2 h_1, \\
&\tilde{c}_{88} = I^*_s.
\end{aligned}
$$

Introducing the values of coefficients (8.41) into general expressions (8.37), (8.40), we can further obtain all the necessary formulas describing the stressed and strained state of the shell. Some results corresponding to these formulas are shown in Figures 8.6-8.8 in the form of curves for n*=2 for bending and n*=1 for torsion. The calculations show that for straight wings, as a first approximation, only the torsional warping can be considered. In this connection, it appears useful to write the solution for a straight-wing type shell with an arbitrary profile, retaining only one approximating power function (8.8) corresponding to torsion.

We obtain for this case

$$\Xi_y = \frac{1}{k_{yy}}\left[\frac{Q_y}{G} - \frac{S_y^*}{I_z^*}\frac{M_z}{G} - \left(\tilde{b}_{y\omega} - \frac{S_y^*}{I_z^*}\tilde{b}_{z\omega}\right)\omega\right],$$

$$\theta_z' = \frac{1}{I_z^*}\left(\frac{M_z}{G} + S_y^*\Xi_y - \tilde{b}_{z\omega}\omega\right), \qquad (8.42)$$

$$\theta_x' = \frac{1-\nu}{2GI_x}M_x,$$

$$-\tilde{b}_{y\omega}\Xi_y - \tilde{b}_{z\omega}\theta_z' + \tilde{a}_{\omega\omega}\omega'' - \tilde{c}_{\omega\omega}\omega = 0. \qquad (8.43)$$

Excluding Σ_y and θ_z' from Eq. (8.43), we obtain

$$\omega'' - k^2\omega = R^*, \qquad (8.44)$$

where

$$k^2 = \frac{1}{\tilde{a}_{\omega\omega}}\left[\tilde{c}_{\omega\omega} + \frac{1}{k_{yy}}\tilde{b}_{y\omega}\left(\frac{S_y^*}{I_z^*}\tilde{b}_{z\omega} - \tilde{b}_{y\omega}\right) - \frac{1}{I_z^*}\tilde{b}_{z\omega}^2\right],$$

$$R^* = \frac{1}{\tilde{a}_{\omega\omega}}\left[\tilde{b}_{y\omega}\frac{Q_y}{Gk_{yy}} + \left(\tilde{b}_{z\omega} - \frac{S_y^*}{k_{yy}}\tilde{b}_{y\omega}\right)\frac{M_z}{GI_z^*}\right]. \qquad (8.45)$$

From (8.44) we get

$$\omega(Z) = C_1e^{kZ} + C_2e^{-kZ} + \frac{1}{k}\int_{Z_0}^{Z} R^*(\zeta)\,\mathrm{sh}\,k(Z-\zeta)\,d\zeta. \qquad (8.46)$$

For the warping obtained w(Z), the components of the law of plane sections $\theta_x(Z)$, $\eta_y^*(Z)$, $\theta_z(Z)$ are determined by expressions (8.40).

The general solutions obtained for the differential resolvents contain arbitrary constants subject to determination from the boundary conditions on the ends.

If the displacements or the external load on the ends are given, the boundary conditions are correspondingly formulated in the form of equality, on the ends of the shell, of the generalized displacements or generalized forces (corresponding to the arpings) and the specified ones.

If an end of the shell is elastically fastened to the end of another shell, the boundary conditions are formulated in the form of equality on this end of the generalized displacements and the generalized forces for both shells.

4. Simplified Analysis of Shells of Variable Cross Section

Let us consider a shell with a cross section symmetrical about both the Ox axis and the Oy axis (Figure 8.22). In this case, the line of centroids is perpendicular to the planes of the ends, so that such a shell according to the terminology adopted above is a slightly conical straight shell. Shells with a cross-sectional profile asymmetric about the Oy a^vis generally belong to the class of slightly oblique shells and will be discussed in the next section.

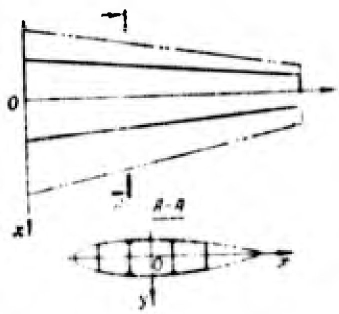


Figure 8.22. Slightly conical caisson with two axes of symmetry.

All the relations necessary for the analysis of a slightly conical straight shell can be readily obtained from more general relations for slightly oblique shells, given in Section 7.2, by setting x0=π/2.

The coordinate functions corresponding to the law of plane sections, in the case of bending in the vertical plane and torsion, will be

i	V_i	φ_{im_z}	φ_{in_z}	φ_{in_n}
2	η_ψ	$-\frac{y_0}{H}$	$\dot{y}_0$	$\dot{x}_0$
4	$\left(1-\frac{Z}{l_0}\right)\theta_x$	y_0	0	0
6	$\left(1-\frac{Z}{l_0}\right)\theta_z$	0	$x_0\dot{y}_0 - \dot{x}_0 y_0$	$x_0\dot{x}_0 + y_0\dot{y}_0$

(8.47)

The approximating functions corresponding to bending and torsion, as in the case of a shell of constant cross section, will be specified as in (8.35):

$$\varphi_1^1(S)=x_0^1(S)y_0(S)+k_{11}y_0(S), \tag{8.48}$$

$$k_{11}=-\frac{\oint x_0^1 y_0^1 h\,dS}{\oint y_0^2 h\,dS}. \tag{8.49}$$

$$\varphi_2^1(S)=x_0(S)y_0(S).$$

Moreover, for the coefficients of differential resolvents, from (7.23) and taking the symmetry of the cross section into account, we will have

$\tilde{a}_{22}=\frac{2}{1-\nu}\frac{I_x}{H^2}+F_{yy}^{*}$	$\tilde{a}_{24}=-\frac{2}{1-\nu}\frac{I_x}{H}$	$\tilde{a}_{26}=0$	$\tilde{a}_{27}=0$	$\tilde{a}_{28}=0$
$\tilde{a}_{42}=-\frac{2}{1-\nu}\frac{I_x}{H}$	$\tilde{a}_{44}=\frac{2}{1-\nu}I_x$	$\tilde{a}_{46}=0$	$\tilde{a}_{47}=0$	$\tilde{a}_{48}=0$
$\tilde{a}_{62}=0$	$\tilde{a}_{64}=0$	$\tilde{a}_{66}=I_z^{*}$	$\tilde{a}_{67}=0$	$\tilde{a}_{68}=0$
$\tilde{a}_{72}=0$	$\tilde{a}_{74}=0$	$\tilde{a}_{76}=0$	$\tilde{a}_{77}=\frac{2}{1-\nu}\oint(\varphi_1^1)^2 h\,dS$	$\tilde{a}_{78}=0$
$\tilde{a}_{82}=0$	$\tilde{a}_{84}=0$	$\tilde{a}_{86}=0$	$\tilde{a}_{87}=0$	$\tilde{a}_{88}=\frac{2}{1-\nu}\oint(\varphi_2^1)^2 h\,dS$

(8.50)

$\tilde{b}_{22}=0$	$\tilde{b}_{24}=F_{yy}^{*}$	$\tilde{b}_{26}=0$	$\tilde{b}_{27}=\oint y_0'(\varphi_1^1)' h\,dS$	$\tilde{b}_{28}=0$
$\tilde{b}_{42}=0$	$\tilde{b}_{44}=0$	$\tilde{b}_{46}=0$	$\tilde{b}_{47}=0$	$\tilde{b}_{48}=0$
$\tilde{b}_{62}=0$	$\tilde{b}_{64}=0$	$\tilde{b}_{66}=\frac{I_z^{*}}{H}$	$\tilde{b}_{67}=0$	$\tilde{b}_{68}=\oint(x_0y_0'-x_0'y_0)\times(\varphi_2^1)' h\,dS$
$\tilde{b}_{72}=0$	$\tilde{b}_{74}=0$	$\tilde{b}_{76}=0$	$\tilde{b}_{77}=0$	$\tilde{b}_{78}=0$
$\tilde{b}_{82}=0$	$\tilde{b}_{84}=0$	$\tilde{b}_{86}=0$	$\tilde{b}_{87}=0$	$\tilde{b}_{88}=0$

(8.51)

$\tilde{c}_{72}=0$	$\tilde{c}_{74}=\oint(\varphi_1^1)' y_0' h\,dS$	$\tilde{c}_{76}=0$	$\tilde{c}_{77}=\oint(\varphi_1^1)'^2 h\,dS$	$\tilde{c}_{78}=0$
$\tilde{c}_{82}=0$	$\tilde{c}_{84}=0$	$\tilde{c}_{86}=\frac{1}{H}\oint(\varphi_2^1)'\times(x_0y_0'-x_0'y_0) h\,dS$	$\tilde{c}_{87}=0$	$\tilde{c}_{88}=\oint(\varphi_2^1)'^2 h\,dS$

(8.52)

In expressions (8.50)-(8.52), the quantities I_x, F^*_{yy}, I^*_z are determined from formulas (6.138), (6.139), (6.141) for the end section $Z=0$.

The differential resolvents (7.13), (7.14), in view of (8.50)-(8.52), have the form

$$\left(\frac{2}{1-\nu}\frac{I_x}{H^2}+F^*_{yy}\right)\left(1-\frac{Z}{H}\right)\eta_y'-\frac{2}{1-\nu}\frac{I_x}{H}\left(1-\frac{Z}{H}\right)\left[\left(1-\frac{Z}{H}\right)\theta_x\right]'+$$
$$+F^*_{yy}\left(1-\frac{Z}{H}\right)\theta_x+\tilde{b}_{27}\omega_1=\frac{1}{G}Q_y, \tag{8.53}$$

$$-\frac{2}{1-\nu}\frac{I_x}{H}\left(1-\frac{Z}{H}\right)\eta_y'+\frac{2}{1-\nu}I_x\left(1-\frac{Z}{H}\right)\left[\left(1-\frac{Z}{H}\right)\theta_x\right]'=\frac{1}{G}M_x, \tag{8.54}$$

$$I^*_z\left(1-\frac{Z}{H}\right)\left[\left(1-\frac{Z}{H}\right)\theta_z\right]'+\frac{I^*_z}{H}\left(1-\frac{Z}{H}\right)\theta_z+\tilde{b}_{88}\omega_2=\frac{1}{G}M_z, \tag{8.55}$$

$$\tilde{a}_{77}\left[\left(1-\frac{Z}{H}\right)\omega_1'\right]'-\tilde{b}_{27}\eta_y'-\tilde{c}_{74}\theta_x-\frac{\tilde{c}_{77}}{1-\frac{Z}{H}}\omega_1=0, \tag{8.56}$$

$$\tilde{a}_{88}\left[\left(1-\frac{Z}{H}\right)\omega_2'\right]'-\tilde{b}_{88}\left[\left(1-\frac{Z}{H}\right)\theta_z\right]'-\tilde{c}_{88}\theta_z-\frac{\tilde{c}_{88}}{1-\frac{Z}{H}}\omega_2=0. \tag{8.57}$$

Thus, the system of differential resolvents for the case at hand decomposes in the same way as for a straight wing of constant cross section. Equations (8.53), (8.54), (8.56) pertain to bending, and Eqs. (8.55), (8.57), to torsion.

Equations (8.53)-(8.57) have variable coefficients. Introducing the new independent variable

$$\ln\left(1-\frac{Z}{H}\right) \tag{8.58}$$

and setting

$$\left(1-\frac{Z}{H}\right)\theta_x=V_{\theta_x},\quad\left(1-\frac{Z}{H}\right)\theta_z=V_{\theta_z}, \tag{8.59}$$

we arrive at the equations with constant coefficients

$$-\frac{1}{H}\left(\frac{2}{1-\nu}\frac{I_x}{H^2}+F^*_{yy}\right)\eta_y'+\frac{2}{1-\nu}\frac{I_x}{H^2}V_{\theta_x}'+F^*_{yy}V_{\theta_x}+\tilde{b}_{27}\omega_1=\frac{Q_y}{G}, \tag{8.60}$$

$$\frac{2}{1-\nu}\frac{I_x}{H^2}\eta_y'-\frac{2}{1-\nu}\frac{I_x}{H}V_{\theta_x}'=\frac{M_x}{G}, \tag{8.61}$$

$$-\frac{I^*_z}{H}V_{\theta_z}'+\frac{I^*_z}{H}V_{\theta_z}+\tilde{b}_{88}\omega_2=\frac{M_z}{G}, \tag{8.62}$$

$$\frac{\tilde{a}_{77}}{H^2}\omega_1''+\frac{\tilde{b}_{27}}{H}\eta_y'-\tilde{c}_{74}V_{\theta_x}-\tilde{c}_{77}\omega_1=0, \tag{8.63}$$

$$\frac{\tilde{a}_{88}}{H^2}\omega_2''+\frac{\tilde{b}_{88}}{H}V_{\theta_z}'-\tilde{c}_{88}V_{\theta_z}-\tilde{c}_{88}\omega_2=0. \tag{8.64}$$

On the basis of (8.51) and (8.52), we have

$$\tilde{b}_{87}=\tilde{c}_{78},\quad \tilde{b}_{88}=H\tilde{c}_{88}. \tag{8.65}$$

Using relations (8.65), we can reduce Eqs. (8.60)-(8.64) to the form

$$\begin{gathered}
HF^{\circ}_{yy}\Sigma_y+H\tilde{b}_{87}\omega_1=H\frac{Q_y}{G}+\frac{M_x}{G},\\
\frac{2}{1-\nu}\frac{I_x}{H}\left(\frac{1}{H}\eta^{\circ}_y-V_{\theta_x}\right)'=\frac{M_x}{G},\\
\frac{\tilde{a}_{77}}{H^2}\omega_1''-\tilde{b}_{87}\Sigma_y-\tilde{c}_{77}\omega_1=0,
\end{gathered} \tag{8.66}$$

$$\begin{gathered}
\frac{I^{\circ}_z}{H}\left(-V'_{\theta_z}+V_{\theta_z}\right)+\tilde{b}_{88}\omega_2=\frac{M_z}{G},\\
\frac{\tilde{a}_{88}}{H^2}\omega_2''+\frac{1}{H}\tilde{b}_{88}\left(V'_{\theta_z}-V_{\theta_z}\right)-\tilde{c}_{88}\omega_2=0,
\end{gathered} \tag{8.67}$$

where

$$\Sigma_y=-\frac{1}{H}\eta^{\circ\prime}_y+V_{\theta_x}. \tag{8.68}$$

The systems of Eqs. (8.66), (8.67) are easily integrated. Eliminating the unknown Σ_y from the first and third equations of system (8.66), and the difference $V'_{\theta_z} - V_{\theta_z}$ from system (8.67), we have

$$\frac{\tilde{a}_{77}}{H^2}\omega_1''-\left(\tilde{c}_{77}-\frac{\tilde{b}^2_{27}}{F^{\circ}_{yy}}\right)\omega_1=\frac{\tilde{b}_{27}}{GF^{\circ}_{yy}}\left(Q_y+\frac{M_x}{H}\right). \tag{8.69}$$

$$\frac{\tilde{a}_{88}}{H^2}\omega_2''-\left(\tilde{c}_{88}-\frac{\tilde{b}^2_{88}}{I^{\circ}_z}\right)\omega_2=\tilde{b}_{88}\frac{M_z}{GI^{\circ}_z}. \tag{8.70}$$

It is of interest to compare Eqs. (8.69) and (8.70) with analogous Eqs. (8.36) for a straight shell. These equations are in fact identical when the transverse force Q_y is replaced by the quantity $Q^*_y=Q_y+M_x/H$. It is easy to see that Q^*_yH is the moment of the external forces, applied to the cut-off portion of the shell, about the cone apex.

From (8.69) and (8.70), we have

$$\omega_1(Z)=C_1e^{k_1Z}+C_2e^{-k_1Z}+\frac{\tilde{b}_{27}H^2}{\tilde{a}_{77}k_1GF^{\circ}_{yy}}\int\limits_{Z_0}^{Z}\left[Q_y(\xi)+\frac{M_x(\xi)}{H}\right]\operatorname{sh}k_1(Z-\xi)d\xi,$$

$$\omega_2(Z)=C_3e^{k_2Z}+C_4e^{-k_2Z}+\frac{\tilde{b}_{84}H^2}{\tilde{a}_{88}k_2GI^{\circ}_z}\int\limits_{Z_0}^{Z}M_z(\xi)\operatorname{sh}k_2(Z-\xi)d\xi. \qquad (8.71)$$

where

$$k_1=H\sqrt{\frac{1}{\tilde{a}_{77}}\left(\tilde{c}_{77}-\frac{1}{F^{\circ}_{yy}}\tilde{b}^2_{27}\right)},$$

$$k_2=H\sqrt{-\frac{1}{\tilde{a}_{88}}\left(\tilde{c}_{88}-\frac{1}{I^{\circ}_z}\tilde{b}^2_{84}\right)}. \qquad (8.72)$$

It is now easy to determine the components of the law of plane sections. From the two equations of system (8.66), in view of (8.68), we have

$$\Xi_y=-\frac{\tilde{b}_{27}}{F^{\circ}_{yy}}\omega_1+\frac{Q_y+\frac{1}{H}M_x}{GF^{\circ}_{yy}},$$

$$-V'_{\theta_x}+V_{\theta_x}-\Xi_y=(1-\nu)H\frac{M_x}{2GI_x}. \qquad (8.73)$$

whence, eliminating Σ_y, we obtain

$$V'_{\theta_x}-V_{\theta_x}=\frac{\tilde{b}_{27}}{F^{\circ}_{yy}}\omega_1-\frac{Q_y+\frac{1}{H}M_x}{GF^{\circ}_{yy}}-(1-\nu)H\frac{M_x}{2GI_x}. \qquad (8.74)$$

Integrating (8.74), we find

$$V_{\theta_x}(t)=V_{\theta_x}(0)e^t+e^t\int\limits_0^t\left[\frac{\tilde{b}_{27}}{F^{\circ}_{yy}}\omega_1(\xi)-\frac{Q_y(\xi)+\frac{1}{H}M_x(\xi)}{GF^{\circ}_{yy}}-\right.$$
$$\left.-(1-\nu)H\frac{M_x(\xi)}{2GI_x}\right]e^{-\xi}d\xi. \qquad (8.75)$$

From the first equation of system (8.73), in view of (8.68), we have

$$\eta_y^{\prime}=HV_{\theta_x}-\frac{HQ_y+M_x}{GF^{\circ}_{yy}}+H\frac{\tilde{b}_{27}}{F^{\circ}_{yy}}\omega_1. \qquad (8.76)$$

whence, in view of (8.75),

$$\eta_y^*(t) = \eta_y^*(0) + HV_{\theta_x}(0)(e^t - 1) - \frac{1}{GF_{yy}^*}\int_0^t [HQ_y(\xi) + M_z(\xi)]\,d\xi +$$

$$+ H\frac{\tilde{b}_{yy}}{F_{yy}^*}\int_0^t \omega_1(\xi)\,d\xi + \int_0^t e^{\zeta}\int_0^{\zeta}\Big[\frac{\tilde{b}_{yy}}{F_{yy}^*}\omega_1(\xi) - \frac{Q_y(\xi) + \frac{1}{H}M_x(\xi)}{GF_{yy}^*} -$$

$$-(1-\nu)H\frac{M_x(\xi)}{2GI_x}\Big]e^{-\xi}\,d\xi\,d\zeta. \tag{8.77}$$

From the first equation of system (8.67), we find

$$V_{\theta_x}(t) = V_{\theta_x}(0)e^t + H\frac{\tilde{b}_{xx}}{I_x^*}e^t\int_0^t \omega_2(\xi)e^{-\xi}\,d\xi - \frac{H}{GI_x^*}e^t\int_0^t M_x(\xi)e^{-\xi}\,d\xi. \tag{8.78}$$

We have thus obtained the general solution (8.71), (8.75), (8.77), (8.78) of the system of differential resolvents (8.53)-(8.57). Now, satisfying the boundary conditions, we can obtain the working formulas for the various cases of fixing and loading.

If the shell has a rectangular cross section, all the coefficients of the resolvents are determined by expressions (8.41), where d_1, d_2 are the dimensions of the end cross section Z=0. Some results for this case are given in Figures 8.14-8.17 in the form of curves for n*=2 for bending and n*=1 for torsion.

8.2. Swept-Wing Type Shells

1. Shells of Constant Cross Section

The resolvents and other relations necessary for the analysis of an oblique cylindrical shell of constant cross section (Figure 6.1) were obtained in Chapter VI from general expressions for a conical shell of arbitrary configuration.*

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*For a simplified consideration of sweepback, see I.F. Obraztsov, Analysis of Swept-Wing Caisson-Type Shells on the Basis of V.Z. Vlasov's Theory. Trudy MAI, No. 59. Moscow, Oborongiz, 1956; I.F. Obraztsov, Some Aspects of Strength Analysis of Thin-Walled Aircraft Structures. Moscow, Oborongiz, 1957.

We will use numerical examples to analyze the character of convegence of the expansions corresponding to warping of the contour Z=const. As in the preceding section, we will consider expansions in power functions as well as piecewise-linear local coordinate functions.

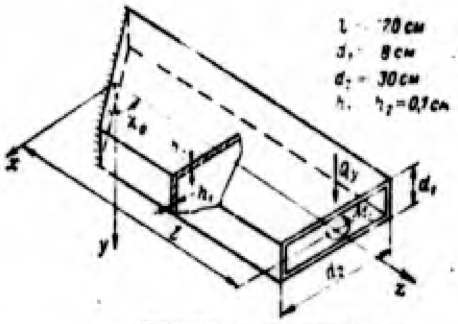
We superpose the Ox axis on the neutral axis of the cross section. As we know, the stressed state of the cantilever portion of a swept wing type shell with the exception of a region adjacent to the root triangle is identical to the stressed state of a straight wing. Therefore, on the basis of relation (8.1), characteristic of wings, the major axis of inertia of the cross section, about which the amount of inertia I_x will be minimum, can be taken with a high degree of accuracy as the neutral axis for any loads typical of wings.

A specific feature of oblique shells of swept-wing type consists in the fact that for any load characteristic of wings, both a bending moment and a twisting moment are set up in the section Z=const. Therefore, in such shells, bending is always associated with torsion, so that strictly speaking, the concept of the rigidity axis of the wing has no meaning. In this connection, in analyzing a swept wing on the basis of the six generalized displacements corresponding to the displacement of the contour Z=const as a solid body, it is necessary to retain η_y, θ_x, θ_z, by adding to them the three remaining ones η_z, η_x, θ_y in the case of an asymmetric profile, and also when the external load does not meet conditions (8.33).

As the coordinate functions corresponding to warpings of the contour Z=const, one can obviously select power function (3.309) as well as piecewise-linear local functions (3.310). Moreover, in selecting the Ox axis in the above-indicated manner, we will specify the power coordinate functions in the form (8.2), and the piecewise-linear functions, so that they satisfy condition (8.3).

Given below are the results of comparative analyses of swept caissons of rectangular cross section (Figure 8.23) with different sweep angles for the action of a force Q_y and moment M_z, applied to the end $Z=Z_1$. The calculations were carried out according to a universal program written for a conical shell of arbitrary configuration on the basis of the fundamental static-geometric model.

Figure 8.24 shows curves of coordinate functions $\phi i m_z$, $\phi i n_z$ corresponding to the displacement of the section Z=const as a solid body in bending and torsion. The diagrams of the power and piecewise-linear local coordinate functions $\phi_k^1(S)$ for the case at hand coincide with the diagrams of the coordinate functions $\phi_k^1(S)$ for a straight caisson, as shown in Figures 8.4 and 8.5.



	I	II	III	IV
χ_0	60°	45°	30°	20°

Figure 8.23. Swept caisson of constant rectangular cross section.

Figure 8.25 shows diagrams of generalized coordinates of the stressed state ψ, $\bar{\psi}$, corresponding to displacements of the contour Z=const as a solid body and to its warpings.

Figures 8.26-8.29 show the results of calculations of swept caissons, illustrated in Figure 8.23, for different numbers n of retained coordinate functions $\phi_k^1(S)$. Here, as before, n* is the number of retained power functions ϕ_k^1; n** is the number of retained local functions ϕ_k^1.

As an illustration, Figure 8.30 shows the results of analysis of a shell of swept-wing type IV (see Figure 8.9) for the action of an air load.

Figures 8.31 and 8.32 compare the results of the analysis with experimental data known from the literature.

2. Shells of Variable Cross Section

The resolvents and other relations necessary for the analysis of a slightly conical oblique shell of variable cross section are given in Section 7.1. We will analyze the character of convergence of the expansions corresponding to warping of the contour Z=const by using numerical examples. Again, we will consider expansions in power functions as well as local coordinate functions, as in Subsection 1. Given below are results of comparative analyses of slightly conical swept caissons of rectangular cross section (Figure 8.33) with different sweep angles for the action of force Q_y and moment M_z, applied at the end section $Z=Z_1$. As in all the preceding subsections, the calculations were carried out according to the universal program for a conical shell of arbitrary configuration.

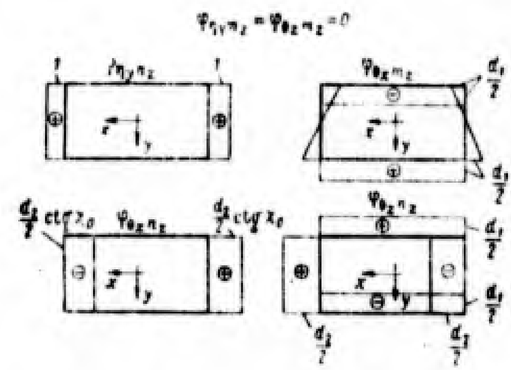


Figure 8.24. Coordinate functions corresponding to the law of plane sections.

Superposing the Ox axis on the neutral axis of the cross section, we must, now, as in the preceding case, retain the three generalized displacements η_y, θ_x, θ_z corresponding to displacement of the contour Z=const as a solid body. Diagrams of the corresponding coordinate functions, plotted by considering the assumptions related to the slight conicity, are shown in Figure 8.34. Diagrams of the power and local coordinate functions ϕ_k^1, as before, coincide with the analogous diagrams for a straight caisson, shown in Figures 8.4 and 8.5. Diagrams of generalized coordinates of the stressed state ψ, $\overline{\psi}$, plotted by considering the assumptions related to slight conicity, are shown in Figure 8.35.

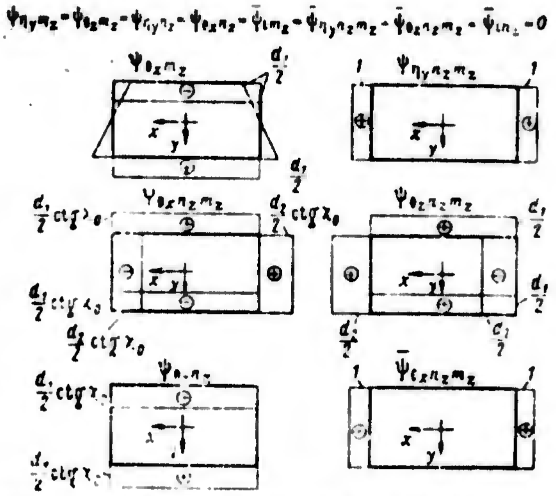


Figure 8.25. Generalized coordinates of stressed state, corresponding to the law of plane sections.

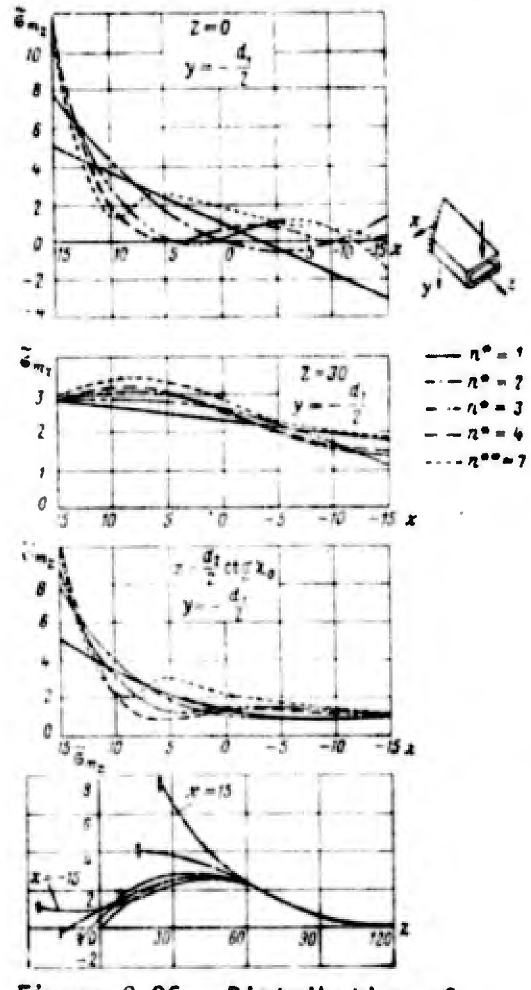


Figure 8.26. Distribution of normal stresses in a swept caisson. (χ_0=30°)

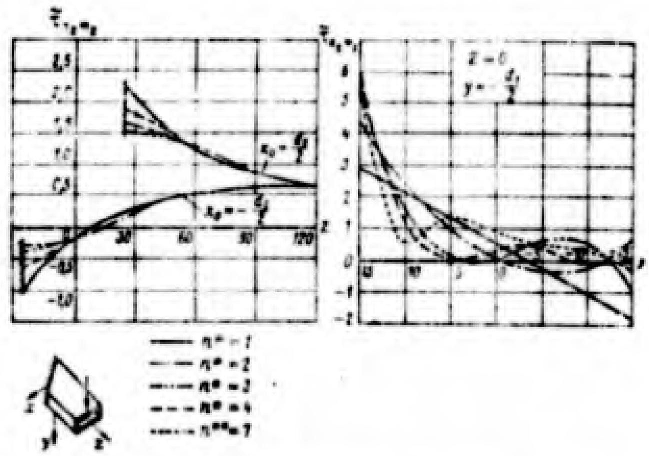


Figure 8.27. Distribution of tangential stresses in a swept caisson. (χ_0=30°)

Figures 8.36-8.39 give the results of analyses of slightly conical swept caissons shown in Figure 8.33 for different numbers of retained coordinate functions ϕ_k^1.

Figure 8.40 illustrates the results of analyses of a shell of swept wing type V (see Figure 8.9) for the action of an air load.

3. Simplified Analysis of Shells of Constant Cross Section with Ribs Perpendicular to the Generatrices

Let a load applied to the surface of the shell be represented by a normal pressure distributed in accordance with an arbitrary law. In addition, some system of external forces can be applied to the end $Z=Z_1$. As was noted above, in a swept wing, at any load, bending is associated with torsion. In this connection, in approximating the warping of the section Z=const by only one function, we must give preference to function $\phi_2^1(S)$, determined by expression (8.8), which corresponds to torsion, since, as we have ascertained on the example of straight shells, torsional warping for not-too-short shells prevails over flexural warping. Thus, by superposing the Ox and Oy axes on the principal central axes of the cross section, we will retain four generalized displacements: η_y, θ_x, θ_z, w. We will examine shells with

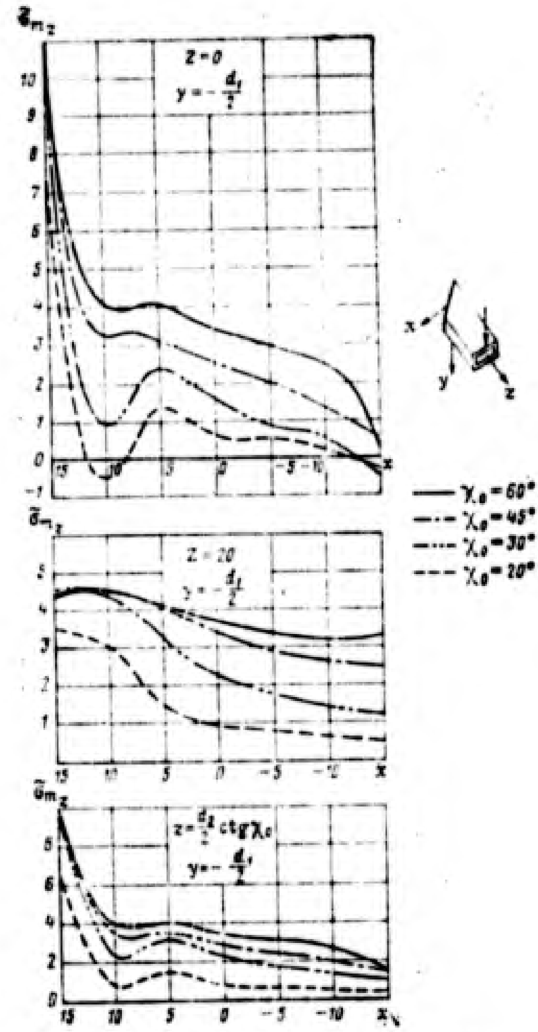


Figure 8.28. Distribution of normal stresses in sections of swept caissons at different sweep angles (n**=7).

symmetric profiles, and also shells similar enough so that the quadratures in the coefficients of differential resolvents, which become exactly zero for a symmetric profile, can be assumed equal to zero without any significant errors in the final result.

The resolvent equations in the desired generalized displacements can be readily obtained from general canonical relations (6.106)-(6.108).

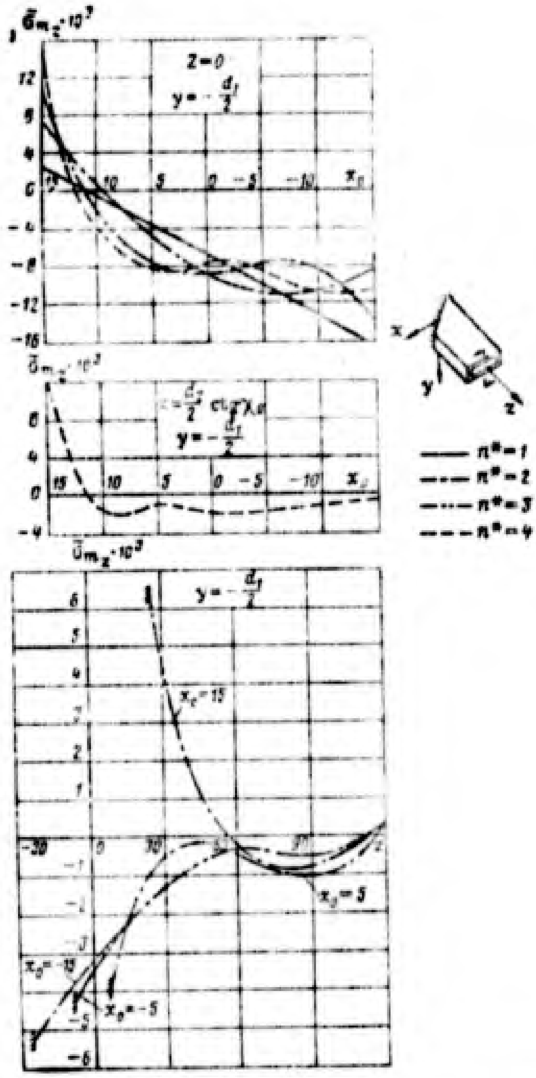


Figure 8.29. Distribution of normal stresses in a swept caisson ($_{\chi}0=30°$) acted on by a twisting moment.

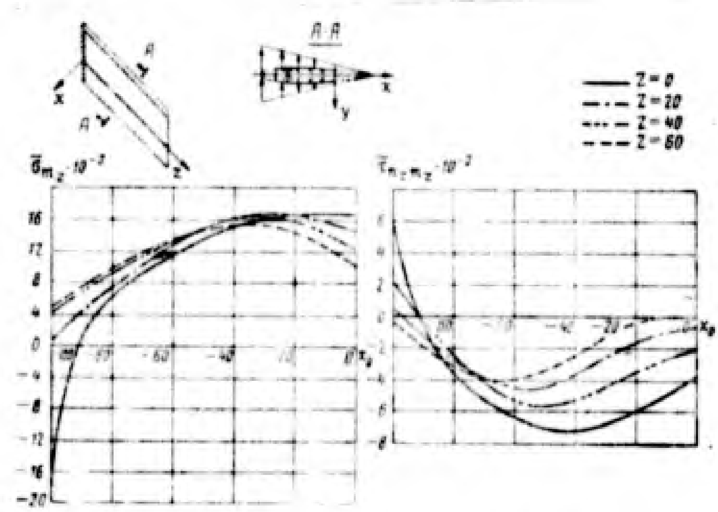


Figure 8.30. Distribution of normal and tangential stresses in a swept wing with curved contour ($n^{**}=7$).

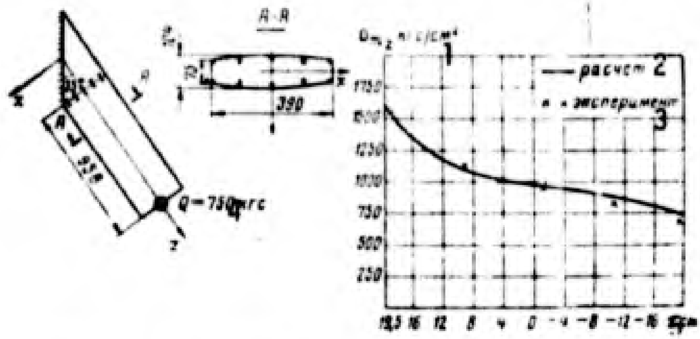


Figure 8.31. Comparison of results of calculation with A.F. Feofanov's experiment. Key: 1) kgs/cm^2, 2) calculated, 3) experimental, 4) kgs, 5) cm.

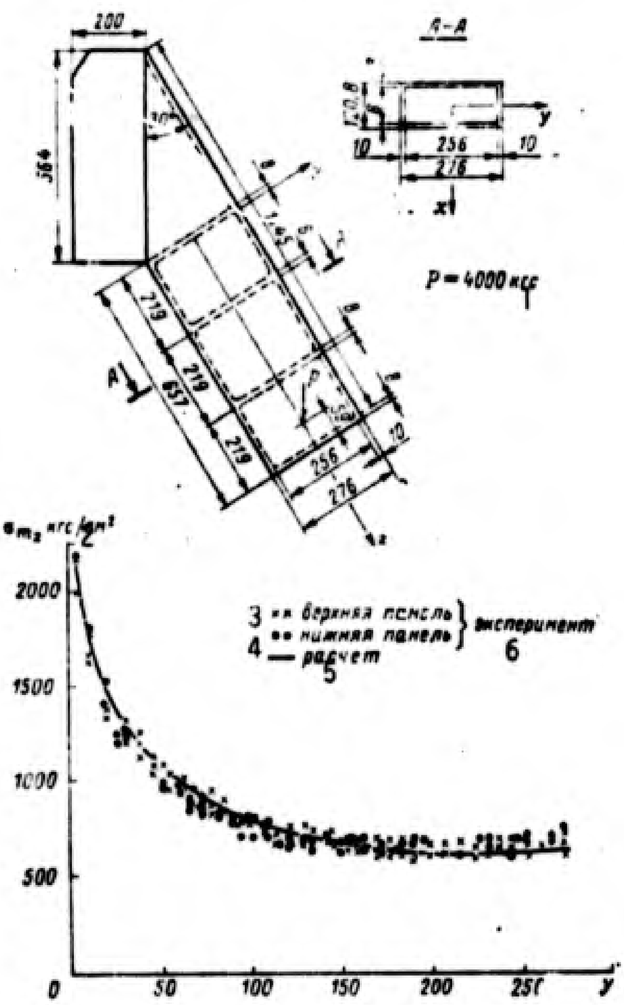


Figure 8.32. Comparison of calculated and experimental results. Key: 1) kgs, 2) kgs/cm^2, 3) upper panel, 4) lower panel, 5) calculated, 6) experimental.

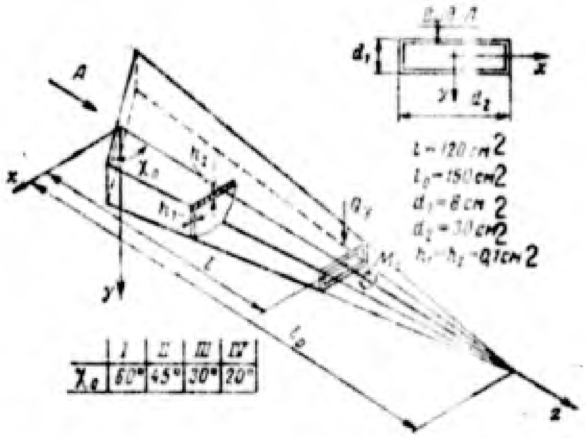


Figure 8.33. Slightly conical oblique caisson of rectangular cross section. Key: 1) view A, 2) cm.

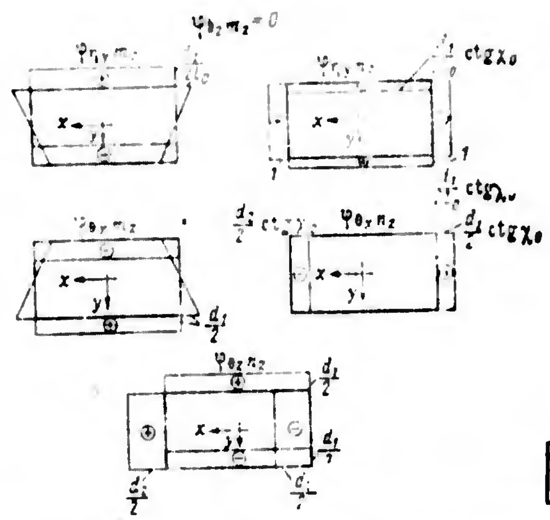


Figure 8.34. Coordinate functions corresponding to the law of plane sections.

Expanding expressions (6.15) and (6.16), we obtain with the aid of formulas (6.8)

i	U_i	ψ_{im_z}	$\psi_{in_zm_z}$	ψ_{in_z}
1	η_x	0	$\dfrac{x'}{\sin\chi}$	$-\dfrac{\cos\chi}{\sin^2\chi}x'$
2	η_y	0	$\dfrac{y'}{\sin\chi}$	$-\dfrac{\cos\chi}{\sin^2\chi}y'$
3	η_z	1	$-\operatorname{ctg}\chi$	0
4	θ_x	y	$-y\operatorname{ctg}\chi-\dfrac{\operatorname{ctg}\chi_0}{\sin\chi}xy'$	$\dfrac{\cos\chi\operatorname{ctg}\chi_0}{\sin^2\chi}xy'$
5	θ_y	$-x$	$2x\operatorname{ctg}\chi$	$-x\operatorname{ctg}^2\chi$
6	θ_z	0	$\dfrac{1}{\sin\chi}(xy'-x'y)$	$-\dfrac{\cos\chi}{\sin^2\chi}(xy'-x'y)$
7	ω	φ_2^1	$-\operatorname{ctg}\chi\,\varphi_2^1$	0

(8.79)

$$\bar{\psi}_{\omega m_z}=0;\quad \bar{\psi}_{\omega n_z}=0;\quad \bar{\psi}_{\omega n_z m_z}=\frac{1}{\sin\chi}(\varphi_2^1)'. \tag{8.80}$$

Taking (8.79), (8.80) into account, and assuming that the contour Z=const is symmetric about the Ox axis, we find from (6.26)

$$\tilde{\mathbf{A}}_{\xi\xi}=\begin{pmatrix}\tilde{a}_{11} & 0 & \tilde{a}_{13}\\ 0 & \tilde{a}_{22} & 0\\ \tilde{a}_{31} & 0 & \tilde{a}_{33}\end{pmatrix},\quad \tilde{\mathbf{A}}_{\xi\theta}=\tilde{\mathbf{A}}'_{\theta\xi}=\begin{pmatrix}0 & \tilde{a}_{15} & 0\\ \tilde{a}_{24} & 0 & \tilde{a}_{26}\\ 0 & \tilde{a}_{35} & 0\end{pmatrix},$$
$$\tilde{\mathbf{A}}_{\theta\theta}=\begin{pmatrix}\tilde{a}_{44} & 0 & \tilde{a}_{46}\\ 0 & \tilde{a}_{55} & 0\\ \tilde{a}_{64} & 0 & \tilde{a}_{66}\end{pmatrix}. \tag{8.81}$$

$$\tilde{\mathbf{A}}_{\xi\varphi}=\tilde{\mathbf{A}}'_{\varphi\xi}=\begin{pmatrix}0\\ \tilde{a}_{27}\\ 0\end{pmatrix},\quad \tilde{\mathbf{A}}_{\theta\varphi}=\tilde{\mathbf{A}}'_{\varphi\theta}=\begin{pmatrix}\tilde{a}_{47}\\ 0\\ \tilde{a}_{67}\end{pmatrix}, \tag{8.82}$$
$$\tilde{\mathbf{A}}_{\varphi\varphi}=(\tilde{a}_{77}).$$

$$\tilde{\mathbf{B}}_{\xi\varphi}=\begin{pmatrix}0\\ \tilde{b}_{27}\\ 0\end{pmatrix},\quad \tilde{\mathbf{B}}_{\theta\varphi}=\begin{pmatrix}\tilde{b}_{47}\\ 0\\ \tilde{b}_{67}\end{pmatrix},\quad \tilde{\mathbf{B}}_{\varphi\varphi}=(\tilde{b}_{77}), \tag{8.83}$$
$$\tilde{\mathbf{C}}_{\varphi\varphi}=(\tilde{c}_{77}).$$

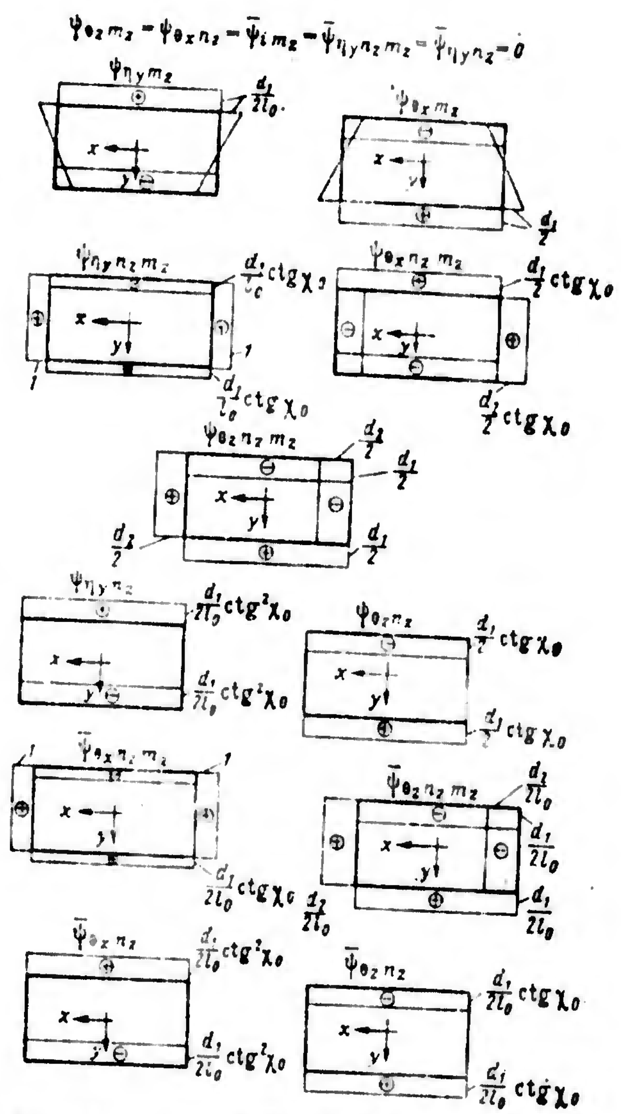


Figure 8.35. Generalized coordinates of stressed state, corresponding to the law of plane sections.

It is now easy to expand matrix equations (6.106)-(6.108). We obtain a system of four differential equations in the canonical unknowns Σ_y, θ_x, θ_z, w

$$\bar{a}_{22}\Xi_y + \bar{a}_{24}\theta_x' + \bar{a}_{26}\theta_z' + \bar{a}_{27}w' + \bar{b}_{27}w = \frac{Q_y}{G},$$

$$\bar{a}_{42}\Xi_y + \bar{a}_{44}\theta_x' + \bar{a}_{46}\theta_z' + \bar{a}_{47}w' + \bar{b}_{47}w = \frac{M_x}{G}, \tag{8.85}$$

$$\bar{a}_{62}\Xi_y + \bar{a}_{64}\theta_x' + \bar{a}_{66}\theta_z' + \bar{a}_{67}w' + \bar{b}_{67}w = \frac{M_z}{G}.$$

$$\bar{a}_{72}\Xi_y' - \bar{b}_{72}\Xi_y + \bar{a}_{74}\theta_x'' + \bar{a}_{76}\theta_z'' - \dots - \bar{b}_{76}\theta_z' + \bar{a}_{77}w'' - \bar{c}_{77}w = 0. \tag{8.86}$$

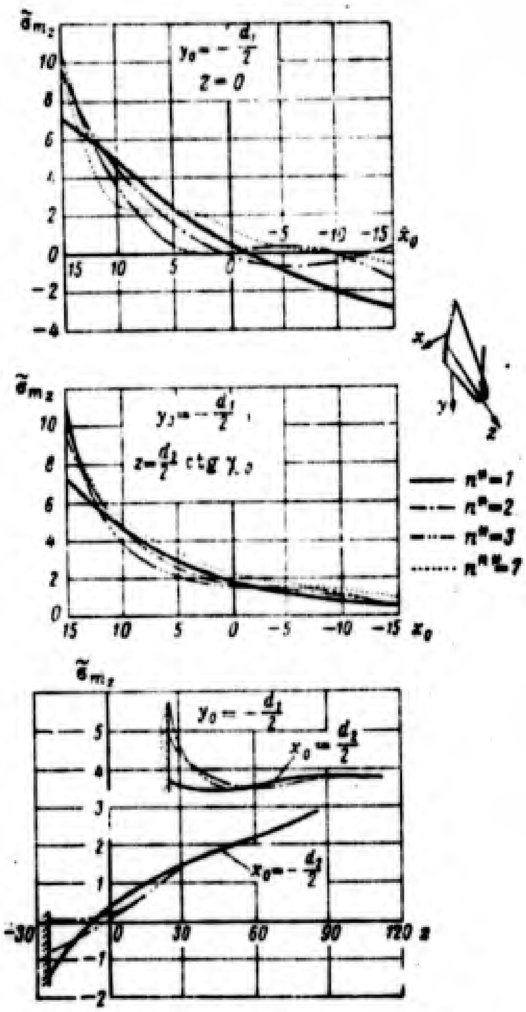


Figure 8.36. Distribution of normal stresses in a slightly conical swept caisson (χ_0=30°) loaded by force Q_y.

In Eqs. (8.85), Q_y, M_x, M_z are the components of the resultant and of the net moment of the external load applied to the end portion of the shell cut off by the section Z=const, in axes passing through point (0,0,Z) parallel to the axes of the fundamental system of Cartesian coordinates.

In view of relation (8.1), the remaining equations for a load typical of wings are in fact satisfied identically. Let us recall that in Eqs. (8.85) and (8.86), the quantity Σ_y is determined by the expression

$$\Sigma_y = \eta'_y - Z\theta'_x, \tag{8.87}$$

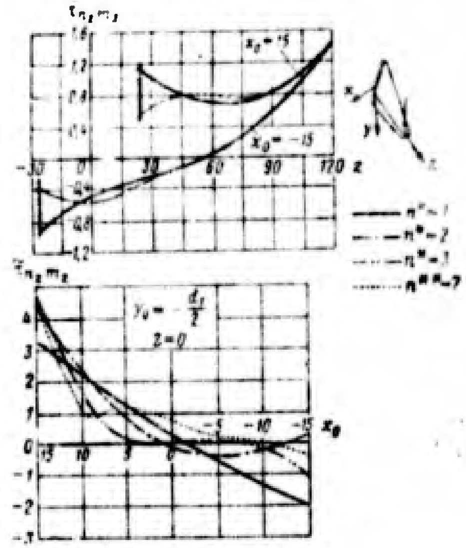


Figure 8.37. Distribution of tangential stresses in a slightly conical swept caisson ($\chi_0=30°$) loaded by force Q_y.

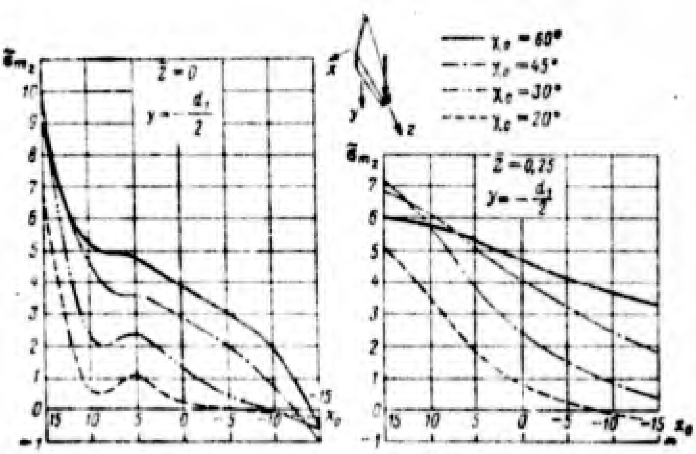


Figure 8.38. Normal stresses in sections of slightly conical swept caissons loaded by force Q_y at different sweep angles ($n^{**}=7$).

where $\eta_y(Z)$ is the displacement in the direction of the Oy axis of the pole rigidly bound to the contour Z=const and coinciding up to the deformation with the origin of the fundamental system of Cartesian coordinates; $\theta_x(Z)$ is the angle of rotation of the section Z=const as a solid about the fixed axis Ox of the fundamental system of Cartesian coordinates.

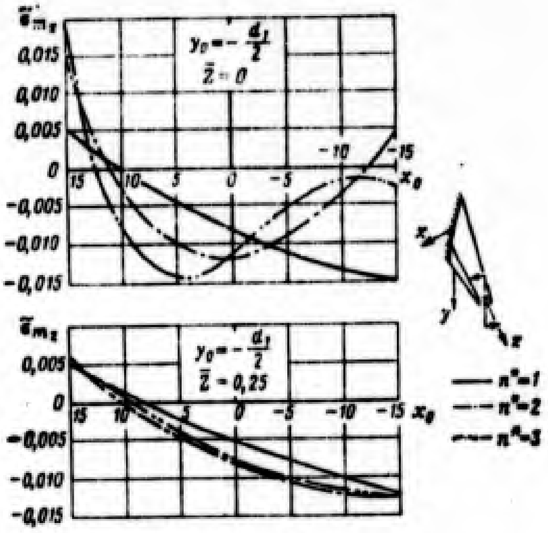


Figure 8.39. Distribution of normal stresses in a slightly conical swept caisson (${}_{x}0$=30°) loaded by -oment M_z.

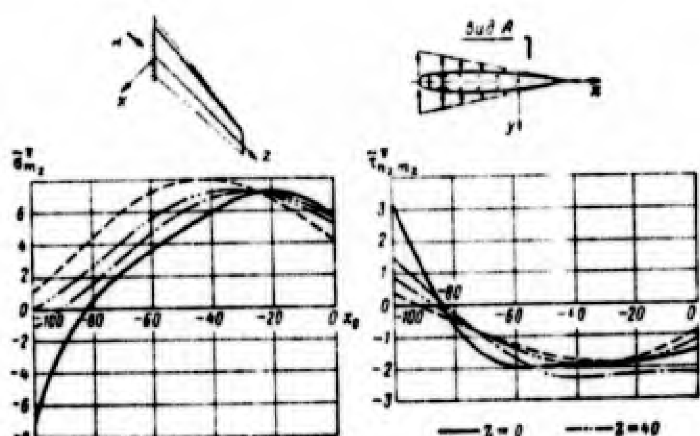


Figure 8.40. Distribution of normal and tangential stresses in slightly conical swept wing with curved contour. Key: 1) view A.

Canonical system (8.85) is algebraic with respect to the unknowns Σ_y, θ'_x, θ'_z. Solving this system, then eliminating Σ_y, Σ'_y, Θ'_x, Θ''_x, Θ'_z, Θ''_z from Eqs. (8.86) with the aid of the expressions obtained, we arrive, as was shown in Section 6.4, at a resolvent of the form

$$\omega'' - k\omega = F(Z). \tag{8.88}$$

Having determined W_1, we can also readily find all the components of the law of plane sections.

To simplify the operations, let us consider a section symmetric about both the Ox and the Oy axes. In this case, in accordance with formulas (8.11)

$$k_1 = k_{12} = k_2 = k_{1x} = k_{2x} = k_{2y} = 0, \tag{8.89}$$

so that we have from (8.8)

$$\bar{\gamma}_2^1(S) = x(S)\,y(S). \tag{8.90}$$

In view of (8.90), taking (8.79) and (8.80) into consideration, we have from (6.26)

$$\tilde{a}_{34} = \tilde{a}_{36} = \tilde{a}_{47} = \tilde{a}_{67} = 0; \quad \tilde{b}_{37} = 0, \tag{8.91}$$

so that the canonical system of resolvents (8.85), (8.86) assumes the form

$$\tilde{a}_{33}\Xi_y + \tilde{a}_{37}\omega' = \frac{Q_y}{G}, \tag{8.92}$$

$$\begin{gathered}
\tilde{a}_{44}\theta_x' + \tilde{a}_{46}\theta_z' + \tilde{b}_{47}\omega = \frac{M_x}{G}, \\
\tilde{a}_{46}\theta_x' + \tilde{a}_{66}\theta_z' + \tilde{b}_{67}\omega = \frac{M_z}{G}, \\
\tilde{a}_{73}\Xi_y' - \tilde{b}_{47}\theta_x' - \tilde{b}_{67}\theta_z' + \tilde{a}_{77}\omega'' - \tilde{c}_{77}\omega = 0.
\end{gathered} \tag{8.93}$$

From (8.92) we have

$$\begin{gathered}
\Xi_y = \frac{Q_y}{\tilde{a}_{33}G} - \tilde{a}_{37}\omega', \\
\theta_x' = \frac{\tilde{a}_{66}}{\Delta}\,\frac{M_x}{G} - \frac{\tilde{a}_{46}}{\Delta}\,\frac{M_z}{G} + \frac{\Delta_x}{\Delta}\omega, \\
\theta_z' = -\frac{\tilde{a}_{46}}{\Delta}\,\frac{M_x}{G} + \frac{\tilde{a}_{44}}{\Delta}\,\frac{M_z}{G} + \frac{\Delta_z}{\Delta}\omega,
\end{gathered} \tag{8.94}$$

where

$$\Delta = \tilde{a}_{44}\tilde{a}_{66} - \tilde{a}_{46}^2, \quad \Delta_x = \tilde{a}_{46}\tilde{b}_{67} - \tilde{a}_{66}\tilde{b}_{47}; \quad \Delta_z = \tilde{a}_{46}\tilde{b}_{47} - \tilde{a}_{44}\tilde{b}_{67} \tag{8.95}$$

Eliminating the unknowns Σ_y', θ_x', θ_z' from (8.93), we obtain with the aid of (8.94)

$$\omega'' - k^2\omega = g_1 \frac{Q_y'}{G} + g_2 \frac{M_x}{G} + g_3 \frac{M_y}{G}, \tag{8.96}$$

where

$$k^2 = \frac{\tilde{c}_{77}\Delta + \tilde{b}_{47}\Delta_x + \tilde{b}_{67}\Delta_y}{\tilde{a}_{77}\Delta}, \tag{8.97}$$

$$g_1 = -\frac{\tilde{a}_{27}}{\tilde{a}_{22}\tilde{a}_{77}}; \quad g_2 = -\frac{\Delta_x}{\tilde{a}_{77}\Delta}; \quad g_3 = -\frac{\Delta_y}{\tilde{a}_{77}\Delta}. \tag{8.98}$$

Integrating Eq. (8.96), we find

$$\omega(Z) = C_1 e^{kZ} + C_2 e^{-kZ} - \frac{g_1}{k}\frac{Q_y(0)}{G}\,\mathrm{sh}\,kZ +$$
$$+\int_0^Z \left[\frac{1}{k}\left(g_2 \frac{M_x(\xi)}{G} + g_3 \frac{M_y(\xi)}{G}\right)\mathrm{sh}\,k(Z-\xi) + g_1 \frac{Q_y(\xi)}{G}\,\mathrm{ch}\,k(Z-\xi)\right] d\xi. \tag{8.99}$$

Considering (8.99) and (8.31), we represent the components of the law of plane sections, determined by Eqs. (8.94), in the form

$$\theta_x(Z) = \theta_x(0) + \int_0^Z \left[\frac{\tilde{a}_{66}}{\Delta}\frac{M_x(\xi)}{G} - \frac{\tilde{a}_{46}}{\Delta}\frac{M_y(\xi)}{G}\right] d\xi + \frac{\Delta_x}{\Delta}\int_0^Z \omega(\xi)\,d\xi,$$

$$\theta_y(Z) = \theta_y(0) + \int_0^Z \left[-\frac{\tilde{a}_{46}}{\Delta}\frac{M_x(\xi)}{G} + \frac{\tilde{a}_{44}}{\Delta}\frac{M_y(\xi)}{G}\right] d\xi + \frac{\Delta_y}{\Delta}\int_0^Z \omega(\xi)\,d\xi, \tag{8.100}$$

$$\eta_y^\circ(Z) = \eta_y^\circ(0) - \theta_x(0)Z + \int_0^Z \left\{\frac{Q_y(\zeta)}{\tilde{a}_{77}G} - \int_0^\zeta \left[\frac{\tilde{a}_{66}}{\Delta}\frac{M_x(\xi)}{G} - \right.\right.$$
$$\left.\left. - \frac{\tilde{a}_{46}}{\Delta}\frac{M_y(\xi)}{G}\right] d\xi - \frac{\Delta_x}{\Delta}\int_0^\zeta \omega(\xi)\,d\xi\right\} d\zeta - \tilde{a}_{27}[\omega(Z) - \omega(0)].$$

Expressions (8.99) and (8.100) represent the general solution of the system of differential resolvents for a section symmetric about the Ox and Oy axes. As an example, let us consider a swept caisson of rectangular cross section (see Figure 8.23). The coefficients of the differential resolvents entering into general expressions (8.99) and (8.100)

$$\tilde{a}_{33}=\oint\left(1+\frac{1+\nu}{1-\nu}\cos^2\chi\right)\frac{y'^2}{\sin^3\chi}h\,dS,$$

$$\tilde{a}_{44}=\oint\left[\left(1+\frac{1+\nu}{1-\nu}\sin^2\chi\right)y^2+\left(1+\frac{1+\nu}{1-\nu}\cos^2\chi\right)\frac{\operatorname{ctg}^2\gamma_0}{\sin^2\chi}x^2y'^2+\right.$$
$$\left.+2\frac{1+\nu}{1-\nu}\operatorname{ctg}\gamma_0\cos\chi\,x\,yy'\right]\frac{h}{\sin\chi}dS+2(1+\nu)\sum_k y_k^2\Delta F_k,$$

$$\tilde{a}_{66}=\oint\left(1+\frac{1+\nu}{1-\nu}\cos^2\chi\right)\frac{(xy'-x'y)^2}{\sin^3\chi}h\,dS,$$

$$\tilde{a}_{46}=\oint\left[\frac{1+\nu}{1-\nu}y^2\frac{\cos^2\chi}{\operatorname{ctg}\gamma_0}-\frac{\operatorname{ctg}\gamma_0}{\sin^2\chi}\left(1+\frac{1+\nu}{1-\nu}\cos^2\chi\right)x^2y'^2+\right. \tag{8.101}$$
$$\left.+\frac{2}{1-\nu}(\operatorname{ctg}^2\chi-\nu)\cos\chi\,xyy'\right]\frac{h}{\sin\chi}dS,$$

$$\tilde{a}_{37}=-\oint y'\varphi_2^1\operatorname{ctg}\chi h\,dS,$$

$$\tilde{a}_{77}=\oint\left(1+\frac{1+\nu}{1-\nu}\sin^2\chi\right)(\varphi_2^1)^2\frac{h}{\sin\chi}dS+2(1+\nu)\sum_k[\varphi_2^1(S_k)]^2\Delta F_k.$$

(8.102)

$$\tilde{b}_{47}=-\oint\left(y\operatorname{ctg}\chi+\frac{\operatorname{ctg}\gamma_0}{\sin\chi}xy'\right)(\varphi_2^1)'h\,dS,$$

$$\tilde{b}_{67}=\oint(xy'-x'y)(\varphi_2^1)'\frac{h}{\sin\chi}dS, \tag{8.103}$$

$$\tilde{c}_{77}=\oint(\varphi_2^1)''\frac{h}{\sin\chi}dS.$$

Computing the quadratures (8.101)-(8.103) on the contour of the rectangular section, we find

$$\tilde{a}_{33}=c;\quad \tilde{a}_{44}=\frac{2}{1-\nu}I_s+f\operatorname{ctg}^2\gamma_0;\quad \tilde{a}_{66}=f+\frac{2}{1-\nu}a\operatorname{ctg}^2\gamma_0,$$
$$\tilde{a}_{46}=\bar{f}\operatorname{ctg}\gamma_0+\frac{2\nu}{1-\nu}a\operatorname{ctg}\gamma_0,\quad \tilde{a}_{37}=0,\quad \tilde{a}_{77}=\frac{2}{1-\nu}I_w, \tag{8.104}$$
$$\tilde{b}_{47}=-f\operatorname{ctg}\gamma_0;\quad \tilde{b}_{67}=-\bar{f},\quad \tilde{c}_{77}=f,$$

where I_s, I_w, f, F, a, c are determined by the formulas

$$I_s=a+b,\quad I_w=\bar{a}+\bar{b}+\frac{1-\nu}{2}\bar{a}\operatorname{ctg}^2\gamma_0,$$
$$f=a+e,\quad \bar{f}=a-e,$$
$$a=\frac{d_1^2d_2}{2}h_2,\quad \bar{a}=\frac{d_1^2d_2^3}{24}h_2,$$
$$b=\frac{d_1^3}{6}h_1+(1-\nu^2)\frac{d_1^2}{4}\sum_k\Delta F_k, \tag{8.105}$$
$$\bar{b}=\frac{d_1^3d_2^2}{24}h_1+(1-\nu^2)\frac{d_1^2}{4}\sum_k x_k^2\Delta F_k,$$
$$e=\frac{d_1d_2^2}{2}h_1,\quad c=2d_1h_1.$$

Let the normal pressure on the surface of the caisson be absent, and the load on the end $Z=Z_1=l$ be reduced to the vertical force P_y and moments M and H about the Ox and Oz axes respectively. In this case

$$Q_y(Z)=P_y,\quad M_x(Z)=M-P_y(l-Z),\quad M_z(Z)=H. \tag{8.106}$$

In view of (8.104), (8.106), general expressions (8.99), (8.100) take the form

$$\begin{aligned}
&\omega(Z)=C_1e^{kZ}+C_2e^{-kZ}+\frac{\Delta_x}{Gk^2\tilde{a}_{77}\Delta}[M-P_y(l-Z)]+\frac{\Delta_z}{Gk^2\tilde{a}_{77}\Delta}H,\\
&\theta_x(Z)=\theta_x(0)+\frac{1}{k}\frac{\Delta_x}{\Delta}[C_1(e^{kZ}-1)-C_2(e^{-kZ}-1)]+\\
&\qquad+\frac{1}{G}\left(\frac{\tilde{a}_{66}}{\Delta}+\frac{1}{k^2\tilde{a}_{77}}\frac{\Delta_x^2}{\Delta^2}\right)\left[(M-P_yl)Z+P_y\frac{Z^2}{2}\right]+\\
&\qquad+\frac{1}{G}\left(-\frac{\tilde{a}_{46}}{\Delta}+\frac{1}{k^2\tilde{a}_{77}}\frac{\Delta_x}{\Delta}\frac{\Delta_z}{\Delta}\right)HZ,\\
&\theta_z(Z)=\theta_z(0)+\frac{1}{k}\frac{\Delta_z}{\Delta}[C_1(e^{kZ}-1)-C_2(e^{-kZ}-1)]+\\
&\qquad+\frac{1}{G}\left(-\frac{\tilde{a}_{46}}{\Delta}+\frac{1}{k^2\tilde{a}_{77}}\frac{\Delta_x}{\Delta}\frac{\Delta_z}{\Delta}\right)\left[(M-P_yl)Z+P_y\frac{Z^2}{2}\right]+\\
&\qquad+\frac{1}{G}\left(\frac{\tilde{a}_{44}}{\Delta}+\frac{1}{k^2\tilde{a}_{77}}\frac{\Delta_z^2}{\Delta^2}\right)HZ,\\
&\eta_y^*(Z)=\eta_y^*(0)-\theta_x(0)Z-\frac{1}{k^2}\frac{\Delta_x}{\Delta}[C_1(e^{kZ}-1)+C_2(e^{kZ}-1)-\\
&\qquad-k(C_1-C_2)Z]+\frac{P_yZ}{\tilde{a}_{77}G}-\frac{1}{G}\left(\frac{\tilde{a}_{66}}{\Delta}+\frac{1}{k^2\tilde{a}_{77}}\frac{\Delta_x^2}{\Delta^2}\right)\Big[(M-\\
&\qquad-P_yl)\frac{Z^2}{2}+P_y\frac{Z^3}{6}\Big]-\frac{1}{G}\left(-\frac{\tilde{a}_{46}}{\Delta}+\frac{1}{k^2\tilde{a}_{77}}\frac{\Delta_x}{\Delta}\frac{\Delta_z}{\Delta}\right)H\frac{Z^2}{2}.
\end{aligned} \tag{8.107}$$

It follows from expressions (8.107) that in an oblique cylindrical shell, bending is inseparably connected with torsion. This always takes place regardless of the boundary conditions, since the expressions for angles θ_x and θ_z explicityly contain both the bending and twisting moments.

Expressions (8.107) are simplified substantially if in calculating the coefficients (8.101)-(8.103) of the differential resolvents, the mutual influence of strains e_{m_z} and e_{n_z} is neglected in the expression for the potential energy. In this case, only the coefficient $\tilde{a}_{46}$, which takes the value

$$\tilde{a}_{46}=\tilde{f}\,\mathrm{ctg}\,\gamma_0, \tag{8.108}$$

undergoes a change.

In view of (8.108), after some transformations, expressions (8.107) take the form

$$w(Z)=C_1e^{kZ}+C_2e^{-kZ}+\operatorname{ctg}\chi_0\frac{M-P_u(l-Z)}{\frac{E}{1-\nu^2}I_x}+\operatorname{ctg}\varkappa\frac{H}{GI_{кр}}.\tag{8.109}$$

$$\theta_x(Z)=\theta_x(0)+k\operatorname{ctg}\chi_0\frac{I_\omega}{I_x}\left[C_1(e^{kZ}-1)-C_2(e^{-kZ}-1)\right]+\frac{(M-P_yl)Z+P_y\frac{Z^2}{2}}{\frac{E}{1-\nu^2}I_x}.\tag{8.110}$$

$$\theta_z(Z)=\theta_z(0)+\frac{2}{1-\nu}k\operatorname{ctg}\varkappa\frac{I_\omega}{I_{кр}}\left[C_1(e^{kZ}-1)-C_2(e^{-kZ}-1)\right]+\operatorname{ctg}\varkappa\frac{J}{\bar{J}}\frac{HZ}{GI_{кр}}.\tag{8.111}$$

$$\eta_y^*(Z)=\eta_y^*(0)-\theta_x(0)Z-\operatorname{ctg}\chi_0\frac{I_\omega}{I_x}\left[C_1(e^{kZ}-1)+C_2(e^{-kZ}-1)-k(C_1-C_2)Z\right]-\frac{(M-P_yl)\frac{Z^2}{2}+P_y\frac{Z^3}{6}}{\frac{E}{1-\nu^2}I_x}+\frac{P_yZ}{cG}.\tag{8.112}$$

Here

$$\operatorname{ctg}\varkappa=\frac{\bar{J}}{a_{66}},\tag{8.113}$$

$$I_{кр}=J-\bar{J}\operatorname{ctg}\varkappa,\tag{8.114}$$

$$k=\frac{1}{\sqrt{\left(\frac{2}{1-\nu}\frac{1}{I_{кр}}+\frac{\operatorname{ctg}^2\chi_0}{I_x}\right)I_\omega}}.\tag{8.115}$$

Let the end Z=0 be completely fixed. In this case, in expressions (8.110)-(8.113) it is necessary to set

$$\eta_y^*(0)=\theta_x(0)=\theta_z(0)=0.\tag{8.116}$$

From expression (8.109), setting w(0)=0, we find

$$C_1+C_2=-\operatorname{ctg}\chi_0\frac{M-P_yl}{\frac{E}{1-\nu^2}I_x}-\operatorname{ctg}\varkappa\frac{H}{GI_{кр}}.\tag{8.117}$$

Relations (8.116) and (8.117) represent the kinematic boundary conditions on the end Z=0. On the end Z=1, static boundary conditions take place, and with respect to the resolvent and the net moment of the external forces applied at this end, these conditions are already satisfied when the resolvents are derived. It is therefore necessary to satisfy only the boundary conditions with respect to the generalized force P_w by setting

$$P_w(l)=0. \tag{8.118}$$

Expanding conditions (8.118), in view of (6.32), (8.91), (8.104) and also (6.8), (8.16) and (8.79), we obtain

$$\dot{\omega}'(l)=0, \tag{8.119}$$

whence on the basis of (8.109) we find

$$k(C_1 e^{kl}-C_2 e^{-kl})+\operatorname{ctg}\chi_0\frac{P_y}{\frac{E}{1-\nu^2}I_x}=0. \tag{8.120}$$

Solving Eqs. (8.117), (8.120) for arbitrary constants C_1 and C_2, we obtain

$$\begin{aligned} C_1&=-\frac{1}{1+e^{2kl}}\left(\operatorname{ctg}\chi_0\frac{M_0+k^{-1}P_y e^{kl}}{\frac{E}{1-\nu^2}I_x}+\operatorname{ctg}\varkappa\,\frac{H}{GI_{np}}\right),\\ C_2&=-\frac{1}{1+e^{-2kl}}\left(\operatorname{ctg}\chi_0\frac{M_0-k^{-1}P_y e^{-kl}}{\frac{E}{1-\nu^2}I_x}+\operatorname{ctg}\varkappa\,\frac{H}{GI_{np}}\right), \end{aligned} \tag{8.121}$$

where $M_0=M-P_y l$ is the moment of the external load about the Ox axis of the fundamental Cartesian coordinate system.

Expressions (8.121) pertain to the case where the end plane of the swept caisson is parallel to the plane Z=const. If however the end plane is perpendicular to the Oz axis, the static boundary conditions can be formulated in the same way as for a straight caisson by orthogonalizing the normal forces T_{m_z} on this end with respect to the generalized coordinate of warpings. For a caisson with a cross section symmetric about the Ox and Oy axes, we have

$$\int T_{m_z}(l-x\operatorname{ctg}\chi_0,\,S)\,x(S)\,y(S)\,dS=0. \tag{8.122}$$

where the integration should be carried out over the contour of the end cross section by including the elements of the longitudinal structure.

For normal forces T_{m_z} (6.19) in the case of a rectangular cross section, using (6.8), (6.15), (6.16), we can readily obtain

$$T_{m_z}(Z, S) = \frac{Eh}{1-\nu^2}\left[\theta'_x(Z) + \theta'_z(Z)\nu \operatorname{ctg}\chi + \omega'(Z)x(S)\right]y(S) \tag{8.123}$$

or, considering the first equation of system (8.92) and relations (8.31)

$$T_{m_z}(Z, S) = \frac{Eh}{1-\nu^2}\left[-\eta''_y + \omega'(Z)x(S) + \nu\theta'_z \operatorname{ctg}\chi\right]y(S). \tag{8.124}$$

Expanding (8.123) with the aid of (8.109)-(8.111), we obtain

$$\sigma_{m_z}(z, S) = \frac{M_{b}(z)\,y}{I_x} + \frac{kE}{1-\nu^2}\left[C_1 e^{kz}(x_M + \nu x_H \operatorname{ctg}\chi + x)e^{-kx \operatorname{ctg}\chi_0} + C_2 e^{-kz}(x_M + \nu x_H \operatorname{ctg}\chi - x)e^{kx \operatorname{ctg}\chi_0}\right]y. \tag{8.125}$$

Here

z is the Cartesian coordinate related to the oblique-angled coordinate Z as follows:

$$z = Z + x \operatorname{ctg}\chi_0; \tag{8.126}$$

$M_b(z)$ is the bending moment in the cross section:

$$M_{b}(z) = M - P_y(l - z). \tag{8.127}$$

$$x_M = k \operatorname{ctg}\chi_0 \frac{I_\omega}{I_x}; \quad x_H = \frac{2}{1-\nu} k \operatorname{ctg}\chi \frac{I_\omega}{I_{kp}}. \tag{8.128}$$

Introducing (8.125) into (8.122), we readily obtain

$$C_1 e^{kl} - C_2 e^{-kl} = 0. \tag{8.129}$$

Now, solving (8.129) together with Eq. (8.117), we obtain

$$C_1 = -\frac{1}{1+e^{2kl}}\left(\operatorname{ctg}\chi_0\frac{M_0}{\frac{E}{1-\nu^2}I_x} + \operatorname{ctg}\varkappa\,\frac{H}{GI_{kp}}\right),$$
$$C_2 = -\frac{1}{1+e^{-2kl}}\left(\operatorname{ctg}\chi_0\frac{M_0}{\frac{E}{1-\nu^2}I_x} + \operatorname{ctg}\varkappa\,\frac{H}{GI_{kp}}\right). \qquad (8.130)$$

Thus, expressions (8.121) determine arbitrary constants for the case where the end plane makes an angle χ_0 with the Oz axis, whereas expressions (8.130) correspond to the case where this plane is perpendicular to the Oz axis. In shells with geometrical dimensions corresponding to real wings

$$e^{2kl} \gg 1. \qquad (8.131)$$

In view of (8.131), expressions (8.121) and (8.130), respectively, assume the form

$$C_1 = -e^{-kl}\left(\operatorname{ctg}\chi_0\frac{M_0 e^{-kl} + k^{-1}P_y}{\frac{E}{1-\nu^2}I_x} + \operatorname{ctg}\varkappa\,\frac{H}{GI_{kp}}e^{-kl}\right),$$
$$C_2 = -e^{-kl}\left(\operatorname{ctg}\chi_0\frac{M_0 e^{kl} - k^{-1}P_y}{\frac{E}{1-\nu^2}I_x} + \operatorname{ctg}\varkappa\,\frac{H}{GI_{kp}}e^{kl}\right). \qquad (8.132)$$

and

$$C_1 = -e^{-2kl}\left(\operatorname{ctg}\chi_0\frac{M_0}{\frac{E}{1-\nu^2}I_x} + \operatorname{ctg}\varkappa\,\frac{H}{GI_{kp}}\right),$$
$$C_2 = -\operatorname{ctg}\chi_0\frac{M_0}{\frac{E}{1-\nu^2}I_x} - \operatorname{ctg}\varkappa\,\frac{H}{GI_{kp}}. \qquad (8.133)$$

In practical calculations, it is completely permissible to adopt a rougher estimate, setting

$$1 \pm e^{-kl} \approx 1. \qquad (8.134)$$

In this case, arbitrary constants (8.132), (8.133), respectively, assume the form

$$C_1 = -\operatorname{ctg}\chi_0 \frac{k^{-1}P_y}{\frac{E}{1-\nu^2}I_x} e^{-kl},$$

$$C_3 = -\operatorname{ctg}\chi_0 \frac{M_0}{\frac{E}{1-\nu^2}I_x} - \operatorname{ctg}\chi \frac{H}{GI_{\kappa p}}, \tag{8.135}$$

and

$$C_1 = 0,$$

$$C_3 = -\operatorname{ctg}\chi_0 \frac{M_0}{\frac{E}{1-\nu^2}I_x} - \operatorname{ctg}\chi \frac{H}{GI_{\kappa p}}. \tag{8.136}$$

It is easy to see that the difference in calculated results, due to the difference in constants (8.135) and (8.136), will be perceptible only in the end portion of the caisson. For this reason, we will give preference to simpler expressions (8.136), since anomalies of the stressed and strained state due to oblique fixing are of chief interest.

Considering (8.136), we have from (8.125)

$$\sigma_{m_z} = \frac{M_{\text{изг}}}{I_x} y + \frac{(M - P_y l)x_M + Hx_H}{I_\omega} e^{-kz}(x - x_M - \nu x_H \operatorname{ctg}\chi) y e^{kx \operatorname{ctg}\chi_0}. \tag{8.137}$$

Expression (8.137) is very clear. The first term of this expression coincides with the elementary solution based on the hypothesis of plane cross sections. The second term represents the self-balanced portion of the solution, reflecting the fixing conditions and determining the edge effect.

The stresses σ_{m_z} determined by expression (8.137) will not be continuous on the cross sectional contour, as a result of the presence of the term $\nu x_H \operatorname{ctg}\chi$. As can be easily ascertained, this term is related to the consideration of the stresses σ_{n_z} in the energy balance, and of the strains e_{n_z} corresponding to them. When these factors are not considered, the stresses σ_{m_z} will be determined by the expression

$$\sigma_{m_z} = \frac{M_{\text{изг}}}{I_x} y + \frac{(M - P_y l)x_M + Hx_H}{I_\omega} e^{-kz}(x - x_M) y e^{kx \operatorname{ctg}\chi_0}, \tag{8.138}$$

where all the coefficients are determined by the same expressions as before, with the exception of the coefficient $\tilde{a}_{66}$ in expression (8.113). In the case under consideration, it is necessary to take

$$\tilde{a}_{66}=f. \tag{8.139}$$

Introducing the values of arbitrary constants obtained into expressions (8.109)-(8.112), one can readily obtain the expanded expressions for displacements and the remaining components of internal forces. We will give the expanded expressions only for the tangential forces $S_{n_z m_z}$. This expression can be obtained by using Hooke's law, but a more accurate expression is obtained by integrating the differential equilibrium equation

$$\frac{\partial T_{m_z}}{\partial z}+\frac{\partial S_{n_z m_z}}{\partial S^*}=0. \tag{8.140}$$

where S* is the cross-sectional arc.

From (8.140)

$$S_{n_z m_z}(z, S^*)=S_{n_z m_z}(z, S_1^*)-\int\limits_{S_1^*}^{S^*}\frac{\partial T_{m_z}}{\partial z}dS^*, \tag{8.141}$$

where S_1^* corresponds to some generatrix taken as the initial one.

In expression (8.141), the tangential forces $S_{n_z m_z}(z, S_1^*)$ are determined from the equilibrium condition

$$\oint S_{n_z m_z}(z, S^*)h^*(S^*)dS^*-H=0, \tag{8.142}$$

where h*, the length of the perpendicular dropped on the tangent to the cross-sectional contour from the point of intersection of the plane of this contour with the Oz axis, is determined by the expression

$$h^*=x\frac{dy}{dS^*}-y\frac{dx}{dS^*}. \tag{8.143}$$

or, passing in (8.143) to derivatives with respect to the S coordinate, by the expression

$$h^* = \frac{1}{\sin\chi}(xy' - x'y). \tag{8.144}$$

Expanding equilibrium conditions (8.142) with the aid of (8.141), we obtain

$$S_{a_2 m_2}(z, S_1^*) = \frac{H + \oint \left[\int\limits_{S_1^*}^{S^*} - \frac{\partial T_{m_2}(z,\xi)}{\partial z} d\xi \right] h^*(S^*)\, dS^*}{\oint h^*(S^*)\, dS^*} \tag{8.145}$$

We will choose a point with coordinates x=0, $y=d_1/2$ as the origin of S_1^* on the cross-sectional contour. For simplicity, we will assume that the walls of the caisson are not subjected to normal stresses. In this case, using (8.138), we can readily obtain

$$\int\limits_{S_1^*}^{S^*} \frac{\partial T_{m_2}(z,\xi)}{\partial z} d\xi = \Big\{ -\frac{P_y}{I_x}\frac{d_1}{2}x + \frac{(M - P_y l)x_M + Hx_H}{I_\omega}\frac{d_1}{2\operatorname{ctg}\chi_0}e^{-kz} \times$$
$$\times \left[\left(x - x_M - \frac{1}{k\operatorname{ctg}\chi_0} \right) e^{kx\operatorname{ctg}\chi_0} + x_M + \frac{1}{k\operatorname{ctg}\chi_0} \right] \Big\} h_2. \tag{8.146}$$

Further, using (8.146), we also find

$$\oint \left[\int\limits_{S_1^*}^{S^*} \frac{\partial T_{m_2}(z,\xi)}{\partial z} d\xi \right] h^*(S^*)\, dS^* = \frac{(M - P_y l)x_M + Hx_H}{I_\omega} e^{-kz}\frac{d_1^2 d_2 h_2}{\operatorname{ctg}\chi_0} A, \tag{8.147}$$

where

$$A = \frac{1}{2}\operatorname{sh}\left(k\frac{d_2}{2}\operatorname{ctg}\chi_0\right)\left[\frac{d_2}{2} - \frac{1}{k\frac{d_2}{2}\operatorname{ctg}\chi_0}\left(x_M + \frac{2}{k\operatorname{ctg}\chi_0}\right)\right] -$$
$$-\frac{1}{2}x_M \operatorname{ch}\left(k\frac{d_2}{2}\operatorname{ctg}\chi_0\right) + x_M + \frac{1}{k\operatorname{ctg}\chi_0}. \tag{8.148}$$

Now, introducing (8.145) and (8.147) into (8.141), we finally obtain

$$S_{n_2m_2}=\frac{P_y}{I_x}\frac{d_1}{2}x+\frac{H}{2A_1d_2}-\frac{(M-P_yl)x_M+Hx_H}{I_\omega}\frac{d_1h_2}{2\,\mathrm{ctg}\,\chi_0}e^{-kz}\times$$
$$\times\left[\left(x-x_M-\frac{1}{k\,\mathrm{ctg}\,\chi_0}\right)e^{kx\,\mathrm{ctg}\,\chi_0}+x_M+\frac{1}{k\,\mathrm{ctg}\,\chi_0}-A\right]. \qquad (8.149)$$

Expression (8.149) is very clear. The first two terms coincide with the elementary solution on the basis of the hypothesis of plane cross sections and determine, respectively, the flux of tangential stresses in bending and the "Bredt" flux in torsion. The third term represents the self-balanced flux, which determines the edge effect, related to the fixing conditions.

The results of the calculation made by using the simplified formulas obtained above are compared with the experimental results* in Figure 8.44.

4. Simplified Analysis of Shells of Constant Cross Sections with Ribs Oriented Parallel to the Flow

As in the preceding subsection, we will approximate the warping by only one function, assuming as a result of the symmetry of the cross section (Figure 8.41) that

$$\varphi^1(S)=x^*(S)y(S), \qquad (8.150)$$

which corresponds to torsional warping of the section Z=const. We will confine ourselves to the case in which the external load reduces to the force P_y, and also to moments M and H applied to the end Z=1.

We will consider an oblique cylindrical shell reinforced with a set of diaphragms oriented parallel to the plane Z=const. The differential resolvents for such shells are given in Section 6.2. These equations are easily reduced to the canonical form

*A.L. Lang and R.L. Bisplinghoff. Some Results of Sweptback Wing Structural Studies. JAS, 1951, vol. 18, No. 11.

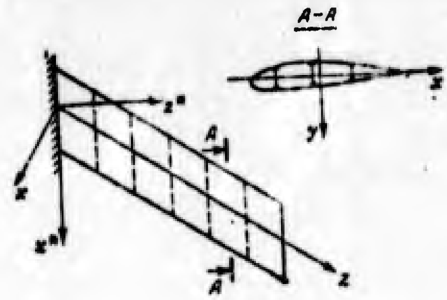


Figure 8.41. Oblique cylindrical shell of swept wing type with ribs oriented parallel to the flow.

by taking the origin of the fundamental system of Cartesian coordinates as the pole. However, we will demonstrate the solution for the case in which the point of intersection of the plane of this contour with the Oz axis is chosen as the pole of the contour Z=const.

As in the preceding case, we will keep four generalized displacements: η_y, θ_{x*}, θ_{z*}, w, denoting by $\eta_y(Z)$ the vertical displacement of the point with coordinates (0,0,z), rigidly bound to the contour Z=const during displacements of the latter as a solid, and by θ_{x*}, θ_{z*}, the angles of rotation of this contour as a solid about the Ox*, Oz* axes, respectively (see Figure 8.41). The generalized strain coordinates for a contour of arbitrary configuration are represented by expressions (6.9). Figure 8.42 shows diagrams of coordinate functions ϕ for the rectangular section under consideration. Figure 8.43 shows diagrams of generalized coordinates of the stressed state ψ, $\overline{\psi}$. Multiplying the corresponding diagrams according to formulas (6.27), we can easily obtain

$\tilde b_{22} = 0$	$\tilde b_{24} = c \sin \chi_0$	$\tilde b_{26} = -c \cos \chi_0$	$\tilde b_{27} = 0$
$\tilde b_{42} = 0$	$\tilde b_{44} = 0$	$\tilde b_{46} = 0$	$\tilde b_{47} = -\frac{2}{1-\nu} a \operatorname{ctg} \chi_0$
$\tilde b_{62} = 0$	$\tilde b_{64} = 0$	$\tilde b_{66} = 0$	$\tilde b_{67} = -\overline{f}$
$\tilde b_{72} = 0$	$\tilde b_{74} = 0$	$\tilde b_{76} = 0$	$\tilde b_{77} = 0$

(8.151)

$\tilde c_{72} = 0$	$\tilde c_{74} = 0$	$\tilde c_{76} = 0$	$\tilde c_{77} = -f \sin \chi_0 - \frac{2}{1-\nu} a \frac{\cos^2 \chi_0}{\sin \chi_0}$,

$\tilde{a}_{22} = c$	$\tilde{a}_{24} = 0$	$\tilde{a}_{26} = 0$	$\tilde{a}_{27} = 0$
$\tilde{a}_{42} = 0$	$\tilde{a}_{44} = \frac{2}{(1-\nu)\sin\chi_0} I_x$	$\tilde{a}_{46} = -\frac{2}{1-\nu} b \sin\chi_0 \cos\chi_0$	$\tilde{a}_{47} = 0$
$\tilde{a}_{62} = 0$	$\tilde{a}_{64} = -\frac{2}{1-\nu} b \sin\chi_0 \cos\chi_0$	$\tilde{a}_{66} = \frac{2}{1-\nu} b \cos^2\chi_0 + \frac{f}{\sin\chi_0}$	$\tilde{a}_{67} = 0$
$\tilde{a}_{72} = 0$	$\tilde{a}_{74} = 0$	$\tilde{a}_{76} = 0$	$\tilde{a}_{77} = \frac{2}{(1-\nu)\sin\chi_0} I_\omega$

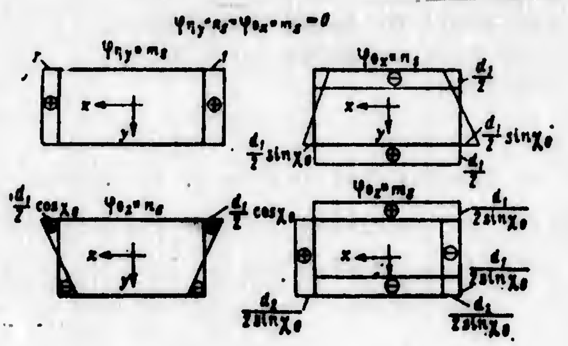


Figure 8.42. Coordinate functions corresponding to the law of plane sections.

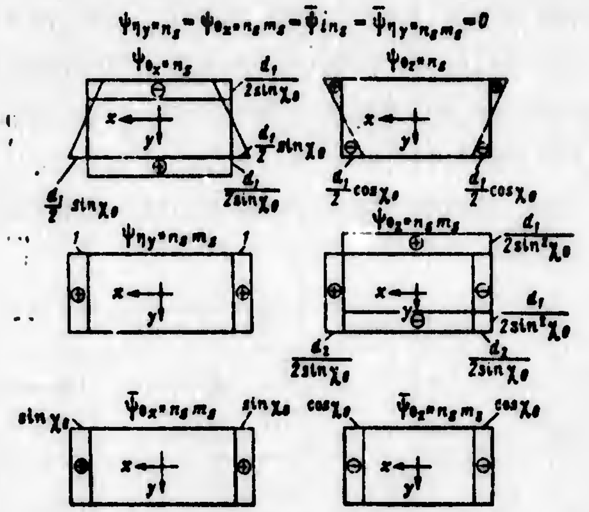


Figure 8.43. Generalized coordinates of stressed state, corresponding to the law of plane sections.

where I_x, I_w, f, $\bar{f}$, a, b, c are determined from the formulas

$$
\begin{aligned}
&I_x = a + b\sin^2\chi_0, \quad I_\omega = \bar{a} + \bar{b}\sin^2\chi_0,\\
&f = a + e\sin\chi_0, \quad \bar{f} = a - e\sin\chi_0,\\
&a = \frac{d_1^2 d_2}{2} h_2, \qquad \bar{a} = \frac{d_1^2 d_2^3}{24} h_2,\\
&b = \frac{d_1^3}{6} h_1 + (1-\nu^2)\frac{d_1^2}{4}\sum_k \Delta F_k, \quad \bar{b} = \frac{d_2^2 d_1^3}{24} h_1 + (1-\nu^2)\frac{d_1^2}{4}\sum_k x_k^{*2}\Delta F_k,\\
&e = \frac{d_2^2 d_1}{2} h_1, \quad c = 2d_1 h_1.
\end{aligned}
\tag{8.152}
$$

Expressions (8.151) being taken into account, the differential resolvents in the desired generalized displacements η_y, θ_{x*}, θ_{z*}, W assume the form

$$
\begin{aligned}
&c\eta_y' + c\sin\chi_0\theta_{x*} - c\cos\chi_0\theta_{z*} = \frac{1}{G}Q_y,\\
&\frac{2I_x}{(1-\nu)\sin\chi_0}\theta_{x*}' - \frac{2}{1-\nu}b\sin\chi_0\cos\chi_0\theta_{z*}' - \frac{2}{1-\nu}a\operatorname{ctg}\chi_0\omega = \frac{1}{G}M_{x*},\\
&-\frac{2}{1-\nu}b\sin\chi_0\cos\chi_0\theta_{x*}' + \left(\frac{2}{1-\nu}b\cos^2\chi_0 + \frac{I}{\sin\chi_0}\right)\theta_{z*}' - \bar{f}\omega = \frac{1}{G}M_{z*},\\
&\frac{2I_\omega}{(1-\nu)\sin\chi_0}\omega'' + \frac{2}{1-\nu}a\operatorname{ctg}\chi_0\theta_{x*}' + \bar{f}\theta_{z*}' - \left(\frac{2}{1-\nu}a\frac{\cos^2\chi_0}{\sin\chi_0} + f\sin\chi_0\right)\omega =\\
&= -\frac{1}{G}\bar{R}_\omega.
\end{aligned}
\tag{8.153}
$$

The first three equations of system (8.153) represent the equilibrium conditions of the cut-off portion of the shell as a solid. The fourth equation represents the equilibrium condition with respect to warping.

The right-hand sides of system (8.153) are represented in the general case by expressions (6.28) and (6.36). In the absence of a surface load, one can readily obtain

$$
\begin{aligned}
&Q_y = \bar{P}_5 = P_y,\\
&M_{x*} = \bar{P}_4 = M - P_y(l-Z)\sin\chi_0,\\
&M_{z*} = \bar{P}_6 = H + P_y(l-Z)\cos\chi_0,\\
&\bar{R}_\omega = \bar{R}_7 = 0.
\end{aligned}
\tag{8.154}
$$

Integrating system (8.153), we find

$$\omega(Z)=AZ+B+C+C_1e^{kz}+C_2e^{-kz},$$

$$\eta_y(Z)=L_{\eta_y}(C_1e^{kz}+C_2e^{-kz})+M_{\eta_y}\left(A\frac{Z^3}{6}+B\frac{Z^2}{2}+C\frac{Z^2}{2}\right)+N_{\eta_y}\frac{Z^2}{2}+$$

$$+(C_3\cos\chi_0-C_4\sin\chi_0)Z+\frac{\mu_1}{cG}AZ+C_5,$$

$$\theta_{x0}(Z)=L_{\theta_x}(C_1e^{kz}-C_2e^{-kz})+M_{\theta_x}\left(A\frac{Z^2}{2}+BZ+CZ\right)+N_{\theta_x}Z+C_6,$$

$$\theta_{z0}(Z)=L_{\theta_z}(C_1e^{kz}-C_2e^{-kz})+M_{\theta_z}\left(A\frac{Z^2}{2}+BZ+CZ\right)+N_{\theta_z}Z+C_7 \qquad (8.155)$$

$A=\frac{Q_{y0}}{\mu_1}$	$B=\frac{M_{x0}-Q_yl\sin\chi_0}{\mu_2}$	$C=\frac{M_{z0}+Q_yl\cos\chi_0}{\mu_3}$
$L_{\eta_y}=-\frac{2}{1-\nu}\frac{aI_\omega\operatorname{ctg}\chi_0}{m+n}$	$M_{\eta_y}=\operatorname{tg}\chi_0(\nu\sin^2\chi_0-2)$	$N_{\eta_y}=N_{\theta_z}\cos\chi_0-$ $-N_{\theta_x}\sin\chi_0$
$L_{\theta_x}=\frac{I^*\cos\chi_0}{kI_x}$	$M_{\theta_x}=$ $=\frac{1}{\cos\chi_0}(1-\nu\sin^2\chi_0)$	$N_{\theta_x}=-\frac{\bar{f}N_{\theta_z}}{\frac{2}{1-\nu}a\operatorname{ctg}\chi_0}$
$L_{\theta_z}=\frac{k^*}{k}\sin\chi_0$	$M_{\theta_z}=-\sin\chi_0$	$N_{\theta_z}=$ $=\frac{M_{x0}\cos\chi_0+M_{z0}\sin\chi_0}{2e\sin\chi_0 G}$

(8.156)

where

(8.157)

$$m=\bar{f}I_x+\frac{2}{1-\nu}ab\cos^2\chi_0\sin\chi_0,$$

$$n=fI_x+\frac{2}{1-\nu}ab\cos^3\chi_0\sin\chi_0,$$

$$k^*=\frac{m}{n},$$

$$k=\sqrt{\frac{2e\sin^2\chi_0(1+k^*)}{\frac{2}{1-\nu}I_\omega}},$$

$$I^*=a+k^*b\sin^2\chi_0,$$

$$\mu_1=\frac{m+n}{a\cos\chi_0}G,$$

$$\mu_2=\frac{\mu_1 a\sin\chi_0(1-k^*)}{I^*},$$

$$\mu_3=\frac{\mu_1 a\cos\chi_0(1-k^*)}{k^*I_x}.$$

Expressions (8.155) represent the general solution of system (8.153), containing four arbitrary constants. These constants are to be determined from the conditions of fixing of the end Z=0 and from the static boundary condition

$$P_{w}=0 \tag{8.158}$$

on the end Z=1.

Let the end Z=0 of the caisson be completely fixed. In this case, one should assume that

$$\eta_{y}(0)=\theta_{x}(0)=\theta_{z}(0)=w(0)=0. \tag{8.159}$$

In view of (6.32) and (8.151), conditions (8.158) take the form

$$w'(l)=0. \tag{8.160}$$

Expanding conditions (8.159) and (8.160) with the aid of (8.155), we obtain

$$\begin{aligned} &L_{y}(C_1+C_2)+C_5=0,\\ &L_{9_x}(C_1-C_2)+C_4=0,\\ &L_{9_z}(C_1-C_2)+C_3=0,\\ &C_1+C_2+B+C=0,\\ &C_1e^{kl}-C_2e^{-kl}+\frac{1}{k}A=0, \end{aligned} \tag{8.161}$$

whence

$$C_1=-\frac{(B+C)e^{-kl}+\frac{1}{k}A}{e^{kl}+e^{-kl}};\quad C_2=-\frac{(B+C)e^{kl}-\frac{1}{k}A}{e^{kl}+e^{-kl}}, \tag{8.162}$$

$$C_3=L_{9_z}(C_2-C_1);\quad C_4=L_{9_x}(C_2-C_1);\quad C_5=L_{y}(B+C).$$

For the values of the arbitrary constants obtained, by using general solution (8.155), one can obtain expanded expressions for the displacements and stresses at any point of the caisson. We will cite here only the expressions for normal stresses σ_{m_z}.

Using formulas (6.9), (6.10) and relations (1.20), we have

i	U_i	φ_{im_z}	φ_{in_z}
2	η_y	0	$\dfrac{y'}{\sin\chi}$
4	θ_{x0}	$y \sin\chi_0$	$-y \sin\chi_0 \operatorname{ctg}\chi$
6	θ_{z0}	$-y\cos\chi_0$	$\dfrac{xy' - x'y}{\sin\chi_0 \sin\chi} + y \operatorname{ctg}\chi \cos\chi_0$
7	ω	xy	$-xy \operatorname{ctg}\chi$

(8.163)

In view of (8.163), the generalized coordinates of the stressed state ψ_{im_z}, ψ_{in_z}, $\overline{\psi}_{in_z}$ assume the following form on the rectangular contour:

i	U_i	ψ_{im_z}	ψ_{in_z}	$\overline{\psi}_{in_z}$
2	η_y	0	0	0
4	θ_{x0}	$y\sin\chi_0$	$y\lvert\cos\chi\rvert\operatorname{ctg}\chi_0$	0
6	θ_{z0}	$-y\cos\chi_0$	$y\lvert\cos\chi\rvert$	0
7	ω	xy	$xy\operatorname{ctg}^2\chi$	$-y\lvert\operatorname{ctg}\chi\rvert$

(8.164)

Introducing (8.164) into (6.19), we have

$$\sigma_{m_z} = \frac{E}{1-\nu^2}\left[\theta'_{x0}(\sin\chi_0 + \nu\operatorname{ctg}\chi_0\lvert\cos\chi\rvert) + \theta'_{z0}(-\cos\chi_0 + \nu\lvert\cos\chi\rvert) + \right.$$
$$\left. + \omega'(1 + \nu\operatorname{ctg}^2\chi)x - \omega\nu\lvert\operatorname{ctg}\chi\rvert\right] y. \tag{8.165}$$

Taking the first equation of system (8.153) into account, we can reduce formula (8.165) to the form

$$\sigma_{m_z} = \frac{E}{1-\nu^2}\left\{-\eta''_y + \omega'x + \nu\left[\lvert\cos\chi\rvert\left(\theta'_{x0}\operatorname{ctg}\chi_0 + \theta'_{z0} - \omega\frac{1}{\sin\chi}\right) + \right.\right.$$
$$\left.\left. + \omega' x \operatorname{ctg}^2\chi\right]\right\} y. \tag{8.166}$$

Introducing expanded expressions (8.155) for the generalized displacements into (8.166), one can calculate the stresses σ_{m_z} at any point of the caisson. For a caisson with a set of ribs oriented parallel to the flow, Figure 8.44 shows the results of calculation with formulas (8.137) and (8.166). This figure also shows the experimental results.

5. Simplified Analysis of Shells of Variable Section

Given below are the fundamental equations and formulas for analyzing slightly conical oblique shells of swept wing type, constituting a generalization of the

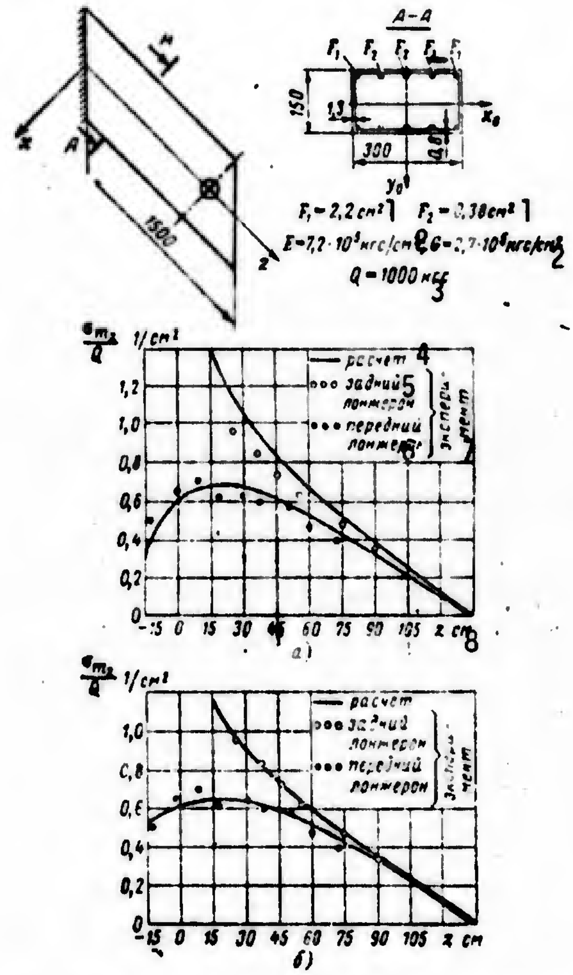


Figure 8.44. Comparison of results of simplified calculation with experiment: a - ribs considered to be perpendicular to generatrices; b - ribs considered to be oriented parallel to the flow. Key: 1) cm^2, 2) kgs/cm^2, 3) kgs, 4) calculation, 5) rear part, 6) front spar, 7) experiment, 8) cm.

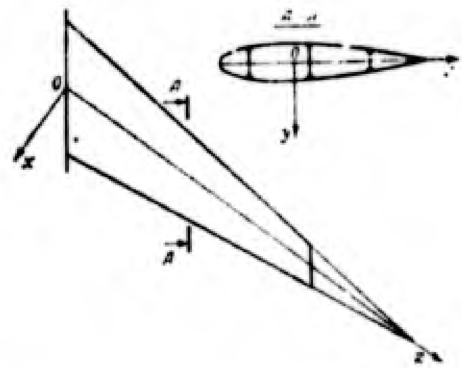


Figure 8.45. Slightly conical oblique shell of swept wing type.

corresponding expressions for shells of constant cross section obtained in Subsection 3. For this reason, our discussion will be in the nature of a survey and will not substantiate again the choice of the retained generalized displacements, coordinate function $\phi^1(S)$, axes, etc., assuming them to be the same as in Subsection 3. We will use here the resolvents constructed only on the basis of the fundamental static-geometric model, since, as comparative calculations have shown, consideration of rib orientation does not introduce any appreciable difference into the results.

Figure 8.45 shows a slightly conical oblique shell of swept-wing type. The wing profile is assumed to be close to symmetric. We superpose the Oz axis on the line of centroids of the cross sections of the shell, the cross sections being understood to mean the lines of intersection of the middle surface by spheres with their center at the cone apex. It is obvious that because of the slight conicity of the surface, these sections may be considered planar and perpendicular to the Oz axis. Superposing the Ox, Oy axes on the major axes of inertia of the cross sections so that the moment of inertia I_x is minimal, we will keep the four generalized displacements η_y, θ_x, θz, w as the coordinate function $\phi_w^1(S)$, just as in the case of sweptback shells of constant cross sections, we will choose the function $\phi_2^1(S)$, determined by expressions (8.8) and (8.10).

The resolvents in the desired generalized displacements can be readily obtained from general canonical relations for a conical shell of arbitrary configuration. In this case, the generalized displacement $\eta_y(Z)$ constitutes the displacement in the direction of the Oy axis of the pole, rigidly bound to the section Z=const and coinciding before deformation with the apex of the conical surface; the generalized displacement $\theta_x(Z)$ is the angle of rotation of the section Z=const as a solid about the axis passing through the apex of the conical surface parallel to the Ox axis.

The coordinate functions corresponding to the canonical kinematic unknowns are represented by expressions (3.6). Since the fundamental system of Cartesian coordinates is chosen so that the Oz axis is always located inside the conical surface, by taking into consideration the slight conicity, we have

$$l_s \approx l_0. \tag{8.167}$$

Angle x between the coordinate lines is determined in the general case by formula (1.11). Taking (8.167) into account, from (1.12) we have

$$\frac{x_0}{l_s},\ \frac{y_0}{l_s} \ll 1. \tag{8.168}$$

and consequently, we find from (1.12)

$$\cos\chi \approx x_0' \operatorname{ctg}\gamma_0, \tag{8.169}$$

whence it follows that the values of the coordinate angles x in slightly conical oblique shells can be assumed to be equal to the coordinates of the angles in oblique cylindrical shells (6.3).

Considering (8.167) and (8.168), we obtain from (3.6)

i	U_i	φ_{im_x}	φ_{im_s}	φ_{in_n}
1	η_x	$-\frac{x_0}{l_0}$	x_0'	$-\frac{y_0'}{\sin\chi}$
2	η_y	$-\frac{y_0}{l_0}$	y_0'	$\frac{x_0'}{\sin\chi}$
3	ϑ	$1-\frac{x_0}{l_0}\operatorname{ctg}\gamma_0$	$x_0'\operatorname{ctg}\gamma_0$	$\frac{x_0 y_0' - x_0' y_0}{l_0\sin\chi}$

(8.170)

4	θ_x	0	$l_0 y_0' + (x_0' y_0 - x_0 y_0') \operatorname{ctg} \gamma_0$	$-\frac{x_0'}{\sin \chi} l_0$
5	θ_y	0	$-l_0 x_0'$	$-\frac{y_0'}{\sin \chi} l_0$
6	θ_z	0	$x_0 y_0' - x_0' y_0$	$\frac{1}{\sin \chi} (x_0 x_0' - y_0 y_0')$
7	w	φ_2^1	$\varphi_2^1 \cos \chi$	0

(8.170)

Expanding expressions (3.17) with the aid of (8.170), we also find

i	U_i	ψ_{im_z}	$\psi_{in_zm_z}$	ψ_{in_z}
1	η_x	$-\frac{x_0}{l_0}$	$2\frac{x_0}{l_0} \operatorname{ctg} \chi + \frac{1}{\sin \chi} x_0'$	$-\operatorname{ctg} \chi \left(\frac{x_0}{l_0} \operatorname{ctg} \chi + \frac{x_0'}{\sin \chi} \right)$
2	η_y	$-\frac{y_0}{l_0}$	$2\frac{y_0}{l_0} \operatorname{ctg} \chi + \frac{1}{\sin \chi} y_0'$	$-\operatorname{ctg} \chi \left(\frac{y_0}{l_0} \operatorname{ctg} \chi + \frac{y_0'}{\sin \chi} \right)$
3	η_z	$1 - \frac{x_0}{l_0} \operatorname{ctg} \chi_0$	$\operatorname{ctg} \chi \left(2\frac{x_0}{l_0} \operatorname{ctg} \chi_0 - 1 \right)$	$-\frac{x_0}{l_0} \operatorname{ctg}^2 \chi \operatorname{ctg} \chi_0$
4	θ_x	0	$\frac{1}{\sin \chi} [l_0 y_0' + (x_0' y_0 - x_0 y_0') \operatorname{ctg} \chi_0]$	$-\frac{\cos \chi}{\sin^2 \chi} [l_0 y_0' + (x_0' y_0 - x_0 y_0') \operatorname{ctg} \chi_0]$
5	θ_y	0	$-l_0 \frac{x_0'}{\sin \chi}$	$l_0 \operatorname{ctg} \chi \frac{x_0'}{\sin \chi}$
6	θ_z	0	$\frac{x_0 y_0' - x_0' y_0}{\sin \chi}$	$\frac{\cos \chi}{\sin^2 \chi} (x_0' y_0 - x_0 y_0')$
7	w	φ_2^1	$-\varphi_2^1 \operatorname{ctg} \chi$	0

(8.171)

$$\bar{\psi}_{wm_z} = 0; \; \bar{\psi}_{wn_zm_z} = \frac{1}{\sin \chi} (\varphi_2^1)'; \; \bar{\psi}_{wn_z} = 0. \qquad (8.172)$$

We will again consider shells of symmetric profile and similar ones in the sense in which this was discussed in Subsection 3 for shells of constant cross section. In this case, taking (8.171) and (8.172) into account, we can conclude from (7.15) that the matrices of the coefficients of the differential resolvents formally coincide with matrices (8.81)-(8.84). Therefore, the canonical resolvents in the desired generalized displacements η_y, θ_x, θ_z, w resulting from general system (7.19) and (7.20) assume the form

$$\tilde{a}_{44}\left(1-\frac{Z}{l_0}\right)\eta'_y+\tilde{a}_{45}\left(1-\frac{Z}{l_0}\right)^2\theta'_x+\tilde{a}_{46}\left(1-\frac{Z}{l_0}\right)^2\theta'_z+\tilde{a}_{47}\left(1-\frac{Z}{l_0}\right)\omega'+$$
$$+\tilde{b}_{47}\omega=\frac{Q_{y0}}{G}.$$
$$\tilde{a}_{54}\left(1-\frac{Z}{l_0}\right)\eta'_y+\tilde{a}_{55}\left(1-\frac{Z}{l_0}\right)^2\theta'_x+\tilde{a}_{56}\left(1-\frac{Z}{l_0}\right)^2\theta'_z+\tilde{a}_{57}\left(1-\frac{Z}{l_0}\right)\omega'+$$
$$+\tilde{b}_{57}\omega=\frac{1}{1-\frac{Z}{l_0}}\cdot\frac{M_{x0}}{G}, \tag{8.173}$$
$$\tilde{a}_{64}\left(1-\frac{Z}{l_0}\right)\eta'_y+\tilde{a}_{65}\left(1-\frac{Z}{l_0}\right)^2\theta'_x+\tilde{a}_{66}\left(1-\frac{Z}{l_0}\right)^2\theta'_z+\tilde{a}_{67}\left(1-\frac{Z}{l_0}\right)\omega'+$$
$$+\tilde{b}_{67}\omega=\frac{1}{1-\frac{Z}{l_0}}\,\frac{M_{z0}}{G}.$$

$$\tilde{a}_{74}\left[\left(1-\frac{Z}{l_0}\right)\eta'_y\right]'+\tilde{a}_{75}\left[\left(1-\frac{Z}{l_0}\right)^2\theta'_x\right]'+\tilde{a}_{76}\left[\left(1-\frac{Z}{l_0}\right)^2\theta'_z\right]'-$$
$$-\tilde{b}_{47}\eta'_y-\tilde{b}_{57}\left(1-\frac{Z}{l_0}\right)\theta'_x-\tilde{b}_{67}\left(1-\frac{Z}{l_0}\right)\theta'_z+\tilde{a}_{77}\left[\left(1-\frac{Z}{l_0}\right)\omega'\right]'-$$
$$-\frac{\tilde{c}_{77}}{1-\frac{Z}{l_0}}\,\omega=0. \tag{8.174}$$

In Eqs. (8.173), Q_{y0}, M_{x0}, M_{z0} are the components of the resultant and net moment of the external load applied to the cut-off portion of the shell, in axes passing through the apex of the conical surface parallel to the axes of the fundamental system of Cartesian coordinates. It is evident that

$$Q_{y0}=Q_y,$$
$$M_{x0}=M_x+Q_y(l_0-Z);\quad M_{z0}=M_z. \tag{8.175}$$

Canonical system (8.173) is algebraic with respect to the unknowns $\left(1-\frac{Z}{l_0}\right)\eta'_y$, $\left(1-\frac{Z}{l_0}\right)^2\theta'_x$, $\left(1-\frac{Z}{l_0}\right)^2\theta'_z$. Solving this system and eliminating the components of the law of plane sections from Eq. (8.174) with the aid of the expressions obtained, we can arrive at a resolvent in the generalized warping w. It is easy to see that as a result of the substitution of variable

$$1-\frac{Z}{l_0}=e^t \tag{8.176}$$

this equation will have constant coefficients. Indeed, in this case, system (8.173) and (8.174) assumes the form

$$
\begin{gathered}
-\tilde{a}_{22}\eta_y' - \tilde{a}_{24}e^t\theta_x' - \tilde{a}_{26}e^t\theta_z' - \tilde{a}_{27}\omega' + l_0\tilde{b}_{27}\omega = \frac{Q_{y_0} l_0}{G}, \\
-\tilde{a}_{42}\eta_y' - \tilde{a}_{44}e^t\theta_x' - \tilde{a}_{46}e^t\theta_z' - \tilde{a}_{47}\omega' + l_0\tilde{b}_{47}\omega = \frac{e^{-t}M_{x_0} l_0}{G}, \\
-\tilde{a}_{62}\eta_y' - \tilde{a}_{64}e^t\theta_x' - \tilde{a}_{66}e^t\theta_z' - \tilde{a}_{67}\omega' + l_0\tilde{b}_{67}\omega = \frac{e^{-t}M_{z_0} l_0}{G},
\end{gathered}
\tag{8.177}
$$

$$
\begin{gathered}
\tilde{a}_{72}\eta_y'' + \tilde{a}_{74}(e^t\theta_x')' + \tilde{a}_{76}(e^t\theta_z')' + l_0\tilde{b}_{27}\eta_y' + l_0\tilde{b}_{47}e^t\theta_x' + l_0\tilde{b}_{67}e^t\theta_z' + \\
+\tilde{a}_{77}\omega'' - l_0^2\tilde{c}_{77}\omega = 0.
\end{gathered}
\tag{8.178}
$$

Solving (8.177) for the quantities η_y', $e^t\theta_x'$, $e^t\theta_z'$ and eliminating these quantities and derivatives from (8.178), we obtain, as was shown in Subsection 3.3, an equation of the type

$$
\frac{d^2\omega}{dt^2} - k\omega = F(t). \tag{8.179}
$$

Having determined w, from Eqs. (8.177) solved for η_y', $e^t\theta_x'$, $e^t\theta_z'$ we can then also find all the components of the law of plane sections.

Equations (8.177), (8.178) are general in character. If we solve them in general form, we will arrive at fairly cumbersome expressions. It is therefore more desirable to follow the procedure presented above, by first calculating the coefficients of system (8.177), (8.178) for each specific case. In particular, for a rectangular contour (see Figure 8.33), calculating the quadratures in accordance with (7.15), we obtain

$$
\begin{gathered}
\tilde{a}_{22} = \frac{2}{1-\nu}\,\frac{1}{\sin^4\chi_0}\,\frac{1}{l_0^2}\,a + c, \\
\tilde{a}_{24} = \frac{2}{1-\nu}\,\frac{\cos^2\chi_0}{\sin^4\chi_0}\,\frac{1}{l_0}\,a + l_0 c, \\
\tilde{a}_{26} = -\frac{2}{1-\nu}\,\frac{\operatorname{ctg}\chi_0}{\sin^2\chi_0}\,\frac{1}{l_0}\,a, \\
\tilde{a}_{44} = \frac{2}{1-\nu}\,a\operatorname{ctg}^4\chi_0 + l_0^2 c + f\operatorname{ctg}^2\chi_0, \\
\tilde{a}_{46} = -\frac{2}{1-\nu}\,a\operatorname{ctg}^3\chi_0 - f\operatorname{ctg}\chi_0,
\end{gathered}
\tag{8.180}
$$

$$
\begin{gathered}
\tilde{a}_{66} = \frac{2}{1-\nu}\,a\operatorname{ctg}^2\chi_0 + f, \\
\tilde{a}_{27} = 0;\ \tilde{a}_{47} = 0;\ \tilde{a}_{67} = 0, \\
\tilde{a}_{77} = \frac{2}{1-\nu}\,I_\omega, \\
\tilde{b}_{27} = 2\operatorname{ctg}\chi_0\,\frac{1}{l_0}\,a;\ \tilde{b}_{47} = \bar{f}\operatorname{ctg}\chi_0;\ \tilde{b}_{67} = -\bar{f},\ \tilde{c}_{77} = f.
\end{gathered}
$$

Here, in calculating $\tilde{a}_{22}$ a minor term was discarded on the basis of the estimate

$$1+\frac{1}{3}\,\frac{d_1}{d_2}\,\frac{h_1}{h_2}\approx 1 \tag{8.181}$$

resulting from a comparison of the moments of inertia of the vertical and horizontal panels, and the following notation is used:

$$\begin{gathered} f=a+e,\quad \bar{f}=a-e, \\ a=I_x=\frac{d_1^2 d_2}{2}h_2,\quad e=\frac{d_1 d_2^2}{2}h_1, \\ I_\omega=\left(1+\frac{1-\nu}{2}\operatorname{ctg}^2\chi_0\right)\frac{d_1^2 d_2^2}{24}h_2+\frac{d_1^2 d_2^2}{24}h_1, \\ c=2d_1 h_1. \end{gathered} \tag{8.182}$$

Solving system (8.177) for η_y', $e^t\theta_x'$, $e^t\theta_z'$, after some transformations taking (8.180) into account, we obtain

$$\begin{gathered} \eta_y'=-l_0^3 L_{11}\frac{Q_{y_0}}{G}+l_0^2 L_{12}\frac{M_{x_0}}{G}e^{-t}-l_0^2 L_{13}\frac{M_{z_0}}{G}e^{-t}+l_0^2 L_{10}\omega, \\ e^t\theta_x'=l_0^2 L_{21}\frac{Q_{y_0}}{G}-l_0 L_{22}\frac{M_{x_0}}{G}e^{-t}+l_0 L_{23}\frac{M_{z_0}}{G}e^{-t}+l_0 L_{20}\omega, \\ e^t\theta_z'=-l_0^2 L_{31}\frac{Q_{y_0}}{G}+l_0 L_{32}\frac{M_{x_0}}{G}e^{-t}-l_0 L_{33}\frac{M_{z_0}}{G}e^{-t}+l_0 L_{30}\omega, \end{gathered} \tag{8.183}$$

where in view of the estimate $\left(\frac{x}{l_0}\right)^2,\left(\frac{y}{l_0}\right)^2 \ll 1$:

$$\begin{gathered} L_{11}=L_{13}=L_{31}=L_{33}=\sin^2\chi_0\left(\frac{\cos^2\chi_0}{f}+\frac{1-\nu}{2a}\sin^2\chi_0\right), \\ L_{22}=\sin^2\chi_0\left(\frac{\sin^2\chi_0}{f}+\frac{1-\nu}{2a}\cos^2\chi_0\right), \\ L_{12}=L_{21}=L_{23}=L_{32}=\sin^3\chi_0\cos\chi_0\left(\frac{1}{f}-\frac{1-\nu}{2a}\right), \end{gathered} \tag{8.184}$$

$$\begin{gathered} L_{10}=-L_{30}=-\sin^3\chi_0\cos\chi_0\left(\frac{\bar{f}}{f}-\frac{1}{\sin^2\chi_0}+\nu\right), \\ L_{20}=\sin^4\chi_0\left(\nu\operatorname{ctg}^2\chi_0-\frac{\bar{f}}{f}\right). \end{gathered} \tag{8.185}$$

Eliminating the components of the law of plane sections from Eq. (8.178) by means of expressions (8.183), we arrive at the equation

$$\omega''-l_0^2k^2\omega=l_0^3 L_{01}\frac{Q_{y_0}}{G}+l_0^2 L_{02}\frac{M_{x_0}}{G}e^{-t}+l_0^2 L_{03}\frac{M_{z_0}}{G}e^{-t}. \tag{8.186}$$

where, in view of (8.180), (8.184), (8.185),

$$k^2=(1-\nu)\frac{I_x}{I_\omega}\sin^2\chi_0\left(1+\nu\cos^2\chi_0-\frac{\bar{f}}{f}\sin^2\chi_0\right).\tag{8.187}$$

$$L_{01}=-L_{02}=\frac{\operatorname{ctg}\chi_0}{2I_x}\left[k^2-\nu(1-\nu)\frac{I_x}{I_\omega}\sin^2\chi_0\right],$$

$$L_{03}=\frac{1}{2I_x}\left[k^2-(1-\nu)\frac{I_x}{I_\omega}\sin^2\chi_0\right].\tag{8.188}$$

From (8.186), taking (8.188) into account, we find

$$\begin{aligned}w(t)=C_1e^{kl_0t}+C_2e^{-kl_0t}+\frac{kl_0}{2GI_x}\left\{\operatorname{ctg}\chi_0\left[1-\nu(1-\nu)\frac{1}{k^2}\,\frac{I_x}{I_\omega}\sin^2\chi_0\right]\times\right.\\ \times\int_0^t[l_0Q_{y_0}(\xi)-M_{x_0}(\xi)e^{-\xi}]\operatorname{sh}[kl_0(t-\xi)]\,d\xi+\left[1-(1-\nu)\frac{1}{k^2}\,\frac{I_x}{I_\omega}\sin^2\chi_0\right]\times\\ \left.\times\int_0^t M_{z_0}(\xi)e^{-\xi}\operatorname{sh}[kl_0(t-\xi)]\,d\xi\right\}.\end{aligned}\tag{8.189}$$

Introducing (8.189) into (8.183), we can find the components of the law of plane sections η_y, θ_x, θ_z then determine the stresses and displacements at any point of the shell.

Let the normal pressure on the surface of the caisson be absent, and to the end $Z=Z_1=l$ be applied a vertical force P_y and moments M and H about axes Ox and Oz, respectively. In this case

$$Q_{y_0}(t)=P_y,\quad M_{x_0}(t)=M+P_y(l_0-l),\quad M_{z_0}(t)=H.\tag{8.190}$$

Considering (8.190) and estimates (8.168), we have from (8.189)

$$w(t)=C_1e^{kl_0t}+C_2e^{-kl_0t}+\frac{1}{k^2G}\left[-L_{01}Q_{y_0}l_0+(L_{01}M_{x_0}-L_{03}H)e^{-t}\right].\tag{8.191}$$

Introducing (8.191) into (8.183), we also find

$$\theta_x(t)=\theta_x(0)+l_0^2\frac{Q_{y_0}}{G}\left(L_{21}-\frac{L_{20}L_{01}}{k^2}\right)(1-e^{-t})+$$
$$+l_0\left[\frac{M_{x_0}}{G}\left(-L_{22}+\frac{L_{20}L_{01}}{k^2}\right)+\frac{M_{z_0}}{G}\left(L_{23}-\frac{L_{20}L_{03}}{k^2}\right)\right]\frac{1-e^{-2t}}{2}+$$
$$+\frac{1}{k}L_{20}[e^{-t}(C_1e^{kl_0t}-C_2e^{-kl_0t})+C_2-C_1],$$
$$\theta_z(t)=\theta_z(0)-l_0^2\frac{Q_{y_0}}{G}\left(L_{31}+\frac{L_{30}L_{01}}{k^2}\right)(1-e^{-t})+$$
$$+l_0\left[\frac{M_{x_0}}{G}\left(L_{32}+\frac{L_{30}L_{01}}{k^2}\right)-\frac{M_{z_0}}{G}\left(L_{33}+\frac{L_{30}L_{03}}{k^2}\right)\right]\frac{1-e^{-2t}}{2}+ \tag{8.192}$$
$$+\frac{1}{k}L_{30}[e^{-t}(C_1e^{kl_0t}-C_2e^{-kl_0t})+C_2-C_1].$$
$$\eta_y(t)=\eta_y(0)-l_0^3\frac{Q_{y_0}}{G}\left(L_{11}+\frac{L_{10}L_{01}}{k^2}\right)t+l_0^2\left[\frac{M_{x_0}}{G}\left(L_{12}+\frac{L_{10}L_{01}}{k^2}\right)-\right.$$
$$\left.-\frac{M_{z_0}}{G}\left(L_{13}-\frac{L_{10}L_{03}}{k^2}\right)\right](1-e^{-t})+\frac{1}{k}l_0L_{10}[C_1(e^{kl_0t}-1)-C_2(e^{-kl_0t}-1)].$$

Let us recall that $\eta_y(T)$ is the vertical displacement of the pole, rigidly bound to the section Z=const and coinciding before deformation with the apex of the conical shell. The vertical displacement $\eta_y^*(t)$ of the section Z=const itself, related to $\eta_y(T)$ by

$$\eta_y^*(t)=\eta_y(t)+\theta_x(t)l_0e^t,$$

will, in view of (8.192), be

$$\eta_y^*(t)=\eta_y^*(0)+l_0(e^t-1)\theta_x(0)+l_0^3\frac{Q_{y_0}}{G}\left(L_{21}-\frac{L_{20}L_{01}}{k^2}\right)(e^t-1-t)+$$
$$+l_0^2\left[\frac{M_{x_0}}{G}\left(-L_{22}+\frac{L_{20}L_{01}}{k^2}\right)+\frac{M_{z_0}}{G}\left(L_{23}-\frac{L_{20}L_{03}}{k_2}\right)\right](\operatorname{ch}t-1)+$$
$$+\frac{1}{k^2}L_{20}[C_1(e^{kl_0t}-1)+C_2(e^{-kl_0t}-1)]. \tag{8.193}$$

Expressions (8.191)-(8.193) for caissons satisfying relation (8.181) constitute a generalization of expressions (8.107) for the case of the cross section. These expressions are considerably simplified if in calculating the coefficients of the resolvents, the forces T_{n_z} are neglected in the expression for the potential energy. In this case, the coefficients of the resolvents

$$\tilde{a}_{33}=\left(\frac{2}{1-\nu}+4\operatorname{ctg}^2\chi_0\right)\frac{a}{l_0^2}+c,$$
$$\tilde{a}_{34}=2\operatorname{ctg}^2\chi_0\frac{a}{l_0}+l_0c,$$
$$\tilde{a}_{36}=-2\operatorname{ctg}\chi_0\frac{a}{l_0};\quad \tilde{a}_{46}=-f\operatorname{ctg}\chi_0,$$
$$\tilde{a}_{44}=f\operatorname{ctg}^2\chi_0+l_0^2c;\quad \tilde{a}_{66}=f, \tag{8.194}$$
$$\tilde{a}_{77}=\frac{2}{1-\nu}l_0,$$
$$\tilde{a}_{37}=\tilde{a}_{47}=\tilde{a}_{67}=0,$$
$$\tilde{b}_{37}=2\operatorname{ctg}\chi_0\frac{a}{l_0};\quad \tilde{b}_{47}=\bar{f}\operatorname{ctg}\chi_0;\quad \tilde{b}_{67}=-\bar{f},$$
$$\tilde{c}_{77}=f,$$

In view of (8.194), expressions (8.183) assume the form

$$\frac{1}{l_0}\eta'_y = -e^t\theta'_x = l_0\frac{f}{\Delta}\left(-\frac{Q_{y_0}l_0}{G} + \frac{M_{x_0}}{G}e^{-t}\right) + \\ -l_0\frac{\bar{f}}{\Delta}\operatorname{ctg}\chi_0\frac{M_{z_0}}{G}e^{-t} + l_0\frac{4ae}{\Delta}\operatorname{ctg}\chi_0\omega, \tag{8.195}$$

$$e^t\theta'_z = l_0\frac{\bar{f}}{\Delta}\operatorname{ctg}\chi_0\left(-\frac{Q_{y_0}l_0}{G} + \frac{M_{x_0}}{G}e^{-t}\right) - \\ -l_0\frac{\frac{2}{1-\nu}a + f\operatorname{ctg}^2\chi_0}{\Delta}\frac{M_{z_0}}{G}e^{-t} - \frac{2}{1-\nu}\frac{a\bar{f}}{\Delta}\omega,$$

where

$$\Delta = a\left(\frac{2}{1-\nu}f + 4ea\operatorname{ctg}^2\chi_0\right). \tag{8.196}$$

Equation (8.186) in the generalized warping assumes the form

$$\omega'' - l_0^2k^2\omega = l_0^2k^2\frac{1-\nu}{2}\frac{1}{a}\operatorname{ctg}\chi_0\left(\frac{Q_{y_0}l_0}{G} - \frac{M_{x_0}}{G}e^{-t}\right) - l_0^2k^2\frac{\bar{f}}{4ae}\frac{M_{z_0}}{G}e^{-t}. \tag{8.197}$$

Integrating Eq. (8.197), then relations (8.195), we find

$$\omega(t) = C_1e^{kl_0t} + C_2e^{-kl_0t} + \operatorname{ctg}\chi_0\frac{M_{x_0}e^{-t} - Q_{y_0}l_0}{\frac{E}{1-\nu^2}I_x} + \operatorname{ctg}\varkappa\frac{M_{z_0}e^{-t}}{GI_{\kappa p}},$$

$$\theta_x(t) = \theta_x(0) + l_0\frac{Q_{y_0}l_0(1-e^{-t}) - \frac{1}{2}M_{x_0}(1-e^{-2t})}{\frac{E}{1-\nu^2}I_x} - \\ -k\operatorname{ctg}\chi_0\frac{I_\omega}{I_x}\left[e^{-t}(C_1e^{kl_0t} - C_2e^{-kl_0t}) + C_2 - C_1\right], \tag{8.198}$$

$$\theta_z(t) = \theta_z(0) - \frac{l_0}{2}\frac{M_{z_0}}{GI_{\kappa p}}(1-e^{-2t}) - \\ -\frac{2}{1-\nu}k\operatorname{ctg}\varkappa\frac{I_\omega}{I_{\kappa p}}\left[e^{-t}(C_1e^{kl_0t} - C_2e^{-kl_0t}) + C_2 - C_1\right],$$

$$\eta_y(t) = \eta_y(0) + l_0^2\frac{-Q_{y_0}l_0t + M_{x_0}(1-e^{-t})}{\frac{E}{1-\nu^2}I_x} + \\ +kl_0\operatorname{ctg}\chi_0\frac{I_\omega}{I_x}\left[C_1(e^{kl_0t} - 1) - C_2(e^{-kl_0t} - 1)\right],$$

where

$$\operatorname{ctg}\varkappa = \frac{\bar{f}}{f}, \tag{8.199}$$

$$I_{\kappa p} = f(1 - \operatorname{ctg}^2\varkappa), \tag{8.200}$$

$$k = \frac{1}{\sqrt{\left(\frac{2}{1-\nu}\frac{1}{I_{\kappa p}} + \frac{\operatorname{ctg}^2\chi_0}{I_x}\right)I_\omega}}. \tag{8.201}$$

Let the end Z=0 of the shell be completely fixed. In this case

$$\eta_v(0)=\theta_r(0)=\vartheta_r(0)=0, \tag{8.202}$$

$$\omega(0)=0. \tag{8.203}$$

From condition (8.203), using (8.198) and (8.190), we obtain

$$C_1+C_2=-\operatorname{ctg}\gamma_0\frac{M-P_y l}{\frac{E}{1-\nu^2}I_x}-\operatorname{ctg}\varkappa\frac{H}{GI_{\kappa p}}. \tag{8.204}$$

From static boundary conditions on the end Z=1, we have

$$P_\omega(l)=0. \tag{8.205}$$

Expanding condition (8.205) and taking (3.71), (3.72), (8.194) as well as (3.18), (8.170), (8.171) into account, we obtain

$$\omega'(l)=0, \tag{8.206}$$

whence, in view of (8.190), (8.198)

$$kl_0(C_1e^{kl_0t_1}-C_2e^{-kl_0t_1})-\left(\operatorname{ctg}\gamma_0\frac{M+P_y(l_0-l)}{\frac{E}{1-\nu^2}I_x}+\operatorname{ctg}\varkappa\frac{H}{GI_{\kappa p}}\right)e^{-t_1}=0, \tag{8.207}$$

where t_1 is determined from the relation

$$1-\frac{l}{l_0}=e^{t_1}. \tag{8.208}$$

Solving Eqs. (8.204), (8.207) for constants C_1 and C_2, we obtain

$$\begin{gathered}C_1=\frac{1}{2\operatorname{ch}kl_0t_1}\left\{\frac{\operatorname{ctg}\chi_0}{\frac{E}{1-\nu^2}I_xk}\left[-M\left(ke^{-kl_0t_1}-\frac{1}{l_0-l}\right)+P_y(kle^{-kl_0t_1}+1)\right]-\right.\\ \left.-\frac{\operatorname{ctg}\varkappa}{GI_{\kappa p}k}H\left(ke^{-kl_0t_1}-\frac{1}{l_0-l}\right)\right\},\\ C_2=\frac{1}{2\operatorname{ch}kl_0t_1}\left\{\frac{\operatorname{ctg}\chi_0}{\frac{E}{1-\nu^2}I_xk}\left[-M\left(ke^{kl_0t_1}+\frac{1}{l_0-l}\right)+P_y(kle^{kl_0t_1}-1)\right]-\right.\\ \left.-\frac{\operatorname{ctg}\varkappa}{GI_{\kappa p}k}H\left(ke^{kl_0t_1}+\frac{1}{l_0-l}\right)\right\}.\end{gathered} \tag{8.209}$$

Expressions (8.209) pertain to the case in which the end plane is parallel to the plane Z=const. If however the end plane is perpendicular to the Oz axis, the static boundary conditions for the case at hand can be formulated as for a swept caisson of constant cross section, by orthogonalizing on this end the normal forces T_{m_z} with respect to the generalized coordinate of warping. For a slightly conical caisson

$$\oint T_{m_z}(l - x_0 \operatorname{ctg} \chi_0, S) x_0(S) y_0(S) ds = 0, \tag{8.210}$$

where the integration is carried out over the contour of the end cross section.

For normal forces (3.23) in the case at hand, taking (3.96), (3.97), (8.171), (8.172) into consideration, we can easily obtain

$$T_{m_z}(t,S) = -\frac{Eh}{1-\nu^2}\frac{e^{-t}}{l_0}\left[\eta_y'(\psi_{\eta_y m_z} + \nu\psi_{\eta_y n_z}) + e^t\theta_x'\nu\psi_{\theta_x n_z} + e^t\theta_z'\nu\psi_{\theta_z n_z} + \omega'\psi_{\omega m_z}\right]. \tag{8.211}$$

For practical purposes, it is sufficient to use a simpler expression, taking

$$T_{m_z} = -\frac{Eh}{1-\nu^2}\frac{e^{-t}}{l_0}(\eta_y'\psi_{\eta_y m_z} + \omega'\psi_{\omega m_z}); \tag{8.212}$$

whence in view of (8.171)

$$T_{m_z}(t, S) = \frac{Eh}{1-\nu^2}\frac{e^{-t}}{l_0}\left[\frac{1}{l_0}\eta_y'(t) - x_0(S)\omega'(t)\right] y_0(S). \tag{8.213}$$

Expanding (8.213) with the aid of expressions (8.198) and considering (8.168), we obtain

$$\sigma_{m_z}(z, S) = \frac{M_{var}(z)\, y_0}{I_x}\frac{1}{\left(1-\frac{z}{l_0}\right)^2} + \frac{H}{I_\omega}\frac{x_H}{kl_0}\frac{x_0 y_0}{\left(1-\frac{z}{l_0}\right)^2} +$$
$$+\frac{kE}{1-\nu^2}\left[C_1 e^{-kz}(x_M - x_0) e^{kx_0 \operatorname{ctg} \chi_0} + C_2 e^{kz}(x_M + x_0) e^{-kx_0 \operatorname{ctg} \chi_0}\right]\frac{y_0}{1-\frac{z}{l_0}}. \tag{8.214}$$

Here

z is a Cartesian coordinate related to the oblique coordinate $Z=l_0 Z$ by relation (1.5); in view of (8.167), (8.168)

$$z = Z + x_0 \operatorname{ctg} \chi_0; \tag{8.215}$$

$$M_{изг}(z) = M - P_y(l - z) \tag{8.216}$$

is the bending moment in the cross section Z=const;
the constant quantities

$$x_M = k \operatorname{ctg} \chi_0 \frac{I_\omega}{I_c}; \quad x_H = \frac{2}{1-\nu} k \operatorname{ctg} \chi \frac{I_\omega}{I_{кр}}. \tag{8.217}$$

Introducing (8.214) into (8.210), we can readily obtain

$$C_1 e^{-kl} - C_2 e^{kl} = \frac{H}{GI_{кр}} \frac{\operatorname{ctg} \chi}{l_0 - l} R. \tag{8.218}$$

where

$$R = \frac{d_2^2}{24}(d_2 h_2 + d_1 h_1)\Big\{\Big[e^{k\frac{d_2}{2}\operatorname{ctg}\chi_0}\Big(d_2\frac{x_M - \frac{d_2}{2}}{2} - \frac{x_M - \frac{d_2}{2}}{k \operatorname{ctg} \chi_0} - \frac{2}{k^2 \operatorname{ctg}^2 \chi_0}\Big) + \frac{x_M}{k \operatorname{ctg} \chi_0} + \frac{2}{k^2 \operatorname{ctg}^2 \chi_0}\Big] h_2 \operatorname{tg} \chi_0 + k_1 e^{k\frac{d_2}{2}\operatorname{ctg}\chi_0}\Big(x_M - \frac{d_2}{2}\Big)\frac{d_2 h_2}{12}\Big\}^{-1}. \tag{8.219}$$

Now, solving Eq. (8.204) together with Eqs. (8.218), we find

$$C_1 = -\frac{1}{1+e^{-2kl}}\left[\operatorname{ctg}\chi_0 \frac{M - P_y l}{\frac{E}{1-\nu^2} I_c} + \operatorname{ctg}\chi \frac{H}{GI_{кр}}\left(1 - \frac{R}{l_0 - l} e^{-kl}\right)\right],$$
$$C_2 = -\frac{1}{1+e^{2kl}}\left[\operatorname{ctg}\chi_0 \frac{M - P_y l}{\frac{E}{1-\nu^2} I_c} + \operatorname{ctg}\chi \frac{H}{GI_{кр}}\left(1 + \frac{R}{l_0 - l} e^{kl}\right)\right]. \tag{8.220}$$

For not-too-short caissons, formulas (8.209), (8.220) in view of (8.131) respectively take the form

$$C_1=e^{kl_0t_1}\left\{\frac{\operatorname{ctg}\chi_0}{\frac{E}{1-\nu^2}I_xk}\left[-M\left(ke^{-kl_0t_1}-\frac{1}{l_0-l}\right)+P_y\left(kle^{-kl_0t_1}+1\right)\right]-\right.$$

$$\left.-\frac{\operatorname{ctg}\varkappa}{GI_{\text{кр}}k}H\left(ke^{-kl_0t_1}-\frac{1}{l_0-l}\right)\right\},$$

$$C_3=e^{kl_0t_1}\left\{\frac{\operatorname{ctg}\chi_0}{\frac{E}{1-\nu^2}I_xk}\left[-M\left(ke^{kl_0t_1}+\frac{1}{l_0-l}\right)+P_y\left(kle^{kl_0t_1}-1\right)\right]-\right. \tag{8.221}$$

and

$$\left.-\frac{\operatorname{ctg}\varkappa}{GI_{\text{кр}}k}H\left(ke^{kl_0t_1}+\frac{1}{l_0-l}\right)\right\},$$

$$C_1=-\operatorname{ctg}\chi_0\frac{M-P_yl}{\frac{E}{1-\nu^2}I_x}-\operatorname{ctg}\varkappa\frac{H}{GI_{\text{кр}}}\left(1-\frac{R}{l_0-l}e^{-kl}\right),$$

$$C_3=-e^{-2kl}\left[\operatorname{ctg}\chi_0\frac{M-P_yl}{\frac{E}{1-\nu^2}I_x}+\operatorname{ctg}\varkappa\frac{H}{GI_{\text{кр}}}\left(1+\frac{R}{l_0-l}e^{kl}\right)\right]. \tag{8.222}$$

For practical calculations, taking the rougher estimate (8.134), from formulas (8.222) we obtain

$$C_1=-\operatorname{ctg}\chi_0\frac{M-P_yl}{\frac{E}{1-\nu^2}I_x}-\operatorname{ctg}\varkappa\frac{H}{GI_{\text{кр}}},$$

$$C_3=-\operatorname{ctg}\varkappa\frac{H}{GI_{\text{кр}}}\frac{Re^{-kl}}{l_0-l}. \tag{8.223}$$

As already noted above, the difference in the results of the calculations, due to the difference in constants (8.221), (8.222) and (8.223), is manifested only in the end portion of the caisson. We will therefore give preference to simpler expressions (8.223).

From (8.214), in view of (8.223), we have

$$\sigma_{m_z}=\left(\frac{M_{\text{изг}}y_0}{I_x}+\frac{2}{1-\nu}\operatorname{ctg}\varkappa\frac{H}{I_{\text{кр}}}\frac{x_0y_0}{l_0}\right)\frac{1}{\left(1-\frac{z}{l_0}\right)^2}-$$

$$-\frac{Hx_H}{I_\omega}\frac{R}{l_0-l}\frac{e^{k(z-l)}}{1-\frac{z}{l_0}}(x_0+x_M)y_0e^{-kx_0\operatorname{ctg}\chi_0}+ \tag{8.224}$$

$$+\frac{(M-P_yl)x_M+Hx_H}{I_\omega}\frac{e^{-kz}}{1-\frac{z}{l_0}}(x_0-x_M)y_0e^{kx_0\operatorname{ctg}\chi_0}.$$

It is of interest to compare expression (8.224) with its analogous expression (8.137) for a swept caisson of constant cross section. The first terms of these expressions coincide with the elementary solutions on the basis of the hypothesis of plane cross sections for a slightly conical and a prismatic caisson. The remaining terms represent the self-balanced part of the solution, reflecting the fixing conditions and determining the edge effect. Moreover, whereas in a prismatic caisson the edge effect arises only in the region of the fixed end, in a slightly conical caisson, as can be seen by analyzing (8.224), there is an additional edge effect near the free end.

Introducing the values obtained for the arbitrary constants into expressions (8.199), one can further write the expanded formulas for the displacements and remaining components of internal forces. We will cite here only the expression for the tangential forces $S_{n_z m_z}$, and, just as for a prismatic swept caisson, we will obtain it not from Hooke's law, but by integrating a differential equilibrium equation.

For a conical shell of arbitrary configuration, the differential equilibrium equation constituting a generalization of Eq. (8.140) can be represented, as was shown in Section 1.4, in the form

$$\frac{\partial}{\partial m_z}\left[(1-\bar{Z})T_{m_z}\right]+(1-\bar{Z})\frac{\partial}{\partial n_z}(S_{n_z m_z})+\frac{1}{l_s}T_{n_z}=0. \qquad (8.225)$$

Neglecting the quantity $1/l_s(T_{n_z})$ for a slightly shell in comparison with other terms and considering that

$$\frac{\partial}{\partial n_z}=\frac{\partial}{\partial S^*}. \qquad (8.226)$$

where S^* is the arc of the cross section, in view of (8.168) we have

$$\frac{\partial}{\partial z}\left[\left(1-\frac{z}{l_0}\right)T_{m_z}\right]+\frac{\partial S_{n_z m_z}}{\partial S_0^*}=0. \qquad (8.227)$$

where S_0^* is the arc of the cross section z=0.

Integrating (8.227), we arrive at the expression

$$S_{n_z m_z}(z, S_0^*) = S_{n_z m_z}(z, S_{01}^*) - \int_{S_{01}^*}^{S_0^*} \frac{\partial}{\partial z}\left[\left(1 - \frac{z}{l_0}\right) T_{m_z}\right] dS_0^*, \tag{8.228}$$

where the tangential forces $S_{n_z m_z}(z, S_{01}^*)$ are determined from the equilibrium conditions

$$\oint S_{n_z m_z} h^*(z, S^*)\, dS^* - H = 0. \tag{8.229}$$

Changing in expression (8.229) to the coordinate S_0^*, measured in the cross section z=0, in view of (8.143) we have

$$\oint S_{n_z m_z}(z, S_0^*)\, h^*(0, S_0^*)\, dS_0^* - \frac{H}{\left(1 - \frac{z}{l_0}\right)} = 0, \tag{8.230}$$

where

$$h^*(0, S_0^*) = x_0 \frac{dy_0}{dS_0^*} - y_0 \frac{dx_0}{dS_0^*}. \tag{8.231}$$

Expanding equilibrium conditions (8.230) with the aid of (8.228), we obtain

$$S_{n_z m_z}(z, S_{01}^*) = \frac{\frac{H}{\left(1 - \frac{z}{l_0}\right)^2} + \oint \left\{ \int_{S_{01}^*}^{S_0^*} \frac{\partial}{\partial z}\left[\left(1 - \frac{z}{l_0}\right) T_{m_z}(z, \xi)\right] d\xi \right\} h^*(0, S_0^*)\, dS_0^*}{\oint h^*(0, S_0^*)\, dS_0^*}. \tag{8.232}$$

As the origin of S^*_{01} on the contour of the cross section, we choose the generatrix $x_0=0$, $y=d_1/2$. As in the case of a prismatic swept caisson, we will assume for simplicity that the walls do not receive normal stresses. In this case, using (8.224), we can obtain

$$\int_{S^*_{01}}^{S^*_0} \frac{\partial}{\partial z}\left[\left(1-\frac{z}{l_0}\right)T_{m_z}(z,\xi)\right]d\xi=\left\{-\left[\frac{M+P_y(l_0-l)}{I_x}\frac{d_1}{2}x_0+\frac{2}{1-\nu}\operatorname{ctg}\varkappa\frac{H}{I_{\text{кр}}}\times\right.\right.$$
$$\left.\times\frac{d_1}{2}\frac{1}{l_0}\frac{x_0^2}{2}\right]\frac{1}{l_0}\frac{1}{\left(1-\frac{z}{l_0}\right)^2}+\frac{Hx_H}{I_\omega}\frac{R}{l_0-l}\frac{d_1}{2\operatorname{ctg}\chi_0}e^{k(z-l)}\times$$
$$\times\left[\left(x_0+x_M+\frac{1}{k\operatorname{ctg}\chi_0}\right)e^{-kx_0\operatorname{ctg}\chi_0}-x_M-\frac{1}{k\operatorname{ctg}\chi_0}\right]+ \tag{8.233}$$
$$+\frac{(M-P_yl)x_M+Hx_H}{I_\omega}\frac{d_1}{2\operatorname{ctg}\chi_0}e^{-kz}\times$$
$$\left.\times\left[\left(x_0-x_M-\frac{1}{k\operatorname{ctg}\chi_0}\right)e^{kx_0\operatorname{ctg}\chi_0}+x_M+\frac{1}{k\operatorname{ctg}\chi_0}\right]\right\}h_2.$$

Further, using (8.223), we also find

$$\oint\left\{\int_{S^*_{01}}^{S^*_0}\frac{\partial}{\partial z}\left[\left(1-\frac{z}{l_0}\right)T_{m_z}(z,\xi)\right]d\xi\right\}h^*(0,S^*_0)\,dS^*_0=$$
$$=-\frac{2}{1-\nu}\operatorname{ctg}\varkappa\frac{H}{I_{\text{кр}}}\frac{d_1^2d_2^3}{12}h_2\frac{1}{l_0^2}\frac{1}{\left(1-\frac{z}{l_0}\right)^2}+$$
$$+A\left[\frac{(M-P_yl)x_M+Hx_H}{I_\omega}e^{-kz}-\frac{Hx_H}{I_\omega}\frac{R}{l_0-l}e^{k(z-l)}\right]\frac{d_1^2d_2}{\operatorname{ctg}\chi_0}h_2, \tag{8.234}$$

where

$$A=\frac{1}{2}\operatorname{sh}\left(k\frac{d_2}{2}\operatorname{ctg}\chi_0\right)\left[\frac{d_2}{2}-\frac{1}{k\frac{d_2}{2}\operatorname{ctg}\chi_0}\left(x_M+\frac{2}{k\operatorname{ctg}\chi_0}\right)\right]-$$
$$-\frac{1}{2}x_M\operatorname{ch}\left(k\frac{d_2}{2}\operatorname{ctg}\chi_0\right)+x_M+\frac{1}{k\operatorname{ctg}\chi_0}. \tag{8.235}$$

Now, introducing (8.232)-(8.234) into (8.228), we finally obtain

$$S_{n_zm_z}=\left[\frac{M+P_y(l_0-l)}{I_x}\frac{d_1}{2l_0}x_0+\frac{H}{2d_1d_2}\right]\frac{1}{\left(1-\frac{z}{l_0}\right)^2}+$$
$$+\frac{2}{1-\nu}\operatorname{ctg}\varkappa\frac{H}{I_{\text{кр}}}\frac{d_1h_2}{4}\frac{1}{l_0^2\left(1-\frac{z}{l_0}\right)^2}\left(x_0^2-\frac{d_2^2}{6}\right)- \tag{8.236}$$
$$-\frac{Hx_H}{I_\omega}\frac{R}{l_0-l}\frac{d_1h_2}{2\operatorname{ctg}\chi_0}e^{k(z-l)}\left[\left(x_0+x_M+\frac{1}{k\operatorname{ctg}\chi_0}\right)e^{-kx_0\operatorname{ctg}\chi_0}-\right.$$
$$\left.-x_M-\frac{1}{k\operatorname{ctg}\chi_0}+A\right]-\frac{(M-P_yl)x_M+Hx_H}{I_\omega}\frac{d_1h_2}{2\operatorname{ctg}\chi_0}e^{-kz}\times$$
$$\times\left[\left(x_0-x_M-\frac{1}{k\operatorname{ctg}\chi_0}\right)e^{kx_0\operatorname{ctg}\chi_0}+x_M+\frac{1}{k\operatorname{ctg}\chi_0}-A\right].$$

It is of interest to compare expression (8.236) with its analogous expression (8.149) for a swept caisson of constant cross section. The first terms of these expressions coincide with elementary solutions based on the hypothesis of plane cross sections for a slightly conical and a prismatic caisson, and determine, respectively, the fluxes of tangential stresses in bending and the "Bredt" fluxes in torsion. The remaining terms represent self-balanced fluxes determining the edge effects. Moreover, as can be seen from expression (8.236), the edge effects in the zones of both ends take place in the slightly conical swept caisson.

8.3. Low-Aspect Wing Type Shells

1. Wing of Arbitrary Outline

The resolvents and other relations necessary for calculating low-aspect wing type conical shells (Figure 8.46) are given in Ch. III. We will use numerical examples to analyze the character of convergence of the expansions corresponding to warping of the contour Z=const. Just as for a straight and a swept wing, we will consider expansions in power functions and piecewise-linear local coordinate functions.

By virtue of relation (8.1), characteristic of any low-aspect wings as well as wings of all other types under any typical loads, we will assume as a first approximation that the neutral axis of the section $\bar{Z}$=const coincides with the major axis of inertia of this section, about which the moment of inertia I of the section is minimal. In this connection, choosing the plane xOz so that it intersects the plane of the directrix along a line parallel to its neutral axis, we will specify the power coordinate functions $\phi_k^1(S)$ in the form (8.2), and the piecewise linear ones, so that they satisfy relation (8.3).

Since in low-aspect wings, bending and torsion are inseparable from each other, as in swept wings, of the six generalized displacements corresponding to the displacement of the contour $\bar{Z}$=const as a solid, we will keep for a symmetric profile η_y, θ_x, θ_z, adding to them the remaining three η_x, η_z, θ_y in the case of

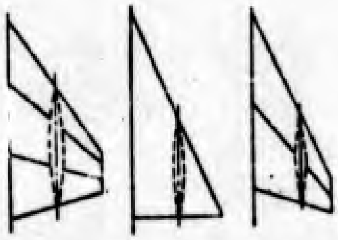
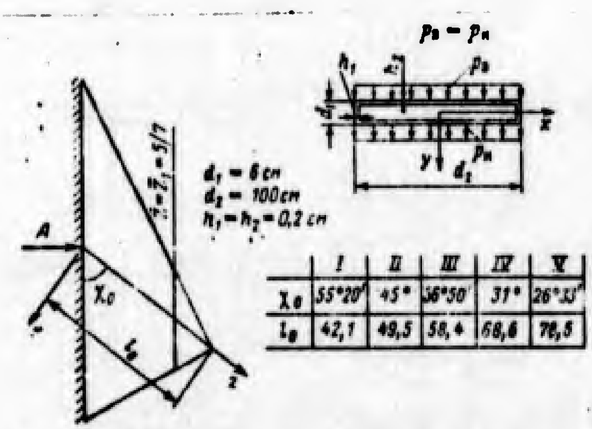


Figure 8.46. Low-aspect wings.



	I	II	III	IV	V
χ_0	55°20'	45°	36°50'	31°	26°33'
l_0	42,1	49,5	58,4	68,6	78,5

Figure 8.47. Low-aspect conical rectangular caisson.

an asymmetric profile, and also when the external load does not satisfy conditions (8.33).*

Given below are the results of comparative analyses of conical caissons of rectangular cross section (Figure 8.47) of different shapes in the plane.

The calculations were carried out according to a universal program written for a conical shell of arbitrary configuration on the basis of the fundamental static-geometric model.

*For certain special low-aspect wing types, the problem in the simplified formulation was discussed by Yu. S. Matyushev. Strength Analysis of a Low-Aspect Wing Type Shell. Izv. VUZov "Aviatsionnaya tekhnika," 1961, No. 3.

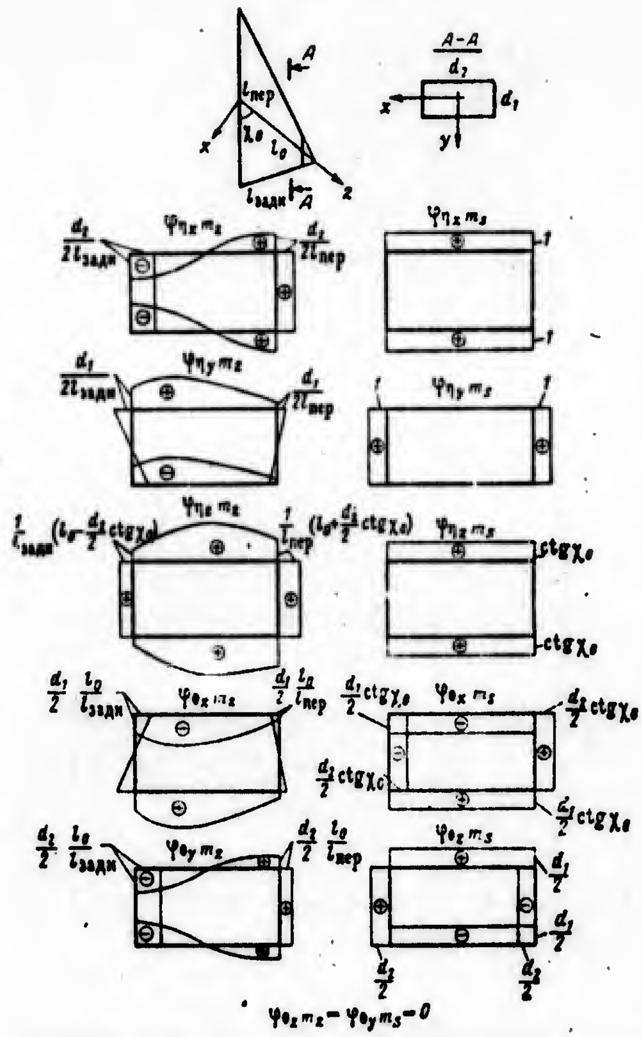


Figure 8.48. Coordinate functions corresponding to the law of plane sections.

For shells of low-aspect wing type, it is characteristic that the coordinate functions ϕ_i and hence the generalized coordinates ψ and $\bar{\psi}$ corresponding to displacement of the contour $\bar{Z}$=const as a solid depend on the shape of the wing in the plane. Figure 8.48 shows a typical form of diagrams of the coordinate functions ϕ_{im_z} and ϕ_{im_s} for a rectangular conical caisson.

Figures 8.49-8.53 show the results of calculation of the action of a uniformly distributed external pressure on conical caissons, represented in Figure 8.47, for different numbers n of retained coordinate functions $\phi_k^1(S)$. As before, n* is the

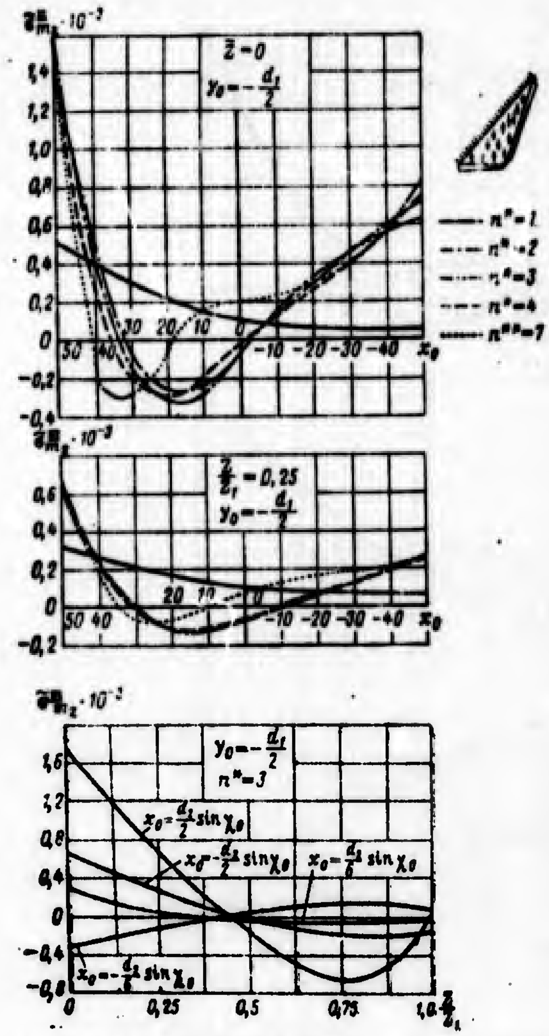


Figure 8.49. Stresses σ_{nz} in caisson III.

number of retained local functions.

Figures 8.55-8.58 illustrate the results of an analysis of caissons (Figure 8.54) of low-aspect wing type with a rhomboidal profile for the action of uniformly distributed pressure.

In Figures 8.59, 8.60, the results of the calculation are compared with experimental data known from literature.

2. Simplified Calculation of a Wing of Arbitrary Outline

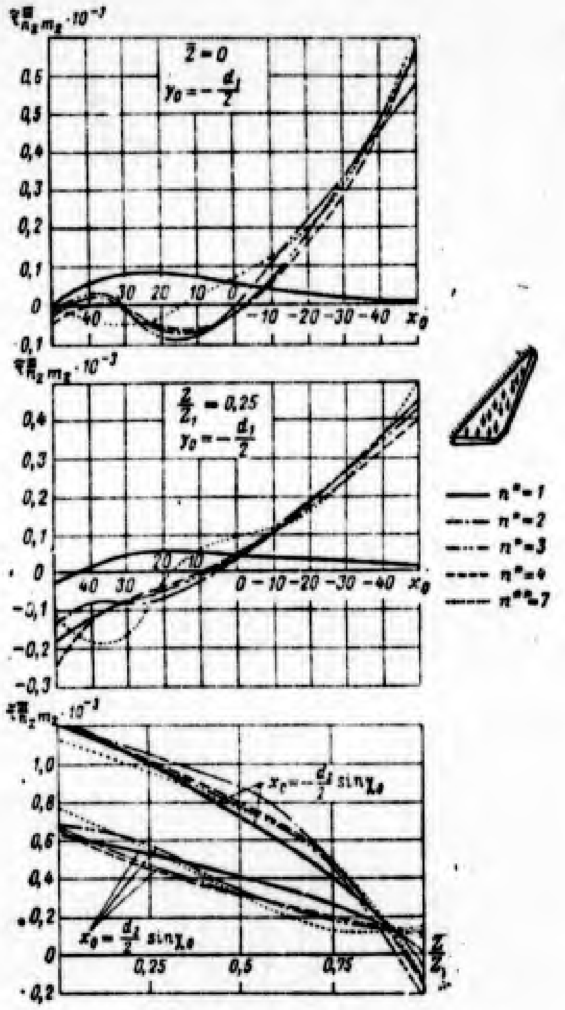


Figure 8.50. Stresses $t_{n_z m_z}$ in caisson III.

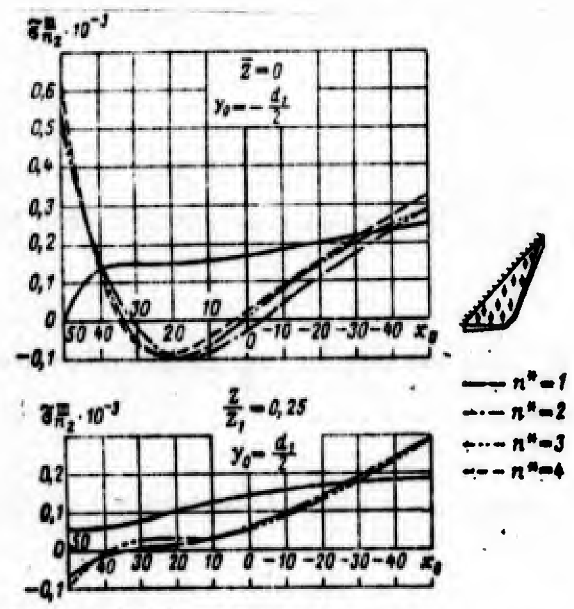


Figure 8.51. Stresses σ_{n_z} in caisson III.

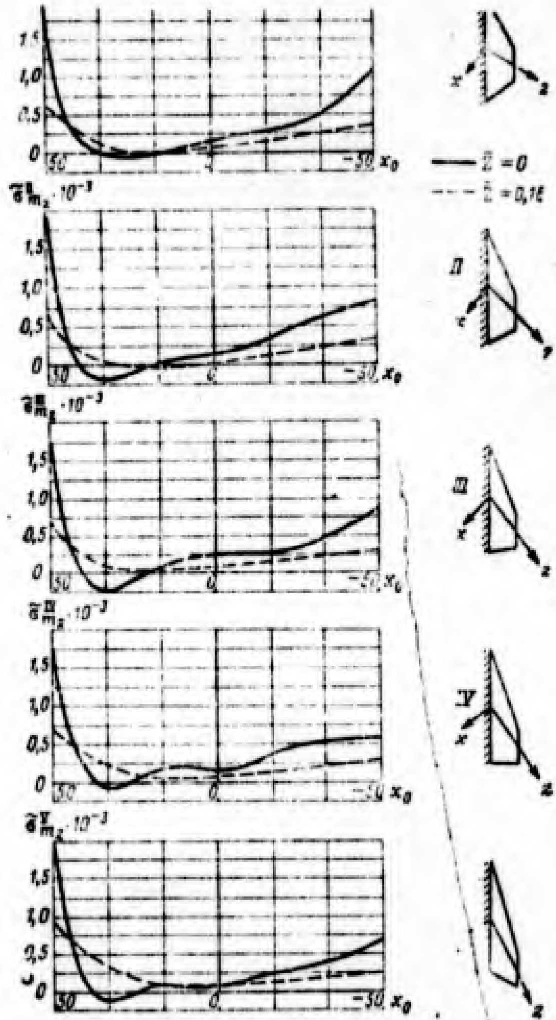


Figure 8.52. Distribution of n[illegible]al stresses in sections of conical [illegible]issons at different angles x_0.

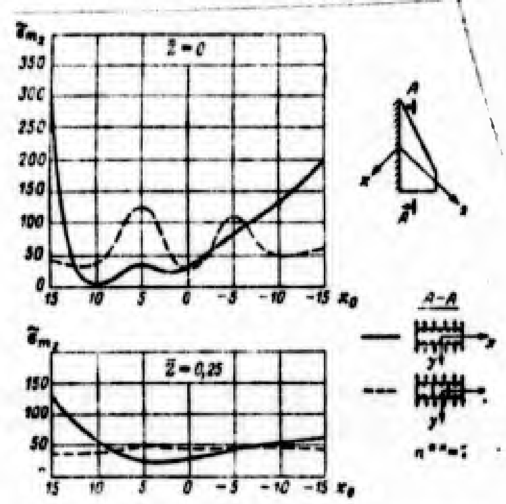
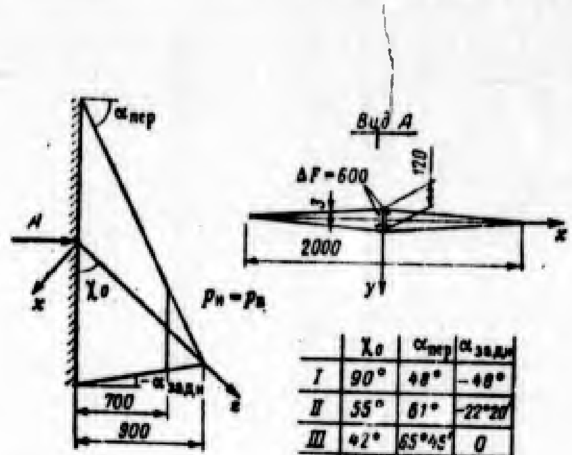


Figure 8.53. Results of comparative calculations of single- and multiclosed conical caissons.



	χ_0	$\alpha_{пер}$	$\alpha_{задн}$
I	90°	48°	-48°
II	55°	61°	-22°20'
III	42°	65°45'	0
IV	35°	68°30'	18°30'

Figure 8.54. Low-aspect wing with rhomboidal profile. Key: 1) view A.

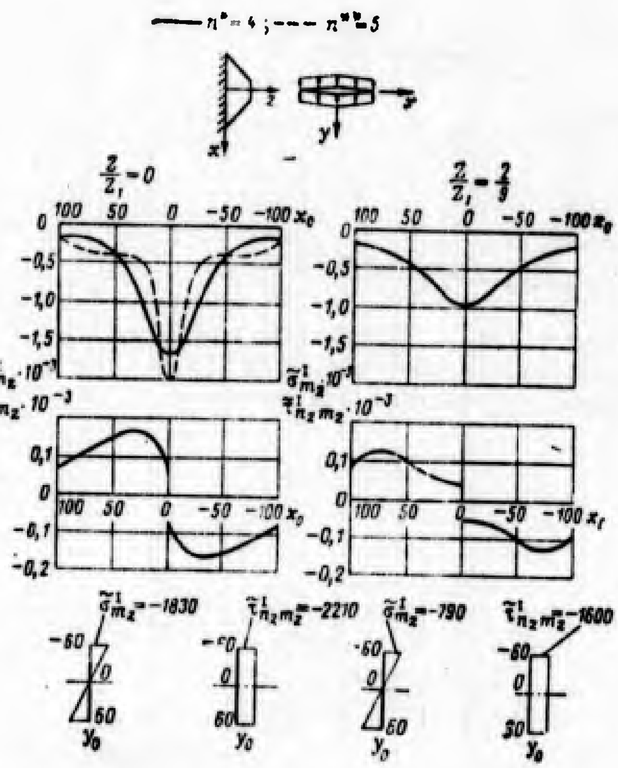


Figure 8.55. Distribution of normal and tangential stresses in the panels and wall of wing I.

number of retained local functions.

Figures 8.55-8.58 illustrate the results of an analysis of caissons (Figure 8.54) of low-aspect wing type with a rhomboidal profile for the action of a uniformly distributed pressure.

In Figures 8.59, 8.60, the results of the calculation are compared with experimental data known from literature.

2. Simplified Calculation of a Wing of Arbitrary Outline

In approximating the warping of the section $\bar{Z}$=const with only one function, we must obviously, as above, give preference to torsional warping. In straight and oblique cylindrical shells, we approximated the torsional warping by the function

$$\varphi_3^1(S)=x(S)\,y(S).$$

This same function can be used for the analysis of conical shells. However, as is shown by the analysis of experimental data, more effective in the latter case is the function

$$\varphi_3^1(S)=x_0(S)\,y_0(S)\left(\frac{l_0}{l_z}\right)^2. \tag{8.237}$$

which corresponds to a choice of the fundamental system of Cartesian coordinates in which on the contour $\bar{Z}$=const, including the elements of the longitudinal structure, the following orthogonality conditions are fulfilled:

$$\oint_L \varphi_3^1\,\varphi_{i,m_z}\,\frac{\sin\chi}{l_z}\,h\,dS=0 \tag{8.238}$$

with respect to the generalized coordinates corresponding to displacement of this contour as a solid (i=1, 2, ..., 6). This form of warping constitutes a definite generalization of torsional warping of cylindrical shells, and is suggested by

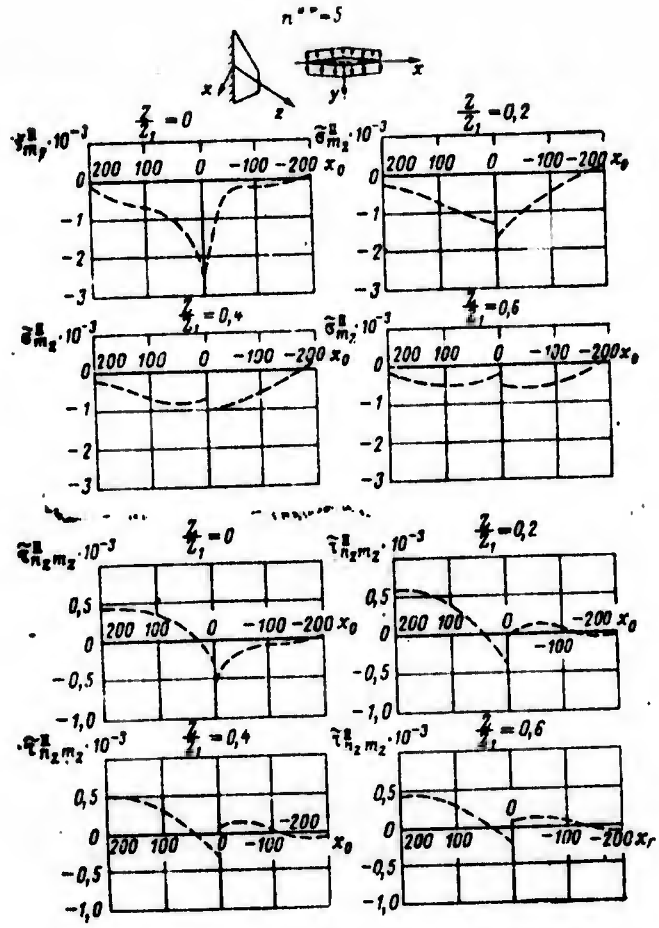


Figure 8.56. Distribution of normal and tangential stresses in wing II.

the analogy to it, as follows from the table below:

	$\varphi_{0_x m_z}$	$\varphi_{0_y m_z}$	φ_2^1
Прямая цилиндрическая оболочка 1	$y(S)$	$-x(S)$	$-\varphi_{0_x m_z}\varphi_{0_y m_z}$
Скошенная цилиндрическая оболочка 2	$y(S)$	$-x(S)$	$-\varphi_{0_x m_z}\varphi_{0_y m_z}$
Коническая оболочка 3	$y_0(S)\frac{l_0}{l_s}$	$-x_0(S)\frac{l_0}{l_s}$	$-\varphi_{0_x m_z}\varphi_{0_y m_z}\frac{l_0}{l_s}$

Key: 1) right cylindrical shell, 2) oblique cylindrical shell, 3) conical shell.

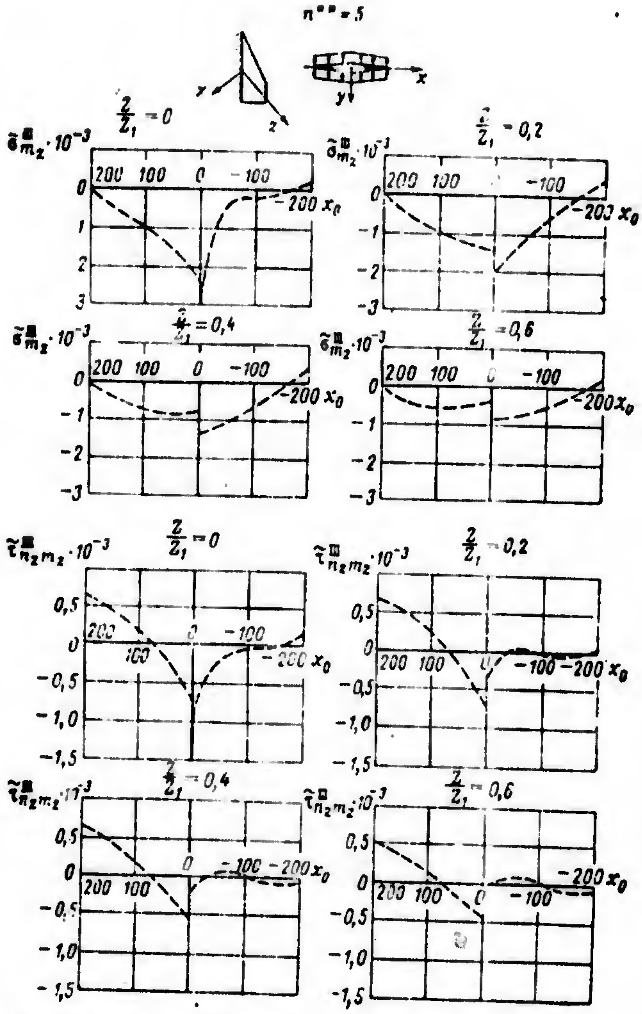


Figure 8.57. Distribution of normal and tangential stresses in wing III.

We will consider shells of symmetric profile and similar shells in the sense discussed for shells of straight and swept wing types.

Let $\tilde{x}\tilde{y}\tilde{z}$ be the initial system of Cartesian coordinates, chosen so that the $O\tilde{z}$ axis is perpendicular to the plane of the directrix, and xyz is the fundamental system of Cartesian coordinates corresponding to conditions (8.238).

As can be readily seen (Figure 8.61)

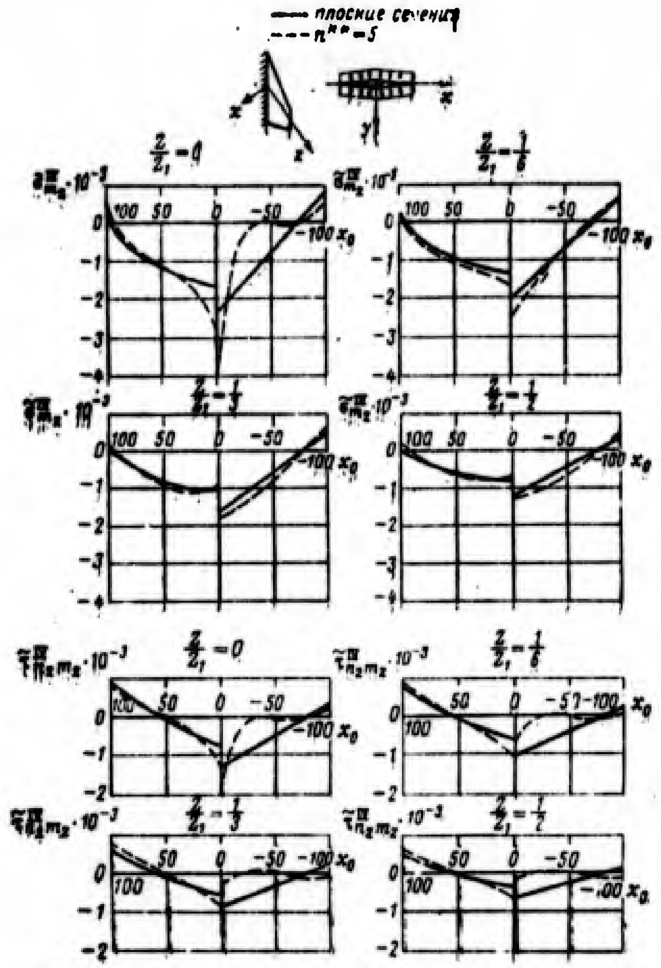


Figure 8.58. Distribution of normal and tangential stresses in wing IV. Key: 1) plane sections.

$$x_0(S) = \frac{\tilde{x}_0(S) - a}{\sqrt{1+\left(\frac{a}{H}\right)^2}},$$
$$\mathrm{ctg}\,\chi_0 = -\frac{a}{H}. \qquad (8.239)$$

Satisfying orthogonality conditions (8.238), we find

$$a = \frac{\oint \tilde{x}_0 \nu_0^2 \left(\frac{l_0}{l_s}\right)^5 \sin\chi\, dF}{\oint \nu_0^2 \left(\frac{l_0}{l_s}\right)^5 \sin\chi\, dF}, \qquad (3.240)$$

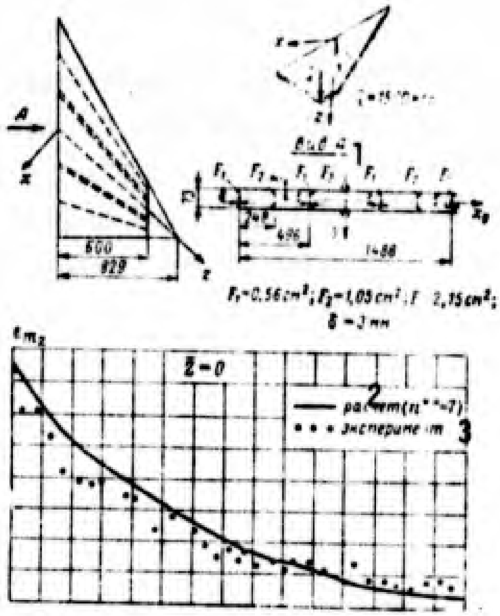


Figure 8.59. Comparison of calculated and experimental results. Key: 1) view A, 2) calculation, 3) experiment.

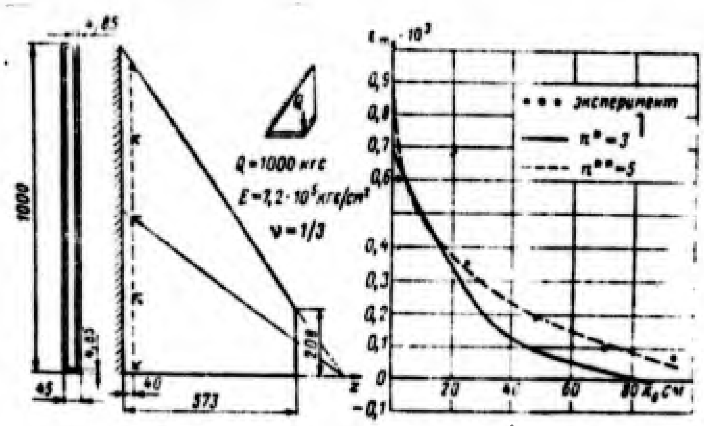


Figure 8.60. Comparison of calculated results with Yu. S. Matyushev's experiment. Key: 1) experiment.

where the differential dF, equal to hdS for s for a smooth shell, signifies that the corresponding quadratures also extend to the elements of the longitudinal structure (Stiltjes integrals).

The use of approximating function (8.237) usually leads to satisfactory results. However, since the validity of such an approximation can only be confirmed experimentally, a more reliable representation of warping is that in the form of an expansion consisting of two terms corresponding to bending and torsion. In this case, superposing the Oz axis on the line of centroids of the contours $\bar{Z}$=const, it is desirable to use as the universal coordinate functions ϕ_1^1, ϕ_2^1 the first even and first odd function, with respect to the x coordinate, of the system of power coordinate functions (8.2)

$$\varphi_1^1(S)=x_0(S)\,y_0(S), \qquad (8.241)$$

$$\varphi_2^1(S)=x_0^2(S)\,y_0(S). \qquad (8.242)$$

We will write the system of differential resolvents in the desired generalized displacements η_y, θ_x, θ_z, w. Considering a conical shell of arbitrary configuration,

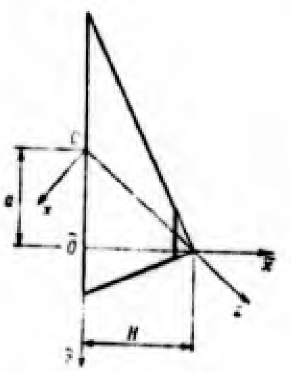


Figure 8.61. In reference to the selection of the approximating function.

we will be able to obtain an analytical solution by using only the canonical variant of the differential solution by using only the canonical variant of the differential resolvents, since otherwise the analytical solution is unacceptably cumbersome.

The coordinate functions corresponding to the canonical kinematic unknowns are represented by expressions (3.6). These expressions contain the parameter l_s, which represents the total length of the generatrix of the conical surface. In contrast to the preceding cases, l_s for low-aspect wings is an essentially variable quantity, which complicates the computation of the coefficients of the differential resolvents. Nevertheless, one can simplify the operation somewhat by using the obvious estimate

$$1 \pm \left(\frac{y_0}{l_s}\right)^2 \approx 1. \tag{8.243}$$

The length l_s of the generatrix and angle x between the coordinate lines are determined in the general case by formulas (1.11), (1.12), whence, in view of (8.243), it follows that

$$l_s^2 \approx x_0^2 + (l_0 - x_0 \operatorname{ctg} \chi_0)^2. \tag{8.244}$$

$$\cos \chi \approx x_0' \frac{l_0}{l_s} \operatorname{ctg} \chi_0 - \frac{x_0'}{\sin^2 \chi_0} \frac{x_0}{l_s}. \tag{8.245}$$

The geometric meaning of the approximate expressions (8.244), (8.245) lies in the fact that the true values of l_s and x on the middle surface of the shell are replaced by the values of the corresponding quantities from the projection of the three-dimensional coordinate grid on the xOz plane. Moreover, angle x on the vertical webs turns out to be equal to $\pi/2$.

Expanding expressions (3.17) with the aid of (3.6), we have

i	U_i	ψ_{im_2}	$\psi_{in_2m_2}$	ψ_{in_2}
1	η_x	$-\frac{x_0}{l_s}$	$2 \operatorname{ctg} \chi \frac{x_0}{l_s} + \frac{x_0'}{\sin \chi}$	$-\operatorname{ctg} \chi \left(\operatorname{ctg} \chi \frac{x_0}{l_s} + \frac{x_0'}{\sin \chi}\right)$
2	η_y	$-\frac{y_0}{l_s}$	$2 \operatorname{ctg} \chi \frac{y_0}{l_s} + \frac{y_0'}{\sin \chi}$	$-\operatorname{ctg} \chi \left(\operatorname{ctg} \chi \frac{y_0}{l_s} + \frac{y_0'}{\sin \chi}\right)$
3	η_z	$-\frac{l_0}{l_s} - \frac{x_0}{l_s} \operatorname{ctg} \chi_0$	$-2 \operatorname{ctg} \chi \left(\frac{l_0}{l_s} - \frac{x_0}{l_s} \operatorname{ctg} \chi_0\right) + \frac{x_0' \operatorname{ctg} \chi_0}{\sin \chi}$	$-\operatorname{ctg} \chi \left[\operatorname{ctg} \chi \left(-\frac{l_0}{l_s} + \frac{x_0}{l_s} \operatorname{ctg} \chi_0\right) + \frac{x_0' \operatorname{ctg} \chi_0}{\sin \chi}\right]$

4	θ_x	0	$\frac{1}{\sin\chi}\left[l_0 y_0' + (x_0' y_0 - x_0 y_0')\,\mathrm{ctg}\,\chi_0\right]$	$-\frac{\cos\chi}{\sin^2\chi}\left[l_0 y_0' + (x_0' y_0 - x_0 y_0')\,\mathrm{ctg}\,\chi_0\right]$
5	θ_y	0	$-\frac{l_0}{\sin\chi}x_0'$	$\frac{\cos\chi}{\sin^2\chi}l_0 x_0'$
6	θ_z	0	$\frac{x_0 y_0' - x_0' y_0}{\sin\chi}$	$-\frac{\cos\chi}{\sin^2\chi}(x_0 y_0' - x_0' y_0)$
⋮	ω	φ^1	$-\varphi^1\,\mathrm{ctg}\,\chi$	0

(8.246)

$$\bar{\psi}_{\omega m_z}=0;\quad \bar{\psi}_{\omega n_z m_z}=\frac{1}{\sin\chi}(\varphi^1)';\quad \bar{\psi}_{\omega n_z}=0. \tag{8.247}$$

Assuming the wing profile to be symmetric, in view of (8.246) and (8.247), we can easily observe that the matrices of coefficients (3.44) of the differential resolvents formally coincide with matrices (8.81)-(8.84), so that the canonical system of resolvents which follows from (3.70), (3.92) will have the form

$$\begin{aligned}
&-a_{33}\eta_y'-a_{34}e^t\theta_x'-a_{36}e^t\theta_z'=a_{37}\omega'-b_{37}\omega+\frac{Q_{y_0}}{G},\\
&-a_{43}\eta_y'-a_{44}e^t\theta_x'-a_{46}e^t\theta_z'=a_{47}\omega'-b_{47}\omega+\frac{e^{-t}M_{x_0}}{G},\\
&-a_{63}\eta_y'-a_{64}e^t\theta_x'-a_{66}e^t\theta_z=a_{67}\omega'-b_{67}\omega+\frac{e^{-t}M_{z_0}}{G},
\end{aligned} \tag{8.248}$$

$$\begin{aligned}
&a_{73}\eta_y''+a_{74}(e^t\theta_x')'+a_{76}(e^t\theta_z')'+b_{37}\eta_y'+b_{47}e^t\theta_x'+\\
&+b_{67}e^t\theta_z'+a_{77}\omega''-c_{77}\omega=0.
\end{aligned} \tag{8.249}$$

Solving (8.248) for the quantities η_y', $e^t\theta_x'$, $e^t\theta_z'$ and eliminating the latter and their derivatives from (8.249), we obtain, as was shown in Section 3.3, an equation of the form

$$\frac{d^2\omega}{dt^2}-k\omega=F(t). \tag{8.250}$$

Having determined w, we can subsequently find all the components of the law of plane sections from Eqs. (8.248), solved for η_y', $e^t\theta_x'$, $e^t\theta_z'$.

Equations (8.248) and (8.249) are general in character, but if we solve them in general form, we arrive at fairly cumbersome expressions. Therefore, it is more desirable to follow the procedure described above, having first computed the coefficients of system (8.248) and (8.249) for each specific case. In particular, for a rectangular contour, having computed the quadratures corresponding to the law of plane sections, according to (3.44), we can obtain

$$
\begin{aligned}
a_{22} &= \frac{1}{l_0}\left(c + \frac{2}{1-\nu}\,\frac{1}{l_0^2}\,\frac{a}{\sin^4\chi_0}\right),\\
a_{24} &= c + \frac{2}{1-\nu}\,\frac{1}{l_0^2}\,\frac{\operatorname{ctg}\chi_0\operatorname{ctg}\chi_{cp}}{\sin^2\chi_0}\,a - \frac{1}{l_0}\,g\operatorname{ctg}\chi_0,\\
a_{26} &= -\frac{2}{1-\nu}\,\frac{1}{l_0^2}\,\frac{\operatorname{ctg}\chi_{cp}}{\sin^2\chi_0}\,a + \frac{1}{l_0}\,g,\\
a_{44} &= l_0 c + \frac{1}{l_0}\operatorname{ctg}^2\chi_0\left(f + \frac{2}{1-\nu}\,a\operatorname{ctg}^2\chi_{cp}\right) - 2g\operatorname{ctg}\chi_0,\\
a_{46} &= -\frac{1}{l_0}\operatorname{ctg}\chi_0\left(f + \frac{2}{1-\nu}\,a\operatorname{ctg}^2\chi_{cp}\right) + g,\\
a_{66} &= \frac{1}{l_0}\left(f + \frac{2}{1-\nu}\,a\operatorname{ctg}^2\chi_{cp}\right).
\end{aligned}
\tag{8.251}
$$

Here, in calculating a_{22}, a minor term was discarded on the basis of estimate (8.181), resulting from a comparison of the moments of inertia of the vertical and horizontal panels, and the following notation was introduced:

$$
\begin{aligned}
f &= \left(1 + \frac{1}{6(1-\nu)}\,\frac{d_2^2}{H^2}\right)a + e,\\
a &= \frac{d_1^2 d_2 \sin\chi_0}{2}\,h_2,\\
c &= d_1\sum_k \left(\frac{l_0}{l_s}\right)_k h_{1k},\\
e &= d_1\sum_k x_{0k}^2\left(\frac{l_0}{l_s}\right)_k h_{1k},\\
g &= d_1\sum_k x_{0k}\left(\frac{l_0}{l_s}\right)_k h_{1k},
\end{aligned}
\tag{8.252}
$$

where $\sum_k$ denotes summation over all vertical webs;

x_m is the angle x of the generatrix passing through the centroid of the directrix.

Using (8.251), for the determinant corresponding to system (8.248), we can obtain

$$
\Delta = \begin{vmatrix} a_{22} & a_{24} & a_{26} \\ a_{42} & a_{44} & a_{46} \\ a_{62} & a_{64} & a_{66} \end{vmatrix} = \frac{2}{1-\nu}\,\frac{1}{H^2}\,\frac{1}{\sin\chi_0}\,acf. \tag{8.253}
$$

Solving system (8.248) for the components of the law of plane sections, we obtain

$$\eta_y' = -L_{22}\frac{Q_{y_0}}{G} + L_{24}\frac{M_{x_0}}{G}e^{-t} - L_{26}\frac{M_{z_0}}{G}e^{-t} + L_{20}^0\omega + L_{20}^1\omega',$$

$$e^t\theta_x' = L_{42}\frac{Q_{y_0}}{G} - L_{44}\frac{M_{x_0}}{G}e^{-t} + L_{46}\frac{M_{z_0}}{G}e^{-t} + L_{40}^0\omega + L_{40}^1\omega', \tag{8.254}$$

$$e^t\theta_z' = -L_{62}\frac{Q_{y_0}}{G} + L_{64}\frac{M_{x_0}}{G}e^{-t} - L_{66}\frac{M_{z_0}}{G}e^{-t} + L_{60}^0\omega + L_{60}^1\omega'.$$

Here

$$L_{ij} = \frac{\Delta_{ji}}{\Delta},$$

$$L_{i0}^{(0)} = \sum_{j=2,4,6}(-1)^{\frac{i+j}{2}} b_{j7}\frac{\Delta_{ji}}{\Delta}, \tag{8.255}$$

$$L_{i0}^{(1)} = -\sum_{j=2,4,6}(-1)^{\frac{i+j}{2}} a_{j7}\frac{\Delta_{ji}}{\Delta},$$

$$(i = 2, 4, 6)$$

where ∇_{jt} is the minor of element a_{ji} of determinant (8.253).

Using expressions (8.254), for the coefficient k and right-hand side of Eq. (8.250) we find

$$k = \frac{c_{77} - \sum\limits_{i=2,4,6} b_{i7}L_{i0}^0}{a_{77}' + \sum\limits_{i=2,4,6} a_{7i}L_{i0}^1}, \tag{8.256}$$

$$F(t) = \frac{\sum\limits_{j=2,4,6} P_j^* \sum\limits_{i=2,4,6}(-1)^{\frac{i+j}{2}} b_{i7}L_{ij} + \sum\limits_{j=2,4,6}\frac{dP_j^*}{dt}\sum\limits_{i=2,4,6}(-1)^{\frac{i+j}{2}} a_{7i}L_{ij}}{a_{77} + \sum\limits_{i=2,4,6} a_{7i}L_{i0}^{(0)}}, \tag{8.257}$$

where

$$P^* \frac{Q_{y_0}}{G},\quad P_4^* = \frac{M_{x_0}}{G}e^{-t},\quad P_6^* = \frac{M_{z_0}}{G}e^{-t}. \tag{8.258}$$

We will now write a canonical system of resolvents for the case in which the warping is represented by two expansion terms. From (3.70) and (3.92), we have

$$-a_{22}\eta_y' - a_{24}e^t\theta_x' - a_{26}e^t\theta_z' = \frac{Q_{y_0}}{G} + \sum_{k=1}^{2}\left[a_{2(6+k)}\omega_k' - b_{2(6+k)}\omega_k\right],$$

$$-a_{42}\eta_y' - a_{44}e^t\theta_x' - a_{46}e^t\theta_z' = \frac{M_{x_0}}{G}e^{-t} + \sum_{k=1}^{2}\left[a_{4(6+k)}\omega_k' - b_{4(6+k)}\omega_k\right], \tag{8.259}$$

$$-a_{64}\eta_y' - a_{64}e^t\theta_x' - a_{66}e^t\theta_z' = \frac{M_{z_0}}{G}e^{-t} + \sum_{k=1}^{2}\left[a_{6(6+k)}\omega_k' - b_{6(6+k)}\omega_k\right], \tag{8.260}$$

$$a_{(6+h)2}\eta_y' + a_{(6+h)4}(e^t\theta_x')' + a_{(6+h)6}(e^t\theta_z')' + b_{2(6+h)}\eta_y' + b_{4(6+h)}e^t\theta_x' +$$

$$+ b_{5(6+h)}e^t\theta_z' + \sum_{k=1}^{2}\left[a_{(6+h)(6+k)}\omega_k' - c_{(6+h)(6+k)}\omega_k\right] = 0$$

$$(h = 1, 2).$$

Solving (8.259) for η_y', $e^t\theta_x'$, $e^t\theta_z'$ and eliminating these quantities and their derivatives from (8.260), we arrive, as follows from (3.187), at equations of the form

$$\begin{aligned}A_{11}^{**}\ddot{\omega}_1+A_{12}^{**}\ddot{\omega}_2+B^{**}\dot{\omega}_2-C_{11}^{**}\omega_1-C_{12}^{**}\omega_2=F_1(t),\\ A_{21}^{**}\ddot{\omega}_1+A_{22}^{**}\ddot{\omega}_2-B^{**}\dot{\omega}_1-C_{21}^{**}\omega_1-C_{22}^{**}\omega_2=F_2(t),\end{aligned} \tag{8.261}$$

where

$$A_{12}=A_{21},\ C_{12}=C_{21}.$$

The coefficients and right-hand sides of system (8.261) are calculated in the same way as for Eq. (8.250).

The integration of system (8.261) is elementary in character. Representing the partial solution of the homogeneous system in the form

$$\omega_1(t)=C_1e^{\lambda t},\ \omega_2(t)=C_2e^{\lambda t},$$

we arrive at the characteristic equation

$$\begin{vmatrix}A_{11}^{**}\lambda^2-C_{11}^{**} & A_{12}^{**}\lambda^2+B^{**}\lambda-C_{12}^{**}\\ A_{21}^{**}\lambda^2-B^{**}\lambda-C_{21}^{**} & A_{22}^{**}\lambda^2-C_{22}^{**}\end{vmatrix}=0,$$

whence

$$(A_{11}^{**}A_{22}^{**}-A_{12}^{**2})\lambda^4-(A_{11}^{**}C_{22}^{**}+A_{22}^{**}C_{11}^{**}-2A_{12}^{**}C_{12}^{**}-B^{**2})\lambda^2+C_{11}^{**}C_{22}^{**}-C_{12}^{**2}=0. \tag{8.262}$$

For the determined values of the roots of the characteristic equation

$$\lambda_1=-\lambda_2,\quad \lambda_3=-\lambda_4$$

the general solution of the homogeneous system has the form

$$\omega_1(t)=\sum_{k=1}^{s}\frac{C_k}{A_{11}^{**}\lambda_k^2-C_{11}^{**}}e^{\lambda_k t},$$

$$\omega_2(t)=\sum_{k=1}^{s}\frac{C_k}{A_{12}^{**}\lambda_k^2+B^{**}\lambda_k-C_{12}^{**}}e^{\lambda_k t}. \tag{8.263}$$

The partial solution of system (8.261) can also be readily found by using the method of variation of arbitrary constants.

In conclusion, we will present the values of the coefficients of the differential resolvents corresponding to warping proper and to the mutual influence of warping and components of displacement of the contour Z=const as a solid for a caisson of rectangular cross section, shown in Figure 8.47.

In approximating the warping by function (8.237), by computing the quadratures in accordance with (3.44), we can obtain

$$a_{67}=-\frac{1}{1-\nu}\,\frac{d_1^2h_2}{4\sin^2\chi_0}\left[\cos^4\chi+\frac{1}{2}\operatorname{ctg}\chi_0\left(\chi-\frac{1}{4}\sin 4\chi\right)\right]\Big|_{\chi_{пер}}^{\chi_{зад}},$$

$$a_{67}=-\frac{1+\nu}{1-\nu}\,\frac{d_1^2h_2}{4}\,l_0\operatorname{ctg}\chi_0\left[\operatorname{ctg}\chi_0\sin^2\chi-\chi-\frac{1}{2}\sin 2\chi\right]\Big|_{\chi_{пер}}^{\chi_{зад}},$$

$$a_{87}=\frac{1+\nu}{1-\nu}\,\frac{d_1^2h_2}{4}\,l_0\left[\operatorname{ctg}\chi_0\sin^2\chi-\chi-\frac{1}{2}\sin 2\chi\right]\Big|_{\chi_{пер}}^{\chi_{зад}},$$

$$\begin{aligned}a_{77}={}&\frac{1}{1-\nu}\,\frac{1}{l_0}\,\frac{d_1^3}{6}\sum_k\left(\frac{l_0}{l_s}\right)_k^7x_{0k}^2h_{1k}+\\&+\frac{d_1^2h_2}{2\sin^2\chi}\,l_0^2\left[\frac{3-\nu+(11-\nu)\operatorname{ctg}^2\chi_0}{16(1-\nu)}\left(\chi-\frac{1}{4}\sin 4\chi\right)-\right.\\&-\frac{1+\nu+(11-\nu)\operatorname{ctg}^2\chi_0}{48(1-\nu)}\sin^3 2\chi-\frac{1}{3}\,\frac{1+\nu}{1-\nu}\operatorname{ctg}\chi_0\sin^6\chi-\\&\left.-\frac{1}{2}\operatorname{ctg}\chi_0\sin^4\chi-\frac{2}{1-\nu}\operatorname{ctg}^2\chi_0\sin^4\chi\cos\chi\right]\Big|_{\chi_{пер}}^{\chi_{зад}},\end{aligned}$$

$$\begin{aligned}b_{87}={}&d_1\sum_k\left(\frac{l_0}{l_s}\right)_k^3x_{0k}h_{1k}+\frac{d_1^2h_2}{2\sin^2\chi_0}\left[\sin^4\chi+\frac{3}{2}\cos^4\chi+\right.\\&\left.+\frac{3}{4}\operatorname{ctg}\chi_0\left(\chi-\frac{1}{4}\sin 4\chi\right)\right]\Big|_{\chi_{пер}}^{\chi_{зад}},\\ b_{67}={}&l_0d_1\sum_k\left(\frac{l_0}{l_s}\right)_k^3x_{0k}h_{1k}-d_1\operatorname{ctg}\chi_0\sum_k\left(\frac{l_0}{l_s}\right)_k^3x_{0k}^2h_{1k}-\\&-\frac{d_1^2h_2}{4}\,l_0\operatorname{ctg}\chi_0\left[\chi-3\operatorname{ctg}\chi_0\sin^2\chi+\frac{3}{2}\sin 2\chi\right]\Big|_{\chi_{пер}}^{\chi_{зад}},\\ b_{77}={}&d_1\sum_k\left(\frac{l_0}{l_s}\right)_k^3x_{0k}^2h_{1k}+\\&+\frac{d_1^2h_2}{4}\,l_0\left[\chi-3\operatorname{ctg}\chi_0\sin^2\chi+\frac{3}{2}\sin 2\chi\right]\Big|_{\chi_{пер}}^{\chi_{зад}}.\end{aligned}\tag{8.264}$$

$$c_{77}=d_1 l_0 \sum_k \left(\frac{l_0}{l_s}\right)_k^3 x_{0k}^2 h_{1k}+\frac{d_1^2 h_2}{2\sin^2\chi_0} l_0^2 \left[\frac{1}{16}(9\,\mathrm{ctg}^2\chi_0+5)\chi-\right.$$

$$-\frac{1}{4}\sin 2\chi-3\left(\mathrm{ctg}\,\chi_0+\frac{1}{32}\right)\sin^4\chi+3\,\mathrm{ctg}\,\chi_0\sin^6\chi-$$

$$\left.-\frac{9}{16}(\mathrm{ctg}^2\chi_0-1)\left(\frac{1}{4}\sin 4\chi-\frac{1}{3}\sin^3 2\chi\right)\right]\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}}. \tag{8.264}$$

In approximating the warping by functions (8.241) and (8.242), we also find

$$a_{37}=-\frac{1}{1-\nu}\frac{1}{l_0^2}\frac{d_1^3}{6}\sum_k\left(\frac{l_0}{l_s}\right)_k^2 x_{0k}h_{1k}-$$

$$-\frac{1}{1-\nu}d_1^2 h_2\sin\chi_0\left(\frac{1}{\sin\chi}+\mathrm{ctg}\,\chi_0\ln\mathrm{tg}\,\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}},$$

$$a_{38}=-\frac{1}{1-\nu}\frac{1}{l_0^2}\frac{d_1^3}{6}\sum_k\left(\frac{l_0}{l_s}\right)_k^2 x_{0k}^2 h_{1k}-\frac{1}{1-\nu}d_1^2 h_2 l_0\sin^3\chi_0\times$$

$$\times\left[2\,\mathrm{ctg}\,\chi_0\frac{1}{\sin\chi}-\frac{1}{2}\frac{\cos\chi}{\sin^2\chi}+\left(\mathrm{ctg}^2\chi_0-\frac{1}{2}\right)\ln\mathrm{tg}\,\frac{\chi}{2}\right]\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}},$$

$$a_{47}=\frac{1+\nu}{1-\nu}\frac{d_1^2 h_2}{2}l_0\sin^3\chi_0\cos\chi_0\left(-\mathrm{ctg}\,\chi_0\frac{1}{\sin\chi}+\frac{1}{2}\frac{\cos\chi}{\sin^2\chi}+\right.$$

$$\left.+\frac{1}{2}\ln\mathrm{tg}\,\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}},$$

$$a_{48}=\frac{1+\nu}{1-\nu}\frac{d_1^2 h_2}{2}l_0^2\sin^4\chi_0\cos\chi_0\left[(1-\mathrm{ctg}^2\chi_0)\frac{1}{\sin\chi}-\frac{1}{3\sin^3\chi}+\right.$$

$$\left.+\mathrm{ctg}\,\chi_0\frac{\cos\chi}{\sin^2\chi}+\mathrm{ctg}\,\chi_0\ln\mathrm{tg}\,\frac{\chi}{2}\right]\Bigg|_{\chi_n}^{\chi_0}, \tag{8.265}$$

$$a_{67}=-\frac{1+\nu}{1-\nu}\frac{d_1^2 h_2}{2}l_0\sin^3\chi_0\left(-\mathrm{ctg}\,\chi_0\frac{1}{\sin\chi}+\frac{1}{2}\frac{\cos\chi}{\sin^2\chi}+\right.$$

$$\left.+\frac{1}{2}\ln\mathrm{tg}\,\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}},$$

$$a_{68}=-\frac{1+\nu}{1-\nu}\frac{d_1^2 h_2}{2}l_0^2\sin^4\chi_0\left[(1-\mathrm{ctg}^2\chi_0)\frac{1}{\sin\chi}-\frac{1}{3\sin^3\chi}+\right.$$

$$\left.+\mathrm{ctg}\,\chi_0\frac{\cos\chi}{\sin^2\chi}+\mathrm{ctg}\,\chi_0\ln\mathrm{tg}\,\frac{\chi}{2}\right]\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}},$$

$$a_{77}=\frac{1}{1-\nu}\frac{d_1^3}{6}\frac{1}{l_0}\sum_k\left(\frac{l_0}{l_s}\right)_k x_{0k}^2 h_{1k}+$$

$$+\frac{d_1^2 h_2}{2}l_0^2\sin^4\chi_0\left[\frac{1+\nu}{1-\nu}(\mathrm{ctg}^2\chi_0-1)\chi-\frac{1}{3}\mathrm{ctg}^3\chi-\right.$$

$$\left.-\left(\frac{1+\nu}{1-\nu}+\mathrm{ctg}^2\chi_0\right)\mathrm{ctg}\,\chi-\frac{4}{1-\nu}\mathrm{ctg}\,\chi_0\ln\sin\chi\right]\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{зам}}},$$

$$a_{78}=\frac{1}{1-\nu}\frac{d_1^3}{6}\frac{1}{l_0}\sum_k\left(\frac{l_0}{l_s}\right)_k x_{0k}^3 h_{1k}+\frac{d_1^2 h_2}{2}l_0^3\sin^6\chi_0\times$$

$$
\begin{aligned}
&\times\left[\frac{1+\nu}{1-\nu}\operatorname{ctg}\chi_0(\operatorname{ctg}^2\chi_0-3)\chi-\operatorname{ctg}\chi_0\operatorname{ctg}\chi\left(3\frac{1+\nu}{1-\nu}+\operatorname{ctg}^2\chi_0\right)+\right.\\
&+\frac{1}{2}\left(\frac{1+\nu}{1-\nu}+3\operatorname{ctg}^2\chi_0\right)\operatorname{ctg}^2\chi-\operatorname{ctg}\chi_0\operatorname{ctg}^3\chi+\\
&\left.+\frac{1}{4}\operatorname{ctg}^4\chi+\frac{1+\nu}{1-\nu}(1-3\operatorname{ctg}^2\chi)\ln\sin\chi\right]\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
a_{88}&=\frac{1}{1-\nu}\,\frac{d_1^3}{6}\,\frac{1}{l_0}\sum_k\left(\frac{l_0}{l_s}\right)_k x_{0k}^3h_{1k}+\\
&+\frac{d_1^2h_2}{2}\,l_0^4\sin^8\chi_0\left\{(1-6\operatorname{ctg}^2\chi_0+\operatorname{ctg}^4\chi_0)\frac{1+\nu}{1-\nu}\chi+\right.\\
&+\left[\frac{1+\nu}{1-\nu}(1-6\operatorname{ctg}^2\chi_0)-\operatorname{ctg}^4\chi_0\right]\operatorname{ctg}\chi+\\
&+2\operatorname{ctg}\chi_0\left(\frac{1+\nu}{1-\nu}+\operatorname{ctg}^2\chi_0\right)\operatorname{ctg}^2\chi-\frac{1}{3}\left(\frac{1+\nu}{1-\nu}+6\operatorname{ctg}^2\chi_0\right)\operatorname{ctg}^3\chi+\\
&+\operatorname{ctg}\chi_0\operatorname{ctg}^4\chi-\frac{1}{5}\operatorname{ctg}^5\chi+\\
&\left.+4\frac{1+\nu}{1-\nu}\operatorname{ctg}\chi_0(1-\operatorname{ctg}^2\chi_0)\ln\sin\chi\right\}\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
b_{57}&=d_1\sum_k x_{0k}h_{1k}-d_1^2h_2\sin\chi_0\,\frac{1}{\sin\chi}\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
b_{58}&=d_1\sum_k x_{0k}^2h_{1k}+2d_1^2\,h_2l_0\sin^3\chi_0\times\\
&\times\left(\frac{1}{2}\,\frac{\cos\chi}{\sin^2\chi}-\operatorname{ctg}\chi_0\frac{1}{\sin\chi}+\frac{1}{2}\ln\operatorname{tg}\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
b_{67}&=l_0d_1\sum_k x_{0k}h_{1k}-\operatorname{ctg}\chi_0d_1\sum_k x_{0k}^2h_{1k}+\\
&+\frac{d_1^2h_2}{4}\,l_0\sin^3\chi_0\cos\chi_0\left(-\frac{\cos\chi}{\sin^2\chi}+\ln\operatorname{tg}\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
b_{68}&=l_0d_1\sum_k x_{0k}^2h_{2k}-\operatorname{ctg}\chi_0d_1\sum_k x_{0k}^3h_{1k}+\frac{d_1^2h_2}{2}\,l_0^2\sin^4\chi_0\cos\chi_0\times\\
&\times\left(-\operatorname{ctg}\chi_0\frac{\cos\chi}{\sin^2\chi}+\frac{2}{3}\,\frac{1}{\sin^3\chi}+\operatorname{ctg}\chi_0\ln\operatorname{tg}\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
b_{77}&=d_1\sum_k x_{0k}^2h_{1k}+\frac{d_1^2h_2}{4}\,l_0\sin^3\chi_0\left(\frac{\cos\chi}{\sin^2\chi}-\ln\operatorname{tg}\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
b_{78}&=d_1\sum_k x_{0k}^3h_{1k}+\frac{d_1^2h_2}{2}\,l_0^2\sin^5\chi\times\\
&\times\left(\operatorname{ctg}\chi_0\frac{\cos\chi}{\sin^2\chi}-\frac{2}{3}\,\frac{1}{\sin^3\chi}-\operatorname{ctg}\chi_0\ln\operatorname{tg}\frac{\chi}{2}\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
c_{77}&=l_0d_1\sum_k\left(\frac{l_s}{l_0}\right)_k x_{0k}^2h_{1k}-\frac{d_1^2h_2}{2}\,l_0^3\sin^4\chi\left(\operatorname{ctg}\chi+\frac{1}{3}\operatorname{ctg}^3\chi\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
c_{78}&=l_0d_1\sum_k\left(\frac{l_s}{l_0}\right)_k x_{0k}^3h_{1k}+\frac{d_1^2h_2}{2}\,l_0^3\sin^6\chi_0\times\\
&\times\left(\frac{1}{4}\,\frac{1}{\sin^4\chi}-\operatorname{ctg}\chi_0\operatorname{ctg}\chi-\frac{1}{3}\operatorname{ctg}\chi_0\operatorname{ctg}^3\chi\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}},\\
c_{88}&=l_0d_1\sum_k\left(\frac{l_s}{l_0}\right)_k x_{0k}^4h_{1k}+2d_1^2h_2l_0^3\sin^8\chi_0\left(\frac{1}{2}\operatorname{ctg}\chi_0\frac{1}{\sin^4\chi}-\right.\\
&\left.-\frac{1}{5}\operatorname{ctg}^5\chi-\frac{1}{3\sin^2\chi_0}\operatorname{ctg}^3\chi-\operatorname{ctg}^2\chi_0\operatorname{ctg}\chi\right)\Bigg|_{\chi_{\text{пер}}}^{\chi_{\text{задн}}}.
\end{aligned}
\tag{8.265}
$$

3. Low-Aspect Wing with an Arbitrarily Oriented Reinforcing Structure

In Subsections 1 and 2, we proceeded from the general resolvents of a conical shell of arbitrary configuration, thereby assuming that the spars of the wings were oriented in the direction of the generatrices of the conical surface. We will now consider a more general case, in which the spars are oriented arbitrarily (Figure 8.62). Various ways of solving such problems are possible. Given below is a

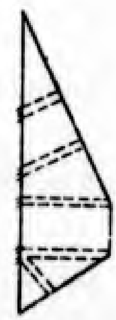


Figure 8.62. Low-aspect wing with arbitrarily oriented spar structure.

general system of resolvents for a conical shell of arbitrary configuration reinforced with an arbitrarily oriented spar structure; a combined method of calculation using the method of forces is presented, and simplified ways of solving the problems are indicated.

A. Differential Resolvents of a Wing-Type Conical Shell with an Arbitrarily Oriented Structure

Let δW_{F_k} be the potential energy variation of a unit length of the kth element of the reinforcing structure. In the general case, we will assume that this element has flanges as well as a web. This makes it possible to consider beam-type elements as well as, assuming a web thickness equal to zero, stringer-type elements. Assuming that the web does not receive normal stresses, we have

$$\delta W_{t_k} = \frac{1}{\cos\beta_u} N_{t_k}^{u} \delta \varepsilon_{t_k}^{0u} + \frac{1}{\cos\beta_u} N_{t_k}^{u} \delta \varepsilon_{t_k}^{0u} + S_{t_k y} d_k \delta \gamma_{t_k y}. \tag{8.266}$$

Here $N_{t_k}^{u}$ are the forces in the upper and lower flanges of the kth element;
$e_{t_k}^{0u}$, $e_{t_k}^{0l}$ are the strains of the upper and lower flanges of the kth element;
$S_{t_k y}$, $\gamma_{t_k y}$ is the flux of tangential forces and shear strain in the web of the kth element;
d_k is the web height of the kth element;
t_k^{u}, t_k^{l}, t_k are unit vectors oriented in the direction of the flanges and axis of the kth element;
β^{u}, β^{u} are the angles made by the flanges of the kth element with the xOz plane.

The relationship between the strain e_{t_k} and the components of the displacement vector can obviously be obtained from general formula (1.53), which determines the linear strain of the middle surface of the shell in an arbitrary tangential direction. The relationship between the shear strain $\gamma_{t_k y}$ and the components of the displacement vector can obviously be represented in the form

$$\gamma_{t_k y} = \frac{\partial u_{t_k}}{\partial y} + \frac{\partial u_y}{\partial t_k}. \tag{8.267}$$

Assuming that the displacement u_{t_k} changes linearly along the web height and considering the web to be incompressible in the direction of the Oy axis, we arrive at a model in which the shear strain $\gamma_{t_k y}$ is constant along the web height, and hence, can be determined from the corresponding displacements in the middle surface of the shell. Consequently, expressions of the following type are obtained for the strains and forces in the reinforcing elements, (3.16) being taken into account:

$$\varepsilon_{t}^{0} = \sum_{i=1}^{6+n} \left[V_i' \frac{1}{l_s} \psi_{it_k} + \frac{1}{1-\bar{z}} V_i \bar{\psi}_{it_k} \right],$$

$$\gamma_{t_k y} = \sum_{i=1}^{6+n} \left[V_i' \frac{1}{l_s} \psi_{it_k y} + \frac{1}{1-\bar{z}} V_i \bar{\psi}_{it_k y} \right]. \tag{8.268}$$

$$N_{t_k} = E_k \Delta F_k \varepsilon_{t_k}^{0}; \quad S_{t_k y} = G_k h_k \gamma_{t_k y}. \tag{8.269}$$

where ψ_{itk}, $\bar{\psi}_{itk}$; ψ_{itky}, $\bar{\psi}_{itky}$ are some functions of the S coordinate;

E_k, G_k are the elastic moduli of the kth element;
ΔF_k, H_k are the areas of the flanges and web thickness of the kth element.

Let ξ_k be the longitudinal coordinate, measured along the projection of the kth element on the xOz plane.

The line of intersection of the kth element with the middle surface of the shell can obviously be represented by parametric equations

$$S=S_k(\zeta_k), \quad \bar{Z}=\bar{Z}_k(\zeta_k). \tag{8.270}$$

Eliminating the parameter ξ_k, we can always find the two-valued dependence $S_k=S_k(\bar{Z})$, which determines the values of the S coordinate on the upper and lower flanges of the kth element.

One can also readily find the derivative (Figure 8.63).

$$\frac{d\zeta_k}{d\bar{Z}}=l_0\frac{\sin\chi_0}{\sin\chi_k}. \tag{8.271}$$

The potential energy variation of the entire structure can obviously be represented in the form

$$\delta U_p=\sum_k\int_0^{\bar{Z}_1}\delta W_{p_k}\frac{d\zeta_k}{d\bar{Z}}d\bar{Z}, \tag{8.272}$$

assuming that

$$\Delta F_k,\ h_k\Big|_{\substack{\bar{Z}<\bar{Z}_{0k}\\ \bar{Z}>\bar{Z}_{1k}}}=0. \tag{8.273}$$

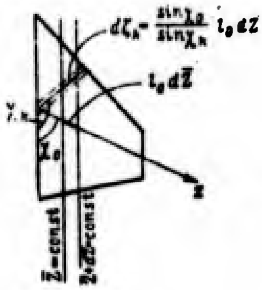


Figure 8.63.

where $\bar{Z}_{ok}$, $\bar{Z}_{1k}$ are the coordinates of the origin and end of the kth element.

Introducing (8.266) into (8.272) and considering (8.268), we obtain

$$\delta W_P = \int_0^{\bar{Z}_1} \sum_{l=1}^{n+m} \left[P_{P_l} \frac{(\delta W_l \lambda_l)'}{\lambda_l} + Q_{P_l} \delta W_l \right] d\bar{Z}. \tag{8.274}$$

where

$$P_{P_l}(\bar{Z}) = \lambda_l \sum_k \left\{ S_{l_k\nu}(\xi, \eta) d_k \bar{\varphi}_{il_k\nu}(\eta) + \right.$$

$$\left. + \left[\frac{N^{\text{B}}_{l_k}(\xi, \eta)}{\cos\beta^{\text{B}}_k(\xi, \eta)} + \frac{N^{\text{H}}_{l_k}(\xi, \eta)}{\cos\beta^{\text{H}}_k(\xi, \eta)} \right] \bar{\varphi}_{il_k}(\eta) \right\} \frac{1}{l_\eta} \frac{d\zeta_k}{d\xi} \Bigg|_{\substack{\xi=\bar{Z} \\ \eta=s_k(\bar{Z})}}, \tag{8.275}$$

$$Q_{P_l}(\bar{Z}) = \frac{\lambda_l}{1-\bar{Z}} \sum_k \left\{ S_{l_k\nu}(\xi, \eta) d_k \bar{\varphi}_{il_k\nu}(\eta) + \right.$$

$$\left. + \left[\frac{N^{\text{B}}_{l_k}(\xi, \eta)}{\cos\beta^{\text{B}}_k(\xi, \eta)} + \frac{N^{\text{H}}_{l_k}(\xi, \eta)}{\cos\beta^{\text{H}}_k(\xi, \eta)} \right] \bar{\varphi}_{il_k}(\eta) \right\} \frac{d\zeta_k}{d\xi} \Bigg|_{\substack{\xi=\bar{Z} \\ \eta=s_k(\bar{Z})}}. \tag{8.276}$$

Considering that for a wing-type shell

$$\cos\beta^{\text{B}}_k \approx \cos\beta^{\text{H}}_k \approx 1,$$

and taking (8.271), into account, we obtain from (8.275) and (8.276)

$$P_{P_l}(\bar{Z}) = \lambda_l \sum_k \left[N^{\text{B}}_{l_k}(\xi, \eta) \bar{\varphi}_{il_k}(\eta) + N^{\text{H}}_{l_k}(\xi, \eta) \bar{\varphi}_{il_k}(\eta) + \right.$$

$$\left. + S_{l_k\nu}(\xi, \eta) d_k \bar{\varphi}_{il_k\nu}(\eta) \right] \frac{l_0}{l_\eta} \frac{\sin\chi_0}{\sin\chi_k(\eta)} \Bigg|_{\substack{\xi=\bar{Z} \\ \eta=s_k(\bar{Z})}}, \tag{8.277}$$

$$Q_{P_l}(\bar{Z}) = \frac{\lambda_l}{1-\bar{Z}} \sum_k \left[N^{\text{B}}_{l_k}(\xi, \eta) \bar{\varphi}_{il_k}(\eta) + N^{\text{H}}_{l_k}(\xi, \eta) \bar{\varphi}_{il_k}(\eta) + \right.$$

$$\left. + S_{l_k\nu}(\xi, \eta) d_k \bar{\varphi}_{il_k\nu}(\eta) \right] l_0 \frac{\sin\chi_0}{\sin\chi_k(\eta)} \Bigg|_{\substack{\xi=\bar{Z} \\ \eta=s_k(\bar{Z})}}. \tag{8.278}$$

The potential energy variation of the system under consideration as a whole will be

$$\delta U_{\Sigma}=\delta U+\delta U_{F}. \tag{8.279}$$

where δU is the potential energy variation of the shell proper, represented by expression (3.35).

Expanding (8.279), in view of (3.35) and (8.274), we have

$$\delta U_{\Sigma}=\int_{0}^{\bar{Z}_1}\sum_{i=1}^{6+n}\left[P_{\Sigma i}\frac{(\delta U_i\lambda_i)'}{\lambda_i}+Q_{\Sigma i}\delta U_i\right]d\bar{Z}, \tag{8.280}$$

where

$$P_{\Sigma i}=P_i+P_{Fi};\quad Q_{\Sigma i}=Q_i+Q_{Fi}. \tag{8.281}$$

Introducing (8.280) into variational Eq. (3.26), we obtain

$$\left(\frac{P_{\Sigma j}}{\lambda_j}\right)'\lambda_j-Q_{\Sigma j}+R_j=0\quad (j=1,2,\ldots,6+n), \tag{8.282}$$

$$(\bar{P}_{\Sigma j}-P_{\Sigma j})\,\delta U_j\Big|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1}=0\quad (j=1,2,\ldots,6+n). \tag{8.283}$$

Here

$$\bar{P}_{\Sigma j}=\bar{P}_j+\bar{P}_{Fj}, \tag{8.284}$$

where $\bar{P}_{Fj}$ is the generalized force corresponding to the work done by external forces applied at the shell ends to the elements of the reinforcing structure.

Let us expand Eq. (8.282). Using (8.268) and (8.269), we obtain

$$\frac{1}{G}P_{Fj}(\bar{Z})=\lambda_j\left[(1-\bar{Z})\sum_{i=1}^{6+n}a_{Fji}V_i'+\sum_{i=1}^{6+n}b_{Fji}V_i\right], \tag{8.285}$$

$$\frac{1}{G}Q_{Fj}(\bar{Z})=\lambda_j\left[\sum_{i=1}^{6+n}b_{Fij}V_i'+\frac{1}{1-\bar{Z}}\sum_{i=1}^{6+n}c_{Fji}V_i\right]. \tag{8.286}$$

Here

$$a_{Fji}(\bar{Z})=\frac{1}{1-\bar{Z}}\sum_{k}\left\{2(1+\nu)\left[\psi_{jlk}(\eta)\psi_{ilk}(\eta)\frac{\Delta F_k^*}{l_\eta}+\right.\right.$$

$$\left.\left.+\psi_{jlk}(\eta)\psi_{ilk}(\eta)\frac{\Delta F_k^*}{l_\eta}\right]+\psi_{jlk\nu}(\eta)\psi_{ilk\nu}(\eta)\frac{h_k d_k}{l_\eta}\right\}\frac{l_0}{l_\eta}\left.\frac{\sin\chi_0}{\sin\chi_k(\eta)}\right|_{\eta=s_k(Z)},$$

$$b_{Fji}(\bar{Z})=\frac{1}{1-\bar{Z}}\sum_{k}\left\{2(1+\nu)\left[\psi_{jlk}(\eta)\bar{\psi}_{ilk}(\eta)\Delta F_k^*+\right.\right.$$

$$\left.\left.+\psi_{jlk}(\bar{\eta})\psi_{ilk}(\eta)\Delta F_k^*\right]+\psi_{jlk\nu}(\eta)\bar{\psi}_{ilk\nu}(\eta)h_k d_k\right\}\frac{l_0}{l_\eta}\left.\frac{\sin\chi_0}{\sin\chi_k(\eta)}\right|_{\eta=s_k(Z)},\tag{8.287}$$

$$c_{Fji}(\bar{Z})=\frac{1}{1-\bar{Z}}\sum_{k}\left\{2(1+\nu)\left[\bar{\psi}_{jlk}(\eta)\bar{\psi}_{ilk}(\eta)\Delta F_k^*+\right.\right.$$

$$\left.\left.+\bar{\psi}_{jlk}(\eta)\bar{\psi}_{ilk}(\eta)\Delta F_k^*\right]+\bar{\psi}_{jlk\nu}(\eta)\bar{\psi}_{ilk\nu}(\eta)h_k d_k\right\}l_0\left.\frac{\sin\chi_0}{\sin\chi_k(\eta)}\right|_{\eta=s_k(Z)}.$$

Introducing expression (3.42), (3.43) and (8.285), (2.286) into (8.281), we have

$$\frac{1}{G}P_j(\bar{Z})=\lambda_j\left[(1-\bar{Z})\sum_{i=1}^{6+n}a_{\Sigma ji}V_i'+\sum_{i=1}^{6+n}b_{\Sigma ji}V_i\right],\tag{8.288}$$

$$\frac{1}{G}Q_j(\bar{Z})=\lambda_j\left[\sum_{i=1}^{6+n}b_{\Sigma ij}V_i'+\frac{1}{1-\bar{Z}}\sum_{i=1}^{6+n}c_{\Sigma ji}V_i\right].\tag{8.289}$$

where

$$a_{\Sigma ji}=a_{ji}+a_{Fji},\quad b_{\Sigma ji}=b_{ji}+b_{Fji},\quad c_{\Sigma ji}=c_{ji}+c_{Fji}.\tag{8.290}$$

Now, using (8.288) and (8.289), we obtain from (8.282)

$$\sum_{i=1}^{6+n}\left\{\left[(1-\bar{Z})a_{\Sigma ji}V_i'+b_{\Sigma ji}V_i\right]'-b_{\Sigma ij}V_i'-\frac{1}{1-\bar{Z}}c_{\Sigma ji}V_i\right\}=-\frac{1}{\lambda_j G}R_j$$

$$(j=1,2,\ldots,6+n).\tag{8.291}$$

Thus, the resolvent systems of ordinary differential equations for shells reinforced in the direction of the generatrices and shells reinforced by an arbitrarily oriented structure are structurally identical. The matrices of their coefficients possess the same symmetry properties. For Eqs. (8.291), represented in mixed form, one can readily find the first six integrals expressing the equilibrium conditions of the cut-off portion of the shell:

$$(1-\bar{Z})\sum_{i=1}^{6+n}a_{\Sigma ji}V_i'+\sum_{i=1}^{6+n}b_{\Sigma ji}V_i=\frac{1}{\lambda_j G}\bar{P}_j$$

$$(j=1,2,\ldots,6+n).\tag{8.292}$$

Equations (8.291), (8.292) pertain equally to the fundamental static-geometric model and to the model with a contour $\bar{Z}$=const nondeformable in its plane. In the first case, the coefficients a_{ji}, b_{ji}, c_{ji} are determined by expressions (3.44), and in the second case, by expressions (3.271). The coefficients a_{Fji}, b_{Fji}, c_{Fji} are determined by expressions (8.287) in both cases. These coefficients are fairly complex in character and are determined by the orientation of the reinforcing structure. To compute the coefficients, using (1.53), (8.267) and (1.32), we can readily obtain

$$
\begin{aligned}
&\psi_{it_0} = \frac{1}{\sin\chi} \mathbf{t}_0 \cdot \mathbf{n}_s \varphi_{it_0}, \\
&\dot{\psi}_{it_0} = \frac{1}{\sin\chi} \mathbf{t}_0 \cdot \mathbf{n}_s \varphi'_{it_0} - \left[(1-\bar{Z}) \frac{\partial}{\partial t_0} (\mathbf{t}_0 \cdot \mathbf{m}_s) + \frac{1}{l_s} (\mathbf{t}_0 \cdot \mathbf{n}_s)^2\right] \varphi_{im_s} - \\
&- \left[(1-\bar{Z}) \frac{\partial}{\partial t_0} (\mathbf{t}_0 \cdot \mathbf{n}_s) - \frac{1}{l_s} (\mathbf{t}_0 \cdot \mathbf{n}_s)(\mathbf{t}_0 \cdot \mathbf{m}_s)\right] \varphi_{in_s} - \frac{1}{R_0} (\mathbf{t}_0 \cdot \mathbf{n}_s)^2 \varphi_{in_\beta}, \\
&\psi_{it_\beta} = \frac{1}{\sin\chi} \mathbf{t}_0 \cdot \mathbf{n}_s \varphi_{i\beta}, \\
&\dot{\psi}_{it_\beta} = \frac{1}{\sin\chi} \mathbf{t}_0 \cdot \mathbf{n}_s \varphi'_{i\beta} + \frac{1}{\rho_1} \varphi_{it_0}.
\end{aligned}
\qquad (8.293)
$$

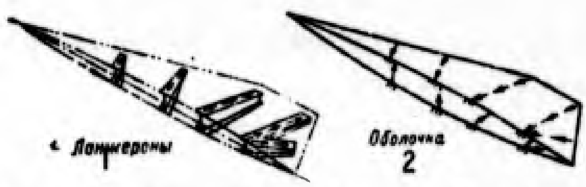


Figure 8.64. Equivalent system. Key: 1) spars, and 2) shell.

B. Combined Method of Calculation

In the combined method of calculation presented below, the arbitrarily oriented reinforcing structure is taken into consideration by means of the method of forces.

We will take as the fundamental system the shell proper, reinforced with a rib structure, as well as a structure of stringers and spars oriented along the generatrices; for brevity, this system will hereinafter be simply referred to as a "shell", and the system of isolated spars oriented arbitrarily, as "spars" (Figure 8.64).

Let q_k be the unknown contact tangential forces of interaction of the kth spar with the shell, directed along the spar contour. Together with the dimensional coordinate ζ_k, measured along the kth spar, we introduce the dimensionless coordinate

$$\bar{\zeta} = \frac{\zeta_k}{l_k}, \tag{8.294}$$

where l_k is the total length of the kth spar, and we represent these forces in the form of the expansions

$$q_k(\bar{\zeta}) = \sum_r q_{kr}\theta_r(\bar{\zeta}) \quad (k=1, 2, \ldots), \tag{8.295}$$

where $\theta_r(\bar{\zeta})$ (r=1, 2, ...) is some system of coordinate functions subject to preselection and q_{kr} (r=1, 2, ...) are the desired coefficients. As the coordinate functions $\{\theta_r\}$ one can adopt any complete system of functions. These can be trigonometric or power functions, polynomials, etc.

Let us note that in the general case $q_k^u \neq q_k^l$, but usually one can nevertheless approximately assume that $q_k^u = q_k^l$, which reduces the number of coefficients q_{kr} in expansions (8.295) by exactly one=half.

The unknown coefficients q_{kr}, in accordance with the method of forces, will be sought from the strin compatibility conditions of the shell and spars. These conditions are conveniently formulated on the basis of the Castigliano principle by minimizing the potential energy of the internal forces of the system with respect to the parameters q_{kr}:

$$\frac{\partial U_\Sigma}{\partial q_{kr}} = 0 \quad (k=1, 2, \ldots, m;\ r=1, 2, \ldots, l). \tag{8.296}$$

Here

$$U_\Sigma = U_0 + \sum_{k=1}^{m} U_k. \tag{8.297}$$

where

U_0 is the potential energy of the shell;
U_k is the potential energy of the kth spar;
m is the number of spars;
t is the number of retained terms of expansion (8.295).

In the general case, the energy

$$U_0=\int_0^{\bar{Z}_1}\oint\frac{1}{E}\left[\frac{1}{2}\sigma^2_{m_z}-\nu\sigma_{m_z}\sigma_{n_z}+\frac{1}{2}\sigma^2_{n_z}+(1+\nu)\tau^2_{n_z m_z}\right]AB\sin\chi h\,dS\,d\bar{Z}, \tag{8.298}$$

where the first term also extends (in the sense of the Stieltjes integral) to the reinforcing elements oriented in the direction of the generatrices.

On condition that $q_k^u = q_k^l$, the energy

$$U_k=l_k\int_0^1\frac{M_k^2(\bar{\zeta})}{2E_kI_k(\bar{\zeta})}\,d\bar{\zeta}, \tag{8.299}$$

where $M_k(\bar{\zeta})$ is the bending moment in the cross section of the kth spar.

In view of the lienarity of the problem, the components of the stressed state of the shell and the bending moments in the spars can be represented in the form

$$\sigma_{m_z}(\bar{Z},S)=\sigma^0_{m_z}(\bar{Z},S)+\sum_{k=1}^{m}\sum_{p=1}^{t}q_{kp}\sigma^{kp}_{m_z}(\bar{Z},S),$$

$$\sigma_{n_z}(\bar{Z},S)=\sigma^0_{n_z}(\bar{Z},S)+\sum_{k=1}^{m}\sum_{p=1}^{t}q_{kp}\sigma^{kp}_{n_z}(\bar{Z},S), \tag{8.300}$$

$$\tau_{n_z m_z}(\bar{Z},S)=\tau^0_{n_z m_z}(\bar{Z},S)+\sum_{k=1}^{m}\sum_{p=1}^{t}q_{kp}\tau^{kp}_{n_z m_z}(\bar{Z},S), \tag{8.301}$$

$$M_k(\bar{\zeta})=M_k^0(\bar{\zeta})+\sum_{p=1}^{t}q_{kp}M_k^p(\bar{\zeta}).$$

where

$\sigma^0_{m_z}$, $\sigma^0_{n_z}$, $\tau^0_{n_z m_z}$ are the stresses in the shell due to a given external load applied directly to the shell;

M^0_k is the bending moment in the kth spar due to a given external load applied directly to the spar;

$\sigma^{kp}_{m_z}$, $\sigma^{kp}_{n_z}$, $\tau^{kp}_{n_z m_z}$, M^p_k are the stresses in the shell and the bending moment in the kth spar due to contact load (8.295), satisfying the conditions

$$q_{kr} = \begin{cases} 1 & (r = p), \\ 0 & (r \neq p). \end{cases} \tag{8.302}$$

Introducing representations (8.300) and (8.301) into Eqs. (8.296), taking (8.298), (8.299) into consideration, we arrive at a system of algebraic equations in the unknowns q_{kr}:

$$\sum_{k=1}^{m} \sum_{r=1}^{t} A^{lp}_{kr} q_{kr} + B^{lp} = 0 \quad (l = 1, 2, \ldots, m), \quad (p = 1, 2, \ldots, t). \tag{8.303}$$

Here

$$A^{lp}_{kr} = \int_0^{Z_1} \oint \frac{1}{E} \left[\sigma^{kr}_{m_z} \sigma^{lp}_{m_z} - \nu \left(\sigma^{kr}_{m_z} \sigma^{lp}_{n_z} + \sigma^{lp}_{m_z} \sigma^{kr}_{n_z} \right) + \sigma^{kr}_{n_z} \sigma^{lp}_{n_z} + 2(1+\nu) \tau^{kr}_{n_z m_z} \tau^{lp}_{n_z m_z} \right] \times$$

$$\times AB \sin \chi \, h \, dS \, d\bar{Z} + \delta_{kl} l_l \int_0^1 \frac{M^r_l(\bar{\zeta}) M^p_l(\bar{\zeta})}{E_l I_l(\bar{\zeta})} d\bar{\zeta}, \tag{8.304}$$

$$B^{lp} = \int_0^{Z_1} \oint \frac{1}{E} \left[\sigma^0_{m_z} \sigma^{lp}_{m_z} - \nu \left(\sigma^0_{m_z} \sigma^{lp}_{n_z} + \sigma^{lp}_{m_z} \sigma^0_{n_z} \right) + \sigma^0_{n_z} \sigma^{lp}_{n_z} + 2(1+\nu) \tau^0_{n_z m_z} \tau^{lp}_{n_z m_z} \right] \times \tag{8.305}$$

where

$$\times AB \sin \chi \, h \, dS \, d\bar{Z} + \delta_{kl} l_l \int_0^1 \frac{M^0_l(\bar{\zeta}) M^p_l(\bar{\zeta})}{E_l I_l(\bar{\zeta})} d\bar{\zeta}, \tag{8.306}$$

$$\delta_{kl} = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

is Kronecker's symbol.

Having solved system (8.303) for the unknown coefficients q_{kr}, one can then, using expressions (8.300) and (8.301), find the stresses in the shell and the bending moments in the spars.

The largest volume of computational work in the procedure described corresponds to the determination of $\sigma^{q}_{m_z}$, $\sigma^{q}_{n_z}$, $\tau^{q}_{n_z m_z}$ and $\sigma^{kr}_{m_z}$, $\sigma^{kr}_{n_z}$, $\tau^{kr}_{n_z m_z}$*. These stressed states are determined by integrating the differential resolvents of the conical shell under the corresponding external load. Use may be made of any of the resolvent variants, obtained above, depending on the specific wing structure. Having represented the general solution of the system of differential resolvents in the form of a general solution of the homogeneous system and partial solution of the inhomogeneous system, we should find a total of $m \cdot t+1$ partial solutions.

C. Simplified Analysis of a Low-Aspect Wing with Arbitrarily Oriented Webs

Certain structures of low-aspect wings can be treated as isotropic conical shells reinforced with an arbitrarily oriented structure of webs (Figure 8.65a), which, being supported by the upper and lower panels of the wing and by the edge rib and edging, are in shear, transforming the normal pressure on the wing into fluxes of tangential contact forces $q_k=q_k(\overline{\zeta})$

With the exception of the edge rib, ribs of such type are absent in wings, and we will therefore assume that the entire air load applied directly to the skin is transmitted further to the webs.

The true distribution of the air load among the webs is unknown, and in this sense the problem is statically indeterminate, but in practical calculations, it is permissible to specify this distribution approximately, on the basis of heuristic representations of the operation of a structure. For example, a satisfactory result can be obtained by dividing the wing into zones adjacent to the webs (see Figure 8.65b). Having first specified the distribution of the air load among the webs, we will separate the kth web from the wing and apply to it the flux of tangential contact forces q_k in addition to the air load (Figure 8.66a). We will neglect the normal stresses in the web cross sections, assuming that the tangential stresses are constant along the web height. From the equilibrium condition of the web element (Figure 8.66 b), we have

*(k=1, 2, ..., m; r=1, 2, ..., t).

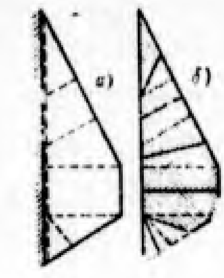


Figure 8.65. Low-aspect wing with arbitrarily oriented webs.

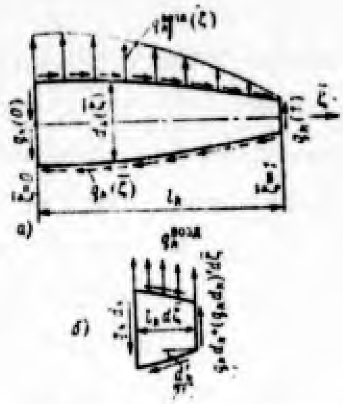


Figure 8.66. Determination of flows q_k in webs. Key: 1) air.

$$d_k \frac{d}{d\zeta} q_k + 2\left(\frac{d}{d\zeta} d_k\right) q_k = -l_k q_k^{возд}. \tag{8.307}$$

Expression (8.307) is a differential equation in the unknown flux of tangential forces q_k. Integrating Eq. (8.307), we can readily obtain

$$q_k(\zeta) = \frac{1}{d_k^2(\zeta)}\left[C - l_k \int_1^{\zeta} q_k^{возд}(\xi) d_k(\xi) d\xi\right], \tag{8.308}$$

where C is an arbitrary constant.

Neglecting the rigidity of the wing edging, we take

$$q_k(1) = 0. \tag{8.309}$$

In view of (8.309), expression (8.308) finally assumes the form

$$q_k(\zeta) = \frac{l_k}{d_k^2(\zeta)} \int_1^{\zeta} q_k^{возд}(\xi) d_k(\xi) d\xi. \tag{8.310}$$

Thus, for a specified distribution of the airload among the webs, the problem of determination of the contact forces q_k turns out to be statically determinate. By combining the determined contact tangential load acting in the direction of the web contours with the air load acting on the wing, we reduce the analysis of the wing to the analysis of a conical shell for a given load. The method described is due to V.I. Figurovskiy.* With a successful distribution of the airload, applying the above-described methods to the analysis of a conical shell, we arrive at completely satisfactory results.

D. Simplified Analysis of a Low-Aspect Wing with a Regular Structure of Spars Perpendicular to the Edge

The method discussed in Par. C can also be applied to the analysis of low-aspect wings reinforced with spars. In this case, by cutting out the spar webs and replacing their action on the wing by tangential fluxes of contact forces determined by expression (8.310), we can, as was done above, reduce the analysis of the wing to that of a conical shell for a given load. The difference from the case discussed in Par. C lies in the fact that we must now deal with a conical shell reinforced with bands of spars. If the spars form a regular structure, by "smearing out" the spars, we can reduce the problem to the analysis of an orthotropic shell. For example, if we deal with a regular structure of spars on a model with a contour $\bar{Z}$=const nondeformable in its plane, it is necessary to assume

$$E_{n_s}=E\left(1+\frac{\Delta F}{hb}\right). \quad (8.311)$$

where

E_{n_s}, E is the reduced modulus and elasticity modulus of the material;

ΔF is the area of the spar band;

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*V.I. Figurovskiy. Method of Analysis of a Conical Shell with a Portion of its Contour Built-in. Izv. VUZov "Aviatsionnaya tekhnika", 1961, No. 2.

b is the distance between the spars;

h is the skin thickness.

Chapter IX. ANALYSIS OF A WING ALLOWING FOR ELASTICITY OF THE RIBS

The differential resolvents obtained in Part One, and along with them all the problems discussed in Ch. VIII, are based on computational models in which the role of ribs, neglecting their actual rigidity, reduces to certain additional constraints imposed on the deformation of the shell proper. Such an approach is approximate, since it does not consider the elastic energy accumulated by the ribs in the course of deformation. The fraction of this energy in the total energy balance of the structure can be appreciable not only when we are dealing with reinforced ribs receiving concentrated loads, but also in the case of normal ribs supporting the wing profile, since normal ribs are subjected to an additional load due to warping of the cross section, as can be readily ascertained by using the example of cylindrical shells.

Let a straight cylindrical shell be completely fixed in section $Z = 0$ and loaded on the opposite end $Z = Z_1$ by an arbitrary system of normal and tangential forces.

From the fixing conditions

$$u(0, S) = 0 \tag{9.1}$$

Considering that the cross section of the shell is nondeformable in its plane, for tangential forces (6.130) on the end $Z = 0$, in view of (9.1), we have

$$S_{n_z m_z}(0, S) = Gh\{\eta_x'(0)x'(S) + \eta_y'(0)y'(S) + \theta_z'(0)[x(S)y'(S) - x'(S)y(S)]\}. \tag{9.2}$$

To within the three constants $\eta_x'(0), \eta_y'(0), \theta_z'(0)$, expression (9.2) determines the law of distribution of tangential forces in the built-in end. Outside a certain region adjacent to it, the tangential forces are known to be distributed in accordance with the law of plane sections, i.e., in a different manner than (9.2). For example, in the case of pure torsion on the end Z = 0, we have from (9.2)

$$S_{n_z m_z}(0) = Gh\theta_z'(0)\left[x(S)y'(S) - x'(S)y(S)\right]. \tag{9.3}$$

and at a certain distance from it, the Bredt distribution

$$S_{n_z m_z}(Z, S) = \mathrm{const}. \tag{9.4}$$

The irregularity of the distribution of tangential forces over the length of the shell is directly related to the warping of the cross section.

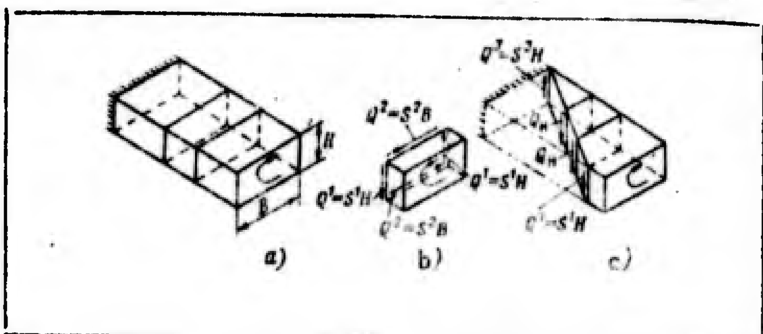

Fig. 9.1. In reference to the analysis of loading of ribs.

Let us now consider a straight prismatic caisson of rectangular cross section, as shown in Fig. 9.1a. Let the system of external forces on the end Z = Z_1 reduce to the twisting moment M_z. We will take the end portion of the caisson, cutoff by a certain cross section Z = const (Fig. 9.1b). With respect to the displacements in the xOy plane, the external twisting moment is obviously balanced by the reactive fluxes of tangential forces $S^1_{n_z m_z}, S^2_{n_z m_z}$ of the cross section Z = const. If the entire separated portion of the caisson lies in the Bredt zone, then

$$S^1_{n_z m_z} = S^2_{n_z m_z} = S = \text{const},$$

so that the transverse forces in the webs

$$Q^1 = \pm SH. \tag{9.5}$$

Let us now consider the end portion of the caisson, cut off by an oblique section (Fig. 9.1c). Let us assume that the internal forces Q_r in the ribs are absent. In view of the fact that in the built-in end, in accordance with expression (9.3), the forces $S_{n_z m_z}$ are constant in each web taken separately, we arrive at a contradiction; this is because in this case, equilibrium in the direction of the Oy axis is impossible, since, as was shown above, $S^3_{n_z m_z} \neq S^1_{n_z m_z}$ hence, $Q^3 = S^3_{n_z m_z} H \neq Q^1$. Equilibrium is obviously possible only in the presence of forces in the ribs that balance the difference $\Delta Q = Q^3 - Q^1$.

Thus, even in the simplest case of torsion of a straight rectangular caisson, forces are set up in the ribs that are related to warping of the cross section and deform the latter in its plane. A similar effect naturally also takes place in conical shells of arbitrary configuration, but in this case the situation becomes much more complex.

Consideration of the above-described phenomenon may be of interest in the case of shells reinforced with a weak structure of normal ribs. For shells reinforced with strengthened ribs receiving large concentrated forces, consideration of the elasticity of the ribs is necessary.

Given below are the differential resolvents of an arbitrary conical wing-type shell reinforced with elastic ribs mounted parallel to the edge section.* Consideration is given to web ribs as well as beam-type ribs which in addition to webs have top and bottom flanges and bracing struts.

The case in which the shell is reinforced with frame-type ribs can be analyzed on the basis of differential resolvents (3.224), allowing for bending of the middle surface of the shell. Here we arrive at a structurally orthtropic model whose local rigidities in the planes parallel to the edge must be calculated by taking into account rigidities of the reinforcing rib frames.

The rib structure is taken into consideration both on the basis of a simplified computational model assuming a continuous distribution of the ribs, and by proceeding strictly from their actual discrete arrangement. In the latter case, the problem reduces to equations with singular coefficients (see Appendices I-IV).

9.1 Geometric Model

Considering the elasticity of the ribs, in contrast to (2.6), we will proceed from the representation of the elastic displacement vector in the form

$$U(M) = U^{0}(M) + U^{1}(M) + U^{2}(M). \tag{9.6}$$

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*For prismatic caissons, see I. F. Obraztsov. Methods of Strength Analysis of Wing-Type Caisson Structures. Moscow, Oborongiz, 1960.

Here U^0 is the vector function corresponding to the displacement of the contour $\bar{Z}$ = const as a solid;

$$U^1(M) = \Omega(\bar{Z}, S)e_3 \tag{9.7}$$

is the vector function corresponding to warping of the contour $\bar{Z}$ = const in the direction of the normal to its plane;

$$U^2(M) = K_1(\bar{Z}, S)e_1 + K_2(\bar{Z}, S)e_2 \tag{9.8}$$

is the vector function corresponding to the deformation of the contour $\bar{Z}$ = const in its plane;

e_1, e_2, e_3 are unit vectors (3.241)

We represent the functions Ω, K_1, K_2 in the form of the expansions

$$\Omega(\bar{Z}, S) = \sum_{k=1}^{n} \omega_k(\bar{Z})\varphi_k^1(S), \tag{9.9}$$

$$K_1(\bar{Z}, S) = \sum_{k=1}^{n_1} \varkappa_k^1(\bar{Z})\varphi_k^{21}(S), \tag{9.10}$$

$$K_2(\bar{Z}, S) = \sum_{k=1}^{n_2} \varkappa_k^2(\bar{Z})\varphi_k^{22}(S), \tag{9.11}$$

where $\{\varphi_k^1\}, \{\varphi_k^{21}\}, \{\varphi_k^{22}\}$ are some preselected systems of coordinate functions; $\omega_k, \varkappa_k^1, \varkappa_k^2$ are the desired generalized displacements.

The functions Ω, K_1, K_2, represented by expansions (9.9) - (9.11), are determined only on the middle surface of the shell. When the elasticity of the ribs is taken into account, it is necessary to extend these functions by determining them also in the region of space occupied by the ribs. For brevity of the treatment, we will consider the ribs to be absolutely flexible out of their plane. In this connection, it is sufficient to extend each of the functions $\{\varphi_k^{21}(S)\}, \{\varphi_k^{22}(S)\}$, by additionally determining them in the region enclosed by the contour of the directrix. In this case, as the curvilinear coordinates of the point $M(x,y,z)$ inside the shell, we can take the coordinates $x_0(M)$, $y_0(M)$ of point M_0 of intersection of the ray passing through apex 0' of the conical surface and through point M with the plane of the directrix, and the curvilinear coordinates $\bar{Z}(M)$, determining the plane passing through point M and parallel to the plane of the directrix. It is obvious that $\bar{Z}(M)$ is the distance from point M to the plane of the directrix, measured along the ray passing through the apex and point M, in fractions of its total length $0'M_0$ (Fig. 9.2).

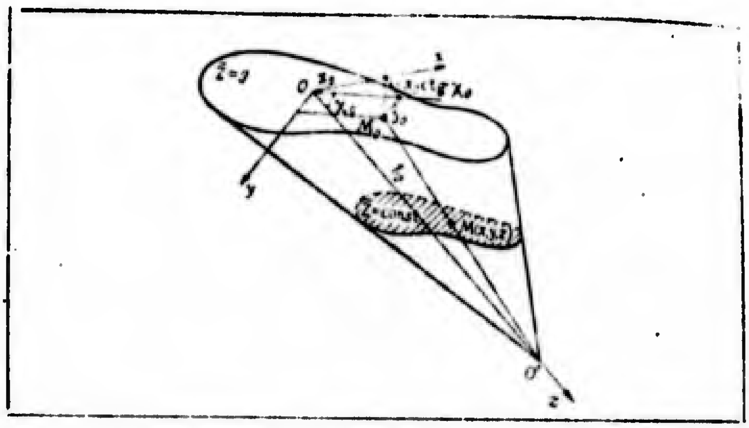

Fig. 9.2. System of curvilinear coordinates $x_0, y_0, \bar{Z}$

The Cartesian coordinates x, y, z of point M and its curvilinear coordinates $x_0, y_0, \bar{Z}$ are related by the obvious relations

$$x = x_0(1-\bar{Z});\; y = y_0(1-\bar{Z});\; z = l_0\bar{Z} + x_0(1-\bar{Z})\operatorname{ctg}\gamma_0. \tag{9.12}$$

For the length ds of an arbitrarily oriented element in coordinates $x_0, y_0, \bar{Z}$, in view of (9.12), we can readily obtain

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \frac{1}{\sin^2\gamma_0}(1-\bar{Z})^2(dx_0)^2 + (1-\bar{Z})^2(dy_0)^2 + \\ + l^2_{M_0}(d\bar{Z})^2 - (1-\bar{Z})\,dl_{M_0}\,d\bar{Z}. \tag{9.13}$$

where l_{M_0} is the length $O'M_0$ of the ray passing through point M:

$$l_{M_0}^2 = x_0^2 + y_0^2 + (l_0 - x_0 \operatorname{ctg} \chi_0)^2; \tag{9.14}$$

$$dl_{M_0} = \frac{\partial l_{M_0}}{\partial x_0} dx_0 + \frac{\partial l_{M_0}}{\partial y_0} dy_0 \tag{9.15}$$

is the total differential.

For the length ds* of an arbitrarily oriented segment lying in the plane $\bar{Z}$ = const, we have from (9.13)

$$ds^* = (1 - \bar{Z}) ds_0^*, \tag{9.16}$$

where ds^*_0 is the length of the corresponding segment in the plane of the directrix:

$$(ds_0^*)^2 = \frac{1}{\sin^2 \chi_0} (dx_0)^2 + (dy_0)^2. \tag{9.17}$$

Extending each of the functions $\{\varphi_k^{21}(S)\}$, $\{\varphi_k^{22}(S)\}$ to the region included within the contour of the directrix, we will proceed from the representation

$$\varphi_k^{21} = \varphi_k^{21}(x_0, y_0), \; \varphi_k^{22} = \varphi_k^{22}(x_0, y_0). \tag{9.18}$$

In view of the continuity of displacements of the shell proper and ribs, coordinate functions (9.18) must be continuous on the contour of the directrix and inside it, and in addition, must satisfy the conditions

$$\varphi_k^{21}[x_0(S), y_0(S)] = \varphi_k^{21}(S), \; \varphi_k^{22}[x_0(S), y_0(S)] = \varphi_k^{22}(S), \tag{9.19}$$

where $x_0(S)$, $y_0(S)$ are the coordinates of a point lying on the contour of the directrix.

All the indicated requirements placed on coordinate functions (9.18) will be fulfilled if power coordinate functions (3.309) are chosen as these functions.

For the chosen coordinate functions (9.18), expressions (9.6)-(9.11) determine the displacements of both the shell proper and the ribs. The strains of the shell proper are represented by expressions (1.55), (1.69). We will write the expressions for the strains of the flanges and webs of the ribs. In so doing, it is necessary to keep in mind that in displacements U^0 the ribs are not deformed at all, and in displacements U^1, they are deformed only out of their plane. In this connection, for the strain ε_{m_s} of the rib flanges, in view of (1.55), we have

$$\varepsilon_{m_s}=\frac{1}{1-\bar{Z}}\frac{\partial u^2_{m_s}}{\partial S}-\frac{1-\frac{1}{2}(l^2_s)'}{(1-\bar{Z})l_s\sin\chi}u^{(2)}_{n_s}-\frac{\sin^2\chi}{R}u^2_{n_n}, \tag{9.20}$$

where

$$u^2_{m_s}=U^2\cdot m_s,\ u^2_{n_s}=U^2\cdot n_s,\ u^2_{n_n}=U^2\cdot n_n. \tag{9.21}$$

The shear strains of the rib webs are determined by the expression

$$\gamma_{e_1e_2}=\frac{\partial u^2_{e_1}}{\partial e_2}+\frac{\partial u^2_{e_2}}{\partial e_1}, \tag{9.22}$$

where

$$u^2_{e_1}=U^2\cdot e_1,\ u^2_{e_2}=U^2\cdot e_2. \tag{9.23}$$

For arbitrary scalar functions along the direction of unit vectors e_1, e_2, in view of (9.16), (9.17), we have

$$\frac{\partial}{\partial e_1}=\frac{\sin\chi_0}{1-\bar{Z}}\frac{\partial}{\partial x_0};\quad\frac{\partial}{\partial e_2}=\frac{1}{1-\bar{Z}}\frac{\partial}{\partial y_0}. \tag{9.24}$$

Considering (9.24), we obtain from (9.22)

$$\gamma_{e_1e_2}=\frac{1}{1-\bar{Z}}\left[\frac{\partial u^2_{e_1}}{\partial y_0}+\sin\chi_0\frac{\partial u^2_{e_2}}{\partial x_0}\right]. \tag{9.25}$$

We will assume that the rib webs are subjected only to shear. It then becomes obvious that the fluxes of tangential forces, and along with them, the shear strains within each section of the rib located between neighboring struts must be considered constant. Thus the shear strain $\gamma_{e_1e_2}$ of the rib webs of the shell can be represented as

$$\gamma_{e_1e_2} = \gamma_{e_1e_2}(x_0, \bar{Z}) = \sum_{k=1}^{p} \gamma_k(\bar{Z})\mu_k(x_0), \tag{9.26}$$

where γk is the shear strain of the kth section of the rib located between struts with coordinates $x_{0,k} - 1$ and $x_{0,k}$, and the functions

$$\mu_k(x_0) = \begin{cases} 1 & x_{0,k-1} \leqslant x_0 \leqslant x_{0,k}, \\ 0 & x_0 < x_{0,k-1}, \\ & x_0 > x_{0,k}. \end{cases} \tag{9.27}$$

Further, introducing the Heaviside unit funtion θ, we will use the representation

$$\mu_k(x_0) = \theta(x_0 - x_{0,k-1}) - \theta(x_0 - x_{0,k}). \tag{9.28}$$

Superposing the x0z plane of the fundamental Cartesian coordinate system on the tentative position of the neutral plane (we took as the latter the plane passing through the principal central axes of the cross sections $\bar{Z}$ = const, about which the moments of inertia $I = I_{min}$), we will assume that the displacements $u^{\dots}_{e_1}$ depend linearly on the y_0 coordinate:

$$u^{?}_{e_1}(x_0, y_0, \bar{Z}) = y_0 \frac{\partial u^{?}_{e_1}}{\partial y_0}(x_0, \bar{Z}). \tag{9.29}$$

Taking (9.26) and (9.29) into account, we have from expression (9.25)

$$u^{?}_{e_1}(x_0, \bar{Z}) = u^{?}_{e_1}(0, \bar{Z}) + \\ + \frac{1}{\sin \chi_0}\left[(1-\bar{Z})\sum_{k=1}^{p}\gamma_k(\bar{Z})\int_0^{x_0}\mu_k(\xi)d\xi - \int_0^{x_0}\frac{\partial u^{?}_{e_1}}{\partial y_0}(\xi, \bar{Z})d\xi\right]. \tag{9.30}$$

It is easy to see that in accordance with (9.30), the total displacements $u_{e_1} = u^0_{e_1} + u^{\kappa}_{e_1}$ are independent of the y_0 coordinate. This is in account with the generally accepted concepts of operation of a wing.

Omitting the function $u^2_{e_2}(0, \bar{Z})$ in (9.30) and using representation (9.28), we find to within the displacements of the contour $\bar{Z}$ = const as a solid

$$u^2_{e_2}(x_0, \bar{Z}) = \frac{1}{\sin \chi_0} \left\{ (1-\bar{Z}) \sum_{k=1}^{p} \gamma_k(\bar{Z}) [(x_0 - x_{0,k-1})_+ - (x_0 - x_{0,k})_+ + C_k] - \right.$$
$$\left. - \int_0^{x_0} \frac{\partial u^2_{e_1}}{\partial y_0} (\xi, \bar{Z}) \, d\xi \right\}, \tag{9.31}$$

where functions of the type $(x_0 - x_{0,i})_+ = (x_0 - x_{0,i}) \theta (x_0 - x_{0,i})$:

$$(x_0 - x_{0,i})_+ = \begin{cases} 0 & x_0 < x_{0,i}, \\ x_0 - x_{0,i}, & x_0 > x_{0,i}, \end{cases} \tag{9.32}$$

and constants C_k can be chosen arbitrarily.

Thus, for example, using the condition

$$u^{(2)}_{e_2}(0, \bar{Z}) = 0 \tag{9.33}$$

we can obtain

$$C_k = (-x_{0,k})_+ - (-x_{0,k-1})_+. \tag{9.34}$$

Let us turn to relations (9.23). Taking (9.8) into account, we have

$$u^{(2)}_{e_1} = K_1, \; u^{(2)}_{e_2} = K_2. \tag{9.35}$$

Thus, expression (9.31) establishes the relationship between the components $K_1 = K_1(x_0, y_0, \bar{Z})$, $K_2 = K_2(x_0, \bar{Z})$ of vector U^2 and the values of shear strain $\gamma_k(\bar{Z})$ in the rib sections.

Considering (9.29), we will choose the coordinate functions $\varphi^{21}_k(x_0, y_0)$ in the form

$$\varphi^{21}_k = x_0^k y_0. \tag{9.36}$$

On the basis of (9.10)

$$K_1(x_0, y_0, \bar{Z}) = y_0 \sum_{k=1}^{n_1} x_k(\bar{Z}) x_0^k. \tag{9.37}$$

where $x_k(\bar{Z})$ are the desired functions.

Introducing representation (9.37) into (9.31), in view of (9.35), we also

$$K_2(x_0, \bar{Z}) = \frac{1}{\sin\chi_0}\left\{(1-\bar{Z})\sum_{k=1}^{p}\gamma_k(\bar{Z})\left[(x_0-x_{0,k-1})_+ - (x_0-x_{0,k})_+ + C_k\right] - \right.$$

$$\left. - \sum_{k=1}^{n_1} x_k(\bar{Z})\frac{x_0^{k+1}}{k+1}\right\}. \tag{9.38}$$

Now, introducing expansions (9.37) and (9.38) into expression (9.8), we represent it in the form

$$\mathbf{U}^{n}(M) = (1-\bar{Z})\sum_{k=1}^{p}\gamma_k(\bar{Z})\,\varphi_k^1(M_0) + \sum_{k=1}^{n_1} x_k(\bar{Z})\,\varphi_k^2(M_0), \tag{9.39}$$

where

$$\varphi_k^1(M_0) = \frac{\mathbf{e}_2}{\sin\chi_0}\left[(x_0-x_{0,k-1})_+ - (x_0-x_{0,k})_+ + C_k\right], \tag{9.40}$$

$$\varphi_k^2(M_0) = \left(y_0\mathbf{e}_1 - \frac{x_0}{k+1}\,\frac{\mathbf{e}_2}{\sin\chi_0}\right)x_0^k. \tag{9.41}$$

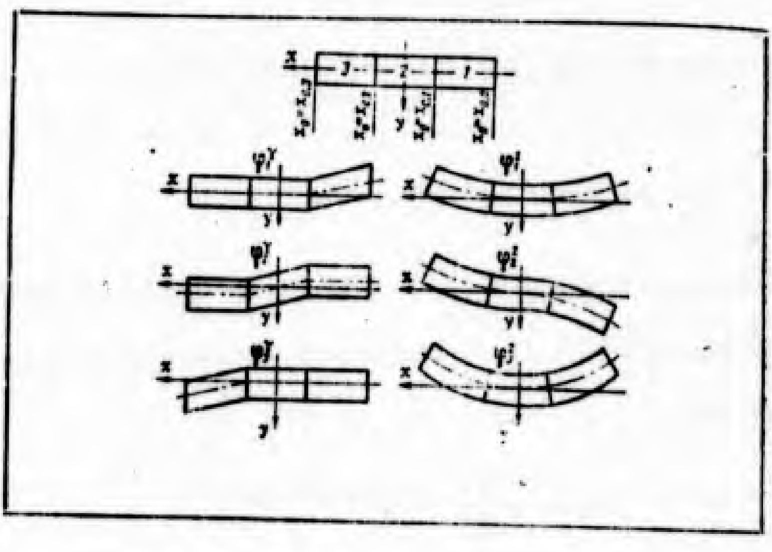

Fig. 9.3. Shearing and bending forms of deformation of a rib in its plane.

Expression (9.39) represents the expansion of vector U^2, which determines the deformation of the ribs in their plane. The functions $\gamma_k, \varkappa_k$ of variable Z are the desired generalized displacements; the vector functions φ_k^1, φ_k^2, determined on the contour of the directrix and inside it, constitute the given coordinate functions. It is evident from the above that the coordinate vector functions φ_k^2 determine the bending forms of deformation of the ribs in their plane, which are not associated with shear of the rib webs. However, the coordinate vector functions φ_k^1 correspond to shear strains of the webs. As an example, Fig. 9.3 shows the forms of deformation of a rib in its plane, corresponding to the coordinate functions φ_k^1, φ_k^2, for a rib of rectangular configuration reinforced with four struts.

We will represent the vector U^1 in the form of an expansion analogous to (9.39). Considering (9.7) and (9.9),

$$U^1(M)=\sum_{k=1}^{n} \omega_k(\bar{Z})\varphi_k^1(M_0), \tag{9.42}$$

where $\omega_k(\bar{Z})$ are the desired generalized displacements;

$\varphi_k^1(M_0)$ are coordinate vector functions determined on the contour of the directrix, which are subject to preselection.

Now, introducing expansions (9.39) and (9.42) into expression (9.6), and considering expansion (3.3), we represent the elastic displacement vector U in the form

$$U(M) = \sum_{i=1}^{6+n+p+n_1} U_i(\bar{Z})\lambda_i(\bar{Z})\varphi_i(M_0). \tag{9.43}$$

Here the first six terms pertain to the vector function U^0:

$U_1(\bar{Z}), U_2(\bar{Z}), U_3(\bar{Z})$ are the components of the translation displacement vector $\eta(\bar{Z})$;

$U_4(\bar{Z}), U_5(\bar{Z}), U_6(\bar{Z})$ are the components of the rotation vector $\theta(\bar{Z})$;

$$\lambda_i(\bar{Z}) = \begin{cases} 1 & (i=1,2,3); \\ 1-\bar{Z} & (i=4,5,6); \end{cases} \tag{9.44}$$

$\varphi_i(M_0)$ are the coordinate vector functions, determined, strictly speaking, both on the contour of the directrix and inside it; they are represented by expressions (2.97) and 3.240).

The next n terms of expansion (9.43) correspond to the vector function U^1:

$U_i(\bar{Z})|_{i=6+k} = w_k(\bar{Z})$ are the generalized warpings;

$$\lambda_i(\bar{Z})|_{i=6+k} = 1; \tag{9.45}$$

$\varphi_i(M_0)|_{i=6+k} = \varphi_k^1(S)$ are the coordinate vector functions, determined on the contour of the directrix.

The next ρ terms of expansion (9.43) correspond to the vector function U^2:

$\dot{U}_i(\bar{Z})|_{i=6+n+k} = \gamma_k(\bar{Z})$ are the shear strains in the kth sections of the rib webs:

$$\lambda_i(\bar{Z})|_{i=6+n+k} = 1-\bar{Z}; \tag{9.46}$$

$\varphi_i(M_0)|_{i=6+n+k}$ are the coordinate vector functions, determined both on the contour of the directrix and inside it; they are represented by expressions (9.40).

The last n_1 terms of expansion (9.43) also correspond to the vector function U^2:

$U_i(\bar{Z})|_{i=6+n+\rho+k} = \dot{x}_k(\bar{Z})$ are the generalized displacements, corresponding to bending of the ribs in their plane;

$$\lambda_i(\bar{Z})|_{i=6+n+p+k} = 1; \tag{9.47}$$

$\varphi_i(M_0)|_{i=6+n+p+k}$ are the coordinate vector functions, determined both on the contour of the directrix and inside it; they are represented by expressions (9.41).

Let t be an arbitrarily oriented unit vector. In accordance with (9.43), the displacements in the direction of t

$$u_t(\bar{Z},\ M_0) = \sum_{i=1}^{N} V_i(\bar{Z})\varphi_{it}(M_0), \tag{9.48}$$

where

(9.49)

$$V_i(\bar{Z}) = U_i(\bar{Z})\lambda_i(\bar{Z}), \tag{9.50}$$

$$\varphi_{it}(M_0) = \varphi_i \cdot \mathbf{t},$$

$$N = 6 + n + p + n_1. \tag{9.51}$$

Superposing t on the unit vectors of the main moving trihedron, we will expand geometric relations (2.2).

For the components of the strain of the shell proper, we obtain

$$\epsilon^0_{m_s} = \sum_{i=1}^{N}\left[V'_i \frac{1}{l_s}\psi_{im_s} + \frac{1}{1-\bar{Z}} V_i\bar{\psi}_{im_s}\right], \tag{9.52}$$

$$\gamma^0_{m_s m_s} = \sum_{i=1}^{N}\left[V'_i \frac{1}{l_s}\psi_{im_s m_s} + \frac{1}{1-\bar{Z}} V_i\bar{\psi}_{im_s m_s}\right],$$

$$\epsilon^0_{n_s} = \sum_{i=1}^{N}\left[V'_i \frac{1}{l_s}\psi_{in_s} + \frac{1}{1-\bar{Z}} V_i\bar{\psi}_{in_s}\right],$$

where ψ, $\bar{\psi}$ are determined by expressions (3.17) and (3.18).

For the strain ϵ_{m_s} of the rib flanges, we obtain an expression of the form

$$\varepsilon_{m_r} = \frac{1}{1-\bar{Z}} \sum_{i=1}^{N} V_i \bar{\varphi}_{im_r}, \tag{9.53}$$

where

$$\bar{\varphi}_{im_r} = \begin{cases} 0 & (i \leqslant n+6), \\ \varphi'_{im_s} - \dfrac{1-\frac{1}{2}(l_s^2)'}{l_s \sin\chi}\varphi_{in_s} - \dfrac{\sin^2\chi}{R_0}\varphi_{in_n} & (i > n+6). \end{cases} \tag{9.54}$$

The shear strain (9.26) of the rib webs can also be represented in the form

$$\gamma_{e_r e_r} = \frac{1}{1-\bar{Z}} \sum_{i=1}^{N} V_i \bar{\varphi}_{ie_r e_r}, \tag{9.55}$$

where, in view of (9.28),

$$\bar{\varphi}_{ie_r e_r} = \begin{cases} 0 & (i \leqslant n+6), \\ \theta(x_e - x_{0,k-1}) - \theta(x_e - x_{0,k}) & (n+6 < i \leqslant n+6+p), \\ 0 & (i > n+6+p). \end{cases} \tag{9.56}$$

The elasticity relations for the shell proper are represented by expressions (2.3). For the rib flanges and webs,

$$\sigma_{m_r} = E_r \varepsilon_{m_r}; \quad \tau_{e_r e_r} = G_r \gamma_{e_r e_r}, \tag{9.57}$$

where E_r, G_r are the elasticity moduli of the rib elements

9.2. Differential Resolvents

We will first assume the distribution of the ribs over the wing span to be continuous. Denoting by ΔF_r and h_r the cross-sectional area of the flanges and web thickness of a rib, we introduce two parameters characterizing the rib structure

$$\begin{gathered} \Delta\bar{F}_r(\bar{Z}, S) = \frac{1}{l_0} \frac{\partial}{\partial \bar{Z}} [\Delta F_r(\bar{Z}, S)], \\ \bar{h}_r(\bar{Z}, x_e) = \frac{1}{l_0} \frac{\partial}{\partial \bar{Z}} [h_r(\bar{Z}, x_e)], \end{gathered} \tag{9.58}$$

constituting the cross sectional area of the flanges and the web thickness of a rib, per unit length of the wing measured along the Oz axis.

Let $\delta\bar{U}_r$ be the potential energy variation of the rib structure per unit length of the wing. Considering that in view of the adopted hypotheses, the rib webs are subjected only to shear, and taking (9.16) into account, we have

$$\delta\bar{U}_{\text{н}}=(1-\bar{Z})\oint \sigma_{m_s}\Delta\bar{F}_r\delta\varepsilon_{m_s}dS_{\text{н}}+(1-\bar{Z})^2\int\limits_{F_0}\tau_{x_0y_0}\bar{h}_{\text{н}}\delta\gamma_{x_0y_0}dx_0dy_0. \tag{9.59}$$

Here

$\oint\ldots dS_{\text{н}}$ pertains to the contour formed by the axial line of the rib flanges in the section $\bar{Z} = 0$;

$\int\limits_{F_0}\ldots dx_0dy_0$ extends to the area of the rib web in the section $\bar{Z} = 0$.

Expanding (9.59) and considering expansions (9.53) and (9.55), we obtain for the rib structure of the entire wing

$$\delta U_{\text{н}}=\int\limits_0^{\bar{Z}_1}\delta\bar{U}_{\text{н}}l_0d\bar{Z}=\int\limits_0^{\bar{Z}_1}\sum_{i=1}^{N}Q_{\text{н}i}\delta U_i d\bar{Z}. \tag{9.60}$$

Here

$$Q_{\text{н}i}(\bar{Z})=\lambda_i\left[\oint \sigma_{m_s}\bar{\varphi}_{im_s}\Delta\bar{F}_{\text{н}}dS_{\text{н}}+(1-\bar{Z})\int\limits_{F_0}\tau_{x_0y_0}\bar{\varphi}_{ix_0y_0}\bar{h}_{\text{н}}dx_0dy_0\right], \tag{9.61}$$

where in view of (9.58)

$$\begin{aligned}\Delta\bar{F}_{\text{н}}&=l_0\Delta\bar{F}_{\text{н}}=\frac{\partial\Delta F_{\text{н}}}{\partial\bar{Z}},\\ \bar{h}_{\text{н}}&=l_0\bar{h}_{\text{н}}=\frac{\partial h_{\text{н}}}{\partial\bar{Z}}.\end{aligned} \tag{9.62}$$

We will assume that within a section, the rib web thickness is constant. Since each of the sections is in the state of pure shear, expression (9.61) in view of (9.56) can be reduced to the form

$$Q_{\text{н}i}(\bar{Z})=\lambda_i\left[\oint \sigma_{m_s}\bar{\varphi}_{im_s}\Delta\bar{F}_{\text{н}}dS_{\text{н}}+(1-\bar{Z})\tau_{x_0y_0k}\bar{h}_{\text{н}k}F_{0k}\right] \tag{9.63}$$
$$(k=i-(6+n)=1, 2,\ldots, p).$$

where

$\tau_{e_1 e_2 k} = \tau_{e_1 e_2 k}(\bar{Z})$ are tangential stresses in the web of the kth rib section;

$h_{k} = h_{k}(\bar{Z})$ is the relative web thickness of the kth rib section;

F_{0k} is the web area of the kth rib section for $\bar{Z} = 0$.

For the system as a whole, the potential energy variation

$$\delta U_{\Sigma} = \delta U + \delta U_{w}, \tag{9.64}$$

where U will designate the sum of potential energies of the shell proper and of the reinforcing structure of spars and stringers.

Expanding (9.64) and considering (3.55) and (9.60), we have

$$\delta U_{\Sigma} = \int_0^{\bar{Z}_1} \sum_{i=1}^{N} \left[P_i \frac{(\delta U_i \lambda_i)'}{\lambda_i} + Q_{\Sigma i} \delta U_i \right] d\bar{Z}. \tag{9.65}$$

Here

P_i is the generalized force, determined by taking into account the reinforcing structure of spars and stringers;

$$Q_{\Sigma i} = Q_i + Q_{w i}, \tag{9.66}$$

where

Q_i is determined for a shell, the spars and stringers being considered;

Q_{ri} is represented by expression (9.63).

Introducing (9.65) into variational Eq. (3.26), we obtain

$$\left(\frac{P_j}{\lambda_j}\right)' \lambda_j - Q_{\Sigma j} + R_j = 0 \quad (j = 1, 2, \ldots, N), \tag{9.67}$$

$$(\bar{P}_j - P_j) \delta U_j \Big|_{\bar{Z}=0}^{\bar{Z}=\bar{Z}_1} = 0 \quad (j = 1, 2, \ldots, N) \tag{9.68}$$

We expand Eqs. (9.67). Using (9.53), (9.55), (9.57), we have

$$\frac{1}{G} Q_{w j}(\bar{Z}) = \frac{\lambda_j}{1 - \bar{Z}} \sum_{i=1}^{N} c_{w ji} V_i. \tag{9.69}$$

where the coefficients c_{rji} are determined by the expressions

$$c_{kji} = \frac{2}{1-\nu} \oint \bar{\psi}_{jm_s} \bar{\psi}_{im_s} \Delta \tilde{F}_k dS_k + (1-\bar{Z}) F_{0k} \tilde{h}_{kk} \delta_{ji}$$
$$(k = j - (6+n) = 1, 2, \ldots, p). \qquad (9.70)$$

Here

$$\delta_{ji} = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases}$$

is Kronecker's symbol.

It is now easy to see that the system of differential resolvents in the case under consideration has the form

$$\sum_{i=1}^{N} \left\{ [(1-\bar{Z}) a_{ji} V_i' + b_{ji} V_i]' - b_{ij} V_i' - \frac{1}{1-\bar{Z}} (c_{ji} + c_{kji}) V_i \right\} = -\frac{1}{\sqrt{G}} R_j$$
$$(j = 1, 2, \ldots, N), \qquad (9.71)$$

where the coefficients a_{ji}, b_{ji}, c_{ji} for both smoother and reinforced shells (the sign Σ in the subscripts of the coefficients is omitted) are determined by the same expressions as when the rib elasticity is neglected, and the coefficients c_{rji}, which allow for the rib elasticity, are determined by expressions (9.70). In this connection, system (9.71), represented in mixed form, has six first integrals expressing the equilibrium conditions of the cut-off portion of the shell as a solid:

$$(1-\bar{Z}) \sum_{i=1}^{N} a_{ji} V_i' + \sum_{i=1}^{N} b_{ji} V_i = -\frac{1}{\sqrt{G}} \bar{P}_j \quad (j = 1, 2, \ldots, 6). \qquad (9.72)$$

In the system considered, the matrices of the coefficients of the resolvents can be represented in the form of block matrices of the type

$$\bar{R}_0 = 0, \quad M_z = \text{const.} \tag{9.119}$$

Integrating system (9.113), in view of (9.119), we obtain

$$\begin{aligned}
\omega(Z) &= \sum_{k=1}^{4} C_k e^{\lambda_k Z} + \frac{1}{4}\,\frac{a-e}{ae}\,\frac{M_z}{G},\\
\gamma(Z) &= -\frac{1}{2}\,\frac{\bar{a}_{77}}{\bar{c}_{r\omega}} \sum_{k=1}^{4} C_k \lambda_k^3 e^{\lambda_k Z},\\
\theta_z(Z) &= \theta_z(0) + \frac{1}{a+e} \sum_{k=1}^{4} C_k \left[a - e - \frac{1}{2} e \frac{\bar{a}_{77}}{\bar{c}_{r\omega\omega}} \lambda_k^4\right] e^{\lambda_k Z} +\\
&+ \frac{a+e}{4ae}\,\frac{M_z}{G}\,Z.
\end{aligned} \tag{9.120}$$

Let us now examine the model with discretely arranged ribs in the sections $Z = Z_r\,(r = 1, 2, \ldots, m)$. In this case, in view of (9.85), (9.86), we have

$$\bar{c}_{r\omega\omega} = \bar{c}_{r\omega\omega}(Z) = d_1 d_2 \sum_{r=1}^{m} h_r \delta(Z - Z_r), \tag{9.121}$$

where h_r is the web thickness of the rth rib.

Reducing system (9.112), (9.113) to a partly degenerate form by taking (9.121) into consideration, we obtain for case (9.119)

$$\theta_z' = \frac{a-e}{a+e}\,\omega - \frac{e}{a+e}\,\gamma' + \frac{1}{a+e}\,\frac{M_z}{G}. \tag{9.122}$$

$$\begin{aligned}
&\bar{a}_{77}\omega'' - 4\,\frac{ae}{a+e}\,\omega - 2\,\frac{ae}{a+e}\,\gamma' = -\frac{a-e}{a+e}\,\frac{M_z}{G},\\
&\frac{ae}{a+e}\,\gamma'' + 2\,\frac{ae}{a+e}\,\omega' = d_1 d_2 \sum_{r=1}^{m} \gamma(Z_r) h_r \delta(Z - Z_r).
\end{aligned} \tag{9.123}$$

From Eqs. (9.123), we have

$$\omega''' = \frac{2 d_1 d_2}{\bar{a}_{77}} \sum_{r=1}^{m} \gamma(Z_r) h_r \delta(Z - Z_r). \tag{9.124}$$

Integrating (9.124), we find

$$\omega(Z) = C_0 + C_1 Z + C_2 Z^2 + \frac{d_1 d_2}{\bar{a}_{77}} \sum_{r=1}^{m} \gamma(Z_r) h_r (Z - Z_r)_+^2. \tag{9.125}$$

From the first two equations of system (9.122) and (9.123), we can easily obtain

$$\theta_z' = -\frac{\tilde{a}_{77}}{2a}\omega'' + \omega + \frac{1}{2a}\frac{M_z}{G}, \tag{9.126}$$

$$(\theta_z + \gamma)' = \frac{\tilde{a}_{77}}{2e}\omega'' - \omega + \frac{1}{2e}\frac{M_z}{G},$$

whence, in view of (9.125)

$$\begin{aligned} \theta_z(Z) = \theta_z(0) + \frac{1}{2a}\frac{M_z}{G}Z + \left(C_0 - \frac{\tilde{a}_{77}}{a}C_2\right)Z + C_1\frac{Z^2}{2} + C_2\frac{Z^3}{3} + \\ + d_1 d_2 \sum_{r=1}^{m} \gamma(Z_r) h_r \left[\frac{1}{3\tilde{a}_{77}}(Z - Z_r)_+^3 - \frac{1}{a}(Z - Z_r)_+\right], \\ \theta_z(Z) + \gamma(Z) = \theta_z(0) + \gamma(0) + \frac{1}{2e}\frac{M_z}{G}Z + \\ + \left(\frac{\tilde{a}_{77}}{e}C_2 - C_0\right)Z - C_1\frac{Z^2}{2} - C_2\frac{Z^3}{3} + \\ + d_1 d_2 \sum_{r=1}^{m} \gamma(Z_r) h_r \left[\frac{1}{e}(Z - Z_r)_+ - \frac{1}{3\tilde{a}_{77}}(Z - Z_r)_+^3\right]. \end{aligned} \tag{9.127}$$

From (9.127), we have

$$\begin{aligned} \gamma_q = \gamma_0 + 2\left(\frac{2}{\lambda_\infty^2}C_2 - C_0\right)Z_q - C_1 Z_q^2 - \frac{2}{3}C_2 Z_q^3 + \frac{1}{2}\frac{a - e}{ae}\frac{M_z}{G}Z_q + \\ + 4\frac{d_1 d_2}{\tilde{a}_{77}} \sum_{r=1}^{m} \gamma_r h_r (Z_q - Z_r)\left[\frac{1}{\lambda_\infty^2} - \frac{1}{6}(Z_q - Z_r)^2\right]\theta(Z_q - Z_r) \\ (q = 1, 2, \ldots, m), \end{aligned} \tag{9.128}$$

where

(9.129)

λ_∞ is root (9.117) of the characteristic equation for the case of absolutely rigid ribs.

Expressions (9.127), (9.128) represent the general solution of the problem in mixed form. Solving system (9.128) for the parameters $\gamma_q (q = 1, 2, \ldots, m)$ and eliminating them from expressions (9.127), we can easily obtain the general solution in explicit form.

Comparing expressions (9.120) and (9.127), we see that in contrast to the continuous rib distribution, where the additional displacements and hence the stresses associated with warping and deformation of the contour are exponential in character, in the case of discrete arrangement of the ribs, the additional part of the solution is piecewise-polynomial, i.e., the distribution of the displacements and stresses in the portion between neighboring ribs is represented by polynomials up to third degree inclusive.

Chapter X. SOME NUMERICAL RESULTS

The present chapter presents the results of computations for certain specific problems involving shells of straight and swept wing type and low-aspect wing type. It discusses the effect of the rigidity of the structure of normal ribs on the stressed state of the wing, shows the distribution of thermal stresses in low-aspect wings of various shapes in the plane, and gives the distribution of normal stresses for local fixing of the wing.

All the calculations were carried out on a computer using a universal program written for a conical shell of arbitrary configuration (Appendix V). Since the calculations are comparative and illustrative in character, we will confine ourselves mainly to caissons with a rectangular cross-sectional contour, which constitute a sufficiently versatile model for qualitative analysis of the operation of wing type shells.

10.1. Effect of Rib Elasticity on the Stressed State of a Wing

Presented below are the results of comparative analyses of wing type shells reinforced with a structure of normal ribs of different rigidities. In view of the fact that normal ribs are usually identical and mounted at fairly close intervals, we proceed from a continuous and uniform distribution of the ribs over the span.

Figures 10.2 and 10.3 show the results of analysis of a straight wing type caisson, illustrated in Fig. 10.1.

Figures 10.5-10.7 show the results of analysis of a swept wing type caisson, illustrated in Fig. 10.4.

Figures 10.9-10.11 show the results of analysis of a low-aspect wing type caisson, illustrated in Fig. 10.8.

The results cited show that raising the rigidity of the rib structure above a certain limit has little effect on the stressed state of the wing, but for very weak ribs, even a slight change of their rigidity is sharply reflected in the nature of the stress distribution in the wing. The computations also show that in high-aspect wings, the deformation of the contour is chiefly determined by the coordinate function φ^{τ}, corresponding to the operation of the rib webs in shear, whereas in low-aspect wings, the deformation of the contour is chiefly determined by the coordinate functions φ^2, corresponding to the operation of the ribs in bending.

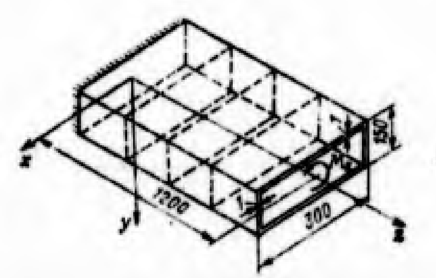

Fig. 10.1. Straight caisson of rectangular cross-section reinforced with a structure of elastic ribs.

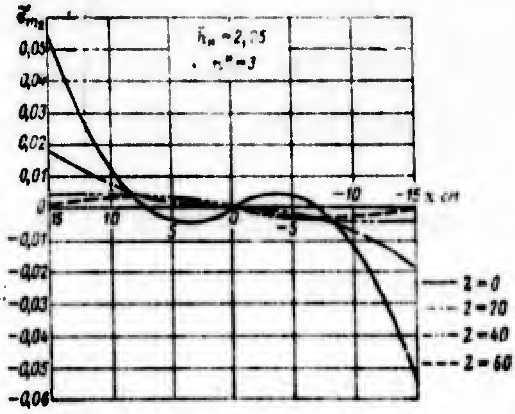

Fig. 10.2. Distribution of normal stresses in the cross sections of a straight caisson with elastic ribs.

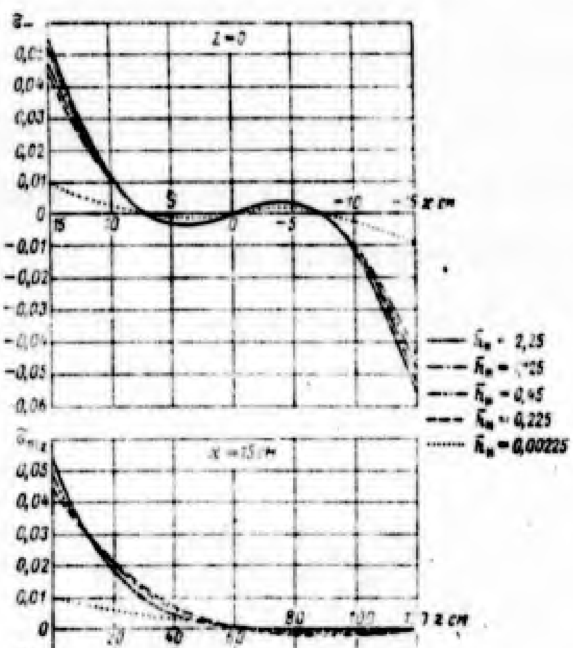

Fig. 10.3. Distribution of normal stresses in a straight caisson for different rib rigidities.

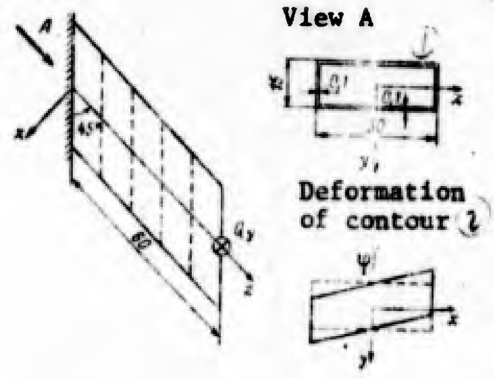

Fig. 10.4. Swept caisson of rectangular cross section reinforced with structure of elastic ribs.

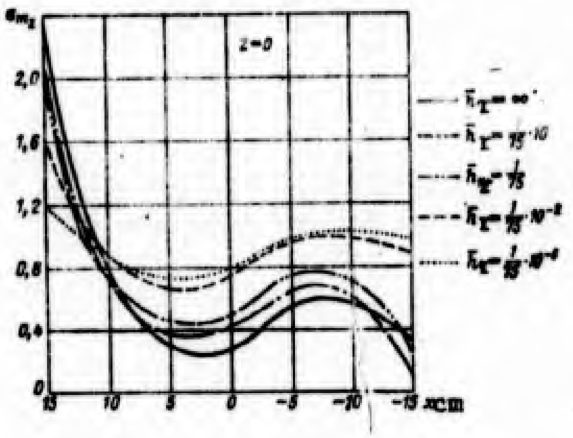

Fig. 10.5. Distribution of normal stresses in the built-in end of a swept caisson with elastic ribs

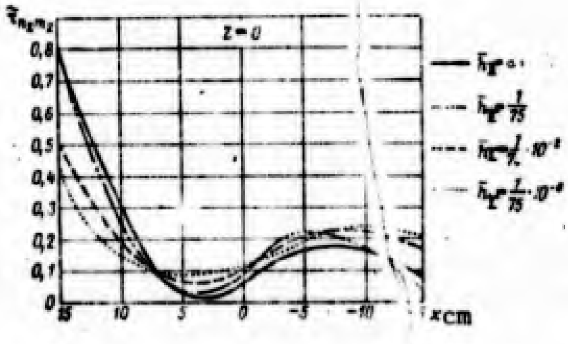

Fig. 10.6. Distribution of tangential stresses in the built-in end of swept caisson with elastic ribs.

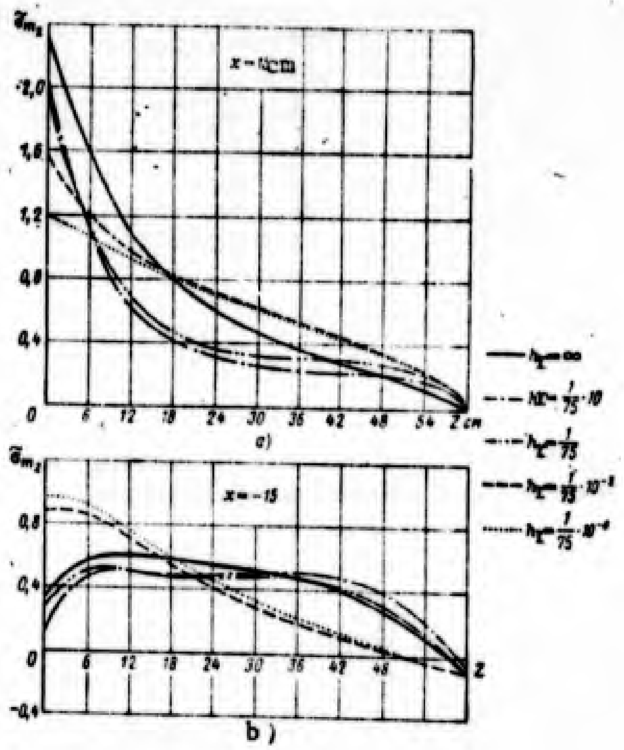

Fig. 10.7. Distribution of normal stresses along the rear web (a) and front web (b).

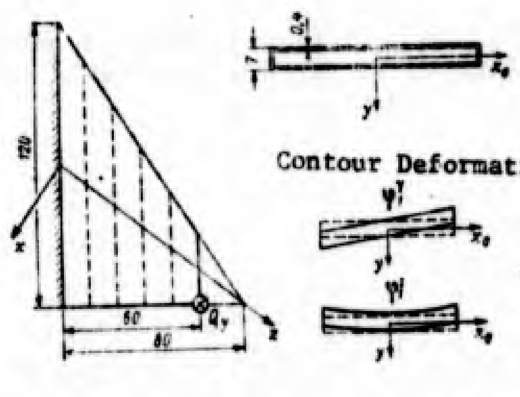

Fig. 10.8. Los-aspect caisson with a rectangular cross section, reinforced with a structure of elastic web ribs.

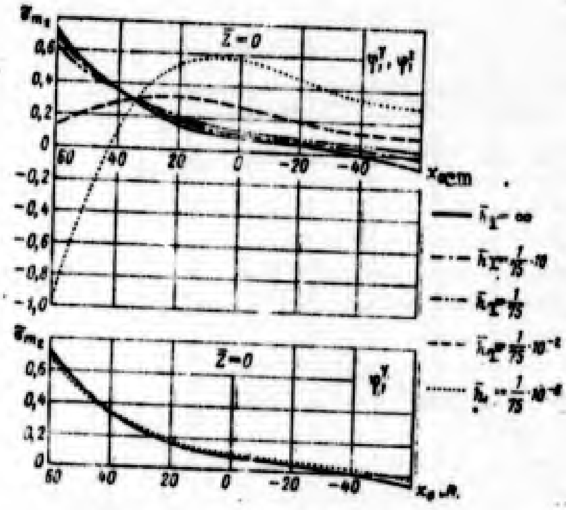

Fig. 10.9. Distribution of normal stresses in the built-in end of a low-aspect caisson with elastic ribs for different functions approximating the contour deformation and different rib rigidities.

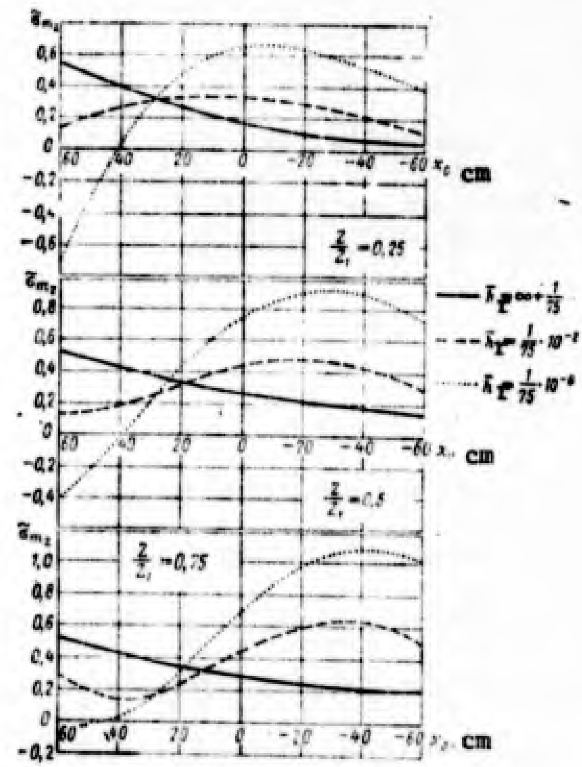

Fig. 10.10. Distribution of normal stressed in the sections of a low-aspect caisson with a deformable contour for different rib rigidities $(\varphi_1^T, \varphi_1^2)$

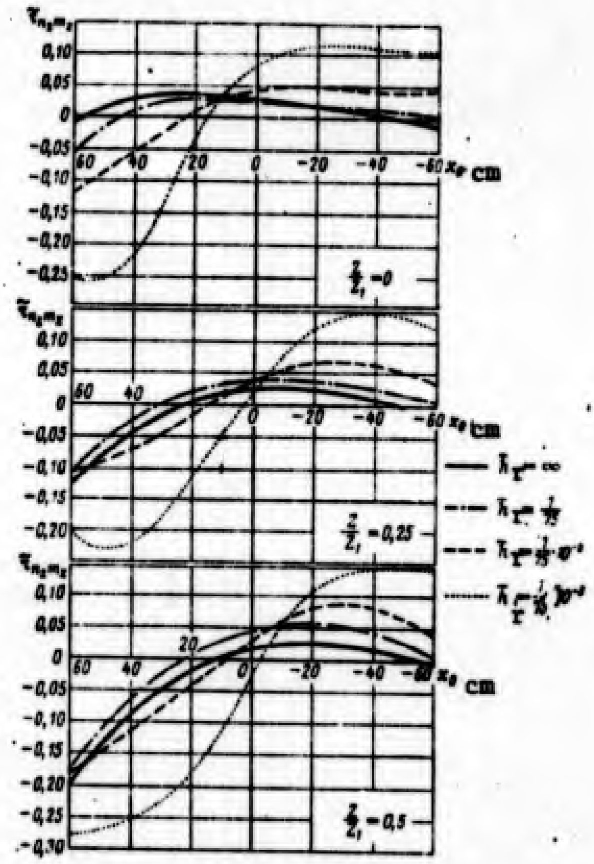

Fig. 10.11. Distribution of tangential stresses in the sections of a low-aspect caisson with a deformable contour for different rib rigidities (φ_i^j, ψ_i^j)

10.2. Thermal Stresses

Given below (Fig. 10.14) are the results of an analysis of low-aspect wings of different shapes in the plane with a rhomboidal profile (Fig. 10.12) for the action of a temperature field* (Fig. 10.13). The temperature distribution over the wing span and skin thickness was assumed to be uniform; the web was considered to be unheated (t = 20°C).

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*S. N. Kan and I. A. Sverdlov. Strength Analysis of Aircraft. Mowcow, Mashinostroyeniye, 1966.

$$\begin{pmatrix} M_{\eta\eta} & M_{\eta 6} & M_{\eta\theta} & M_{\eta\gamma} & M_{\eta\kappa} \\ M_{6\eta} & M_{66} & M_{6\theta} & M_{6\gamma} & M_{6\kappa} \\ M_{\theta\eta} & M_{\theta 6} & M_{\theta\theta} & M_{\theta\gamma} & M_{\theta\kappa} \\ M_{\gamma\eta} & M_{\gamma 6} & M_{\gamma\theta} & M_{\gamma\gamma} & M_{\gamma\kappa} \\ M_{\kappa\eta} & M_{\kappa 6} & M_{\kappa\theta} & M_{\kappa\gamma} & M_{\kappa\kappa} \end{pmatrix}.$$

where the blocks $M_{\eta\eta}, M_{\eta 6}, M_{6\eta}, M_{66}$ are identical to the corresponding blocks of matrices written without considering the rib elasticity. Consequently, the matrices of the coefficients of system (9.71) have the same general properties as when the rib elasticity is neglected, so that it is possible to change to the canonical form for any choice of the coordinate function $\varphi_i (i>6)$. Moreover, all the canonical relations retain their form when sums of the form $\sum_{i=7}^{6+n} A_i = \sum_{k=1}^{n} A_{k+6}$ are replaced by sums of the form $\sum_{i=7}^{N} A_i = \sum_{k=1}^{n} A_{6+k} + \sum_{k=1}^{p} A_{6+n+k} + \sum_{k=1}^{n_1} A_{6+n+p+k}$.

Let us also note that the matrix of coefficients c_{rji} is symmetric and has the form

$$C_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{rx\gamma} & C_{rx\gamma} \\ 0 & C_{rx\gamma} & C_{rx\gamma} \end{pmatrix}. \tag{9.73}$$

where the block $C_{rx\gamma}$ is diagonal. In addition, for ribs whose webs have rectangular sections, the blocks $C_{rx\gamma}$ $C_{rx\gamma}$, which constitute p x n_1 and n_1 x p matrices, turn out to be null.

In the general case, Esq. (9.71) have variable coefficients. If the thickness of the shell proper depends only on the contour coordinate

$$h = h(S), \tag{9.74}$$

and the area of the cross sections of the stringers changes linearly along the length

$$ \tag{9.75} $$

(these conditions were sufficient for the equations without consideration of the elasticity of the ribs) and if, in addition, the relative cross-sectional area of the rib flanges $\Delta\check{F}_r$ also depends only on the contour coordinate

$$\Delta\check{F}_r = \Delta\check{F}_r(S), \tag{9.76}$$

and the relative thickness of the rib webs changes along a hyperbola

$$\tilde{h}_r = \frac{\tilde{h}^0_r}{1-\bar{Z}}, \tag{9.77}$$

then, introducing the new independent variable

$$t = \ln(1-\bar{Z}),$$

we arrive at equations with constant coefficients

$$\sum_{i=1}^{N} \left[a_{ji}V_i'' + (b_{ij} - b_{ji})V_i' - (c_{ji} + c_{aji})V_i\right] = -\frac{l'}{\lambda_j G} R_j \tag{9.78}$$

$$(j = 1, 2, \ldots, N)$$

In Eqs. (9.71) and (9.78), the load terms are determined by the expressions

$$\frac{1}{\lambda_j G} R_j = \frac{1}{G}(1-\bar{Z}) \oint \left[p^0_{n_z}(\bar{Z}, S)\varphi_{jm_z}(S) + p^0_{n_s}(\bar{Z}, S)\varphi_{jn_s}(S) + \right. \\ \left. + p^0_{n_n}(\bar{Z}, S)\varphi_{jn_n}(S)\right] l_s \sin\chi\, dS. \tag{9.79}$$

9.3. <u>Discrete Rib Structure</u>

In the preceding section, we assumed the distribution of the ribs over the wing span to be continuous. Such a model, in which the ribs are "smeared out" over the wing span, is unconditionally acceptable if the ribs form a sufficiently close and regular structure, and in addition, the external load is smooth in character. When the rib structure is sparse, and also in the presence of reinforced ribs and concentrated force factors, the "smearing out" concept is inadmissible. However, all the results obtained above can be transposed to the case of discrete ribs by using an idealized representation of the intensity of the concentrated quantities, i.e., by employing the device of generalized functions (see Appendix I).

Let in the wing section $\bar{Z} = \bar{Z}_r$ be mounted a rib with web thickness

$$h_r = h_r(x_0), \tag{9.80}$$

constant within each section, and with a cross-sectional area of the flanges

$$\Delta F_r = \Delta F_r(S). \tag{9.81}$$

Distributions (9.62) of these quantities, determined by the conditions

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_{\bar{Z}-\epsilon}^{\bar{Z}+\epsilon} \tilde{h}_r(\xi, x_0)\, d\xi = \begin{cases} h_r(x_0), & \bar{Z} = \bar{Z}_r, \\ 0, & \bar{Z} \neq \bar{Z}_r, \end{cases} \\ &\lim_{\epsilon \to 0} \int_{\bar{Z}-\epsilon}^{\bar{Z}+\epsilon} \Delta \tilde{F}_r(\xi, S)\, d\xi = \begin{cases} \Delta F_r(S), & \bar{Z} = \bar{Z}_r, \\ 0, & \bar{Z} \neq \bar{Z}_r, \end{cases} \end{aligned} \tag{9.82}$$

will be

$$\begin{aligned} &\tilde{h}_r(\bar{Z}, x_0) = h_r(x_0)\,\delta(\bar{Z} - \bar{Z}_r), \\ &\Delta \tilde{F}_r(\bar{Z}, S) = \Delta F_r(S)\,\delta(\bar{Z} - \bar{Z}_r). \end{aligned} \tag{9.83}$$

Let the ribs be mounted at m arbitrary wing sections $\bar{Z} = \bar{Z}_r$ ($r = 1, 2, \ldots, m$) For the entire rib structure, in view of (9.83), the distributions

$$\begin{aligned} &\tilde{h}(\bar{Z}, x_0) = \sum_{r=1}^{m} h_r(x_0)\,\delta(\bar{Z} - \bar{Z}_r), \\ &\Delta \tilde{F}(\bar{Z}, S) = \sum_{r=1}^{m} \Delta F_r(S)\,\delta(\bar{Z} - \bar{Z}_r). \end{aligned} \tag{9.84}$$

Introducing distributions (9.84) into expression (9.70) for the coefficients allowing for rib elasticity, we obtain

$$c_{jl}(\overline{Z}) = \sum_{r=1}^{m} c_{rjl}\delta(\overline{Z} - \overline{Z}_r). \tag{9.85}$$

Here

$$c_{rjl} = \frac{2}{1-\nu}\oint \bar{\psi}_{jm_r}\bar{\psi}_{lm_r}\Delta F_r dS_n + (1 - \overline{Z}_r) F_{ok} h_{rk}\delta_{jl}, \tag{9.86}$$

where h_{rk} is the web thickness of the kth section of the rth rib ($k = j - 6 - n$).

Introducing (9.85) into (9.71), we have

$$\sum_{l=1}^{N}\left\{[(1-\overline{Z})a_{jl}V_l' + b_{jl}V_l]' - b_{lj}V_l' - \frac{1}{1-\overline{Z}}\left[c_{jl} + \sum_{r=1}^{m} c_{rjl}\delta(\overline{Z} - \overline{Z}_r)\right]V_l\right\} = -\frac{1}{\lambda_j G}R_j \tag{9.87}$$

$$(j = 1, 2, \ldots, N).$$

Thus, taking the discrete arrangement of the ribs into account, we arrive at differential equations with singular variable coefficients. A general method of solving such equations is described in Appendix II. Let us consider the scheme used to derive the solution.

We reduce the differential resolvents to a partly degenerate form. Using first integrals (9.72) and considering (9.73), we obtain

$$(1-\overline{Z})\sum_{l=1}^{N} a_{jl}V_l' + \sum_{l=1}^{N} b_{jl}V_l = \frac{1}{\lambda_j G}\overline{P}_j \tag{9.88}$$

$$(j = 1, 2, \ldots, 6),$$

$$\sum_{l=1}^{N}\left\{[(1-\overline{Z})a_{jl}V_l' + b_{jl}V]' - b_{lj}V_l' - \frac{1}{1-\overline{Z}}c_{jl}V_l\right\} = -\frac{1}{G}R_j \tag{9.89}$$

$$(j = 7, 8, \ldots, 6+n),$$

$$\sum_{l=1}^{N}\left\{[(1-\overline{Z})a_{jl}V_l' + b_{jl}V]' - b_{lj}V_l' - \frac{1}{1-\overline{Z}}c_{jl}V_l\right\} =$$

$$= -\frac{1}{\lambda_j G}R_j + \sum_{l=1}^{N}\sum_{r=1}^{m} c^{\circ}_{rjl}V_l(\overline{Z}_r)\delta(\overline{Z} - \overline{Z}_r) \tag{9.90}$$

$$(j = n+7, n+8, \ldots, N),$$

where

$$c^{*}_{rji} = \frac{c_{rji}}{1-\bar{Z}_r}. \tag{9.91}$$

Equations (9.88) - (9.90) lead to the important conclusion that for a discrete rib structure, the differential resolvents reduce to equations with constant coefficients with conditions (9.74) and (9.75), i.e., with the same conditions as the equations neglecting rib elasticity.

The solution of system (9.88) - (9.90)

$$\mathbf{V} = \begin{pmatrix} V_\eta \\ V_\xi \\ V_\theta \\ V_\gamma \\ V_* \end{pmatrix} \tag{9.92}$$

can be represented in the form

$$V(\bar{Z}) = V_0(\bar{Z}) + \sum_{i=1}^{N} \sum_{r=1}^{m} V_i(\bar{Z}_r)\, V_{ir}(\bar{Z}), \tag{9.93}$$

where V_0 is the general solution of this system for a given external load:

$$P_j\ (j=1, 2, \ldots, 6);\ R_j (j=7, 8, \ldots, N);$$

V_{ir} are partial solutions for singular load terms

$$c^{*}_{rji}\,\delta(\bar{Z}-\bar{Z}_r)\ (j=n+7,\ n+8, \ldots, N). \tag{9.94}$$

In expression (9.93), vectors V_o and V_{ir} may be interpreted as solutions in the absence of ribs for a given external load and reactive load (9.94).

Expression (9.93) is more conveniently represented in the form

$$V(\bar{Z}) = V_0(\bar{Z}) + \sum_{r=1}^{m} V_r(\bar{Z})\, V(\bar{Z}_r), \tag{9.95}$$

where

$$V_r = (V_{\eta r}, V_{\xi r}, V_{\theta r}, V_{\gamma r}, V_{*r}). \tag{9.96}$$

From (9.95), we have

$$V(\bar{Z}_q) = V_0(\bar{Z}_q) + \sum_{r=1}^{m} V_r(\bar{Z}_q) V(\bar{Z}_r) \qquad (9.97)$$
$$(q = 1, 2, \ldots, m).$$

Expressions (9.95) and (9.97) represent the solution of Eqs. (9.87) in mixed form. Determining the vectors $V(\bar{Z}_r)$ (r = 1, 2, 111, m) from (9.97), then eliminating them from (9.95), we can readily obtain the solution of Eq. (9.87) in explicit form.

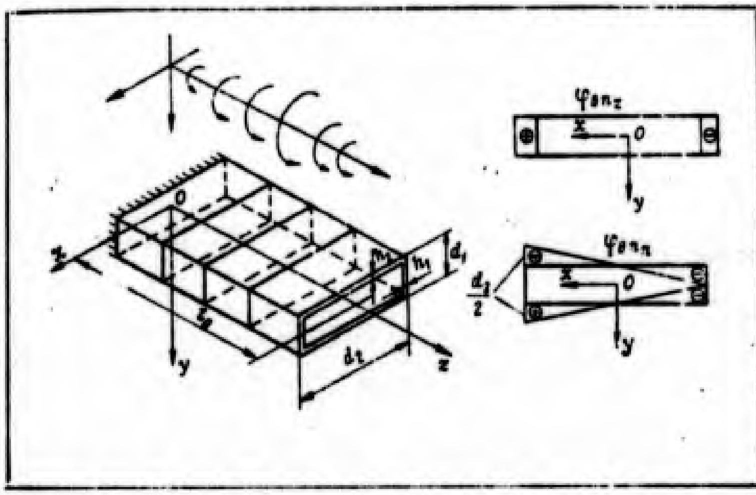

Fig. 9.4. In reference to the problem of torsion of a rectangular caisson, allowing for rib elasticity.

9.4. Torsion of a Straight-Wing Type Cylindrical Shell With Continuous and Discrete Ribs

To illustrate the above, we will consider the problem of torsion of a straight caisson with a rectangular cross section, symmetric about both the Ox axis and the Oy axis (Fig. 9.4 a).

Assuming that bending of the ribs in their plane is absent, we will retain the three generalized displacements:

$$U_6(Z)=\theta_z(Z),\ U_7(Z)=\omega(Z),\ U_8(Z)=\gamma(Z). \tag{9.98}$$

Changing in resolvents (9.71) and (9.72) from the relative coordinate $\bar{Z}$ to the dimensional coordinate Z, we obtain in the case at hand

$$\begin{gathered}\tilde{a}_{66}\theta_z'+\tilde{a}_{67}\omega'+\tilde{a}_{68}\gamma'+\tilde{b}_{66}\theta_z+\tilde{b}_{67}\omega+\tilde{b}_{68}\gamma=\frac{1}{G}M_z,\\ \tilde{a}_{76}\theta_z''+\tilde{a}_{77}\omega''+\tilde{a}_{78}\gamma''+(\tilde{b}_{76}-\tilde{b}_{67})\theta_z'+(\tilde{b}_{78}-\tilde{b}_{87})\gamma'-\tilde{c}_{76}\theta_z-\tilde{c}_{77}\omega-\tilde{c}_{78}\gamma=0,\\ \tilde{a}_{86}\theta_z''+\tilde{a}_{87}\omega''+\tilde{a}_{88}\gamma''+(\tilde{b}_{86}-\tilde{b}_{68})\theta_z'+(\tilde{b}_{87}-\tilde{b}_{78})\omega'-\\ -\tilde{c}_{86}\theta_z-\tilde{c}_{87}\omega-(\tilde{c}_{88}+\tilde{c}_{r88})\gamma=-\frac{1}{G}\tilde{R}_8.\end{gathered} \tag{9.99}$$

Here the coefficients

$$\tilde{a}_{ji}=\lim_{l_0\to\infty} l_0 a_{ji},\quad \tilde{b}_{ji}=\lim_{l_0\to\infty} b_{ji},\quad \tilde{c}_{ji}=\lim_{l_0\to\infty}\frac{1}{l_0}c_{ji}$$

are determined by general expressions (6.133) for cylindrical shells; in view of (9.70), the coefficient $\tilde{c}_{r88}=\lim\limits_{l_0\to\infty}\frac{1}{l_0}c_{r88}$ will be:

$$\tilde{c}_{r88}=\frac{2}{1-\nu}\oint \varphi^2_{8m_s}\Delta\bar{F}_r\,dS_r+\bar{h}_r F_0, \tag{9.100}$$

where

$\Delta\bar{F}_r$, $\bar{h}_r$ is the cross-sectional area of the flanges and the rib web thickness, referred to a unit length of the wing;

F_0 is the area of the rib web.

In view of (9.79), the load term $\tilde{R}_8=\lim\limits_{l_0\to\infty}\frac{1}{l_0}R_8$ will be

$$\tilde{R}_8=\oint\left(p^0_{m_s}\varphi_{8m_s}+p^0_{n_s}\varphi_{8n_s}+p^0_{n_n}\varphi_{8n_n}\right)dS. \tag{9.101}$$

We will take the coordinate function corresponding to torsional warping $\omega(z)$ in the form

$$\varphi_1^1(S)=x(S)\,y(S). \tag{9.102}$$

In this case, the coefficients of system (9.99) for $i, j\neq 8$ coincide with the corresponding coefficients calculated in Section 8.3 for a caisson with nondeformable contour

$$
\begin{aligned}
&\tilde{a}_{\theta\theta}=I_s^{\circ}=\frac{d_1d_2}{2}(d_2h_1+d_1h_2),\\
&\tilde{a}_{\theta\gamma}=0,\\
&\tilde{a}_{\gamma\gamma}=\frac{2}{1-\nu}\;\frac{d_1^2d_2^2}{24}(d_1h_1+d_2h_2),\\
&\tilde{b}_{\theta\theta}=\tilde{b}_{\gamma\theta}=\tilde{b}_{\gamma\gamma}=0,\\
&\tilde{b}_{\theta\gamma}=\frac{d_1d_2}{2}(d_2h_1-d_1h_2),\\
&\tilde{c}_{\theta\theta}=\tilde{c}_{\theta\gamma}=0,\\
&\tilde{c}_{\gamma\gamma}=I_s^{\circ}.
\end{aligned}
\qquad (9.103)
$$

The coordinate function Υ, corresponding to the generalized displacement $\gamma(Z)$, is determined by expression (9.40). Selecting the constant C_1 in accordance with (9.34), we can easily obtain

$$\varphi_1^{\gamma}(M_0)=x\,\mathbf{e}_3, \qquad (9.104)$$

whence

$$
\begin{aligned}
&\varphi_{\theta m_s}(S)=0,\\
&\varphi_{\theta n_s}(S)=x(S)\,y'(S),\\
&\varphi_{\theta n_n}(S)=x(S)\,x'(S).
\end{aligned}
\qquad (9.105)
$$

The diagrams of coordinate functions (9.105) are shown in Fig. 9.4 b. Calculating quadrature (6.133) while taking (9.105) into account, we find

$$
\begin{aligned}
&\tilde{a}_{\theta\theta}=\frac{d_1d_2^2}{2}h_1;\ \tilde{a}_{\gamma\theta}=0;\ \tilde{a}_{\theta\theta}=\frac{d_1d_2^2}{2}h_1,\\
&\tilde{b}_{\theta\theta}=\tilde{b}_{\gamma\theta}=\tilde{b}_{\theta\theta}=\tilde{b}_{\theta\theta}=0;\ \tilde{b}_{\theta\gamma}=\frac{d_1d_2^2}{2}h_1,\\
&\tilde{c}_{\theta\theta}=\tilde{c}_{\gamma\theta}=\tilde{c}_{\theta\theta}=0.
\end{aligned}
\qquad (9.106)
$$

Passing to the limit in expression (9.54) when $l_{\theta}\to\infty$, we find for a straight cylindrical shell

$$
\tilde{\psi}_{im_s}=\tilde{\psi}_{in_s}=\begin{cases} 0 & (i\leqslant n+6),\\ \varphi'_{in_s}-\dfrac{1}{R_0}\varphi_{in_n} & (i>n+6),\end{cases}
\qquad (9.107)
$$

whence, in view of (9.105),

$$\bar{\Psi}_{\theta m_z} = 0. \tag{9.108}$$

Considering (9.108), we have from (9.100)

$$\tilde{c}_{\kappa\theta\theta} \quad d_1 d_2 \bar{h}_\kappa. \tag{9.109}$$

For the values obtained for coefficients (9.103), (9.106), (9.109), system (9.99) becomes

$$\begin{aligned}
&(a+e)\theta_z' + e\gamma' - (a-e)\omega = \frac{1}{G} M_z,\\
&\tilde{a}_{\gamma\gamma}\omega'' + (a-e)\theta_z' - e\gamma' - (a+e)\omega = 0,\\
&e\theta_z' + e\gamma'' + e\omega' - \tilde{c}_{\kappa\theta\theta}\gamma = -\frac{1}{G}\bar{R}_\theta,
\end{aligned} \tag{9.110}$$

where

$$a = \frac{d_1^2 d_2}{2} h_2, \quad e = \frac{d_1 d_2^2}{2} h_1. \tag{9.111}$$

From the first equation of system (9.110), we have

$$\theta_z' = \frac{a-e}{a+e}\omega - \frac{e}{a+e}\gamma' + \frac{1}{a+e}\frac{M_z}{G}. \tag{9.112}$$

Eliminating θ_z', θ_z'' from the other two equations of system (9.110), with the aid of (9.112) we obtain

$$\begin{aligned}
&\tilde{a}_{\gamma\gamma}\omega'' - 4\frac{ae}{a+e}\omega - 2\frac{ae}{a+e}\gamma' = -\frac{a-e}{a+e}\frac{M_z}{G},\\
&\frac{ae}{a+e}\gamma'' + 2\frac{ae}{a+e}\omega' - \tilde{c}_{\kappa\theta\theta}\gamma = -\frac{\bar{R}_\theta + \frac{e}{a+e}M_z'}{G}.
\end{aligned} \tag{9.113}$$

Let us consider the model with a continuous rib distribution. We will assume that all the ribs are identical and uniformly distributed over the length of the caisson. In this case, on the basis of (9.58), (9.109)

$$\tilde{c}_{\varphi\theta\theta} = d_1 d_2 \frac{h_r}{\Delta l} = \text{const}, \tag{9.114}$$

where

h_r is the rib web thickness;

Δl is the distance between the ribs.

System (9.113) now has constant coefficients. It is easy to see that this system is associated with the following biquadratic characteristic equation

$$\lambda^4 - \tilde{c}_{r88} \frac{a+e}{ae} \lambda^2 + 4 \frac{\tilde{c}_{r88}}{\tilde{a}_{77}} = 0, \tag{9.115}$$

whence

$$\lambda_{1,2,3,4} = \pm \sqrt{\frac{1}{2}\left[\tilde{c}_{r88} \frac{a+e}{ae} \pm \sqrt{\tilde{c}^2_{r88}\left(\frac{a+e}{ae}\right)^2 - 16 \frac{\tilde{c}_{r88}}{\tilde{a}_{77}}}\right]} \tag{9.116}$$

We will analyze Eq. (9.115). Let $\tilde{c}_{r88} \to \infty$, which corresponds to a caisson with a cross-sectional contour nondeformable in its plane. In this case, from (9.115) it follows in the limit that

$$\lambda^2 = \lambda_*^2 = \frac{4}{\tilde{a}_{77}} \frac{ae}{a+e}. \tag{9.117}$$

It is easy to see that expression (9.117) coincides with expression (8.45), written directly for a cylindrical shell with a nondeformable cross-sectional contour if this section has a rectangular configuration symmetric about the Ox, Oy axes.

Let $\tilde{c}_{r88} = 0$, which corresponds to the case where the ribs are absent. It then follows from (9.115) that

$$\lambda_{1,2,3,4} = 0, \tag{9.118}$$

i.e., we obtain an undamped solution.

Finally, it follows from (9.116) that for small values of $\tilde{c}_{r88}$ (weak rib structure), the roots of the characteristic equation will be complex, and for large values of $\tilde{c}_{r88}$ (strong structure), they will be real.

Let us consider a case in which the caisson is loaded only by a twisting moment applied to the end $Z = l$. In this case

10.3. Local Fixing of the Wing

Results of analysis of wing type shells with point fixing of the edge section are cited below. The rib elasticity is not considered. For this reason, the edge rib is considered to be fixed along the entire contour, i.e., the point character of the fixing is assumed only with respect to longitudinal displacements.

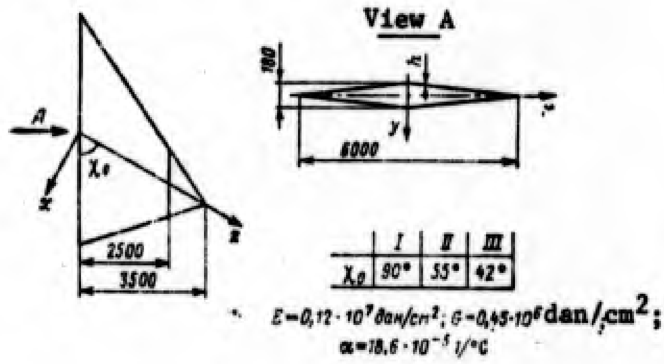

Fig. 10.12. Low-aspect wing with rhomboidal profile.

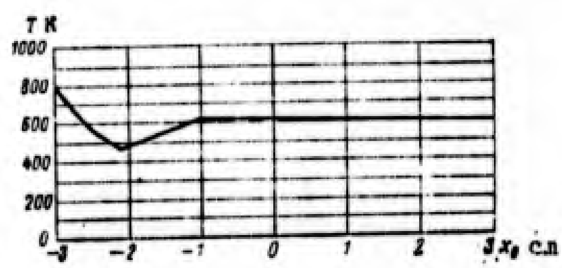

Fig. 10.13. Distribution of temperature over the wing chord at zero incidence.

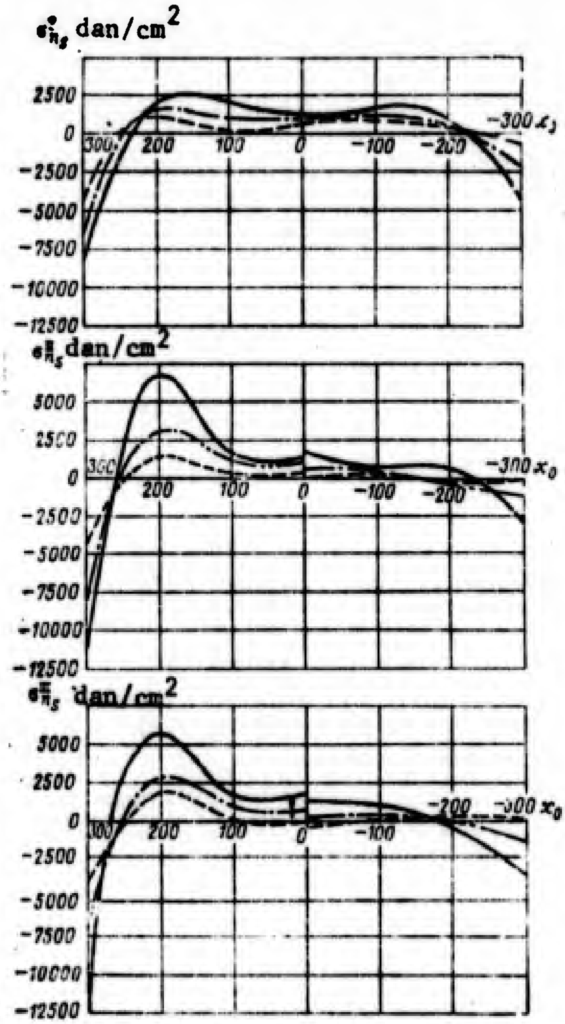

Fig. 10.14. Thermal stresses in low-aspect wings of rhomboidal profile.

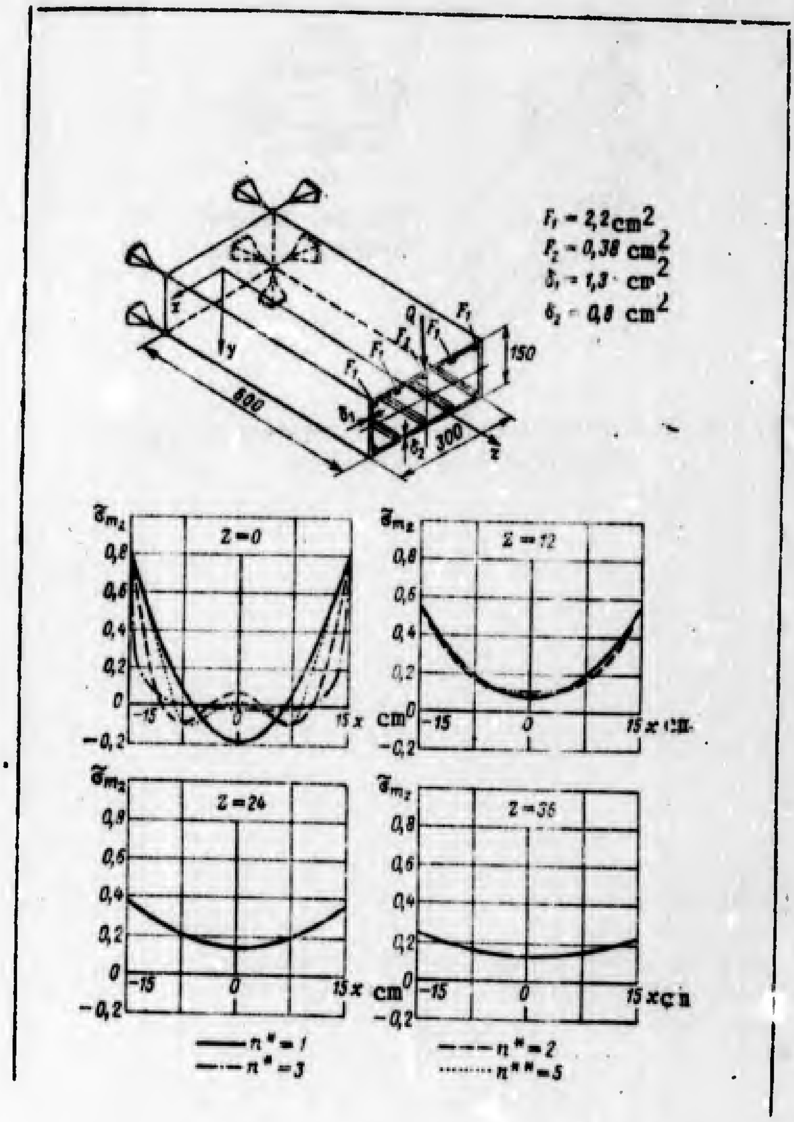

Fig. 10.15. Distribution of normal stresses during bending of a straight caisson fixed at points.

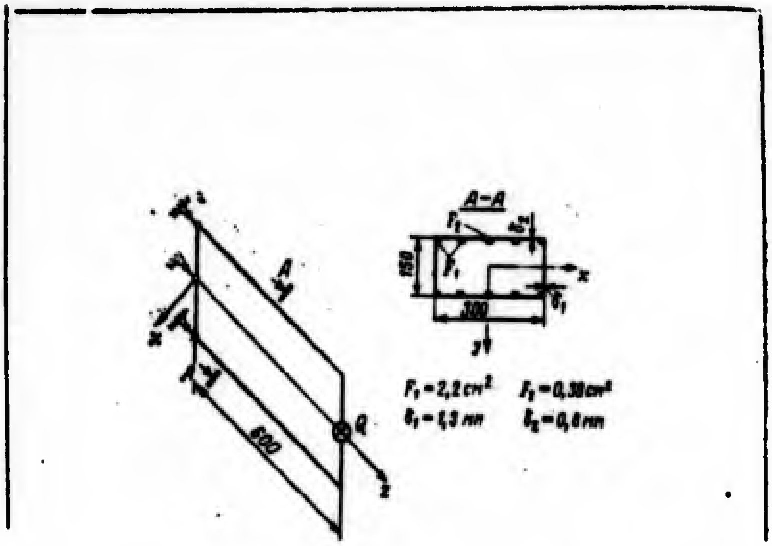

Fig. 10.16. Swept caisson of rectangular cross section fixed at points.

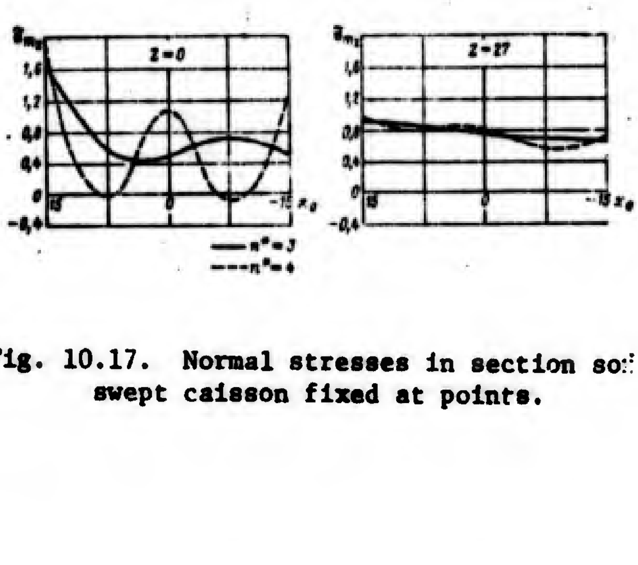

Fig. 10.17. Normal stresses in section of swept caisson fixed at points.

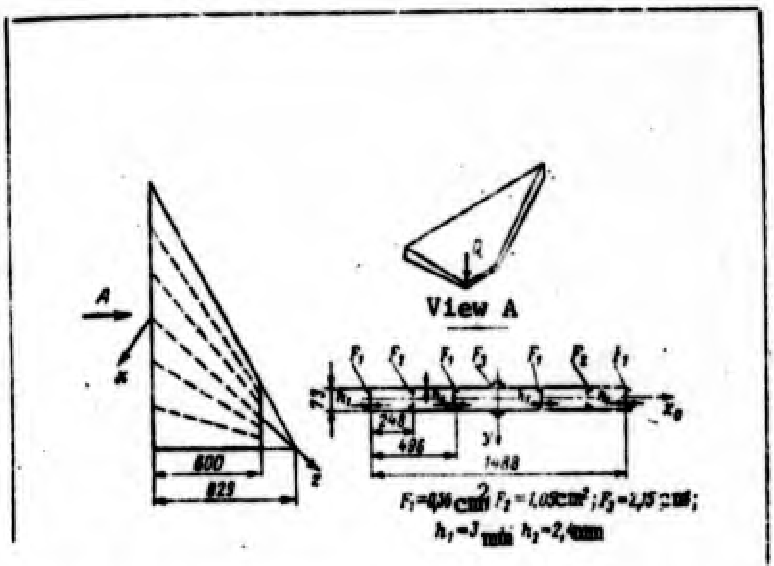

Fig. 10.18. Los-aspect wing type caisson.

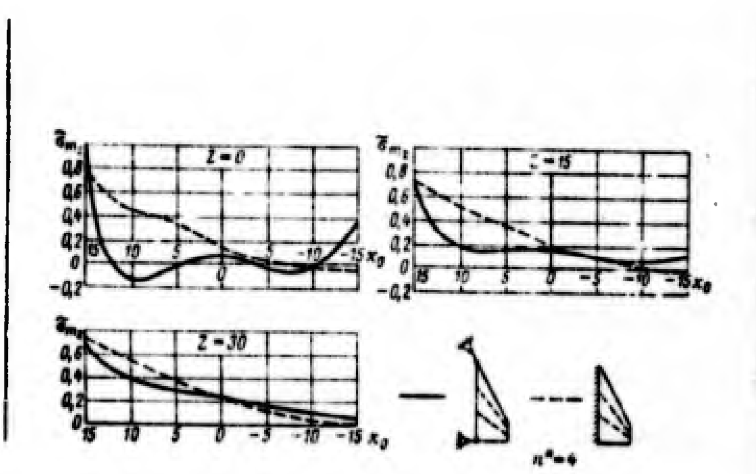

Fig. 10.19. Comparison of normal stresses for wing fixed at the two extreme spars and along the entire edge.

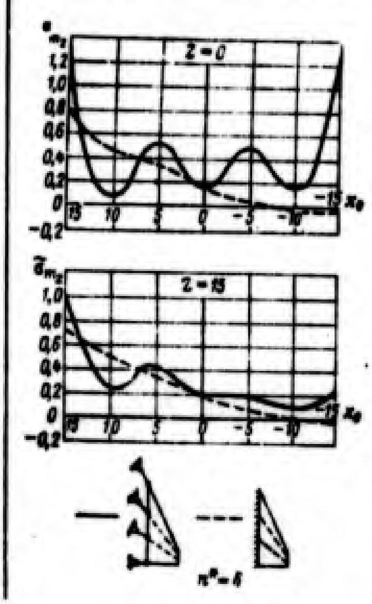

Fig. 10.20. Comparison of normal stresses for wing fastened at four spars and along the entire edge.

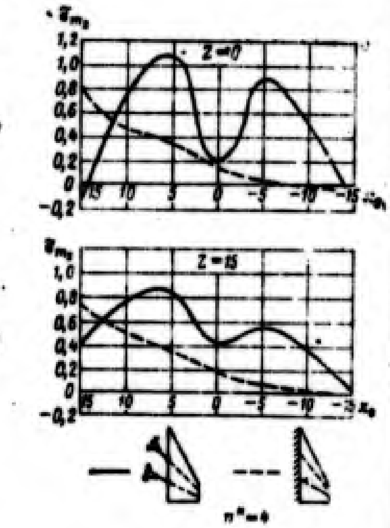

Fig. 10.21. Comparison of normal stresses for a wing fixed at two middle spars and along the entire edge.

Figures 10.15-10.17 show the results of analyses of straight and swept wing type shells. For comparison, they also show the results of analyses for the edge rib fastened along the entire contour.

Figures 10.19-10.21 give the results of analyses of a low-aspect wing type shell illustrated in Fig. 10.18, for different variants of point fixing and continuous fixing of the edge rib.

Chapter XI. VIBRATIONS OF WINGS

11.1. Differential Resolvents

In considering the nonsteady motion of a wing, we will proceed from the representation of the elastic displacement vector in the form

$$\mathbf{U}(M,t)=\sum_{i=1}^{N} U_i(\bar{Z},t)\lambda_i(\bar{Z})\varphi_i(M_0), \tag{11.1}$$

where, in contrast to the above, the desired generalized displacements U_i depend on both the coordinate $\bar{Z}$ and time t. For a suitable choice of coordinate functions $\varphi_i (i>6)$ and number N, this expansion corresponds to any of the geometrical models examined above. As in all previous cases, the first six coordinate functions correspond to displacement of the contour $\bar{Z}$ = const as a solid.

The differential resolvents in the desired generalized displacements can be obtained from the corresponding equations for static problems if aerodynamic and inertial forces are added to the given external load. Then, the surface load referred to the area of the middle surface

$$\mathbf{p}=\mathbf{p}^0(M)+\mathbf{p}^a\mathbf{U}+\mathbf{p}^u\mathbf{U}. \tag{11.2}$$

Here p^0 is the given external load;

p^a is the operator of nonconservative (aerodynamic) forces;

$$\mathbf{p}^i=\varrho(M)h\frac{\partial^2}{\partial t^2} \tag{11.3}$$

is the operator of inertial forces (Q is the density of the shell material).

In view of (11.1)

$$p^{i}U = -\varrho h \sum_{i=1}^{N} \frac{\partial^2 U_i}{\partial t^2} \lambda_i \varphi_i. \tag{11.4}$$

If the loads are located within the wing, the corresponding inertial forces must be added to the external load. Referring these forces to a unit volume enclosed within the wing, we have

$$p^{*i} = -\varrho^*(M) \frac{\partial^2}{\partial t^2}, \tag{11.5}$$

where p* is the load distribution density within the wing.

Extending expressions (11.1) to the region enclosed within the wing, we have

$$p^{*i}U = -\varrho^* \sum_{i=1}^{N} \frac{\partial^2 U_i}{\partial t^2} \lambda_i \varphi_i^*. \tag{11.6}$$

where the functions $\varphi_i^* = \varphi_i^*(\bar{Z}, M_0)$ constitute an extension (allowing for the character of fixing of the loads) of the coordinate functions $\varphi_i(M_0) = \varphi_i(S$ to the region enclosed within the wing.

Usually, owing to the character of fixing of the loads, an inertial load $p^{*i}U$ is determined only by wing displacements for which the contour $\bar{Z}$ = const undergoes displacement as a solid. In this case

$$\varphi_i^*(M) = \begin{cases} \varphi_i(M_0) & (i = 1, 2, \ldots, 6), \\ 0 & (i > 6), \end{cases} \tag{11.7}$$

where the coordinate functions $\varphi_i(M_0)$ $(i = 1, 2, \ldots, 6)$ are determined over the entire plane $\bar{Z}$ = const.

Let us turn to variational Eq. (3.26). In the case at hand, the work done by the external forces, represented earlier by expression (3.27), will be

$$\delta T = \int_0^{\overline{Z}_1} [(1-\overline{Z}) \oint \mathbf{p}\cdot\delta\mathbf{U} l_0 \sin\chi\, dS +$$

$$+ (1-\overline{Z})^2 \int_{F_0} (\mathbf{p}^{*1}\mathbf{U})\cdot\delta\mathbf{U} l_0 \sin\chi_0\, dx_0\, dy_0]\, d\overline{Z} +$$

$$+ (1-\overline{Z}) \oint \overline{\mathbf{q}}^0\cdot\delta\mathbf{U}\, dS\Big|_{\overline{Z}=0} + (1-\overline{Z}) \oint \overline{\mathbf{q}}^1\cdot\delta\mathbf{U}\, dS\Big|_{\overline{Z}=\overline{Z}_1}. \tag{11.8}$$

We conclude from (11.8) that in the differential resolvents, to the load p and $p^{*1}U$ there will correspond a load term

$$R_j(\overline{Z}) = R_j^p(\overline{Z}) + R_j^{p^*}(\overline{Z}), \tag{11.9}$$

where on the basis of (11.1)

$$R_j^p = (1-\overline{Z})\lambda_j \oint \mathbf{p}\cdot\boldsymbol{\varphi}_j l_0 \sin\chi\, dS, \tag{11.10}$$

$$R_j^{p^*} = (1-\overline{Z})^2\lambda_j \int_{F_0} (\mathbf{p}^*\,\mathbf{U})\cdot\boldsymbol{\varphi}_j^0 l_0 \sin\chi_0\, dx_0\, dy_0. \tag{11.11}$$

From (11.10), in view of (11.2), we have

$$R_j^p = R_j^0 + R_j^* + R_j, \tag{11.12}$$

where

$$R_j^0 = (1-\overline{Z})\lambda_j \oint \mathbf{p}^0\cdot\boldsymbol{\varphi}_j l_0 \sin\chi\, dS, \tag{11.13}$$

$$R_j^* = (1-\overline{Z})\lambda_j \oint (\mathbf{p}^*\mathbf{U})\cdot\boldsymbol{\varphi}_j l_0 \sin\chi\, dS, \tag{11.14}$$

$$R_j^{**} = (1-\overline{Z})\lambda_j \oint (\mathbf{p}^{**}\mathbf{U})\cdot\boldsymbol{\varphi}_j l_0 \sin\chi\, dS. \tag{11.15}$$

Expanding expressions (11.11) and (11.15) with the aid of (11.4), (11.6), we find

$$R_j^{p^*} = -l_0 \sin\chi_0 (1-\overline{Z})^2\lambda_j \sum_{l=1}^{N} \frac{\partial^2 U_l}{\partial t^2}\lambda_l \int_{F_0} \boldsymbol{\varphi}_j^0\cdot\boldsymbol{\varphi}_l^0\rho^0\, dx_0\, dy_0, \tag{11.16}$$

$$R_j^{**} = -(1-\overline{Z})\lambda_j \sum_{l=1}^{N} \frac{\partial^2 U_l}{\partial t^2}\lambda_l \oint \boldsymbol{\varphi}_j\cdot\boldsymbol{\varphi}_l \rho l_0 \sin\chi\, h\, dS. \tag{11.17}$$

In view of (11.9), (11.112) - (11.14) and (11.16), (11.17), the differential resolvents resulting from variational Eq. (3.26) assume the form

$$\sum_{l=1}^{N} \Big\{ [(1-\overline{Z}) a_{jl} V_l' + b_{jl} V_l]' - b_{lj} V_l' - \frac{1}{1-\overline{Z}} c_{jl} V_l -$$

$$-(1-\overline{Z}) d_{jl} \frac{\partial^2 V_l}{\partial t^2} - (1-\overline{Z}) \overline{R}_j^* [V_1, V_2, \ldots, V_N] \Big\} = -\frac{1}{\lambda_j G} R_j^0 \tag{11.18}$$

$$(j = 1, 2, \ldots, N).$$

Here the coefficients a_{jk}, b_{ji}, c_{ji} for any of the above-discussed computational shell models are determined by the same expressions as before; the coefficients

$$d_{ji}=\frac{1}{a}\oint(\varphi_{jm_x}\varphi_{im_x}+\varphi_{jn_x}\varphi_{in_x}+\varphi_{jn_n}\varphi_{in_n})\varrho l_s\sin\chi h\,dS+$$
$$+(1-\bar{Z})\frac{l_0\sin\chi_0}{a}\int\limits_{F_0}(\varphi^{*}_{jm_x}\varphi^{*}_{im_x}+\varphi^{*}_{jn_x}\varphi^{*}_{in_x}+\varphi^{*}_{jn_n}\varphi^{*}_{in_n})\varrho^{*}\,dx_1\,dy_1, \tag{11.19}$$

where the curvilinear integral should also be extended in the sense of a Stieltjes integral to the concentrated reinforcing elements

$$R^{a}_{j}=\oint(\mathrm{p}^{a}\mathrm{U})\cdot\varphi_{j}l_s\sin\chi\,dS \tag{11.20}$$

corresponds to the operator of nonconservative forces; as before, the right-hand side corresponding to the given external load

$$R^{0}_{j}=(1-\bar{Z})\lambda_{j}\mathrm{p}^{0}\cdot\varphi_{j}l_s\sin\chi\,dS. \tag{11.21}$$

Expressions (11.18) represent a general system of differential resolvents describing the nonsteady motion of a conical shell of arbitrary configuration acted on by nonconservative (aerodynamic) forces and a given external load, which may be both steady and dependent on time according to a given law.

Equations (11.18) may be taken as the basis for a study of the dynamic behavior of wings of any types: aeroelasticity (flutter, divergence, etc.), natural and forced vibrations, and vibrations of wings partly filled with liquid. For example, in solving natural vibration problems, by discarding the terms R^{a}_{j} and $\bar{R}^{a}_{j}$ in Eqs. (11.18) and taking

$$V_{i}(\bar{Z},t)=V^{*}_{i}(\bar{Z})\sin pt \quad (i=1,2,\ldots,N), \tag{11.22}$$

considering the vibrations to be harmonic, we obtain

$$\sum_{i=1}^{N}\left\{[(1-\bar{Z})a_{ji}V^{*\prime}_{i}+b_{ji}V^{*}_{i}]'-b_{ij}V^{*\prime}_{i}+\left[(1-\bar{Z})d_{ji}p^{2}-\frac{1}{1-\bar{Z}}c_{ji}\right]V^{*}_{i}\right\}=0$$
$$(j=1,2,\ldots,N). \tag{11.23}$$

Thus, the determination of the modes and frequencies of natural vibrations of a wing of arbitrary outline reduces to finding the eigenfunctions and eigenvalues of a system of homogeneous differential Eqs. (11.23). This problem is solved by numerical methods in the general case, but simplified analytical solutions analogous to the simplified solutions of static problems discussed in Ch. VIII are also possible.

11.2. Natural Vibrations of Straight Wing Type Shells

In the present section, in selecting the retained generalized displacements, coordinate functions and positions of the axes, we will proceed from the same premises as in the simplified analysis of straight wing type shells for a given static load (Section 8.1., Subsection 3). Considering the cross section to be close to symmetric, we will retain five generalized displacements: $\eta_y, \theta_x, \theta_z$, corresponding to displacement of the cross section as a solid, and ω_1, ω_2, corresponding to bending and torsional warpings.

Changing in Eqs. (11.23) to the dimensional coordinate $z = l_0 Z$, in the limit when $l_0 \to \infty$, we obtain for shells of constant cross section

$$\sum_{i=1}^{N} [\tilde{a}_{ji} U_i'' + (\tilde{b}_{ji} - \tilde{b}_{ij}) U_i' + (\tilde{d}_{ji} p^2 - \tilde{c}_{ji}) U_i] = 0$$

$$(j = 1, 2, \ldots, N), \tag{11.24}$$

where on the basis of (11.19)

$$\tilde{d}_{ji} = \frac{1}{G} \oint (\varphi_{jm_z} \varphi_{im_z} + \varphi_{jn_z} \varphi_{in_z} + \varphi_{jn_n} \varphi_{in_n}) \varrho \sin \chi h \, dS +$$

$$+ \frac{\sin \chi_0}{G} \int_{F_0} (\varphi^*_{jm_z} \varphi^*_{im_z} + \varphi^*_{jn_z} \varphi^*_{in_z} + \varphi^*_{jn_n} \varphi^*_{in_n}) \varrho^* \, dx_0 \, dy_0, \tag{11.25}$$

and the coefficients $\tilde{a}_{ji}, \tilde{b}_{ji}, \tilde{c}_{ji}$ are determined from formulas corresponding to the chosen geometric model.

Equations (11.24) and coefficients (11.25) correspond to oblique cylindrical shells of arbitrary configuration. On passing to straight cylindrical shells, we must set $\dot{a}_{jk}, \bar{b}_{jk}, \bar{c}_{jk}$ in the expressions for the coefficients $\chi_0=\pi/2$. and in expression (11.25). In the case at hand, Eqs. (11.24) assume the form

$$
\begin{aligned}
&F^{\circ}_{yy}\eta^{\circ\prime\prime}_{y}+S^{\circ}_{y}\theta^{\circ\prime\prime}_{z}+F^{\circ}_{yy}\theta^{\circ\prime}_{x}+\tilde{b}_{37}\omega^{\circ\prime}_{1}+\tilde{b}_{38}\omega^{\circ\prime}_{2}+p^{2}(\tilde{d}_{33}\eta^{\circ}_{y}+\tilde{d}_{36}\theta^{\circ}_{z})=0,\\
&\frac{2}{1-\nu}I_{x}\theta^{\circ\prime\prime}_{x}-F^{\circ}_{yy}\eta^{\circ\prime}_{y}-S^{\circ}_{y}\theta^{\circ\prime}_{z}-F^{\circ}_{yy}\theta^{\circ}_{x}-\tilde{c}_{67}\omega^{\circ}_{1}-\tilde{c}_{68}\omega^{\circ}_{2}+\\
&\qquad+p^{2}(\tilde{d}_{44}\theta^{\circ}_{x}+\tilde{d}_{67}\omega^{\circ}_{1}+\tilde{d}_{68}\omega^{\circ}_{2})=0,\\
&S^{\circ}_{y}\eta^{\circ\prime\prime}_{y}+I^{\circ}_{z}\theta^{\circ}_{z}+S^{\circ}_{y}\theta^{\circ\prime}_{z}+\tilde{b}_{67}\omega^{\circ\prime}_{1}+\tilde{b}_{68}\omega^{\circ\prime}_{2}+p^{2}(\tilde{d}_{63}\eta^{\circ}_{y}+\tilde{d}_{66}\theta^{\circ}_{z})=0\\
&\tilde{a}_{77}\omega^{\circ\prime\prime}_{1}-\tilde{b}_{37}\eta^{\circ\prime}_{y}-\tilde{b}_{67}\theta^{\circ\prime}_{z}-\tilde{c}_{76}\theta^{\circ}_{x}-\tilde{c}_{77}\omega^{\circ}_{1}-\tilde{c}_{78}\omega^{\circ}_{2}+\\
&\qquad+p^{2}(\tilde{d}_{76}\theta^{\circ}_{x}+\tilde{d}_{77}\omega^{\circ}_{1}+\tilde{d}_{78}\omega^{\circ}_{2})=0,\\
&\tilde{a}_{88}\omega^{\circ\prime\prime}_{2}-\tilde{b}_{38}\eta^{\circ\prime}_{y}-\tilde{b}_{68}\theta^{\circ\prime}_{z}-\tilde{c}_{86}\theta^{\circ}_{x}-\tilde{c}_{87}\omega^{\circ}_{1}-\tilde{c}_{88}\omega^{\circ}_{2}+\\
&\qquad+p^{2}(\tilde{d}_{86}\theta^{\circ}_{x}+\tilde{d}_{87}\omega^{\circ}_{1}+\tilde{d}_{88}\omega^{\circ}_{2})=0,
\end{aligned}
\tag{11.26}
$$

where the coefficients $F^{\circ}_{\mu\nu}, S^{\circ}_{\nu}, I^{\circ}_{z}$ are determined by expressions (6.139)-(6.141); I_x is the cross-sectional moment of inertia. Let us also note that

$$\tilde{b}_{37}=\tilde{c}_{67},\quad \tilde{b}_{38}=\tilde{c}_{68}. \tag{11.27}$$

System (11.26) is general in character,* and the problem of finding the eigenvalues and eigenfunctions of this system is fairly cumbersome. In special cases, this problem is somewhat simplified. For example, if loads within the wing are absent, then in view of orthogonality conditions (8.10),

$$\tilde{d}_{67}=\tilde{d}_{68}=\tilde{d}_{78}=0. \tag{11.28}$$

Moreover, for wing type shells, in determining the lower frequencies and modes of natural vibrations, it is permissible to neglect the rotatory

*For equations for straight prismatic caissons, see I. F. Obraztsov. Variational Methods of Analysis of Thin-Walled Aeronautical Structures. Moscow, Mashinostroyeniye, 1966.

inertia of the cross section about the Ox axis, and the inertia of its warping, i.e., to assume that

$$\tilde{d}_{44}=\tilde{d}_{77}=\tilde{d}_{88}=0. \tag{11.29}$$

In this case, system (11.26) assumes the form

$$\begin{gathered}
F_{yy}^{\circ}\eta_y^{\circ\prime\prime}+S_y^{\circ}\theta_z^{\circ\prime\prime}+F_{yy}^{\circ}\theta_x^{\circ\prime}+\tilde{b}_{27}\omega_1^{\circ\prime}+\tilde{b}_{28}\omega_2^{\circ\prime}+p^2(\tilde{d}_{22}\eta_y^{\circ}+\tilde{d}_{26}\theta_z^{\circ})=0,\\
\frac{2}{1-\nu}I_x\theta_x^{\circ\prime\prime}-F_{yy}^{\circ}\eta_y^{\circ\prime}-S_y^{\circ}\theta_z^{\circ\prime}-F_{yy}^{\circ}\theta_x^{\circ}-\tilde{b}_{27}\omega_1^{\circ}-\tilde{b}_{28}\omega_2^{\circ}=0,\\
S_y^{\circ}\eta_y^{\circ\prime\prime}+I_z^{\circ}\theta_z^{\circ\prime\prime}+S_y^{\circ}\theta_x^{\circ\prime}+\tilde{b}_{67}\omega_1^{\circ\prime}+\tilde{b}_{68}\omega_2^{\circ\prime}+p^2(\tilde{d}_{62}\eta_y^{\circ}+\tilde{d}_{66}\theta_z^{\circ})=0,\\
\tilde{a}_{77}\omega_1^{\circ\prime\prime}-\tilde{b}_{27}\eta_y^{\circ\prime}-\tilde{b}_{67}\theta_z^{\circ\prime}-\tilde{b}_{27}\theta_x^{\circ}-\tilde{c}_{77}\omega_1^{\circ}-\tilde{c}_{78}\omega_2^{\circ}=0,\\
\tilde{a}_{88}\omega_2^{\circ\prime\prime}-\tilde{b}_{28}\eta_y^{\circ\prime}-\tilde{b}_{68}\theta_z^{\circ\prime}-\tilde{b}_{28}\theta_x^{\circ}-\tilde{c}_{87}\omega_1^{\circ}-\tilde{c}_{88}\omega_2^{\circ}=0.
\end{gathered} \tag{11.30}$$

The problem is simplified even more if we assume the cross section to be symmetric about both the Ox axis and the Oy axis. For this case, we can readily see that

$$\tilde{d}_{26}=0. \tag{11.31}$$

Considering also (8.34), we have

$$\begin{gathered}
F_{yy}^{\circ}\eta_y^{\circ\prime\prime}+F_{yy}^{\circ}\theta_x^{\circ\prime}+\tilde{b}_{27}\omega_1^{\circ\prime}+p^2\tilde{d}_{22}\eta_y^{\circ}=0,\\
\frac{2}{1-\nu}I_x\theta_x^{\circ\prime\prime}-F_{yy}^{\circ}\eta_y^{\circ\prime}-F_{yy}^{\circ}\theta_x^{\circ}-\tilde{b}_{27}\omega_1^{\circ}=0,\\
\tilde{a}_{77}\omega_1^{\circ\prime\prime}-\tilde{b}_{27}\eta_y^{\circ\prime}-\tilde{b}_{27}\theta_x^{\circ}-\tilde{c}_{77}\omega_1^{\circ}=0,
\end{gathered} \tag{11.32}$$

$$\begin{gathered}
I_z^{\circ}\theta_z^{\circ\prime\prime}+\tilde{b}_{68}\omega_2^{\circ\prime}+p^2\tilde{d}_{66}\theta_z^{\circ}=0,\\
\tilde{a}_{88}\omega_2^{\circ\prime\prime}-\tilde{b}_{68}\theta_z^{\circ\prime}-\tilde{c}_{88}\omega_2^{\circ}=0.
\end{gathered} \tag{11.33}$$

Thus, for a cross section symmetric about both axes, the coupled system of five differential equations decomposes. Equations (11.32) determine bending vibrations, and Eqs. (11.33), torsional vibrations.

As an example, let us consider a caisson with a rectangular cross section (see Fig. 8.2).

Expressions (8.41) represent the values of all the coefficients of Eqs. (11.32) and (11.33) with the exception of the coefficients $\tilde{d}_{22}$, $\tilde{d}_{66}$, which determine the inertial characteristics of the wing. For these coefficients, computing quadratures (11.25), we find

$$\begin{aligned} \tilde{d}_{22} &= \frac{Q}{G} F^{*}_{yy}\left(1+\frac{d_2 h_2}{d_1 h_1}\right), \\ \tilde{d}_{66} &= \frac{Q}{G} I^{*}_{z}\left[1+\frac{1}{6}\,\frac{d_2^3 h_2 + d_1^3 h_1}{d_1 d_2 (d_1 h_2 + d_2 h_1)}\right]. \end{aligned} \tag{11.34}$$

1. <u>Bending Vibrations</u>

Let us turn to Eqs. (11.32). Representing the partial solution in the form

$$\eta^{*}_{y}(Z)=C_{\eta}e^{\lambda Z}, \quad \theta^{*}_{x}(Z)=C_{\theta}e^{\lambda Z}, \quad \omega^{*}_{1}(Z)=C_{\omega}e^{\lambda Z}, \tag{11.35}$$

we easily arrive at the characteristic equation

$$\zeta^{3}+(\mu\xi+\bar{p}^{2})\zeta^{2}-(\vartheta+\varkappa\xi)\bar{p}^{2}\zeta-\vartheta\mu\xi\bar{p}^{2}=0, \tag{11.36}$$

where

$$\zeta=\lambda^{2}, \quad \bar{p}^{2}=p^{2}\frac{\tilde{d}_{22}}{F^{*}_{yy}}, \quad \mu=\frac{\bar{b}^{2}_{27}}{F^{*}_{yy}}-\bar{c}_{77}, \quad \varkappa=\bar{c}_{77}, \quad \vartheta=\frac{1-\nu}{2}\,\frac{F^{*}_{yy}}{I_{x}}, \quad \xi=\frac{1}{a_{77}}. \tag{11.37}$$

Equation (11.36) represents the dependence of the characteristic indices λ on the frequencies of natural vibrations p. Since the solution of a cubic equation in general form is fairly cumbersome, we will confine ourselves to an approximate symptotic solution.

Treating (11.36) as a multivalued (three-valued) function $\zeta(\xi)$, given in the implicit form

$$\Phi(\zeta, \xi)=0, \tag{11.38}$$

we have

$$\frac{d\zeta}{d\xi}=-\frac{\Phi'_{\xi}}{\Phi'_{\zeta}}=-\frac{\mu\zeta^2-\varkappa\bar{p}^2\zeta-\vartheta\mu\bar{p}^2}{3\zeta^2+2(\mu\xi+\bar{p}^2)\zeta-(\vartheta+\varkappa\xi)\bar{p}^2}. \tag{11.39}$$

Expanding $\zeta(\xi)$ in a Taylor series near $\xi = \xi_o$, we have

$$\zeta(\xi)=\zeta_0-\frac{\mu\zeta_0^2-\varkappa\bar{p}^2\zeta_0-\vartheta\mu\bar{p}^2}{3\zeta_0^2+2(\mu\xi_0+\bar{p}^2)\zeta_0-(\vartheta+\varkappa\xi_0)\bar{p}^2}(\xi-\xi_0)+\ldots, \tag{11.40}$$

where $\zeta_0=\zeta(\xi_0)$ is the root of Eq. (11.36) for $\xi = \xi_o$.

Since the coefficient $\xi=\frac{1}{a_\eta}$ of Eq. (11.36) is small, we will assume that

$$\xi_0=0. \tag{11.41}$$

In this case, from (11.36), we have

$$(\zeta_0)_1=0,\quad (\zeta_0)_{2,3}=\frac{\bar{p}^2}{2}\left(-1\pm\sqrt{1+4\frac{\vartheta}{\bar{p}^2}}\right). \tag{11.42}$$

It is easy to see that the values of ξ represented by expressions (11.42) are exact when warping of the cross section is not considered. It follows that these values are close to the exact values of the roots of Eq. (11.36), since the influence of warping on the lower frequencies is obviously slight. In this connection, it is sufficient to confine onself to the linear approximation in expansion (11.40).

Introducing (11.42) into (11.40), we obtain the asymptotic values of the roots of Eq. (11.36)

$$\zeta_1=-\mu\xi, \tag{11.43}$$

$$\zeta_{2,3}=\bar{p}\left(-\frac{\bar{p}}{2}\pm\sqrt{\frac{\bar{p}^2}{4}+\vartheta}\pm\frac{\mu+\varkappa}{2}\,\frac{\xi}{\sqrt{\frac{\bar{p}^2}{4}+\vartheta}}\right). \tag{11.44}$$

In order to evaluate the error of approximate solution (11.43), (11.44), we reconstruct the corresponding cubic equation

$$(\zeta-\zeta_1)(\zeta-\zeta_2)(\zeta-\zeta_3)=0. \tag{11.45}$$

Expanding (11.45), we obtain

$$\zeta^3+(\mu\xi+\bar{p}^2)\zeta^2-\left(\theta+\nu\xi+\xi^2\,\frac{\mu+\varkappa}{\bar{p}^2+4\theta}\right)\bar{p}^2\zeta-$$
$$-\theta\mu\xi\bar{p}^2-\xi^2\bar{p}^2\mu(\mu+\varkappa)\left(1+\xi\frac{\mu+\varkappa}{\bar{p}^2+4\theta}\right)=0. \tag{11.46}$$

Comparing Eqs. (11.36), (11.46) we see that Eq. (11.46) differs from the exact equation only in some minor additions of the order of ξ^2 and ξ^2 in the coefficient on ξ and the free term.

For wing type shells, usually

$$\theta\ll 1. \tag{11.47}$$

Therefore, for all natural frequencies with the possible exception of the lowest

(11.48)

Taking (11.48) into consideration, we have from (11.44)

$$\zeta_2=\theta+\xi\,\frac{\mu+\varkappa}{1+2\dfrac{\theta}{\bar{p}^2}},$$
$$\zeta_3=-\bar{p}^2-\theta-\xi\,\frac{\mu+\varkappa}{1+2\dfrac{\theta}{\bar{p}^2}}. \tag{11.49}$$

For practical purposes, the accuracy of approximate solution (11.45), (11.49) is complete sufficient. The corresponding cubic equation

$$\zeta^3+(\mu\xi+\bar{p}^2)\zeta^2-\left[\theta+\varkappa\xi+\frac{\theta^2}{\bar{p}^2}+\xi^2\bar{p}^2\,\frac{(\mu+\varkappa)^2}{(\bar{p}^2+2\theta)^2}\right]\bar{p}^2\zeta-$$
$$-\theta\mu\xi\bar{p}^2-\xi^2\bar{p}^2\mu\,\frac{\mu+\varkappa}{1+2\dfrac{\theta}{\bar{p}^2}}-\xi\mu\left(\theta+\xi\,\frac{\mu+\varkappa}{1+2\dfrac{\theta}{\bar{p}^2}}\right)^2=0$$

differs from the exact equation only in minor additions of second- and third-order small terms in the coefficients on ξ and the free term.

To solution (11.43), (11.49) of Eq. (11.36) there corresponds six values of the characteristic index λ, two of which are real, and the remaining ones, purely imaginary:

$$\lambda_{1,2} = \pm\sqrt{\theta + \xi\frac{\mu + \varkappa}{1 + 2\frac{\theta}{\bar{p}^2}}}, \tag{11.50}$$

$$\lambda_{3,4} = \pm i\sqrt{\bar{p}^2 + \theta + \xi\frac{\mu + \varkappa}{1 + 2\frac{\theta}{\bar{p}^2}}}, \tag{11.51}$$

$$\lambda_{5,6} = \pm i\sqrt{\mu\xi}. \tag{11.52}$$

To the λ values obtained there corresponds a general solution of system (11.32) of the form

$$\begin{aligned}
\eta_y^\circ(Z) = {} & \frac{\sqrt{\zeta_2}}{\bar{p}^2}\,\frac{\mu\xi + \zeta_2}{\mu + \varkappa}\left(C_1 \operatorname{sh}\sqrt{\zeta_2}Z + C_2 \operatorname{ch}\sqrt{\zeta_2}Z\right) + \\
& + \frac{\sqrt{-\zeta_1}}{\bar{p}^2}\,\frac{\mu\xi + \zeta_1}{\mu + \varkappa}\left(C_3 \sin\sqrt{-\zeta_1}Z + C_4 \cos\sqrt{-\zeta_1}Z\right) + \\
& + \frac{\sqrt{-\zeta_3}}{\bar{p}^2}\,\frac{\mu\xi + \zeta_3}{\mu + \varkappa}\left(C_5 \sin\sqrt{-\zeta_3}Z + C_6 \cos\sqrt{-\zeta_3}Z\right), \\
\theta_x^\circ(Z) = {} & \frac{\bar{p}^2(\varkappa\xi - \zeta_2) - \zeta_2(\mu\xi + \zeta_2)}{\bar{p}^2(\mu + \varkappa)}\left(C_1 \operatorname{ch}\sqrt{\zeta_2}Z + C_2 \operatorname{sh}\sqrt{\zeta_2}Z\right) + \\
& + \frac{\bar{p}^2(\varkappa\xi - \zeta_1) - \zeta_1(\mu\xi + \zeta_1)}{\bar{p}^2(\mu + \varkappa)}\left(-C_3 \cos\sqrt{-\zeta_1}Z + C_4 \sin\sqrt{-\zeta_1}Z\right) + \\
& + \frac{\bar{p}^2(\varkappa\xi - \zeta_3) - \zeta_3(\mu\xi + \zeta_3)}{\bar{p}^2(\mu + \varkappa)}\left(-C_5 \cos\sqrt{-\zeta_3}Z + C_6 \sin\sqrt{-\zeta_3}Z\right), \\
\omega_1^\circ(Z) = {} & -\frac{F_{yy}^\circ\xi}{\bar{b}_{27}}\left(C_1 \operatorname{ch}\sqrt{\zeta_2}Z + C_2 \operatorname{sh}\sqrt{\zeta_2}Z - C_3 \cos\sqrt{-\zeta_1}Z + \right. \\
& \left. + C_4 \sin\sqrt{-\zeta_1}Z - C_5 \cos\sqrt{-\zeta_3}Z + C_6 \sin\sqrt{-\zeta_3}Z\right).
\end{aligned} \tag{11.53}$$

It is now necessary to satisfy the boundary conditions when Z = 0 and Z = l.

On the end Z = 0, from the fixing conditions we have

$$\eta_y^\circ(0) = \theta_x^\circ(0) = \omega_1^\circ(0) = 0. \tag{11.54}$$

On the end Z = l, from the conditions

$$Q_y(l) = 0 \text{ и } M_x(l) = P_\omega(l) = 0, \tag{11.55}$$

where P_ω is the generalized force corresponding to warping ω_1, it follows that

$$\begin{aligned}
& \eta_y^{\circ\prime}(l) + \theta_x^\circ(l) + \frac{\bar{b}_{27}}{F_{yy}^\circ}\omega_1^\circ(l) = 0, \\
& \theta_x^{\circ\prime}(l) = \omega_1^{\circ\prime}(l) = 0.
\end{aligned} \tag{11.56}$$

Expanding conditions (11.54) and (11.56) with the aid of general solution (11.53), we arrive at a homogeneous system of six algebraic equations in constants $C_1, C_2, \ldots, C_6$. On the basis of the condition of nontriviality of the solution, equating the determinant of this system to zero, we arrive at a transcendental equation in the eigenvalues of the parameter $\bar{p}$.

To simplify the operations, let us assume, for example, that for roots (11.49) of Eq. (11.36), because of the smallness of the parameters and ξ, we have the following rough approximation values:

$$\begin{aligned} \zeta_2 &= \theta + \xi(\mu + \varkappa), \\ \zeta_1 &= -\bar{p}^2. \end{aligned} \tag{11.57}$$

In view of (11.43), (11.57), general solution (11.53) takes the form

$$\begin{aligned} \eta_y^*(Z) &= \frac{\sqrt{\zeta_2}}{\bar{p}^2}\left(\xi + \frac{\mu\xi + \theta}{\mu + \varkappa}\right)(C_1 \operatorname{sh}\sqrt{\zeta_2}Z + C_2 \operatorname{ch}\sqrt{\zeta_2}Z) - \\ &\quad - \frac{\bar{p}}{(\mu + \varkappa)}(C_5 \sin\bar{p}Z + C_6 \cos\bar{p}Z), \\ \theta_x^*(Z) &= -\frac{\mu\xi + \theta}{\mu + \varkappa}(C_1 \operatorname{ch}\sqrt{\zeta_2}Z + C_2 \operatorname{sh}\sqrt{\zeta_2}Z) + \\ &\quad + \xi(-C_3 \cos\sqrt{-\zeta_1}Z + C_4 \sin\sqrt{-\zeta_1}Z - \\ &\quad - C_5 \cos\bar{p}Z + C_6 \sin\bar{p}Z), \\ \omega_1^*(Z) &= -\frac{F_{yy}^*\xi}{\bar{b}_{27}}(C_1 \operatorname{ch}\sqrt{\zeta_2}Z + C_2 \operatorname{sh}\sqrt{\zeta_2}Z - \\ &\quad - C_3 \cos\sqrt{-\zeta_1}Z + C_4 \sin\sqrt{-\zeta_1}Z - C_5 \cos\bar{p}Z + C_6 \sin\bar{p}Z). \end{aligned} \tag{11.58}$$

Satisfying boundary conditions (11.54), (11.56) and 11.58) into consideration, we find

$$\begin{aligned} k_1 C_2 - \bar{p}^2 C_6 &= 0, \\ -k_2 C_1 - C_3 - C_5 &= 0, \\ C_1 - C_3 - C_5 &= 0, \end{aligned} \tag{11.59}$$

$$\frac{k_1}{\sqrt{\zeta_2}}(\zeta_2 - \bar{p}^2)(C_1 \operatorname{ch}\sqrt{\zeta_2}l + C_2 \operatorname{sh}\sqrt{\zeta_2}l) - p^4(C_5 \cos\bar{p}l - C_6 \sin\bar{p}l) = 0, \tag{11.60}$$

$$\begin{aligned} &-k_2\sqrt{\zeta_2}(C_1 \operatorname{sh}\sqrt{\zeta_2}l + C_2 \operatorname{ch}\sqrt{\zeta_2}l) + \sqrt{-\zeta_1}(C_3 \sin\sqrt{-\zeta_1}l + \\ &+ C_4 \cos\sqrt{-\zeta_1}l) + \bar{p}(C_5 \sin\bar{p}l + C_6 \cos\bar{p}l) = 0, \\ &\sqrt{\zeta_2}(C_1 \operatorname{sh}\sqrt{\zeta_2}l + C_2 \operatorname{ch}\sqrt{\zeta_2}l) + \sqrt{-\zeta_1}(C_3 \sin\sqrt{-\zeta_2}l + \\ &+ C_4 \cos\sqrt{-\zeta_1}l) + \bar{p}(C_5 \sin\bar{p}l + C_6 \cos\bar{p}l) = 0. \end{aligned} \tag{11.61}$$

where

$$k_1 = \sqrt{\zeta_2}\,(\xi\varkappa + 2\mu\xi + \theta),$$
$$k_2 = -\frac{1}{\xi}\,\frac{\mu\xi + \theta}{\mu + \varkappa}. \qquad (11.62)$$

From (11.59) we have

$$C_1 = 0, \qquad (11.63)$$
$$C_2 + C_6 = 0, \qquad (11.64)$$
$$C_6 = \frac{k_1}{p^2} C_2. \qquad (11.65)$$

From (11.61) in view of (11.59), we find

$$C_2 = 0, \qquad (11.66)$$

$$C_3 \sin \sqrt{-\zeta_1}\,l + C_4 \cos \sqrt{-\zeta_1}\,l + \frac{\bar{p}}{\sqrt{-\zeta_1}}(C_5 \sin \bar{p}l + C_6 \cos \bar{p}l) = 0. \qquad (11.67)$$

From (11.65), on the basis of (11.66),

$$C_6 = 0. \qquad (11.68)$$

Now from (11.60) and (11.67), taking (11.63) - (11.66) and (11.68 into consideration, we find

$$C_5 \cos \bar{p}l = 0, \qquad (11.69)$$

$$C_4 = C_5\,\frac{\sin \sqrt{-\zeta_1}\,l - \frac{\bar{p}}{\sqrt{\zeta_1}} \sin \bar{p}l}{\cos \sqrt{-\zeta_1}\,l}. \qquad (11.70)$$

Thus, the nontrivial solution will take place only when C_5 is different from zero. In this connection, it follows from (11.69) that

$$\cos \bar{p}l = 0, \qquad (11.71)$$

whence in view of (11.37),

$$p = \frac{\pi}{l}\left(k + \frac{1}{2}\right)\sqrt{\frac{F^{\circ}_{zy}}{d_{22}}} \quad (k = 0, 1, 2, \ldots). \qquad (11.72)$$

Taking into consideration the relations obtained for the arbitrary constants, and also (11.72), we obtain from (11.58) to within the constant multiplier

$$\eta_y^*(Z) = \sin\left(k+\frac{1}{2}\right)\frac{\pi Z}{l},$$

$$\theta_x^*(Z) = k_\theta\left[\frac{-\sin\alpha l + (-1)^k \frac{\pi l}{\alpha}\left(k+\frac{1}{2}\right)}{\cos\alpha l}\sin\alpha Z - \cos\alpha Z + \right.$$
$$\left. + \cos\left(k+\frac{1}{2}\right)\frac{\pi Z}{l}\right], \tag{11.73}$$

$$\omega^*(Z) = k_\omega\left[\frac{\sin\alpha l - (-1)^k \frac{\pi l}{\alpha}\left(k+\frac{1}{2}\right)}{\cos\alpha l}\sin\alpha Z + \cos\alpha Z - \right.$$
$$\left. - \cos\left(k+\frac{1}{2}\right)\frac{\pi Z}{l}\right],$$

where

$$\alpha = \sqrt{\frac{1}{\tilde{a}_{77}}\left(\frac{\tilde{b}_{37}^2}{F_{yy}^*} - \tilde{c}_{77}\right)},$$
$$k_\theta = \frac{\tilde{b}_{37}^2}{\tilde{a}_{77}F_{yy}^*}\;\frac{l}{\pi\left(k+\frac{1}{2}\right)}, \tag{11.74}$$
$$k_\omega = \frac{\tilde{b}_{37}}{\tilde{a}_{77}}\;\frac{l}{\pi\left(k+\frac{1}{2}\right)}.$$

Expressions (11.72), (11.73) represent the frequencies and modes of natural bending vibrations. In view of the adopted assumptions, the solution obtained is approximate in character. Frequency Eq. (11.71) actually coincides with the frequency equation for a cantilever beam starting with the frequency of the second tone. However, the solution obtained is not a beam solution, since the cross section warps, and in addition, as follows from (11.73), transverse shear strains take place. More exact results can be obtained from general solution (11.53) by assuming the refined values (11.43), (11.49) for the roots of characteristic Eq. (11.36). In that case, the solution can be obtained only with the aid of a computer.*

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* See I. F. Obraztsov. Variational Methods of Analysis of Thin-Walled Aeronautical Structures. Izd-vo Mashinostroyeniye, 1966.

2. Torsional Vibrations

Let us turn to Eqs. (11.33). Representing the partial solution in the form

$$\theta_z^{\circ}(Z)=C_1 e^{\lambda Z}, \quad \omega_z^{\circ}(Z)=C_2 e^{\lambda Z},$$

we arrive at the characteristic equation

$$\zeta^2+(\bar{p}^2+\mu\xi)\zeta-\varkappa\xi\bar{p}^2=0, \tag{11.75}$$

where

$$\zeta=\lambda^2, \quad \bar{p}^2=p^2\frac{\tilde{a}_{66}}{I_z^{\circ}}, \quad \mu=\frac{\tilde{b}_{66}^2}{I_z^{\circ}}-\tilde{c}_{66}, \quad \varkappa=\tilde{r}_{66}, \quad \xi=\frac{1}{\tilde{a}_{66}}. \tag{11.76}$$

From (11.75)

$$\zeta_{1,2}=\frac{\bar{p}^2+\mu\xi}{2}\left[-1\pm\sqrt{1+4\varkappa\xi\left(\frac{\bar{p}}{\bar{p}^2+\mu\xi}\right)^2}\right]. \tag{11.77}$$

We thus have two real and two purely imaginary values of the characteristic index λ. To the λ values obtained, there corresponds a general solution of system (11.33) of the form

$$\begin{aligned}\theta_z^{\circ}(Z)=&\frac{1}{\zeta_1+\bar{p}^2}(C_1\operatorname{sh}\sqrt{\zeta_1}Z+C_2\operatorname{ch}\sqrt{\zeta_1}Z)+\\&+\frac{1}{\zeta_2+\bar{p}^2}(C_3\sin\sqrt{-\zeta_2}Z+C_4\cos\sqrt{-\zeta_2}Z).\end{aligned} \tag{11.78}$$

$$\begin{aligned}\omega_z^{\circ}(Z)=&\frac{I_z^{\circ}}{\tilde{b}_{66}}\left[\frac{-1}{\sqrt{\zeta_1}}(C_1\operatorname{ch}\sqrt{\zeta_1}Z+C_2\operatorname{sh}\sqrt{\zeta_1}Z)+\right.\\&\left.+\frac{1}{\sqrt{-\zeta_2}}(C_3\cos\sqrt{-\zeta_2}Z-C_4\sin\sqrt{-\zeta_2}Z)\right].\end{aligned} \tag{11.79}$$

We will satisfy the boundary conditions on the ends of the shell.

On the end Z = 0

$$\theta_z^{\circ}(0)=\omega_z^{\circ}(0)=0. \tag{11.80}$$

On the end Z = l, from the conditions

$$M_z(l)=P_{\omega}(l)=0 \tag{11.81}$$

it follows that

$$\theta_2^{\prime\prime}(l)+\frac{\bar{b}_{02}}{l_z^2}\omega_2^{\prime}(l)=0,$$
$$\omega_2^{\prime\prime}(l)=0. \tag{11.82}$$

Expanding conditions (11.80) and (11.82) with the aid of general solution (11.78) and (11.79), we obtain

$$\begin{gathered}
\frac{C_2}{\zeta_1+\bar{p}^2}+\frac{C_4}{\zeta_2+\bar{p}^2}=0,\\
\frac{C_1}{\sqrt{\zeta_1}}-\frac{C_3}{\sqrt{-\zeta_2}}=0,\\
\left(\frac{\sqrt{\zeta_1}}{\zeta_1+\bar{p}^2}-\frac{1}{\sqrt{\zeta_1}}\right)(C_1\operatorname{ch}\sqrt{\zeta_1}l+C_2\operatorname{sh}\sqrt{\zeta_1}l)+\\
+\left(\frac{\sqrt{-\zeta_2}}{\zeta_2+\bar{p}^2}+\frac{1}{\sqrt{-\zeta_2}}\right)(C_3\cos\sqrt{-\zeta_2}l-C_4\sin\sqrt{-\zeta_2}l)=0,\\
C_1\operatorname{sh}\sqrt{\zeta_1}l+C_2\operatorname{ch}\sqrt{\zeta_1}l+C_3\sin\sqrt{-\zeta_2}l+C_4\cos\sqrt{-\zeta_2}l=0.
\end{gathered} \tag{11.83}$$

Setting to zero the determinant of system (11.83), we arrive at the transcendental frequency equation

$$\frac{\bar{p}^2+\mu\xi}{\bar{p}\sqrt{\varkappa\xi}}\operatorname{sh}\sqrt{\zeta_1}l\sin\sqrt{-\zeta_2}l-\left[2+\frac{(\bar{p}^2-\mu\xi)^2}{\bar{p}^2\xi(\mu+\varkappa)}\right]\operatorname{ch}\sqrt{\zeta_1}l\cos\sqrt{-\zeta_2}l-2=0. \tag{11.84}$$

System (11.83) has a solution to within the arbitrary multiplier

$$\begin{aligned}
C_1&=C,\\
C_2&=\frac{\operatorname{sh}\sqrt{\zeta_1}l+\dfrac{\sqrt{-\zeta_2}}{\sqrt{\zeta_1}}\sin\sqrt{-\zeta_2}l}{\dfrac{\zeta_2+\bar{p}^2}{\zeta_1+\bar{p}^2}\cos\sqrt{-\zeta_2}l-\operatorname{ch}\sqrt{\zeta_1}l}C,\\
C_3&=\frac{\sqrt{-\zeta_2}}{\sqrt{\zeta_1}}C,\\
C_4&=\frac{\operatorname{sh}\sqrt{\zeta_1}l+\dfrac{\sqrt{-\zeta_2}}{\sqrt{\zeta_1}}\sin\sqrt{-\zeta_2}l}{\dfrac{\zeta_1+\bar{p}^2}{\zeta_2+\bar{p}^2}\operatorname{ch}\sqrt{\zeta_1}l-\cos\sqrt{-\zeta_2}l}C.
\end{aligned} \tag{11.85}$$

The roots of Eq. (11.84) determine the values of natural frequencies p. Introducing constants (11.85) into general solution (11.78), (11.79), one can also easily obtain the expressions for the modes of natural torsional vibrations, which will be determined by both trigonometric and hyperbolic functions.

The solution obtained above can be simplified to some extent if by analogy with the previous problems, the estimate $\xi \ll 1$ is adopted. In that case, we have from (11.77)

$$\zeta_1 = \varkappa\xi,$$
$$\zeta_2 = -\bar{p}^2. \tag{11.86}$$

In view of (11.86), general solution (11.78), (11.79) assumes the form

$$\theta_z^*(Z) = \frac{\sqrt{\varkappa\xi}}{\bar{p}^2}(C_1 \operatorname{sh}\sqrt{\varkappa\xi}Z + C_2 \operatorname{ch}\sqrt{\varkappa\xi}Z) + C_3 \sin\bar{p}Z + C_4 \cos\bar{p}Z. \tag{11.87}$$

Satisfying boundary conditions (11.80), (11.82), we obtain

$$\begin{gathered}
\frac{\sqrt{\varkappa\xi}}{\bar{p}^2}C_2 + C_4 = 0, \\
C_1 = 0, \\
\left(\frac{\varkappa\xi}{\bar{p}^2} - 1\right)(C_1 \operatorname{ch}\sqrt{\varkappa\xi}l + C_2 \operatorname{sh}\sqrt{\varkappa\xi}l) + \bar{p}(C_3 \cos\bar{p}l - C_4 \sin\bar{p}l) = 0, \\
C_1 \operatorname{sh}\sqrt{\varkappa\xi}l + C_2 \operatorname{ch}\sqrt{\varkappa\xi}l = 0,
\end{gathered} \tag{11.88}$$

whence

$$C_1 = C_2 = C_4 = 0, \tag{11.89}$$

and the partial equation takes the form

$$\cos\bar{p}l = 0. \tag{11.90}$$

In view of (11.89), (11.90), to within the multiplier, we find from (11.87)

$$\theta_z^*(Z) = \sin\bar{p}Z,$$
$$\omega_z^*(Z) = 0. \tag{11.91}$$

Thus, the simplified solution is in exact agreement with the elementary solution based on the hypothesis of plane sections.

11.3. Natural Vibrations of Swept Wing Type Shells

In selecting the generalized displacements retained, the coordinate functions, and the positions of the axes, we will proceed from the same premise as in the simplified analysis of swept wing type shells for a given static load (Section 8.2, Subsection 3). Assuming the cross section to be close to symmetric, we will retain four gereralized displacements: $\eta_y, \theta_x, \theta_z$ corresponding to displacement of the contour Z = const as a solid, and ω, the torsional warping.

Equations (11.24) for shells of constant cross section take the following form in the case at hand:

$$
\begin{aligned}
&\tilde{a}_{22}\eta_y^{*''}+\tilde{a}_{24}\theta_x^{*''}+\tilde{a}_{26}\theta_z^{*''}+\tilde{a}_{27}\omega^{*''}+\tilde{b}_{24}\theta_x^{*'}+\tilde{b}_{27}\omega^{*'}+\\
&\quad+p^2(\tilde{d}_{22}\eta_y^{*}+\tilde{d}_{24}\theta_x^{*}+\tilde{d}_{26}\theta_z^{*})=0,\\
&\tilde{a}_{42}\eta_y^{*''}+\tilde{a}_{44}\theta_x^{*''}+\tilde{a}_{46}\theta_z^{*''}+\tilde{a}_{47}\omega^{*''}-\tilde{b}_{24}\eta_y^{*'}-\tilde{b}_{46}\theta_z^{*'}+\\
&\quad+(\tilde{b}_{47}-\tilde{b}_{74})\omega^{*'}-\tilde{c}_{44}\theta_x^{*}-\tilde{c}_{47}\omega^{*}+\\
&\quad+p^2(\tilde{d}_{42}\eta_y^{*}+\tilde{d}_{44}\theta_x^{*}+\tilde{d}_{46}\theta_z^{*}+\tilde{d}_{47}\omega^{*})=0,\\
&\tilde{a}_{62}\eta_y^{*''}+\tilde{a}_{64}\theta_x^{*''}+\tilde{a}_{66}\theta_z^{*''}+\tilde{a}_{67}\omega^{*''}+\tilde{b}_{64}\theta_x^{*'}+\tilde{b}_{67}\omega^{*'}+\\
&\quad+p^2(\tilde{d}_{62}\eta_y^{*}+\tilde{d}_{64}\theta_x^{*}+\tilde{d}_{66}\theta_z^{*})=0,\\
&\tilde{a}_{72}\eta_y^{*''}+\tilde{a}_{74}\theta_x^{*''}+\tilde{a}_{76}\theta_z^{*''}+\tilde{a}_{77}\omega^{*''}-\tilde{b}_{27}\eta_y^{*'}+(\tilde{b}_{74}-\tilde{b}_{47})\theta_x^{*'}-\\
&\quad-\tilde{b}_{67}\theta_z^{*'}-\tilde{c}_{74}\theta_x^{*}-\tilde{c}_{77}\omega^{*}+p^2(\tilde{d}_{74}\theta_x^{*}+\tilde{d}_{77}\omega^{*})=0,
\end{aligned}
\tag{11.92}
$$

where the coefficients $\tilde{a}_{ji}, \tilde{b}_{ji}, \tilde{c}_{ji}$ are determined by expressions (6.26), and the coefficients $\tilde{d}_{ji}$, by expression (11.25). Let us also note that

$$\tilde{a}_{22}=\tilde{b}_{24}=\tilde{c}_{44}, \quad \tilde{b}_{64}=\tilde{a}_{67}; \quad \tilde{b}_{74}=\tilde{a}_{77}; \quad \tilde{b}_{27}=\tilde{c}_{47}. \tag{11.93}$$

System (11.92) is general in character. The problem of finding the eigenvalues and eigenfunctions of this system is rather cumbersome in the general case. For simplicity, let us consider the case in which the loads within the wing are absent, and the cross section is symmetric about both the 0x and the 0y axis. In that case, from (11.25) and considering (8.10),

$$\tilde{d}_{24}=\tilde{d}_{26}=\tilde{d}_{47}=0. \tag{11.94}$$

Moreover, for wing type shells, it is permissible to neglect the rotatory inertia of the section Z = const with respect to the neutral axis, and also the inertia of its warping, assuming

$$\tilde{d}_{44}=\tilde{d}_{66}\,\mathrm{ctg}^2\chi_0;\quad \tilde{d}_{77}=0. \tag{11.95}$$

Taking (11.93) - (11.95) into account with (8.91), from (11.92) we obtain

$$\begin{aligned}
&\tilde{a}_{22}\eta_y^{\circ''}+\tilde{a}_{27}\omega^{\circ''}+\tilde{a}_{23}\theta_x^{\circ'}+p^2\tilde{d}_{22}\eta_y^{\circ}=0,\\
&\tilde{a}_{44}\theta_x^{\circ''}+\tilde{a}_{46}\theta_z^{\circ''}-\tilde{a}_{23}\eta_y^{\circ'}+(\tilde{b}_{67}-\tilde{a}_{37})\omega^{\circ}-\tilde{a}_{33}\theta_x^{\circ}+\\
&\qquad+p^2(\mathrm{ctg}^2\chi_0\tilde{d}_{66}\theta_x^{\circ}+\tilde{d}_{64}\theta_z^{\circ})=0,\\
&\tilde{a}_{64}\theta_x^{\circ''}+\tilde{a}_{66}\theta_z^{\circ''}+\tilde{b}_{67}\omega^{\circ'}+p^2(\tilde{d}_{64}\theta_x^{\circ}+\tilde{d}_{66}\theta_z^{\circ})=0,\\
&\tilde{a}_{72}\eta_y^{\circ''}+\tilde{a}_{77}\omega^{\circ''}+(\tilde{a}_{73}-\tilde{b}_{47})\theta_x^{\circ'}-\tilde{b}_{67}\theta_z^{\circ'}-\tilde{c}_{77}\omega^{\circ}=0.
\end{aligned} \tag{11.96}$$

Thus, in contrast to the corresponding equations for a nonoblique shell, the differential equations describing the natural vibrations of a swept shell form a coupled system. Bending is not separated from torsion, and this is the characteristic feature of the problem at hand.

All the coefficients of system (11.96) with the exception of the inertial ones are represented by expressions (8.101). For inertial coefficients, neglecting the moments of inertia about the neutral axis, we have from (11.25)

$$\begin{aligned}
&\tilde{d}_{22}=\frac{1}{G}\oint\varrho\sin\chi h\,dS,\\
&\tilde{d}_{64}=-\frac{2}{G}\mathrm{ctg}\,\chi_0\oint x^2y'^2\frac{\varrho}{\sin\chi}h\,dS,\\
&\tilde{d}_{66}=\frac{1}{G}\oint x^2\varrho\sin\chi h\,dS.
\end{aligned} \tag{11.97}$$

As an example, let us consider a swept caisson of rectangular cross section. In this case, $\tilde{a}_{27}=0$. The remaining rigidity coefficients are represented by expressions (8.104), (8.105). For inertial coefficients (11.97), we find

$$\begin{aligned}
&\tilde{d}_{22}=\frac{\varrho}{G}c\left(1+\frac{d_2h_2}{d_1h_1}\right),\\
&\tilde{d}_{64}=-\frac{\varrho}{G}\mathrm{ctg}\,\chi_0 d_2^2d_1h_1,\\
&\tilde{d}_{66}=\frac{\varrho}{G}\frac{d_2^2}{2}\left(d_1h_1+\frac{1}{3}d_2h_2\right).
\end{aligned} \tag{11.98}$$

Representing the partial solution of system (11.96) in the form

$$\eta_y^*(Z)=C_\eta e^{\lambda Z},\quad \theta_x^*(Z)=C_{\theta_x} e^{\lambda Z},\quad \theta_z^*(Z)=C_{\theta_z} e^{\lambda Z},\quad \omega^*(Z)=C_\omega e^{\lambda Z},$$

we arrive at a characteristic equation of eighth degree:

$$\begin{aligned}&(\tilde{a}_{22}\zeta+p^2\tilde{d}_{22})\{(\tilde{a}_{44}\zeta-\tilde{a}_{22}+\mu p^2\,\mathrm{ctg}^2\chi_0)\left[(\tilde{a}_{66}\zeta+\mu p^2)(\zeta-\xi\tilde{c}_{77})+\xi\tilde{b}_{67}^2\zeta\right]-\\&-\left(\tilde{a}_{64}\zeta+\mu\frac{\tilde{d}_{64}}{\tilde{d}_{66}}\right)\left[\left(\tilde{a}_{46}\zeta+\mu p^2\frac{\tilde{d}_{46}}{\tilde{d}_{66}}\right)(\zeta-\xi\tilde{c}_{77})+\xi\tilde{b}_{47}\tilde{b}_{67}\zeta\right]-\\&-\xi\tilde{b}_{47}\zeta\left[\left(\tilde{a}_{46}\zeta+\mu p^2\frac{\tilde{d}_{46}}{\tilde{d}_{66}}\right)\tilde{b}_{67}-(\tilde{a}_{66}\zeta+\mu p^2)\tilde{b}_{47}\right]\}+\\&+\tilde{a}_{22}^2\zeta\left[(\tilde{a}_{66}\zeta+\mu p^2)(\zeta-\xi\tilde{c}_{77})+\xi\tilde{b}_{67}^2\zeta\right]=0,\end{aligned} \tag{11.99}$$

where

$$\zeta=\lambda^2,\quad \xi=\frac{1}{\tilde{a}_{77}},\quad \mu=\tilde{d}_{44}. \tag{11.100}$$

Equation (11.99) represents the dependence of the characteristic indices λ on the natural vibration frequencies p. Since the solution of an equation of fourth degree in general form is very cumbersome, we will again confine ourselves to an asymptotic solution. We will treat (11.99) as a multivalued (four-valued) function $\zeta=\zeta(\mu,\xi)$, given in the implicit form

$$\Phi(\zeta,\mu,\xi)=0. \tag{11.101}$$

The partial derivatives of this function

$$\frac{\partial\zeta}{\partial\mu}=-\frac{\frac{\partial\Phi}{\partial\mu}}{\frac{\partial\Phi}{\partial\zeta}},\quad \frac{\partial\zeta}{\partial\xi}=-\frac{\frac{\partial\Phi}{\partial\xi}}{\frac{\partial\Phi}{\partial\zeta}}. \tag{11.102}$$

Expanding $\zeta(\mu,\xi)$ in a Taylor series in the vicinity of $\xi=\xi_0$, $\mu=\mu_0$, we have

$$\zeta(\mu,\xi)=\zeta_0+\left.\frac{\partial\zeta}{\partial\mu}\right|_{\substack{\zeta=\zeta_0\\ \mu=\mu_0\\ \xi=\xi_0}}(\mu-\mu_0)+\left.\frac{\partial\zeta}{\partial\xi}\right|_{\substack{\zeta=\zeta_0\\ \mu=\mu_0\\ \xi=\xi_0}}(\xi-\xi_0)+\ldots, \tag{11.103}$$

where $\zeta_0=\zeta(\mu_0,\xi_0)$ is the root of Eq. (11.99) when $\mu=\mu_0$, $\xi=\xi_0$.

We take

$$\mu_0=\xi_0=0. \tag{11.104}$$

In this case, Eq. (11.99) assumes the form

$$\zeta^4+\zeta^3\bar{p}^2-\vartheta\bar{p}^2\zeta^2=0. \tag{11.105}$$

where

$$\bar{p}^2 = p^2 \frac{\tilde{d}_{22}}{\tilde{a}_{22}}; \quad \theta = \frac{\tilde{a}_{22}\tilde{a}_{66}}{\tilde{a}_{44}\tilde{a}_{66} - \tilde{a}_{64}^2}. \tag{11.106}$$

We find from (11.105)

$$(\zeta_0)_{1,2} = 0; \quad (\zeta_0)_{3,4} = \frac{\bar{p}^2}{2}\left(-1 \pm \sqrt{1 + 4\frac{\theta}{\bar{p}^2}}\right). \tag{11.107}$$

It is easy to see that expressions (11.107) are exact for the case where warping of the cross section and the inertia of the rotations are not considered. This obviously corresponds to a model in which a swept caisson, constituting a three-dimensional system, is interpreted as a material axis fixed at an angle $\frac{\pi}{2} - \chi_0$ to the cross section, with the corresponding bending and torsional rigidities. Hence it may be concluded that roots (11.107) are sufficiently close to the exact values of the roots of Eq. (11.99). In this connection, in expansion (11.103) it is sufficient to confine oneself to the linear approximation. Then for the asymptotic values of the roots of Eq. (11.99), by using (11.102), (11.99) and (11.107), it is easy to obtain

$$\zeta_1 = \zeta_2 = -\mu k_1 \bar{p}^2 - \varepsilon k_2,$$

$$\zeta_3 = \frac{1}{2}\bar{p}^2\left(-1 + \sqrt{1 + 4\frac{\theta}{\bar{p}^2}}\right) - \frac{\mu k_3 \bar{p}^2 + \varepsilon\left(k_5 + k_6\sqrt{1 + \frac{4\theta}{\bar{p}^2}}\right)}{2 - k_4\bar{p}^2\left(-1 + \sqrt{1 + \frac{4\theta}{\bar{p}^2}}\right)},$$

$$\zeta_4 = \frac{1}{2}\bar{p}^2\left(-1 - \sqrt{1 + 4\frac{\theta}{\bar{p}^2}}\right) - \frac{\mu k_3 \bar{p}^2 + \varepsilon\left(k_5 - k_6\sqrt{1 + \frac{4\theta}{\bar{p}^2}}\right)}{2 - k_4\bar{p}^2\left(-1 - \sqrt{1 + 4\frac{\theta}{\bar{p}^2}}\right)}. \tag{11.108}$$

where

$$k_1 = \frac{\tilde{a}_{22}}{2\tilde{d}_{22}\tilde{a}_{66}}; \quad k_2 = \frac{\tilde{a}_{66}^2 + \tilde{b}_{67}^2}{2\tilde{a}_{66}},$$

$$k_3 = \frac{\left(\tilde{a}_{66}\operatorname{ctg}^2\chi_0 + \frac{\tilde{a}_{64}^2}{\tilde{a}_{66}} - \frac{\tilde{a}_{64}\tilde{d}_{46}}{\tilde{d}_{66}}\right)\frac{\tilde{a}_{22}}{\tilde{d}_{22}} - \tilde{a}_{46}\frac{\tilde{d}_{64}}{\tilde{d}_{66}}}{\tilde{a}_{44}\tilde{a}_{66} - \tilde{a}_{64}^2}; \quad k_4 = \frac{\tilde{a}_{44}\tilde{a}_{66} - \tilde{a}_{64}^2}{2\tilde{c}_{22}\tilde{a}_{66}},$$

$$k_5 = \frac{\tilde{b}_{67}^2\tilde{a}_{64}^2}{\tilde{a}_{44}\tilde{a}_{66} - \tilde{a}_{64}^2}\,\frac{1}{\tilde{a}_{66}}; \quad k_6 = \frac{\tilde{a}_{66}\tilde{b}_{47}^2 - 2\tilde{a}_{64}\tilde{b}_{27}\tilde{b}_{67}}{2\tilde{a}_{22}^2\tilde{a}_{66}}. \tag{11.109}$$

Expressions (11.108) are considerably simplified on the basis of estimate (11.48):

$$
\begin{aligned}
\zeta_1 &= \zeta_2 = -\mu k_1 \bar{p}^2 - \xi k_2, \\
\zeta_3 &= \vartheta - \mu k_3 \bar{p}^2 - \xi (k_5 + k_6), \\
\zeta_4 &= -\bar{p}^2 - \vartheta \left[1 + \mu k_3 - \frac{\xi}{\bar{p}^2} (k_5 - k_6) \right].
\end{aligned}
\tag{11.110}
$$

Eight values of the characteristic index λ correspond to solution (11.110) of Eq. (11.99). Six purely imaginary values correspond to $\zeta_1, \zeta_2, \zeta_4$:

$$
\begin{aligned}
\lambda_1 &= \lambda_2 = i \sqrt{\mu k_1 \bar{p}^2 + \xi k_2}, \\
\lambda_3 &= \lambda_4 = -i \sqrt{\mu k_1 \bar{p}^2 + \xi k_2}, \\
\lambda_{5,6} &= \pm i \sqrt{\bar{p}^2 + \vartheta \left[1 + \mu k_3 + \frac{\xi}{\bar{p}^2} (k_5 - k_6) \right]}.
\end{aligned}
\tag{11.111}
$$

The characteristic indices $\lambda_{7,8}$, corresponding to ζ_3, can be both real and purely imaginary. For large frequency values $\bar{p}$,

$$\lambda_{7,8} = \pm i \sqrt{\mu k_3 \bar{p}^2 + \xi (k_5 + k_6) - \vartheta}, \tag{11.112}$$

and for sufficiently small ones.

$$\lambda_{7,8} = \pm \sqrt{\vartheta - \mu k_3 \bar{p}^2 - \xi (k_5 + k_6)}. \tag{11.113}$$

Thus, an explicit relationship has been established between the characteristic indices λ and the relative frequency values $\bar{p}$. Now, having written down the general solution of system (11.96) and satisfied the homogeneous boundary conditions, one can obtain the partial equation from the condition of nontriviality of the solution. It should be noted that the partial equation obtained from expressions (11.110), which constitute rational dependences between λ^2 and $\bar{p}^2$, is considerably simpler in structure than the equation obtained from irrational relations (11.108). Indeed, the system of algebraic equations in constants $C_{\eta_y}, C_{\theta_x}, C_{\theta_z}, C_{\bullet}$ may be represented in the form

$$
\begin{gathered}
(\lambda_i^2+\bar{p}^2)C_{\eta_y}+\lambda_i C_{\theta_x}=0,\\
-\tilde{a}_{55}\lambda_i C_{\eta_y}+\left(\tilde{a}_{44}\lambda_i^2-\tilde{a}_{55}+\bar{p}^2\operatorname{ctg}^2\chi_0\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\mu\right)C_{\theta_x}+\\
+\left(\tilde{a}_{46}\lambda_i^2+\bar{p}^2\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\frac{\tilde{d}_{46}}{\tilde{d}_{66}}\mu\right)C_{\theta_z}+\tilde{b}_{47}\lambda_i C_w=0,\\
\left(\tilde{a}_{64}\lambda_i^2+\bar{p}^2\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\frac{\tilde{d}_{64}}{\tilde{d}_{66}}\mu\right)C_{\theta_x}+\left(\tilde{a}_{66}\lambda_i^2+\bar{p}^2\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\mu\right)C_{\theta_z}+\tilde{b}_{67}\lambda_i C_w=0,\\
-\xi\tilde{b}_{47}\lambda_i C_{\theta_x}-\xi\tilde{b}_{67}\lambda_i C_{\theta_z}+(\lambda_i^2-\xi\tilde{c}_{77})C_w=0\\
(i=1,2,\ldots,8).
\end{gathered}
\tag{11.114}
$$

Discarding one of the equations of system (11.114), for each of the characteristic indices $\lambda = \lambda_i$ one can easily find the values of constants $C^i_{\eta_y}, C^i_{\theta_x}, C^i_{\theta_z}, C^i_w$ to within the multiplier C_i. Let us note that the result will be somewhat different depending on which of the equations is discarded, since approximate expressions are used for the roots of Eq. (11.99). Therefore, in view of the smallness of the parameter ξ, it is desirable to discard the fourth equation of system (11.114).

Eliminating the constant C_{η_y} from the second Eq. (11.114), with the aid of the first equation, one can easily reduce the first three equations of system (11.114) to the form

$$
\begin{gathered}
\lambda_i C_{\eta_y}=-\frac{\zeta_i}{\zeta_i+\bar{p}^2}C_{\theta_x},\\
\left(-\tilde{a}_{55}\frac{\bar{p}^2}{\zeta_i+\bar{p}^2}+\tilde{a}_{44}\zeta_i+\bar{p}^2\operatorname{ctg}^2\chi_0\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\mu\right)C_{\theta_x}+\\
+\left(\tilde{a}_{46}\zeta_i+\bar{p}^2\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\frac{\tilde{d}_{46}}{\tilde{d}_{66}}\mu\right)C_{\theta_z}=C_i,\\
\left(\tilde{a}_{64}\zeta_i+\bar{p}^2\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\frac{\tilde{d}_{64}}{\tilde{d}_{66}}\mu\right)C_{\theta_x}+\left(\tilde{a}_{66}\zeta_i+\bar{p}^2\frac{\tilde{a}_{22}}{\tilde{d}_{22}}\mu\right)C_{\theta_z}=\frac{\tilde{b}_{67}}{\tilde{b}_{47}}C_i,\\
\lambda_i C_w=-\frac{1}{\tilde{b}_{47}}C_i.
\end{gathered}
\tag{11.115}
$$

System (11.115) is easily solved for the unknowns $\lambda_i C_{\eta_y}, C_{\theta_x}, C_{\theta_z}, \lambda_i C_w$. Moreover, using approximate relations (11.110), we obtain expressions which are rational with respect to the desired frequencies $\bar{p}^2$, and this substantially simplifies the frequency equation resulting from the boundary conditions. In the case under consideration, these conditions have the form

$$\eta_y^*(0)=\theta_x^*(0)=\theta_z^*(0)=\omega^*(0)=0 \qquad (11.116)$$

and

$$\theta_y(l)=M_x(l)=M_z(l)=P_\omega(l)=0, \qquad (11.117)$$

whence

$$\begin{aligned}
&\eta_y^{*\prime}(l)+\theta_x^*(l)=0,\\
&\tilde{a}_{44}\theta_x^{*\prime}(l)+\tilde{a}_{46}\theta_z^{*\prime}(l)+\tilde{b}_{47}\omega^*(l)=0,\\
&\tilde{a}_{64}\theta_x^{*\prime}(l)+\tilde{a}_{66}\theta_z^{*\prime}(l)+\tilde{b}_{67}\omega^*(l)=0,\\
&\omega^{*\prime}(l)=0.
\end{aligned} \qquad (11.118)$$

Expanding conditions (11.116) and (11.118), we arrive at a homogeneous system of eight equations in the constants $C_i\ (i=1,\ 2, \ldots,\ 8)$; equating the determinant of this system to zero, we obtain a transcendental frequency equation that is very cumbersome in general form, but with the coefficients $\tilde{a}_{jt}, \tilde{b}_{jt}, \tilde{c}_{jt}, \tilde{d}_{jt}$ obtained, its notation and numerical solution are not difficult.

Chapter XII. VIBRATIONS OF A WING TYPE SHELL PARTLY FILLED WITH LIQUID

Wings of large airplanes usually contain tanks with liquid fuel whose mobility in the tanks causes additional loads on the wing and alters its dynamic characteristics, for example, an important characteristic such as the critical flutter speed.

The present chapter is devoted to the general problem of vibrations of elastic caisson structures, partly filled with liquid, of swept wing type and low-aspect wing types of arbitrary shape in the plane. The three-dimensional hydrodynamic problem of vibrations of an ideal incompressible liquid inside an elastic caisson reduces to a system of ordinary differential equations in functions characterizing the displacements of the liquid along the wing span. These equations in combination with the corresponding differential resolvents pertaining to the wing proper (their right-hand sides include the generalized hydrodynamic forces) form a closed system of ordinary differential equations. The boundary conditions on the ends of each section reduce to a system of algebraic equations.

The results given in this chapter can be used as the basis of studies of a broad range of dynamic problems. However, because the hydrodynamic problem is treated in a very general manner, the general relationships determining the interaction of an elastic structure with a liquid are complex, and this complicates their direct application in practical studies. However, these relationships can be appreciably simplified by taking into consideration the relationships of the rigidities and geometric dimensions characteristic of wings. For example, assuming that the interaction of the liquid with the wing takes place only in displacements corresponding to displacements of the contour $\bar{Z}$ = const as a solid, very simple relations suited for practical work can be obtained.

The fundamental results pertaining to the solution of the hydrodynamic problem were obtained by F. N. Shklyarchuk.

12.1. Formulation of the Hydrodynamic Problem

Figure 12.1 shows a caisson type conical shell partly filled with liquid. In solving the hydrodynamic problem, we will introduce the system of Cartesian coordinates Oxyz in such a way that the Oxz plane coincides with the free liquid surface in the section under consideration, the Oy axis is directed downward parallel to the gravitational vector, and the Oz axis is perpendicular to the plane of the ends.

In the general case, we will assume that all the webs of the section under consideration are movable, and their displacements consist of small displacements characterizing the motion of the section as a solid, and small elastic displacements.

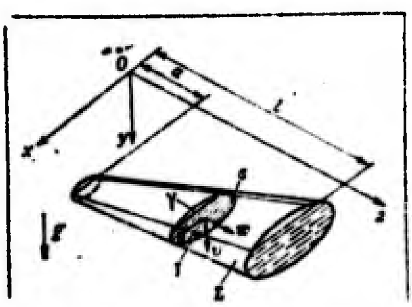


Fig. 12.1. In reference to the problem of vibrations of a shell partly filled with liquid.

When the problem is solved in the Cartesian coordinate system, the motion of the liquid in a volume τ must obey the continuity equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (12.1)$$

and linear equations of small vibrations

$$\frac{\partial p}{\partial x}+\rho\ddot{u}=0,$$
$$\frac{\partial p}{\partial y}-\rho g+\rho\ddot{v}=0, \qquad (12.2)$$
$$\frac{\partial p}{\partial z}+\rho\ddot{w}=0,$$

where $u=u(x, y, z, t)$, $v=v(x, y, z, t)$, $w=w(x, y, z, t)$ are the displacements of the liquid particles in the direction of coordinate axes Ox, Oy, Oz, respectively;

$p=p(x, y, z, t)$ is the pressure in the liquid;

ρ is the density of the liquid;

g is the free-fall acceleration.

On the wetted web surface Σ, the kinematic boundary conditions of undetached motion of the shell webs and adjacent liquid particles must be fulfilled. For this purpose, it is necessary that the displacements of the liquid in the direction of the normal n_n to the wall be equal to the given normal displacements of the shell:

$$ul_{n_n}+vm_{n_n}+wn_{n_n}=W \text{ на } \Sigma. \qquad (12.3)$$

Here l_{n_n}, m_{n_n}, n_{n_n} are the direction cosines of the unit vector of the normal n_n to the shell surface;

W is the normal displacement of the web, positive in the direction of n_n.

On the disturbed free surface of the liquid, when $y = v(x, 0, z, t)$, the pressure of the liquid $P(x, y, z, t)$ should be equal to the pressure of the gas occupying the space above the liquid. If the pressure of the gas remains constant, without decreasing the generality, one can assume it to be equal to zero. Then the dynamic boundary conditions on the free surface can be written in the form

$$p=0 \quad \text{when} \quad y=v(x, 0, z, t). \qquad (12.4)$$

It follows from Eqs. (12.2) that the displacements u, v, w can be represented in the form

$$u=\psi_{1u}+\psi_{2u}t+\frac{\partial\Psi}{\partial x},\quad v=\psi_{1v}+\psi_{2v}t+\frac{\partial\Psi}{\partial y},\quad w=\psi_{1w}+\psi_{2w}t+\frac{\partial\Psi}{\partial z},$$

where $\psi_1(x, y, z)$, $\psi_2(x, y, z)$ depend on the initial conditions, and $\Psi(x, y, z, t)$ is some continuous function. Hence we conclude that it at the initial instant the motion of the liquid is potential, it will also be potential at any other instant. Then

$$u=\frac{\partial\Phi}{\partial x},\quad v=\frac{\partial\Phi}{\partial y},\quad w=\frac{\partial\Phi}{\partial z}, \tag{12.5}$$

where $\Phi=\Phi(x, y, z, t)$ is some function referred to as the displacement potential.

Integrating system (12.2), in view of (12.5), with respect to the pressure $p=p(x, y, z, t)$, we can obtain

$$p=\varrho g y-\varrho\ddot{\Phi}. \tag{12.6}$$

In the case of potential motion of the liquid, continuity Eq. (12.1) reduces to the Laplace equation in the potential Φ:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \text{ in } \tau. \tag{12.7}$$

Kinematic boundary conditions (12.3) can then be written as

$$\frac{\partial\Phi}{\partial n_n}=W \text{ on } \Sigma, \tag{12.8}$$

and dynamic boundary conditions (12.4), transferred to the plane y = 0 from the disturbed surface $y=\left.\frac{\partial\Phi}{\partial y}\right|_{y=0}$, assume the following form in the case of small vibrations:

$$\varrho g\frac{\partial\Phi}{\partial y}-\varrho\ddot{\Phi}=0 \quad \text{when } y=0. \tag{12.9}$$

Thus, the hydrodynamic problem reduces to the determination of the displacement potential Φ from Laplace Eq. (12.7) with boundary conditions (12.8) and (12.9). This problem can be solved exactly only in certain special cases for the simplest regions occupied by the liquid. For instance, the hydrodynamic problem has a simple solution in double trigonometric series when the region τ occupied by the liquid is in the shape of a rectangular parallelepiped. In the case under consideration, when the region τ is in the form of a conical caisson of arbitrary shape in the plane, the hydrodynamic problem can be solved approximately only.

To reduce the three-dimensional hydrodynamic problem to a one-dimensional problem, we represent Eq. (12.7) and boundary conditions (12.8), (12.9) taking (12.5) into account, in the form

$$\left.\begin{aligned} &\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=-\frac{\partial w}{\partial z} \qquad (12.10)\\ &\frac{\partial\Phi}{\partial z}-w=0 \qquad (12.11)\end{aligned}\right\}\text{in } \tau,$$

$$\sin\beta\frac{\partial\Phi}{\partial\nu}+\cos\beta\, w=W \text{ on } \Sigma, \qquad (12.12)$$

$$g\frac{\partial\Phi}{\partial y}-\ddot{\Phi}=0 \text{ when } y=0. \qquad (12.13)$$

Here

β is the angle between the positive direction of the normal n_n to the shell surface and the Oz axis;

ν is the unit vector of the normal contour of the section of the shell by the plane Z = const, making an acute angle with the normal n_n.

Let $\sigma=\sigma(z)$ be the section of the volume τ of the liquid by the plane z = const, bounded by the contour $\Gamma^*(z)=\Gamma(z)+\gamma(z)$, where Γ is the portion of the contour Γ^* on the wetted surface of the shell, and γ, on the free surface of the liquid (see Fig. 12.1).

We will seek the displacement w in the form of the expansion

$$w(x, y, z, t)=\sum_{i=1}^{M}Z_i(z, t)\varphi_i(x, y), \qquad (12.14)$$

where

$\varphi_i(x, y)$ are the given coordinate functions, determining the different forms of possible displacement of the liquid particles in the section $\sigma(z)$ in the direction of the Oz axis;

$Z_l(z, t)$ are the unknown generalized displacements to be determined. Expansion (12.14) represents the explicit dependence of the displacement w on the variables x and y. This makes it possible to establish an explicit relation between the potential Φ on the one hand, and on the other hand the functions Z_1 and their derivatives, while satisfying Eq. 12.10) in the arbitrary section $\sigma(z)$ and conditions (12.2), (12.13) on the corresponding contour $\Gamma^*(z)$. It is easy to see that

$$\Phi(x, y, z, t)=L[Z_1, Z_2, \ldots, Z_M], \tag{12.15}$$

where L is some linear integral operator.

Satisfying Eq. (12.11) by using the Bubnov-Galerkin method in the section $\sigma(z)$ on the set of coordinate functions $\varphi_l(x, y)$, we arrive at a system of linear differential equations in the desired generalized displacements $Z_l(z, t)\ (l=1, 2, \ldots, M)$:

$$\iint\limits_{\sigma(z)}\left(\frac{\partial\Phi}{\partial z}-w\right)\varphi_k\, d\sigma(z)=0 \quad (k=1, 2, \ldots, M). \tag{12.16}$$

For the case of harmonic vibrations, resolvents (12.16) of the hydrodynamic problem form a linear system of ordinary differential equations in the generalized displacements $Z_l(z)\ (l=1, 2, \ldots, M)$.

The solution of Eqs. (12.16) must obviously satisfy the initial conditions, and also the kinematic boundary conditions when z = a and z = 1, expressed in the equality of the normal displacements of the end ribs and adjacent liquid particles:

$$\begin{aligned} w(x, y, a, t)&=W_n^a(x, y, t), \\ w(x, y, l, t)&=W_n^l(x, y, t), \end{aligned} \tag{12.17}$$

where W_r^a, W_r^l are the functions of the deflections of the end ribs, positive in the direction of the Oz axis.

In accordance with expansion (12.14), conditions (12.17) are also satisfied by the Bubnov-Galerkin method on the set of coordinate functions $\varphi_i(x, y)$:

$$\int_{\sigma(z)} (w - W_r)\varphi_k d\sigma(z) = 0 \quad (z = a, \ z = l, \ k = 1, 2, \ldots, M). \tag{12.18}$$

In the case of harmonic vibrations, expressions (12.18) constitute a linear system of algebraic equations in the integration constants of the system of ordinary differential Eqs. (12.16).

12.2. Variational Formulation of the Hydrodynamic Problem

We will apply Hamilton's variational principle to the motion of a liquid:

$$\int_0^{t_1} (\delta\Pi - \delta T - \delta A)\, dt = 0, \tag{12.19}$$

where T, Π are the kinetic and potential energy of the liquid;

A is the work done by external forces.

According to this principle, the varied motion of the liquid, like the true motion, must satisfy continuity Eq. (12.1) and kinematic boundary conditions (12.3). As we know, equations of motion (12.2) and dynamic boundary conditions (12.4) follow from (12.19) in this case.

We will assume that the varied motion of the liquid in the plane z = const is potential motion. Then the kinetic and potential energy of the liquid can be represented by the expressions

$$T = \frac{\rho}{2} \int_0^l \iint_\sigma \left[\left(\frac{\partial \dot{\Phi}}{\partial x} \right)^2 + \left(\frac{\partial \dot{\Phi}}{\partial y} \right)^2 + \dot{w}^2 \right] d\sigma\, dz,$$

$$\Pi = \frac{\rho g}{2} \int_0^l \left[\int_\gamma \left(\frac{\partial \Phi}{\partial y} \right)^2 d\gamma - \int_\Gamma W^2 m_{n_n}\, d\Gamma \right] dz. \qquad (12.20)$$

Let the potential Φ of the displacements in the plane z = const and the displacement w satisfy continuity Eq. (12.10) in τ and kinematic boundary conditions (12.12) on Σ.

We will assume that the displacements of the side and end webs of the caisson are given. Therefore, considering also that the pressure on the free surface of the liquid is absent, we have

$$\delta A = 0,$$
$$\delta \Pi = \rho g \int_0^l \int_\gamma \frac{\partial \Phi}{\partial y}\, \delta \frac{\partial \Phi}{\partial y}\, d\gamma\, dz, \qquad (12.21)$$
$$\delta T = \rho \int_0^l \iint_\sigma [\nabla \dot{\Phi} \delta \nabla \dot{\Phi} + \dot{w} \delta \dot{w}]\, d\sigma\, dz,$$

where ∇ is the Hamiltonian operator:

$$\nabla = \frac{\partial}{\partial x} \mathbf{i}_x + \frac{\partial}{\partial y} \mathbf{i}_y.$$

We transform the first term in the expression for δT by first using Green's formula, then the kinematic boundary conditions and continuity equation in the form (12.12) and (12.10). We obtain

$$\iint_\sigma \nabla \dot{\Phi} \delta \nabla \dot{\Phi}\, d\sigma = \int_{\Gamma^*} \dot{\Phi} \delta \frac{\partial \dot{\Phi}}{\partial \nu}\, d\Gamma^* - \iint_\sigma \dot{\Phi} \delta \nabla^2 \dot{\Phi}\, d\sigma =$$
$$= -\int_\Gamma \dot{\Phi} \operatorname{ctg} \beta\, \delta \dot{w}\, d\Gamma - \int_\gamma \dot{\Phi} \delta \frac{\partial \dot{\Phi}}{\partial y}\, d\gamma + \iint_\sigma \dot{\Phi} \delta \frac{\partial \dot{w}}{\partial z}\, d\sigma.$$

Since

$$\iint_\sigma \dot{\Phi} \delta \frac{\partial \dot{w}}{\partial z}\, d\sigma = \frac{\partial}{\partial z} \iint_\sigma \dot{\Phi} \delta \dot{w}\, d\sigma + \int_\Gamma \operatorname{ctg} \beta\, \dot{\Phi} \delta \dot{w}\, d\Gamma - \iint_\sigma \frac{\partial \dot{\Phi}}{\partial z} \delta \dot{w}\, d\sigma,$$

the variation δT can be finally represented in the form

$$\delta T = \varrho \int_a^l \Big[-\int_\gamma \Phi \delta \frac{\partial \dot{\Phi}}{\partial y} d\gamma + \frac{\partial}{\partial z} \iint_\sigma \Phi \delta \dot{w} d\sigma +$$

$$+ \iint_\sigma \left(\dot{w} - \frac{\partial \dot{\Phi}}{\partial z} \right) \delta \dot{w} d\sigma \Big] dz. \quad (12.22)$$

By expanding variational Eq. (12.19) with the aid of (12.21) and (12.22), we transform the time integral pertaining to the variation δT by parts. Considering that the variations $\delta w, \delta \frac{\partial \dot{\Phi}}{\partial y}$ at the beginning and end of the time interval considered should be assumed equal to zero, and postulating in view of (12.17) that when z = a, l the variations $\delta w = \delta W_n = 0$, we can readily obtain

$$\int_0^{t_1} \varrho \int_a^l \left[\iint_\sigma \left(\ddot{w} - \frac{\partial \ddot{\Phi}}{\partial z} \right) \delta w d\sigma + \int_\gamma \left(g \frac{\partial \Phi}{\partial y} - \ddot{\Phi} \right) \delta \frac{\partial \Phi}{\partial y} d\gamma \right] dz dt = 0. \quad (12.23)$$

Hence, as a result of the arbitrariness of the variations $\delta\omega$ in τ and $\delta \frac{\partial \Phi}{\partial y}$ when y = 0, it follows that

$$\ddot{w} - \frac{\partial \ddot{\Phi}}{\partial z} = 0 \text{ in } \tau, \quad (12.23A)$$

$$g \frac{\partial \Phi}{\partial y} - \ddot{\Phi} = 0 \text{ when } y = 0. \quad (12.23B)$$

Expressions (12.23a), (12.23b) represent the equations of motion in the direction of the Oz axis and dynamic boundary conditions (12.13) on the free surface of the liquid. It is easy to show that if the initial conditions satisfy the conditions of potentiality of motion, then Eqs. (12.23a) and (12.11) can be considered equivalent.

We will represent the displacement w in the form of expansion (12.14). If the corresponding expansion (12.15) for Φ satisfies continuity Eq. (12.10), kinematic boundary conditions (12.12), and dynamic boundary conditions (12.13), then from variational Eq. (12.23) there results system of Eqs. (12.16), which is equivalent to (12.11).

If expansion (12.15) for Φ satisfies only continuity equation (12.10) and kinematic boundary conditions (12.12), then from variational

Eq. (12.23) there results a system of partial differential equations in generalized displacements $Z_i(z, t)$, equivalent simultaneously to Eq. (12.11) and to conditions (12.13).

Let us note that resolvents in Z_1 can also be obtained directly from variational Eq. (12.19) without reducing it to the form (12.23). However, the expressions thus obtained for the coefficients are so cumbersome that such a method is practically unacceptable.

Reducing variational Eq. (12.19) to the form (12.23), we assume that kinematic boundary conditions (12.12) on Σ are fulfilled in advance, including the end ribs of the section. In the general case, this is not possible for arbitrary displacements $W_r(x, y, t)$ of the end ribs. However, conditions (12.17) on the ends can be satisfied on the basis of the variational principle by using the method of indeterminate Lagrangian multipliers. In this case, in variational Eq. (2.19), δA should be taken to mean the variation of the work of unknown reactions $\lambda(x, y, t)$ on the ends of the segment, expanded in maintaining kinematic constraint (12.17):

$$\delta A = \delta \iint_{\sigma(z)} \lambda (W_n - w)\, d\sigma(z) \Big|_{z=0}^{z=l}. \tag{12.24}$$

In view of (12.24), variational Eq. (12.19) for $\delta W_r = 0$ reduces to the form

$$\begin{gathered} \int_0^{t_1} \Big\{ \rho \int_0^l \Big[\iint_\sigma \Big(\ddot{w} - \frac{\partial \ddot{\Phi}}{\partial z} \Big) \delta w \, d\sigma + \int_\gamma \Big(g \frac{\partial \psi}{\partial y} - \ddot{\psi} \Big) \delta \frac{\partial \Phi}{\partial y} d\gamma \Big] dz - \\ - \Big[\iint_\sigma (W_n - w) \delta\lambda \, d\sigma - \iint_\sigma (\lambda + \rho \ddot{\Phi}) \delta w \, d\sigma \Big] \Big|_{z=0}^{z=l} \Big\} dt = 0. \end{gathered} \tag{12.25}$$

Since the variations $\delta\lambda$ and $\delta\omega$ on the ends of the segment are arbitrary and independent, then in addition to the equations resulting from (12.23), kinematic boundary conditions (12.17) and the following relations result from variational Eq. (12.25):

$$\lambda_l = -\varrho\dot{\Phi}, \quad z = l,$$
$$\lambda_a = -\varrho\dot{\Phi}, \quad z = a, \tag{12.26}$$

whence we can conclude that the Lagrangian multipliers constitute the disturbed part of the pressure of the end ribs on the liquid.

Variational Eq. (12.25) also gives rise to kinematic conditions (12.17), and, as can be demonstrated, to Eqa. (12.18) equivalent to them, if the displacement w is represented in the form of expansion (12.14).

Indeed, it follows from the third of Eqs. (12.2) that in this case, the corresponding expansion for the disturbed pressure of the liquid should have the form

$$p(x, y, z, t) = \sum_{i=1}^{N} p_i(z, t)\varphi_i(x, y). \tag{12.27}$$

In this connection, reactions λ_l, λ_a should be represented in the form

$$\lambda(x, y, t) = \sum_{m=1}^{M} p_m(t)\varphi_m(x, y), \tag{12.27a}$$

where $p_m(t)$ are unknown functions.

Introducing expansions of the form (12.27a) into variational Eq. (12.25), we obviously arrive at system of Eqs. (12.18), and also at a system of ordinary differential equations in the unknowns $p_m(t)$. These equations will have a symmetric structure. If however relations (12.26) are used instead of expansions of the form 12.27a), the symmetry is disturbed.

Thus, expression (12.27) represents an expansion of the function $\varrho\Phi$ as a series in the coordinate functions $\varphi_i(x, y)$.

12.3. Determination of Hydrodynamic Pressure for a Tank Caisson of Arbitrary Outline

The problem of construction of expansion (12.15), which in $\sigma(z)$ satisfies Eq. (12.10) and conditions (12.12), (12.13) on $\Gamma(z)$ and $\gamma(z)$, respectively, can be solved exactly only in certain special cases, when the configuration of the section $\sigma(z)$ admits an exact solution of Poisson's equation.

In most cases, all the webs of wing tanks may be considered planar, since the lateral webs are usually formed by the webs of the spars and ribs, and the curvature of the upper and lower panels can be neglected in solving a hydrodynamic problem. Therefore, without limiting the outline of the caisson in the plane, we will assume that the region $\sigma(z)$ is rectangular (Fig. 12.2a). This assumption narrows the scope of practical application of the solution only slightly, and at the same time is very important, since it is well-known that a rectangular region permits the solution of the second boundary-value problem for the Poisson equation by means of separation of variables.

We write boundary conditions (12.12) separately for each of the three webs of the rectangular section (Fig. 12.2b):

for the front web

$$\frac{\partial\Phi}{\partial x} = -\frac{W_1}{\cos\alpha_1} + w\,\mathrm{tg}\,\alpha_1 \text{ when } x = c_1, \tag{12.28}$$

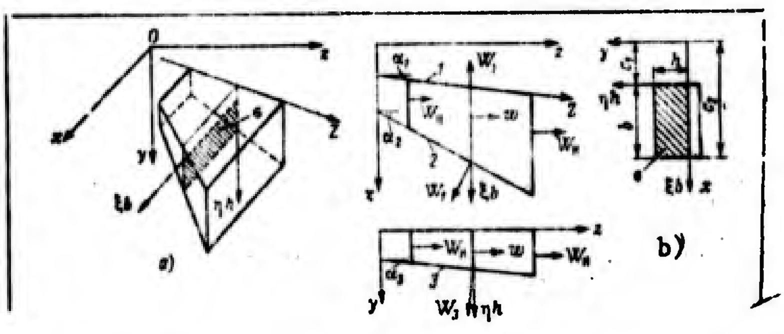


Fig. 12.2. Section of conical caisson of rectangular cross section containing liquid.

for the rear web

$$\frac{\partial\Phi}{\partial x}=\frac{W_2}{\cos\alpha_2}+w\operatorname{tg}\alpha_2\ \text{when}\quad x=c_2, \tag{12.29}$$

for the bottom web

$$\frac{\partial\Phi}{\partial y}=\frac{W_3}{\cos\alpha_3}+w\operatorname{tg}\alpha_3\ \text{when}\quad y=h. \tag{12.30}$$

Expansion (12.15) for the displacement potential satisfying Eq. (12.10) and boundary conditions (12.13) and (12.28) - (12.30), will be sought in the form of the sum

$$\Phi=\Phi_1+\Phi_2+\Phi_3+\Phi_4 \tag{12.31}$$

of four expansions, so that each can be obtained by applying the method of separation of variables.

Let the component Φ_1 of the potential Φ correspond to the motion of the liquid in the section $\sigma(z)$, caused by displacements W of the bottom web; Φ_2, to the motion caused by displacements W of the front and rear webs; Φ_3, to the motion due to the component, normal to the section $\sigma(z)$, of the displacement gradient w; Φ_4, to the motion due to gravitational waves on the free surface of the liquid. Then the components of the potential should satisfy the following equations and boundary conditions:

$$\frac{\partial^2\Phi_i}{\partial x^2}+\frac{\partial^2\Phi_i}{\partial y^2}=0\quad (i=1,2,4), \tag{12.32}$$

$$\frac{\partial^2\Phi_3}{\partial x^2}+\frac{\partial^2\Phi_3}{\partial y^2}=-\frac{\partial w}{\partial z}, \tag{12.33}$$

$$\left.\frac{\partial\Phi_1}{\partial x}\right|_{x=c_1}=0,\quad \left.\frac{\partial\Phi_1}{\partial x}\right|_{x=c_2}=0,$$

$$\Phi_1|_{y=0}=0,\quad \frac{\partial\Phi_1}{\partial y}=\frac{W_3}{\cos\alpha_3}+w\operatorname{tg}\alpha_3\ \text{when}\ y=h, \tag{12.34}$$

$$\frac{\partial\Phi_2}{\partial x}=-\frac{W_1}{\cos\alpha_1}+w\operatorname{tg}\alpha_1\ \text{when}\ x=c_1,$$

$$\frac{\partial\Phi_2}{\partial x}=\frac{W_2}{\cos\alpha_2}+w\operatorname{tg}\alpha_2\ \text{when}\ x=c_2. \tag{12.35}$$

$$\Phi_2\Big|_{y=0}=0,\quad \frac{\partial\Phi_2}{\partial y}\Big|_{y=h}=0,$$

$$\frac{\partial\Phi_3}{\partial x}\Big|_{x=c_1}=0,\quad \frac{\partial\Phi_3}{\partial x}\Big|_{x=c_2}=0,\qquad (12.36)$$

$$\Phi_3\Big|_{y=0}=0,\quad \frac{\partial\Phi_3}{\partial y}\Big|_{y=h}=0,$$

$$\frac{\partial\Phi_4}{\partial x}\Big|_{x=c_1}=0,\quad \frac{\partial\Phi_4}{\partial x}\Big|_{x=c_2}=0,\quad \frac{\partial\Phi_4}{\partial y}\Big|_{y=h}=0,\qquad (12.37)$$

$$g\frac{\partial\Phi_4}{\partial y}-\ddot{\Phi}_4+g\left(\frac{\partial\Phi_1}{\partial y}+\frac{\partial\Phi_2}{\partial y}+\frac{\partial\Phi_3}{\partial y}\right)=0 \text{ when } y=0.$$

It is easy to see that in set (12.31), the components Φ_i of the potential Φ satisfy Eq. (12.10) and boundary conditions (12.13) and (12.28)-(12.30).

Using the method of separation of variables, we find

$$\Phi_1=\sum_{n=0}^{\infty}\frac{b\varepsilon_n\int\limits_0^1\left(\frac{W_3}{\cos\alpha_3}+w\Big|_{\eta=1}\operatorname{tg}\alpha_3\right)\cos\mu_n\xi\,d\xi}{\mu_n\operatorname{ch}\mu_n\bar{h}}\cos\mu_n\xi\operatorname{sh}\mu_n\bar{h}\eta,\qquad (12.38)$$

$$\Phi_2=\sum_{n=1}^{\infty}\frac{2\bar{h}b}{\lambda_n\operatorname{sh}(\lambda_n/\bar{h})}\Bigg[\operatorname{ch}\lambda_n\frac{1}{\bar{h}}(1-\xi)\int\limits_0^1\left(\frac{W_1}{\cos\alpha_1}-w\Big|_{\xi=0}\operatorname{tg}\alpha_1\right)\sin\lambda_n\eta\,d\eta+$$

$$+\operatorname{ch}\lambda_n\frac{1}{\bar{h}}\xi\int\limits_0^1\left(\frac{W_2}{\cos\alpha_2}+w\Big|_{\xi=1}\operatorname{tg}\alpha_2\right)\sin\lambda_n\eta\,d\eta\Bigg]\sin\lambda_n\eta,\qquad (12.39)$$

$$\Phi_3=\sum_{n=0}^{\infty}b^2\bar{h}\varepsilon_n\cos\mu_n\xi\int\limits_0^1G_n(\eta,\eta_1)\int\limits_0^1\frac{\partial w}{\partial z}\cos\mu_n\xi_1\,d\xi_1\,d\eta_1.\qquad (12.40)$$

The following dimensionless coordinates and symbols are introduced into expressions (12.38)-(12.40):

$$\mu_n=n\pi,\quad \lambda_n=\frac{2n-1}{2}\pi,\quad \varepsilon_n=\begin{cases}1 \text{ when } n=0,\\ 2 \text{ when } n\geqslant 1,\end{cases}\qquad (12.41)$$

$$\xi=\frac{x-c_1}{b},\quad \eta=\frac{y}{h},\quad \bar{h}=\frac{h}{b},$$

$$G_n(\eta,\eta_1)=\begin{cases}\operatorname{sh}\mu_n\bar{h}\eta_1\cdot\operatorname{ch}\mu_n\bar{h}(1-\eta)/(\mu_n\operatorname{ch}\mu_n\bar{h}) \text{ when } \eta_1\leqslant\eta,\\ \operatorname{sh}\mu_n\bar{h}\eta\cdot\operatorname{ch}\mu_n\bar{h}(1-\eta_1)/(\mu_n\operatorname{ch}\mu_n\bar{h}) \text{ when } \eta_1\geqslant\eta,\end{cases}\qquad (12.42)$$

and the function G_n can also be represented in the form of the expansion

$$G_n(\eta, \eta_1)=\sum_{m=1}^{\infty} \frac{2\bar{h}}{\lambda_m^2+\bar{h}^2\mu_n^2} \sin \lambda_m\eta \sin \lambda_m\eta_1. \tag{12.43}$$

The potential Φ_4, which corresponds to gravitational waves, can be constructed in the most general case for a motion arbitrary in time, of the container walls and free surface of the liquid. Moreover, as follows from Eq. (12.32) and boundary conditions (12.37), the initial conditions are formulated only on the free surface. Of greatest practical interest, however, are vibrations according to the harmonic law. On the basis of the solution for harmonic vibrations, one determines the amplitude-frequency characteristics and also the frequencies and modes of natural vibrations of an elastic tank containing liquid. However, to solve the problem of motion of a tank containing liquid, aribtrary in time with arbitrary initial conditions, one can always subsequently use the method of decomposition of the motion in normal modes.

In this connection, we will hereinafter consider only steady harmonic vibrations of frequency ω, assuming $\ddot{\Phi}=-\omega^2\Phi$. Instead of the symbol $\bar{U}_j$ for the peak values of generalized displacements used in Ch. XI, we will use U_j.

Applying the method of separation of variables to Eq. (12.32) and satisfying the first three boundary conditions (12.37), we obtain

where the unknown constants A_n for the case of harmonic vibrations $(\ddot{\Phi}_4=\omega^2\Phi_4)$ will be determined by satisfying boundary conditions (12.37).

$$\Phi_4=\sum_{n=0}^{\infty} A_n \cos \frac{\mu_n}{b}(x-c_1) \operatorname{ch} \frac{\mu_n}{b}(h-y),$$

Expanding this condition with the aid of expression (12.38)-(12.40) for the components of Φ_1, Φ_2, Φ_3 of the potential Φ, we successively multiply the right- and left-hand sides of the expression obtained by the functions $\cos\frac{\mu_n}{b}(x-c_1)\ (n=0, 1, 2, \ldots)$, which form an orthogonal system on the segment $c_1<x<c_2$. Integrating each term with respect to x from $x = c_1$ to $x = c_2$, then folding the unary series in functions $\sin\lambda_m\eta$ on the basis of the expansion

$$\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}=\sum_{m=1}^{\infty}\frac{2\lambda_m}{\lambda_m^2+\mu_n^2\bar{h}^2}\sin\lambda_m\eta, \tag{12.44}$$

we finally obtain

$$\begin{gathered}\Phi_s=-\sum_{n=0}^{\infty}\frac{b\varepsilon_n\cos\mu_n\xi\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}\left(b\omega^2/g-\mu_n\operatorname{th}\mu_n\bar{h}\right)}\left\{\int_0^1\left(\frac{W_3}{\cos\alpha_3}+\operatorname{tg}\alpha_3 w\Big|_{\eta=1}\right)\cos\mu_n\xi\,d\xi+\right.\\+\bar{h}\int_0^1\left[\left(\frac{W_1}{\cos\alpha_1}-\operatorname{tg}\alpha_1 w\Big|_{\xi=0}\right)+(-1)^n\left(\frac{W_2}{\cos\alpha_2}+\operatorname{tg}\alpha_2 w\Big|_{\xi=1}\right)+\right.\\\left.\left.+b\int_0^1\frac{\partial w}{\partial z}\cos\mu_n\xi\,d\xi\right]\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}\,d\eta\right\}.\end{gathered} \tag{12.45}$$

Thus, for a tank caisson of arbitrary outline in the plane with a rectangular region $\sigma(z)$ of potential (12.31) for arbitrary normal displacements W_1, W_2, W_3 of the webs and arbitrary displacements of the liquid w in the direction of the Oz axis (see Fig. 12.2).

The expansion obtained for the potential Φ can be introduced into Eq. (12.16) to calculate the vibrations of the liquid in any elastic structure with an arbitrary law of change of the rectangular region $\sigma(z)$, determined by functions $b(z)$, $\bar{h}(z)$, $\alpha_1(z)$, $\alpha_2(z)$, $\alpha_3(z)$.

Let us now consider a conical caisson of arbitrary outline in the plane with a directrix of rectangular form. We introduce the oblique coordinate Z, measured along the line of intersection of the front web with the free surface of the liquid (see Fig. 12.2). We will assume that in the sections parallel to the plane of the directrix, Z = const. In this case, the derivative

$$\frac{\partial}{\partial z} = \frac{1}{\cos\alpha_1}\frac{\partial}{\partial Z} - \frac{\mathrm{tg}\,\alpha_1}{b}\frac{\partial}{\partial \xi}. \tag{12.46}$$

The displacements of the liquid along the Oz axis will be represented as the expansion

$$w(\xi, \eta, Z) = \sum_{l=1}^{M} Z_l(Z)\varphi_l(\xi, \eta), \tag{12.47}$$

where $\varphi_l(\xi, \eta)$ are specified functions.

We develop expansion (12.31) for the potential Φ by taking (12.38) - (12.40) and (12.45) - (12.47) into consideration. Using the expansions

$$\begin{gathered}\frac{\mathrm{ch}\left[(\lambda_n/\bar{h})(1-\xi)\right]}{\lambda_n\,\mathrm{sh}\,(\lambda_n/\bar{h})} = \sum_{m=0}^{\infty}\frac{\varepsilon_m\bar{h}}{\lambda_n^2+\bar{h}^2\mu_m^2}\cos\mu_m\xi,\\ \frac{\mathrm{ch}\left[(\lambda_n/\bar{h})\,\xi\right]}{\lambda_n\,\mathrm{sh}\,(\lambda_n/\bar{h})} = \sum_{m=0}^{\infty}\frac{(-1)^m\varepsilon_m\bar{h}}{\lambda_n^2+\bar{h}^2\mu_m^2}\cos\mu_m\xi,\end{gathered} \tag{12.48}$$

and also expansion (12.43) for the function $G_n(\eta, \eta_1)$, we can see that in expressions (12.39) for Φ_2 and (12.40) for Φ_3, there are similar terms which cancel each other out in the sum.

After some operations, we obtain

$$\begin{gathered}\Phi = \sum_{n=0}^{\infty}\frac{b\varepsilon_n\cos\mu_n\xi\,\mathrm{sh}\,\mu_n\bar{h}\eta}{\mu_n\,\mathrm{ch}\,\mu_n\bar{h}}\left[\frac{W_{3n}}{\cos\alpha_3}+\mathrm{tg}\,\alpha_3\sum_{l=1}^{M}Z_l a_{ln}(1)\right]+\\ +\sum_{n=1}^{\infty}\frac{2\bar{h}b\sin\lambda_n\eta}{\lambda_n\,\mathrm{sh}\,(\lambda_n/\bar{h})}\left[\frac{W_{1n}}{\cos\alpha_1}\mathrm{ch}\frac{\lambda_n}{\bar{h}}(1-\xi)+\frac{W_{2n}}{\cos\alpha_2}\mathrm{ch}\frac{\lambda_n}{\bar{h}}\xi+\right.\\ \left.+(\mathrm{tg}\,\alpha_2-\mathrm{tg}\,\alpha_1)\sum_{l=1}^{M}Z_l\gamma_{ln}(1)\,\mathrm{ch}\frac{\lambda_n}{\bar{h}}\xi\right]+\\ +\sum_{n=0}^{\infty}\bar{h}b\varepsilon_n\cos\mu_n\xi\sum_{l=1}^{M}\left[\frac{bZ_l'}{\cos\alpha_1}\int_0^1 G_n(\eta, \eta_1)\alpha_{ln}(\eta_1)\,d\eta_1-\right.\\ \left.-\mathrm{tg}\,\alpha_1 Z_l\mu_n\int_0^1 G_n(\eta, \eta_1)\beta_{ln}(\eta_1)\,d\eta_1\right]-\\ -\sum_{n=0}^{\infty}\frac{b\varepsilon_n\cos\mu_n\xi\,\mathrm{ch}\,\mu_n\bar{h}(1-\eta)}{\mathrm{ch}\,\mu_n\bar{h}\,(b\omega^2/g-\mu_n\,\mathrm{th}\,\mu_n\bar{h})}\left(\frac{W_{3n}}{\cos\alpha_3}+\frac{\bar{h}\dot{W}_{1n}}{\cos\alpha_1}+\right.\\ +(-1)^n\frac{\bar{h}\dot{W}_{2n}}{\cos\alpha_2}+\sum_{l=1}^{M}\left[\frac{\bar{h}b\dot{Z}_l}{\cos\alpha_1}d_{ln}+Z_l\left(\mathrm{tg}\,\alpha_3\,a_{ln}(1)+\right.\right.\\ \left.\left.\left.+(\mathrm{tg}\,\alpha_2-\mathrm{tg}\,\alpha_1)\,\bar{h}(-1)^n\theta_{ln}-\bar{h}\,\mathrm{tg}\,\alpha_1 r_{ln}\right)\right]\right).\end{gathered} \tag{12.49}$$

Here

$$
\begin{aligned}
W_{3n}(Z) &= \int_0^1 W_3(\xi, Z)\cos\mu_n\xi d\xi, \\
W_{1n}(Z) &= \int_0^1 W_1(\eta, Z)\sin\lambda_n\eta d\eta, \\
W_{2n}(Z) &= \int_0^1 W_2(\eta, Z)\sin\lambda_n\eta d\eta, \\
W^*_{1n}(Z) &= \int_0^1 W_1(\eta, Z)\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}d\eta, \\
W^*_{2n}(Z) &= \int_0^1 W_2(\eta, Z)\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}d\eta, \\
\alpha_{in}(\eta) &= \int_0^1 \varphi_i(\xi, \eta)\cos\mu_n\xi d\xi, \\
\beta_{in}(\eta) &= \int_0^1 \varphi_i(\xi, \eta)\sin\mu_n\xi d\xi, \\
\gamma_{in}(\xi) &= \int_0^1 \varphi_i(\xi, \eta)\sin\lambda_n\eta d\eta, \\
d_{in} &= \int_0^1 \alpha_{in}(\eta)\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}d\eta, \\
r_{in} &= \mu_n\int_0^1 \beta_{in}(\eta)\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}d\eta, \\
\vartheta_{in} &= \int_0^1 \varphi_i(1, \eta)\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}d\eta.
\end{aligned}
\qquad (12.50)
$$

12.4. Vibrations of a Swept Wing Type Shell

We will consider the harmonic vibrations of an elastic swept caisson of constant rectangular cross section, partly filled with liquid (Fig. 12.3). The motion of the elastic webs of the caisson will be considered aribtrary - they can participate in the general flexural-torsional motion of the entire caisson and, in addition, can execute transverse vibrations due to bending of the plates. For a swept caisson with parallel webs, it is necessary to assume $b=\text{const}, \bar{h}=\text{const}, a_2=a_1, a_3=0$ in expression (12.59) for the displacement potential.

Let us turn to equations of motion (12.16) of the liquid. Multiplying these equations by $\sin \alpha_1$, so that they pertain to the layer $dZ = 1$, and omitting the multiplier $b^2\bar{h}$, we obtain

$$\cos \alpha_1 \int_0^1 \int_0^1 \left(\frac{\partial \Phi}{\partial z} - w\right) \varphi_k(\xi, \eta)\, d\xi\, d\eta = 0 \quad (k = 1, \ldots, M), \tag{12.51}$$

where the derivative of the displacement potential Φ, represented by general expansion (12.49), is calculated in accordance with (12.46).

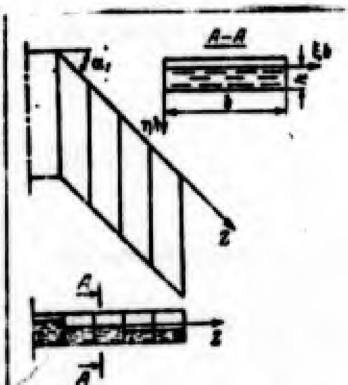


Fig. 12.3. Swept caisson of rectangular cross section partly filled with liquid.

Developing (12.51) by using expansions (12.47), (12.49), we arrive at a system of ordinary differential equations of the form

$$\sum_{l=1}^{M} (A_{kl} b^2 \ddot{Z}_l + \bar{B}_{kl} b \dot{Z}_l - C_{kl} Z_l) + Q_k = 0 \quad (k = 1, \ldots, M). \tag{12.52}$$

The coefficients of these equations are determined by the expressions

$$\begin{gathered}
A_{kl} = \frac{\bar{h}}{\cos \alpha_1} \sum_{n=0}^{\infty} \varepsilon_n \left(a_{kl.n} - \frac{d_{kn} d_{ln}}{b\omega^2/g - \mu_n \operatorname{th} \mu_n \bar{h}}\right), \\
\bar{B}_{kl} = B_{lk} - B_{kl}, \\
B_{kl} = 2\bar{h} \operatorname{tg} \alpha_1 \sum_{n=1}^{\infty} \left(b_{kl.n} - \frac{d_{kn} r_{ln}}{b\omega^2/g - \mu_n \operatorname{th} \mu_n \bar{h}}\right), \\
C_{kl} = \cos \alpha_1 S_{kl} + 2\bar{h} \operatorname{tg}^2 \alpha_1 \cos \alpha_1 \sum_{n=1}^{\infty} \left(c_{kl.n} - \frac{r_{kn} r_{ln}}{b\omega^2/g - \mu_n \operatorname{tg} \mu_n \bar{h}}\right).
\end{gathered} \tag{12.53}$$

Here use was made of notation (12.50) and in addition, the following symbols were introduced:

$$
\begin{aligned}
a_{kl,n} &= \int_0^1\int_0^1 G_n(\eta, \eta_1)\alpha_{kn}(\eta)\alpha_{ln}(\eta_1)\,d\eta\,d\eta_1, \\
b_{kl,n} &= \mu_n \int_0^1\int_0^1 G_n(\eta, \eta_1)\alpha_{kn}(\eta)\beta_{ln}(\eta_1)\,d\eta\,d\eta_1, \\
c_{kl,n} &= \mu_n^2 \int_0^1\int_0^1 G_n(\eta, \eta_1)\beta_{kn}(\eta)\beta_{ln}(\eta_1)\,d\eta\,d\eta_1, \\
S_{kl} &= \int_0^1\int_0^1 \varphi_k(\xi, \eta)\varphi_l(\xi, \eta)\,d\xi\,d\eta.
\end{aligned} \tag{12.54}
$$

It is evident from (12.53), (12.54) that the coefficients A_{kl} and C_{kl} of Eqs. (12.52) are symmetric, and the coefficients $\bar{B}_{kl}$ are skew-symmetric, i.e., the following conditions are fulfilled:

$$
A_{kl} = A_{lk}, \quad \bar{B}_{kl} = -\bar{B}_{lk}, \quad C_{kl} = C_{lk}. \tag{12.55}
$$

The free terms Q_k in Eqs. (12.52) depend on the normal displacements of the bottom web W_3 and lateral webs W_1 and W_2 of the swept caisson (see Fig. 12.3) and are determined by the expressions

$$
\begin{aligned}
Q_k(Z) &= 2\frac{\bar{h}b}{\cos\alpha_1}\sum_{n=1}^{\infty}(k_{kn}W'_{1n} + l_{kn}W'_{2n}) + \\
&+ \frac{\bar{h}b}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n\left[\frac{\cos\alpha_1}{\bar{h}} f_{kn}W'_{3n} - \frac{d_{kn}}{b\omega^2/g - \mu_n \operatorname{th}\mu_n\bar{h}}\left(\frac{\cos\alpha_1}{\bar{h}} W'_{3n} + \right.\right. \\
&\left.\left. + \dot{W}'_{1n} + (-1)^n \dot{W}'_{2n}\right)\right] + 2\operatorname{tg}\alpha_1\sum_{n=1}^{\infty}(p_{kn}W_{1n} - q_{kn}W_{2n}) + \\
&+ \bar{h}\operatorname{tg}\alpha_1\sum_{n=0}^{\infty}\varepsilon_n\left[\frac{\cos\alpha_1}{\bar{h}} h_{kn}W_{3n} - \frac{r_{kn}}{b\omega^2/g - \mu_n \operatorname{th}\mu_n\bar{h}}\times\right. \\
&\left.\times\left(\frac{\cos\alpha_1}{\bar{h}} W_{3n} + \dot{W}_{1n} + (-1)^n\dot{W}_{2n}\right)\right].
\end{aligned} \tag{12.56}
$$

Here

$$
\begin{aligned}
k_{kn} &= \int_0^1 \gamma_{kn}(\xi)\frac{\operatorname{ch}[(\lambda_n/\bar{h})(1-\xi)]}{\lambda_n \operatorname{sh}(\lambda_n/\bar{h})}\,d\xi, \\
l_{kn} &= \int_0^1 \gamma_{kn}(\xi)\frac{\operatorname{ch}[(\lambda_n/\bar{h})\xi]}{\lambda_n \operatorname{sh}(\lambda_n/\bar{h})}\,d\xi, \\
f_{kn} &= \int_0^1 \alpha_{kn}(\eta)\frac{\operatorname{sh}\mu_n\bar{h}\eta}{\mu_n \operatorname{ch}\mu_n\bar{h}}\,d\eta, \\
p_{kn} &= \int_0^1 \gamma_{kn}(\xi)\frac{\operatorname{sh}[(\lambda_n/\bar{h})(1-\xi)]}{\operatorname{sh}(\lambda_n/\bar{h})}\,d\xi, \\
q_{kn} &= \int_0^1 \gamma_{kn}(\xi)\frac{\operatorname{sh}[(\lambda_n/\bar{h})\xi]}{\operatorname{sh}(\lambda_n/\bar{h})}\,d\xi, \\
h_{kn} &= \int_0^1 \beta_{kn}(\eta)\frac{\operatorname{sh}\mu_n\bar{h}\eta}{\operatorname{ch}\mu_n\bar{h}}\,d\eta,
\end{aligned} \tag{12.57}
$$

and in addition, notations (12.50) and (12.54) are employed.

If the normal displacements of the webs W_1, W_2, and W_3 are given, then the functions $\bar{Z}_i(Z)$ are determined from Eqs. (12.52) under the corresponding boundary conditions on the ends. Then, having determined w and Φ from (12.47), (12.49), we can also find the displacements u, v in the direction of the Ox, Oy axes and the hydrodynamic pressure $\rho\omega^2\Phi$ on the shell, then the coefficients of added liquid masses corresponding to the given motion of the webs. Such an approach immediately achieves its objective if the tank is assumed to be nondeformable or if the displacements of the elastic tank are given in the form of expansions in given functions with unknown coefficients, i.e., in cases where the displacements of the tank webs are characterized by a finite number of degrees of freedom.

Let us now consider a more general case, where the normal displacements of the tank webs are represented by an expansion of the form

$$W(Z, S)=\sum_{i=1}^{N} U_i(Z)\varphi_{in_n}(S),$$

where $\varphi_{in_n}(S)$ is a given system of functions of the S coordinate, measured along the arc of the contour Z = const, and the functions $U_i(Z)$ are the desired generalized displacements. In this case

$$\begin{aligned} W_1(Z, \eta)&=\sum_{i=1}^{N} U_i(Z)\varphi^1_{in_n}(\eta), \\ W_2(Z, \eta)&=\sum_{i=1}^{N} U_i(Z)\varphi^2_{in_n}(\eta), \\ W_3(Z, \xi)&=\sum_{i=1}^{N} U_i(Z)\varphi^3_{in_n}(\xi), \end{aligned} \qquad (12.58)$$

where

$$\varphi^1_{in_n}(\eta)=\varphi_{in_n}\big|_{\xi=0},\quad \varphi^2_{in_n}(\eta)=\varphi_{in_n}\big|_{\xi=1},\quad \varphi^3_{in_n}(\xi)=\varphi_{in_n}\big|_{\eta=1}.$$

Taking (12.58) into account, expression (12.56) for Q_k may be written in the form

$$Q_k=\sum_{l=1}^{N}(D_{kl}bU_l'+E_{kl}U_l), \tag{12.59}$$

where

$$D_{kl}=\frac{2\bar h}{\cos\alpha_1}\sum_{n=1}^{\infty}(k_{kn}\bar\gamma^1_{ln}+l_{kn}\bar\gamma^2_{ln})+$$
$$+\frac{\bar h}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n\left(\frac{\cos\alpha_1}{\bar h}f_{kn}\bar g_{ln}-\frac{d_{kn}\bar i_{ln}}{b\omega^2/g-\mu_n\,\mathrm{th}\,\mu_n\bar h}\right), \tag{12.60}$$
$$E_{kl}=2\,\mathrm{tg}\,\alpha_1\sum_{n=1}^{\infty}(p_{kn}\bar\gamma^1_{ln}-q_{kn}\bar\gamma^2_{ln})+$$
$$+\bar h\,\mathrm{tg}\,\alpha_1\sum_{n=0}^{\infty}\varepsilon_n\left(\frac{\cos\alpha_1}{\bar h}h_{kn}\bar g_{ln}-\frac{r_{kn}\bar i_{ln}}{b\omega^2/g-\mu_n\,\mathrm{th}\,\mu_n\bar h}\right),$$

$$\bar\gamma^r_{ln}=\int_0^1\varphi^r_{ln_n}(\eta)\sin\lambda_n\eta\,d\eta\qquad(r=1,\,2),$$
$$\bar g_{ln}=\int_0^1\varphi^3_{ln_n}(\xi)\cos\mu_n\xi\,d\xi,$$
$$\bar i_{ln}=\frac{\cos\alpha_1}{\bar h}\,\bar g_{ln}+\bar d^1_{ln}+(-1)^n\bar d^2_{ln}, \tag{12.61}$$
$$\bar d^r_{ln}=\int_0^1\varphi^r_{ln_n}(\eta)\frac{\mathrm{ch}\,\mu_n\bar h(1-\eta)}{\mathrm{ch}\,\mu_n\bar h}\,d\eta\;(r=1,\,2).$$

To determine the generalized forces exerted on the shell by the liquid in the presence of harmonic vibrations, we will consider the variation

$$\delta A=\int_\Gamma p\,\delta W\,\frac{d\Gamma\,dz}{\sin\beta} \tag{12.62}$$

of the work of hydrodynamic pressure on a strip of shell of width dz.

For a rectangular swept caisson, considering that

$$\alpha_2=\alpha_1,\quad \alpha_3=0,\quad dz=dZ\cos\alpha_1,$$

we have

$$\delta A=b\left[\bar h\int_0^1 p_1\delta W_1d\eta+\bar h\int_0^1 p_2\delta W_2d\eta+\cos\alpha_1\int_0^1 p_3\delta W_3d\xi\right]dZ.$$

On the basis of (12.6), the disturbed pressure of the liquid on the webs

$$p_1=\varrho\omega^2\Phi|_{\xi=0},\quad p_2=\varrho\omega^2\Phi|_{\xi=1},\quad p_3=\varrho g W_3+\varrho\omega^2\Phi|_{\eta=1}. \tag{12.63}$$

In view of (12.58) and (12.63), the linear generalized force, equal to the work of the pressure on the strip dZ = 1 on displacements corresponding to the generalized displacement $\delta U_j=1$, will be

$$R_j(Z)=\varrho\omega^2 b\bar{h}\left[\int_0^1\Phi\Big|_{\xi=0}\varphi^1_{jn_n}d\eta+\int_0^1\Phi\Big|_{\xi=1}\varphi^2_{jn_n}d\eta+\right.$$
$$\left.+\frac{\cos\alpha_1}{\bar{h}}\int_0^1\Phi\Big|_{\eta=1}\varphi^3_{jn_n}d\xi+\frac{g}{\omega^2}\frac{\cos\alpha_1}{\bar{h}}\int_0^1 W_3\varphi^3_{jn_n}d\xi\right]. \tag{12.64}$$

Developing expression (12.64) by considering expansion (12.49) for the potential Φ when $a_2=a_1,\ a_3=0$ and expansions (12.58) for the displacements W of the shell, after some transformations, we obtain

$$R_j=\varrho\omega^2 b\bar{h}\left[\sum_{i=1}^{N}F_{ji}U_i+\sum_{l=1}^{M}(D_{lj}bZ'_l-E_{lj}Z_l)\right]. \tag{12.65}$$

Here the symmetric coefficients

$$F_{ji}=\frac{\cos\alpha_1}{\bar{h}}\left[\frac{g}{l\omega^2}\bar{e}_{ji}+\sum_{n=0}^{\infty}\varepsilon_n\left(\frac{\operatorname{th}\mu_n\bar{h}}{\mu_n}\bar{g}_{jn}\bar{g}_{in}-\frac{\bar{t}_{jn}\bar{t}_{in}}{b\omega^2/g-\mu_n\operatorname{th}\mu_n\bar{h}}\right)\right]+$$
$$+2\sum_{n=1}^{\infty}\left[\bar{g}_{jn}(\bar{f}^1_{in}+(-1)^n\bar{f}^2_{in})+\bar{g}_{in}(\bar{f}^1_{jn}+(-1)^n\bar{f}^2_{jn})+\right.$$
$$\left.+\frac{\bar{h}}{\cos\alpha_1}\left(\frac{\bar{\gamma}^1_{jn}\bar{\gamma}^1_{in}+\bar{\gamma}^2_{jn}\bar{\gamma}^2_{in}}{\lambda_n\operatorname{th}(\lambda_n/\bar{h})}+\frac{\bar{\gamma}^1_{jn}\bar{\gamma}^2_{in}+\bar{\gamma}^2_{jn}\bar{\gamma}^1_{in}}{\lambda_n\operatorname{sh}(\lambda_n/\bar{h})}\right)\right], \tag{12.66}$$
$$\bar{e}_{ji}=\int_0^1\varphi^3_{jn_n}(\xi)\varphi^3_{in_n}(\xi)d\xi;\quad \bar{f}^r_{jn}=\int_0^1\frac{\operatorname{sh}\mu_n\bar{h}\eta}{\mu_n\operatorname{ch}\mu_n\bar{h}}\varphi^r_{jn_n}(\eta)d\eta,$$

and the coefficients D_{lj} and E_{lj}, as in expression (12.59) for Q_k, are determined from formulas (12.60).

In order to reduce the coefficients F_{ji} to symmetric form and to show that the coefficients D_{lj} and E_{lj} or the functions Z'_l and Z_l in expression (12.65) are symmetric with respect to the coefficients D_{ki} and E_{ki} of the functions U'_i and U_i in expression (12.59) for Q_k, we had to establish the identify of certain terms represented in the form of unary series in various functions of ξ and η. Use was made of expressions (12.43), (12.44) and (12.48), and also of the expansion

$$\frac{\operatorname{sh}\mu_n\bar{h}\eta}{\mu_n\operatorname{ch}\mu_n\bar{h}}=-\sum_{m=1}^{\infty}\frac{2\bar{h}(-1)^m}{\lambda_m^2+\bar{h}^2\mu_n^2}\sin\lambda_m\eta. \tag{12.67}$$

The generalized forces R_j (12.65) are included in the differential equations of the shell proper, so that the problem reduces to the integration of a simultaneous system of N + M ordinary differential equations in N generalized displacements $U_i(Z)$ of the shell and M generalized displacements $Z_i(Z)$ of the liquid.

We will finally write out the equations for the vibrations of a single-closed swept wing type caisson.

Introducing expressions (12.65) for the generalized forces R_j into Eqs. (11.24) and adding Eqs. (12.55) to this system, we obtain in view of (12.59)

$$\begin{gathered}\sum_{i=1}^{N}[a_{ji}U_i''+(b_{ji}-b_{ij})U_i'+(-c_{ji}+\omega^2 d_{ji}+\Omega^2F_{ji})U_i]+\\+\Omega^2\sum_{i=1}^{M}[D_{ij}bZ_i'-E_{ij}Z_i]=0\\(j=1,\ldots,N),\\\sum_{i=1}^{M}[A_{ki}b^2Z_i''+(B_{ik}-B_{ki})bZ_i'-C_{ki}Z_i]+\\+\sum_{i=1}^{N}[D_{ki}bU_i'+E_{ki}U_i]=0\\(k=1,\ldots,M),\end{gathered} \tag{12.68}$$

where

$$\Omega^2=\frac{\omega^2\varrho b^2\bar{h}}{G}.$$

If in system (12.68), each of the equations of the second group is multiplied by Ω^2, the matrices of the coefficients of this system will be symmetric for unknown functions and their second derivatives, and anti-symmetric for the first derivatives. It is easy to show that such a system of equations is self-conjugate. This property follows from the conservativeness of the problem at hand.

12.5. Vibrations of a Low-Aspect Wing Type Shell of Arbitrary Shape in the Plane

Let us consider the vibrations of an elastic conical caisson with a rectangular single-closed contour of the directrix, partly filled with liquid (Fig. 12.4.).

To keep the expressions for the coefficients of ordinary differential equations from being too cumbersome, we will confine our discussion to the case in which the free surface of the liquid passes through the apex of the cone. Then $k = h(Z)/\bar{b}(Z) = \mathrm{const}$, so that the derivative $\partial\Phi/\partial z$ is substantially simplified, since dependence (12.49) of the potential Φ on $\bar{h}$ is very complex.

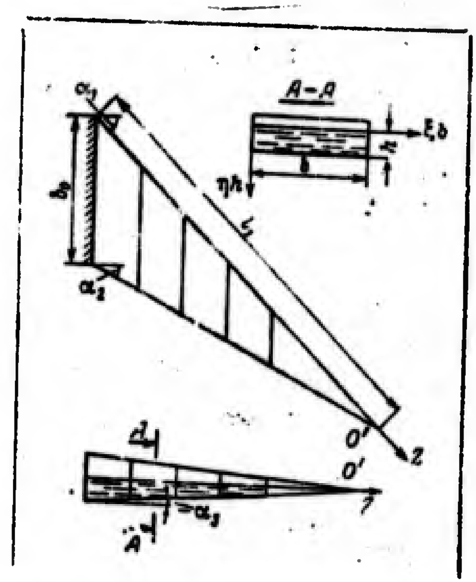


Fig. 12.4. Conical caisson of rectangular cross section, partly filled with liquid.

In addition, setting g = 0, we will not consider the influence of gravitation, since this influence is negligibly small in vibrations at frequencies of the same order as the frequencies of natural elastic vibrations of the structure, since the latter frequencies are usually much lower than the lowest natural frequencies of gravitational vibrations of the free surface. This is valid provided

$$\omega^2 \gg \frac{g}{b_{cp}} \mu_1 \,\mathrm{th}\, \mu_1 \bar{h}.$$

when b_m is the mean dimension of the chord $b(Z)$ of the section under consideration.

For a conical caisson, the equations of harmonic vibrations of the liquid inside the caisson for arbitrary displacements of elastic webs, as well as for a swept caisson, can be written by introducing into Eqs. (12.51) expansion (12.47) of the displacements of the liquid w and derivative (12.46) of the displacement potential Φ, represented by expansion (12.49).

These equations reduce to the form

$$\sum_{l=1}^{M}\left[A^{*}_{kl}\,(b^{2}Z'_{l})'+\bar{B}^{*}_{kl}\,bZ'_{l}-C^{*}_{kl}Z_{l}\right]+Q^{*}_{k}=0 \qquad (12.69)$$

$$(k=1,\ldots,M).$$

Here b is a linear function of the Z coordinate. If the free surface of the liquid passes through the cone apex, it is evident that

$$\operatorname{tg}\alpha_3=\bar{h}\,(\operatorname{tg}\alpha_2-\operatorname{tg}\alpha_1), \qquad (12.70)$$

so that the derivative

$$\frac{db}{dZ}=(\operatorname{tg}\alpha_2-\operatorname{tg}\alpha_1)\cos\alpha_1=\frac{\operatorname{tg}\alpha_3}{\bar{h}}\cos\alpha_1. \qquad (12.71)$$

The coefficients in Eqs. (12.69) are determined by expressions

$$A^{*}_{kl}=\frac{\bar{k}}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n a_{kl,n},$$

$$
\begin{aligned}
B_{ki}^{\circ} &= 2\bar{h}\,\mathrm{tg}\,\alpha_1 \sum_{n=1}^{\infty}(b_{ik.n} - b_{ki.n}) + \mathrm{tg}\,\alpha_3 \sum_{n=0}^{\infty} \varepsilon_n f_{kn} \alpha_{in}(1) + \\
&\quad + 2\,\mathrm{tg}\,\alpha_3 \sum_{n=1}^{\infty} l_{kn} \gamma_{in}(1), \\
C_{ki}^{\circ} &= \cos\alpha_1 S_{ki} + 2\bar{h}\,\mathrm{tg}^2\alpha_1 \cos\alpha_1 \sum_{n=1}^{\infty} c_{ik.n} + \\
&\quad + \cos\alpha_1\,\mathrm{tg}\,\alpha_3 \Bigg\{ 2 \sum_{n=1}^{\infty} \left[b_{ki.n} + \left(\mathrm{tg}\,\alpha_1 q_{kn} + \frac{\mathrm{tg}\,\alpha_3}{\bar{h}} l_{kn} \right) \gamma_{in}(1) \right] - \\
&\quad - \sum_{n=0}^{\infty} \varepsilon_n \left(\mathrm{tg}\,\alpha_1 h_{kn} + \frac{\mathrm{tg}\,\alpha_3}{\bar{h}} f_{kn} \right) \alpha_{in}(1) \Bigg\}.
\end{aligned}
\qquad (12.72)
$$

The free terms dependent on the normal displacements of the caisson webs

$$
\begin{aligned}
Q_k^{\circ}(Z) &= \frac{1}{\cos\alpha_3} \sum_{n=0}^{\infty} \varepsilon_n b W_{3n}' f_{kn} + 2\bar{h} \sum_{n=1}^{\infty} \left(\frac{bW_{1n}'}{\cos\alpha_1} h_{kn} + \frac{bW_{2n}'}{\cos\alpha_2} l_{kn} \right) + \\
&\quad + \frac{\cos\alpha_1}{\cos\alpha_3} \sum_{n=0}^{\infty} \varepsilon_n W_{3n} \left(\mathrm{tg}\,\alpha_1 h_{kn} + \frac{\mathrm{tg}\,\alpha_3}{\bar{h}} f_{kn} \right) + 2 \sum_{n=1}^{\infty} \Big[W_{1n} (\mathrm{tg}\,\alpha_1 p_{kn} + \mathrm{tg}\,\alpha_2 h_{kn}) - \\
&\quad - \frac{\cos\alpha_1}{\cos\alpha_2} W_{2n} (\mathrm{tg}\,\alpha_1 q_{kn} - \mathrm{tg}\,\alpha_2 l_{kn}) \Big].
\end{aligned}
\qquad (12.73)
$$

The symbols in expressions (12.72), (12.73) are the same as the corresponding symbols in expressions (12.53), (12.56) of the preceding section.

In differential Eqs. (12.69) for a conical caisson, the coefficients b are variable. However, it is easy to see that these equations are Euler type equations. When b = const, Eqs. (12.69) change into Eqs. (12.52) for a swept caisson of constant cross section, gravity being neglected ($g = 0$).

Let the normal displacements of the webs be represented in the form of expressions (3.13). Then

$$
\begin{aligned}
W_1(\eta, Z) &= \sum_{i=1}^{N} V_i(Z) \varphi_{in_n}^{1}(\eta), \\
W_2(\eta, Z) &= \sum_{i=1}^{N} V_i(Z) \varphi_{in_n}^{2}(\eta), \\
W_3(\xi, Z) &= \sum_{i=1}^{N} V_i(Z) \varphi_{in_n}^{3}(\xi).
\end{aligned}
\qquad (12.74)
$$

where $V_i(Z)=U_i(Z)\lambda_i(Z)$.

Introducing these expansions into (12.73), we obtain

$$Q_k^{\bullet}(Z)=\sum_{i=1}^{N}(D_{ki}^{\bullet}bV_i'+E_{ki}^{\bullet}V_i), \tag{12.75}$$

where

$$\begin{gathered}
D_{ki}^{\bullet}=\frac{1}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n f_{kn}\bar{g}_{in}+\frac{2\bar{h}}{\cos\alpha_1}\sum_{n=1}^{\infty}\left(k_{kn}\bar{\gamma}_{in}^{1}+\frac{\cos\alpha_1}{\cos\alpha_2}l_{kn}\bar{\gamma}_{in}^{2}\right), \\
E_{ki}^{\bullet}=\frac{\cos\alpha_1}{\cos\alpha_3}\sum_{n=0}^{\infty}\varepsilon_n\left(\operatorname{tg}\alpha_1 h_{kn}+\frac{\operatorname{tg}\alpha_3}{\bar{h}}f_{kn}\right)\bar{g}_{in}+ \\
+2\sum_{n=1}^{\infty}\left[\operatorname{tg}\alpha_1\left(p_{kn}\bar{\gamma}_{in}^{1}-\frac{\cos\alpha_1}{\cos\alpha_2}q_{kn}\bar{\gamma}_{in}^{2}\right)+\operatorname{tg}\alpha_3\left(k_{kn}\bar{\gamma}_{in}^{1}+\frac{\cos\alpha_1}{\cos\alpha_2}l_{kn}\bar{\gamma}_{in}^{2}\right)\right].
\end{gathered} \tag{12.76}$$

Proceeding from expression (12.62), we determine the generalized linear force as the work of hydrodynamic pressure $p=\rho\omega^2\Phi$ on a strip of shell dZ = 1 for displacements corresponding to the generalized displacement $\delta U_j=1$. In view of (12.74)

$$\begin{gathered}
R_j(Z)=\rho\omega^2 b\bar{h}\lambda_j\left[\int_0^1\Phi\bigg|_{\xi=0}\varphi_{jn_n}^{1}d\eta+\frac{\cos\alpha_1}{\cos\alpha_2}\int_0^1\Phi\bigg|_{\xi=1}\varphi_{jn_n}^{2}d\eta+\right. \\
\left.+\frac{1}{\bar{h}}\frac{\cos\alpha_1}{\cos\alpha_3}\int_0^1\Phi\bigg|_{\eta=1}\varphi_{jn_n}^{3}d\xi\right].
\end{gathered} \tag{12.77}$$

Introducing expansion (12.49) for the potential Φ into (12.77) without considering the gravity (g = 0), after some transformations we obtain

$$R_j=\rho\omega^2 b^2\bar{h}\lambda_j\left[\sum_{i=1}^{N}L_{ji}V_i+\sum_{l=1}^{M}(M_{jl}bZ_l'-N_{jl}Z_l)\right]_j, \tag{12.78}$$

where

$$L_{jl}=\frac{\cos\alpha_1}{\bar h\cos^2\alpha_3}\sum_{n=0}^{\infty}\varepsilon_n\frac{\operatorname{th}\mu_n\bar h}{\mu_n}\bar g_{ln}\bar g_{jn}+$$

$$+\frac{2}{\cos\alpha_3}\sum_{n=1}^{\infty}\left[\left(\bar f^1_{jn}+(-1)^n\frac{\cos\alpha_1}{\cos\alpha_2}\bar f^2_{jn}\right)\bar g_{ln}+\left(\bar f^1_{ln}+(-1)^n\frac{\cos\alpha_1}{\cos\alpha_2}\bar f^2_{jn}\right)\bar g_{jn}\right]+$$

$$+\frac{2\bar h}{\cos\alpha_1}\sum_{n=1}^{\infty}\left[\frac{\bar\gamma^1_{jn}\bar\gamma^1_{ln}}{\lambda_n\operatorname{th}(\lambda_n/\bar h)}+\frac{\cos^2\alpha_1}{\cos^2\alpha_2}\frac{\bar\gamma^2_{jn}\bar\gamma^2_{ln}}{\lambda_n\operatorname{th}(\lambda_n/\bar h)}+\right. \tag{12.79}$$

$$\left.+\frac{\cos\alpha_1}{\cos\alpha_2}\frac{\bar\gamma^1_{jn}\bar\gamma^2_{ln}+\bar\gamma^2_{jn}\bar\gamma^1_{ln}}{\lambda_n\operatorname{sh}(\lambda_n/\bar h)}\right],$$

$$M_{jl}=\frac{1}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n\bar g_{jn}f_{ln}+2\frac{\bar h}{\cos\alpha_1}\sum_{n=1}^{\infty}\left(\bar\gamma^1_{jn}k_{ln}+\frac{\cos\alpha_1}{\cos\alpha_2}\bar\gamma^2_{jn}l_{ln}\right),$$

$$N_{jl}=\sum_{n=0}^{\infty}\varepsilon_n\left[\operatorname{tg}\alpha_1\frac{\cos\alpha_1}{\cos\alpha_1}\bar g_{jn}h_{ln}-\operatorname{tg}\alpha_3\left(\frac{\cos\alpha_1}{\bar h\cos\alpha_1}\frac{\operatorname{th}\mu_n\bar h}{\mu_n}\bar g_{jn}+\bar f^1_{jn}+\right.\right.$$

$$\left.\left.+(-1)^n\frac{\cos\alpha_1}{\cos\alpha_3}\bar f^2_{jn}\right)a_{ln}(1)\right]+2\sum_{n=1}^{\infty}\left[\operatorname{tg}\alpha_1\left(\bar\gamma^1_{jn}p_{ln}-\frac{\cos\alpha_1}{\cos\alpha_2}\bar\gamma^2_{jn}q_{ln}\right)-\right.$$

$$\left.-\operatorname{tg}\alpha_3\left(\frac{\bar\gamma^1_{jn}}{\lambda_n\operatorname{th}(\lambda_n/\bar h)}+\frac{\cos\alpha_1}{\cos\alpha_2}\frac{\bar\gamma^2_{jn}}{\lambda_n\operatorname{th}(\lambda_n/\bar h)}-(-1)^n\frac{\cos\alpha_1}{\bar h\cos\alpha_3}\bar l_{jn}\right)\right].$$

In addition to the symbols adopted in the preceding section, the following symbol has been introduced:

$$\bar l_{jn}=\int_0^1\varphi^2_{jn_n}(\xi)\frac{\operatorname{ch}[(\lambda_n/\bar h)\xi]}{\lambda_n\operatorname{sh}\lambda_n/\bar h)}d\xi. \tag{12.80}$$

The coefficients L_{jl}, M_{jk} and N_{jl} for a caisson of constant cross section $(\alpha_2=\alpha_1,\ \alpha_3=0)$ change into coefficients F_{ji}, D_{lj} and E_{lj}, respectively, when $g = 0$.

Equations (12.69), where the last terms are given by expressions (12.75), and the corresponding equations of the shell proper, which include generalized forces (12.78), form a closed system of N + M ordinary differential equations in N generalized displacements $V_i(Z)$ of the shell and M generalized displacements $Z_l(Z)$ of the liquid.

Since equations of motion of the liquid (12.69) and generalized hydrodynamic force (12.78) were written for the most general case of motion of the webs of a tank caisson, by using these coordinates, we can study both bending-torsional vibrations on the basis of the static-geometric model or a model with a nondeformable contour, and general type vibrations, allowing for rib elasticity and bending strain of the middle surface.

We will finally write out the equations of vibrations of a low-aspect wing type conical caisson of arbitrary shape in the plane, whose sections are partly filled with liquid (see Fig. 12.4). Since in the equations pertaining to the shell proper, $\bar{Z}$ is a dimensionless coordinate representing the relative distance measured along the generatrices in fractions of their total length l_s, then in Eqs. (12.69) and in expression (12.78) for R_j, it is necessary to switch to the new variable

$$\bar{Z}=\frac{z}{l_1}, \tag{12.81}$$

where l_1 is the distance from the origin of the coordinates to the apex of the cone along the generatrix coinciding with the line of intersection of the front web and the free surface of the liquid (see Fig. 12.4).

In this case, in the current section $\bar{Z}$ = const, the chord length

$$b=b_0(1-\bar{Z}), \tag{12.82}$$

where b_0 is the chord length in the edge section (see Fig. 12.4).

Introducing expressions (12.78) for generalized forces R_j into equations of shell vibrations (11.23), and adding to this system the equations of vibrations of the liquid (12.69), (12.75), after switching to the dimensionless coordinate $\bar{Z}$, we obtain

$$\sum_{i=1}^{N}\{[(1-\bar{Z})a_{ji}V_i'+b_{ji}V_i]'-b_{ij}V_i'+$$
$$+\left[-\frac{1}{1-\bar{Z}}c_{ji}+\omega^2(1-\bar{Z})d_{ji}+\Omega^2(1-\bar{Z})^2L_{ji}\right]V_i\}+$$
$$+\sum_{l=1}^{M}\left[\frac{b_0}{l_1}M_{jl}(1-\bar{Z})Z_l'-N_{jl}Z_l\right]=0$$
$$(j=1,\ldots,N),\qquad (12.83)$$
$$\sum_{l=1}^{M}\left\{\frac{b_0^2}{l_1^2}A_{kl}^*[(1-\bar{Z})^2Z_l']'+\frac{b_0}{l_1}(B_{lk}^*-B_{kl}^*)(1-\bar{Z})Z_l'-C_{kl}^*Z_l\right\}+$$
$$+\sum_{i=1}^{N}\left[\frac{b_0}{l_1}D_{ki}^*(1-\bar{Z})V_i'+E_{ki}^*V_i\right]=0$$

where

$$(k=1,\ldots,M),$$
$$\Omega^2=\frac{\omega^2\varrho b_0^2\bar{h}}{G},\quad \frac{b_0}{l_1}=\cos\alpha_1(\operatorname{tg}\alpha_1-\operatorname{tg}\alpha_2).$$

In these equations, the prime denotes the derivative with respect to the $\bar{Z}$ coordinate.

12.6. Boundary Conditions Allowing for Rib Elasticity

Thus, problems involving harmonic vibrations of a swept caisson of constant cross section and a conical caisson of arbitrary outline, partly filled with liquid, reduce to the integration of systems of ordinary differential equations (12.68) and (12.83).

We will now formulate the corresponding boundary conditions. Usually, the shell of a wing caisson is divided into compartments by ribs which are very pliable with respect to bending out of their plane. Therefore, in formulating the boundary conditions on the function $\ddot{Z}_l(\bar{Z})$, it is necessary to consider the elasticity of the ribs in each section.

We will consider the ribs as thin plates in bending. For the sake of generality, we will assume that in neighboring compartments separated by an elastic rib, the level $\bar{h}$ of the liquid varies, and also that the webs of these compartments abut in the plane of the rib at different angles. In that case, it will be possible to apply Eqs. (12.68), (12.83) to the analysis of caisson structures of different sweep angles in different portions, for example, to the analysis of a swept caisson with a center section, which are separated by an elastic edge rib (see Fig. 12.3).

By satisfying the dynamic boundary conditions on the end rib, the hydrodynamic pressure on the rib may be assumed equal to $\rho\omega^2\Phi$, or, was shown in Section 2 of the present chapter, to the Lagrangian multiplier $\lambda(\xi, \eta)$, which constitutes an expansion of the function $2S$ as a series of functions $\varphi_i(\xi, \eta)$ $(i = 1, \ldots, M)$. Taking the Lagrangian multiplier as the pressure, we arrive at a symmetric system of kinematic and dynamic boundary conditions, but in this case it is necessary to determine the coefficients p_m of expansion (12.27a) on the basis of the system which follows from variational Eq. (12.25).

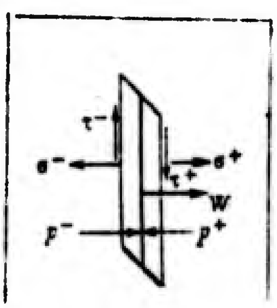


Fig. 12.5. In reference to derivation of boundary conditions of conjugation of two compartments filled with liquid, allowing for rib elasticity.

As the pressure, we will take the function $\rho\omega^2\Phi$ directly. All the quantities and functions pertaining to the left-hand compartment will be denoted by the superscript $-$, and those pertaining to the right-hand compartment, by the superscript $+$ (Fig. 12.5).

The kinematic boundary conditions of compatability of the displacements of an elastic rib and liquid in the left- and right-hand compartments may be written, on the basis of Eqs. (12.17), in the form

$$\begin{aligned}&\int_0^1\int_0^1 (W_n - w^-)\varphi_k^- d\xi d\eta^- = 0 \quad (k=1,\ldots, M),\\ &\int_0^1\int_0^1 (W_n - w^+)\varphi_k^+ d\xi d\eta^+ = 0 \quad (k=1,\ldots, M).\end{aligned} \tag{12.84}$$

At the same time, we assume that in each compartment, the displacements of the liquid are decomposed in the same functions $v_l(\xi, \eta)$ $(l=1,\ldots, M)$. Since the dimensionless coordinates $\eta = y/h$ in each of two neighboring compartments generally do not coincide because the degrees of filling h- and h+ are different, then in Eqs. (12.84) the functions $\varphi_k^- = \varphi_k(\xi, \eta^-)$ и $\varphi_k^+ = \varphi_k(\xi, \eta^+)$.

To formulate the dynamic boundary conditions on an elastic rib and calculate the concentrated generalized forces exerted by the rib on the shell, we will apply the Lagrange principle to an elementary strip of the shell, isolated together with the rib (see Fig. 12.5). We have

$$\begin{aligned}\delta U_r - \omega^2 \rho_r \delta_r \iint_S W_r \delta W_r dS_r - \iint_{\sigma^-} p^- \delta W_r d\sigma + \iint_{\sigma^+} p^+ \delta W_r d\sigma +\\ + \sum_{i=1}^{N} (P_i^- - P_i^+)\delta U_i = 0,\end{aligned} \tag{12.85}$$

where P_i^-, P_i^+ - are generalized forces (3.36) in the sections of the shell proper, to the left and right of the rib;

ρ_r, δ_r, S_r are the density, thickness and area of the rib.

The first term in (12.85) represents the variation of the potential energy of bending of the rib as a plate, the second represents the variation of the work done by the inertial forces acting directly on the rib, and the third and fourth terms represent the variation of the work of hydrodynamic pressure; finally, the last term represents the variation of the work of internal forces in the sections of the shell proper to the left and right of the rib (see Fig. 12.5.).

We represent the normal displacements of the elastic rib in the form

$$W_{n}(\xi,\eta)=\sum_{i=1}^{N} V_i^r \varphi_i(\xi,\eta)+\sum_{s=1}^{R} c_s \psi_s(\xi,\eta), \tag{12.86}$$

where $V_i^r=\lambda_i(Z_r)U_i(Z_r)$ are values of the function $V_i(Z)$ in the section $\bar{Z}=\bar{Z}_r$ along the rib; c_s are unknown coefficients; the functions $\varphi_i(\xi,\eta)$, $\psi_s(\xi,\eta)$ are chosen so that expansion (12.86) for any values of V_i^r and c_s satisfies the conditions of conjugation of the rib with the shell.

We will assume that the plate (rib) is hinged on the shell along the entire contour. In this case, it is necessary to require that each of the functions φ_i and ψ_s on the contour $\xi=\xi(S)$, $\eta=\eta(S)$ of the rib satisfy the conditions

$$\psi_s[\xi(S),\ \eta(S)]=0,$$
$$\varphi_i[\xi(S),\ \eta(S)]=\varphi_{ie_3}(S),$$

where $\varphi_{ie_3}(S)$ is the coordinate function of the shell, corresponding to displacements in the direction of the unit vector e_3, perpendicular to the plane $\bar{Z}$ = const.

Introducing expansion (12.86) into Eq. (12.85) and equating the coefficients of the variations δU_j and δC_s to zero, we obtain

$$\sum_{i=1}^{N} k_{ji}^{11} V_i^r+\sum_{s=1}^{R} k_{js}^{12} c_s-\iint_{\sigma^-} p^- \varphi_{je_3} d\sigma+\iint_{\sigma^+} p^+ \varphi_{je_3} d\sigma+ \tag{12.87}$$
$$+\frac{1}{\lambda_j(\bar{Z}_0)_r}(P_j^- - P_j^+)=0 \qquad (j=1,\ldots,N),$$
$$\sum_{i=1}^{N} k_{ri}^{12} V_i^r+\sum_{s=1}^{R} k_{rs}^{22} c_s-\iint_{\sigma^-} p^- \psi_r d\sigma+\iint_{\sigma^+} p^+ \psi_r d\sigma=0$$
$$(r=1,\ldots,R),$$

where the expressions for the coefficients k can be easily obtained by using the known expression for the potential energy of bending of plates.

Introducing the notation

$$(w_m, w_n) = D \iint_{S_m} \left\{ \nabla^2 w_m \nabla^2 w_n - (1-\nu)\left[\frac{\partial^2 w_m}{\partial x^2}\frac{\partial^2 w_n}{\partial y^2} + \frac{\partial^2 w_m}{\partial y^2}\frac{\partial^2 w_n}{\partial x^2} - 2\frac{\partial^2 w_m}{\partial x\,\partial y}\frac{\partial^2 w_n}{\partial x\,\partial y}\right]\right\} dx\,dy - \omega^2 \rho_r \delta_r \iint_{S_r} w_m w_n\,dx\,dy,$$

we have

$$k_{jl}^{11} = (\varphi_j;\ \varphi_l),\quad k_{rs}^{22} = (\psi_r;\ \psi_s),\quad k_{js}^{12} = (\varphi_j;\ \psi_s). \tag{12.88}$$

From the first group of equations of system (12.87), we determined the concentrated generalized forces

$$\Delta P_j = P_j^- - P_j^+ \qquad (j = 1, \ldots, N), \tag{12.89}$$

exerted on the shell by the elastic ribs.

The second group of equations of system (12.87) represents the equilibrium equations of the elastic rib. If the rib is considered to be absolutely rigid in bending, all the coefficients c_s must be equated to zero, and only the first six generalized displacements corresponding to the displacements of the contour $\bar{Z}$ = const as a solid should be retained in expansion (12.86).

Substituting rib displacements (12.86) and liquid displacements (12.87) into kinematic conditions (12.84), we obtain the system of equations

$$\begin{gathered}\sum_{i=1}^{N} \theta_{ki}^- V_i^0 + \sum_{s=1}^{R} \theta_{ks}^- c_s - \sum_{l=1}^{M} S_{kl}^- Z_l^- = 0 \\ (k = 1, \ldots, M), \\ \sum_{i=1}^{N} \theta_{ki}^+ V_i^0 + \sum_{s=1}^{R} \theta_{ks}^+ c_s - \sum_{l=1}^{M} S_{kl}^+ Z_l^+ = 0 \\ (k = 1, \ldots, M),\end{gathered} \tag{12.90}$$

where

$$\theta_{ki} = \int_0^1\int_0^1 \varphi_k \varphi_i \, d\xi \, d\eta; \quad \tilde{\theta}_{ks} = \int_0^1\int_0^1 \varphi_k \tilde{\varphi}_s \, d\xi \, d\eta.$$

The superscripts - and + show that all the coefficients are calculated for the compartments to the left and right of the rib respectively.

The generalized liquid displacements Z_i for different degrees of filling of neighboring compartments have different values Z_i^- and Z_i^+ to the left and right of the rib. The generalized shell displacements V_i in the section coinciding with the rib are continuous functions. If the webs of the shell in the rib section abut at different angles, the continuity of the functions V_i is ensured by a special choice of the coordinate functions $\varphi_{ti}(S)$ to the left and right of the rib.

Dynamic boundary conditions (12.87) will be written separately for a swept caisson of constant cross section and for a conical caisson.

Introducing into Eq. (12.87) the expansion for the pressure $p = \varrho\omega^2\Phi$, developed with the aid of expansion (12.49) for the potential Φ and expression (3.42) for generalized forces P_j, we obtain

$$\begin{gathered}
\sum_{i=1}^{N}\left[\left(\frac{1}{G}k_{ji}^{11} - G_{ji}^- + G_{ji}^+ + b_{ji}^- - b_{ji}^+\right)U_i + a_{ji}^- U_i^{-\prime} - a_{ji}^+ U_i^{+\prime}\right] + \\
+ \sum_{i=1}^{M}\left[H_{ji}^+ Z_i^+ + T_{ji}^+ b Z_i^{+\prime} - H_{ji}^- Z_i^- - T_{ji}^- b Z_i^{-\prime}\right] + \frac{1}{G}\sum_{s=1}^{R} k_{js}^{12} c_s = 0 \\
(j = 1, \ldots, N),
\end{gathered} \tag{12.91}$$

$$\begin{gathered}
\sum_{i=1}^{N}\left[\frac{1}{G}k_{ji}^{12} - \tilde{G}_{ri}^- + \tilde{G}_{ri}^+\right]U_i + \sum_{i=1}^{M}\left[\tilde{H}_{ri}^+ Z_i^+ + \tilde{T}_{ri}^+ b Z_i^{+\prime} - \right. \\
\left. - \tilde{H}_{ri}^- Z_i^- - \tilde{T}_{ri}^- b Z_i^{-\prime}\right] + \frac{1}{G}\sum_{s=1}^{R} k_{rs}^{22} c_r = 0 \quad (r = 1, \ldots, R).
\end{gathered}$$

In these equations

$$G_{jl} = b\Omega^2 \frac{\bar{h}}{\cos\alpha_1}\left[\sum_{n=0}^{\infty}\varepsilon_n\left(\frac{\cos\alpha_1}{\bar{h}}\varkappa_{jn}\bar{g}_{ln} - \frac{Q_{jn}\bar{t}_{ln}}{b\omega^2/g - \mu_n \operatorname{th}\mu_n\bar{h}}\right) + \right.$$

$$\left. + 2\sum_{n=1}^{\infty}\left(\nu_{jn}\bar{\gamma}^1_{ln} + \pi_{jn}\bar{\gamma}^2_{ln}\right)\right],$$

$$H_{jl} = -b\Omega^2\bar{h}\operatorname{tg}\alpha_1\sum_{n=0}^{\infty}\varepsilon_n\left(\tau_{jl,n} - \frac{Q_{jn}r_{ln}}{b\omega^2/g - \mu_n \operatorname{th}\mu_n\bar{h}}\right), \tag{12.92}$$

$$T_{jl} = b\Omega^2\frac{\bar{h}}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n\left(\sigma_{jl,n} - \frac{Q_{jn}d_{ln}}{b\omega^2/g - \mu_n \operatorname{th}\mu_n\bar{h}}\right),$$

where

$$\varkappa_{jn} = \int_0^1\int_0^1 \varphi_{jn_s}(\xi,\eta)\cos\mu_n\xi\,\frac{\operatorname{sh}\mu_n\bar{h}\eta}{\mu_n\operatorname{ch}\mu_n\bar{h}}\,d\xi d\eta,$$

$$Q_{jn} = \int_0^1\int_0^1 \varphi_{jn_s}(\xi,\eta)\cos\mu_n\xi\,\frac{\operatorname{ch}\mu_n\bar{h}(1-\eta)}{\operatorname{ch}\mu_n\bar{h}}\,d\xi d\eta,$$

$$\nu_{jn} = \int_0^1\int_0^1 \varphi_{jn_s}(\xi,\eta)\,\frac{\operatorname{ch}[(\lambda_n/\bar{h})(1-\xi)]}{\lambda_n\operatorname{sh}(\lambda_n/\bar{h})}\sin\lambda_n\eta\,d\xi d\eta, \tag{12.93}$$

$$\pi_{jn} = \int_0^1\int_0^1 \varphi_{jn_s}(\xi,\eta)\,\frac{\operatorname{ch}[(\lambda_n/\bar{h})\xi]}{\lambda_n\operatorname{sh}(\lambda_n/\bar{h})}\sin\lambda_n\eta\,d\xi d\eta,$$

$$\tau_{jl,n} = \mu_n\int_0^1\int_0^1 \varphi_{jn_s}(\xi,\eta)\cos\mu_n\xi\left[\int_0^1 G_n(\eta,\eta_1)\beta_l(\eta_1)d\eta_1\right]d\xi d\eta,$$

$$\sigma_{jl,n} = \int_0^1\int_0^1 \varphi_{jn_s}(\xi,\eta)\cos\mu_n\xi\left[\int_0^1 G_n(\eta,\eta_1)\alpha_l(\eta_1)d\eta_1\right]d\xi d\eta.$$

In Eqs. (12.91), the superscripts - and + denote the fact that the coefficients (12.92), (12.93) are calculated for the left- and right-hand compartments, respectively. The difference lies in the fact that each compartment in these formulas will have its own values of the parameters $\bar{h}^-$, α_1^- и $\bar{h}^+$, α_1^+: the dimensionless coordinates η^- and η^+ are also distinguished.

The coefficients G_{rl}, H_{rl} and T_{rl} are calculated from formulas (12.92) in the same way as the coefficients G_{jl}, H_{jl}, T_{jl}, but in (12.93), the functions $\varphi_r(\xi,\eta)$ must be substituted for the functions $\psi_r(\xi,\eta)$ in (12.93).

The simultaneous system 2M + N + R of linear algebraic equations (12.90), (12.91) is used to "sew" together the solutions of system of differential Eqs. (12.68) for each of the neighboring compartments separated by an elastic rib. If a numerical method, for example, the Runge-Kutta method, is used to solve system of differential Eqs. (12.68), system of algebriac Eqs. (12.90), (12.91) is used to determine the values of $Z_i^+, Z_i^{+\prime}$ $(i=1,\ldots, M)$ and $U_j^{+\prime}$ $(j=1,\ldots, N)$, which constitute the initial conditions for the next compartment, in terms of the values of $Z_i^-, Z_i^{-\prime}$ $(i=1,\ldots,M)$ and $U_j^{-\prime}$ $(j=1,\ldots,N)$, pertaining to the preceding compartment. Simultaneously, from system (12.90), (12.91) are determined the unknown coefficients $C_r(r = 1,\ldots, R)$, characterizing the bending of the rib in the specified coordinate functions $\psi_r(\xi, \eta)$.

For the rib bounding the last compartment, all the terms with superscript + must be equated to zero in Eqs. (12.90) and (12.91); these equations will then correspond to the boundary conditions on the free edge of the shell, bounded by an elastic rib.

Equations (12.90), (12.91) are substantially simplified if neighboring compartments have the same sweep angle $(\alpha_i^- = \alpha_i^+)$ and the same degree of filling $(h^- = h^+)$. Then, all the coefficients with superscript - are equal to the corresponding coefficients with superscript +. It follows from system (12.90) that

$$Z_i^+ = Z_i^- = Z_i,$$

so that we will have only M first equations.

Thus, general system (12.90), (12.91) for a swept caisson with a constant degree of filling reduces to a system of M + N + R equations:

$$\sum_{l=1}^{N} \theta_{kl} U_l + \sum_{s=1}^{R} \bar{\theta}_{ks} c_s - \sum_{l=1}^{M} S_{kl} Z_l = 0 \qquad (k=1,\ldots,M),$$

$$\sum_{l=1}^{N}\left[\frac{1}{G} k_{jl}^{11} U_l + a_{jl}(U_l^{-\prime} - U_l^{+\prime})\right] + \sum_{l=1}^{M} T_{jl} b (Z_l^{+\prime} - Z_l^{-\prime}) +$$

$$+ \frac{1}{G}\sum_{s=1}^{R} k_{js}^{12} c_s = 0 \quad (j=1,\ldots,N), \tag{12.94}$$

$$\frac{1}{G}\sum_{l=1}^{N} k_{jl}^{12} U_l + \sum_{l=1}^{M} \bar{T}_{rl} b (Z_l^{+\prime} - Z_l^{-\prime}) + \frac{1}{G}\sum_{s=1}^{R} k_{rs}^{22} c_s = 0$$

$$(r=1,\ldots,R).$$

From this system of equations, the values of $Z_l^{+\prime} (l=1,\ldots,M)$, $U_l^{+\prime}$ $(i=1,\ldots,N)$ and $c_r (r=1,\ldots,R)$ are determined as a function of the values of Z_l, Z_l^{-} $(l=1,\ldots,M)$ and $U_l, U_l^{-\prime}$ $(i=1,\ldots,N)$.

We will expand the dynamic boundary conditions (12.87) for a conical caisson. As in the derivation of differential Eqs. (12.83), we will neglect the influence of gravity.

Introducing into (12.87) the expression for the generalized forces P_j and pressure $p=\varrho\omega^2\Phi$, developed with the aid of the expansion for the potential Φ when g = 0, and changing to the dimensionless coordinate $\bar{Z}$ (12.81), we obtain

$$\sum_{l=1}^{N}\left[\left(\frac{1}{G} k_{jl}^{11} - G_{jl}^{-} + G_{jl}^{+} + b_{jl}^{-} - b_{jl}^{+}\right) V_l + (1-\bar{Z})(a_{jl}^{-} V_l^{-\prime} - a_{jl}^{+} V_l^{+\prime})\right] +$$

$$+ \sum_{l=1}^{M}\left[H_{jl}^{+} Z_l^{+} - H_{jl}^{-} Z_l^{-} + \frac{b_0}{l_1}(1-\bar{Z})(T_{jl}^{+} Z_l^{+\prime} - T_{jl}^{-} Z_l^{-\prime})\right] +$$

$$+ \frac{1}{G}\sum_{s=1}^{R} k_{js}^{12} c_s = 0 \qquad (j=1,\ldots,N). \tag{12.95}$$

$$\sum_{l=1}^{N}\left[\frac{1}{G} k_{jr}^{12} - \bar{G}_{rl}^{-} + \bar{G}_{rl}^{+}\right] V_l + \sum_{l=1}^{M}\left[\bar{H}_{rl}^{+} Z_l^{+} - \bar{H}_{rl}^{-} Z_l^{-} + \frac{b_0}{l_1}(1-\bar{Z}) \times\right.$$

$$\left.\times (\bar{T}_{rl}^{+} Z_l^{+\prime} - \bar{T}_{rl}^{-} Z_l^{-\prime})\right] + \frac{1}{G}\sum_{s=1}^{R} k_{rs}^{22} c_s = 0 \qquad (r=1,\ldots,R).$$

Here in contrast to Eqs. (12.91), the prime denotes the derivative with respect to the $\bar{Z}$ coordinate, and the coefficients G_{jl}, H_{jl}, T_{jl} are determined by the expressions

$$G_{jl}=b_0\Omega^2(1-\bar{Z})^3\left[\frac{1}{\cos\alpha_3}\sum_{n=0}^{\infty}\varepsilon_n\varkappa_{jn}\bar{g}_{ln}+\right.$$

$$\left.+\frac{2\bar{h}}{\cos\alpha_1}\sum_{n=1}^{\infty}\left(\nu_{jn}\bar{\gamma}_{ln}^1+\frac{\cos\alpha_1}{\cos\alpha_2}\pi_{jn}\bar{\gamma}_{ln}^2\right)\right],$$

$$H_{jl}=b_0\Omega^2(1-\bar{Z})^3\left[\operatorname{tg}\alpha_3\sum_{n=0}^{\infty}\varepsilon_n\varkappa_{jn}\alpha_{ln}(1)+\right. \tag{12.96}$$

$$\left.+2\operatorname{tg}\alpha_3\sum_{n=1}^{\infty}\pi_{jn}\gamma_{ln}(1)-\bar{h}\operatorname{tg}\alpha_1\sum_{n=0}^{\infty}\varepsilon_n\tau_{jl,n}\right],$$

$$T_{jl}=b_0\Omega^2(1-\bar{Z})^3\frac{\bar{h}}{\cos\alpha_1}\sum_{n=0}^{\infty}\varepsilon_n\sigma_{jl,n},$$

where the coefficients $\varkappa_{jn},\ \nu_{jn},\ \pi_{jn},\ \tau_{jl,n},\ \sigma_{jl,n}$ are determined from formulas (12.93).

In eqs. (12.94), (12.95), the superscripts - and + signify that coefficients (12.96), like coefficients (12.93) in Eqs. (12.91), are calculated for the left- and right-hand compartments: for the left-hand compartment, the values $\bar{h}^-,\ \alpha_1^-,\ \alpha_2^-,\ \alpha_3^-$, and for the right-hand compartment, $\bar{h}^+,\ \alpha_1^+,\ \alpha_2^+,\ \alpha_3^+$, are taken.

If the degree of filling and the angles made by the webs of the left- and right-hand compartments with the rib are the same, then in Eqs. (12.95) all the coefficients with superscript - are equal to the corresponding coefficients with superscript +. Then, only M equations remain in (12.90), and the remaining ones are replaced by the conditions $Z_l^+=Z_l^-$.

The coefficients $G_{rl},\ H_{rl}$ and T_{rl} are calculated from formulas (12.96) in the same way as the coefficients $G_{jl},\ H_{jl},\ T_{jl}$, but in this case, instead of the functions $\varphi_j(\xi,\eta)$, the functions $\psi_r(\xi,\eta)$ are substituted into (12.93).

Equations (12.95) together with Eqs. (12.90) are used in the numerical integration of a system of differential equations of vibrations of an elastic conical caisson separated by elastic ribs into compartments partly filled with liquid. From the system of linear algebraic Eqs. (12.90), (12.95) one determines the values of $V_i^{+\prime}$ $(i=1,\ldots,N)$ and Z_i^{+}, $Z_i^{\prime}$ $(i=1,\ldots,M)$ behind the elastic rib by means of the values of these functions and their derivatives in front of the rib.

From Eqs. (12.90), (12.95) there also follow the boundary conditions for the free end of a conical caisson bounded by a rib elastic to bending.

Part Three. CIRCULAR CONICAL SHELLS

Part Three is devoted to the analysis of conical shells of revolution on the basis of methods discussed in Part One.

In contrast to the problems examined in Part Four, in the analysis of shells of revolution, the number of the degrees of freedom of the cross section $\bar{Z}$ = const with respect to displacements from the plane is not limited, and therefore all the solutions obtained within the framework of the computational model are exact.

Both smooth and reinforced shells are discussed, and the presence of a reinforcing structure is considered exactly, with the revelation of the concentrations generated by it, not on the basis of "smearing out" concepts. The device of generalized functions is widely employed.

Principal attention is given to problems of concentrated action. Shells with discrete fixing conditions are examined, as well as shells subjected to the action of concentrated forces applied in the direction of the generatrices.

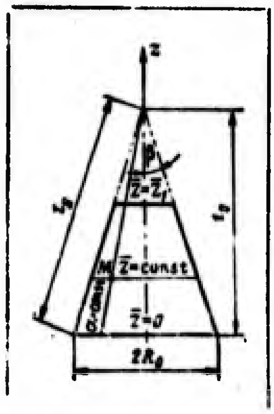
All the results obtained for conical shells are extended to cylindrical shells by means of passage to the limit.

For the conical shell of revolution shown in the figure,

$$\chi = \frac{\pi}{2}, \qquad \text{(a)}$$
$$l_1 = l_0/\cos\beta = R_0/\sin\beta.$$

In the analysis of shells of revolution, it is desirable to change from the arc coordinate S to the central angle α:

$$S = R_0 \alpha. \tag{b}$$



A - Circulat Conical Shell

Then the parametric equations of the directrix assume the form

$$x_0 = x_0(\alpha) = R_0 \cos \alpha, \quad y_0 = y_0(\alpha) = R_0 \sin \alpha. \tag{c}$$

Hence

$$\frac{dx_0}{dS} = -\sin \alpha; \quad \frac{dy_0}{dS} = \cos \alpha. \tag{d}$$

The displacement vestor U(M) of point M of the middle surface will be represented as before in the form

$$U = U^0 + U^1, \tag{e}$$

where U^0 corresponds to displacement of the contour $\bar{Z}$ = const as a solid, and U^1, to warpings of this contour.

The vector function

$$U^0(M) = \sum_{i=1}^{6} U_i(\bar{Z}) \lambda_i(\bar{Z}) \varphi_i(\alpha). \tag{f}$$

Here

$$U_1 = \eta_x, \quad U_2 = \eta_y, \quad U_3 = \eta_z, \quad U_4 = \theta_x, \quad U_5 = \theta_y, \quad U_6 = \theta_z, \tag{g}$$

where η_x, η_y, η_z are the components of translational displacement of the contour $\bar{Z}$ = const along the Ox, Oy, Oz axes;

$\theta_x, \theta_y, \theta_z$ are the components of slight rotation of the contour $\bar{Z}$ = const about the Ox, Oy, Ox axes;

where

$$\lambda_i(\bar{Z}) = (1 - \bar{Z})^{\nu_i}, \quad \nu_i = \begin{cases} 0 & (i = 1, 2, 3) \\ 1 & (i = 4, 5, 6) \end{cases}; \tag{h}$$

$\varphi_i(\alpha)$ are vectors represented by (2.15), whose components in axes of the main moving trihedron on the basis of (a)-(d) for conical shells of revolution have the form

i	U_i	φ_{im_z}	φ_{in_t}	φ_{in_n}
1	η_x	$-\sin\beta\cos\alpha$	$-\sin\alpha$	$-\cos\beta\cos\alpha$
2	η_y	$-\sin\beta\sin\alpha$	$\cos\alpha$	$-\cos\beta\sin\alpha$
3	η_z	$\cos\beta$	0	$-\sin\beta$
4	θ_x	$R_0\cos\beta\sin\alpha$	0	$-R_0\sin\beta\sin\alpha$
5	θ_y	$-R_0\cos\beta\cos\alpha$	0	$R_0\sin\beta\cos\alpha$
6	θ_z	0	R_0	0

Without limiting the number of degrees of freedom of the contour $\bar{Z}$ = const, we will represent the warping displacements in the form of the infinite expansion

$$U^1 \cdot m_z = \sum_{i=7}^{\infty} U_i(\bar{Z})\varphi_{im_z}(S). \tag{i}$$

For nonreinforced shells of revolution of constant thickness, it is natural to represent these displacements in the form of an infinite trigonometric series

$$U^1 \cdot m_z = \sum_{k=2}^{\infty}\left[\omega_{1k}(\bar{Z})\sin k\alpha + \omega_{2k}(\bar{Z})\cos k\alpha\right], \tag{j}$$

where the free term and the terms containing $\sin\alpha$, $\cos\alpha$, have been omitted in view of the fact that the warpings are associated with self-balanced internal forces.

The normal and tangential stresses, (a)-(d) being taken into account, will be determined by the expressions

$$\sigma_{m_z} = \frac{E}{R_0}\sin\beta\sum_{i=1}^{\infty}(U_i\lambda_i)'\varphi_{im_z}, \tag{k}$$

$$\tau_{n_z m_z} = \frac{G}{R}\sin\beta\sum_{i=1}^{\infty}\left[(U_i\lambda_i)'\varphi_{in_t} + U_i\lambda_i\frac{\frac{1}{\sin\beta}\varphi'_{im_z} + \varphi_{in_t}}{1-\bar{Z}}\right]. \tag{l}$$

The desired generalized displacements $U_i(\bar{Z})$ satisfy a system of ordinary differential equations resulting from the principle of possible Lagrangian displacements (Part One). In Part Three, we will proceed from the equations corresponding to the fundamental static-geometric model, but will neglect the stresses σ_{n_2} in comparison with σ_{m_1} and $\tau_{n_2 m_1}$. It is evident that for a shell of revolution, this will cause the fundamental static-geometric model and the model with the contour $\bar{Z}$ = const nondeformable in its plane to be essential identical, except for the fact that in the latter case, the modulus E should be replaced by $E/(1-\nu^2)$. Studies show that the refinement of the results on the basis of more complex Eqs. (3.224) as well as Eqs. (9.71), which describe a model with a deformable contour, is unimportant for the problems considered if the cone angle β is not too large.

Considering (a)-(d), from (3.45), (3.65) and (3.44), (3.62), (3.64) we have for both smooth shells and shells reinforced with stringers:

$$(1-\bar{Z}) \sum_{i=1}^{\infty} a_{ji} (U_i \lambda_i)' + \sum_{i=1}^{\infty} b_{ji} U_i \lambda_i = \frac{P_j(\bar{Z})}{\lambda_j G} \tag{m}$$

$$(j = 1, 2, \ldots, 6),$$

$$\left[(1-\bar{Z}) \sum_{i=1}^{\infty} a_{ji} (U_i \lambda_i)'\right]' + \sum_{i=1}^{\infty} (b_{ji} - b_{ij})(U_i \lambda_i)' - \tag{n}$$

$$-\frac{1}{1-\bar{Z}} \sum_{i=1}^{\infty} c_{ji} U_i \lambda_i = -\frac{R_j(\bar{Z})}{G} \qquad (j = 7, 8, \ldots, \infty),$$

where the coefficients

$$a_{ji} = \gamma \sin\beta \oint \varphi_{jm_2} \varphi_{im_2} h\,d\alpha + \sin\beta \int \varphi_{jm_2} \varphi_{im_2} h\,d\alpha + \tag{o}$$

$$+ \frac{\gamma \sin\beta}{R_0} \sum_{s} \varphi_{jm_2} \varphi_{im_2} \Delta F_s,$$

$$b_{ji} = \sin\beta \oint \varphi_{jm_1} \left(\frac{1}{\sin\beta} \varphi'_{im_2} + \varphi_{in_2}\right) h\,d\alpha,$$

$$c_{ji} = \sin\beta \oint \left(\frac{1}{\sin\beta} \varphi'_{jm_2} + \varphi_{jn_2}\right) \left(\frac{1}{\sin\beta} \varphi'_{im_2} + \varphi_{in_2}\right) h\,d\alpha.$$

and the load terms

$$R_j(\bar{Z}) = l_0^2 \frac{\sin\beta}{\cos^2\beta} \lambda_j (1-\bar{Z}) \oint (p_{n_n}\varphi_{jn_n} + p_{n_z}\varphi_{jn_z} + p_{m_z}\varphi_{jm_z})\, d\alpha,$$

$$\bar{P}_j(\bar{Z}) = \bar{P}_j^e - \int_{\bar{Z}_e}^{\bar{Z}} R_j(\bar{Z})\, d\bar{Z} \qquad (j = 1, 2, 3, 6), \tag{p}$$

$$\bar{P}_{\theta_x}(\bar{Z}) = \bar{P}_{\theta_x}^e + \int_{\bar{Z}_e}^{\bar{Z}} [l_0 \bar{P}_{\theta_y}(\bar{Z}) - R_{\theta_x}(\bar{Z})]\, d\bar{Z},$$

$$\bar{P}_{\theta_y}(\bar{Z}) = \bar{P}_{\theta_y}^e - \int_{\bar{Z}_e}^{\bar{Z}} [l_0 \bar{P}_{\theta_x}(\bar{Z}) + R_{\theta_y}(\bar{Z})]\, d\bar{Z},$$

where $\bar{P}_j^e$ are the generalized forces given in the end section of the shell, $\bar{Z} = \bar{Z}_e$, and

$p_{n_n}, p_{m_z}, p_{n_z}$ are the components of the surface load.

In expressions (o) and hereinafter, the following notation is used:

$$\gamma = \frac{2}{1-\nu}. \tag{q}$$

Chapter XIII. CONICAL SHELL UNDER DISCRETE FIXING CONDITIONS

The analysis of discretely fixed shells is of major practical interest. This problem is encountered in the design of various thin-walled structures and, in the case of a shell of revolution, is very essential in the development of butt joints of airframe compartments, which are cylindrical and slightly conical shells of revolution.

The joining of both smooth and reinforced shell compartments is usually accomplished by means of an end ring, whose structural design determines the model of the butt, discrete with respect to displacements along the shell generatrices and continuous with respect to transverse displacements. The end ring also ensures the nondeformability of the cross section in its plane, even in the absence of an intermediate set of rings, since the external load on the air frame is smooth in character. The problem consists in studying the deviations from the law of plane sections due to discrete fixing.

The solutions for stringerless shells given in the present chapter* and analogous solutions for stringer shells given in Chapter XV may be considered rigorous if the butt joints in the direction of the generatrices are discrete not only in tension, but also in compression. If however the joining of the compartments in the compressed zone is achieved through

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*G. G. Onanov. Bending of a Cyclically Fixed Circular Cylindrical Shell. In: Strength and Stability of Thin-Walled Aeronautical Structures, edited by I. F. Obraztsov. Trudy MAI, No. 180. Moscow, Mashinostroyeniye. 1970.

some contact portion, these solutions are provisional and can be used only for tentative calculations of the zone in tension. In the rigorous formulation, such a problem reduces to a nonlinear integral equation in the reactive end forces. The kernel of this equation is readily obtained from the closed solution presented in this chapter.

13.1. Equilibrium Equations of a Circular Conical Shell. General Solution

Assuming the external load to be arbitrary, in expansion (f) we will retain all six components of displacement of the contour $\bar{Z}$ = const as a solid, and in expansion (j), the generalized displacements ω_{1k} and ω_{2k}.

Using formulas (o), we will calculate the coefficients of the system of differential resolvents (m), (n).

The coefficients a_{ji}, b_{ji}, c_{ji} and i, j = 1, 2,..., are given in Tables 13.1-13.3.

The coefficients a_{ji} and c_{ji} for $j > 6$ or $i > 6$ are, by virtue of the orthogonality of the trigonometric functions, equal to zero if $j \neq i$.

For i = j = 5+k, in view of (j), we have

$$a_{jj} = \pi h \gamma \sin \beta, \tag{13.1}$$

$$c_{jj} = \pi h \frac{k^2}{\sin \beta}. \tag{13.2}$$

The coefficients b_{ji} for $i > 6$ or $j > 6$ are always equal to zero.

After the substitution of derivatives $1 - \bar{Z} = \rho'$, Eqs. (m), (n) in the desired generalized displacements will assume the form

Table 13.1 Coefficients a_{ji}.

j \ i	1	2	3	4	5	6
1	$\pi h \sin\beta\,(1+\gamma\sin^2\beta)$	0	0	0	$\pi h\gamma R_0 \sin^2\beta\cos\beta$	0
2	0	$\pi h \sin\beta\,(1+\gamma\sin^2\beta)$	0	$-\pi h\gamma R_0\sin^2\beta\cos\beta$	0	0
3	0	0	$2\pi h\gamma\sin\beta\cos^2\beta$	0	0	0
4	0	$-\pi h\gamma R_0\sin^2\beta\cos\beta$	0	$\pi h\gamma R_0^2\sin\beta\cos^2\beta$	0	0
5	$\pi h\gamma R_0\sin^2\beta\cos\beta$	0	0	0	$\pi h\gamma R_0^2\sin\beta\cos^2\beta$	0
6	0	0	0	0	0	$2\pi h R_0^2\sin\beta$

Table 13.2 Coefficients b_{ji}.

j \ i	1, 2, 3	4	5	6
1	0	0	$-\pi h R_0\cos\beta$	0
2	0	$\pi h R_0\cos\beta$	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	$2\pi h R_0^2\sin\beta$

Table 13.3 Coefficients c_{ji}.

j \ i	1, 2, 3	4	5	6
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	$\pi h\,\dfrac{R_0^2\cos^2\beta}{\sin\beta}$	0	0
5	0	0	$\pi h\,\dfrac{R_0^2\cos^2\beta}{\sin\beta}$	0
6	0	0	0	$2\pi h R_0^2\sin\beta$

$$-a_{11}\eta_x' - a_{18}(e^t\theta_y)' + b_{18}e^t\theta_y = \frac{Q_x(t)}{G}, \tag{13.3}$$

$$-a_{22}\eta_y' - a_{26}(e^t\theta_x)' + b_{26}e^t\theta_x = \frac{Q_y(t)}{G}, \tag{13.4}$$

$$-a_{33}\eta_z' = \frac{N_z(t)}{G}, \tag{13.5}$$

$$-a_{62}\eta_y' - a_{66}(e^t\theta_x)' = \frac{M_x(t)e^{-t}}{G}, \tag{13.6}$$

$$-a_{81}\eta_x' - a_{88}(e^t\theta_y)' = \frac{M_y(t)e^{-t}}{G}, \tag{13.7}$$

$$-a_{99}(e^t\theta_z)' + b_{99}e^t\theta_z = \frac{M_z(t)e^{-t}}{G}, \tag{13.8}$$

$$a_{kk}\dot{\omega}_{1k} - c_{kk}\omega_{1k} = -\frac{R_{1k}(t)e^t}{G} \quad (k=2,\ 3,\ldots), \tag{13.9}$$

$$a_{kk}\dot{\omega}_{2k} - c_{kk}\omega_{2k} = -\frac{R_{2k}(t)e^t}{G} \quad (k=2,\ 3,\ldots), \tag{13.10}$$

where

$$Q_x = \bar{P}_{\eta_x}, \quad Q_y = \bar{P}_{\eta_y}, \quad N_z = \bar{P}_{\eta_z} \tag{13.11}$$

are the transverse forces and axial force;

$$M_x = \bar{P}_{\theta_x}, \quad M_y = \bar{P}_{\theta_y}, \quad M_z = \bar{P}_{\theta_z} \tag{13.12}$$

are the bending moments and twisting moment;

the linear bimoments determined according to (p) from the formulas

$$\begin{aligned} R_{1k} &= l_0^2 \frac{\sin\beta}{\cos^2\beta} e^t \oint p_{m_z}(t, \alpha) \sin k\alpha d\alpha, \\ R_{2k} &= l_0^2 \frac{\sin\beta}{\cos^2\beta} e^t \oint p_{m_z}(t, \alpha) \cos k\alpha d\alpha. \end{aligned} \tag{13.13}$$

Thus, the infinite system of differential equations in the generalized displacements has been decomposed almost completely. An exception are only coupled Eqs. (13.3), (13.7) and (13.4), (13.6).

Having eliminated η_y', from Eqs. (13.4), (13.5) and η_x', we obtain two independent equations in θ_x, θ_y

$$(a_{66}a_{22} - a_{26}a_{62})(e^t\theta_x)' + a_{62}b_{26}e^t\theta_x = a_{62}\frac{Q_y}{G} - a_{22}\frac{M_x e^{-t}}{G}, \tag{13.14}$$

$$(a_{11}a_{88} - a_{18}a_{81})(e^t\theta_y)' + a_{81}b_{18}e^t\theta_y = a_{81}\frac{Q_x}{G} - a_{11}\frac{M_y e^{-t}}{G}. \tag{13.15}$$

Integrating Eqs. (13.5), (13.8), (13.14), (13.15) then Eqs. (13.6), (13.7) for the $\theta_x(t), \theta_y(t)$ obtained, we get the general solution of system

(13.3)-(13.8). Considering the values of the coefficients listed in Table 13.1-13.3 after some transformations we obtain

$$\eta_x(t)=\eta_x^0-l_0\theta_y^0(e^t-1)-\frac{1}{a_{15}G}\left\{e^t\int_0^t[\gamma\sin^2\beta(l_0e^{\xi}Q_x(\xi)-M_y(\xi))-\right.$$
$$\left.-M_y(\xi)]e^{-2\xi}d\xi+\int_0^t M_y(\xi)e^{-\xi}d\xi\right\}.\tag{13.16}$$

$$\eta_y(t)=\eta_y^0+l_0\theta_x^0(e^t-1)+\frac{1}{a_{24}G}\left\{e^t\int_0^t[\gamma\sin^2\beta(l_0e^{\xi}Q_y(\xi)+M_x(\xi))+\right.$$
$$\left.+M_x(\xi)]e^{-2\xi}d\xi-\int_0^t M_x(\xi)e^{-\xi}d\xi\right\}.\tag{13.17}$$

$$\eta_z(t)=\eta_z^0-\frac{1}{a_{33}G}\int_0^t N_z(\xi)d\xi.\tag{13.18}$$

$$\theta_x(t)=\theta_x^0-\frac{1}{a_{44}G}\int_0^t[\gamma\sin^2\beta(l_0e^{\xi}Q_y(\xi)+M_x(\xi))-M_x(\xi)]e^{-2\xi}d\xi.\tag{13.19}$$

$$\theta_y(t)=\theta_y^0+\frac{1}{a_{55}G}\int_0^t[\gamma\sin^2\beta(l_0e^{\xi}Q_x(\xi)-M_y(\xi))-M_y(\xi)]e^{-2\xi}d\xi.\tag{13.20}$$

$$\theta_z(t)=\theta_z^0-\frac{1}{a_{66}G}\int_0^t M_z(\xi)e^{-2\xi}d\xi.\tag{13.21}$$

where $\eta_x^0, \eta_y^0, \eta_z^0, \theta_x^0, \theta_y^0, \theta_z^0$ are the components of translational displacement and rotation of the end section of the shell (t = 0).

Integrating differential Eqs. (13.9), (13.10) by the method of variation of arbitrary constants, we obtain

$$\omega_{1k}(t)=C_{1k}e^{k\bar t}+\bar C_{1k}e^{-k\bar t}+\frac{\sqrt{\gamma}\sin\beta}{ka_{kk}G}\int_0^t \operatorname{sh}k(\bar\xi-\bar t)R_{1k}(\xi)e^{\xi}d\xi.\tag{13.22}$$

$$\omega_{2k}(t)=C_{2k}e^{k\bar t}+\bar C_{2k}e^{-k\bar t}+\frac{\sqrt{\gamma}\sin\beta}{ka_{kk}G}\int_0^t \operatorname{sh}k(\bar\xi-\bar t)R_{2k}(\xi)e^{\xi}d\xi.\tag{13.23}$$

where

$$\bar t=\frac{t}{\sqrt{\gamma}\sin\beta},\quad \bar\xi=\frac{\xi}{\sqrt{\gamma}\sin\beta}\tag{13.24}$$

are relative coordinates;

$C_{1k}, \bar C_{1k}, C_{2k}, \bar C_{2k}$ are arbitrary constants.

Expressions (13.6) - (13.23) represent the general solution of the system of differential resolvents (13.3) - (13.10). The arbitrary constants are determined from the conditions on the shell ends $t = 0$, $t = t_k$. If the end $t = 0$ is fixed over the entire contour, it is necessary to set

$$\eta_x^0 = \eta_y^0 = \eta_z^0 = \theta_x^0 = \theta_y^0 = \theta_z^0 = 0,$$

and also

$$\omega_{1k}(0) = \omega_{2k}(0) = 0,$$

whence, using (13.22), (13.23), we obtain

$$\begin{aligned} C_{1k} + \bar{C}_{1k} = 0, \\ C_{2k} + \bar{C}_{2k} = 0. \end{aligned} \tag{13.25}$$

If a given external load is applied to the end $t = 0$, the equations in arbitrary constants $\bar{C}_{1k}, C_{1k}, C_{2k}, \bar{C}_{2k}$ should be obtained from the conditions of equality of the expansion coefficients of the external load T_{m_z} to the corresponding expansion coefficients of internal forces $\sigma_{m_z}(0, \alpha)h$.

Similarly satisfying the conditions of loading of the second end $t = t_1$, we obtain a complete system of equations for determining the arbitrary constants.

Let us note that when $t_1 \to -\infty$, which corresponds to a nontruncated conical shell, the generalized displacements ω_{1k}, ω_{2k}, represented by expressions (13.22), (13.23) for arbitrary values of the integration constants increase indefinitely. In this case, on the basis of considerations of boundedness of the displacements at the apex of the nontruncated shell, it is necessary to impose certain requirements on the arbitrary constants. This can be conveniently done by representing ω_{1k}, ω_{2k} as follows:

$$w_{1k}(t)=C'_{1k}e^{kt}+\overline{C}'_{1k}e^{-kt}+\frac{\sqrt{\gamma}\sin\beta}{ka_{kk}G}\int_{t_1}^{t}\operatorname{sh}k(\xi-t)\,R_{1k}(\xi)\,e^{\xi}d\xi, \tag{13.26}$$

$$w_{2k}(t)=C'_{2k}e^{kt}+\overline{C}'_{2k}e^{-kt}+\frac{\sqrt{\gamma}\sin\beta}{ka_{kk}G}\int_{t_1}^{t}\operatorname{sh}k(\xi-t)\,R_{2k}(\xi)\,e^{\xi}d\xi. \tag{13.27}$$

Now, as can be seen from (13.13), at a sufficient rate of attentuation of the component $p_{m}(t,\alpha)$ of the external load in the direction of the t coordinate, commensurate with the degree of its variability in the circumferential direction, the improper integrals entering into (13.26), (13.27) will converge. We will therefore satisfy the requirement of boundedness of the displacements at the apex by setting in (13.26), (13.27).

$$\overline{C}'_{1k}=0,\quad \overline{C}'_{2k}=0. \tag{13.28}$$

The arbitrary constants in expressions (13.22), (13.23) are related to the constants in expressions (13.26), (13.27) as follows:

$$\begin{gathered}
C_k=C'_k+\frac{\sqrt{\gamma}\sin\beta}{2ka_{kk}G}\int_{0}^{t_1}R_k(\xi)e^{-k\xi}d\xi,\\
\overline{C}_k=\overline{C}'_k-\frac{\sqrt{\gamma}\sin\beta}{2ka_{kk}G}\int_{0}^{t_1}R_k(\xi)e^{\xi+k\xi}d\xi.
\end{gathered} \tag{13.29}$$

Introducing (13.28) into (13.29), we obtain the values of $\overline{C}_{1k}$, $\overline{C}_{2k}$ corresponding to the condition of boundedness of the displacement at the apex of a nontruncated conical shell.

It should be noted that in studying the stressed state of not too short truncated conical shells near the end t = 0, one can set $\overline{C}'_{1k}=\overline{C}'_{2k}=0$ with a high degree of accuracy in (13.26), (13.27), since the mutual influence of the ends is insignificant in this case.

Developing expansions (f) and (j) with the aid of the expressions obtained from (13.16) - (13.23) and formulas shown in the table on p. 453, one can determine the displacement of an arbitrary point of the shell in an arbitrary direction. After some simple transformations, we obtain

$$
\begin{aligned}
u_{m_z}(l,\alpha) = & -\left[\eta_x^0 + \theta_y^0 R_0 \operatorname{ctg}\beta - \frac{1}{a_{15}G}\int_0^l M_y(\xi)e^{-\xi}d\xi\right]\sin\beta\cos\alpha - \\
& -\left[\eta_y^0 - \theta_x^0 R_0 \operatorname{ctg}\beta - \frac{1}{a_{24}G}\int_0^l M_x(\xi)e^{-\xi}d\xi\right]\sin\beta\sin\alpha + \\
& +\left[\eta_z^0 - \frac{1}{a_{33}G}\int_0^l N_z(\xi)d\xi\right]\cos\beta + \\
& +\sum_{k=2}^{\infty}\left\{\sin k\alpha\left[C_{1k}e^{kl} + \bar{C}_{1k}e^{-kl} + \frac{\sqrt{\gamma}\sin\beta}{k a_{kk}G}\int_0^l \operatorname{sh} k(\bar{\xi}-\bar{l})\, R_{1k}(\xi)e^{\xi}d\xi\right] + \right. \\
& \left. +\cos k\alpha\left[C_{2k}e^{kl} + \bar{C}_{2k}e^{-kl} + \frac{\sqrt{\gamma}\sin\beta}{k a_{kk}G}\int_0^l \operatorname{sh} k(\bar{\xi}-\bar{l})\, R_{2k}(\xi)e^{\xi}d\xi\right]\right\}. \qquad (13.30)
\end{aligned}
$$

$$
\begin{aligned}
u_{s_z}(l,\alpha) = & -\left\{\eta_x^0 - \theta_y^0 R_0 \operatorname{ctg}\beta\,(e^l - 1) - \frac{1}{a_{15}G}\left[e^l\int_0^l (\gamma\sin^2\beta\,[l_0 e^{\xi}Q_x(\xi) - M_y(\xi)] - \right.\right. \\
& \left.\left. - M_y(\xi))\,e^{-2\xi}d\xi + \int_0^l M_y(\xi)e^{-\xi}d\xi\right]\right\}\sin\alpha + \\
& +\left\{\eta_y^0 + \theta_x^0 R_0 \operatorname{ctg}\beta\,(e^l - 1) + \frac{1}{a_{24}G}\left[e^l\int_0^l (\gamma\sin^2\beta\,[l_0 e^{\xi}Q_y(\xi) + M_x(\xi)] + \right.\right. \\
& \left.\left. + M_x(\xi))\,e^{-2\xi}d\xi - \int_0^l M_x(\xi)e^{-\xi}d\xi\right]\right\}\cos\alpha + \\
& +\left[\theta_z^0 - \frac{1}{a_{66}G}\int_0^l M_z(\xi)e^{-2\xi}d\xi\right]R_0. \qquad (13.31)
\end{aligned}
$$

$$
\begin{aligned}
u_{n_n}(l,\alpha) = & \left\{-\eta_x^0\cos\beta + \theta_y^0 R_0 \frac{e^l - \cos^2\beta}{\sin\beta} + \frac{1}{a_{15}G}\left[\frac{e^l}{\cos\beta}\int_0^l (\gamma\sin^2\beta\,[l_0 e^{\xi}Q_x(\xi) - \right.\right. \\
& \left.\left. - M_y(\xi)] - M_y(\xi))\,e^{-2\xi}d\xi + \cos\beta\int_0^l M_y(\xi)e^{-\xi}d\xi\right]\right\}\cos\alpha + \\
& +\left\{-\eta_y^0\cos\beta - \theta_x^0 R_0 \frac{e^l - \cos^2\beta}{\sin\beta} - \frac{1}{a_{24}G}\left[\frac{e^l}{\cos\beta}\int_0^l (\gamma\sin^2\beta\,[l_0 e^{\xi}Q_y(\xi) + \right.\right. \\
& \left.\left. + M_x(\xi)] + M_x(\xi))\,e^{-2\xi}d\xi - \cos\beta\int_0^l M_x(\xi)e^{-\xi}d\xi\right]\right\}\sin\alpha - \\
& -\left[\eta_z^0 - \frac{1}{a_{33}G}\int_0^l N_z(\xi)d\xi\right]\sin\beta. \qquad (13.32)
\end{aligned}
$$

The stresses are represented by formulas (k) and (l). Using relations (13.30) and (13.31), we obtain

$$\sigma_{m_z}(t,\alpha)=\frac{N_z(t)\,e^{-t}}{2\pi R_0 h\cos\beta}+\frac{M_x(t)\,e^{-2t}}{\pi h R_0^2\cos\beta}\sin\alpha-\frac{M_y(t)\,e^{-2t}}{\pi h R_0^2\cos\beta}\cos\alpha-$$

$$-\frac{E}{\sqrt{\gamma}R_0}e^{-t}\sum_{k=2}^{\infty}\Big\{k\left[\left(C_{1k}e^{k\bar{t}}-\bar{C}_{1k}e^{-k\bar{t}}\right)\sin k\alpha+\left(C_{2k}e^{k\bar{t}}-\bar{C}_{2k}e^{-k\bar{t}}\right)\cos k\alpha\right]-$$

$$-\frac{\sqrt{\gamma}\sin\beta}{a_{kk}G}\int_0^t\left[R_{1k}(\xi)\sin k\alpha+R_{2k}(\xi)\cos k\alpha\right]e^{\xi}\operatorname{ch}k(\bar{\xi}-\bar{t})\,d\xi\Big\}, \tag{13.33}$$

$$\tau_{n_z\tau_z}(t,\alpha)=\frac{e^{-2t}}{\pi h R_0^2\operatorname{ctg}\beta}\Big\{\left[-l_0e^tQ_x(t)+M_y(t)\right]\sin\alpha+$$

$$+\left[l_0e^tQ_y(t)+M_x(t)\right]\cos\alpha\Big\}+\frac{e^{-2t}}{2\pi R_0^2h}M_z(t)+$$

$$+\frac{G}{R_0}e^{-t}\sum_{k=2}^{\infty}\Big\{k\left[\left(C_{1k}e^{k\bar{t}}+\bar{C}_{1k}e^{-k\bar{t}}\right)\cos k\alpha-\left(C_{2k}e^{k\bar{t}}+\bar{C}_{2k}e^{-k\bar{t}}\right)\sin k\alpha\right]+$$

$$+\frac{\sqrt{\gamma}\sin\beta}{a_{kk}G}\int_0^t\left[R_{1k}(\xi)\cos k\alpha-R_{2k}(\xi)\sin k\alpha\right]e^{\xi}\operatorname{sh}k(\bar{\xi}-\bar{t})\,d\xi\Big\}. \tag{13.34}$$

13.2. Boundary Conditions for Discrete Fixing

Let the shell end t = 0 be fixed completely with respect to tangential displacements and in the direction of the generatrices, only at an arbitrary number n of equidistant points (Fig. 13.1).

In support to satisfy the point boundary conditions, we will fictitiously disregard the supports preventing longitudinal displacements, and replace their action by unknown reactions. Considering the reactions to be given, we will examine the corresponding static boundary conditions.

Let P_m be the reaction at the mth point with the coordinate

$$\alpha_m=\frac{2\pi}{n}m\qquad (m=1,\,2,\ldots,\,n).$$

We will assume for the time being that each of the reactions is uniformly distributed over the segment $\Delta S=2\varepsilon$ of the contour t = 0. Then the contour linear load corresponding to these reactions

$$\overline{T}(\alpha)=\begin{cases}\frac{P_m}{2\varepsilon}, & \frac{2\pi}{n}m-\frac{\varepsilon}{R_0}\leqslant\alpha\leqslant\frac{2\pi}{n}m+\frac{\varepsilon}{R_0},\\ 0, & \frac{2\pi}{n}(m-1)+\frac{\varepsilon}{R_0}<\alpha<\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}\end{cases} \tag{13.35}$$

$$(m=1,2,\ldots,n).$$

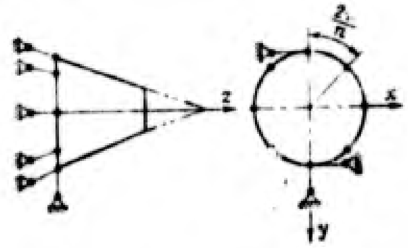


Fig. 13.1. Cyclically fixed shell.

We will represent this load in the form of the trigonometric series

$$\overline{T}(\alpha)=\frac{a_0}{2}+\sum_{k=1}^{\infty}[a_k\cos k\alpha+b_k\sin k\alpha], \tag{13.36}$$

where the coefficients

$$a_k=\frac{1}{\pi}\int_0^{2\pi}\overline{T}(\alpha)\cos k\alpha d\alpha,\quad b_k=\frac{1}{\pi}\int_0^{2\pi}\overline{T}(\alpha)\sin k\alpha d\alpha.$$

Using (13.35), we have

$$a_k=\frac{1}{\pi}\sum_{m=1}^{n}\int_{\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}}^{\frac{2\pi}{n}m+\frac{\varepsilon}{R_0}}\frac{P_m}{2\varepsilon}\cos k\alpha d\alpha=\frac{1}{2\pi\varepsilon}\sum_{m=1}^{n}P_m\int_{\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}}^{\frac{2\pi}{n}m+\frac{\varepsilon}{R_0}}\cos k\alpha d\alpha=$$

$$=\frac{1}{2\pi\varepsilon}\sum_{m=1}^{n}P_m\frac{\sin k\alpha}{k}\Bigg|_{\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}}^{\frac{2\pi}{n}m+\frac{\varepsilon}{R_0}}=\frac{1}{k\pi\varepsilon}\sin\frac{k\varepsilon}{R_0}\sum_{m=1}^{n}P_m\cos\frac{2\pi m}{n}k, \tag{13.37}$$

$$b_k=\frac{1}{\pi}\sum_{m=1}^{n}\int_{\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}}^{\frac{2\pi}{n}m+\frac{\varepsilon}{R_0}}\frac{P_m}{2\varepsilon}\sin k\alpha d\alpha=\frac{1}{2\pi\varepsilon}\sum_{m=1}^{n}P_m\int_{\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}}^{\frac{2\pi}{n}m+\frac{\varepsilon}{R_0}}\sin k\alpha d\alpha=$$

$$=-\frac{1}{2\pi\varepsilon}\sum_{m=1}^{n}P_m\frac{\cos k\alpha}{k}\Bigg|_{\frac{2\pi}{n}m-\frac{\varepsilon}{R_0}}^{\frac{2\pi}{n}m+\frac{\varepsilon}{R_0}}=\frac{1}{k\pi\varepsilon}\sin\frac{k\varepsilon}{R_0}\sum_{m=1}^{n}P_m\sin\frac{2\pi m}{n}k. \tag{13.38}$$

Let us note that when $k = 0$, expression (13.37) contains an indeterminacy. Directly calculating a_0 or revealing the indeterminacy, we find

$$a_0 = \frac{1}{\pi R_1} \sum_{m=1}^{n} P_m. \tag{13.39}$$

Expansion (13.36) with coefficients (13.37) and (13.38) represents a system of external forces $\bar{T}(\alpha)$ applied to the end $t = 0$ in the direction of the generatrices. The corresponding internal forces

$$T(\alpha) = \sigma_{m_t}(0, \alpha) h.$$

Setting $t = 0$ in (13.33), we obtain

$$T(0, \alpha) = \frac{N_t(0)}{2\pi R_0 \cos\beta} + \frac{M_1(0)}{\pi R_0^2 \cos\beta}\sin\alpha - \frac{M_\nu(0)}{\pi R_0^2 \cos\beta}\cos\alpha -$$
$$- \frac{Eh}{\sqrt{\gamma} R_0} \sum_{k=2}^{\infty} k\left[(C_{1k} - \bar{C}_{1k})\sin k\alpha + (C_{2k} - \bar{C}_{2k})\cos k\alpha\right]. \tag{13.40}$$

From the condition

$$T(0, \alpha) = \bar{T}(\alpha),$$

comparing the coefficients (13.37), (13.38), (13.39) of expansion (13.36) with the corresponding coefficients of expression (13.40), we obtain

$$\sum_{m=1}^{n} P_m = \frac{N_t(0)}{\cos\beta}, \tag{13.41}$$

$$\sum_{m=1}^{n} P_m \sin\frac{2\pi}{n} m = \frac{M_1(0)}{R_0 \cos\beta}\,\frac{\epsilon/R_0}{\sin\epsilon/R_0}, \tag{13.42}$$

$$\sum_{m=1}^{n} P_m \cos\frac{2\pi}{n} m = -\frac{M_\nu(0)}{R_0 \cos\beta}\,\frac{\epsilon/R_0}{\cos\epsilon/R_0}, \tag{13.43}$$

. . . .

$$\sum_{m=1}^{n} P_m \cos\frac{2\pi k}{n} m = -\frac{Eh\pi}{\sqrt{\gamma}} k \frac{k\epsilon/R_0}{\sin k\epsilon/R_0}(C_{2k} - \bar{C}_{2k}), \tag{13.44}$$

$$\sum_{m=1}^{n} P_m \sin\frac{2\pi k}{n} m = -\frac{Eh\pi}{\sqrt{\gamma}} k \frac{k\epsilon/R_0}{\sin k\epsilon/R_0}(C_{1k} - \bar{C}_{1k}), \tag{13.45}$$

. . . .

$$(k = 2, 3, \ldots).$$

Let $\bar{\bar{T}}(\alpha)$ be a system of external forces on the end $t = t_1$, applied in the direction of the generatrices. Equating the generalized forces corresponding to the external forces with the generalized forces corresponding to the internal forces, or, what amounts to the same thing, equating the coefficients of expansions of external and internal forces on $\sin k\alpha$, $\cos k\alpha$, for each value of k we can obtain two equations in the corresponding arbitrary constants. These equations will have the form

$$
\begin{aligned}
&-\frac{Eh\pi}{\sqrt{\gamma}R_0} k e^{-t_1} \left(C_{1k} e^{k\bar{t}_1} - \bar{C}_{1k} e^{-k\bar{t}_1}\right) = \\
&= -\frac{Eh\pi}{R_0} \frac{e^{-t_1} \sin\beta}{a_{kk} G} \int_0^{t_1} \operatorname{ch} k(\bar{\xi} - \bar{t}_1) R_{1k}(\xi) e^{\xi} d\xi + \\
&+ \int_0^{2\pi} \bar{\bar{T}}(\alpha) \sin k\alpha d\alpha \quad (k = 2, 3, \ldots),
\end{aligned}
\tag{13.46}
$$

$$
\begin{aligned}
&-\frac{Eh\pi}{\sqrt{\gamma}R_0} k e^{-t_1} \left(C_{2k} e^{k\bar{t}_1} - \bar{C}_{2k} e^{-k\bar{t}_1}\right) = \\
&= -\frac{Eh\pi}{R_0} \frac{e^{-t_1} \sin\beta}{a_{kk} G} \int_0^{t_1} \operatorname{ch} k(\bar{\xi} - \bar{t}_1) R_{2k}(\xi) e^{\xi} d\xi + \\
&+ \int_0^{2\pi} \bar{\bar{T}}(\alpha) \cos k\alpha d\alpha \quad (k = 2, 3, \ldots).
\end{aligned}
\tag{13.47}
$$

Solving system (13.44) - (13.47) for the integration constants, we obtain

$$
\begin{aligned}
C_{1k} &= \frac{1}{2 \operatorname{sh} k\bar{t}_1} \left(\frac{\sqrt{\gamma}}{Eh\pi} \frac{1}{k} \frac{\sin k\varepsilon/R_0}{k\varepsilon/R_0} e^{-k\bar{t}_1} \sum_{m=1}^{n} P_m \sin \frac{2\pi k}{r} m + L_{1k} \right), \\
\bar{C}_{1k} &= \frac{1}{2 \operatorname{sh} k\bar{t}_1} \left(\frac{\sqrt{\gamma}}{Eh\pi} \frac{1}{k} \frac{\sin k\varepsilon/R_0}{k\varepsilon/R_0} e^{k\bar{t}_1} \sum_{m=1}^{n} P_m \sin \frac{2\pi k}{n} m + L_{1k} \right), \\
C_{2k} &= \frac{1}{2 \operatorname{sh} k\bar{t}_1} \left(\frac{\sqrt{\gamma}}{Eh\pi} \frac{1}{k} \frac{\sin k\varepsilon/R_0}{k\varepsilon/R_0} e^{-k\bar{t}_1} \sum_{m=1}^{n} P_m \cos \frac{2\pi k}{n} n + L_{2k} \right), \\
\bar{C}_{2k} &= \frac{1}{2 \operatorname{sh} k\bar{t}_1} \left(\frac{\sqrt{\gamma}}{Eh\pi} \frac{1}{k} \frac{\sin k\varepsilon/R_0}{k\varepsilon/R_0} e^{k\bar{t}_1} \sum_{m=1}^{n} P_m \cos \frac{2\pi k}{n} m + L_{2k} \right),
\end{aligned}
\tag{13.48}
$$

where

$$L_{1k} = -\frac{\sqrt{\gamma}R_0}{Eh\pi}\frac{1}{k}e^{l_1}\int_0^{2\pi}\bar{\bar{T}}(\alpha)\sin k\alpha\,d\alpha +$$

$$+\frac{\sin\beta}{a_{33}G}\frac{\sqrt{\gamma}}{k}\int_0^{l_1}\operatorname{ch}k(\bar{\xi}-\bar{l})R_{1k}(\xi)e^{\xi}d\xi,$$

$$L_{2k} = -\frac{\sqrt{\gamma}R_0}{Eh\pi}\frac{1}{k}e^{l_1}\int_0^{2\pi}\bar{\bar{T}}(\alpha)\cos k\alpha\,d\alpha + \tag{13.49}$$

$$+\frac{\sin\beta}{a_{33}G}\frac{\sqrt{\gamma}}{k}\int_0^{l_1}\operatorname{ch}k(\bar{\xi}-\bar{l})R_{2k}(\xi)e^{\xi}d\xi.$$

In relations (13.48), the problem of determination of the integration constants reduces to finding n unknown forces P_m. These forces must be determined from the fixing conditions of the end section.

We set t = 0 in (13.30):

$$u_{m_z}(0,\alpha) = -(\eta_x^0+\theta_y^0 R_0\operatorname{ctg}\beta)\sin\beta\cos\alpha -$$
$$-(\eta_y^0-\theta_x^0 R_0\operatorname{ctg}\beta)\sin\beta\sin\alpha+\eta_z^0\cos\beta+$$
$$+\sum_{k=2}^{\infty}[(C_{1k}+\bar{C}_{1k})\sin k\alpha+(C_{2k}+\bar{C}_{2k})\cos k\alpha]. \tag{13.50}$$

Substituting expressions (13.48) into (13.50), we obtain

$$u_{m_z}(0,\alpha) = -(\eta_x^0+\theta_y^0 R_0\operatorname{ctg}\beta)\sin\beta\cos\alpha -$$
$$-(\eta_y^0-\theta_x^0 R_0\operatorname{ctg}\beta)\sin\beta\sin\alpha+\eta_z^0\cos\beta+$$

$$+\sum_{k=2}^{\infty}\frac{1}{\operatorname{sh}k\bar{l}_1}\left\{\frac{\sqrt{\gamma}}{Eh\pi}\frac{1}{k}\frac{\sin k\epsilon/R_0}{k\epsilon/R_0}\operatorname{ch}k\bar{l}_1\left[\sin k\alpha\sum_{m=1}^{n}P_m\sin\frac{2\pi k}{n}m+\right.\right. \tag{13.51}$$
$$\left.\left.+\cos k\alpha\sum_{m=1}^{n}P_m\cos\frac{2\pi k}{n}m\right]+L_{1k}\sin k\alpha+L_{2k}\cos k\alpha\right\}.$$

In accordance with cyclic fixing of the shell in the direction of the generatrices

$$u_{m_z}(0,\alpha_l)=0,\quad \alpha_l=\frac{2\pi}{n}l\quad (l=1,2,\ldots,n), \tag{13.52}$$
$$\eta_x(0)=\eta_y(0)=0, \tag{13.53}$$
$$\theta_z(0)=0. \tag{13.54}$$

Expanding conditions (13.52) - (13.54) with the aid of expressions (13.16), (13.17), (13.21) and (13.51), and adding expressions (13.41) - (13.43), we obtain a complete system of 6+n equations in the six arbitrary constants $\eta_x^0, \eta_y^0, \eta_z^0, \vartheta_x^0, \vartheta_y^0, \vartheta_z^0$ and n unknown forces P_m:

$$\eta_x^0 = 0, \tag{13.55}$$

$$\eta_y^0 = 0, \tag{13.56}$$

$$\sum_{m=1}^{n} P_m = \frac{N_z(0)}{\cos\beta}, \tag{13.57}$$

$$\sum_{m=1}^{n} P_m \sin\frac{2\pi}{n} m = \frac{M_x(0)}{R_0\cos\beta}\,\frac{s/R_0}{\sin s/R_0}, \tag{13.58}$$

$$\sum_{m=1}^{n} P_m \cos\frac{2\pi}{n} m = -\frac{M_y(0)}{R_0\cos\beta}\,\frac{s/R_0}{\sin s/R_0}, \tag{13.59}$$

$$\vartheta_z^0 = 0, \tag{13.60}$$

$$\eta_z^0 - \vartheta_y^0 R_0 \cos\frac{2\pi}{n} l + \vartheta_x^0 R_0 \sin\frac{2\pi}{n} l +$$

$$+ \sum_{k=1}^{\infty} T_k \left[\sin\frac{2\pi k}{n} l \sum_{m=1}^{n} P_m \sin\frac{2\pi k}{n} m + \cos\frac{2\pi k}{n} l \sum_{m=1}^{n} P_m \cos\frac{2\pi k}{n} m\right] = L_l$$

$$(l = 1, 2, \ldots, n), \tag{13.61}$$

where

$$L_l = \frac{1}{\cos\beta} \sum_{k=1}^{\infty} [L_{1k}\sin k\alpha_l + L_{2k}\cos k\alpha_l]\frac{1}{\operatorname{sh} k\bar{t}_1}. \tag{13.62}$$

is the sum of the convergent series

$$T_k = \frac{\sqrt{\gamma}}{Eh\pi\cos\beta}\,\frac{\sin ks/R_0}{ks/R_0}\,\frac{1}{k}\operatorname{cth} k\bar{t}_1. \tag{13.63}$$

13.3. Determination of Reactions of the Supports

The desired reactions P_m together with the displacement components $\eta_z^0, \vartheta_x^0, \vartheta_y^0$ are related by a system of n+3, Eqs. (13.57) - (13.59), (13.61).

We will give the solution of this system in general form for an arbitrary n. Such a solution can be constructed because the shell is fixed cyclically; an analogous solution can be constructed in more complex, but also regular cases. When the shell is fixed in an arbitrary system of points, the reactions of the support can be found only numerically.

Let $\{f_m\}$ $(m=1, 2, \ldots, n)$ be a certain set of given numbers. A correspondence can be established between these numbers and some function $f(\alpha)$, which at the points $\alpha=\alpha_m=\frac{2\pi}{n}m$ assumes the values

$$f(\alpha_m)=f_m \quad (m=1, 2, \ldots, n). \tag{13.64}$$

It is obvious that this function can be represented in the form of a finite interpolation polynomial of nth order

$$f(\alpha)=\sum_{m=1}^{n} c_m F_m(\alpha), \tag{13.65}$$

whose coefficients c_m must satisfy the system of algebraic equations

$$\sum_{m=1}^{n} c_m F_m\left(\frac{2\pi}{n} l\right)=f_l \qquad (l=1, 2, \ldots, n). \tag{13.66}$$

The functions $F_m(\alpha)$ can be chosen in various ways. For odd n, a representation in the form of the following trigonometric polynomial is known*

$$f(\alpha)=A_0+\sum_{r=1}^{\frac{n-1}{2}} (A_r \cos r\alpha+B_r \sin r\alpha), \tag{13.67}$$

where

$$\begin{aligned} A_0&=\frac{1}{n}\sum_{m=1}^{n} f_m, \\ A_r&=\frac{2}{n}\sum_{m=1}^{n} f_m \cos\frac{2\pi r}{n}m, \\ B_r&=\frac{2}{n}\sum_{m=1}^{n} f_m \sin\frac{2\pi r}{n}m. \end{aligned} \tag{13.68}$$

*G. M. Fikhtengol'ts. A Course in Differential and Integral Calculus. Moscow, Nauka, 1966; F. Atkinson. Discrete and Continuous Boundary Value Problems. Moscow, Mir, 1968.

For even n, the corresponding polynomial has the form

$$f(\alpha)=A_0+\sum_{r=1}^{\frac{n}{2}-1}(A_r\cos r\alpha+B_r\sin r\alpha)+(-1)^{\frac{n}{2}\frac{\alpha}{2\pi}}A_{\frac{n}{2}}, \tag{13.69}$$

where

$$A_0=\frac{1}{n}\sum_{m=1}^{n}f_m,$$
$$A_r=\frac{2}{n}\sum_{m=1}^{n}f_m\cos\frac{2\pi r}{n}m, \tag{13.70}$$
$$B_r=\frac{2}{n}\sum_{m=1}^{n}f_m\sin\frac{2\pi r}{n}m,$$
$$A_{\frac{n}{2}}=\frac{1}{n}\sum_{m=1}^{n}(-1)^m f_m.$$

The validity of formulas (13.68) and (13.70) can be readily checked by substituting them directly into expressions (13.67) and (13.69).

For odd n, when $\alpha=\alpha_l=\frac{2\pi}{n}l$, we have from (13.67) and (13.68)

$$f(\alpha_l)=\frac{1}{n}\sum_{m=1}^{n}f_m\left[1+2\sum_{r=1}^{\frac{n-1}{2}}\left(\cos\frac{2\pi r}{n}m\cos\frac{2\pi r}{n}l+\right.\right.$$
$$\left.\left.+\sin\frac{2\pi r}{n}m\sin\frac{2\pi r}{n}l\right)\right]=\frac{1}{n}\sum_{m=1}^{n}f_m\left[1+2\sum_{r=1}^{\frac{n-1}{2}}\cos\frac{2\pi(m-l)}{n}r\right] \tag{13.71}$$
$$(l=1,2,\ldots,n).$$

For even n when $\alpha=\alpha_l=\frac{2\pi}{n}l$, we correspondingly obtain from (13.69) and (13.70)

$$f(\alpha_l)=\frac{1}{n}\sum_{m=1}^{n}f_m\left[1+2\sum_{r=1}^{\frac{n}{2}-1}\cos\frac{2\pi(m-l)}{n}r+(-1)^{m+l}\right] \tag{13.72}$$
$$(l=1,2,\ldots,n).$$

To transform expressions (13.71), (13.72), we calculate

$$\sum_{k=1}^{p}\cos kx=\frac{1}{2}\left(\sum_{k=1}^{p}e^{ikx}+\sum_{k=1}^{p}e^{-ikx}\right). \tag{13.73}$$

Having calculated the sums entering into the right-hand side of (13.73) as sums of geometric progressions, we obtain

$$\sum_{k=1}^{p}\cos kx=\frac{1}{2}\left[\frac{e^{ix}(1-e^{ipx})}{1-e^{ix}}+\frac{e^{-ix}(1-e^{-ipx})}{1-e^{-ix}}-1\right]=$$

$$=\frac{1}{2}\left[\frac{e^{ix\left(p+\frac{1}{2}\right)}-e^{-ix\left(p+\frac{1}{2}\right)}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}-1\right],$$

whence

$$\sum_{k=1}^{p}\cos kx=\frac{1}{2}\left[\frac{\sin\left(p+\frac{1}{2}\right)x}{\sin\frac{x}{2}}-1\right]. \qquad (13.74)$$

On the basis of formula (13.74), with $m \neq 1$, we have

$$\sum_{r=1}^{\frac{n-1}{2}}\cos\frac{2\pi(m-l)}{n}r=\frac{1}{2}\left[\frac{\sin\pi(m-l)}{\sin\frac{\pi(m-l)}{n}}-1\right]=-\frac{1}{2}, \qquad (13.75)$$

$$\sum_{r=1}^{\frac{n}{2}-1}\cos\frac{2\pi(m-l)}{n}r=\frac{1}{2}\left[\frac{\sin\left[\pi(m-l)-\frac{\pi}{n}(m-l)\right]}{\sin\frac{\pi(m-l)}{n}}-1\right]=$$

$$=\frac{1}{2}\left[-\cos\pi(m-l)-1\right]=-\frac{1}{2}\left[(-1)^{m-l}+1\right]. \qquad (13.76)$$

When $m = 1$, we find directly

$$\sum_{r=1}^{\frac{n-1}{2}}\cos\frac{2\pi(m-l)}{n}r=\frac{n-1}{2},$$

$$\sum_{r=1}^{\frac{n}{2}-1}\cos\frac{2\pi(m-l)}{n}r=\frac{n}{2}-1. \qquad (13.77)$$

Now, using (13.75) - (13.77), from expressions (13.71) and (13.72), we see that

$$f(\alpha_l)=f_l$$

for any l, q.e.d.

Thus, expressions (13.67), (13.68) for odd n and expressions (13.69), (13.70) for even n establish a mutually single-valued correspondence between the set n of numbers f_m and n coefficients of the corresponding interpolation polynomials.

In their structure, the trigonometic interpolation polynomials are unique discrete analogs of trigonometric Fourier series. The Fourier coefficients corresponding to the series expansion on the segment $[0, 2\pi]$ can be obtained from the coefficients of these polynomials by a formal passage to the limit when $n \to \infty$.

We will establish a correspondence between the polynomials of the form (13.67) or (13.69) and the desired reactions of the supports.

For an odd number n of fixing points

$$P_l = A_0 + \sum_{r=1}^{\frac{n-1}{2}} \left(A_r \cos \frac{2\pi l}{n} r + B_r \sin \frac{2\pi l}{n} r \right) \quad (l = 1, 2, \ldots, n), \tag{13.78}$$

for an even number n of fixing points

$$P_l = A_0 + \sum_{r=1}^{\frac{n}{2}-1} \left(A_r \cos \frac{2\pi l}{n} r + B_r \cos \frac{2\pi l}{n} r \right) + (-1)^l A_{\frac{n}{2}} \quad (l = 1, 2, \ldots, n). \tag{13.79}$$

Expressions (13.78) and (13.79) make it possible to reduce the problem of determination of the reactions of the supports P_m to finding the coefficients

$$\begin{aligned} A_0 &= \frac{1}{n} \sum_{m=1}^{n} P_m, \\ A_r &= \frac{2}{n} \sum_{m=1}^{n} P_m \cos \frac{2\pi r}{r} m, \\ B_r &= \frac{2}{n} \sum_{m=1}^{n} P_m \sin \frac{2\pi r}{n} m, \\ A_{\frac{n}{2}} &= \frac{1}{n} \sum_{m=1}^{n} (-1)^m P_m. \end{aligned} \tag{13.80}$$

Substituting expressions (13.78) or (13.79) into (13.57) - (13.59) and (13.61), we obtain a system of n+3 equations in the three components $\eta_x^0, \theta_x^0, \theta_y^0$ and n coefficients $A_0, A_r, B_r, (A_{\frac{n}{2}})$.

From (13.57) - (13.59), in view of (13.80), we have:

$$A_0 = -\frac{N_r(0)}{n\cos\beta}, \tag{13.81}$$

$$A_1 = -\frac{M_y(0)}{\frac{nR_0}{2}\beta}\,\frac{\iota/R_0}{\sin\iota/R_0}, \tag{13.82}$$

$$B_1 = \frac{M_x(0)}{\frac{nR_0}{2}\cos\beta}\,\frac{\iota/R_0}{\sin\iota/R_0}. \tag{13.83}$$

To determine the remaining unknowns, we first transform Eqs. (13.61). Decomposing the natural series k = 1, 2,... in accordance with the schemes presented in Fig. 13.2, for infinite sums in k we obtain:

for odd n

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{r=1}^{\frac{n-1}{2}} a_r + \sum_{q=1}^{\infty}\sum_{r=-\left(\frac{n-1}{2}\right)}^{\frac{n-1}{2}} a_{qn+r} = \\ &= \sum_{r=1}^{\frac{n-1}{2}} a_r + \sum_{q=1}^{\infty}\left(\sum_{r=1}^{\frac{n-1}{2}} a_{qn-r} + a_{qn} + \sum_{r=1}^{\frac{n-1}{2}} a_{qn+r}\right) = \\ &= \sum_{r=1}^{\frac{n-1}{2}} a_r + \sum_{q=1}^{\infty} a_{qn} + \sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n-1}{2}} (a_{qn-r} + a_{qn+r}).\end{aligned} \tag{13.84}$$

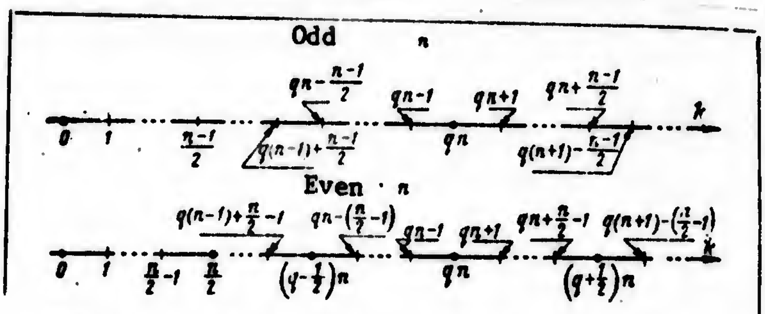


Fig. 13.2. Decomposition of natural series.

for even n

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{r=1}^{\frac{n}{2}} a_r + \sum_{q=1}^{\infty}\sum_{r=-\left(\frac{n}{2}-1\right)}^{\frac{n}{2}} a_{qn+r} = \\ &= \sum_{r=1}^{\frac{n}{2}-1} a_r + a_{\frac{n}{2}} + \sum_{q=1}^{\infty}\left[\sum_{r=1}^{\frac{n}{2}-1} a_{qn-r} + a_{qn} + \sum_{r=1}^{\frac{n}{2}-1} a_{qn+r} + a_{\left(q+\frac{1}{2}\right)n}\right] = \\ &= \sum_{r=1}^{\frac{n}{2}-1} a_r + \sum_{q=1}^{\infty} a_{qn} + \sum_{q=1}^{\infty} a_{\left(q-\frac{1}{2}\right)n} + \sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n}{2}-1} (a_{qn-r} + a_{qn+r}).\end{aligned} \tag{13.85}$$

Decomposition (13.84), (13.85) of the infinite sums permits a radical simplication of Eqs. (13.61).

For odd n, we can obtain:

$$\eta_z^0-\theta_y^0R_0\cos\frac{2\pi}{n}l+\theta_x^0R_0\sin\frac{2\pi}{n}l+$$

$$+\sum_{r=1}^{\frac{n-1}{2}}T_r\left(\sin\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\sin\frac{2\pi r}{n}m+\cos\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\cos\frac{2\pi r}{n}m\right)+$$

$$+\sum_{q=1}^{\infty}T_{qn}\sum_{m=1}^{n}P_m+\sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n-1}{2}}(T_{qn-r}+T_{qn+r})\left(\sin\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\sin\frac{2\pi r}{n}m+\right.$$

$$\left.+\cos\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\cos\frac{2\pi r}{n}m\right)=L_l. \qquad (13.86)$$

$$(l=1,2,\ldots,n).$$

For even n, we have

$$\eta_z^0-\theta_y^0R_0\cos\frac{2\pi}{n}l+\theta_x^0R_0\sin\frac{2\pi}{n}l+$$

$$+\sum_{r=1}^{\frac{n}{2}-1}T_r\left(\sin\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\sin\frac{2\pi r}{n}m+\cos\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\cos\frac{2\pi r}{n}m\right)+$$

$$+\sum_{q=1}^{\infty}T_{qn}\sum_{m=1}^{n}P_m+\sum_{q=1}^{\infty}T_{\left(q-\frac{1}{2}\right)n}(-1)^l\sum_{m=1}^{n}(-1)^mP_m+$$

$$+\sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n}{2}-1}(T_{qn-r}+T_{qn+r})\left(\sin\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\sin\frac{2\pi r}{n}m+\right.$$

$$\left.+\cos\frac{2\pi r}{n}l\sum_{m=1}^{n}P_m\cos\frac{2\pi r}{n}m\right)=L_l \qquad (13.87)$$

$$(l=1,2,\ldots,n).$$

Now, in view of (13.80, Eqs. (13.86), (13.87) take the form:

for odd n

$$\eta_z^0-\theta_y^0R_0\cos\frac{2\pi}{n}l+\theta_x^0R_0\sin\frac{2\pi}{n}l+$$

$$+\frac{n}{2}\sum_{r=1}^{\frac{n-1}{2}}T_r\left(B_r\sin\frac{2\pi r}{n}l+A_r\cos\frac{2\pi r}{n}l\right)+A_0n\sum_{q=1}^{\infty}T_{qn}+$$

$$+\frac{n}{2}\sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n-1}{2}}(T_{qn-r}+T_{qn+r})\left(B_r\sin\frac{2\pi r}{n}l+A_r\cos\frac{2\pi r}{n}l\right)=L_l \qquad (13.88)$$

$$(l=1,2,\ldots,n),$$

for even n

$$\eta_z^0 - \theta_y^0 R_0 \cos\frac{2\pi}{n} l + \theta_x^0 R_0 \sin\frac{2\pi}{n} l +$$
$$+ \frac{n}{2}\sum_{r=2}^{\frac{n}{2}-1} T_r \left(B_r \sin\frac{2\pi r}{n} l + A_r \cos\frac{2\pi r}{n} l\right) + A_0 n \sum_{q=1}^{\infty} T_{qn} +$$
$$+(-1)^l A_{\frac{n}{2}} n \sum_{q=1}^{\infty} T_{\left(q-\frac{1}{2}\right)n} + \frac{n}{2}\sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n}{2}-1}(T_{qn-r} + T_{qn+r}) \times$$
$$\times \left(B_r \sin\frac{2\pi r}{n} l + A_r \cos\frac{2\pi r}{n} l\right) = L_l \tag{13.89}$$
$$(l = 1, 2, \ldots, n).$$

For the obtained values of the coefficients A_0, A_1, B_1, determined by formulas (13.81) - (13.83), Eqs. (13.88), /(13.89)/ form a complete system of n equations in the unknown components $\eta_z^0, \theta_x^0, \theta_y^0$ and coefficients $A_r, B_r \left(r=2, 3, \ldots, \frac{n-1}{2}\right)$ for odd n or $A_r, B_r \left(r=2, 3, \ldots, \frac{n}{2}-1\right)$ and $A_{\frac{n}{2}}$ for even n.

Using decomposition (13.84), (13.85) of the infinite sums, we transform expression (13.62) and obtain

for odd n

$$L_l = \sum_{r=2}^{\frac{n-1}{2}}\left(Z_{1r} \sin\frac{2\pi r}{n} l + Z_{2r} \cos\frac{2\pi r}{n} l\right) +$$
$$+ \sum_{q=1}^{\infty} Z_{2(qn)} + \sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n-1}{2}}\left[(-Z_{1(qn-r)} + Z_{1(qn+r)}) \sin\frac{2\pi r}{n} l +\right.$$
$$\left.+ (Z_{2(qn-r)} + Z_{2(qn+r)}) \cos\frac{2\pi r}{n} l\right], \tag{13.90}$$

for even n

$$L_l = \sum_{r=2}^{\frac{n}{2}-1}\left(Z_{1r} \sin\frac{2\pi r}{n} l + Z_{2r} \cos\frac{2\pi r}{n} l\right) +$$
$$+ \sum_{q=1}^{\infty} Z_{2(qn)} + \sum_{q=1}^{\infty}(-1)^l Z_{2\left(q-\frac{1}{2}\right)n} + \sum_{q=1}^{\infty}\sum_{r=1}^{\frac{n}{2}-1}\left[(-Z_{1(qn-r)} +\right.$$
$$\left.+ Z_{1(qn+r)}) \sin\frac{2\pi r}{n} l + (Z_{2(qn-r)} + Z_{2(qn+r)}) \cos\frac{2\pi r}{n} l\right], \tag{13.91}$$

where

$$Z_{1k} = \frac{L_{1k}}{\cos\beta\, \mathrm{sh}\, k\bar{l}_1}; \quad Z_{2k} = \frac{L_{2k}}{\cos\beta\, \mathrm{sh}\, k\bar{l}_1}. \tag{13.92}$$

Now, using expressions (13.90), (13.91) and changing the order of summation, we will reduce similar terms in Eqs. (13.88) and (13.89), and obtain

for odd n

$$\begin{gathered}\eta_z^0+\sum_{q=1}^{\infty}\left(A_0 nT_{qn}-\tilde{L}_{2(qn)}\right)+\Big\{-\theta_y^0R_0+\sum_{q=1}^{\infty}\Big[\frac{n}{2}A_1(T_{qn-1}+T_{qn+1})-\\-\tilde{L}_{2(qn-1)}-\tilde{L}_{2(qn+1)}\Big]\Big\}\cos\frac{2\pi l}{n}+\Big\{\theta_x^0R_0+\sum_{q=1}^{\infty}\Big[\frac{n}{2}B_1(T_{qn-1}+T_{qn+1})+\\+\tilde{L}_{1(qn-1)}-\tilde{L}_{1(qn+1)}\Big]\Big\}\sin\frac{2\pi l}{n}+\sum_{r=2}^{\frac{n-1}{2}}\Big\{\Big[\frac{n}{2}T_rA_r+\\+\frac{n}{2}A_r\sum_{q=1}^{\infty}(T_{qn-r}+T_{qn+r})-\tilde{L}_{2r}-\sum_{q=1}^{\infty}(\tilde{L}_{2(qn-r)}+\tilde{L}_{2(qn+r)})\Big]\cos\frac{2\pi l}{n}r+\\+\Big[\frac{n}{2}T_rB_r+\frac{n}{2}B_r\sum_{q=1}^{\infty}(T_{qn-r}-T_{qn+r})-\tilde{L}_{1r}+\\+\sum_{q=1}^{\infty}(\tilde{L}_{1(qn-r)}-\tilde{L}_{1(qn+r)})\Big]\sin\frac{2\pi l}{n}r\Big\}=0\\(l=1,2,\ldots,n),\end{gathered}\tag{13.93}$$

for even n

$$\begin{gathered}\eta_z^0+\sum_{q=1}^{\infty}\left(A_0 nT_{qn}-\tilde{L}_{2(qn)}\right)+\Big\{-\theta_y^0R_0+\sum_{q=1}^{\infty}\Big[\frac{n}{2}A_1(T_{qn-1}+T_{qn+1})-\\-\tilde{L}_{2(qn-1)}-\tilde{L}_{2(qn+1)}\Big]\Big\}\cos\frac{2\pi l}{n}+\Big\{\theta_x^0R_0+\sum_{q=1}^{\infty}\Big[\frac{n}{2}B_1(T_{qn-1}+T_{qn+1})+\\+\tilde{L}_{1(qn-1)}-\tilde{L}_{1(qn+1)}\Big]\Big\}\sin\frac{2\pi l}{n}+\sum_{r=2}^{\frac{n}{2}-1}\Big\{\Big[\frac{n}{2}T_rA_r+\frac{n}{2}A_r\times\\\times\sum_{q=1}^{\infty}(T_{qn-r}+T_{qn+r})-\tilde{L}_{2r}-\sum_{q=1}^{\infty}(\tilde{L}_{2(qn-r)}+\tilde{L}_{2(qn+r)})\Big]\cos\frac{2\pi l}{n}r+\\+\Big[\frac{n}{2}T_rB_r+\frac{n}{2}B_r\sum_{q=1}^{\infty}(T_{qn-r}+T_{qn+r})-\tilde{L}_{1r}+\sum_{q=1}^{\infty}(\tilde{L}_{1(qn-r)}-\tilde{L}_{1(qn+r)})\Big]\times\\\times\sin\frac{2\pi l}{n}r\Big\}+(-1)^l\sum_{q=1}^{\infty}\Big[nA_{\frac{n}{2}}T_{\left(qn-\frac{n}{2}\right)}-L_{2\left(qn-\frac{n}{2}\right)}\Big]=0\\(l=1,2,\ldots,n).\end{gathered}\tag{13.94}$$

The solution of system of Eqs. (13.93) / (13.94)/ is obvious. It is easy to observe, comparing (13.93) and (13.67) or (13.94) and (13.69), that these equations can be placed in correspondence with an interpolation polynomial that takes on zero values at the points $u = \frac{2\pi l}{n}$ $(l = 1, 2, \ldots, n)$. However, in this case it follows from relations (13.68)/(13.70)/ that all the coefficients of such a polynomial should be equal to zero. In view of these considerations, equating to zero in (13.93), (13.94) the coefficients on $\sin\frac{2\pi l}{n}r$, $\cos\frac{2\pi l}{n}r$, $(-1)^l$, and the free term, and taking (13.81) - (13.83) into account, we find

$$\eta_z^0 = -\frac{N_z(0)}{\cos\beta}\sum_{q=1}^{\infty} T_{qn} + \sum_{q=1}^{\infty} \bar{L}_{2(qn)}. \tag{13.95}$$

$$\theta_x^0 = -\frac{M_x(0)}{R_0^2\cos\beta}\,\frac{s/R_0}{\sin s/R_0}\sum_{q=1}^{\infty}(T_{qn-1}+T_{qn+1}) - \frac{1}{R_0}\sum_{q=1}^{\infty}(\bar{L}_{1(qn-1)} - \bar{L}_{1(qn+1)}). \tag{13.96}$$

$$\theta_y^0 = -\frac{M_y(0)}{R_0^2\cos\beta}\,\frac{s/R_0}{\sin s/R_0}\sum_{q=1}^{\infty}(T_{qn-1}+T_{qn+1}) - \frac{1}{R_0}\sum_{q=1}^{\infty}(\bar{L}_{2(qn-1)} + \bar{L}_{2(qn+1)}). \tag{13.97}$$

$$A_r = \frac{2}{n}\,\frac{\bar{L}_{2r} + \sum_{q=1}^{\infty}(\bar{L}_{2(qn-r)} + \bar{L}_{2(qn+r)})}{T_r + \sum_{q=1}^{\infty}(T_{qn-r} + T_{qn+r})} \quad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right], \tag{13.98}$$

$$B_r = \frac{2}{n}\,\frac{\bar{L}_{1r} + \sum_{q=1}^{\infty}(-\bar{L}_{1(qn-r)} + \bar{L}_{2(qn+r)})}{T_r + \sum_{q=1}^{\infty}(T_{qn-r} + T_{qn+r})} \quad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right], \tag{13.99}$$

$$A_{\frac{n}{2}} = \frac{1}{n}\,\frac{\sum_{q=1}^{\infty}\bar{L}_{2\left(qn-\frac{n}{2}\right)}}{\sum_{q=1}^{\infty}T_{qn-\frac{n}{2}}}, \tag{13.100}$$

where

$$E\left(\frac{n-1}{2}\right) = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} - 1 & \text{for even } n \end{cases}$$

is the integral part of the number $\frac{n-1}{2}$.

Expressions (13.98) - (13.100) and (13.81) - (13.83) represent the coefficients of the interpolation polynomial (13.78)/(13.79)/, which at the points $\alpha = \alpha_l \ (l = 1, 2, \ldots, n)$ takes on values equal to the values of the desired reactions of the supports. The problem has thus been solved. By determining the integration constants in accordance with formulas (13.48), one can readily compute the displacements and stresses at any point of the shell.

Let us turn to expressions (13.48). Decomposing the natural series according to the schemes of Fig. 13.2, by using relations (13.80), we can obtain the following formulas:

$$\sum_{m=1}^{n} P_m \sin \frac{2\pi k}{n} m = \left| \begin{array}{ll} 0 & (k = qn), \\ \pm \frac{n}{2} B_r & (k = qn \pm r), \\ 0 & \left(k = qn - \frac{n}{2}\right). \end{array} \right. \tag{13.101}$$

$$\sum_{m=1}^{n} P_m \cos \frac{2\pi k}{n} m = \left| \begin{array}{ll} nA_0 & (k = qn), \\ \frac{n}{2} A_r & (k = qn \pm r), \\ nA_{\frac{n}{2}} & \left(k = qn - \frac{n}{2}\right), \end{array} \right. \tag{13.102}$$

where q is an integer; $r = 1, 2, \ldots, E\left(\frac{n-1}{2}\right)$.

In view of (13.101), (13.102) expressions (13.48) take the form

$$\begin{gathered} C_{1(qn)} = \frac{1}{2} \cos\beta \tilde{L}_{1(qn)}, \\ C_{1(qn \pm r)} = \frac{1}{2} \cos\beta \left(\pm n\tilde{T}_{qn+r} B_r + \tilde{L}_{1(qn \pm r)}\right), \\ C_{1\left(qn - \frac{n}{2}\right)} = \frac{1}{2} \cos\beta \tilde{L}_{1\left(qn - \frac{n}{2}\right)}, \end{gathered} \tag{13.103}$$

$$\begin{gathered} \bar{C}_{1(qn)} = \frac{1}{2} \cos\beta \tilde{L}_{1(qn)}, \\ \bar{C}_{1(qn \pm r)} = \frac{1}{2} \cos\beta \left(\pm n\bar{\tilde{T}}_{qn \pm r} B_r + \tilde{L}_{1(qn \pm r)}\right), \end{gathered} \tag{13.104}$$

$$\bar{C}_{1\left(qn-\frac{n}{2}\right)}=\frac{1}{2}\cos\beta \bar{L}_{1\left(qn-\frac{n}{2}\right)},$$

$$C_{2(qn)}=\cos\beta\left(n\bar{T}_{qn}A_0+\frac{1}{2}L_{2(qn)}\right),$$

$$C_{2(qn\pm r)}=\frac{1}{2}\cos\beta\left(n\bar{T}_{qn\pm r}A_r+L_{2(qn\pm r)}\right), \tag{13.105}$$

$$C_{2\left(qn-\frac{n}{2}\right)}=\cos\beta\left(n\bar{T}_{qn-\frac{n}{2}}A_{\frac{n}{2}}+\frac{1}{2}L_{2\left(qn-\frac{n}{2}\right)}\right).$$

$$\bar{C}_{2(qn)}=\cos\beta\left(n\bar{\bar{T}}_{qn}A_0+\frac{1}{2}\bar{L}_{2(qn)}\right),$$

$$\bar{C}_{2(qn\pm r)}=\cos\beta\left(n\bar{\bar{T}}_{qn\pm r}A_r+\bar{L}_{2(qn\pm r)}\right), \tag{13.106}$$

$$\bar{C}_{2\left(qn-\frac{n}{2}\right)}=\cos\beta\left(n\bar{\bar{T}}_{qn-\frac{n}{2}}A_{\frac{n}{2}}+\frac{1}{2}\bar{L}_{2\left(qn-\frac{n}{2}\right)}\right),$$

where

$$\bar{T}_k=\frac{T_k}{1+e^{2k\bar{l}_1}};\qquad \bar{\bar{T}}_k=\frac{T_k}{1+e^{-2k\bar{l}_1}}. \tag{13.107}$$

Now, introducing into expressions (13.30) - (13.32) and (13.33), (13.34) formulas (13.103) - (13.106) for the arbitrary constants, where the coefficients of the interpolation polynomial are represented by relations (13.81) - (13.83) and (13.98) - (13.100), and also the values of the components of displacement of the end t = 0 as a solid, determined in accordance with (13.55), (13.56), (13.60) and (13.95) - (13.97), we obtain the general solution of the problem under consideration, corresponding to a magnitude of the segment $\Delta S=2\varepsilon$ as small as desired, but different from zero.

Passing to the limit in (13.63), we find

$$\lim_{\varepsilon\to 0}T_k=\frac{\sqrt{3}}{Eh\pi\cos\beta}\,\frac{1}{k}\operatorname{cth}k\bar{l}_1. \tag{13.108}$$

By virtue of (13.108), considering the divergence of the harmonic series, we have from expressions (13.98) - (13.100)

$$\lim_{\varepsilon\to 0}A_r=\lim_{\varepsilon\to 0}B_r=\lim_{\varepsilon\to 0}A_{\frac{n}{2}}=0, \tag{13.109}$$

where $r=2,3,\ldots,E\left(\frac{n-1}{2}\right)$.

Now, from relations (13.78) and (13.79), taking (13.109) into account, we arrive at the important conclusion that the reactions of the supports P_m are distributed in accordance with the law of the plane outside the dependence on the number of fixing points and on the character of the external load.

Introducing (13.81) - (13.83) and (13.109) into polynomials (13.78), (13.79) and passing to the limit when $\varepsilon \to 0$, we obtain the final expression for the reactions of the supports

$$P_m = \frac{N_z(0)}{n \cos\beta} + \frac{M_x(0)}{\frac{nR_0}{2}\cos\beta} \sin\frac{2\pi}{n} m - \frac{M_y(0)}{\frac{nR_0}{2}\cos\beta} \cos\frac{2\pi}{n} m \qquad (13.110)$$

$$(m = 1, 2, \ldots, n).$$

Let us note that in the sense of the conclusion, expression (13.110) is valid for a number of fixing points n 3. The cases n = 1, 2 are trivial.

13.4. Stressed State of a Shell, Closed Form of the Solution

In view of (13.107) - (13.109), the expressions for arbitrary constants (13.103) - (13.106) assume the form

$$C_{1k} = \begin{cases} \frac{1}{2}\cos\beta L_{1k} & (k \neq qn \pm 1), \\ \pm \frac{\sqrt{1}}{Eh\pi R_0 \cos\beta} \frac{1}{k} \frac{M_x(0)}{e^{2k\bar{\varepsilon}_*} - 1} + \frac{1}{2}\cos\beta L_{1k} & (k = qn \mp 1), \end{cases} \qquad (13.111)$$

$$\bar{C}_{1k} = \begin{cases} \frac{1}{2}\cos\beta \bar{L}_{1k} & (k \neq qn \pm 1), \\ \pm \frac{\sqrt{1}}{Eh\pi R_0 \cos\beta} \frac{1}{k} \frac{M_x(0)}{1 - e^{-2k\bar{\varepsilon}_*}} + \frac{1}{2}\cos\beta \bar{L}_{1k} & (k = qn \pm 1), \end{cases} \qquad (13.112)$$

$$C_{2k}=\begin{cases}\frac{1}{2}\cos\beta\bar{L}_{2k} & (k\neq qn;\ k\neq qn\pm 1),\\ \frac{\sqrt{\gamma}}{Eh\pi\cos\beta}\,\frac{1}{k}\,\frac{N_z(0)}{e^{2k\bar{t}_1}-1}+\frac{1}{2}\cos\beta\bar{L}_{2k} & (k=qn),\\ -\frac{\sqrt{\gamma}}{Eh\pi R_0\cos\beta}\,\frac{1}{k}\,\frac{M_y(0)}{e^{2k\bar{t}_1}-1}+\frac{1}{2}\cos\beta\bar{L}_{2k} & (k=qn\pm 1).\end{cases}\tag{13.113}$$

$$\bar{C}_{2k}=\begin{cases}\frac{1}{2}\cos\beta\bar{L}_{2k} & (k\neq qn;\ k\neq qn\pm 1),\\ \frac{\sqrt{\gamma}}{Eh\pi\cos\beta}\,\frac{1}{k}\,\frac{N_z(0)}{1-e^{-2k\bar{t}_1}}+\frac{1}{2}\cos\beta\bar{L}_{2k} & (k=qn),\\ -\frac{\sqrt{\gamma}}{Eh\pi R_0\cos\beta}\,\frac{1}{k}\,\frac{M_y(0)}{1-e^{-2k\bar{t}_1}}+\frac{1}{2}\cos\beta\bar{L}_{2k} & (k=qn\pm 1).\end{cases}\tag{13.114}$$

Formulas (13.111) - (13.114) are general in character and permit one to determine the integration constants for a conical shell of arbitrary configuration under an arbitrary external load. In particular, for a truncated cone, $\bar{Z}_1 = 1$. Therefore, passing to the limit at $\bar{t}_1=\ln(1-\bar{Z}_1)\longrightarrow-\infty$, from expressions (13.11) - (13.114) we obtain

$$C_{1k}=\begin{cases}\bar{L}^{\bullet}_{1k} & (k\neq qn\pm 1),\\ \mp\frac{\sqrt{\gamma}}{Eh\pi R_0\cos\beta}\,\frac{1}{k}M_x(0)+\bar{L}^{\bullet}_{1k} & (k=qn\pm 1),\end{cases}\tag{13.115}$$

$$\bar{C}_{1k}=\bar{L}^{\bullet}_{1k},\tag{13.116}$$

$$C_{2k}=\begin{cases}\bar{L}^{\bullet}_{2k} & (k\neq qn;\ k\neq qn\pm 1),\\ -\frac{\sqrt{\gamma}}{Eh\pi\cos\beta}\,\frac{1}{k}N_z(0)+\bar{L}^{\bullet}_{2k} & (k=qn),\\ \frac{\sqrt{\gamma}}{Eh\pi R_0\cos\beta}\,\frac{1}{k}M_y(0)+\bar{L}^{\bullet}_{2k} & (k=qn\pm 1),\end{cases}\tag{13.117}$$

$$\bar{C}_{2k}=\bar{L}^{\bullet}_{2k},\tag{13.118}$$

where

$$\bar{L}^{\bullet}_{1k}=\frac{1}{2\pi h\sqrt{\gamma}Gk}\int_{-\infty}^{0}R_{1k}(\xi)e^{k\bar{\xi}+\xi}d\xi,$$

$$\bar{L}^{\bullet}_{2k}=\frac{1}{2\pi h\sqrt{\gamma}Gk}\int_{-\infty}^{0}R_{2k}(\xi)e^{k\bar{\xi}+\xi}d\xi.\tag{13.119}$$

Formulas (13.115)-(13.118) obviously can also be used with a sufficient degree of accuracy for truncated conical shells ($\bar{Z}_1 < 1$) in the study of the stressed state near the end $\bar{Z} = 0$, since the mutual influence of the conditions on the ends is insignificant. In this case, the limits of applicability of the approximate solution on the basis of formulas (13.115)-(13.118) can be established by comparing it with the exact solution corresponding to formulas (13.111)-(13.114)

Formulas (13.115)-(13.118) can be easily obtained directly by omitting general expressions (13.111)-(13.114). In this case, from the boundedness of the displacements at the apex of a nontruncated conical shell, in accordance with expressions (13.26) and (13.27), it is necessary to set at once

$$\bar{C}'_{1k} = \bar{C}'_{2k} = 0. \tag{13.120}$$

The arbitrary constants C'_{1k} and C'_{2k} are determined from the conditions of cyclic fixing of the end $\bar{Z} = 0$, as was explained above. Then, using (13.29), we again arrive at expressions (13.115)-(13.118).

For truncated shells, an approximate solution with wider applicability limits can be obtained by taking, instead of $\bar{Z}_1 = 0$, the estimate

$$(1 - \bar{Z}_1)^{\frac{2k}{\sqrt{1 \sin\beta}}} \ll 1, \tag{13.121}$$

whence

$$1 - e^{2k\bar{t}_1} \approx 1; \qquad 1 - e^{-2k\bar{t}_1} \approx e^{-2k\bar{t}_1}. \tag{13.122}$$

Considering estimates (13.122), we obtain from (13.111)-(13.114)

$$C_{1k} = \begin{cases} \frac{1}{2} \cos\beta Z_{1k} & (k \neq qn \pm 1), \\ \mp \frac{\sqrt{\bar{1}}}{Eh\pi R_0 \cos\beta} \frac{1}{k} M_z(0) + \frac{1}{2} \cos\beta Z_{1k} & (k = qn \pm 1). \end{cases} \tag{13.123}$$

$$\bar{C}_{1k} = \begin{cases} \frac{1}{2}\cos\beta\bar{L}_{1k} & (k \neq qn \pm 1), \\ \mp \frac{\sqrt{\gamma}}{Eh\pi R_0 \cos\beta} \frac{1}{k} M_x(0) e^{2k\bar{l}_1} + \frac{1}{2}\cos\beta\bar{L}_{1k} & (k = qn \pm 1), \end{cases} \tag{13.124}$$

$$C_{2k} = \begin{cases} \frac{1}{2}\cos\beta L_{2k} & (k \neq qn,\ k \neq qn \pm 1), \\ -\frac{\sqrt{\gamma}}{Eh\pi\cos\beta} \frac{1}{k} N_z(0) + \frac{1}{2}\cos\beta L_{2k} & (k = qn), \\ \frac{\sqrt{\gamma}}{Eh\pi R_0 \cos\beta} \frac{1}{k} M_y(0) + \frac{1}{2}\cos\beta L_{2k} & (k = qn \pm 1), \end{cases} \tag{13.125}$$

$$\bar{C}_{2k} = \begin{cases} \frac{1}{2}\cos\beta\bar{L}_{2k} & (k \neq qn;\ k \neq qn \pm 1), \\ -\frac{\sqrt{\gamma}}{Eh\pi\cos\beta} \frac{1}{k} N_z(0) e^{2k\bar{l}_1} + \frac{1}{2}\cos\beta\bar{L}_{2k} & (k = qn), \\ \frac{\sqrt{\gamma}}{Eh\pi\cos\beta R_0} \frac{1}{k} M_y(0) e^{2k\bar{l}_1} + \frac{1}{2}\cos\beta\bar{L}_{2k} & (k = qn \pm 1). \end{cases} \tag{13.126}$$

Let us turn to expressions (13.33), (13.33), (13.34). For the values of the arbitrary constants represented by relations (13.123)-(13.126), we have

$$\sigma_{m_z}(t,\alpha) = \sigma^0_{m_z}(t,\alpha) + \sigma^R_{m_z}(t,\alpha) + \frac{e^{-t}}{\pi h R_0^2 \cos\beta}\Big\{N_z(0) R_0 \sum_{q=1}^{\infty} (e^{qn\bar{t}} - e^{qn(2\bar{l}_1 - \bar{t})}) \cos qn\alpha + M_x(0) \sum_{q=1}^{\infty} [(-e^{(qn-1)\bar{t}} + e^{(qn-1)(2\bar{l}_1 - \bar{t})}) \times \sin(qn-1)\alpha + (e^{(qn+1)\bar{t}} - e^{(qn+1)(2\bar{l}_1 - \bar{t})}) \sin(qn+1)\alpha] + M_y(0) \sum_{q=1}^{\infty} [(-e^{(qn-1)\bar{t}} + e^{(qn-1)(2\bar{l}_1 - \bar{t})}) \cos(qn-1)\alpha + (-e^{(qn+1)\bar{t}} + e^{(qn+1)(2\bar{l}_1 - \bar{t})}) \cos(qn+1)\alpha]\Big\}, \tag{13.127}$$

where the stresses corresponding to displacements of the cross section of the shell as a solid (law of plane sections)

$$\sigma^0_{m_z}(t,\alpha) = \frac{N_z(t) e^{-t}}{2\pi h R_0 \cos\beta} + \frac{M_x(t) e^{-2t}}{\pi h R_0^2 \cos\beta} \sin\alpha - \frac{M_y(t) e^{-2t}}{\pi h R_0^2 \cos\beta} \cos\alpha, \tag{13.128}$$

and the self-balanced stresses corresponding to the self-balanced components of the external load

$$\sigma^R_{m_z}(t,\alpha) = -\frac{E}{\sqrt{\gamma} R_0} e^{-t} \sum_{k=1}^{\infty} \Big\{\sin k\alpha \Big[k\cos\beta L_{1k} \operatorname{sh} k\bar{t} - \frac{\sqrt{\gamma}\sin\beta}{a_{kk} G} \int_0^t \operatorname{ch} k(\bar{\xi} - \bar{t}) R_{1k}(\xi) e^{\xi} d\xi\Big] + \cos k\alpha \Big[k\cos\beta L_{2k} \operatorname{sh} k\bar{t} - \frac{\sqrt{\gamma}\sin\beta}{a_{kk} G} \int_0^t \operatorname{ch} k(\bar{\xi} - \bar{t}) R_{2k}(\xi) e^{\xi} d\xi\Big]\Big\}. \tag{13.129}$$

Similarly, the tangential stresses

$$
\begin{aligned}
\tau_{n_z m_z}(t, \alpha) = {} & \tau^0_{n_z m_z}(t, \alpha) + \tau^R_{n_z m_z}(t, \alpha) + \frac{e^{-t}}{\pi \sqrt{\gamma} h R_0^2 \cos\beta} \times \\
& \times \Big\{ N_z(0) R_0 \sum_{q=1}^{\infty} \left(e^{qn\bar{t}} + e^{qn(2\bar{t}_1 - \bar{t})} \right) \sin qn\alpha + M_x(0) \sum_{q=1}^{\infty} \Big[\left(e^{(qn-1)\bar{t}} + \right. \\
& \left. + e^{(qn-1)(2\bar{t}_1 - \bar{t})} \right) \cos(qn-1)\alpha - \left(e^{(qn+1)\bar{t}} + e^{(qn+1)(2\bar{t}_1 - \bar{t})} \right) \cos(qn+1)\alpha \Big] - \\
& - M_y(0) \sum_{q=1}^{\infty} \Big[\left(e^{(qn-1)\bar{t}} + e^{(qn-1)(2\bar{t}_1 - \bar{t})} \right) \sin(qn-1)\alpha + \\
& + \left(e^{(qn+1)\bar{t}} + e^{(qn+1)(2\bar{t}_1 - \bar{t})} \right) \sin(qn+1)\alpha \Big] \Big\},
\end{aligned}
\tag{13.130}
$$

where the stresses corresponding to the displacements of the cross section of the shell as a solid (law of place sections)

$$
\begin{aligned}
\tau^0_{n_z m_z}(t, \alpha) = {} & \frac{e^{-2t}}{2\pi R_0^3 h} M_z(t) + \frac{-l_0 e^t Q_x(t) + M_y(t)}{\pi h R_0^2 \operatorname{ctg}\beta} e^{-2t} \sin\alpha + \\
& + \frac{l_0 e^t Q_y(t) + M_x(t)}{\pi h R_0^2 \operatorname{ctg}\beta} e^{-2t} \cos\alpha,
\end{aligned}
\tag{13.131}
$$

and the self-balanced tangential stresses corresponding to the self-balanced components of the external load

$$
\begin{aligned}
\tau^R_{n_z m_z}(t, \alpha) = {} & \frac{G}{R_0} e^{-t} \sum_{k=2}^{\infty} \Big\{ \cos k\alpha \Big[k \cos\beta \bar{L}_{1k} \operatorname{ch} k\bar{t} + \\
& + \frac{\sqrt{\gamma} \sin\beta}{a_{kk} G} \int_0^t \operatorname{sh} k(\bar{\xi} - \bar{t}) R_{1k}(\xi) e^{\xi} d\xi \Big] - \sin k\alpha \Big[k \cos\beta \bar{L}_{2k} \operatorname{ch} k\bar{t} + \\
& + \frac{\sqrt{\gamma} \sin\beta}{a_{kk} G} \int_0^t \operatorname{sh} k(\bar{\xi} - \bar{t}) R_{2k}(\xi) e^{\xi} d\xi \Big] \Big\}.
\end{aligned}
\tag{13.132}
$$

The infinite series in expressions (13.127), (13.130) are easy to sum up.

Using Euler's formulas, we have

$$
\begin{aligned}
\sum_{q=1}^{\infty} e^{qy} \sin qx &= \frac{1}{2i} \left[\sum_{q=1}^{\infty} e^{q(y+ix)} - \sum_{q=1}^{\infty} e^{q(y-ix)} \right], \\
\sum_{q=1}^{\infty} e^{qy} \cos qx &= \frac{1}{2} \left[\sum_{q=1}^{\infty} e^{q(y+ix)} + \sum_{q=1}^{\infty} e^{q(y-ix)} \right].
\end{aligned}
\tag{13.133}
$$

It is easy to see that the infinite series in expressions (13.133) represent for $y < 0$ the sum of the terms of infinitely decreasing geometric progressions with denominators e^{y+ix} and e^{y-ix}. Therefore

$$
\begin{aligned}
&\sum_{q=1}^{\infty} e^{qy} \sin qx = \frac{1}{2} \frac{\sin x}{\operatorname{ch} y - \cos x} \quad (y < 0),\\
&\sum_{q=1}^{\infty} e^{qy} \cos qx = -\frac{1}{2} \left[1 + \frac{\operatorname{sh} y}{\operatorname{ch} y - \cos x}\right] \quad (y < 0).
\end{aligned} \tag{13.134}
$$

In expressions (13.127), (13.130)

$$
\begin{aligned}
&\bar t = \frac{1}{\sqrt{\bar\gamma} \sin\beta} \ln(1 - \bar Z) < 0,\\
&2\bar t_1 - \bar t = \ln \frac{(1 - \bar Z_1)^2}{1 - \bar Z} < 0.
\end{aligned}
$$

Therefore, using relation (13.134), we obtain

$$
\begin{aligned}
&2 \sum_{q=1}^{\infty} \left(e^{qn\bar t} - e^{qn(2\bar t_1 - \bar t)}\right) \cos qn\alpha = -\frac{\operatorname{sh} n\bar t}{\operatorname{ch} n\bar t - \cos n\alpha} + \frac{\operatorname{sh} n(2\bar t_1 - \bar t)}{\operatorname{ch} n(2\bar t_1 - \bar t) - \cos n\alpha},\\
&2 \sum_{q=1}^{\infty} \left(e^{qn\bar t} + e^{qn(2\bar t_1 - \bar t)}\right) \sin qn\alpha = \frac{\sin n\alpha}{\operatorname{ch} n\bar t - \cos n\alpha} + \frac{\sin n\alpha}{\operatorname{ch} n(2\bar t_1 - \bar t) - \cos n\alpha}.
\end{aligned} \tag{13.135}
$$

Transforming the remaining sums entering into expressions (13.127), (13.130), and using expansions (13.134), after some simple operations we obtain

$$
\begin{aligned}
&\sum_{q=1}^{\infty} \left[\left(-e^{(qn-1)\bar t} + e^{(qn-1)(2\bar t_1 - \bar t)}\right) \sin(qn-1)\alpha + \left(e^{(qn+1)\bar t} -\right.\right.\\
&\left.\left. - e^{(qn+1)(2\bar t_1 - \bar t)}\right) \sin(qn+1)\alpha\right] = 2\cos\alpha \left[\operatorname{sh}\bar t \sum_{q=1}^{\infty} e^{qn\bar t} \sin qn\alpha - \right.\\
&\left. - \operatorname{sh}(2\bar t_1 - \bar t) \sum_{q=1}^{\infty} e^{qn(2\bar t_1 - \bar t)} \sin qn\alpha\right] + 2\sin\alpha \left[\operatorname{ch}\bar t \sum_{q=1}^{\infty} e^{qn\bar t} \cos qn\alpha - \right.\\
&\left. - \operatorname{ch}(2\bar t_1 - \bar t) \sum_{q=1}^{\infty} e^{qn(2\bar t_1 - \bar t)} \cos qn\alpha\right] = \cos\alpha \left[\operatorname{sh}\bar t \frac{\sin n\alpha}{\operatorname{ch} n\bar t - \cos n\alpha} - \right.\\
&\left. - \operatorname{sh}(2\bar t_1 - \bar t) \frac{\sin n\alpha}{\operatorname{ch} n(2\bar t_1 - \bar t) - \cos n\alpha}\right] - \sin\alpha \left[\operatorname{ch}\bar t \frac{\operatorname{sh} n\bar t}{\operatorname{ch} n\bar t - \cos n\alpha} - \right.\\
&\left. - \operatorname{ch}(2\bar t_1 - \bar t) \frac{\operatorname{sh} n(2\bar t_1 - \bar t)}{\operatorname{ch} n(2\bar t_1 - \bar t) - \cos n\alpha} - 2\operatorname{sh}\bar t_1 \operatorname{sh}(\bar t_1 - \bar t)\right].
\end{aligned} \tag{13.136}
$$

$$\sum_{q=1}^{\infty}\left[\left(-e^{(qn-1)\bar{i}}+e^{(qn-1)(2\bar{i}_1-\bar{i})}\right)\cos(qn-1)\alpha+\left(-e^{(qn+1)\bar{i}}+\right.\right.$$

$$\left.\left.+e^{(qn+1)(2\bar{i}_1-\bar{i})}\right)\cos(qn+1)\alpha\right]=2\cos\alpha\left[-\operatorname{ch}\bar{i}\sum_{q=1}^{\infty}e^{qn\bar{i}}\cos qn\alpha+\right.$$

$$\left.+\operatorname{ch}(2\bar{i}_1-\bar{i})\sum_{q=1}^{\infty}e^{qn(2\bar{i}_1-\bar{i})}\cos qn\alpha\right]+2\sin\alpha\left[\operatorname{sh}\bar{i}\sum_{q=1}^{\infty}e^{qn\bar{i}}\sin qn\alpha-\right.$$

$$\left.-\operatorname{sh}(2\bar{i}_1-\bar{i})\sum_{q=1}^{\infty}e^{qn(2\bar{i}_1-\bar{i})}\sin qn\alpha\right]=\cos\alpha\left[\operatorname{ch}\bar{i}\frac{\operatorname{sh}n\bar{i}}{\operatorname{ch}n\bar{i}-\cos n\alpha}-\right.$$

$$\left.-\operatorname{ch}(2\bar{i}_1-\bar{i})\frac{\operatorname{sh}n(2\bar{i}_1-\bar{i})}{\operatorname{ch}n(2\bar{i}_1-\bar{i})-\cos n\alpha}-2\operatorname{sh}\bar{i}_1\operatorname{sh}(\bar{i}_1-\bar{i})\right]+$$

$$+\sin\alpha\left[\operatorname{sh}\bar{i}\frac{\sin n\alpha}{\operatorname{ch}n\bar{i}-\cos n\alpha}-\operatorname{sh}(2\bar{i}_1-\bar{i})\frac{\sin n\alpha}{\operatorname{ch}n(2\bar{i}_1-\bar{i})-\cos n\alpha}\right]. \qquad (13.137)$$

$$\sum_{q=1}^{\infty}\left[\left(e^{(qn-1)\bar{i}}+e^{(qn-1)(2\bar{i}_1-\bar{i})}\right)\cos(qn-1)\alpha-\left(e^{(qn+1)\bar{i}}+\right.\right.$$

$$\left.\left.+e^{(qn+1)(2\bar{i}_1-\bar{i})}\right)\cos(qn+1)\alpha\right]=-2\cos\alpha\left[\operatorname{sh}\bar{i}\sum_{q=1}^{\infty}e^{qn\bar{i}}\cos qn\alpha+\right.$$

$$\left.+\operatorname{sh}(2\bar{i}_1-\bar{i})\sum_{q=1}^{\infty}e^{qn(2\bar{i}_1-\bar{i})}\cos qn\alpha\right]+2\sin\alpha\left[\operatorname{ch}\bar{i}\sum_{q=1}^{\infty}e^{qn\bar{i}}\sin qn\alpha+\right.$$

$$\left.+\operatorname{ch}(2\bar{i}_1-\bar{i})\sum_{q=1}^{\infty}e^{qn(2\bar{i}_1-\bar{i})}\sin qn\alpha\right]=\cos\alpha\left[\operatorname{sh}\bar{i}\frac{\operatorname{sh}n\bar{i}}{\operatorname{ch}n\bar{i}-\cos n\alpha}+\right.$$

$$\left.+\operatorname{sh}(2\bar{i}_1-\bar{i})\frac{\operatorname{sh}n(2\bar{i}_1-\bar{i})}{\operatorname{ch}n(2\bar{i}_1-\bar{i})-\cos n\alpha}+2\operatorname{sh}\bar{i}_1\operatorname{ch}(\bar{i}_1-\bar{i})\right]+$$

$$+\sin\alpha\left[\operatorname{ch}\bar{i}\frac{\sin n\alpha}{\operatorname{ch}n\bar{i}-\cos n\alpha}+\operatorname{ch}(2\bar{i}_1-\bar{i})\frac{\sin n\alpha}{\operatorname{ch}n(2\bar{i}_1-\bar{i})-\cos n\alpha}\right]. \qquad (13.138)$$

$$\sum_{q=1}^{\infty}\left[\left(e^{(qn-1)\bar{i}}+e^{(qn-1)(2\bar{i}_1-\bar{i})}\right)\sin(qn-1)\alpha+\left(e^{(qn+1)\bar{i}}+\right.\right.$$

$$\left.\left.+e^{(qn+1)(2\bar{i}_1-\bar{i})}\right)\sin(qn+1)\alpha\right]=2\cos\alpha\left[\operatorname{ch}\bar{i}\sum_{q=1}^{\infty}e^{qn\bar{i}}\sin qn\alpha+\right.$$

$$\left.+\operatorname{ch}(2\bar{i}_1-\bar{i})\sum_{q=1}^{\infty}e^{qn(2\bar{i}_1-\bar{i})}\sin qn\alpha\right]+2\sin\alpha\left[\operatorname{sh}\bar{i}\sum_{q=1}^{\infty}e^{qn\bar{i}}\cos qn\alpha+\right.$$

$$\left.+\operatorname{sh}(2\bar{i}_1-\bar{i})\sum_{q=1}^{\infty}e^{qn(2\bar{i}_1-\bar{i})}\cos qn\alpha\right]=\cos\alpha\left[\operatorname{ch}\bar{i}\frac{\sin n\alpha}{\operatorname{ch}n\bar{i}-\cos n\alpha}+\right.$$

$$\left.+\operatorname{ch}(2\bar{i}_1-\bar{i})\frac{\sin n\alpha}{\operatorname{ch}n(2\bar{i}_1-\bar{i})-\cos n\alpha}\right]-\sin\alpha\left[\operatorname{sh}\bar{i}\frac{\operatorname{sh}n\bar{i}}{\operatorname{ch}n\bar{i}-\cos n\alpha}+\right.$$

$$\left.+\operatorname{sh}(2\bar{i}_1-\bar{i})\frac{\operatorname{sh}n(2\bar{i}_1-\bar{i})}{\operatorname{ch}n(2\bar{i}_1-\bar{i})-\cos n\alpha}+2\operatorname{sh}\bar{i}_1\operatorname{ch}(\bar{i}_1-\bar{i})\right]. \qquad (13.139)$$

Now, introducing the expressions (13.127) and (13.130) the values of the sums (13,136)-(13.139), we finally obtain

$$\sigma_{m_z}(\bar{t},\alpha)=\sigma^0_{m_z}(\bar{t},\alpha)+\sigma^R_{m_z}(\bar{t},\alpha)+\sigma^P_{m_z}(\bar{t},\alpha), \tag{13.140}$$

where $\sigma^0_{m_z}$ and $\sigma^R_{m_z}$ are determined by relations (13.128), (13.129), and the self-balanced normal stresses due to discrete fixing of the shell

$$\sigma^P_{m_z}(\bar{t},\alpha)=\frac{N_z(0)}{2\pi hR_0\cos\beta}e^{-\bar{t}}\left[-\frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t}-\cos n\alpha}+\frac{\operatorname{sh} n(2\bar{t}_1-\bar{t})}{\operatorname{ch} n(2\bar{t}_1-\bar{t})-\cos n\alpha}\right]+$$
$$+\frac{M_{x_\alpha}(0)}{\pi hR_0^2\cos\beta}e^{-\bar{t}}\left[\operatorname{sh}\bar{t}\frac{\sin n\alpha}{\operatorname{ch} n\bar{t}-\cos n\alpha}-\operatorname{sh}(2\bar{t}_1-\bar{t})\frac{\sin n\alpha}{\operatorname{ch} n(2\bar{t}_1-\bar{t})-\cos n\alpha}\right]+$$
$$+\frac{M_{y_\alpha}(0)}{\pi hR_0^2\cos\beta}e^{-\bar{t}}\left[\operatorname{ch}\bar{t}\frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t}-\cos n\alpha}-\right.$$
$$\left.-\operatorname{ch}(2\bar{t}_1-\bar{t})\frac{\operatorname{sh} n(2\bar{t}_1-\bar{t})}{\operatorname{ch} n(2\bar{t}_1-\bar{t})-\cos n\alpha}-2\operatorname{sh}\bar{t}_1\operatorname{sh}(\bar{t}_1-\bar{t})\right]. \tag{13.141}$$

Here

$$M_{x_\alpha}=M_x\cos\alpha+M_y\sin\alpha;\quad M_{y_\alpha}=-M_x\sin\alpha+M_y\cos\alpha \tag{13.142}$$

are the bending moments in axes x_α, y_α, rotated through angle α relative to the initial axes x, y.

Considering (13.142), we have from expression (13.128)

$$\sigma^0_{m_z}(\bar{t},\alpha)=\frac{N_z(\bar{t})}{2\pi hR_0\cos\beta}e^{-\bar{t}}-\frac{M_{y_\alpha}(\bar{t})}{\pi hR_0^2\cos\beta}e^{-2\bar{t}}. \tag{13.143}$$

For tangential stresses, we correspondingly obtain

$$\tau_{n_zm_z}(\bar{t},\alpha)=\tau^0_{n_zm_z}(\bar{t},\alpha)+\tau^R_{n_zm_z}(\bar{t},\alpha)+\tau^P_{n_zm_z}(\bar{t},\alpha), \tag{13.144}$$

where $\tau^0_{n_zm_z}$ and $\tau^R_{n_zm_z}$ are determined by relations (13.131), (13.132), and the self-balanced tangential stresses due to discrete fixing of the shell

$$\tau^P_{n_zm_z}(\bar{t},\alpha)=\frac{N_z(0)}{2\pi\sqrt{\gamma}hR_0\cos\beta}e^{-\bar{t}}\left[\frac{\sin n\alpha}{\operatorname{ch} n\bar{t}-\cos n\alpha}+\frac{\sin n\alpha}{\operatorname{ch} n(2\bar{t}_1-\bar{t})-\cos n\alpha}\right]+$$
$$+\frac{M_{x_\alpha}(0)}{\pi\sqrt{\gamma}hR_0^2\cos\beta}e^{-\bar{t}}\left[\operatorname{sh}\bar{t}\frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t}-\cos n\alpha}+\operatorname{sh}(2\bar{t}_1-\bar{t})\frac{\operatorname{sh} n(2\bar{t}_1-\bar{t})}{\operatorname{ch} n(2\bar{t}_1-\bar{t})-\cos n\alpha}+\right.$$
$$\left.+2\operatorname{sh}\bar{t}_1\operatorname{ch}(\bar{t}_1-\bar{t})\right]-\frac{M_{y_\alpha}(0)}{\pi\sqrt{\gamma}hR_0^2\cos\beta}e^{-\bar{t}}\left[\operatorname{ch}\bar{t}\frac{\sin n\alpha}{\operatorname{ch} n\bar{t}-\cos n\alpha}+\right.$$
$$\left.+\operatorname{ch} n(2\bar{t}_1-\bar{t})\frac{\sin n\alpha}{\operatorname{ch} n(2\bar{t}_1-\bar{t})-\cos n\alpha}\right]. \tag{13.145}$$

Considering (13.142), from (13.131) we also have

$$\tau^{0}_{n_z m_z}(t,\alpha) = \frac{M_z(t)}{2\pi h R_1^2} e^{-2t} + \frac{M_{x_{e0}}(t)}{\pi h R_1^2 \operatorname{ctg}\beta} e^{-2t}. \tag{13.146}$$

Here $M_{x_{e0}}(t)$ is the moment of the external forces applied to the cut-off portion of the shell about axis x passing through the cone apex:

$$M_{x_{e0}}(t) = M_{x_e}(t)\cos\alpha + M_{y_e}(t)\sin\alpha, \tag{13.147}$$

where

$$M_{x_e}(t) = l_0 c' Q_y(t) + M_x(t); \quad M_{y_e}(t) = -l_0 c' Q_x(t) + M_y(t) \tag{13.148}$$

are the moments of the external forces applied to the cut-off portion of the shell about the initial axes x, y passing through the cone apex.

Expressions (13.140) and (13.144) completely determine the stressed state of a circular conical shell for cyclic fixing at an arbitrary number of points and for an arbitrary external load.

These expressions are suitable for determining the stresses at any point of a shell of arbitrary length. Limitation (13.121) is generally nonessential, since even when it is not observed, a perceptible error is introduced only into the first terms of the series determining the self-balanced stresses $\sigma^{P}_{m_z}$, $\tau^{P}_{n_z m_z}$.

Let us also note that the self-balanced stresses $\sigma^{R}_{m_z}$, $\tau^{R}_{n_z m_z}$ in expressions (13.140), (13.144) vanish, as follows from relations (13.13), (13.49) and (13.129), (13.152), if the component $p_{m_z}(t,\alpha)$ of the external surface load and the external forces $\bar{\bar{N}}(\alpha)$ applied to the end $\bar{Z} = \bar{Z}_1$ contain no self-balanced components. This takes place in the majority of cases of practical importance.

For the nontruncated conical shell ($\bar{Z}_1 = 1$), general expressions (13.140), (13.144) are simplified. According to (13.24), in this case

$\bar{t}_1=-\infty$. The corresponding formulas can be obtained from expressions (13.140), (13.144) by means of passage to the limit at $\bar{t}_1\to-\infty$. However, However, the same result can be obtained much more simply by using the values obtained above for arbitrary constants (13.115)-(13.118). Then, successively assuming at all the stages of previous transformations $\bar{t}_1=-\infty$, we can easily obtain

$$\sigma^P_{m_z}(t,\alpha)=-\frac{N_z(0)}{2\pi h R_0\cos\beta}e^{-t}\left[1+\frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t}-\cos n\alpha}\right]+\frac{M_{x_a}(0)}{\pi h R_0^2\cos\beta}\times$$
$$\times e^{-t}\operatorname{sh}\bar{t}\frac{\sin n\alpha}{\operatorname{ch} n\bar{t}-\cos n\alpha}+\frac{M_{y_a}(0)}{\pi h R_0^2\cos\beta}e^{-t}\operatorname{ch}\bar{t}\left[1+\frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t}-\cos n\alpha}\right], \quad (13.149)$$

$$\tau^P_{n_z m_z}(t,\alpha)=\frac{N_z(0)}{2\pi h\sqrt{\gamma}R_0\cos\beta}e^{-t}\frac{\sin n\alpha}{\operatorname{ch} n\bar{t}-\cos n\alpha}+\frac{M_{x_a}(0)}{\pi h\sqrt{\gamma}R_0^2\cos\beta}\times$$
$$\times e^{-t}\operatorname{sh}\bar{t}\left[1+\frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t}-\cos n\alpha}\right]+\frac{M_{y_a}(0)}{\pi h\sqrt{\gamma}R_0^2\cos\beta}e^{-t}\operatorname{ch}\bar{t}\frac{\sin n\alpha}{\operatorname{ch} n\bar{t}-\cos n\alpha}. \quad (13.150)$$

As in the case of a truncated conical shell, the stresses $\sigma^0_{m_z}$, $\tau^0_{n_z m_z}$ are determined by expressions (13.143), (13.146).

By passing to the limit in the expressions (13.129), (13.132) when $t_1\to-\infty$, the stresses $\sigma^R_{m_z}$, $\tau^R_{n_z m_z}$ can be reduced to the form

$$\sigma^R_{m_z}(t,\alpha)=\frac{R_0}{\pi h\sin\beta}e^{-t}\sum_{k=2}^{\infty}\left\{\oint\cos k(\alpha-\zeta)\left[-e^{k\bar{t}}\int_{-\infty}^{0}p_{m_z}(\xi,\zeta)\operatorname{ch}k\bar{\xi}e^{2\xi}d\xi+\right.\right.$$
$$\left.\left.+\int_{-\infty}^{t}p_{m_z}(\xi,\zeta)\operatorname{ch}k(\bar{\xi}-\bar{t})e^{2\xi}d\xi\right]d\zeta\right\}. \quad (13.151)$$

$$\tau^R_{n_z m_z}(t,\alpha)=-\frac{R_0}{\pi h\sqrt{\gamma}\sin\beta}e^{-t}\sum_{k=2}^{\infty}\left\{\oint\sin k(\alpha-\zeta)\left[e^{k\bar{t}}\int_{-\infty}^{0}p_{m_z}(\xi,\zeta)\times\right.\right.$$
$$\left.\left.\times\operatorname{ch}k\bar{\xi}e^{2\xi}d\xi+\int_{-\infty}^{t}p_{m_z}(\xi,\zeta)\operatorname{sh}k(\bar{\xi}-\bar{t})e^{2\xi}d\xi\right]d\zeta\right\}. \quad (13.152)$$

It should be noted that in studying deviations from the law of plane sections caused by point fixing of the end t = 0, expressions (13.149)-(13.152) together with (13.143), (13.146) can also be used for a truncated conical shell. Then the discrepancy of the results in comparison with the solution considering the actual boundary conditions on the end

$t = t_1$ will be practially imperceptible everywhere with the exception of the edge zone near the end $t = t_1$.

The stresses $\sigma^R_{m_z}$ and $\tau^R_{n_z m_z}$, as already noted, are absent in the majority of cases of practical importance. However, even in cases where these stresses do not vanish, they can be neglected in the study of point fixing of the shell by assuming

$$\sigma_{m_z} \approx \sigma^0_{m_z} + \sigma^P_{m_z};\ \tau_{n_z m_z} \approx \tau^0_{n_z m_z} + \tau^P_{n_z m_z}. \tag{13.153}$$

Passing to linear forces $T = \sigma_{m_z} h,\ S = \tau_{n_z m_z} h$, on the basis of (13.153) we have for normal forces

$$e^{\bar{t}} T(t, \alpha) = (T_N)_0 \left[\frac{N_z(t)}{N_z(0)} - \Phi_{11}(n\bar{t}, n\alpha)\right] + e^{-\bar{t}}\left[(T_{M_x})_0 \frac{M_x(t)}{M_x(0)} \sin\alpha - \right.$$
$$\left. - (T_{M_y})_0 \frac{M_y(t)}{M_y(0)} \cos\alpha\right] + \operatorname{sh}\bar{t}\left[(T_{M_x})_0 \cos\alpha + (T_{M_y})_0 \sin\alpha\right]\Phi_1(n\bar{t}, n\alpha) +$$
$$+ \operatorname{ch}\bar{t}\left[-(T_{M_x})_0 \sin\alpha + (T_{M_y})_0 \cos\alpha\right]\Phi_{11}(n\bar{t}, n\alpha), \tag{13.154}$$

where

$$(T_N)_0 = \frac{N_z(0)}{2\pi R_0 \cos\beta} \tag{13.155}$$

are the beam forces in the end section $t = 0$ due to the axial force N_z;

$$(T_{M_x})_0 = \frac{M_x(0)}{\pi R_0^2 \cos\beta};\quad (T_{M_y})_0 = \frac{M_y(0)}{\pi R_0^2 \cos\beta} \tag{13.156}$$

are the maximum beam forces in the end section $t = 0$ due to the bending moments M_x and M_y

$$\Phi_1(n\bar{t}, n\alpha) = \frac{\sin n\alpha}{\operatorname{ch} n\bar{t} - \cos n\alpha}, \tag{13.157}$$

$$\Phi_{11}(n\bar{t}, n\alpha) = 1 + \frac{\operatorname{sh} n\bar{t}}{\operatorname{ch} n\bar{t} - \cos n\alpha} \tag{13.158}$$

are function whose dependence on $n\bar{t}$ is shown in Figs. 13.3 and 13.4;

for tangential forces

$$\sqrt{\gamma}\, e^{\bar{t}} S(t, \alpha) = \sqrt{\gamma}\, e^{-\bar{t}}\left[(S_{M_z})_0 \frac{M_z(t)}{M_z(0)} + (S_{M_{x_0}})_0 \frac{M_{x_0}(t)}{M_{x_0}(0)} \cos\alpha + \right.$$
$$\left. + (S_{M_{y_0}})_0 \frac{M_{y_0}(t)}{M_{y_0}(0)} \sin\alpha\right] + (T_N)_0 \Phi_1(n\bar{t}, n\alpha) + \operatorname{sh}\bar{t}\left[(T_{M_x})_0 \cos\alpha + \right.$$
$$\left. + (T_{M_y})_0 \sin\alpha\right]\Phi_{11}(n\bar{t}, n\alpha) + \operatorname{ch}\bar{t}\left[(T_{M_x})_0 \sin\alpha - (T_{M_y})_0 \cos\alpha\right]\Phi_1(n\bar{t}, n\alpha). \tag{13.159}$$

where

$$(S_{M_z})_0 = \frac{M_z(0)}{2\pi R_0^2} \tag{13.160}$$

are the tangential forces in the end section t = 0 due to the twisting moment M_z in free torsion

$$(S_{M_{x_0}})_0 = \frac{M_{x_0}(0)}{\pi R_0^2 \operatorname{ctg} \beta}; \quad (S_{M_{y_0}})_0 = \frac{M_{y_0}(0)}{\pi R_0^2 \operatorname{ctg} \beta} \tag{13.161}$$

are the maximum bean tangential forces in the end section t = 0, corresponding in accordance with (13.148) to the moments of the external forces about the Ox_0, Oy_0, axes passing through the cone apex.

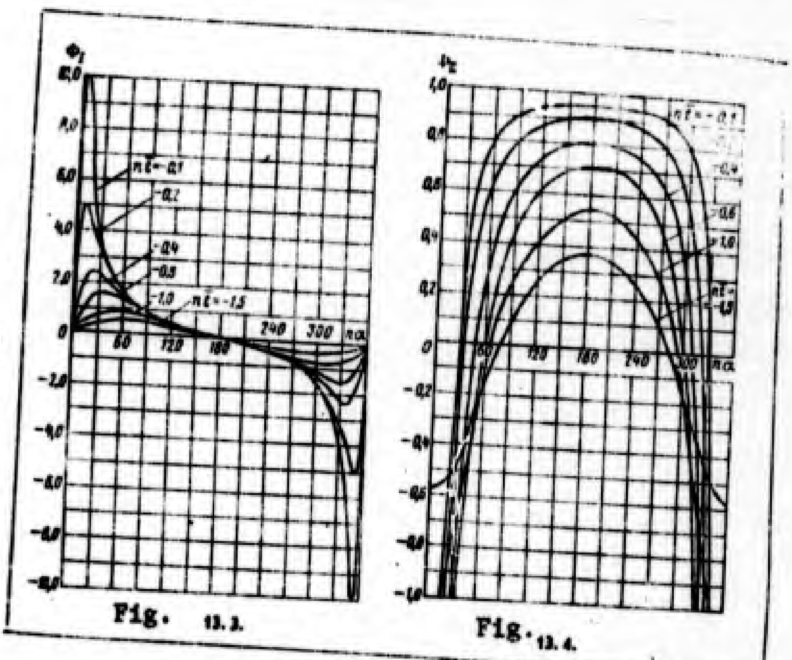


Fig. 13.3. Fig. 13.4.

Formulas (13.154) and (13.159) are very clear. Their structure leads to the conclusion that secondary stresses related to point firing of a conical shell are independent of the nature of the distribution of the external load under the assumptions made above. These stresses are determined only by the vector and point of application of the resultant of the external load, knowing which one can readily calculate the peak values of the beam forces in the section t = 0 (13.155), (13.156), (13.160), (13.161), then, for a given number of fixing points and remaining geometric parameters, find the secondary stresses in accordance with (13.154), (13.159).

Considering that the secondary stresses damp out rapidly, approximately over a length equal to the distance between the fixing points, one can assume for this zone

$$
\begin{gathered}
Q_x(t) \approx Q_x(0); \quad Q_y(t) \approx Q_y(0), \\
N_z(t) \approx N_z(0), \\
M_x(t) = M_x(0) + Q_y(0) l_0 \bar{Z} = M_x(0) + Q_y(0) l_0 (1 - e^t), \\
M_y(t) = M_y(0) - Q_x(0) l_0 \bar{Z} = M_y(0) - Q_x(0) l_0 (1 - e^t) \\
M_z(t) \approx M_z(0).
\end{gathered}
\tag{13.162}
$$

Introducing (13.162) into (13.154) and (13.159), we obtain

$$
\begin{gathered}
e^t T(t, \alpha) = (T_N)_0 [1 - \Phi_{II}(n\bar{t}, n\alpha)] + (T_{M_x})_0 [e^{-t}(1 + k_1) - k_1] \sin \alpha - \\
- (T_{M_y})_0 [e^{-t}(1 - k_2) + k_2] \cos \alpha + \operatorname{sh} \bar{t} [(T_{M_x})_0 \cos \alpha + (T_{M_y})_0 \sin \alpha] \times \\
\times \Phi_I(n\bar{t}, n\alpha) + \operatorname{ch} \bar{t} [-(T_{M_x})_0 \sin \alpha + (T_{M_y})_0 \cos \alpha] \Phi_{II}(n\bar{t}, n\alpha),
\end{gathered}
\tag{13.163}
$$

where

$$
k_1 = \frac{Q_y(0)}{M_x(0)} l_0, \quad k_2 = \frac{Q_x(0)}{M_y(0)} l_0.
\tag{13.164}
$$

For tangential forces, we correspondingly have

$$
\begin{gathered}
\sqrt{\gamma} e^t S(t, \alpha) = \sqrt{\gamma} e^{-t} [(S_{M_z})_0 + (S_{M_{x_0}})_0 \cos \alpha + (S_{M_{y_0}})_0 \sin \alpha] + \\
+ (T_N)_0 \Phi_I(n\bar{t}, n\alpha) + \operatorname{sh} \bar{t} [(T_{M_x})_0 \cos \alpha + (T_{M_y})_0 \sin \alpha] \Phi_{II}(n\bar{t}, n\alpha) + \\
+ \operatorname{ch} \bar{t} [(T_{M_x})_0 \sin \alpha - (T_{M_y})_0 \cos \alpha] \Phi_I(n\bar{t}, n\alpha).
\end{gathered}
\tag{13.165}
$$

Formulas (13.163) and (13.165) represent an explicit dependence of the forces T and S on the coordinates t and α near the end t = 0. These formulas are very convenient for the analysis of anomalies of the stressed state, generated by point fixing of the shell. Figures 13.5-13.9 show the distribution curves of normal and tangential forces T and S, calculated from formulas (13.163) and (13.165) for different cases of loading.

We will determine the character of the distribution of the forces T and S near the singular points, i.e., in the vicinity of the fixing points of the shell. For this purpose, in expressions (13.163) and (13.165), we will change to the coordinates ξ and η, measured along the arc of the cross section and along the generatrix, respectively. Placing the origin of the coordinates ξ and η at one of the fixing points, with coordinates $t=0,\ \alpha=\frac{2\pi}{n}m$, we will assume that

$$\zeta=\left(\alpha-\frac{2\pi}{n}m\right)R_0,$$
$$\eta=\frac{l_0\bar{Z}}{\cos\beta}=\frac{l_0}{\cos\beta}(1-e^t), \tag{13.166}$$

or, changing to dimensionless coordinates,

$$\bar{\zeta}=\frac{\zeta}{R_0}=\alpha-\frac{2\pi}{n}m,$$
$$\bar{\eta}=\frac{\eta}{R_0}=\frac{1}{\sin\beta}(1-e^t). \tag{13.167}$$

Having replaced the variables in expressions (13.163) and (13.165), we then expand the trigonometric and hyperbolic functions as series in powers of the coordinates ξ and η.

We have:

$$e^{\pm t}=1\mp\sin\beta\bar{\eta}+\ldots,$$
$$\operatorname{ch} n\bar{t}=1+\frac{n^2}{2\gamma}\bar{\eta}^2+\ldots,$$
$$\operatorname{sh} n\bar{t}=-\frac{n}{\sqrt{\gamma}}\bar{\eta}+\ldots,$$
$$\sin\alpha=\sin\frac{2\pi}{n}m+\bar{\zeta}\cos\frac{2\pi}{n}m, \tag{13.168}$$
$$\cos\alpha=\cos\frac{2\pi}{n}m-\bar{\zeta}\sin\frac{2\pi}{n}m,$$
$$\sin n\alpha=n\bar{\zeta}+\ldots,$$
$$\cos n\alpha=1-\frac{n^2\bar{\zeta}^2}{2}+\ldots$$

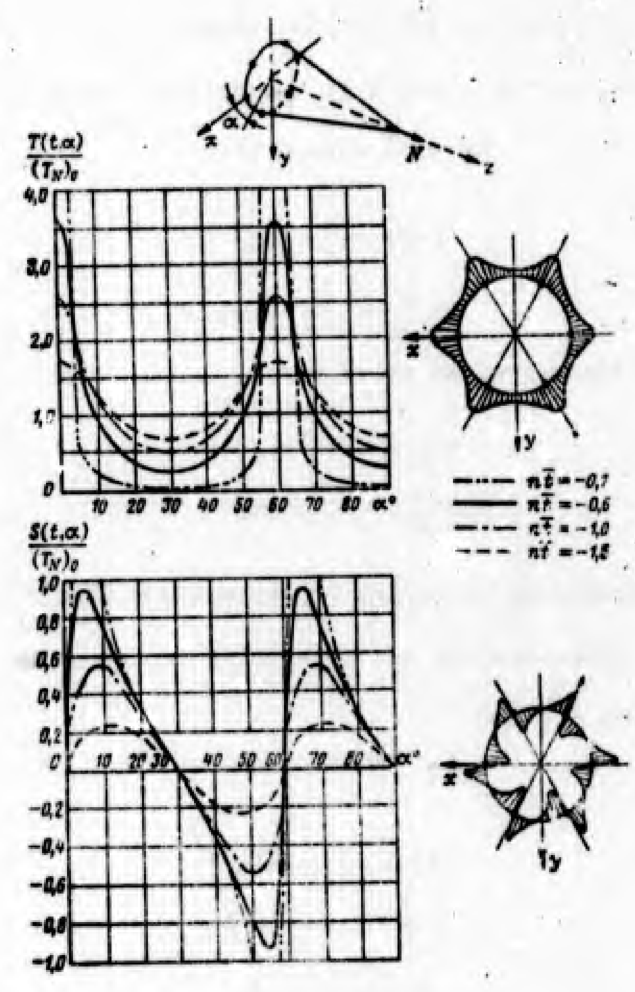

x
y
N
z
α
T(t,α)
(T_N)_0
4,0
3,0
2,0
0
10 20 30 40 50 60 70 80 α°
S(t,α)
(T_N)_0
1,0
0,8
0,6
0,4
0,2
-0,2
-0,4
-0,6
-0,8
-1,0

Fig. 13.5

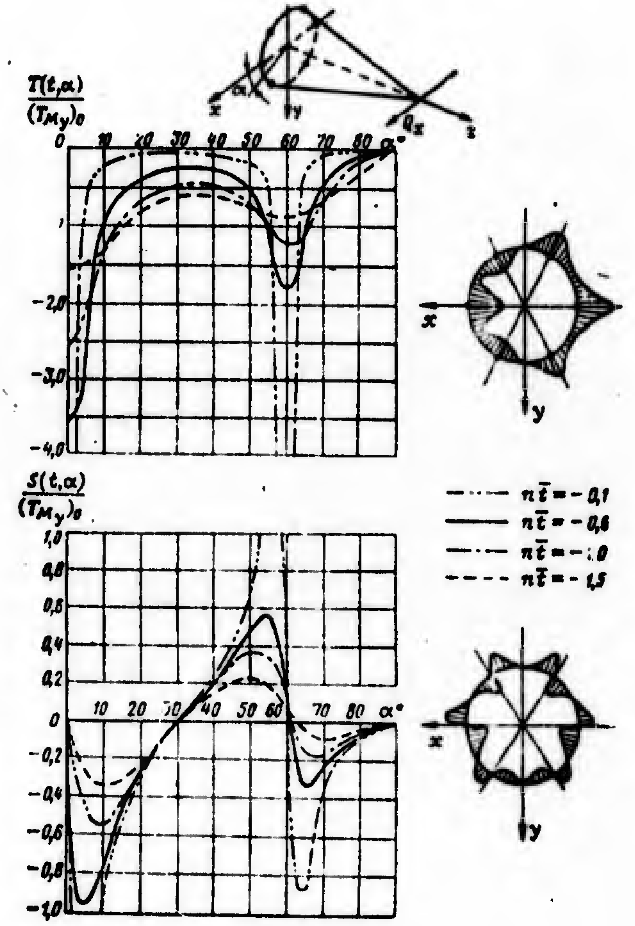

$\frac{T(t,\alpha)}{(T_{M_y})_0}$
0
10 20 30 40 50 60 70 80 $\alpha°$
-1
-2,0
-3,0
-4,0
x
y
z
α
Q_z
$\frac{S(t,\alpha)}{(T_{M_y})_0}$
1,0
0,8
0,6
0,4
0,2
0
-0,2
-0,4
-0,6
-0,8
-1,0
$n\bar{t} = -0,1$
$n\bar{t} = -0,6$
$n\bar{t} = -1,0$
$n\bar{t} = -1,5$

Fig. 13.6

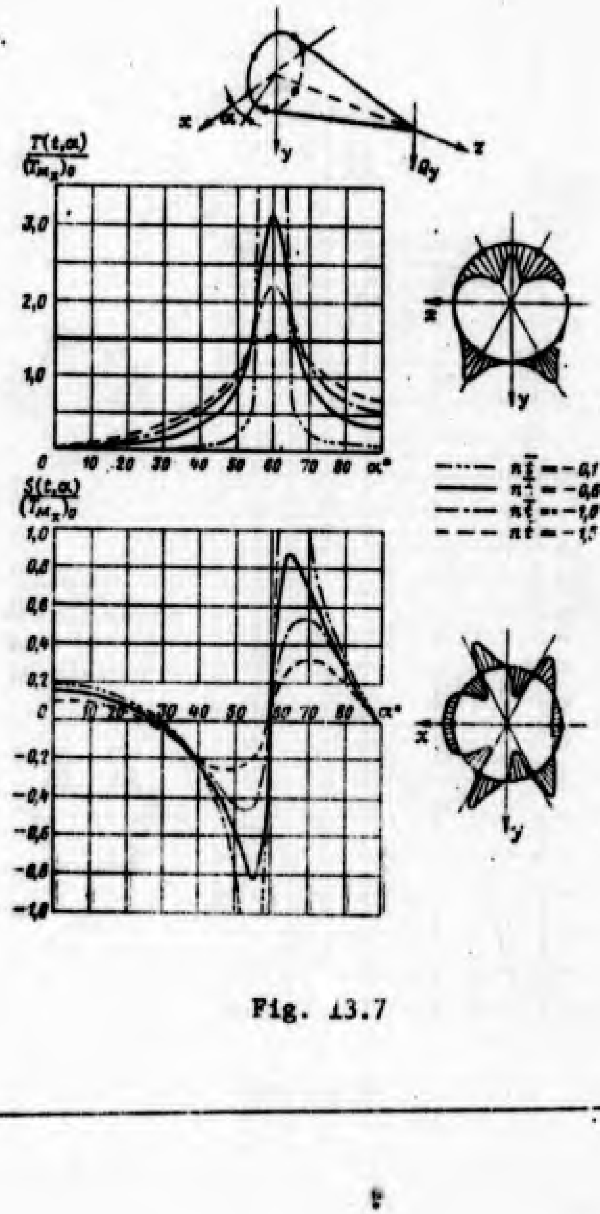
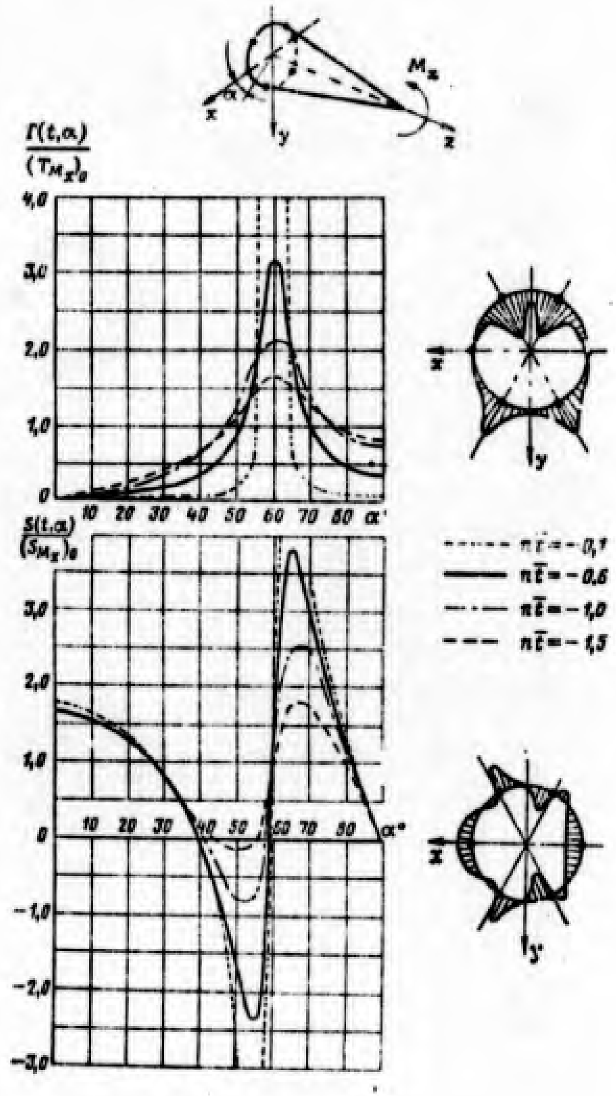


Fig. 13.7



M_z
x
y
z
α
$\frac{\Gamma(t,\alpha)}{(T_{M_x})_0}$
4,0
3,0
2,0
1,0
0
10 20 30 40 50 60 70 80 $\alpha°$
$\frac{S(t,\alpha)}{(S_{M_x})_0}$
3,0
2,0
1,0
0
-1,0
-2,0
-3,0
$n\bar{t} = -0,1$
$n\bar{t} = -0,6$
$n\bar{t} = -1,0$
$n\bar{t} = -1,5$

Fig. 13.8

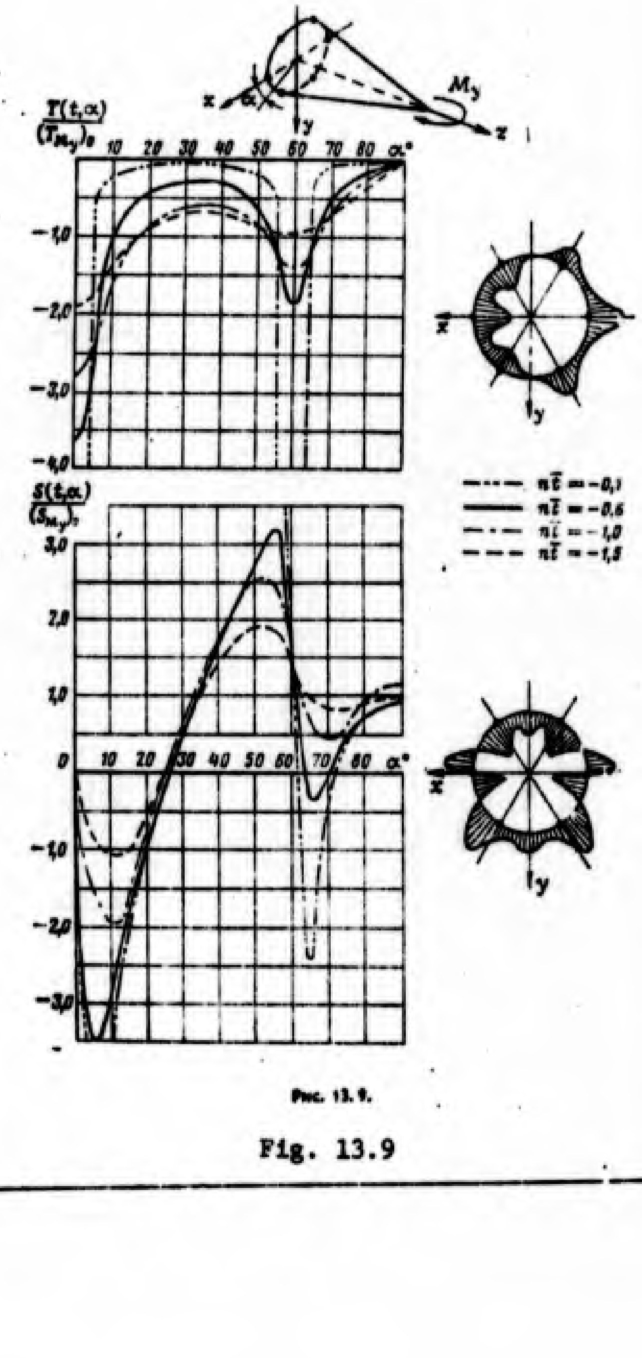


Рис. 13.9.

Fig. 13.9

Using (13.168), for functions (13.157), (13.158) we obtain

$$\Phi_{I}(\bar{\eta}, \bar{\zeta}) \approx \frac{2}{n} \frac{\bar{\zeta}}{\bar{\eta}^2/\gamma + \bar{\zeta}^2}, \qquad (13.169)$$

$$\Phi_{II}(\bar{\eta}, \bar{\zeta}) \approx 1 - \frac{2}{n\sqrt{\gamma}} \frac{\bar{\eta}}{\bar{\eta}^2/\gamma + \bar{\zeta}^2}.$$

Substituting (13.168), (13.169) into (13.163) and (13.165) and neglecting small and finite terms, we obtain the asymptotic formulas

$$T(\bar{\eta}, \bar{\zeta}) \approx \frac{2}{n\sqrt{\gamma}} \left[(T_N)_0 + (T_{M_x})_0 \sin\frac{2\pi}{n} m - (T_{M_y})_0 \cos\frac{2\pi}{n} m \right] \frac{\bar{\eta}}{\bar{\eta}^2/\gamma + \bar{\zeta}^2},$$

$$S(\bar{\eta}, \bar{\zeta}) \approx \frac{2}{n\sqrt{\gamma}} \left[(T_N)_0 + (T_{M_x})_0 \sin\frac{2\pi}{n} m - (T_{M_y})_0 \cos\frac{2\pi}{n} \cdot m \right] \frac{\bar{\zeta}}{\bar{\eta}^2/\gamma + \bar{\zeta}^2},$$

whence, in view of (13.155), (13.156) and (13.110), we finally obtain

$$T(\bar{\eta}, \bar{\zeta}) \approx \frac{P_m}{\pi R_0} \frac{\frac{\bar{\eta}}{\sqrt{\gamma}}}{\left(\frac{\bar{\eta}}{\sqrt{\gamma}}\right)^2 + \bar{\zeta}^2}, \qquad (13.170)$$

$$S(\bar{\eta}, \bar{\zeta}) \approx \frac{1}{\sqrt{\gamma}} \frac{P_m}{\pi R_0} \frac{\bar{\zeta}}{\left(\frac{\bar{\eta}}{\sqrt{\gamma}}\right)^2 + \bar{\zeta}^2}.$$

13.5. Cylindrical Shell Under Discrete Fixing Conditions

A cylindrical shell may be treated as the limiting case of a conical shell when $l_0 \to \infty$, and therefore all the results obtained above for a conical shell can be extended to a cylindrical shell by directly passing to the limit.

Let us note that the model of a cylindrical shell with a cross sectional contour nondeformable in its plane is used fairly often in analyses of flying vehicles. This model is very fruitful for the problem under consideration, since it makes it possible to obtain a closed solution and at the same time reflects the actual stressed and strained state of the shell with sufficient accuracy. Even if the solution of this problem on the basis of the equations of general shell theory could be brought to completion, it would be extremely cumbersome. It should also be noted that the deformation of the contour is actually very slight, since geometrical considerations show that it has a multiwave character with the number of half-waves equal to the number of fixing points. Moreover, in actual structures, the possibility of deformation of the contour is practically completely eliminated by the presence of the transverse reinforcing structure.

We will first change to the longitudinal coordinate

$$\bar{Z} = \frac{\bar{z} l_s}{R_0} = \frac{1}{\sin\beta} \bar{Z}, \tag{13.171}$$

constituting the relative distance from the end $\bar{Z} = 0$ to the current point, measured along the generatrix in fractions of the radius R_0. In view of (13.171), for a conical surface we have

$$t = \ln(1 - \sin\beta\bar{Z}),$$

$$\bar{t} = \frac{1}{\sqrt{\gamma}} \frac{\ln(1 - \sin\beta\bar{Z})}{\sin\beta},$$

whence, passing to the limit when $l_0 \to \infty$ $(\sin\beta \to 0)$, for a cylindrical shell we have

$$t = 0; \quad \bar{t} = -\frac{1}{\sqrt{\gamma}} \bar{Z}, \tag{13.172}$$

where

$$\bar{Z} = \frac{z}{R_0}.$$

Now, performing the substitution of coordinates according to (13.172) in the expressions obtained earlier for a conical shell, we can readily find the expressions describing the stressed and strained states of a cylindrical shell for discrete fixing. We will cite the most important ones.

From expressions (13.128), (13.129), we have

$$\sigma^0_{m_z}(\bar{Z}, \alpha) = \frac{N_z(\bar{Z})}{2\pi R_0 h_0} + \frac{M_x(\bar{Z})}{\pi R_0^2 h} \sin\alpha - \frac{M_y(\bar{Z})}{\pi R_0^2 h} \cos\alpha, \tag{13.173}$$

$$\sigma^R_{m_z}(\bar{Z}, \alpha) = \frac{E}{\sqrt{\gamma} R_0} \sum_{k=1}^{\infty} \left\{ \sin k\alpha \left[k\bar{L}_{1k} \operatorname{sh} \frac{k}{\sqrt{\gamma}} \bar{Z} - \frac{\sqrt{\gamma}}{\pi h E} \int_0^{\bar{Z}} \operatorname{ch}\left(\frac{k}{\sqrt{\gamma}}(\bar{Z} - \xi)\right) \times \right.\right.$$

$$\left.\left. \times R_{1k}(\xi) d\xi \right] + \cos k\alpha \left[k\bar{L}_{2k} \operatorname{sh} \frac{k}{\sqrt{\gamma}} \bar{Z} - \frac{\sqrt{\gamma}}{\pi h E} \int_0^{\bar{Z}} \operatorname{ch}\left(\frac{k^2}{\sqrt{\gamma}}(\bar{Z} - \xi)\right) R_{2k}(\xi) d\xi \right] \right\}, \tag{13.174}$$

where

$$\bar{L}_{1k} = \frac{1}{\operatorname{sh} \frac{k}{\sqrt{\gamma}} \bar{Z}_1} \frac{1}{\pi h E k} \left\{ \sqrt{\gamma} R_0 \int_0^{2\pi} \bar{\bar{N}}(\alpha) \sin k\alpha d\alpha + \right.$$

$$\left. + \int_0^{\bar{Z}_1} \operatorname{ch}\left(\frac{k}{\sqrt{\gamma}}(\bar{Z}_1 - \xi)\right) R_{1k}(\xi) d\xi \right\}, \tag{13.175}$$

$$\bar{L}_{2k} = \frac{1}{\operatorname{sh} \frac{k}{\sqrt{\gamma}} \bar{Z}_1} \frac{1}{\pi h E k} \left\{ \sqrt{\gamma} R_0 \int_0^{2\pi} \bar{\bar{N}}(\alpha) \cos k\alpha d\alpha + \right.$$

$$\left. + \int_0^{\bar{Z}_1} \operatorname{ch}\left(\frac{k}{\sqrt{\gamma}}(\bar{Z}_1 - \xi)\right) R_{2k}(\xi) d\xi \right\},$$

$$R_{1k}(\bar{Z}) = R_0^2 \oint p_{m_z}(\bar{Z}, \alpha) \sin k\alpha d\alpha,$$

$$R_{2k}(\bar{Z}) = R_0^2 \oint p_{m_z}(\bar{Z}, \alpha) \cos k\alpha d\alpha. \tag{13.176}$$

Correspondingly, for tangential stresses we obtain from (13.131), (13.132)

$$\tau^0_{n_z m_z} = \frac{M_z(\bar{Z})}{2\pi R_0^2 h} - \frac{Q_x(\bar{Z})}{\pi h R_0} \sin\alpha + \frac{Q_y(\bar{Z})}{\pi h R_0} \cos\alpha, \tag{13.177}$$

$$\tau^R_{n_z m_z} = \frac{G}{R_0} \sum_{k=1}^{\infty} \left\{ \cos k\alpha \left[k\bar{L}_{1k} \operatorname{ch} \frac{k}{\sqrt{\gamma}} \bar{Z} - \frac{\sqrt{\gamma}}{\pi h E} \int_0^{\bar{Z}} \operatorname{sh}\left(\frac{k}{\sqrt{\gamma}}(\bar{Z} - \xi)\right) \times \right.\right.$$

$$\left.\left. \times R_{1k}(\xi) d\xi \right] - \sin k\alpha \left[k\bar{L}_{2k} \operatorname{ch} \frac{k}{\sqrt{\gamma}} \bar{Z} - \frac{\sqrt{\gamma}}{\pi h E} \int_0^{\bar{Z}} \operatorname{sh}\left(\frac{k}{\sqrt{\gamma}}(\bar{Z} - \xi)\right) R_{2k}(\xi) d\xi \right] \right\}. \tag{13.178}$$

Expressions (13.173)-(13.178) are valid for cylindrical shells of finite length $l_1=\bar Z_1 R_0$. For not too short shells satisfying the relation

(13.179)

the self-balanced stresses caused by point fixing of the shell may be represented in closed form. From expressions (13.141), (13.145), we obtain

$$\sigma^p_{m_z}(\bar Z,\alpha)=\frac{N_z(0)}{2\pi R_0 h}\left[-\Phi_{II}\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)+\Phi_{II}\left(-n\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right]-$$
$$-\frac{M_{x_0}(0)}{\pi h R_0^2}\left[\operatorname{sh}\frac{\bar Z}{\sqrt{\bar\gamma}}\Phi_I\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)-\operatorname{sh}\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}}\Phi_I\left(-n\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right]+$$
$$+\frac{M_{y_0}(0)}{\pi h R_0^2}\left[\operatorname{ch}\frac{\bar Z}{\sqrt{\bar\gamma}}\Phi_{II}\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)-\operatorname{ch}\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}}\Phi_{II}\left(-n\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right]. \tag{13.180}$$

$$\tau^p_{n_z m_z}(\bar Z,\alpha)=\frac{N_z(0)}{2\pi\sqrt{\bar\gamma}hR_0}\left[\Phi_I\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)+\Phi_I\left(-n\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right]-$$
$$-\frac{M_{x_0}(0)}{\pi\sqrt{\bar\gamma}hR_0^2}\left[\operatorname{sh}\frac{\bar Z}{\sqrt{\bar\gamma}}\Phi_{II}\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)+\operatorname{sh}\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}}\Phi_{II}\left(-n\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right]-$$
$$-\frac{M_{y_0}(0)}{\pi\sqrt{\bar\gamma}hR_0^2}\left[\operatorname{ch}\frac{\bar Z}{\sqrt{\bar\gamma}}\Phi_I\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)+\operatorname{ch}\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}}\Phi_I\left(-n\frac{2\bar Z_1-\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right], \tag{13.181}$$

where M_{x_0}, M_{y_0} are determined in accordance with relations (13.142).

In studying the edge effect caused by the point fixing of the shell, it is desirable to simplify expressions (13.180), (13.181) by switching to a semiinfinite cylindrical shell. As follows from (13.172), if $\bar Z\to\infty$, then $\bar t\to-\infty$, i.e., the solution for a semiinfinite cylinder can be obtained from the corresponding solution for a nontruncated cone.

Switching to expressions (13.154), (13.159) and passing to the limit when $\beta\to 0$, we obtain

$$T(\bar Z,\alpha)=(T_N)_0\left[\frac{N_z(\bar Z)}{N_z(0)}-\Phi_{II}\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)\right]+(T_{M_x})_0\frac{M_x(\bar Z)}{M_x(0)}\sin\alpha-$$
$$-(T_{M_y})_0\frac{M_y(\bar Z)}{M_y(0)}\cos\alpha-\operatorname{sh}\frac{\bar Z}{\sqrt{\bar\gamma}}\left[(T_{M_x})_0\cos\alpha+(T_{M_y})_0\sin\alpha\right]\times$$
$$\times\Phi_I\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right)+\operatorname{ch}\frac{\bar Z}{\sqrt{\bar\gamma}}\left[(T_{M_x})_0\sin\alpha+(T_{M_y})_0\cos\alpha\right]\Phi_{II}\left(-n\frac{\bar Z}{\sqrt{\bar\gamma}},\ n\alpha\right), \tag{13.82}$$

$$\sqrt{\gamma}S(\bar{Z},\alpha)=\sqrt{\gamma}\left[(S_{M_z})_0\frac{M_z(\bar{Z})}{M_z(0)}+(S_{Q_y})_0\frac{Q_y(\bar{Z})}{Q_y(0)}\cos\alpha+\right.$$
$$\left.+(S_{Q_x})_0\frac{Q_x(\bar{Z})}{Q_x(0)}\sin\alpha\right]+(T_N)_0\Phi_I\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right)-\operatorname{sh}\frac{\bar{Z}}{\sqrt{\gamma}}[(T_{M_x})_0\cos\alpha+$$
$$+(T_{M_y})_0\sin\alpha]\Phi_{II}\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right)+\operatorname{ch}\frac{\bar{Z}}{\sqrt{\gamma}}[(T_{M_x})_0\sin\alpha-$$
$$-(T_{M_y})_0\cos\alpha]\Phi_I\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right) \qquad (13.183)$$

where the peak values of the tangential forces

$$(S_{Q_x})_0=-\frac{Q_x(0)}{\pi R_0 h};\ (S_{Q_y})_0=\frac{Q_y(0)}{\pi R_0 h}. \qquad (13.184)$$

and the peak values $(T_N)_0$, $(T_{M_x})_0$, $(T_{M_y})_0$ of normal forces and the Bredt tangential force $(S_{M_z})_0$ in the section $\bar{Z} = 0$ are determined by expressions (13.155), (13.156), (13.160).

Expressions (13.82), (13.183) are valid over the entire length of the cylindrical shell with the exception of the region near the end $Z=Z_1$. Near the end Z = 0, these expressions determine the concentration of normal and tangential forces which is generated by point fixing. With increasing distance from the end, the concentration vanishes, and the stressed state represented by expressions (13.182), (13.183) corresponds to the beam theory.

Taking approximate relations (13.162) for the zone of the edge effect, we obtain from expressions (13.163), (13.165)

$$T(\bar{Z},\alpha)=(T_N)_0\left[1-\Phi_{II}\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right)\right]+(T_{M_x})_0(1+\bar{k}_1\bar{Z})\sin\alpha-(T_{M_y})_0\times$$
$$\times(1-\bar{k}_2\bar{Z})\cos\alpha-\operatorname{sh}\frac{\bar{Z}}{\sqrt{\gamma}}[(T_{M_x})_0\cos\alpha+(T_{M_y})_0\sin\alpha]\Phi_I\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right)+$$
$$+\operatorname{ch}\frac{\bar{Z}}{\sqrt{\gamma}}[-(T_{M_x})_0\sin\alpha+(T_{M_y})_0\cos\alpha]\Phi_{II}\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right) \qquad (13.185)$$

$$\sqrt{\gamma}S(\bar{Z},\alpha)=\sqrt{\gamma}[(S_{M_z})_0+(S_{Q_y})_0\cos\alpha+(S_{Q_x})_0\sin\alpha]+$$
$$+(T_N)_0\Phi_I\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right)-\operatorname{sh}\frac{\bar{Z}}{\sqrt{\gamma}}[(T_{M_x})_0\cos\alpha+(T_{M_y})_0\sin\alpha]\times$$
$$\times\Phi_{II}\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right)+\operatorname{ch}\frac{\bar{Z}}{\sqrt{\gamma}}[(T_{M_x})_0\sin\alpha-(T_{M_y})_0\cos\alpha]\times$$
$$\times\Phi_I\left(-n\frac{\bar{Z}}{\sqrt{\gamma}},n\alpha\right). \qquad (13.186)$$

where

$$\tilde{k}_1 = \frac{Q_N(0)}{M_x(0)} R_0; \quad \tilde{k}_2 = \frac{Q_x(0)}{M_p(0)} R_0. \tag{13.187}$$

Finally, comparing (13.166) and (13.171), we note that

$$\tilde{\eta} = \tilde{Z}$$

and hence, asymptotic formulas (13.170), obtained for a cone, remain valid for a cylinder as well.

13.6. Conical Shell Reinforced with an End Ring

Let the shell in the section $\tilde{Z} = 0$ be reinforced with an end ring. In this case, separating the ring from the shell, one must examine its equilibrium under the action of the unknown contact forces and of an external load applied to the ring.

An analysis of a conical shell for an arbitrary end load is given in Section 1. Therefore, by further determining the unknown contact forces from the strain compatibility conditions of the shell and ring, one can use the general solution for a conical shell, represented by expressions (13.30)-(13.34).

In the adopted computational scheme, the contact forces are reduced to normal forces $T_c(\alpha) = T(0, \alpha)$ and tangential forces $S_c(\alpha) = S(0, \alpha)$ (Fig. 13.10, a). Let us note that in examining the problem on the basis of moment theory, it would be necessary, in addition to the components noted above, to take into consideration another three components of contact forces: the transverse force Q, and bending and twisting moments M and H (Fig. 13.10 b). However, it is obvious that the rigidity of the

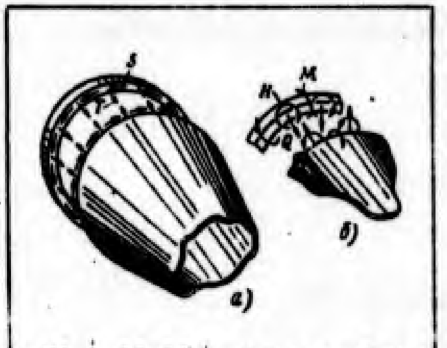


Fig. 13.10. Contact forces

shell in relation to the reception of these components is immeasurably smaller than the tangential forces T and S, and hence, the forces themselves should stand in a similar proportion. For this reason, consideration of transverse forces and moments, which would make the calculation extremely cumbersome in this case, still would not yield any appreciable refinements.

In accordance with general expressions (13.33), (13.34), the contact forces assume the form

$$T_{\kappa}(\alpha)=\frac{N_z(0)}{2\pi R_0\cos\beta}+\frac{M_x(0)}{\pi R_0^2\cos\beta}\sin\alpha-\frac{M_y(0)}{\pi R_0^2\cos\beta}\cos\alpha-$$

$$-\frac{Eh}{\sqrt{\gamma}R_0}\sum_{k=1}^{\infty}k\left[(C_{1k}-\overline{C}_{1k})\sin k\alpha+(C_{2k}-\overline{C}_{2k})\cos k\alpha\right], \qquad (13.188)$$

$$S_{\kappa}(\alpha)=\frac{M_z(0)}{2\pi R_0^2}+\frac{-l_0Q_x(0)+M_y(0)}{\pi R_0^2\operatorname{ctg}\beta}\sin\alpha+\frac{l_0Q_y(0)+M_x(0)}{\pi R_0^2\operatorname{ctg}\beta}\cos\alpha+$$

$$+\frac{Gh}{R_0}\sum_{k=1}^{\infty}k\left[(C_{1k}+\overline{C}_{1k})\cos k\alpha-(C_{2k}+\overline{C}_{2k})\sin k\alpha\right], \qquad (13.189)$$

where the arbitrary constants are determined from the strain compatibility conditions of the shell and ring and from the end loading conditions $\overline{Z} = \overline{Z}_1$.

The strain compatibility conditions are formulated in the form of equality of the displacements of the shell in the section $\overline{Z} = 0$ to the displacements of the ring. The displacements of the ring are obviously determined to within its displacement as a solid. Therefore, in the framework of the adopted computational scheme, the compatibility condition assumes the form

$$[u^*_{m_z}(0,\alpha)]\ \text{shell} = [u^*_{m_z}(\alpha)]\ \text{ring} \qquad (13.190)$$

where $u^{*}_{m_z}$ are the displacements in the direction of the generatrices, corresponding to warpings of the contour $\bar{Z} = 0$.

For a shell, using (13.30), we have

$$u^{*}_{m_z}(0, \alpha) = \sum_{k=2}^{\infty} [(C_{1k} + \bar{C}_{1k}) \sin k\alpha + (C_{2k} + \bar{C}_{2k}) \cos k\alpha]. \tag{13.191}$$

The corresponding displacements of the ring may be represented in the form

$$u^{*}_{m_z}(\alpha) = u^{0}_{m_z}(\alpha) + u^{T}_{m_z}(\alpha) + u^{S}_{m_z}(\alpha), \tag{13.192}$$

where $u^{0}_{m_z}$ are the displacements of the ring due to a given external load to within its displacement as a solid, determined from the analysis of an isolated ring;

$u^{T}_{m_z}$, $u^{S}_{m_z}$ are the displacements of the ring due to self-balanced components of the contact forces T and S.

Without particularizing the ring as an elastic system, on the basis of the superposition principle, we will represent the displacements $u^{T}_{m_z}$, $u^{S}_{m_z}$ as a sum of harmonics of the corresponding harmonics of the external load.

For rings of constant cross section with an arbitrary orientation of the principal central axes upon loading by the forces

$$T_{1k} = \sin k\alpha, \quad T_{2k} = \cos k\alpha \tag{13.193}$$

and

$$S_{1k} = \sin k\alpha, \quad S_{2k} = \cos k\alpha$$

from symmetry considerations we obtain

$$(u^{T}_{m_z})_{1k} = \frac{1}{d^{T}_{k}} \sin k\alpha, \quad (u^{T}_{m_z})_{2k} = \frac{1}{d^{T}_{k}} \cos k\alpha \tag{13.194}$$

and

$$(u^{S}_{m_z})_{1k} = \frac{1}{d^{S}_{k}} \cos k\alpha, \quad (u^{S}_{m_z})_{2k} = -\frac{1}{d^{S}_{k}} \sin k\alpha,$$

where d_k^T and d_k^S are certain generalized rigidities dependent on the number of the harmonic of the corresponding load, on the orientation of the major axes of inertia of the cross section of the rings and its rigidity characteristics, and also on the position of its axis relative to the middle surface of the shell and on the cone angle of the shell.

Using (13.194), in view of (13.188), (13.189) and (13.193), and on the basis of the superposition principle, we can write the expanded expressions for $u_{m_z}^T(\alpha)$, $u_{m_z}^S(\alpha)$. Introducing the latter into (13.192), we obtain

$$u_{m_z}^*(\alpha)=u_{m_z}^0(\alpha)-\frac{Gh}{R_0}\sum_{k=2}^{\infty}k\left\{\left[\frac{V\overline{\gamma}}{d_k^T}(C_{1k}-\bar{C}_{1k})-\frac{1}{d_k^S}(C_{1k}+\bar{C}_{1k})\right]\sin k\alpha+\right.$$
$$\left.+\left[\frac{V\overline{\gamma}}{d_k^T}(C_{2k}-\bar{C}_{2k})+\frac{1}{d_k^S}(C_{2k}+\bar{C}_{2k})\right]\cos k\alpha\right\}. \qquad (13.195)$$

Now, satisfying strain compatibility condition (13.190), we equate the Fourier coefficients of displacements $u_{m_z}^*$, represented by expressions (13.191), (13.195). We obtain

$$\begin{aligned}(1+\lambda_k^*)C_{1k}+(1-\lambda_k)\overline{C}_{1k}&=\frac{1}{\pi}\int_0^{2\pi}u_{m_z}^0(\alpha)\sin k\alpha\, d\alpha,\\(1+\lambda_k)C_{2k}+(1-\lambda_k^*)\overline{C}_{2k}&=\frac{1}{\pi}\int_0^{2\pi}u_{m_z}^0\cos k\alpha\, d\alpha\end{aligned} \qquad (13.196)$$

$$(k=2,3,\ldots),$$

where

$$\begin{aligned}\lambda_k&=k\frac{Gh}{R_0}\left(\frac{V\overline{\gamma}}{d_k^T}+\frac{1}{d_k^S}\right),\\\lambda_k^*&=k\frac{Gh}{R_0}\left(\frac{V\overline{\gamma}}{d_k^T}-\frac{1}{d_k^S}\right)\end{aligned} \qquad (13.197)$$

represent certain relative rigidities.

The boundary conditions on the end $\bar{Z}=\bar{Z}_1$ are represented by Eqs. (13.46), (13.47). Adding them to Eqs. (13.196), we obtain a system of algebraic equations for determining the arbitrary constants. Solving this system, we find

$$C_{1k}=\frac{\frac{1}{\pi(1-\lambda_k)}\int\limits_0^{2\pi} u^0_{m_z}(\alpha)\sin k\alpha d\alpha + L_{1k}e^{k\bar{l}_1}}{e^{2k\bar{l}_1}+\frac{1+\lambda_k^*}{1-\lambda_k}},$$

$$\bar{C}_{1k}=\frac{\frac{1}{\pi(1+\lambda_k^*)}\int\limits_0^{2\pi} u^0_{m_z}(\alpha)\sin k\alpha d\alpha - L_{1k}e^{-k\bar{l}_1}}{e^{-2k\bar{l}_1}+\frac{1-\lambda_k}{1+\lambda_k^*}},$$

$$C_{2k}=\frac{\frac{1}{\pi(1-\lambda_k^*)}\int\limits_0^{2\pi} u^0_{m_z}(\alpha)\cos k\alpha d\alpha + L_{2k}e^{k\bar{l}_1}}{e^{2k\bar{l}_1}+\frac{1+\lambda_k}{1-\lambda_k^*}}, \qquad (13.198)$$

$$\bar{C}_{2k}=\frac{\frac{1}{\pi(1+\lambda_k)}\int\limits_0^{2\pi} u^0_{m_z}(\alpha)\cos k\alpha d\alpha - L_{2k}e^{-k\bar{l}_1}}{e^{-2k\bar{l}_1}+\frac{1-\lambda_k^*}{1+\lambda_k}},$$

$$(k=2,\ 3,\ \ldots).$$

Having computed the arbitrary constants from formulas (13.198), one can then determine the stresses and displacements at any point of the shell, using general expressions (13.30)-(13.34). Moreover, introducing the computed values of the arbitrary constants into expressions (13.188), (13.189), one can determine the contact forces, and then, adding to them the external load applied to the ring, study the stressed and strained states of the ring by taking into account its specific structural characteristics.

It should be noted that representation (13.194) of the displacements of the ring due to elementary loads (13.193) is very general in character and defines a ring of constant cross section of any structure. This may be a frame type ring whose cross section obeys the hypothesis of plane sections, a thin-walled bar type ring whose cross section has additional degrees of freedom in comparison with the frame, and finally, a shell-of-revolution type ring of arbitrary profile. In each of these cases, having determined the generalized rigidities d'_k, d''_k, one can use the solution given above.

When the cross section of the ring is variable, the solution of this problem is very cumbersome. In this case, the contact problem reduces to an infinite system of equations in arbitrary constants, since the harmonics of the contact load (13.188), (13.189) corresponding to different values of k are not determined independently of one another. The solution of the infinite system in such problems can be sought only numerically. The simplest method consists in a successive computation of the solutions of truncated systems of increasingly higher order. Use may also be made of certain iteration processes of successive refinement of the contact load. In the latter case, for the same rigidity relations, a uniformly distributed load should be chosen as the initial approximation, and for other relations, a given external load applied to the ring should be chosen. Practical calculations show that for any relationships of the geometric dimensions and rigidities of the ring and shell, at least one of the indicated methods is sufficiently effective.

For a nontruncated conical shell, expressions (13.198) are simplified. Passing to the limit when $t_1 \to -\infty$, we obtain the following expressions for the arbitrary constants:

$$C_{1k} = \frac{1}{\pi(1+\lambda_k^*)} \int_0^{2\pi} u_{m_z}^0(\alpha) \sin k\alpha \, d\alpha - \\ - \frac{1-\lambda_k}{1+\lambda_k^*} \frac{\sqrt{\gamma}}{2\pi hkE} \int_{-\infty}^{0} R_{1k}(\xi) e^{\left(1+\frac{k}{\sqrt{\gamma}\sin\beta}\right)\xi} d\xi,$$

$$\bar{C}_{1k} = \frac{\sqrt{\gamma}}{2\pi hkE} \int_{-\infty}^{0} R_{1k}(\xi) e^{\left(1+\frac{k}{\sqrt{\gamma}\sin\beta}\right)\xi} d\xi, \tag{13.199}$$

$$C_{2k} = \frac{1}{\pi(1+\lambda_k)} \int_0^{2\pi} u_{m_z}^0(\alpha) \cos k\alpha \, d\alpha - \\ - \frac{1-\lambda_k^*}{1+\lambda_k} \frac{\sqrt{\gamma}}{2\pi hkE} \int_{-\infty}^{0} R_{2k}(\xi) e^{\left(1+\frac{k}{\sqrt{\gamma}\sin\beta}\right)\xi} d\xi,$$

$$\bar{C}_{2k} = \frac{\sqrt{\gamma}}{2\pi hkE} \int_{-\infty}^{0} R_{2k}(\xi) e^{\left(1+\frac{k}{\sqrt{\gamma}\sin\beta}\right)\xi} d\xi.$$

Formulas (13.199) are obviously also applicable to the analysis of not too short truncated conical shells. In this case, the corresponding solution gives appreciable errors only near the end $t = t_1$. Let us note also that in the presence of self-balanced components of the external load, i.e., when R_{1k}, R_{2k} are different from zero, the corresponding integrals in formulas (13.199) are very small, and they can be neglected in studying the edge effect in the ring zone by assuming that

$$
\begin{aligned}
&C_{1k} \approx \frac{1}{\pi(1+\lambda_k^{*})} \int_0^{2\pi} u^0_{m_z}(\alpha) \sin k\alpha \, d\alpha, \\
&\bar{C}_{1k} \approx 0, \\
&C_{2k} \approx \frac{1}{\pi(1+\lambda_k)} \int_0^{2\pi} u^0_{m_z}(\alpha) \cos k\alpha \, d\alpha, \\
&\bar{C}_{2k} \approx 0.
\end{aligned}
\tag{13.200}
$$

13.7. Conical Shell Reinforced with an End Ring Under Discrete Fixing Conditions

Let an end ring be fixed against displacements u_{m_z} along the generatrices at n equidistant points. Imagining that the supports are discarded, we will replace their action on the ring by unknown reactions P_m. Uniformly distributing each of them in the vicinity of the point of application on the portion $\Delta S = 2\varepsilon$, we represent the contour linear forces $\bar{N}(\alpha)$ in the form of a double trigonometric series

$$
\bar{N}(\alpha) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\alpha + b_k \sin k\alpha), \tag{13.201}
$$

where on the basis of (13.37)-(13.39)

$$
\begin{aligned}
&a_k = \frac{1}{\pi R_0} \frac{\sin k\varepsilon/R_0}{k\varepsilon/R_0} \sum_{m=1}^{n} P_m \cos \frac{2\pi k}{n} m, \\
&b_k = \frac{1}{\pi R_0} \frac{\sin k\varepsilon/R_0}{k\varepsilon/R_0} \sum_{m=1}^{n} P_m \sin \frac{2\pi k}{n} m, \\
&a_0 = \frac{1}{\pi R_0} \sum_{m=1}^{n} P_m.
\end{aligned}
\tag{13.202}
$$

We can now use the solution, obtained in the previous section, to the problem of analysis of a conical shell reinforced with an end ring for a given external load, including in this load the forces $\bar{N}(\alpha)$ as well. Then, satisfying the fixing conditions at n points, we obtain a system of equations for determining the unknown reactions P_m.

Let us note that in addition to these reactions, a certain given system of external forces may also be applied to the ring. Another possible case involves a ring fixed in relation to the tangential displacements u_{n_z} and normal ones u_{n_n}. Then, discarding the corresponding constraints, one must apply to the ring the reactive forces corresponding to it. These forces are generally unknown and are to be determined together with the reactions P_m. However, in most cases, in contrast to P_m, their distribution is smooth in character. Therefore, they can be determined with sufficient accuracy only on the basis of the equilibrium conditions of the shell, by specifying a certain distribution law of these forces on the basis of simplified computational schemes. The reactive forces determined in this manner and corresponding to the discarded tangential and normal constraints can then be referred to given external loads. Therefore, we will hereinafter assume all the external forces applied to the ring to be known, with the exception of P_m.

The stressed and strained states of a conical shell reinforced with an end ring are described by general expressions (13.30)-(13.32) and (13.33), (13.34), where the arbitrary constants should be determined from formulas obtained in Section 6. An external load applied to the ring is considered in these formulas by the terms

$$\int_0^{2\pi} u^0_{m_z}(\alpha)\sin k\alpha\, d\alpha; \quad \int_0^{2\pi} u^0_{m_z}(\alpha)\cos k\alpha\, d\alpha \quad (k=2, 3, \ldots),$$

where $u^0_{m_z}(\alpha)$ are displacements of the line of contact of the ring with the shell due to a given external load, and calculated for an isolated ring to within its displacements as a solid. We represent these displacements in the form

$$u^0_{m_z}(\alpha)=\tilde{u}^0_{m_z}(\alpha)+u^P_{m_z}(\alpha), \tag{13.203}$$

where $\tilde{u}^0_{m_z}$ - are the displacements due to a known external load applied to the ring;

$u^P_{m_z}$ are displacements due to the unknown reactions of the supports.

The displacement $\tilde{u}^0_{m_z}$ determined by analyzing an isolated ring, allowing for its specific characteristics, will be considered unknown. However, the displacements $u^P_{m_z}$ will be represented on the basis of the superposition principle in the form of a sum of the harmonics corresponding to expansion (13.201). Considering relations (13.193) and (13.194), we have

$$u^P_{m_z}(\alpha)=-\sum_{k=2}^{\infty}\frac{1}{d^T_k}(a_k\cos k\alpha+b_k\sin k\alpha), \tag{13.204}$$

where $a_k, b_k,$ determined by relations (13.202), depend on the desired reactions of the supports. The minus sign before the summation sign in this formula signifies that the positive directions of the reactions P_m and displacements u_{m_z} are opposite.

Let us note that now in the expressions for coefficients (13.202) we can achieve passage to the limit when $\varepsilon\to 0$, since in contrast to series (13.51), series (13.204), which describes the displacements of the ring, will obviously converge when $\varepsilon = 0$. Therein lies a certain fundamental difference between the problem at hand and the analysis of a shell without a ring, where the passage to the limit is possibly only in the expressions for stresses.

Introducing (13.203) into the most general expressions for integration constants (13.198), corresponding to the loading of the end ring by a given external load, and considering (13.204), (13.202), as well as notation (13.197), after passage to the limit when $\varepsilon \to 0$, we can obtain:

$$
\begin{aligned}
C_{1k} &= C_{1k}^{*} - \frac{1}{1+\lambda_k^{*}+(1-\lambda_k)e^{2k\bar{\gamma}_1}}\,\frac{1}{\pi R_0 d_k^{T}}\sum_{m=1}^{n} P_m \sin\frac{2\pi k}{n}m,\\
\bar{C}_{1k} &= \bar{C}_{1k}^{*} - \frac{1}{1-\lambda_k+(1+\lambda_k^{*})e^{-2k\bar{\gamma}_1}}\,\frac{1}{\pi R_0 d_k^{T}}\sum_{m=1}^{n} P_m \sin\frac{2\pi k}{n}m,\\
C_{2k} &= C_{2k}^{*} - \frac{1}{1+\lambda_k+(1-\lambda_k^{*})e^{2k\bar{\gamma}_1}}\,\frac{1}{\pi R_0 d_k^{T}}\sum_{m=1}^{n} P_m \cos\frac{2\pi k}{n}m,\\
\bar{C}_{2k} &= \bar{C}_{2k}^{*} - \frac{1}{1-\lambda_k^{*}+(1+\lambda_k)e^{-2k\bar{\gamma}_1}}\,\frac{1}{\pi R_0 d_k^{T}}\sum_{m=1}^{n} P_m \cos\frac{2\pi k}{n}m,
\end{aligned}
\qquad (13.205)
$$

where the constants C*, independent of the desired reactions of the supports, constitute, as is evident from (13.205), the integration constants for a shell not fixed at points:

$$
\begin{aligned}
C_{1k}^{*} &= \frac{\frac{1}{\pi}\int_0^{2\pi}\bar{u}_{m_z}^{0}(\alpha)\sin k\alpha\,d\alpha + L_{1k}(1-\lambda_k)e^{k\bar{\gamma}_1}}{1+\lambda_k^{*}+(1-\lambda_k)e^{2k\bar{\gamma}_1}},\\
\bar{C}_{1k}^{*} &= \frac{\frac{1}{\pi}\int_0^{2\pi}\bar{u}_{m_z}^{0}(\alpha)\sin k\alpha\,d\alpha - L_{1k}(1+\lambda_k^{*})e^{-k\bar{\gamma}_1}}{1-\lambda_k+(1+\lambda_k^{*})e^{-2k\bar{\gamma}_1}},\\
C_{2k}^{*} &= \frac{\frac{1}{\pi}\int_0^{2\pi}\bar{u}_{m_z}^{0}(\alpha)\cos k\alpha\,d\alpha + L_{2k}(1-\lambda_k^{*})e^{k\bar{\gamma}_1}}{1+\lambda_k+(1-\lambda_k^{*})e^{2k\bar{\gamma}_1}},\\
\bar{C}_{2k}^{*} &= \frac{\frac{1}{\pi}\int_0^{2\pi}\bar{u}_{m_z}^{0}(\alpha)\cos k\alpha\,d\alpha - L_{2k}(1+\lambda_k)e^{-k\bar{\gamma}_1}}{1-\lambda_k^{*}+(1+\lambda_k)e^{-2k\bar{\gamma}_1}}.
\end{aligned}
\qquad (13.206)
$$

Thus, the problem of determination of the integration constants has been reduced, according to expressions (13.205), (13.206), to finding the unknown reactions of the supports P_m. These reactions should be determined from the fixing conditions of the ring.

We will assume that the points of attachment of the ring to an immovable support with coordinates $\alpha=\alpha_m=\frac{2\pi}{n}m(m=1, 2,\ldots, n)$ are located on the circle of contact of the ring with the shell. Then the geometric boundary conditions can be written in the form

$$\begin{aligned} u_{m_z}(\alpha_m) = -(\eta_x^0 \sin\beta + \theta_y^0 R_0 \cos\beta)\cos\alpha_m - (\eta_y^0 \sin\beta - \theta_x^0 R_0 \cos\beta)\sin\alpha_m + \\ + \eta_z^0 \cos\beta + u^*_{m_z}(\alpha_m) = 0 \quad (m=1, 2,\ldots, n), \end{aligned} \tag{13.207}$$

where η_x^0; η_y^0, η_z^0 are the translational displacements of the ring, θ_x^0, θ_y^0 are its angles of rotation, and $u^*_{m_z}(\alpha)$ are the displacements of the line of contact of the shell with the ring, corresponding to warpings of the butt section.

The components η_x^0, η_y^0 depend on the normal and tangential constraints placed on the ring, and can usually be considered specified.

The components η_z^0, θ_x^0, θ_y^0, as before, are determined jointly with the reactions P_m. The displacements $u^*_{m_z}(\alpha)$ are then determined by expression (13.195), but considering compatibility condition (13.190), simpler expression (13.191) can also be employed.

Expanding the ring fixing conditions (13.207) with the aid of expression (13.191), for values of arbitrary constants represented by relations (13.205), we obtain

$$(\eta_x^0 \sin\beta + \theta_y^0 R_0 \cos\beta)\cos\frac{2\pi}{n} l + (\eta_y^0 \sin\beta - \theta_x^0 R_0 \cos\beta)\sin\frac{2\pi}{n} l -$$

$$-\eta_z^0\cos\beta + \frac{1}{\pi R_0}\sum_{k=2}^{\infty}\frac{1}{d_k^T}\left[\frac{\sum_{m=1}^{n} P_m \sin\frac{2\pi k}{n} m}{1+\frac{\lambda_k^*}{1+e^{2k\bar{t}_1}}-\frac{\lambda_k}{1-e^{-2k\bar{t}_1}}}\sin\frac{2\pi l}{n}k + \right.$$

$$\left. + \frac{\sum_{m=1}^{n} P_m \cos\frac{2\pi k}{n} m}{1+\frac{\lambda_k}{1+e^{2k\bar{t}_1}}-\frac{\lambda_k^*}{1-e^{-2k\bar{t}_1}}}\cos\frac{2\pi l}{n} k\right] =$$

$$= \sum_{k=2}^{\infty}\left[(C_{1k}^* + \bar{C}_{1k}^*)\sin\frac{2\pi l}{n}k + (C_{2k}^* + \bar{C}_{2k}^*)\cos\frac{2\pi l}{n}k\right] \quad (13.208)$$

$$(l = 1, 2, \ldots, n).$$

Expressions (13.208) represent a system of n equations in n unknown reactions P_m and three unknown components of displacement of the ring as a solid $\eta_z^0, \theta_x^0, \theta_y^0$, if the components η_x^0, η_y^0, which as was noted above are dependent on the ring fixing conditions in the directions n_z and n_n, are assumed to be specified. The missing three equations are obtained, by considering the equilibrium of the ring, from the conditions

$$\sum Z = 0, \quad \sum M_x = 0, \quad \sum M_y = 0.$$

We have

$$\sum_{m=1}^{n} P_m = \frac{1}{\cos\beta}[N_z(0) - \Delta N_z],$$

$$\sum_{m=1}^{n} P_m \sin\frac{2\pi}{n} m = \frac{1}{R_0\cos\beta}[M_x(0) - \Delta M_x], \quad (13.209)$$

$$\sum_{m=1}^{n} P_m \cos\frac{2\pi}{n} m = -\frac{1}{R_0\cos\beta}[M_y(0) - \Delta M_y].$$

Here $\Delta N_z, \Delta M_x, \Delta M_y$ are the additional axial force and moments due to the reactive forces applied to the ring instead of the discarded transverse constraints. These additional terms depend on the cone angle of the shell and are usually very insignificant. For a cylindrical shell, these terms vanish when $\beta = 0$.

To solve the system of n+3 Eqs. (13.208), (13.209), we will represent the reactions of the supports, as before, with the aid of interpolation polynomial (13.78), (13.79):

for odd n

$$P_m = A_0 + \sum_{r=1}^{\frac{n-1}{2}} \left(A_r \cos\frac{2\pi m}{n} r + B_r \sin\frac{2\pi m}{n} r \right), \tag{13.210}$$

for even n

$$P_m = A_0 + \sum_{r=1}^{\frac{n}{2}-1} \left(A_r \cos\frac{2\pi m}{n} r + B_r \sin\frac{2\pi m}{n} r \right) + (-1)^m A_{\frac{n}{2}}, \tag{13.211}$$

where the coefficients $A_0, A_r, B_r, A_{\frac{n}{2}}$ are determined by expressions (13.80).

Comparing (13.209) and (13.80), when r = 0, 1, we have

$$\begin{aligned} A_0 &= \frac{1}{n\cos\beta}\,[N_x(0) - \Delta N_x], \\ A_1 &= -\frac{1}{\frac{nR_0}{2}\cos\beta}\,[M_y(0) - \Delta M_y], \\ B_1 &= \frac{1}{\frac{nR_0}{2}\cos\beta}\,[M_z(0) - \Delta M_z]. \end{aligned} \tag{13.212}$$

Equations (13.208) should be transformed by using decompositions (13.84) and (13.85) of infinite sums:

for odd n

$$\sum_{k=1}^{\infty} a_k = \sum_{r=1}^{\frac{n-1}{2}} a_r + \sum_{q=1}^{\infty} a_{qn} + \sum_{q=1}^{\infty} \sum_{r=1}^{\frac{n-1}{2}} (a_{qn-r} + a_{qn+r}), \tag{13.213}$$

for even n

$$\sum_{k=1}^{\infty} a_k = \sum_{r=1}^{\frac{n}{2}-1} a_r + \sum_{q=1}^{\infty} a_{qn} + \sum_{q=1}^{\infty} a_{\left(q-\frac{1}{2}\right)n} + \sum_{q=1}^{\infty} \sum_{r=1}^{\frac{n}{2}-1} (a_{qn-r} + a_{qn+r}). \tag{13.214}$$

Then, using expressions (13.80) for the coefficients of the polynomial and reducing similar terms on $\cos\frac{2\pi m}{n}r$, $\sin\frac{2\pi m}{n}r$, as was done in Section 3, after some transformations, we obtain:

for odd n

$$
\begin{aligned}
&-\eta_z^0\cos\beta+\frac{1}{2}\sum_{q=1}^{\infty}\left[\frac{nA_0}{\pi R_0 D_{q0}^2}-\overline{L}_{q0}^2\right]+\\
&+\cos\frac{2\pi}{n}l\left[\eta_x^0\sin\beta+\theta_y^0R_0\cos\beta+\sum_{q=1}^{\infty}\left(\frac{nA_1}{2\pi R_0 D_{q1}^2}-\overline{L}_{q1}^2\right)\right]+\\
&+\sin\frac{2\pi}{n}l\left[\eta_y^0\sin\beta-\theta_x^0R_0\cos\beta+\sum_{q=1}^{\infty}\left(\frac{nB_1}{2\pi R_0 D_{q1}^1}-\overline{L}_{q1}^1\right)\right]+\\
&+\sum_{r=2}^{\frac{n-1}{2}}\left[\cos\frac{2\pi r}{n}l\sum_{q=0}^{\infty}\left(\frac{nA_r}{2\pi R_0 D_{qr}^2}-\overline{L}_{qr}^2\right)+\right.\\
&\left.+\sin\frac{2\pi r}{n}l\sum_{q=0}^{\infty}\left(\frac{nB_r}{2\pi R_0 D_{qr}^1}-\overline{L}_{qr}^1\right)\right]=0 \\
&\qquad (l=1,\ 2,\ldots,\ n),
\end{aligned}
\tag{13.215}
$$

for even n

$$
\begin{aligned}
&-\eta_z^0\cos\beta+\frac{1}{2}\sum_{q=1}^{\infty}\left[\frac{nA_0}{\pi R_0 D_{q0}^2}-\overline{L}_{q0}^2\right]+\\
&+\cos\frac{2\pi}{n}l\left[\eta_x^0\sin\beta+\theta_y^0R_0\cos\beta+\sum_{q=1}^{\infty}\left(\frac{nA_1}{2\pi R_0 D_{q1}^2}-\overline{L}_{q1}^2\right)\right]+\\
&+\sin\frac{2\pi}{n}l\left[\eta_y^0\sin\beta-\theta_x^0R_0\cos\beta+\sum_{q=1}^{\infty}\left(\frac{nB_1}{2\pi R_0 D_{q1}^1}-\overline{L}_{q1}^1\right)\right]+\\
&+\sum_{r=2}^{\frac{n}{2}-1}\left[\cos\frac{2\pi r}{n}l\sum_{q=0}^{\infty}\left(\frac{nA_r}{2\pi R_0 D_{qr}^2}-\overline{L}_{qr}^2\right)+\right.\\
&\left.+\sin\frac{2\pi r}{n}l\sum_{q=0}^{\infty}\left(\frac{nB_r}{2\pi R_0 D_{qr}^1}-\overline{L}_{qr}^1\right)\right]+\\
&+(-1)^l\frac{1}{2}\sum_{q=0}^{\infty}\left(\frac{nA_{\frac{n}{2}}}{\pi R_0 D^2_{q\frac{n}{2}}}-\overline{L}^2_{q\frac{n}{2}}\right)=0\\
&\qquad (l=1,\ 2,\ldots,\ n),
\end{aligned}
\tag{13.216}
$$

where

$$\frac{1}{D^1_{qr}} = \frac{1}{d^T_{qn-r}\left(1 + \frac{\lambda^*_{qn-r}}{1+e^{2(qn-r)\bar{i}_1}} - \frac{\lambda_{qn-r}}{1-e^{-2(qn-r)\bar{i}_1}}\right)} + \frac{1}{d^T_{qn+r}\left(1 + \frac{\lambda^*_{qn+r}}{1+e^{2(qn+r)\bar{i}_1}} - \frac{\lambda_{qn+r}}{1-e^{-2(qn+r)\bar{i}_1}}\right)},$$

$$\frac{1}{D^2_{qr}} = \frac{1}{d^T_{qn-r}\left(1 + \frac{\lambda_{qn-r}}{1+e^{2(qn-r)\bar{i}_1}} - \frac{\lambda^*_{qn-r}}{1-e^{-2(qn-r)\bar{i}_1}}\right)} + \frac{1}{d^T_{qn+r}\left(1 + \frac{\lambda_{qn+r}}{1+e^{2(qn+r)\bar{i}_1}} - \frac{\lambda^*_{qn+r}}{1-e^{-2(qn+r)\bar{i}_1}}\right)}, \tag{13.217}$$

$$\begin{aligned} \bar{L}^1_{qr} &= -C^*_{1(qn-r)} - \bar{C}^*_{1(qn-r)} + C^*_{1(qn+r)} + \bar{C}^*_{1(qn+r)}, \\ \bar{L}^2_{qr} &= C^*_{2(qn-r)} + \bar{C}^*_{2(qn-r)} + C^*_{2(qn+r)} + \bar{C}^*_{2(qn+r)}, \end{aligned} \tag{13.218}$$

and the quantities $C^*_{1(-r)}, \bar{C}^*_{1(-r)}, C^*_{2(-r)}, \bar{C}^*_{2(-r)}$, as well as $\frac{1}{d^T_{-r}}$ should be taken equal to zero.

Treating the left-hand sides of (13.215), (13.216) as interpolation polynomials, on the basis of formulas (13.80) and in view of (13.209) we have

$$\begin{aligned} \eta^0_x &= \frac{N_x(0) - \Delta N_x}{2\pi R_0 \cos^2\beta} \sum_{q=1}^{\infty} \frac{1}{D^2_{q0}} - \frac{1}{2\cos\beta} \sum_{q=1}^{\infty} \bar{L}^2_{q0}, \\ \theta^0_y &= -\frac{\operatorname{tg}\beta}{R_0}\eta^0_x + \frac{M_y(0) - \Delta M_y}{\pi R_0^3 \cos^2\beta} \sum_{q=1}^{\infty} \frac{1}{D^2_{q1}} + \frac{1}{R_0\cos\beta} \sum_{q=1}^{\infty} \bar{L}^2_{q1}, \\ \theta^0_x &= \frac{\operatorname{tg}\beta}{R_0}\eta^0_y + \frac{M_x(0) - \Delta M_x}{\pi R_0^3 \cos^2\beta} \sum_{q=1}^{\infty} \frac{1}{D^1_{q1}} - \frac{1}{R_0\cos\beta} \sum_{q=1}^{\infty} \bar{L}^1_{q1}, \end{aligned} \tag{13.219}$$

$$\begin{aligned} A_r &= \frac{2\pi R_0}{n} \frac{\sum_{q=0}^{\infty} \bar{L}^2_{qr}}{\sum_{q=0}^{\infty} \frac{1}{D^2_{qr}}}, \\ B_r &= \frac{2\pi R_0}{n} \frac{\sum_{q=0}^{\infty} \bar{L}^1_{qr}}{\sum_{q=0}^{\infty} \frac{1}{D^1_{qr}}}, \\ A_{\frac{n}{2}} &= \frac{\pi R_0}{n} \frac{\sum_{q=0}^{\infty} \bar{L}^2_{q\frac{n}{2}}}{\sum_{q=0}^{\infty} \frac{1}{D^2_{q\frac{n}{2}}}}, \end{aligned} \tag{13.220}$$

where $r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)$.

On the basis of (13.220), from expressions (13.210), (13.211) there follows the important conclusion that if a conical shell is reinforced with an end ring, then in contrast to a nonreinforced shell, the distribution of the reactions of the supports under discrete fixing conditions does not follow the law of the plane. In order to establish the character of the distribution of the reactions, it is necessary for each specific case to calculate the coefficients of interpolation polynomials from formulas (13.212) and (13.220), and introduce them into (13.210) when the number of fixing points is odd and into (13.211) when it is even.

For the reactions of the supports obtained, the problem reduces to an analysis of a conical shell with an edge ring for a given load. This problem is examined in Section 6. For this reason, we will cite here only the values of the integration constants corresponding to the case at hand. These values will be found from expressions (13.205) by considering relations (13.80) and also (13.101), (13.102):

$$C_{1k}=\begin{cases} C_{1k}^{\bullet} & (k=qn), \\ C_{1k}^{\bullet}\mp\dfrac{\dfrac{n}{d_k^T}\,\dfrac{B_r}{2\pi R_0}}{1+\lambda_k^{\bullet}+(1-\lambda_k)\,e^{-2k\bar{t}_1}} & (k=qn\pm r), \\ C_{1k}^{\bullet} & \left(k=qn-\dfrac{n}{2}\right), \end{cases}$$

$$\bar{C}_{1k}=\begin{cases} \bar{C}_{1k}^{\bullet} & (k=qn), \\ \bar{C}_{1k}^{\bullet}\mp\dfrac{\dfrac{n}{d_k^T}\,\dfrac{B_r}{2\pi R_0}}{1-\lambda_k+(1+\lambda_k^{\bullet})\,e^{-2k\bar{t}_1}} & (k=qn\pm r), \\ \bar{C}_{1k}^{\bullet} & \left(k=qn-\dfrac{n}{2}\right). \end{cases} \qquad (13.221)$$

$$C_{2k}=\begin{cases} C^{\circ}_{2k}-\dfrac{\dfrac{n}{d^{T}_{k}}\dfrac{A_0}{\pi R_0}}{1+\lambda_k+(1-\lambda^{\circ}_k)e^{2k\bar{l}_1}} & (k=qn),\\[2ex] C^{\circ}_{2k}-\dfrac{\dfrac{n}{d^{T}_{k}}\dfrac{A_r}{2\pi R_0}}{1+\lambda_k+(1-\lambda^{\circ}_k)e^{2k\bar{l}_1}} & (k=qn\pm r),\\[2ex] C^{\circ}_{2k}-\dfrac{\dfrac{n}{d^{T}_{k}}\dfrac{A_{\frac{n}{2}}}{\pi R_0}}{1+\lambda_k+(1-\lambda^{\circ}_k)e^{2k\bar{l}_1}} & \left(k=qn-\dfrac{n}{2}\right).\end{cases}$$

$$\bar{C}_{2k}=\begin{cases} \bar{C}^{\circ}_{2k}-\dfrac{\dfrac{n}{d^{T}_{k}}\dfrac{A_0}{\pi R_0}}{1-\lambda^{\circ}_k+(1+\lambda_k)e^{-2k\bar{l}_1}} & (k=qn),\\[2ex] \bar{C}^{\circ}_{2k}-\dfrac{\dfrac{n}{d^{T}_{k}}\dfrac{A_r}{2\pi R_0}}{1-\lambda^{\circ}_k+(1+\lambda_k)e^{-2k\bar{l}_1}} & (k=qn\pm r),\\[2ex] \bar{C}^{\circ}_{2k}-\dfrac{\dfrac{n}{d^{T}_{k}}\dfrac{A_{\frac{n}{2}}}{\pi R_0}}{1-\lambda^{\circ}_k+(1+\lambda_k)e^{-2k\bar{l}_1}} & \left(k=qn-\dfrac{n}{2}\right),\end{cases} \tag{13.221}$$

where rae

$$q=1,\ 2,\ldots;\qquad r=1,\ 2,\ldots,\ E\left(\frac{n-1}{2}\right).$$

Now, using general expressions (13.30)-(13.32) and (13.33), (13.34), for values of arbitrary constants calculated from formulas (13.221), one can study the stressed and strained state of the shell.

For a nontruncated conical shell, passing in (13.206) to the limit when $\bar{l}_1\to-\infty$, we obtain

$$C^{\circ}_{1k}=\frac{\bar{b}^{0}_{k}}{1+\lambda^{\circ}_{k}}-\frac{1-\lambda_k}{1+\lambda^{\circ}_{k}}\bar{L}^{\circ}_{1k};\qquad \bar{C}^{\circ}_{1k}=\bar{L}^{\circ}_{1k},$$
$$C^{\circ}_{2k}=\frac{\bar{a}^{0}_{k}}{1+\lambda_k}-\frac{1-\lambda^{\circ}_{k}}{1+\lambda_k}\bar{L}^{\circ}_{2k},\qquad \bar{C}^{\circ}_{2k}=\bar{L}^{\circ}_{2k}, \tag{13.222}$$

where

$$\bar{a}^{0}_{k}=\frac{1}{\pi}\int_{0}^{2\pi}\bar{u}^{0}_{m_z}(\alpha)\cos k\alpha d\alpha,$$
$$\bar{b}^{0}_{k}=\frac{1}{\pi}\int_{0}^{2\pi}\bar{u}^{0}_{m_z}(\alpha)\sin k\alpha d\alpha \tag{13.223}$$

are the Fourier coefficients of the known displacements $\tilde{u}^0_{m_s}$ and Z^{*}_{1k}, Z^{*}_{2k} are determined by expressions(13.118).

From (13.217), when $t_1 \to -\infty$, we have

$$\frac{1}{D^1_{qr}} = \frac{1}{d^T_{qn-r}(1+\lambda^*_{qn-r})} + \frac{1}{d^T_{qn+r}(1+\lambda^*_{qn+r})},$$
$$\frac{1}{D^2_{qr}} = \frac{1}{d^T_{qn-r}(1+\lambda_{qn-r})} + \frac{1}{d^T_{qn+r}(1+\lambda_{qn+r})}. \tag{13.224}$$

The integration constants, represented by expressions (13.221), will assume the following form when $t_1 \to -\infty$

$$C_{1k} = \begin{cases} C^*_{1k} & (k = qn), \\ C^*_{1k} \mp \dfrac{nB_r}{d^T_k(1+\lambda^*_k)2\pi R_0} & (k = qn \pm r), \\ C^*_{1k} & \left(k = qn - \dfrac{n}{2}\right), \end{cases}$$
$$\bar{C}_{1k} = \bar{C}^*_{1k}, \tag{13.225}$$
$$C_{2k} = \begin{cases} C^*_{2k} - \dfrac{nA_0}{d^T_k(1+\lambda_k)\pi R_0} & (k = qn), \\ C^*_{2k} - \dfrac{nA_r}{d^T_k(1+\lambda_k)2\pi R_0} & (k = qn \pm r), \\ C^*_{2k} - \dfrac{nA_{\frac{n}{2}}}{d^T_k(1+\lambda_k)\pi R_0} & \left(k = qn - \dfrac{n}{2}\right); \end{cases}$$
$$\bar{C}_{2k} = \bar{C}^*_{2k}$$
$$\left[q = 1, 2, \ldots; \qquad r = 1, 2, \ldots, E\left(\frac{n-1}{2}\right)\right].$$

As was frequently mentioned above, expressions (13.225) for a nontruncated conical shell can also be used in the analysis of truncated shells at some distance from the end $t = t_1$. Moreover, if the external surface load does not contain any self-balanced components in the direction of the generatrices, then on the basis of (13.13) and (13.19), $Z^*_{1k} = Z^*_{2k} = 0$, and all the expressions are considerably simplified. In this case, outside the zone of the edge effect related to point fixing of the ring, the solutions obtained coincide with the solution based on the hypothesis of plane sections.

If the problem involves only a study of the stress redistribution caused by point fixing of the ring, the influence of self-balanced components of the external surface load applied in the direction of the shell generatrices may be neglected. In addition, for most cases, one can assume $\tilde{u}^0_{m_2} \approx 0$, since ring displacements (13.203) in the direction of the shell generatrices are chiefly determined by concentrated reactions P_m directed along the generatrices.

Making the indicated assumptions, on the basis of (13.13), (13.119), (13.223) we should set

$$\tilde{L}^{\bullet}_{1k} = \tilde{L}^{\bullet}_{2k} = 0,$$
$$\tilde{a}^0_k = \tilde{b}^0_k = 0.$$

Then from (13.222) we obtain

$$C^{\bullet}_{1k} = \tilde{C}^{\bullet}_{1k} = C^{\bullet}_{2k} = \tilde{C}^{\bullet}_{2k} = 0$$

and considering (13.218), we have

$$\tilde{L}^1_{\varphi r} = \tilde{L}^2_{\varphi r} = 0.$$

Now, expressions (13.220) yield

$$A_r = B_r = 0 \quad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right],$$
$$A_{\frac{n}{2}} = 0,$$

and hence, the distribution of the reactions of the supports P_m in accordance with (13.210), (13.211) obeys the law of the plane, as in the case of non-reinforced shells.

In addition to the adopted assumptions, we set $\frac{1}{d_k^2} = 0$, in expressions (13.197), since in most cases, displacements of the ring in the direction of the generatrices, due to tangential contact forces, are very small and can be neglected. Moreover, considering (13.212) and in accordance with the adopted assumption of smallness of $\tilde{u}^0_{m_z}$, neglecting ΔN_z, ΔM_y, ΔM_z in comparison with $N_z(0), M_y(0), M_z(0)$, from expressions (13.225) for the integration constants we have

$$
\begin{gathered}
C_{1k}=\begin{cases} 0 & (k \neq qn \pm 1) \\ \mp \dfrac{M_z(0)}{\pi R_0^2 \cos\beta\, d_k^r (1+\bar{\lambda}_k)} & (k=qn \pm 1), \end{cases} \\
\bar{C}_{1k}=0, \\
C_{2k}=\begin{cases} -\dfrac{N_z(0)}{\pi R_0 \cos\beta\, d_k^r (1+\bar{\lambda}_k)} & (k=qn), \\ \dfrac{M_y(0)}{\pi R_0^2 \cos\beta\, d_k^r (1+\bar{\lambda}_k)} & (k=qn \pm 1), \\ 0 & (k \neq qn), \\ 0 & (k \neq qn \pm 1), \end{cases} \\
\bar{C}_{2k}=0,
\end{gathered}
\tag{13.226}
$$

where

$$\bar{\lambda}_k=\frac{Ehk}{\sqrt{\gamma}\, R_0 d_k^r} \tag{13.227}$$

is the relative rigidity parameter.

Expressions (13.226) are very clear and make it possible to analyze the influence of the end ring on the character of the edge effect in discrete fixing. When $d_k^r=0$ it is easy to see that corresponding expressions for a nonreinforced shell result from these expressions. Introducing (13.226) into the general expressions for stresses (13.33) and (13.34), we obtain

$$\sigma_{m_z}(t, \alpha)=\sigma^0_{m_z}+\sigma^p_{m_z},$$

where the stresses $\sigma^0_{m_z}$, corresponding to the hypothesis of plane section are determined by expression (13.128), and the self-balanced stresses due to the reactions of the supports

$$\sigma^p_{m_z}(t, \alpha)=\frac{N_z(0)}{\pi R_0 h \cos\beta} e^{-t} \sum_{q=1}^{\infty} \frac{e^{qn\bar{t}} \cos qn\alpha_m}{1+\frac{1}{\bar{\lambda}_{qn}}} \tag{13.228}$$

$$-\frac{M_z(0)}{\pi R_0^2 h\cos\beta}e^{-\bar{t}}\sum_{q=1}^{\infty}\left(\frac{e^{(qn-1)\bar{t}}\sin(qn-1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn-1}}}-\frac{e^{(qn+1)\bar{t}}\sin(qn+1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn+1}}}\right)-$$

$$-\frac{M_y(0)}{\pi R_0^2 h\cos\beta}e^{-\bar{t}}\sum_{q=1}^{\infty}\left(\frac{e^{(qn-1)\bar{t}}\cos(qn-1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn-1}}}+\frac{e^{(qn+1)\bar{t}}\cos(qn+1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn+1}}}\right). \quad (13.228)$$

The tangential stresses

$$\tau_{n_z m_z}(t,\alpha)=\tau^0_{n_z m_z}+\tau^P_{n_z m_z},$$

where the stresses $\tau^0_{n_z m_z}$ corresponding to the hypothesis of plane sections are determined by expression (13.131), and the self-balanced tangential stresses due to the reactions of the supports

$$\tau^P_{n_z m_z}(t,\alpha)=\frac{N_z(0)}{\pi R_0 h\sqrt{\bar{\gamma}}\cos\beta}e^{-\bar{t}}\sum_{q=1}^{\infty}\frac{e^{qn\bar{t}}\sin qn\alpha}{1+\frac{1}{\bar{\lambda}_{qn}}}+$$

$$+\frac{M_x(0)}{\pi\sqrt{\bar{\gamma}}R_0^2 h\cos\beta}e^{-\bar{t}}\sum_{q=1}^{\infty}\left(\frac{e^{(qn-1)\bar{t}}\cos(qn-1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn-1}}}-\frac{e^{(qn+1)\bar{t}}\cos(qn+1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn+1}}}\right)-$$

$$-\frac{M_y(0)}{\pi\sqrt{\bar{\gamma}}R_0^2 h\cos\beta}e^{-\bar{t}}\sum_{q=1}^{\infty}\left(\frac{e^{(qn-1)\bar{t}}\sin(qn-1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn-1}}}+\frac{e^{(qn+1)\bar{t}}\sin(qn+1)\alpha}{1+\frac{1}{\bar{\lambda}_{qn+1}}}\right). \quad 13.229)$$

Formulas (13.228) and (13.229) are most useful for practical computations. The values

$$\frac{1}{\bar{\lambda}_k}=\frac{\sqrt{\bar{\gamma}}R_0}{Eh}\frac{d^T_k}{k}$$

increase rapidly with increasing k, so that the infinite series in (13.228), (13.229) converge much faster than the analogous series for a nonreinforced shell, for which $1/\bar{\lambda}_k=0$.

Using relations (13.171) and (13.172), by passing to the limit, it is not difficult to extend all the results obtained above to a cylindrical shell reinforced with an edge ring under cyclic fixing conditions. We will cite here only the expressions (13.228), (13.229):

$$\sigma^{P}_{m_z}(\bar{Z},\alpha)=\frac{N_z(0)}{\pi R_0 h}\sum_{q=1}^{\infty}\frac{e^{-qn\frac{\bar{Z}}{\sqrt{\gamma}}}\cos qn\alpha}{1+\frac{1}{\lambda_{qn}}}-$$

$$-\frac{M_x(0)}{\pi R_0^2 h}\sum_{q=1}^{\infty}\left(\frac{e^{-(qn-1)\frac{\bar{Z}}{\sqrt{\gamma}}}\sin(qn-1)\alpha}{1+\frac{1}{\lambda_{qn-1}}}-\frac{e^{-(qn+1)\frac{\bar{Z}}{\sqrt{\gamma}}}\sin(qn+1)\alpha}{1+\frac{1}{\lambda_{qn+1}}}\right)-$$

$$-\frac{M_y(0)}{\pi R_0^2 h}\sum_{q=1}^{\infty}\left(\frac{e^{-(qn-1)\frac{\bar{Z}}{\sqrt{\gamma}}}\cos(qn-1)\alpha}{1+\frac{1}{\lambda_{qn-1}}}+\frac{e^{-(qn+1)\frac{\bar{Z}}{\sqrt{\gamma}}}\cos(qn+1)\alpha}{1+\frac{1}{\lambda_{qn+1}}}\right); \qquad (13.230)$$

$$\tau^{P}_{s_z m_z}(\bar{Z},\alpha)=\frac{N_z(0)}{\pi R_0 h\sqrt{\gamma}}\sum_{q=1}^{\infty}\frac{e^{-qn\frac{\bar{Z}}{\sqrt{\gamma}}}\sin qn\alpha}{1+\frac{1}{\lambda_{qn}}}+$$

$$+\frac{M_x(0)}{\pi\sqrt{\gamma}R_0^2 h}\sum_{q=1}^{\infty}\left(\frac{e^{-(qn-1)\frac{\bar{Z}}{\sqrt{\gamma}}}\cos(qn-1)\alpha}{1+\frac{1}{\lambda_{qn-1}}}-\frac{e^{-(qn+1)\frac{\bar{Z}}{\sqrt{\gamma}}}\cos(qn+1)\alpha}{1+\frac{1}{\lambda_{qn+1}}}\right)-$$

$$-\frac{M_y(0)}{\pi\sqrt{\gamma}R_0^2 h}\sum_{q=1}^{\infty}\left(\frac{e^{-(qn-1)\frac{\bar{Z}}{\sqrt{\gamma}}}\sin(qn-1)\alpha}{1+\frac{1}{\lambda_{qn-1}}}+\frac{e^{-(qn+1)\frac{\bar{Z}}{\sqrt{\gamma}}}\sin(qn+1)\alpha}{1+\frac{1}{\lambda_{qn+1}}}\right). \qquad (13.231)$$

As was noted above, the computational model which we adopted for a conical shell, after passage to the limit when $\beta\to 0$, determines the model of a cylindrical shell with a nondeformable contour of the cross section. If the contact of the cylindrical shell with the ring takes place along the axis of its rigidity, and one of the major axes of inertia of the cross section of the ring lies in the plane of the cross section of the shell, the approximate relation $\tilde{u}^0_{m_z}\approx 0$ obviously becomes rigorous. In this case, in the absence of self-balanced components of the longitudinal component of the external load, expansions (13.230), (13.231) constitute the exact solution for a semiinfinite cylindrical shell with a rigid contour, and when $\eta^0_x=\eta^0_y=\theta^0_z=0$, the boundary conditions formulated above correspond to complete fixing of the edge ring against tangential and normal displacements and fixing at n equidistant points against longitudinal displacements.

For frame type rings, one of whose major cross-sectional axes of inertia lies in the plane of the ring, the displacements from the plane of the ring satisfy the equation

$$\frac{d^6 u}{d\alpha^6} + 2\frac{d^4 u}{d\alpha^4} + \frac{d^2 u}{d\alpha^2} = f(\alpha). \tag{13.232}$$

Here

$$f(\alpha) = \frac{R^4}{EI}\left\{\left(\frac{d^2}{d\alpha^2} - \mu\right)\left(pR + \frac{dm_{изг}}{d\alpha}\right) + (1+\mu)\frac{d^2 m_{кр}}{d\alpha^2}\right\},$$

where R is the radius of the ring axis;

EI is the flexural rigidity of the cross section of the ring relative to the major axis of inertia lying in its plane;

μ is the ratio of the flexural rigidity EI to the torsional rigidity of the ring GI_t;

p is a linear load, perpendicular to the plane of the ring;

m_b, m_t are linear bending and twisting moments.

If the contact of a cylindrical shell with the ring takes place along the axis of its rigidity, one must assume that

$$m_b = m_t = 0.$$

Applying to the ring, according to (13.192), the load

$$p_{1k} = \sin k\alpha, \quad p_{2k} = \cos k\alpha,$$

in accordance with (13.193), we obtain from differential Eq. (13.232)

$$d_k^T = \frac{1}{R^4}\,\frac{k^2(k^2-1)^2}{\frac{1}{GI_{кр}} + k^2\frac{1}{EI}}. \tag{13.233}$$

Introducing (13.233) into (13.227), we have

$$\frac{1}{\bar{\lambda}_{n}} = \frac{\sqrt{\bar{\gamma}}}{R^{3}h} \frac{k(k^{2}-1)^{2}}{\frac{1}{I_{np}} + k^{2}\frac{1}{I}}, \tag{13.234}$$

Considering (13.234) and the structure of series (13.230), (13.231), one can conclude that they converge very rapidly, not only when $\bar{Z}>0$, but also in the section of contact with the ring when $\bar{Z} = 0$. A similar conclusion can also be drawn regarding series (13.228), (13.229).

Chapter XIV. CONICAL SHELL WITH A REGULAR LONGITUDINAL STRUCTURE

We consider a conical shell (Fig. 14.1) reinforced by a regular set of stringers oriented in the direction of the generatrices $\alpha=\alpha_m=\frac{2\pi}{n}m\ (m=1,2,\ldots,n)$. The number of stringers n is arbitrary. The problem is solved by taking into consideration the discrete arrangement of the stringers exactly, not on the basis of smearing out concepts.* Accordingly, the corresponding quadratures in the coefficients of the system of differential resolvents (m), (n) of Part Three are understood in the sense of Stieltjes integrals. As in Chapter XIII, the number of degrees of freedom of the cross section is not limited, so that the solution obtained may be considered exact within the limits of the adopted computational model.

14.1. Differential Resolvents of a Stringer Shell

Considering the problem under arbitrary loading, we should retain in expansion (f) of Part Three all six components of displacement of the cross section as a solid, and represent the warping displacements by the complete trigonometric series (j).

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*For simplified solutions, see I. F. Obraztsov. Variational Methds of Analysis of Thin-Walled Aeronautical Structures. Moscow, Mashinostroyeniye, 1966, and also Ye. V. Binkevich, L. V. Vergeychik, and V. I. Mossakovskiy, Rational Distribution of the Material in Elements of the Longitudinal Structure of a Shell. In: Hydroaeromechanics and Elasticity Theory, edited by Yu. A. Shevlayakov, Khar'kov University, No. 8, 1968.

We will calculate the coefficients (o) of the system of differential resolvents (m), (n). It is easy to see that the coefficients b_{ji} and c_{ji} in the problem at hand coincide with the corresponding coefficients for a smooth shell, indicated in Tables 13.2 and 13.3.

In accordance with (o), the coefficients a_{ji} may be represented in the form

$$a_{ji}=a^0_{ji}+\Delta a_{ji}, \tag{14.1}$$

where a^0_{ji}, the coefficients for a smooth shell, are represented in Table 13.1 and by formula (13.1), and the coefficients allowing for the longitudinal structure

$$\Delta a_{ji}=\frac{\gamma\sin\beta}{R_0}\Delta F\sum_{m=1}^{n}\varphi_{jm_s}\varphi_{im_s}\Big|_{s=\frac{2\pi}{n}m}. \tag{14.2}$$

To calculate coefficients (14.2), we will need the values of the sums

$$\begin{aligned} S_1&=\sum_{m=1}^{n}\sin\frac{2\pi m}{n}k\sin\frac{2\pi m}{n}l,\\ S_2&=\sum_{m=1}^{n}\cos\frac{2\pi m}{n}k\cos\frac{2\pi m}{n}l,\\ S_3&=\sum_{m=1}^{n}\cos\frac{2\pi m}{n}k\sin\frac{2\pi m}{n}l \end{aligned} \tag{14.3}$$

for any integral k and l.

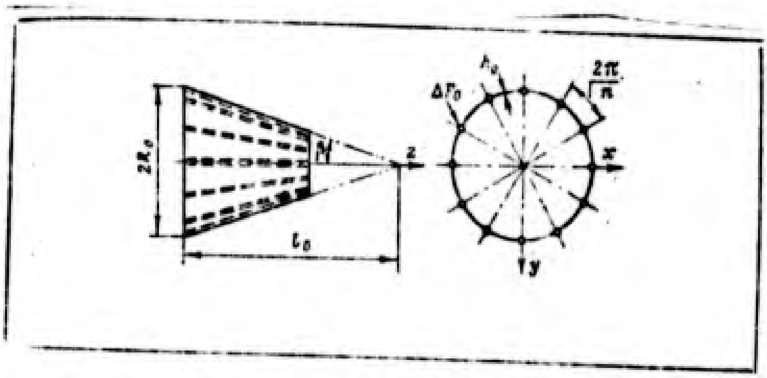

Fig. 14.1. Circular conical shell reinforced with a regular set of stringers.

In Section 13.3, formula (13.74) was obtained

$$\sum_{k=1}^{p} \cos kx = \frac{1}{2}\left[\frac{\sin\left(p+\frac{1}{2}\right)x}{\sin\frac{x}{2}} - 1\right].$$

Similarly, it is also easy to obtain

$$\sum_{k=1}^{p} \sin kx = \frac{1}{2}\,\frac{\cos\frac{x}{2} - \cos\left(p+\frac{1}{2}\right)x}{\sin\frac{x}{2}}. \tag{14.4}$$

Setting p = n and $x=\frac{2\pi k}{n}$ in (13.74) and (14.4), we will have

$$\sum_{m=1}^{n} \cos\frac{2\pi k}{n}m = \frac{1}{2}\left[\frac{\sin\frac{\pi k}{n}}{\sin\frac{\pi k}{n}} - 1\right], \tag{14.5}$$

$$\sum_{m=1}^{n} \sin\frac{2\pi k}{n}m = \frac{1}{2}\,\frac{\cos\frac{\pi k}{n} - \cos\frac{\pi k}{n}}{\sin\frac{\pi k}{n}}, \tag{14.6}$$

whence it follows that sums (14.5), (14.6) may be different from zero only when k is a multiple of n. In this case, introducing $k=qn(q=1,2,\ldots)$ directly under the summation signs, we obtain

$$\sum_{m=1}^{n} \cos\frac{2\pi k}{n}m = \begin{cases} 0 & (k \neq qn), \\ n & (k=qn), \end{cases} \tag{14.7}$$

$$\sum_{m=1}^{n} \sin\frac{2\pi k}{n}m = 0. \tag{14.8}$$

We transform the expressions for S_1, S_2, S_3. We have

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{m=1}^{n}\left[\cos\frac{2\pi m}{n}(k-l) - \cos\frac{2\pi m}{n}(k+l)\right], \\ S_2 &= \frac{1}{2}\sum_{m=1}^{n}\left[\cos\frac{2\pi m}{n}(k+l) + \cos\frac{2\pi m}{n}(k-l)\right], \\ S_3 &= \frac{1}{2}\sum_{m=1}^{n}\left[\sin\frac{2\pi m}{n}(k+l) - \sin\frac{2\pi m}{n}(k-l)\right]. \end{aligned} \tag{14.9}$$

On the basis of (14.7), (14.8), we conclude that the sum S_3 is equal to zero for any k and l, and the sums S_1 and S_2 can be different from zero only when at least one of the numbers $k+l$, $k-l$ is a multiple of n.

We represent k and l in the following form:

$$\begin{aligned} k &= q_k n + r_k, \\ l &= q_l n + r_l, \end{aligned} \tag{14.10}$$

where

$$\begin{aligned} q_k, q_l &= 0, 1, 2, \ldots \\ r_k, r_l &= 0, 1, \ldots, n-1. \end{aligned}$$

It is obvious that for suitable values of q and r, any whole positive number can be represented in the form (14.10).

Considering (14.10), we have

$$\begin{aligned} k+l &= (q_k+q_l)\,n + r_k + r_l, \\ k-l &= (q_k-q_l)\,n + r_k - r_l. \end{aligned} \tag{14.11}$$

From relations (14.11), it is easy to establish the conditions for which at least one of the numbers $k+l$, $k-l$ is a multiple of n. We have:

$$\begin{aligned} k+l &= qn \quad \text{при} \quad r_k = r_l = 0, \quad r_k + r_l = n, \\ k-l &= qn \quad \text{при} \quad r_k = r_l, \end{aligned}$$

where q is an integer.

It follows that when $r_k = r_l = 0$ and when $r_k = r_l = \frac{n}{2}$, both numbers $k+l$ and $k-l$ are multiples of n. In addition, when $r_k + r_l = n$ but $r_k \neq r_l$, only $k+l$ is a multiple of n, and when $r_k = r_l \neq 0, \frac{n}{2}$, only $k-l$ is a multiple of n. Therefore, the values of the sums S_1 and S_2 represented by expressions (14.9) will be

$$S_1=\begin{cases} \frac{n}{2} & \left(r_k=r_l\neq 0,\ \frac{n}{2}\right), \\ -\frac{n}{2} & (r_k+r_l=n,\ r_k\neq r_l), \\ 0 & \begin{cases} r_k\pm r_l\neq 0,\ n, \\ r_k=r_l=0,\ \frac{n}{2}, \end{cases} \end{cases} \tag{14.12}$$

$$S_2=\begin{cases} n & \left(r_k=r_l=0,\ \frac{n}{2}\right), \\ \frac{n}{2} & \begin{cases} r_k=r_l\neq 0,\ \frac{n}{2}, \\ r_k+r_l=n,\ r_k\neq r_l, \end{cases} \\ 0 & (r_k\pm r_l\neq 0,\ n) \end{cases} \tag{14.13}$$

$$S_3=0. \tag{14.14}$$

Using (14.12)-(14.14) as well as the formulas given in the table on p. 453, one can calculate the coefficients $\Delta\alpha_{ji}$ (14.2). For clarity, we will represent them in the form of the square block matrix

$$\Delta\mathbf{A}=\left(\begin{array}{c|c} \Delta\mathbf{A}_{ji} & \Delta\mathbf{A}_{jk} \\ \hline \Delta\mathbf{A}_{li} & \Delta\mathbf{A}_{lk} \end{array}\right).$$

Here

$\Delta\mathbf{A}_{ji}$, **the matrix of coefficients Δa_{ji} $(j,\ i=1,\ 2,\dots,\ 6)$ (Table 14.1), corresponds to displacements of the contour $\bar{Z}$ = const as a solid;**

$\Delta\mathbf{A}_{lk}$, **the matrix of coefficients Δa_{lk} $(l,\ k=2,\ 3,\dots)$, corresponds to warping of the contour $\bar{Z}$ = const;**

$\Delta\mathbf{A}_{li}$, $\Delta\mathbf{A}_{jk}$ -, **the matrices of coefficients Δa_{li}, Δa_{jk}, reflect the mutual influence of the displacements corresponding to displacements of the contour $\bar{Z}$ = const as a solid and to the warping of this contour.**

The $\Delta\mathbf{A}$ is obviously symmetric. Therefore

$$\Delta\mathbf{A}_{li}=\Delta\mathbf{A}'_{jk},$$

where $\Delta\mathbf{A}'_{jk}$ is the transposed matrix.

It is easy to see that the system of differential resolvents (m), (n) decomposes into two independent subsystems, one of which corresponds to the displacements symmetric with respect to the horizontal plane, and the other, to the vertical plane. This makes it possible in turn to represent the matrices $\Delta A_{jk}, \Delta A_{ll}, \Delta A_{lk}$ in the form of diagonal block matrices:

$$\Delta A_{jk}=\left(\begin{array}{c|c}\Delta A^1_{jk} & 0\\ \hline 0 & \Delta A^2_{jk}\end{array}\right),\quad \Delta A_{ll}=\left(\begin{array}{c|c}\Delta A^1_{ll} & 0\\ \hline 0 & \Delta A^2_{ll}\end{array}\right),$$
$$\Delta A_{lk}=\left(\begin{array}{c|c}\Delta A^1_{lk} & 0\\ \hline 0 & \Delta A^2_{lk}\end{array}\right). \tag{14.15}$$

According to (14.2), (14.12)-(14.14) and the formulas of Table 14.1, the elements of the blocks of matrices (14.15) will be

$$\Delta a^1_{jk}=0 \quad (j=1,\ 3,\ 5,\ 6),$$
$$\frac{\Delta a^1_{2k}}{-\sin\beta}=\frac{\Delta a^1_{4k}}{R_0\cos\beta}=\gamma\sin\beta\frac{\Delta F}{R_0}\times\begin{cases}\frac{n}{2} & (k=q_kn+1),\\ -\frac{n}{2} & (k=(q_k+1)n-1),\\ 0 & \begin{cases}k\neq q_kn+1,\\ k\neq(q_k+1)n-1,\end{cases}\end{cases} \tag{14.16}$$

$$\Delta a^2_{jk}=0 \quad (j=2,\ 4,\ 6),$$
$$\frac{\Delta a^2_{1k}}{-\sin\beta}=\frac{\Delta a^2_{5k}}{-R_0\cos\beta}=\gamma\sin\beta\frac{\Delta F}{R_0}\times\begin{cases}\frac{n}{2} & \begin{cases}k=q_kn+1,\\ k=(q_k+1)n-1,\end{cases}\\ 0 & \begin{cases}k\neq q_kn+1,\\ k\neq(q_k+1)n-1,\end{cases}\end{cases} \tag{14.17}$$
$$\frac{\Delta a^2_{3k}}{\cos\beta}=\gamma\sin\beta\frac{\Delta F}{R_0}\times\begin{cases}n & (k=q_kn),\\ 0 & (k\neq q_kn).\end{cases}$$

$$\Delta a^1_{ll}=\Delta a^1_{jk};\quad \Delta a^2_{ll}=\Delta a^2_{jk},\quad (j=l;\ \ k=l). \tag{14.18}$$

$$\Delta a^1_{(q_ln+r_l)k}=\gamma\sin\beta\frac{\Delta F}{R_0}\times\begin{cases}\frac{n}{2} & (k=q_kn+r_l),\\ -\frac{n}{2} & (k=(q_k+1)n-r_l),\\ 0 & (r_k\pm r_l\neq 0,\ n)\end{cases}$$
$$r_l\neq 0,\ \frac{n}{2}$$
$$\Delta a^1_{(q_ln)k}=0,$$
$$\Delta a^1_{\left(q_ln+\frac{n}{2}\right)k}=0, \tag{14.19}$$

$$\Delta a^{2}_{(q_1 n + r_1)n} = \gamma \sin\beta \frac{\Delta F}{R_0} \times \begin{cases} \frac{n}{2} & (k = q_2 n + r_1), \\ \frac{n}{2} & (k = (q_2 + 1) n - r_1), \\ 0 & (r_2 \neq r_1 \neq 0,\ n), \end{cases} \qquad r_1 \neq 0,\ \frac{n}{2} \tag{14.20}$$

$$\Delta a^{2}_{(q_1 n)n} = \gamma \sin\beta \frac{\Delta F}{R_0} \times \begin{cases} n & (k = q_2 n), \\ 0 & (k \neq q_2 n), \end{cases}$$

$$\Delta a^{2}_{\left(q_1 n + \frac{n}{2}\right)n} = \gamma \sin\beta \frac{\Delta F}{R_0} \times \begin{cases} n & \left(k = \left(q + \frac{1}{2}\right) n\right), \\ 0 & \left(k \neq \left(q + \frac{1}{2}\right) n\right). \end{cases}$$

Table 14.1

Coefficients Δa_{ji}

$j \backslash i$	1	2	3	4	5	6
1	$\gamma \sin^3\beta \frac{n\Delta F}{2R_0}$	0	0	0	$\frac{1}{2}\gamma \sin^2\beta \cos\beta\, n\Delta F$	0
2	0	$\gamma \sin^3\beta \frac{n\Delta F}{2R_0}$	0	$-\frac{1}{2}\gamma \sin^2\beta \cos\beta\, n\Delta F$	0	0
3	0	0	$\gamma \sin\beta \cos^2\beta \frac{n\Delta F}{R_0}$	0	0	0
4	0	$-\frac{1}{2}\gamma \sin^2\beta \cos\beta\, n\Delta F$	0	$\frac{1}{2}\gamma R_0 \sin\beta \cos^2\beta\, n\Delta F$	0	0
5	$\frac{1}{2}\gamma \sin^2\beta \cos\beta\, n\Delta F$	0	0	0	$\frac{1}{2}\gamma R_0 \sin\beta \cos^2\beta\, n\Delta F$	0
6	0	0	0	0	0	0

For values of coefficients (14.1) determined by the formulas given in Tables 13.1, 14.1, and also formulas (13.1) and (14.16)-(14.20), and the coefficients shown in Tables 13.2, 13.3 and formula (13.2), the system of differential resolvents (m), (n) after the substitution of variables $1-Z=e^t$, the notation

$$\Delta a=\gamma\frac{n\Delta F}{2R_0} \tag{14.21}$$

being used will assume the following form:

$$-(a_{11}^0+\Delta a\sin^2\beta)\,\eta_x'-(a_{15}^0+\Delta aR_0\sin^2\beta\cos\beta)(e^t\theta_y)'+b_{15}e^t\theta_y+$$
$$+\Delta a\sin^2\beta\left\{\omega_{2(n-1)}'+\sum_{q_2=1}^{\infty}\left[\omega_{2(q_2n+1)}'+\omega_{2(q_2n+n-1)}'\right]\right\}=\frac{Q_x(t)}{G}, \tag{14.22}$$

$$-(a_{22}^0+\Delta a\sin^2\beta)\,\eta_y'-(a_{24}^0-\Delta aR_0\sin^2\beta\cos\beta)(e^t\theta_x)'+b_{24}e^t\theta_x+$$
$$+\Delta a\sin^2\beta\left\{-\omega_{1(n-1)}'+\sum_{q_2=1}^{\infty}\left[\omega_{1(q_2n+1)}'-\omega_{1(q_2n+n-1)}'\right]\right\}=\frac{Q_y(t)}{G}, \tag{14.23}$$

$$-(a_{33}^0+2\Delta a\sin\beta\cos^2\beta)\,\eta_z'-2\Delta a\sin\beta\cos\beta\sum_{q_2=1}^{\infty}\omega_{3(q_2n)}'=\frac{N_z(t)}{G}, \tag{14.24}$$

$$-(a_{42}^0-\Delta aR_0\sin^2\beta\cos\beta)\,\eta_y'-(a_{44}^0+\Delta aR_0^2\sin\beta\cos^2\beta)(e^t\theta_x)'-$$
$$-\Delta aR_0\sin\beta\cos\beta\left\{-\omega_{1(n-1)}'+\sum_{q_2=1}^{\infty}\left[\omega_{1(q_2n+1)}'-\omega_{1(q_2n+n-1)}'\right]\right\}=\frac{M_x(t)\,e^{-t}}{G}, \tag{14.25}$$

$$-(a_{51}^0+\Delta aR_0\sin^2\beta\cos\beta)\,\eta_x'-(a_{55}^0+\Delta aR_0^2\sin\beta\cos^2\beta)(e^t\theta_y)'+$$
$$+\Delta aR_0\sin\beta\cos\beta\left\{\omega_{2(n-1)}'+\sum_{q_2=1}^{\infty}\left[\omega_{2(q_2n+1)}'+\omega_{2(q_2n+n-1)}'\right]\right\}=-\frac{M_y(t)\,e^{-t}}{G}, \tag{14.26}$$

$$-a_{66}^0(e^t\theta_z)'+b_{66}e^t\theta_z=\frac{M_z(t)\,e^{-t}}{G}, \tag{14.27}$$

. . . .

$$\pi h\gamma\sin\beta\,\omega_{1(q_1n)}'-\frac{\pi h}{\sin\beta}(q_1n)^2\omega_{1(q_1n)}=-\frac{R_{1(q_1n)}(t)\,e^t}{G}$$
$$(q_1=1,\ 2,\ldots), \tag{14.28}$$

. . . .

$$\pi h\gamma\sin\beta\,\omega_{1\left(q_1n+\frac{n}{2}\right)}'-\frac{\pi h}{\sin\beta}\left(q_1n+\frac{n}{2}\right)^2\omega_{1\left(q_1n+\frac{n}{2}\right)}=-\frac{R_{1\left(q_1n+\frac{n}{2}\right)}(t)\,e^t}{G}$$
$$(q_1=0,\ 1,\ 2,\ldots) \tag{14.29}$$

. . . .

$$
-\Delta a \sin^2\beta\,\dot{\eta}_y + \Delta a R_0 \sin\beta\cos\beta\,(e^t\theta_x)'' + \pi h\gamma \sin\beta\,\dot{\omega}_{1(q_l n+1)} -
$$

$$
-\frac{\pi h}{\sin\beta}(q_l n+1)^2\,\omega_{1(q_l n+1)} + \Delta a \sin\beta\Big\{-\dot{\omega}_{1(n-1)} + \sum_{q_k=1}^{\infty}\big[\dot{\omega}_{1(q_k n+1)} - \dot{\omega}_{1(q_k n+n-1)}\big]\Big\} =
$$

$$
= -\frac{R_{1(q_l n+1)}(t)\,e^t}{G} \qquad (q_l = 1,\ 2, \ldots), \tag{14.30}
$$

. . . .

$$
\Delta a \sin^2\beta\,\dot{\eta}_y - \Delta a R_0 \sin\beta\cos\beta\,(e^t\theta_x)'' + \pi h\gamma \sin\beta\,\dot{\omega}_{1(q_l n+n-1)} -
$$

$$
-\frac{\pi h}{\sin\beta}(q_l n+n-1)^2\,\omega_{1(q_l n+n-1)} - \Delta a \sin\beta\Big\{-\dot{\omega}_{1(n-1)} +
$$

$$
+\sum_{q_k=1}^{\infty}\big[\dot{\omega}_{1(q_k n+1)} - \dot{\omega}_{1(q_k n+n-1)}\big]\Big\} = -\frac{R_{1(q_l n+n-1)}(t)\,e^t}{G}
$$

$$
(q_l = 0,\ 1,\ 2, \ldots), \tag{14.31}
$$

. . . .

$$
\pi h\gamma \sin\beta\,\dot{\omega}_{1(q_l n+r_l)} - \frac{\pi h}{\sin\beta}(q_l n+r_l)^2\,\omega_{1(q_l n+r_l)} +
$$

$$
+\Delta a \sin\beta \sum_{q_k=0}^{\infty}\big[\dot{\omega}_{1(q_k n+r_l)} - \dot{\omega}_{1(q_k n+n-r_l)}\big] = -\frac{R_{1(q_l n+r_l)}(t)\,e^t}{G}
$$

$$
(q_l = 0,\ 1,\ 2, \ldots;\ r_l = 2,\ 3, \ldots,\ n-2). \tag{14.32}
$$

$$
2\Delta a \sin\beta\cos\beta\,\dot{\eta}_z + \pi h\gamma \sin\beta\,\dot{\omega}_{2(q_l n)} - \frac{\pi h}{\sin\beta}(q_l n)^2\,\omega_{2(q_l n)} +
$$

$$
+2\Delta a \sin\beta \sum_{q_k=1}^{\infty}\dot{\omega}_{2(q_k n)} = -\frac{R_{2(q_l n)}(t)\,e^t}{G}
$$

$$
(q_l = 1,\ 2, \ldots), \tag{14.33}
$$

. . . .

$$
\pi h\gamma \sin\beta\,\dot{\omega}_{2\left(q_l n+\frac{n}{2}\right)} - \frac{\pi h}{\sin\beta}\left(q_l n + \frac{n}{2}\right)^2 \omega_{2\left(q_l n+\frac{n}{2}\right)} +
$$

$$
+2\Delta a \sin\beta \sum_{q_k=0}^{\infty}\dot{\omega}_{2\left(q_k n+\frac{n}{2}\right)} = -\frac{R_{2\left(q_l n+\frac{n}{2}\right)}(t)\,e^t}{G}
$$

$$
(q_l = 0,\ 1,\ 2, \ldots). \tag{14.34}
$$

. . . .

$$
-\Delta a \sin^2\beta\,\dot{\eta}_x - \Delta a R_0 \sin\beta\cos\beta\,(e^t\theta_y)'' + \pi h\gamma \sin\beta\,\dot{\omega}_{2(q_l n+1)} -
$$

$$
-\frac{\pi h}{\sin\beta}(q_l n+1)^2\,\omega_{2(q_l n+1)} + \Delta a \sin\beta\Big\{\dot{\omega}_{2(n-1)} + \sum_{q_k=1}^{\infty}\big[\dot{\omega}_{2(q_k n+1)} + \dot{\omega}_{2(q_k n+n-1)}\big]\Big\} =
$$

$$
= -\frac{R_{2(q_l n+1)}(t)\,e^t}{G}
$$

$$
(q_l = 1,\ 2, \ldots), \tag{14.35}
$$

. . . .

$$-\Delta a \sin^2\beta\, \eta_z^{\cdot} - \Delta a R_0 \sin\beta\cos\beta (e^t\theta_z)^{\cdot} + \pi h \sin\beta\, \omega^{\cdot}_{2(q_l n+n-1)} -$$

$$-\frac{\pi h}{\sin\beta}(q_l n + n - 1)^2 \omega_{2(q_l n+n-1)} + \Delta a \sin\beta \left\{\omega^{\cdot}_{2(n-1)} + \right.$$

$$\left. + \sum_{q_k=1}^{\infty} \left[\omega^{\cdot}_{2(q_k n+1)} + \omega^{\cdot}_{2(q_k n+n-1)}\right]\right\} = -\frac{R_{2(q_l n+n-1)}(t)\, e^t}{G}$$

$$(q_l = 0, 1, 2, \ldots), \qquad (14.36) \qquad (14.36)$$

. . . .

$$\pi h \sin\beta\, \omega^{\cdot}_{2(q_l n+r_l)} - \frac{\pi h}{\sin\beta}(q_l n + r_l)^2 \omega_{2(q_l n+r_l)} +$$

$$+ \Delta a \sin\beta \sum_{q_k=0}^{\infty} \left[\omega^{\cdot}_{2(q_k n+r_l)} + \omega^{\cdot}_{2(q_k n+n-r_l)}\right] = -\frac{R_{2(q_l n+r_l)}(t)\, e^t}{G}$$

$$(q_l = 0, 1, 2, \ldots; \; r_l = 2, 3, \ldots, n-2) \qquad (14.37) \qquad (14.37)$$

. . . .

Equations (14.22)-(14.37) represent the general system of differential resolvents of a conical shell of constant thickness with a regular set of stringers. The load terms on the right-hand sides of this system are determined, as in the case of a smooth shell, by expressions (13.11)-(13.13).

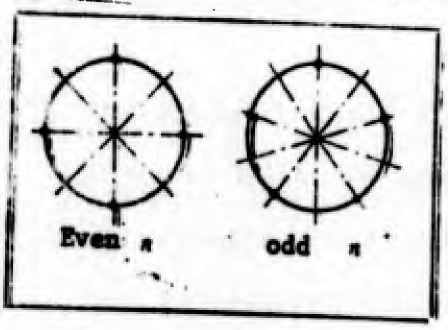

Fig. 14.2. Cyclic symmetry

As can be readily ascertained, the infinite system of differential Eqs. (14.22)-(14.37) decomposes into a series of independent subsystems.

Subsystem (14.24), (1433.) determines the deformation of the shell in the form of a series in cos qnα, cyclically symmetric about the axes of symmetry of the cross section of the shell (Fig. 14.2). A similar deformation takes place if the external load reduces to an axial force or to a self-balanced system of forces with the indicated symmetry properties, applied in the direction of the generatrices.

Subsystem (14.28) determines the deformation of the shell in the form of a series in sin qnα, inversely symmetric about the axes of symmetry of the cross section of the shell. Such deformation can arise only under the influence of self-balanced forces with the same symmetry, applied in the direction of the generatrices.

Subsystems (14.29) and (14.34) take place only when the number n of stringers is even. Subsystem (14.29) determines the deformation of the shell in the form of a series in $\sin\left(q+\frac{1}{2}\right)n\alpha$, inversely symmetric about the axes of symmetry of the cross section passing through the stringers, and symmetric about the axes passing through the middle of the intervals between the stringers.

Subsystem (14.34) corresponds to deformation according to the law $\cos\left(q+\frac{1}{2}\right)n\alpha$, symmetric about the axes of symmetry of the cross section of the shell passing through the stringers, and inversely symmetric about the axes that do not pass through them. Both forms of deformation can arise only under the influence of self-balanced forces with the corresponding symmetry properties, applied along the generatrices.

Subsystems (14.22), (14.26), (14.35), (14.36) and (14.23), (14.25), (14.30), (14.31) determine the deformations of the shell in the form of series in $\cos(qn\pm 1)\alpha$ and correspondingly in $\sin(qn\pm 1)\alpha$. As follows from the expressions for the corresponding load terms, such deformation takes place if the external load reduces to transverse forces and bending moments Q_x, M_y and correspondingly Q_y, M_x, and also in the presence of a self-balanced load with components of the form $\cos(qn\pm 1)\alpha$, $\sin(qn\pm 1)\alpha$, acting in the direction of the generatrices of the shell.

Subsystems (14.32) and (14.37) determine the deformations of the shell in the form of expansions in $\sin(qn\pm r)\alpha$ and correspondingly in

$\cos(qn \pm r)\alpha$ for the values $r=2, 3, \ldots, n-2\left(r \neq \frac{n}{2}\right)$ Such deformation can arise only in the presence of a self-balanced load of the same type acting in the direction of the generatrices.

Finally, Eq. (14.27) in θ_z represents the equilibrium condition of the cut-off portion of the shell in the form of equality to zero of the sum of the moments about the Oz axis.

Thus, the general system of differential resolvents of a conical shell with a stringer structure consists of a single independent equation in θ_z and eight unrelated infinite subsystems of differential equations. Each of these infinite subsystems, with the exception of completely decomposed (14.28) and (14.29) is coupled. This is related to the fundamental analytical content of the problem under consideration, consisting in the construction of solutions of infinite subsystems.

14.2. Homogeneous Problem. Transcendental Characteristic Equations

Setting all the load terms equal to zero, we will consider a homogeneous problem.

Of the first six equations pertaining to displacements of the cross sectional contour $\bar{Z}$ = const as a solid, only Eq. (14.27) in θ_z is independent. The general solution of the homogeneous equation corresponding to it will be

$$\theta_z(t) = \theta_z^0. \tag{14.38}$$

Let us now turn to subsystem (14.24), (14.33) in η_z, $\omega_{z(q,n)}$. The specific feature of the structure of this infinite coupled subsystem lies in the fact that the coupling is determined by the derivatives

$\sum_{q_k=1}^{\infty}\omega_{2(q_k n)}$ in the same way for all the equations of the system. A similar property is exhibited by all the coupled infinite systems of the problem under consideration. We will apply the following method to such systems.

Eliminating η_s^* from system (14.33) with the aid of Eq. (14.24) and expanding the expressions for the coefficients a_{33}^0 (Table 13.1) and Δa (14.21), we obtain

$$\omega_{2(q_i n)}^{**} - \left(\frac{q_i n}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{2(q_i n)} = -\frac{2}{1+\varkappa}\sum_{q_k=1}^{\infty}\omega_{2(q_k n)}^{**} \quad (q_i = 1, 2, \ldots), \tag{14.39}$$

where

$$\varkappa = \frac{2\pi R_0 h}{n\Delta F^0} \tag{14.40}$$

is the ratio of the area of the skin to the total area of the stringers in the section $\bar{Z} = 0$.

We will seek the solution of system (14.39) in the form

$$\omega_{2(q_i n)}(t) = C_{2(q_i n)} e^{\lambda t} \quad (q_i = 1, 2, \ldots). \tag{14.41}$$

Introducing (14.41) into (14.39), we obtain

$$C_{2(q_i n)} = \frac{2}{1+\varkappa}\,\frac{\bar{\lambda}^2}{q_i^2 - \bar{\lambda}^2}\,C_{2(n)}^* \quad (q_i = 1, 2, \ldots), \tag{14.42}$$

where

$$C_{2(n)}^* = \sum_{q_k=1}^{\infty} C_{2(q_k n)}, \tag{14.43}$$

$$\bar{\lambda} = \frac{\sqrt{\gamma}\sin\beta}{n}\lambda. \tag{14.44}$$

We have obtained an infinite homogeneous system of algebraic equations in arbitrary constants $C_{2(q_i n)}$, containing the unknown parameter $\bar{\lambda}$. This parameter must be sought from the condition of nontriviality of the solution of this system.

It follows from (14.42) that all the constants $C_{2(n)}$ are expressed in terms of one arbitrary constant $C^{*}_{2(n)}$. Hence, the condition of nontriviality of the solution of system (14.42) will be $C^{*}_{2(n)} \neq 0$.

Introducing (14.42) into the right-hand side of (14.43) and reducing by $C^{*}_{2(n)}$, we have

$$2\bar{\lambda}^{2} \sum_{q_k=1}^{\infty} \frac{1}{q_k^2 - \bar{\lambda}^2} = 1 + \nu. \tag{14.45}$$

On the basis of the well-known summation formula

$$\sum_{k=1}^{\infty} \frac{1}{x^2 - k^2} = \frac{\pi}{2x}\left(\operatorname{ctg} \pi x - \frac{1}{\pi x}\right) \tag{14.46}$$

we finally obtain

$$\lambda^{*} \operatorname{ctg} \lambda^{*} = -\nu, \tag{14.47}$$

where

$$\lambda^{*} = \pi\bar{\lambda} = \frac{\pi \sqrt{\gamma \sin \beta}}{n} \lambda. \tag{14.48}$$

Expression (14.47) constitutes the transcendental characteristic equation of infinite system of differential Eqs. (14.24), (14.33). This equation has an infinite number of pairs of roots $\lambda^{*} = \pm \lambda^{*}_{s}$. Therefore, the general solution of honogeneous system (14.39) will be

$$\omega_{2(q_i,n)}(t) = \sum_{s=1}^{\infty} \frac{\bar{\lambda}_s^2}{q_i^2 - \bar{\lambda}_s^2} \left[C^{s}_{2(n)} e^{\lambda_s t} + C^{-s}_{2(n)} e^{-\lambda_s t}\right], \tag{14.49}$$

where $C^{s}_{2(n)}$, $C^{-s}_{2(n)}$ are arbitrary constants.

From Eq. (14.24) it is easy to obtain

$$\eta_z(t) = \eta_z^0 - \frac{1}{1+\nu} \sum_{q_s=1}^{\infty} \omega_z(q_s, n) \Big|_0^t, \tag{14.50}$$

where $\eta_z^0 = \eta_z(0)$.

Introducing (14.49) into (14.50) and using summation formula (14.46) as well as characteristic Eq. (14.47), we finally find

$$\eta_s(t)=\eta_s^0+\frac{1}{2\cos\beta}\sum_{s=1}^{\infty}\left[C_{2(n)}^{+}\left(1-e^{\lambda_s t}\right)+C_{2(n)}^{-}\left(1-e^{-\lambda_s t}\right)\right]. \tag{14.51}$$

Let us turn to homogeneous subsystem (14.34) in $\omega_{2\left(q_l n+\frac{n}{2}\right)}$. Using (14.21) and (14.40), we have

$$\ddot{\omega}_{2\left(q_l n+\frac{n}{2}\right)}-\left(\frac{q_l n+\frac{n}{2}}{l\,\bar{\gamma}\sin\beta}\right)^2\omega_{2\left(q_l n+\frac{n}{2}\right)}=-\frac{\varepsilon}{\pi}\sum_{q_k=0}^{\infty}\ddot{\omega}_{2\left(q_k n+\frac{n}{2}\right)} \tag{14.52}$$
$$(q_l=0,\ 1,\ 2,\ldots).$$

Representing the partial solution of system (14.52) in the form

$$\omega_{2\left(q_l n+\frac{n}{2}\right)}(t)=C_{2\left(q_l n+\frac{n}{2}\right)}e^{\lambda t}\qquad (q_l=0,\ 1,\ 2,\ldots),$$

we find

$$C_{2\left(q_l n+\frac{n}{2}\right)}=\frac{2}{\pi}\,\frac{\bar{\lambda}^2}{\left(q_l+\frac{1}{2}\right)^2-\bar{\lambda}^2}\,C_{2\left(\frac{n}{2}\right)}^{*}, \tag{14.53}$$

where

$$C_{2\left(\frac{n}{2}\right)}^{*}=\sum_{q_k=0}^{\infty}C_{2\left(q_k n+\frac{n}{2}\right)}. \tag{14.54}$$

Introducing (14.53) into the right-hand side of (14.54) and reducing by $C_{2\left(\frac{n}{2}\right)}^{*}$, we have

$$2\bar{\lambda}^2\sum_{q_k=0}^{\infty}\frac{1}{\left(q_k+\frac{1}{2}\right)^2-\bar{\lambda}^2}=\nu. \tag{14.55}$$

The infinite series in (14.55) is easy to sum up by using expansion (14.46). We have

$$S=2\bar{\lambda}^2\sum_{q_k=0}^{\infty}\frac{1}{\left(q_k+\frac{1}{2}\right)^2-\bar{\lambda}^2}=8\bar{\lambda}^2\sum_{q_k=0}^{\infty}\frac{1}{(2q_k+1)^2-(2\bar{\lambda})^2}=$$
$$=8\bar{\lambda}^2\left[\sum_{k=1}^{\infty}\frac{1}{k^2-(2\bar{\lambda})^2}-\sum_{k=2,4,6,\ldots}^{\infty}\frac{1}{k^2-(2\bar{\lambda})^2}\right]=$$
$$=8\bar{\lambda}^2\left[\sum_{k=1}^{\infty}\frac{1}{k^2-(2\bar{\lambda})^2}-\sum_{k=1}^{\infty}\frac{1}{(2k)^2-(2\bar{\lambda})^2}\right]=$$
$$=8\bar{\lambda}^2\left[\sum_{k=1}^{\infty}\frac{1}{k^2-(2\bar{\lambda})^2}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2-\bar{\lambda}^2}\right].$$

Now, using formula (14.46) twice, after some simple transformations we obtain

$$S=\lambda^{*}\operatorname{tg}\lambda^{*}. \tag{14.56}$$

Introducing (14.56) into (14.55), we obtain the following transcendental characteristic equation corresponding to infinite system of differential Eqs. (14.34):

$$\lambda^{*}\operatorname{tg}\lambda^{*}=\varkappa. \tag{14.57}$$

Equation (14.57) has an infinite number of pairs of roots $\lambda^{*}=\pm\lambda_s^{*}$. The general solution of homogenous system (14.34) will be

$$w_{2\left(q_1 n+\frac{n}{2}\right)}(t)=\sum_{s=1}^{\infty}\frac{\lambda_s^{2}}{\left(q_1+\frac{1}{2}\right)^2-\lambda_s^2}\left[C^{t}_{2\left(\frac{n}{2}\right)}e^{\lambda_s t}+C^{-t}_{2\left(\frac{n}{2}\right)}e^{-\lambda_s t}\right], \tag{14.58}$$

where $C^{t}_{2\left(\frac{n}{2}\right)}$, $C^{-t}_{2\left(\frac{n}{2}\right)}$ are arbitrary constants.

Expressions (14.49), (14.51) and (14.58) represent the desired generalized displacements corresponding to expansions in cos qnα and $\cos\left(qn+\frac{n}{2}\right)\alpha$.

Let us turn to subsystems (14.28), (14.29), corresponding to expansions in sin qnα and $\sin\left(qn+\frac{n}{2}\right)\alpha$. Each of these subsystems has decomposed completely into independent second-order equations. This is explained by the fact that both sin qnα and $\sin\left(qn+\frac{n}{2}\right)\alpha$ vanish at the points of location of the stringers $\alpha=\frac{2\pi}{n}m\ (m=1,2,\ldots,n)$ for any q. The general solution of homogenous Eqs. (14.28) and (14.29) will be

$$w_{1(q_1 n)}(t)=C^{t}_{1(n)}e^{\frac{q_1 n}{\sqrt{\gamma}\sin\theta}t}+C^{-t}_{1(n)}e^{-\frac{q_1 n}{\sqrt{\gamma}\sin\theta}t} \tag{14.59}$$

$$(q_1=1,2,\ldots),$$

$$w_{1\left(q_1 n+\frac{n}{2}\right)}(t)=C^{t}_{1\left(\frac{n}{2}\right)}e^{\frac{q_1 n+\frac{n}{2}}{\sqrt{\gamma}\sin\theta}t}+C^{-t}_{1\left(\frac{n}{2}\right)}e^{-\frac{q_1 n+\frac{n}{2}}{\sqrt{\gamma}\sin\theta}t} \tag{14.60}$$

$$(q_1=1,2,\ldots).$$

We now turn to infinite subsystem (14.22), (14.26), (14.35), (14.36). Setting the load terms equal to zero and expanding the coefficients with the aid of Tables 13.1 and 13.2 and (14.21), (14.40), we obtain

$$
\begin{aligned}
&-\varkappa \sin\beta\, \eta_x' - \gamma \sin^2\beta (1+\varkappa)\left[\sin\beta\, \eta_x' + R_0 \cos\beta\, (e^t\theta_y)'\right] - \\
&\qquad - R_0\varkappa \cos\beta\, e^t\theta_y + \gamma \sin^2\beta\, S_2'(t) = 0,
\end{aligned} \tag{14.61}
$$

$$
-(\varkappa+1)\left[\sin\beta\, \eta_x' + R_0\cos\beta\,(e^t\theta_y)'\right] + S_2'(t) = 0, \tag{14.62}
$$

$$
\begin{aligned}
&-\frac{1}{\varkappa}\left[\sin\beta\, \dot{\eta}_x + R_0\cos\beta\,(e^t\theta_y)''\right] + \dot{\omega}_{2(q_l n+1)} - \\
&\qquad -\left(\frac{q_l n+1}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{2(q_l n+1)} + \frac{1}{\varkappa}\dot{S}_2(t) = 0 \\
&\qquad\qquad (q_l = 1, 2, \ldots),
\end{aligned} \tag{14.63}
$$

$$
\begin{aligned}
&-\frac{1}{\varkappa}\left[\sin\beta\, \dot{\eta}_x + R_0\cos\beta\,(e^t\theta_y)''\right] + \dot{\omega}_{2(q_l n+n-1)} - \\
&\qquad -\left(\frac{q_l n+n-1}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{2(q_l n+n-1)} + \frac{1}{\varkappa}\dot{S}_2(t) = 0 \\
&\qquad\qquad (q_l = 0, 1, 2, \ldots),
\end{aligned} \tag{14.64}
$$

where

$$
S_2(t) = \omega_{2(n-1)} + \sum_{q_l=1}^{\infty}\left[\omega_{2(q_l n+1)} + \omega_{2(q_l n+n-1)}\right]. \tag{14.65}
$$

Differentiating (14.62) and introducing the result into Eqs. (14.63), (14.64), we eliminate $\dot{\eta}_x$ and $\dot{\theta}_y$ from them. We obtain

$$
\begin{aligned}
&\dot{\omega}_{2(q_l n+1)} - \left(\frac{q_l n+1}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{2(q_l n+1)} = -\frac{1}{1+\varkappa}\dot{S}_2(t) \\
&\qquad (q_l = 1, 2, \ldots), \\
&\dot{\omega}_{2(q_l n+n-1)} - \left(\frac{q_l n+n-1}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{2(q_l n+n-1)} = -\frac{1}{1+\varkappa}\dot{S}_2(t) \\
&\qquad (q_l = 0, 1, 2, \ldots).
\end{aligned} \tag{14.66}
$$

We will seek the solution of infinite system (14.66) in the form

$$
\begin{aligned}
&\omega_{2(q_l n+1)}(t) = C_{2(q_l n+1)} e^{\lambda t} \quad (q_l = 1, 2, \ldots), \\
&\omega_{2(q_l n+n-1)}(t) = C_{2(q_l n+n-1)} e^{\lambda t} \quad (q_l = 0, 1, 2, \ldots).
\end{aligned} \tag{14.67}
$$

Introducing (14.67) into (14.66), we obtain

$$
C_{2(q_l n+1)} = \frac{1}{1+\varkappa}\,\frac{\bar{\lambda}^2}{\left(q_l + \frac{1}{n}\right)^2 - \bar{\lambda}^2}\, C^*_{2(1)} \tag{14.68}
$$

$$
(q_l = 1, 2, \ldots),
$$

$$C_{2(q_l n+n-1)} = \frac{1}{1+\varkappa} \frac{\bar{\lambda}^2}{\left(q_l+1-\frac{1}{n}\right)^2-\bar{\lambda}^2} C^*_{2(1)} \tag{14.69}$$

$$(q_l=0,\ 1,\ 2,\dots)$$

where

$$C^*_{2(1)} = C_{2(n-1)} + \sum_{q_k=1}^{\infty}\left[C_{2(q_k n+1)} + C_{2(q_k n+n-1)}\right]. \tag{14.70}$$

According to (14.68), (14.69), all the constants $C_{2(q_l n+1)}, C_{2(q_l n+n-1)}$ are expressed in terms of $C^*_{2(1)}$. Introducing (14.68), (14.69) into the right-hand side of (14.70), and reducing by $C^*_{2(1)} \neq 0$, we have

$$\frac{\bar{\lambda}^2}{\left(1-\frac{1}{n}\right)^2-\bar{\lambda}^2} + \bar{\lambda}^2 \sum_{q_k=1}^{\infty}\left[\frac{1}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2} + \frac{1}{\left(q_k+1-\frac{1}{n}\right)^2-\bar{\lambda}^2}\right] = 1+\varkappa. \tag{14.71}$$

The infinite series in (14.71) can be easily summed up by using expansion (14.46). We have

$$\begin{aligned} S &= \frac{\bar{\lambda}^2}{\left(1-\frac{1}{n}\right)^2-\bar{\lambda}^2} + \bar{\lambda}^2 \sum_{q_k=1}^{\infty}\left[\frac{1}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2} + \frac{1}{\left(q_k+1-\frac{1}{n}\right)^2-\bar{\lambda}^2}\right] = \\ &= \bar{\lambda}^2 \sum_{q_k=1}^{\infty}\left[\frac{1}{\left(q_k-\frac{1}{n}\right)^2-\bar{\lambda}^2} + \frac{1}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2}\right]. \end{aligned} \tag{14.72}$$

Expanding the expression under the summation sign into simple fractions and rearranging the terms, we obtain

$$\begin{aligned} S &= \bar{\lambda}^2 \sum_{q_k=1}^{\infty}\left\{\frac{1}{2\bar{\lambda}}\left[\frac{1}{q_k+\frac{1}{n}-\bar{\lambda}} - \frac{1}{q_k+\frac{1}{n}+\bar{\lambda}}\right] + \right. \\ &\left. + \frac{1}{2\bar{\lambda}}\left[\frac{1}{q_k-\frac{1}{n}-\bar{\lambda}} - \frac{1}{q_k-\frac{1}{n}+\bar{\lambda}}\right]\right\} = \\ &= \frac{\bar{\lambda}}{2}\left\{\sum_{q_k=1}^{\infty}\left[\frac{1}{q_k+\left(\frac{1}{n}-\bar{\lambda}\right)} - \frac{1}{q_k-\left(\frac{1}{n}-\bar{\lambda}\right)}\right] - \right. \\ &\left. - \sum_{q_k=1}^{\infty}\left[\frac{1}{q_k+\left(\frac{1}{n}+\bar{\lambda}\right)} - \frac{1}{q_k-\left(\frac{1}{n}+\bar{\lambda}\right)}\right]\right\} = \\ &= -\bar{\lambda}\left(\frac{1}{n}-\bar{\lambda}\right)^2 \sum_{q_k=1}^{\infty}\frac{1}{q_k^2-\left(\frac{1}{n}-\bar{\lambda}\right)^2} + \\ &+ \bar{\lambda}\left(\frac{1}{n}+\bar{\lambda}\right) \sum_{q_k=1}^{\infty}\frac{1}{q_k^2-\left(\frac{1}{n}+\bar{\lambda}\right)^2}. \end{aligned}$$

Now, using summation formula (14.46) twice, we find

$$S=\frac{\lambda^*}{2}\left[\operatorname{ctg}\left(\frac{\pi}{n}-\lambda^*\right)-\operatorname{ctg}\left(\frac{\pi}{n}+\lambda^*\right)-\frac{2\lambda^*}{\left(\frac{\pi}{n}\right)^2-\lambda^{*2}}\right]. \tag{14.73}$$

Introducing sum (14.72), represented by expression (14.73), into (14.71), after some transformations we obtain

$$\frac{\lambda^*\sin 2\lambda^*}{\cos 2\lambda^*-\cos\frac{2\pi}{n}}+\frac{1}{\left(\frac{n\lambda^*}{\pi}\right)^2-1}=\varkappa. \tag{14.74}$$

Expression (14.74) represents the transcendental characteristic equation of infinite subsystem (14.22), (14.26), (14.35), (14.36). This equation has an infinite number of pairs of roots $\lambda^*_s=\pm\bar{\lambda}_s$. Therefore, in accordance with (14.67)-(14.69), the general solution will have the form

$$\omega_{2(q_in+1)}=\sum_{s=1}^{\infty}\frac{\bar{\lambda}_s^2}{\left(q_i+\frac{1}{n}\right)^2-\bar{\lambda}_s^2}\left[C^s_{2(1)}e^{\lambda_s t}+C^{-s}_{2(1)}e^{-\lambda_s t}\right]$$

$$(q_i=1, 2, \ldots),$$

$$\omega_{2(q_in+n-1)}=\sum_{s=1}^{\infty}\frac{\bar{\lambda}_s^2}{\left(q_i+1-\frac{1}{n}\right)^2-\bar{\lambda}_s^2}\left[C^s_{2(1)}e^{\lambda_s t}+C^{-s}_{2(1)}e^{-\lambda_s t}\right] \tag{14.75}$$

$$(q_i=0, 1, 2, \ldots).$$

From Eqs. (14.22), (14.26), we find

$$(e^t\theta_y)'-e^t\theta_y-\frac{1}{R_0\cos\beta(1+\varkappa)}S_2'(t)=0, \tag{14.76}$$

$$\eta_x=-R_0\operatorname{ctg}\beta\, e^t\theta_y. \tag{14.77}$$

We expand (14.65) with the aid of general solution (14.75). Changing the order of the summation and considering (14.71), we obtain

$$S_2(t)=(1+\varkappa)\sum_{s=1}^{\infty}\left[C^s_{2(1)}e^{\lambda_s t}+C^{-s}_{2(1)}e^{-\lambda_s t}\right]. \tag{14.78}$$

We integrate Eqs. (14.76), (14.77). Using the method of variation of arbitrary constants, we obtain

$$\theta_y(t)=\theta_y^0+\frac{1}{R_0\cos\beta(1+\varkappa)}\int_0^t e^{-\xi}S_3(\xi)d\xi. \tag{14.79}$$

Eliminating θ_y from (14.77) with the aid of (14.76), we also get

$$\eta_x(t)=\eta_x^0-R_0\operatorname{ctg}\beta\left[e^t\theta_y(t)-\theta_y^0\right]+\frac{1}{\sin\beta(1+\varkappa)}\left[S_3(t)-S_3(0)\right]. \tag{14.80}$$

Expanding (14.79), (14.80) with the aid of (14.78), we finally obtain

$$\theta_y(t)=\theta_y^0+\frac{1}{R_0\cos\beta}\sum_{s=1}^{\infty}\left\{\frac{\lambda_s}{\lambda_s-1}C_{3(1)}^{s}\left[e^{(\lambda_s-1)t}-1\right]+\right.$$
$$\left.+\frac{\lambda_s}{\lambda_s+1}C_{3(1)}^{-s}\left[e^{-(\lambda_s+1)t}-1\right]\right\}, \tag{14.81}$$

$$\eta_x(t)=\eta_x^0+\theta_y^0R_0\operatorname{ctg}\beta(1-e^t)-$$
$$-\frac{1}{\sin\beta}\sum_{s=1}^{\infty}\left[C_{3(1)}^{s}\left(\frac{e^{\lambda_s t}-\lambda_s e^t}{\lambda_s-1}+1\right)-C_{3(1)}^{-s}\left(\frac{e^{-\lambda_s t}+\lambda_s e^t}{\lambda_s+1}-1\right)\right]. \tag{14.82}$$

Thus, expressions (14.75), (14.81), (14.82) represent in final form the general solution of infinite subsystem (14.22), (14.26), (14.35), (14.36) determining the deformation of the shell in the form of series in $\cos(qn\pm1)\alpha$.

Let us now turn to the analogous subsystem (14.23), (14.25), (14.30), (14.31), corresponding to series in $\sin(qn\pm1)\alpha$. Setting the load terms equal to zero and expanding the coefficients with the aid of the expressions given in Table 13.1, 13.2 and expressions (14.21), (14.40), we obtain

$$-\varkappa\sin\beta\eta_y'-\gamma\sin^2\beta(\varkappa+1)\left[\sin\beta\,\eta_y'-R_0\cos\beta(e^t\theta_x)'\right]+$$
$$+R_0\varkappa\cos\beta e^t\theta_x+\gamma\sin^2\beta\,S_1'(t)=0, \tag{14.83}$$

$$(\varkappa+1)\left[\sin\beta\,\eta_y'-R_0\cos\beta(e^t\theta_x)'\right]-S_1'(t)=0, \tag{14.84}$$

$$-\frac{1}{\varkappa}\left[\sin\beta\,\eta_y''-R_0\cos\beta(e^t\theta_x)''+\omega_{1(q_ln+1)}''-\right.$$
$$\left.-\left(\frac{q_ln+1}{\sqrt{\gamma}\sin\beta}\right)^2\omega_{1(q_ln+1)}+\frac{1}{\varkappa}S_1''(t)=0\right.$$
$$(q_l=1,2,\ldots), \tag{14.85}$$

$$\frac{1}{\varkappa}\left[\sin\beta\,\eta_y''-R_0\cos\beta(e^t\theta_x)''\right]+\omega_{1(q_ln+n-1)}''-$$
$$-\left(\frac{q_ln+n-1}{\sqrt{\gamma}\sin\beta}\right)^2\omega_{1(q_ln+n-1)}-\frac{1}{\varkappa}S_1''(t)=0$$
$$(q_l=0,1,2,\ldots), \tag{14.86}$$

where

$$S_1(t) = -\omega_{1(n-1)} + \sum_{q_k=1}^{\infty}\left[\omega_{1(q_k n+1)} - \omega_{1(q_k n+n-1)}\right] \tag{14.87}$$

Differentiating (14.84) and introducing the result into Eqs. (14.85), (14.87), we eliminate η_v^*, θ_x^* from them. We obtain

$$\omega^{\cdot\cdot}_{1(q_l n+1)} - \left(\frac{q_l n+1}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{1(q_l n+1)} = -\frac{1}{1+\varkappa} S_1^{\cdot\cdot}$$
$$(q_l = 1, 2, \ldots).$$
$$\omega^{\cdot\cdot}_{1(q_l n+n-1)} - \left(\frac{q_l n+n-1}{\sqrt{\gamma}\sin\beta}\right)^2 \omega_{1(q_l n+n-1)} = \frac{1}{1+\varkappa} S_1^{\cdot\cdot}$$
$$(q_l = 0, 1, 2, \ldots). \tag{14.88}$$

Representing the solution in the form

$$\omega_{1(q_l n+1)}(t) = C_{1(q_l n+1)} e^{\lambda t} \quad (q_l = 1, 2, \ldots),$$
$$\omega_{1(q_l n+n-1)}(t) = C_{1(q_l n+n-1)} e^{\lambda t} \quad (q_l = 0, 1, 2, \ldots), \tag{14.89}$$

we find from (14.88)

$$C_{1(q_l n+1)} = \frac{1}{1+\varkappa}\,\frac{\bar\lambda^2}{\left(q_l+\frac{1}{n}\right)^2 - \bar\lambda^2}\, C^*_{1(1)}$$
$$(q_l = 1, 2, \ldots),$$
$$C_{1(q_l n+n-1)} = -\frac{1}{1+\varkappa}\,\frac{\bar\lambda^2}{\left(q_l+1-\frac{1}{n}\right)^2 - \bar\lambda^2}\, C^*_{1(1)}$$
$$(q_l = 0, 1, 2, \ldots), \tag{14.90}$$

where

$$C^*_{1(1)} = -C_{1(n-1)} + \sum_{q_k=1}^{\infty}\left[C_{1(q_k n+1)} - C_{1(q_k n+n-1)}\right]. \tag{14.91}$$

Introducing (14.90) into the right-hand size of (14.91) and reducing by $C^*_{1(1)} \neq 0$, we obtain

$$\frac{\bar\lambda^2}{\left(1-\frac{1}{n}\right)^2 - \bar\lambda^2} + \bar\lambda^2 \sum_{q_k=1}^{\infty}\left[\frac{1}{\left(q_k+\frac{1}{n}\right)^2 - \bar\lambda^2} + \frac{1}{\left(q_k+1-\frac{1}{n}\right)^2 - \bar\lambda^2}\right] = 1+\varkappa. \tag{14.92}$$

Expression (14.92) coincides completely with expression (14.71). Hence, the characteristic equation of infinite subsystem (14.23), (14.25), (14.30), (14.31), corresponding to expansions in $\sin(qn \pm 1)\alpha$, coincides with the characteristic equation

$$\frac{\lambda^* \sin 2\lambda^*}{\cos 2\lambda^* - \cos \frac{2\pi}{n}} + \frac{1}{\left(\frac{n\lambda^*}{\pi}\right)^2 - 1} = \varkappa,$$

corresponding to expansions in $\cos(qn \pm 1)\alpha$.

The general solution has the form

$$w_{1(q_l n+1)} = \sum_{s=1}^{\infty} \frac{\bar{\lambda}_s^2}{\left(q_l + \frac{1}{n}\right)^2 - \bar{\lambda}_s^2} \left[C_{1(1)}^s e^{\lambda_s t} + C_{\overline{1(1)}}^s e^{-\lambda_s t}\right] \quad (q_l = 1, 2, \ldots), \tag{14.93}$$

$$w_{1(q_l n+n-1)} = -\sum_{s=1}^{\infty} \frac{\bar{\lambda}_s^2}{\left(q_l + 1 - \frac{1}{n}\right)^2 - \bar{\lambda}_s^2} \left[C_{1(1)}^s e^{\lambda_s t} + C_{\overline{1(1)}}^s e^{-\lambda_s t}\right] \quad (q_l = 0, 1, 2, \ldots)$$

From Eqs. (14.83), (14.84), we find

$$(e^t \theta_z)' - e^t \theta_z + \frac{1}{R_0 \cos\beta\,(1+\varkappa)} S_1'(t) = 0, \tag{14.94}$$

$$\eta_y' = R_0 \operatorname{ctg}\beta\, e^t \theta_z. \tag{14.95}$$

We expand (14.87) with the aid of (14.93). Changing the summation order and considering (14.92), we obtain

$$S_1(t) = (1+\varkappa) \sum_{s=1}^{\infty} \left[C_{1(1)}^s e^{\lambda_s t} + C_{\overline{1(1)}}^s e^{-\lambda_s t}\right]. \tag{14.96}$$

Integrating (14.94), (14.95), we obtain

$$\theta_z(t) = \theta_z^0 - \frac{1}{R_0 \cos\beta\,(1+\varkappa)} \int_0^t e^{-\xi} S_1'(\xi)\, d\xi, \tag{14.97}$$

$$\eta_y(t) = \eta_y^0 + R_0 \operatorname{ctg}\beta \left[e^t \theta_z(t) - \theta_z^0\right] + \frac{1}{\sin\beta\,(1+\varkappa)} \left[S_1(t) - S_1(0)\right]. \tag{14.98}$$

Introducing (14.96) into (14.97), (14.98), we finally obtain

$$\theta_z(t) = \theta_z^0 - \frac{1}{R_0 \cos\beta} \sum_{s=1}^{\infty} \left\{ \frac{\lambda_s}{\lambda_s - 1} C_{1(1)}^s \left[e^{(\lambda_s - 1)t} - 1\right] + \right.$$

$$\left. + \frac{\lambda_s}{\lambda_s + 1} C_{\overline{1(1)}}^s \left[e^{-(\lambda_s + 1)t} - 1\right] \right\}. \tag{14.99}$$

$$\eta_s(t)=\eta_s^0-\theta_s^0 R_s \operatorname{ctg}\beta(1-e^{-t})-\frac{1}{\sin\beta}\sum_{s=1}^{\infty}\left[C_{1(s)}^{+}\left(\frac{e^{\lambda_s t}-\lambda_s e^{t}}{\lambda_s-1}+1\right)-\right.$$

$$\left.-C_{1(s)}^{-}\left(\frac{e^{-\lambda_s t}+\lambda_s e^{t}}{\lambda_s+1}-1\right)\right]. \tag{14.100}$$

Let us consider subsystems (14.32), (14.37). It can be readily seen that each of them in turn decomposes into independent infinite systems in $\omega_{1(q_k n+r)}, \omega_{1(q_k n+n-r)}$ and consequently $\omega_{2(q_k n+r)}, \omega_{2(q_k n+n-r)}$, corresponding to fixed values of r. Obviously, for an even n, the number of such independent systems in each of subsystems (14.32), (14.37) is equal to $\frac{n-3}{2}$, and for odd n, to $\frac{n-1}{2}$.

Setting the load terms equal to zero in (14.32) and considering (14.21), we obtain

$$\omega_{1(q_l n+r)}^{\cdot\cdot}-\left(\frac{q_l n+r}{\sqrt{\gamma}\sin\beta}\right)^2\omega_{1(q_l n+r)}=-\frac{1}{\kappa}S_{1r}^{\cdot\cdot} \qquad (q_l=0,1,2,\ldots); \tag{14.101}$$

when $r_l=n-r$

$$\omega_{1(q_l n+n-r)}^{\cdot\cdot}-\left(\frac{q_l n+n-r}{\sqrt{\gamma}\sin\beta}\right)^2\omega_{1(q_l n+n-r)}=\frac{1}{\kappa}S_{1r}^{\cdot\cdot} \qquad (q_l=0,1,2,\ldots), \tag{14.102}$$

where

$$S_{1r}=\sum_{q_k=0}^{\infty}\left[\omega_{1(q_k n+r)}-\omega_{1(q_k n+n-r)}\right]. \tag{14.103}$$

Taking

$$\omega_{1(q_l n+r)}(t)=C_{1(q_l n+r)}e^{\lambda t},$$

$$\omega_{1(q_l n+n-r)}(t)=C_{1(q_l n+n-r)}e^{\lambda t} \qquad (q_l=0,1,2,\ldots), \tag{14.104}$$

we find from (14.101), (14.102)

$$C_{1(q_l n+r)}=\frac{1}{\kappa}\frac{\bar{\lambda}^2}{\left(q_l+\frac{r}{n}\right)^2-\bar{\lambda}^2}C_{1(r)}^{*} \qquad (q_l=0,1,2,\ldots),$$

$$C_{1(q_l n+n-r)}=-\frac{1}{\kappa}\frac{\bar{\lambda}^2}{\left(q_l+1-\frac{r}{n}\right)^2-\bar{\lambda}^2}C_{1(r)}^{*} \qquad (q_l=0,1,2,\ldots), \tag{14.105}$$

where

$$C^{*}_{1(r)} = \sum_{q_k=0}^{\infty} [C_{1(q_k n+r)} - C_{1(q_k n+n-r)}]. \tag{14.106}$$

Introducing (14.105) into the right-hand side of (14.106) and reducing by $C^{*}_{1(r)} \neq 0$, we obtain

$$\bar{\lambda}^2 \sum_{q_k=0}^{\infty} \left[\frac{1}{\left(q_k + \frac{r}{n}\right)^2 - \bar{\lambda}^2} + \frac{1}{\left(q_k + 1 - \frac{r}{n}\right)^2 - \bar{\lambda}^2} \right] = x. \tag{14.107}$$

The infinite series

$$S = \bar{\lambda}^2 \sum_{q_k=0}^{\infty} \left[\frac{1}{\left(q_k + \frac{r}{n}\right)^2 - \bar{\lambda}^2} + \frac{1}{\left(q_k + 1 - \frac{r}{n}\right)^2 - \bar{\lambda}^2} \right]$$

in (14.107) is easy to sum up by reducing it first to the form

$$S = \frac{\bar{\lambda}^2}{\left(\frac{r}{n}\right)^2 - \bar{\lambda}^2} + \bar{\lambda}^2 \sum_{q_k=1}^{\infty} \left[\frac{1}{\left(q_k + \frac{r}{n}\right)^2 - \bar{\lambda}^2} + \frac{1}{\left(q_k - \frac{r}{n}\right)^2 - \bar{\lambda}^2} \right]. \tag{14.108}$$

Transforming the sum in (14.108) as we did sum (14.72), we find

$$S = \frac{\lambda^*}{2} \left[\operatorname{ctg}\left(\frac{\pi r}{n} - \lambda^*\right) - \operatorname{ctg}\left(\frac{\pi r}{n} + \lambda^*\right) \right]. \tag{14.109}$$

Introducing (14.109) into (14.107), we obtain a transcendental characteristic equation corresponding to the independent infinite system of subsystem (14.32) for a fixed r. After some transformations, we have

$$\frac{\lambda^* \sin 2\lambda^*}{\cos 2\lambda^* - \cos \frac{2\pi r}{n}} = x. \tag{14.110}$$

Like the preceding equation, characteristic Eq. (14.110) has an infinite number of pairs of roots of the form $\lambda^* = \pm\lambda^*_s$. Therefore, in accordance with (14.104), (14.105), the general solution has the form

$$w_{1(q_k n+r)}(t) = \sum_{s=1}^{\infty} \frac{\bar{\lambda}^2_s}{\left(q_k + \frac{r}{n}\right)^2 - \bar{\lambda}^2_s} [C^{s}_{1(r)} e^{\lambda_s t} + C^{-s}_{1(r)} e^{-\lambda_s t}]. \tag{14.111}$$

$$w_{1(q_1 n+n-r)}(t) = -\sum_{s=1}^{\infty} \frac{r_s}{\left(q_1+1-\frac{r}{n}\right)^2 - r_s} \left[C^{s}_{1(r)} e^{\lambda_s t} + C^{-s}_{1(r)} e^{-\lambda_s t}\right] \qquad (14.111)$$

$$(q_1 = 0, 1, 2, \ldots).$$

Expressions (14.111) represent the general solution of the independent infinite system of differential equations of subsystem (14.32) for a fixed r, corresponding to series in sin $(qn+r)\alpha$ and sin $(qn+n-r)\alpha$. Successively assuming in these expressions and in Eq. (14.110)

$$r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right),$$

where $E\left(\frac{n-1}{2}\right)$ is the integral part of the number $\frac{n-1}{2}$, we obtain a complete spectrum of solutions of subsystem (14.32).

Let us turn to subsystem (14.37). Setting the load terms equal to zero and considering (14.21), we obtain

for $r_1 = r$

$$\ddot{w}_{2(q_1 n+r)} - \left(\frac{q_1 n + r}{\sqrt{\gamma}\sin\beta}\right)^2 w_{2(q_1 n+r)} = -\frac{1}{\varkappa} \ddot{S}_{2r} \qquad (14.112)$$

$$(q_1 = 0, 1, 2, \ldots);$$

for $r_1 = r - n$

$$\ddot{w}_{2(q_1 n+n-r)} - \left(\frac{q_1 n + n - r}{\sqrt{\gamma}\sin\beta}\right)^2 w_{2(q_1 n+n-r)} = -\frac{1}{\varkappa} \ddot{S}_{2r}$$

$$(q_1 = 0, 1, 2, \ldots),$$

where

$$S_{2r} = \sum_{q_2=0}^{\infty} \left[w_{2(q_2 n+r)} + w_{2(q_2 n+n-r)}\right]. \qquad (14.113)$$

Representing the solution in the form

$$w_{2(q_1 n+r)}(t) = C_{2(q_1 n+r)} e^{\lambda t} \quad (q_1 = 0, 1, 2, \ldots),$$
$$w_{2(q_1 n+n-r)}(t) = C_{2(q_1 n+n-r)} e^{\lambda t} \quad (q_1 = 0, 1, 2, \ldots). \qquad (14.114)$$

we find from (14.112)

$$C_{2(q_l n+r)} = \frac{1}{\pi}\,\frac{\bar{\lambda}^2}{\left(q_l + \frac{r}{n}\right)^2 - \bar{\lambda}^2}\,C^*_{2(r)} \quad (q_l = 0, 1, 2 \ldots),$$

$$C_{2(q_l n+n-r)} = \frac{1}{\pi}\,\frac{\bar{\lambda}^2}{\left(q_l + 1 - \frac{r}{n}\right)^2 - \bar{\lambda}^2}\,C^*_{2(r)} \quad (q_l = 0, 1, 2, \ldots), \qquad (14.115)$$

where

$$C^*_{2(r)} = \sum_{q_k=0}^{\infty}\left[C_{2(q_k n+r)} + C_{2(q_k n+n-r)}\right]. \qquad (14.116)$$

Introducing (14.115) into the right-hand side of (14.116) and reducing by $C^*_{2(r)} \neq 0$, we obtain

$$\bar{\lambda}^2 \sum_{q_k=0}^{\infty}\left[\frac{1}{\left(q_k + \frac{r}{n}\right)^2 - \bar{\lambda}^2} + \frac{1}{\left(q_k + 1 - \frac{r}{n}\right)^2 - \bar{\lambda}^2}\right] = \pi. \qquad (14.117)$$

It is evident that expression (14.117) coincides completely with expression (14.107). Consequently, the characteristic equation corresponding to the independent infinite system of subsystem (14.37) for a fixed r coincides with Eq. (14.110). According to (14.114), (14.115), the general solution of (14.37) has the form

$$\varphi_{2(q_l n+r)}(t) = \sum_{s=1}^{\infty} \frac{\bar{\lambda}_s^2}{\left(q_l + \frac{r}{n}\right)^2 - \bar{\lambda}_s^2}\left[C^s_{2(r)} e^{\lambda_s t} + C^{-s}_{2(r)} e^{-\lambda_s t}\right],$$

$$\varphi_{2(q_l n+n-r)}(t) = \sum_{s=1}^{\infty} \frac{\bar{\lambda}_s^2}{\left(q_l + 1 - \frac{r}{n}\right)^2 - \bar{\lambda}_s^2}\left[C^s_{2(r)} e^{\lambda_s t} + C^{-s}_{2(r)} e^{-\lambda_s t}\right] \qquad (14.118)$$

$$q_l = 0, 1, 2, \ldots;\quad r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right).$$

Thus, expressions (14.38), (14.49), (14.51), (14.58), (14.59), (14.60), (14.75), (14.81), (14.82), (14.93), (14.99), (14.100), (14.111), (14.118) represent the general solution of a homogeneous system of differential resolvents of a stringer conical shell, which correspond to the expansion of warpings in an infinite trigonometric series. Such an expansion is not orthogonal for a stringer shell. Therefore, the system of differential resolvents consists of infinite coupled subsystems, and the corresponding characteristic equations are transcendental. As a result, the desired generalized displacements are represented in the form of infinite series in roots of these equations. The corresponding integration constants are subject to determination from the conditions at the edges of the shell. In view of the nonorthognolity of a trigonometric expansion, the problem reduces to an infinite system of algebraic equations in constants whose determination pertains to the fundamental characteristics of the problem under consideration.

14.3. Special Coordinate Functions

The displacements $u_{m_z}(l, \alpha)$ in the direction of the shell generatrices in accordance with the table on p. 453 and expressions (f), (j) of Part Three will be

$$u_{m_z}(l,\alpha)=\eta_z\cos\beta-(\eta_x\sin\beta+e^l\theta_y R_0\cos\beta)\cos\alpha+$$
$$+(-\eta_y\sin\beta+e^l\theta_x R_0\cos\beta)\sin\alpha+\sum_{k=1}^{\infty}[\omega_{1k}(l)\sin k\alpha+\omega_{2k}(l)\cos k\alpha]. \quad (14.119)$$

For the generalized displacements found in the preceding section, expression (14.119) may be represented in the form

$$u_{m_z}(l,\alpha)=\eta_z^0\cos\beta-(\eta_x^0\sin\beta+\theta_y^0 R_0\cos\beta)\cos\alpha-(\eta_y^0\sin\beta-\theta_x^0 R_0\cos\beta)\times$$
$$\times\sin\alpha+\frac{1}{2}\sum_{s=1}^{\infty}\left[C_{2(n)}^{s}(1-e^{\lambda_{sn}l})+C_{2(n)}^{-s}(1-e^{-\lambda_{sn}l})\right]+$$
$$+\cos\alpha\sum_{s=1}^{\infty}\left[C_{2(1)}^{s}(1-e^{\lambda_{s1}l})+C_{2(1)}^{-s}(1-e^{-\lambda_{s1}l})\right]+$$
$$+\sin\alpha\sum_{s=1}^{\infty}\left[C_{1(1)}^{s}(1-e^{\lambda_{s1}l})+C_{1(1)}^{-s}(1-e^{-\lambda_{s1}l})\right]+$$
$$+\sum_{q_k=1}^{\infty}\cos q_k n\alpha\sum_{s=1}^{\infty}\frac{\lambda_{sn}^2}{q_k^2-\lambda_{sn}^2}\left[C_{2(n)}^{s}e^{\lambda_{sn}l}+C_{2(n)}^{-s}e^{-\lambda_{sn}l}\right]+$$

$$+\sum_{q_k=0}^{\infty}\cos\left(q_k n+\frac{n}{2}\right)\alpha\ \sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{s\frac{n}{2}}}{\left(q_k+\frac{1}{2}\right)^2-\bar{\lambda}^2_{s\frac{n}{2}}}\left[C^{s}_{2\left(\frac{n}{2}\right)}e^{\lambda_{s\frac{n}{2}}t}+\right.$$

$$\left.+C^{-s}_{2\left(\frac{n}{2}\right)}e^{-\lambda_{s\frac{n}{2}}t}\right]+\sum_{q_k=1}^{\infty}\cos(q_k n+1)\alpha\ \sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{s1}}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}\times$$

$$\times\left[C^{s}_{2(1)}e^{\lambda_{s1}t}+C^{-s}_{2(1)}e^{-\lambda_{s1}t}\right]+\sum_{q_k=0}^{\infty}\cos(q_k n+n-1)\alpha\times$$

$$\times\sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{s1}}{\left(q_k+1-\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}\left[C^{s}_{2(1)}e^{\lambda_{s1}t}+C^{-s}_{2(1)}e^{-\lambda_{s1}t}\right]+$$

$$+\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\ \sum_{q_k=0}^{\infty}\cos(q_k n+r)\alpha\ \sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{sr}}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}\times$$

$$\times\left[C^{s}_{2(r)}e^{\lambda_{sr}t}+C^{-s}_{2(r)}e^{-\lambda_{sr}t}\right]+\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\ \sum_{q_k=0}^{\infty}\cos(q_k n+n-r)\alpha\times$$

$$\times\sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{sr}}{\left(q_k+1-\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}\left[C^{s}_{2(r)}e^{\lambda_{sr}t}+C^{-s}_{2(r)}e^{-\lambda_{sr}t}\right]+$$

$$+\sum_{q_k=1}^{\infty}\sin q_k n\alpha\left[C^{s}_{1(n)}e^{\frac{q_k n}{\sqrt{I\sin\theta}}t}+C^{-s}_{1(n)}e^{-\frac{q_k n}{\sqrt{I\sin\theta}}t}\right]+$$

$$+\sum_{q_k=0}^{\infty}\sin\left(q_k n+\frac{n}{2}\right)\alpha\left[C^{s}_{1\left(\frac{n}{2}\right)}e^{\frac{q_k n+\frac{n}{2}}{\sqrt{I\sin\theta}}t}+C^{-s}_{1\left(\frac{n}{2}\right)}e^{-\frac{q_k n+\frac{n}{2}}{\sqrt{I\sin\theta}}t}\right]+$$

$$+\sum_{q_k=1}^{\infty}\sin(q_k n+1)\alpha\ \sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{s1}}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}\left[C^{s}_{1(1)}e^{\lambda_{s1}t}+\right.$$

$$\left.+C^{-s}_{1(1)}e^{-\lambda_{s1}t}\right]-\sum_{q_k=0}^{\infty}\sin(q_k n+n-1)\alpha\times$$

$$\times\sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{s1}}{\left(q_k+1-\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}\left[C^{s}_{1(1)}e^{\lambda_{s1}t}+C^{-s}_{1(1)}e^{-\lambda_{s1}t}\right]+$$

$$+\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\ \sum_{q_k=0}^{\infty}\sin(q_k n+r)\alpha\ \sum_{s=1}^{\infty}\frac{\bar{\lambda}^2_{sr}}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}\times$$

$$\times [C^{\prime}_{1(r)} e^{\lambda_{sr} t} + C^{-s}_{1(r)} e^{-\lambda_{sr} t}] - \sum_{r=2}^{E\left(\frac{n-1}{2}\right)} \sum_{q_k=0}^{\infty} \sin (q_k n + n - r) \alpha \times$$

$$\times \sum_{s=1}^{\infty} \frac{\bar{\lambda}^2_{sr}}{\left(q_k + 1 - \frac{r}{n}\right)^2 - \bar{\lambda}^2_{sr}} [C^{s}_{1(r)} e^{\lambda_{sr} t} + C^{-s}_{1(r)} e^{-\lambda_{sr} t}]. \qquad (14.120)$$

Here C^s, C^{-s}, C^k, C^{-k} are arbitrary constants, and λ and $\bar{\lambda}$ are determined by the relations

$$\lambda = \frac{n}{\pi \sqrt{\bar{\gamma}} \sin \beta} \lambda^*; \quad \bar{\lambda} = \frac{1}{n} \lambda^*,$$

where the values of the parameter λ^* are the roots of the transcendental equations:

$\{\lambda^*_{s_n}\}$ are the roots of the equation

$$\lambda^* \operatorname{ctg} \lambda^* = -\varkappa;$$

$\{\lambda^*_{s\frac{n}{2}}\}$ are the roots of the equation

$$\lambda^* \operatorname{tg} \lambda^* = \varkappa;$$

$\{\lambda^*_{s1}\}$ are the roots of the equation

$$\frac{\lambda^* \sin 2\lambda^*}{\cos 2\lambda^* - \cos \frac{2\pi}{n}} + \frac{1}{\left(\frac{n\lambda^*}{\pi}\right)^2 - 1} = \varkappa;$$

$\{\lambda^*_{sr}\}$ are the roots of the equation

$$\frac{\lambda^* \sin 2\lambda^*}{\cos 2\lambda^* - \cos \frac{2\pi r}{n}} = \varkappa$$

$$\left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right].$$

Changing the order of summation and reducing similar terms in (14.120), we obtain

$$u_{m_s}(t, \alpha) = \eta^0_z \cos \beta + \frac{1}{2} \sum_{s=1}^{\infty} (C^s_{2(n)} + C^{-s}_{2(n)}) - [\eta^0_x \sin \beta + \theta^0_z R_0 \cos \beta -$$

$$- \sum_{s=1}^{\infty} (C^s_{2(1)} + C^{-s}_{2(1)})] \cos \alpha + [-\eta^0_y \sin \beta + \theta^0_x R_0 \cos \beta + \sum_{s=1}^{\infty} (C^s_{1(1)} +$$

$$+ C^{-s}_{1(1)})] \sin \alpha + \sum_{s=1}^{\infty} U_{2sn}(t) \left(-\frac{1}{2} + \bar{\lambda}^2_{sn} \sum_{q_k=1}^{\infty} \frac{\cos q_k n \alpha}{q_k^2 - \bar{\lambda}^2_{sn}}\right) +$$

$$+\sum_{s=1}^{\infty} U_{2s\frac{n}{2}}(t)\,\bar{\lambda}^2_{s\frac{n}{2}} \sum_{q_k=0}^{\infty} \frac{\cos\left(q_k n+\frac{n}{2}\right)\alpha}{\left(q_k+\frac{1}{r}\right)^2-\bar{\lambda}^2_{s\frac{n}{2}}} -$$

$$-\sum_{s=1}^{\infty} U_{2s1}(t)\left[\cos\alpha-\bar{\lambda}^2_{s1}\sum_{q_k=1}^{\infty}\left(\frac{\cos(q_k n+1)\alpha}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}+\frac{\cos(q_k n-1)\alpha}{\left(q_k-\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}\right)\right]+$$

$$+\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\sum_{s=1}^{\infty} U_{2sr}(t)\bar{\lambda}^2_{sr}\left[\frac{\cos r\alpha}{\left(\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}+\sum_{q_k=1}^{\infty}\left(\frac{\cos(q_k n+r)\alpha}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}+\right.\right.$$

$$\left.\left.+\frac{\cos(q_k n-r)\alpha}{\left(q_k-\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}\right)\right]+$$

$$+\sum_{q_k=1}^{\infty}\sin q_k n\alpha\left[C^{k}_{1(n)}e^{\frac{q_k n}{\sqrt{\gamma}\sin\beta}t}+C^{-k}_{1(n)}e^{-\frac{q_k n}{\sqrt{\gamma}\sin\beta}t}\right]+$$

$$+\sum_{q_k=0}^{\infty}\sin\left(q_k n+\frac{n}{2}\right)\alpha\left[C^{k}_{1\left(\frac{n}{2}\right)}e^{\frac{q_k n+\frac{n}{2}}{\sqrt{\gamma}\sin\beta}t}+C^{-k}_{1\left(\frac{n}{2}\right)}e^{-\frac{q_k n+\frac{n}{2}}{\sqrt{\gamma}\sin\beta}t}\right]-$$

$$-\sum_{s=1}^{\infty}U_{1s1}(t)\left[\sin\alpha-\bar{\lambda}^2_{s1}\sum_{q_k=1}^{\infty}\left(\frac{\sin(q_k n+1)\alpha}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}-\frac{\sin(q_k n-1)\alpha}{\left(q_k-\frac{1}{n}\right)^2-\bar{\lambda}^2_{s1}}\right)\right]-$$

$$+\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\sum_{s=1}^{\infty}U_{1sr}(t)\bar{\lambda}^2_{sr}\left[\frac{\sin r\alpha}{\left(\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}+\right. \qquad (14.121)$$

$$\left.+\sum_{q_k=1}^{\infty}\left(\frac{\sin(q_k n+r)\alpha}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}-\frac{\sin(q_k n-r)\alpha}{\left(q_k-\frac{r}{n}\right)^2-\bar{\lambda}^2_{sr}}\right)\right],$$

where the functions $U_{1sr}(t)$, $U_{2sr}(t)$ are determined by the expressions

$$U_{1sr}=C^{s}_{1(r)}e^{\frac{n\dot{\lambda}_{sr}}{\pi\sqrt{\gamma}\sin\beta}t}+C^{-s}_{1(r)}e^{\frac{-r\dot{\lambda}_{sr}}{\pi\sqrt{\gamma}\sin\beta}t}$$
$$\left[r=1,2,\ldots,E\left(\frac{n-1}{2}\right)\right].$$
$$U_{2sr}=C^{s}_{2(r)}e^{\frac{n\dot{\lambda}_{sr}}{\pi\sqrt{\gamma}\sin\beta}t}+C^{-s}_{2(r)}e^{-\frac{n\dot{\lambda}_{sr}}{\pi\sqrt{\gamma}\sin\beta}t} \qquad (14.122)$$
$$\left[r=1,2,\ldots,E\left(\frac{n-1}{2}\right);\ \frac{n}{2},\ n\right].$$

In expression (14.121), all the series of the form $\sum_{q_k}$ can be summed up with the aid of the known expansions*

$$\sum_{k=1}^{\infty} \frac{k \sin kn\alpha}{k^2-\alpha^2} = \pi \frac{\sin \{\alpha [(2\zeta+1)\pi - n\alpha]\}}{2 \sin \alpha\pi}, \tag{14.123}$$

$$\sum_{k=1}^{\infty} \frac{\cos kn\alpha}{k^2-\alpha^2} = \frac{1}{2\alpha^2} - \pi \frac{\cos \{\alpha [(2\zeta+1)\pi - n\alpha]\}}{2\alpha \sin \alpha\pi}, \tag{14.124}$$

where

$$\frac{2\pi}{n}\zeta \leqslant \alpha \leqslant \frac{2\pi}{n}(\zeta+1),\ \zeta \text{ is an integer.} \tag{14.125}$$

Applying (14.124) directly, we have

$$\Phi_{s n}(\alpha) = -\frac{1}{2} + \bar{\lambda}_{sn}^2 \sum_{q_k=1}^{\infty} \frac{\cos q_k n\alpha}{q_k^2 - \bar{\lambda}_{sn}^2} = -\frac{\pi\bar{\lambda}_{sn}}{2 \sin \pi\bar{\lambda}_{sn}} \cos (\bar{\lambda}_{sn} [(2\zeta+1)\pi - n\alpha]). \tag{14.126}$$

After certain transformations, we also obtain

$$\Phi_{s\frac{n}{2}}(\alpha) = \bar{\lambda}^2_{s\frac{n}{2}} \sum_{q_k=0}^{\infty} \frac{\cos\left(q_k n + \frac{n}{2}\right)\alpha}{\left(q_k + \frac{1}{2}\right)^2 - \bar{\lambda}^2_{s\frac{n}{2}}} = 4\bar{\lambda}^2_{s\frac{n}{2}} \sum_{q_k=0}^{\infty} \frac{\cos (2q_k+1) n \frac{\alpha}{2}}{(2q_k+1)^2 - \left(2\bar{\lambda}_{s\frac{n}{2}}\right)^2} =$$

$$= 4\bar{\lambda}^2_{s\frac{n}{2}} \left[\sum_{k=1}^{\infty} \frac{\cos kn \frac{\alpha}{2}}{k^2 - \left(2\bar{\lambda}_{s\frac{n}{2}}\right)^2} - \sum_{k=2,4,6,\ldots}^{\infty} \frac{\cos kn \frac{\alpha}{2}}{k^2 - \left(2\bar{\lambda}_{s\frac{n}{2}}\right)^2}\right] =$$

$$= 4\bar{\lambda}^2_{s\frac{n}{2}} \left[\sum_{k=1}^{\infty} \frac{\cos kn \frac{\alpha}{2}}{k^2 - \left(2\bar{\lambda}_{s\frac{n}{2}}\right)^2} - \sum_{k=1}^{\infty} \frac{\cos kn\alpha}{(2k)^2 - \left(2\bar{\lambda}_{s\frac{n}{2}}\right)^2}\right] =$$

$$= \bar{\lambda}^2_{s\frac{n}{2}} \left[4 \sum_{k=1}^{\infty} \frac{\cos kn \frac{\alpha}{2}}{k^2 - \left(2\bar{\lambda}_{s\frac{n}{2}}\right)^2} - \sum_{k=1}^{\infty} \frac{\cos kn\alpha}{k^2 - \bar{\lambda}^2_{s\frac{n}{2}}}\right]. \tag{14.127}$$

Now, applying formula (14.124) to (14.127), we have

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*See I. S. Gradshteyn and I. M. Ryzhik. Tables of Integrals, Sums, Series, and Products. Moscow, Fiz. Matgiz., 1963.

$$\Phi_{s\varepsilon\frac{n}{2}}(\alpha)=\pi\bar{\lambda}_{s\frac{n}{2}}\left[-\frac{\cos\left\{\bar{\lambda}_{s\frac{n}{2}}[2\pi(2\zeta_1+1)-n\alpha]\right\}}{\sin 2\pi\bar{\lambda}_{s\frac{n}{2}}}+\right.$$

$$\left.+\frac{\cos\left\{\bar{\lambda}_{s\frac{n}{2}}[\pi(2\zeta+1)-n\alpha]\right\}}{2\sin\pi\bar{\lambda}_{s\frac{n}{2}}}\right], \tag{14.128}$$

where according to (14.125), ζ_1 is determined by the relation

$$\frac{2\pi}{n}\zeta_1\leqslant\frac{\alpha}{2}\leqslant\frac{2\pi}{n}(\zeta_1+1). \tag{14.129}$$

We consider the infinite series

$$S_3(\alpha)=\sum_{q_k=1}^{\infty}\left[\frac{\cos(q_k n+r)\alpha}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}_{sr}^2}+\frac{\cos(q_k n-r)\alpha}{\left(q_k-\frac{r}{n}\right)^2-\bar{\lambda}_{sr}^2}\right] \tag{14.130}$$

Using simple transformations, we reduce (14.130) to a form permitting the use of summation formulas (14.123), (14.124). We have

$$S_3(\alpha)=\frac{1}{2\bar{\lambda}_{sr}}\sum_{q_k=1}^{\infty}\left[\frac{\cos\left(q_k+\frac{r}{n}\right)n\alpha}{q_k+\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)}-\frac{\cos\left(q_k+\frac{r}{n}\right)n\alpha}{q_k+\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)}+\right.$$

$$\left.+\frac{\cos\left(q_k-\frac{r}{n}\right)n\alpha}{q_k-\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)}-\frac{\cos\left(q_k-\frac{r}{n}\right)n\alpha}{q_k-\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)}\right]=$$

$$=\frac{1}{2\bar{\lambda}_{sr}}\left\{\sum_{q_k=1}^{\infty}\left[\frac{\cos\left(q_k+\frac{r}{n}\right)n\alpha}{q_k+\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)}-\frac{\cos\left(q_k-\frac{r}{n}\right)n\alpha}{q_k-\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)}\right]-\right.$$

$$\left.-\sum_{q_k=1}^{\infty}\left[\frac{\cos\left(q_k+\frac{r}{n}\right)n\alpha}{q_k+\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)}-\frac{\cos\left(q_k-\frac{r}{n}\right)n\alpha}{q_k-\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)}\right]\right\}=$$

$$=\frac{1}{\bar{\lambda}_{sr}}\left\{-\sin r\alpha\sum_{k=1}^{\infty}\frac{k\sin kn\alpha}{k^2-\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)^2}-\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)\cos r\alpha\times\right.$$

$$\times \sum_{k=1}^{\infty} \frac{\cos k n\alpha}{k^2 - \left(\frac{r}{n} - \bar{\lambda}_{sr}\right)^2} + \sin r\alpha \sum_{k=1}^{\infty} \frac{k \sin k n\alpha}{k^2 - \left(\frac{r}{n} + \bar{\lambda}_{sr}\right)^2} +$$

$$+ \left(\frac{r}{n} + \bar{\lambda}_{sr}\right) \cos r\alpha \sum_{k=1}^{\infty} \frac{\cos k n\alpha}{k^2 - \left(\frac{r}{n} + \bar{\lambda}_{sr}\right)^2} \Bigg\}.$$

Now, using formulas (14.123), (14.124), we can easily obtain

$$S_1(\alpha) = \frac{\cos r\alpha}{\bar{\lambda}_{sr}^2 - \left(\frac{r}{n}\right)^2} + \frac{\pi}{2\bar{\lambda}_{sr}} \left[\frac{\cos\left\{\left(\frac{r}{n} - \bar{\lambda}_{sr}\right) [(2\zeta + 1)\pi - n\alpha] + r\alpha\right\}}{\sin\left(\frac{r}{n} - \bar{\lambda}_{sr}\right)\pi} - \right.$$

$$\left. - \frac{\cos\left\{\left(\frac{r}{n} + \bar{\lambda}_{sr}\right) [(2\zeta + 1)\pi - n\alpha] + r\alpha\right\}}{\sin\left(\frac{r}{n} + \bar{\lambda}_{sr}\right)\pi} \right]. \tag{14.131}$$

Setting r = 1 in (14.131), we find

$$\Phi_{s r 1}(\alpha) = -\cos\alpha + \bar{\lambda}_{s1}^2 \sum_{q_k=1}^{\infty} \left[\frac{\cos(q_k n + 1)\alpha}{\left(q_k + \frac{1}{n}\right)^2 - \bar{\lambda}_{s1}^2} + \frac{\cos(q_k n - 1)\alpha}{\left(q_k - \frac{1}{n}\right)^2 - \bar{\lambda}_{s1}^2} \right] =$$

$$= -\cos\alpha + \bar{\lambda}_{s1}^2 [S_1(\alpha)]_{r=1} = \frac{\cos\alpha}{(n\bar{\lambda}_{s1})^2 - 1} +$$

$$+ \frac{n\bar{\lambda}_{s1}}{2} \left[\frac{\cos\left\{\left(\frac{1}{n} - \bar{\lambda}_{s1}\right) [(2\zeta + 1)\pi - n\alpha] + \alpha\right\}}{\sin\left(\frac{1}{n} - \bar{\lambda}_{s1}\right)\pi} - \right.$$

$$\left. - \frac{\cos\left\{\left(\frac{1}{n} + \bar{\lambda}_{s1}\right) [(2\zeta + 1)\pi - n\alpha] + \alpha\right\}}{\sin\left(\frac{1}{n} + \bar{\lambda}_{s1}\right)\pi} \right]; \tag{14.132}$$

Using (14.131), we also have

$$\Phi_{s r}(\alpha) = \bar{\lambda}_{sr}^2 \left\{ \frac{\cos r\alpha}{\left(\frac{r}{n}\right)^2 - \bar{\lambda}_{sr}^2} + \sum_{q_k=1}^{\infty} \left[\frac{\cos(q_k n + r)\alpha}{\left(q_k + \frac{r}{n}\right)^2 - \bar{\lambda}_{sr}^2} + \right.\right.$$

$$\left.\left. + \frac{\cos(q_k n - r)\alpha}{\left(q_k - \frac{r}{n}\right)^2 - \bar{\lambda}_{sr}^2} \right] \right\} = \frac{n\bar{\lambda}_{sr}}{2} \left[\frac{\cos\left\{\left(\frac{r}{n} - \bar{\lambda}_{sr}\right) [(2\zeta + 1)\pi - n\alpha] + r\alpha\right\}}{\sin\left(\frac{r}{n} - \bar{\lambda}_{sr}\right)\pi} - \right.$$

$$\left. - \frac{\cos\left\{\left(\frac{r}{n} + \bar{\lambda}_{sr}\right) [(2\zeta + 1)\pi - n\alpha] + r\alpha\right\}}{\sin\left(\frac{r}{n} + \bar{\lambda}_{sr}\right)\pi} \right] \tag{14.133}$$

$$\left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right) \right].$$

We now consider the infinite series

$$S_1(\alpha)=\sum_{q_k=1}^{\infty}\left[\frac{\sin(q_kn+r)\alpha}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}_{sr}^2}-\frac{\sin(q_kn-r)\alpha}{\left(q_k-\frac{r}{n}\right)^2-\bar{\lambda}_{sr}^2}\right]. \tag{14.134}$$

As was done in summing up infinite series (14.130), we find

$$\begin{aligned}S_1(\alpha)={}&\frac{\sin r\alpha}{\bar{\lambda}_{sr}^2-\left(\frac{r}{n}\right)^2}+\frac{\pi}{2\bar{\lambda}_{sr}}\left[\frac{\sin\left\{\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)[(2\zeta+1)\pi-n\alpha]+r\alpha\right\}}{\sin\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)\pi}-\right.\\&\left.-\frac{\sin\left\{\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)[(2\zeta+1)\pi-n\alpha]+r\alpha\right\}}{\sin\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)\pi}\right];\end{aligned} \tag{14.135}$$

Assuming that r = 1 in (14.135), we find

$$\begin{aligned}\Phi_{1n}(\alpha)={}&-\sin\alpha+\bar{\lambda}_{s1}^2\sum_{q_k=1}^{\infty}\left[\frac{\sin(q_kn+1)\alpha}{\left(q_k+\frac{1}{n}\right)^2-\bar{\lambda}_{s1}^2}-\frac{\sin(q_kn-1)\alpha}{\left(q_k-\frac{1}{n}\right)^2-\bar{\lambda}_{s1}^2}\right]=\\&=-\sin\alpha+\bar{\lambda}_{s1}^2[S_1(\alpha)]_{r=1}=\frac{\sin\alpha}{(n\bar{\lambda}_{s1})^2-1}+\\&+\frac{\pi\bar{\lambda}_{s1}}{2}\left[\frac{\sin\left\{\left(\frac{1}{n}-\bar{\lambda}_{s1}\right)[(2\zeta+1)\pi-n\alpha]+\alpha\right\}}{\sin\left(\frac{1}{n}-\bar{\lambda}_{s1}\right)\pi}-\right.\\&\left.-\frac{\sin\left\{\left(\frac{1}{n}+\bar{\lambda}_{s1}\right)[(2\zeta+1)\pi-n\alpha]+\alpha\right\}}{\sin\left(\frac{1}{n}+\bar{\lambda}_{s1}\right)\pi}\right].\end{aligned} \tag{14.136}$$

Using (14.135), we also have

$$\begin{aligned}\Phi_{1sr}(\alpha)={}&\bar{\lambda}_{sr}^2\left\{\frac{\sin r\alpha}{\left(\frac{r}{n}\right)^2-\bar{\lambda}_{sr}^2}+\sum_{q_k=1}^{\infty}\left[\frac{\sin(q_kn+r)\alpha}{\left(q_k+\frac{r}{n}\right)^2-\bar{\lambda}_{s}^2}-\right.\right.\\&\left.\left.-\frac{\sin(q_kn-r)\alpha}{\left(q_k-\frac{r}{n}\right)^2-\bar{\lambda}_{sr}^2}\right]\right\}=\frac{\pi\bar{\lambda}_{sr}}{2}\left[\frac{\sin\left\{\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)[(2\zeta+1)\pi-n\alpha]+r\alpha\right\}}{\sin\left(\frac{r}{n}-\bar{\lambda}_{sr}\right)\pi}-\right.\\&\left.-\frac{\sin\left\{\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)[(2\zeta+1)\pi-n\alpha]+r\alpha\right\}}{\sin\left(\frac{r}{n}+\bar{\lambda}_{sr}\right)\pi}\right]\\&\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right].\end{aligned} \tag{14.137}$$

Transposing the reference point for angle α to the point $\alpha = \alpha_p = \frac{2\pi}{n} p$, where $p = E\left(\frac{\alpha}{2\pi/n}\right)$ is integral part of the number $\frac{\alpha}{2\pi/n}$, we obtain

$$\alpha = \frac{2\pi}{n}(p+\xi), \tag{14.138}$$

where

$$0 < \xi < 1.$$

Carrying out a substitution of variables in accordance with (14.138) in (14.126), and considering (14.48) and (14.125), we finally obtain

$$\Phi_{s1n}(\alpha) = -\frac{\lambda^{*}_{sn}}{2\sin\lambda^{*}_{sn}} \cos 2\lambda^{*}_{sn}\left(\xi - \frac{1}{2}\right). \tag{14.139}$$

Let us turn to expression (14.128). Considering (14.125), (14.130) and (14.138), we have

$$\alpha = \frac{4\pi}{n}\zeta_1 + [1-(-1)^p]\frac{\pi}{n} + \frac{2\pi}{n}\xi. \tag{14.140}$$

Comparing (14.138) and (14.140), we find

$$\zeta_1 = \frac{1}{2}p - \frac{1}{4}[1-(-1)^p]. \tag{14.141}$$

Introducing (14.141) into (14.128) and considering that $\xi = p$, we finally obtain

$$\Phi_{2s\frac{n}{2}}(\alpha) = \frac{\lambda^{*}_{s\frac{n}{2}}}{2\cos\lambda^{*}_{s\frac{n}{2}}}(-1)^p \sin 2\lambda^{*}_{s\frac{n}{2}}\left(\frac{1}{2} - \xi\right) \tag{14.142}$$

Let us turn to expressions (14.132), (14.133). Performing the substitution of variables in accordance with (14.138), after some transformations we finally obtain

$$\Phi_{s\,s1}(\alpha) = \frac{\cos\alpha}{(n\bar{\lambda}_{s1})^2 - 1} + \frac{\lambda^{*}_{s1}}{\cos 2\lambda^{*}_{s1} - \cos\frac{2\pi}{n}} \times$$

$$\times\left[\cos\frac{2\pi}{n}p \sin 2\lambda^{*}_{s1}(1-\xi) + \cos\frac{2\pi}{n}(p+1)\sin 2\lambda^{*}_{s1}\xi\right]. \tag{14.143}$$

$$\Phi_{2sr}(\alpha)=\frac{\lambda^*_{sr}}{\cos 2\lambda^*_{sr}-\cos\frac{2\pi r}{n}}\times$$

$$\times\left[\cos\frac{2\pi r}{n}p\sin 2\lambda^*_{sr}(1-\xi)+\cos\frac{2\pi r}{n}(p+1)\sin 2\lambda^*_{sr}\xi\right]$$

$$\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right]. \qquad (14.144)$$

After the substitution of variables in expressions (14.136), (14.137), we also obtain

$$\Phi_{1s1}(\alpha)=\frac{\sin\alpha}{(n\lambda_{s1})^2-1}+\frac{\lambda^*_{s1}}{\cos 2\lambda^*_{s1}-\cos\frac{2\pi}{n}}\times$$

$$\times\left[\sin\frac{2\pi}{n}p\sin 2\lambda^*_{s1}(1-\xi)+\sin\frac{2\pi}{n}(p+1)\sin 2\lambda^*_{s1}\xi\right]. \qquad (14.145)$$

$$\Phi_{1sr}(\alpha)=\frac{\lambda^*_{sr}}{\cos 2\lambda^*_{sr}-\cos\frac{2\pi r}{n}}\times$$

$$\times\left[\sin\frac{2\pi r}{n}p\sin 2\lambda^*_{sr}(1-\xi)+\sin\frac{2\pi r}{n}(p+1)\sin 2\lambda^*_{sr}\xi\right] \qquad (14.146)$$

$$\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right].$$

We introduce the notation

$$\eta^0_z\cos\beta+\frac{1}{2}\sum_{s=1}^{\infty}\left[C^s_{2(n)}+C^{-s}_{2(n)}\right]=C_0,$$

$$-\eta^0_y\sin\beta+\theta^0_z R_0\cos\beta+\sum_{s=1}^{\infty}\left[C^s_{1(1)}+C^{-s}_{1(1)}\right]=C_1,$$

$$-\eta^0_z\sin\beta-\theta^0_y R_0\cos\beta+\sum_{s=1}^{\infty}\left[C^s_{2(1)}+C^{-s}_{2(1)}\right]=C_2, \qquad (14.147)$$

$$\sin sn\alpha=\Phi_{1s0}(\alpha),$$

$$\sin\left(sn+\frac{n}{2}\right)\alpha=\Phi_{1s\frac{n}{2}}(\alpha),$$

$$U_{2sn}=U_{2s0},\quad \Phi_{2sn}=\Phi_{2s0}.$$

Now, general expression (14.121) for $u_{m_2}(t,\alpha)$ may be represented in the form

$$u_{m_2}(t,\alpha)=C_0+C_1\sin\alpha+C_2\cos\alpha+\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left[U_{1sr}(t)\Phi_{1sr}(\alpha)+U_{2sr}(t)\Phi_{2sr}(\alpha)\right]. \qquad (14.148)$$

It follows from the above that the functions $\Phi_{1sr}(\alpha)$, $\Phi_{2sr}(\alpha)$ form a complete system of special coordinate functions characteristic of a circular conical shell of constant thickness with a regular set of stringers. It will be shown below that this system is orthogonal on the circle if the corresponding quadratures are understood in the sense of Stieltjes integrals, extended to both the shell proper and the stringers. Hence it may be concluded that the special coordinate functions are normal. In solving the problem in these functions, the system of differential resolvents of a stringer shell decomposes completely as in the case of the solution of a smooth shell in trigonometric functions.

The system of coordinate functions obtained is applicable to the analysis of both conical and cylindrical shells with a longitudinal structure. In the latter case, it is necessary to perform the corresponding passage to the limit, as was shown in Ch. XIII.

Chapter XV ANALYSIS OF A SHELL WITH A LONGITUDINAL STRUCTURE IN SPECIAL COORDINATE FUNCTIONS

As was noted in Part One, the resolvents for the adopted computational model of a shell can be obtained both on the basis of variational treatment and by differential formulation of the corresponding boundary value problem. These approaches are fundamentally equivalent, but when using the variational method, it is first necessary to specify a system of coordinate functions, whereas in the differential approach, the solution can be sought in the form of a series in the eigenfunctions of the corresponding homogeneous problem. Therefore, while in the former case the result depends substantially on the choice of the coordinate functions, in the latter case these functions are automatically the "best."

In the case under consideration, the differential formulation of the boundary value problem is complicated by the presence of the reinforcing structure, which generates discontinuities of the first kind in the tangential forces $S_{n_z m_z}$. In the framework of classical analysis, this requires the extremely cumbersome procedure of "sewing" of the solution in portions. However, this can be avoided by using the device of generalized functions. This device is most convenient when studying systems whose model is based on the synthesis of discrete and continuous elements. Such systems include, first of all, bars, plates and shells with discrete elements concentrated at points and on lines and individual surfaces - lumped masses and moments of inertia, reinforcing structure, supporting lawyers of zero thickness, etc.

All the necessary data pertaining to the device of generalized functions are presented in Appendices I and II. Appendix III discusses examples illustrating the proposed general method of integrating differential equations with singularities of the type of the delta function and its derivatives. For convenience, further treatment includes references to the Appendix. It is suggested that the reader who is not familiar with the device of generalized functions first become familiar with the contents of Appendix I.

In the present chapter, the boundary value problem under consideration is reduced to equations in partial derivatives with variable delta-function type coefficients. Separating the variables by the Fourier method, we arrive at a homogeneous equation with variable delta-function type coefficients whose eigenfunctions coincide to within the multiplier with the normal coordinate functions of the stringer shell, obtained in the preceding chapter. Analysis of this homogeneous equation makes it possible to establish the orthogonality properties as well as other important properties of these functions. Constructing the solution in the form of expansions in these functions, we will consider certain specific problems under different boundary conditions.

15.1. Boundary Value Problem. Integro-Differential Resolvent

The hypothesis of nondeformability of the cross section of a shell in its plane is primarily due to the presence of transverse constraints such as rings and diaphragms. Therefore, eliminating from consideration the forces in the transverse constraints, we will use the differential equilibrium equation of an element of the middle surface of the shell in the direction of the axis of revolution

$$\frac{\partial}{\partial \bar{Z}}[(1-\bar{Z})T_{m_z}]+\frac{1}{\sin\beta}\frac{\partial}{\partial\alpha}S_{n_z m_z}-\frac{1}{\cos\beta}\frac{\partial}{\partial\alpha}Q_{n_z}+$$
$$+\frac{R_0(1-\bar{Z})}{\sin\beta\cos\beta}p_z(\bar{Z},\alpha)=0, \quad (p_z=p_{m_z}\cos\beta-p_{n_n}\sin\beta). \tag{15.1}$$

Neglecting in this equation the term containing Q_{n_z}, in comparison with the others, we will hereinafter proceed from the equation

$$\frac{\partial}{\partial \bar{Z}}[(1-\bar{Z})T_{m_z}]+\frac{1}{\sin\beta}\frac{\partial}{\partial\alpha}S_{n_z m_z}=-\frac{R_0(1-\bar{Z})}{\sin\beta\cos\beta}p_z. \tag{15.2}$$

Equation (15.2) will obviously be exact in the case of not only a cylindrical but also a conical shell if the shell proper has zero moment.

Making the substitution of variables

$$1-\bar{Z}=e^t, \tag{15.3}$$

we obtain

$$-\frac{\partial}{\partial t}(e^t T_{m_z})+\frac{e^t}{\sin\beta}\frac{\partial}{\partial\alpha}S_{n_z m_z}=-e^{2t}R_0\frac{p_z(t,\alpha)}{\sin\beta\cos\beta}. \tag{15.4}$$

The elasticity relations will be taken in the form

$$T_{m_z}=Eh_1\varepsilon_{m_z}, \tag{15.5}$$
$$S_{n_z m_z}=Gh_0\gamma_{n_z m_z}, \tag{15.6}$$

where the equivalent thickness h_1 pertains to elements subjected to normal stresses σ_{m_z}, and h_0, to elements subjected to tangential stresses $\tau_{n_z m_z}$.

Geometrical relations (2.2) for a circular conical shell assume the form

$$\varepsilon_{m_z}=\frac{\sin\beta}{R_0}\frac{\partial u_{m_z}}{\partial \bar{Z}}=-\frac{\sin\beta e^{-t}}{R_0}\frac{\partial u_{m_z}}{\partial t}, \tag{15.7}$$

$$\gamma_{\alpha_z m_z} = \frac{1}{R_0(1-\bar{Z})}\frac{\partial u_{m_z}}{\partial \alpha} + \frac{\sin\beta(1-\bar{Z})}{R_0}\frac{\partial}{\partial \bar{Z}}\left(\frac{u_{\alpha_z}}{1-\bar{Z}}\right) =$$

$$= \frac{1}{R_0}e^{-t}\frac{\partial u_{m_z}}{\partial \alpha} - \frac{\sin\beta}{R_0}\frac{\partial}{\partial t}(e^{-t}u_{\alpha_z}), \tag{15.8}$$

Expanding (15.5), (15.6) with the aid of (15.7), (15.8) and introducing the result into (15.4), we obtain

$$\gamma\sin^2\beta\frac{\partial}{\partial t}\left(h_1\frac{\partial u_{m_z}}{\partial t}\right) + \frac{\partial}{\partial \alpha}\left(h_0\frac{\partial u_{m_z}}{\partial \alpha}\right) - \sin\beta e^t\frac{\partial}{\partial \alpha}\left[h_0\frac{\partial}{\partial t}(e^{-t}u_{\alpha_z})\right] =$$
$$= -\frac{R_0^2}{G\cos\beta}e^{2t}p_z. \tag{15.9}$$

Equation (15.9) contains two unknown functions.

The longitudinal displacement $u_{m_z} = u_{m_z}(t, \alpha)$ is the desired function of two variables.

In view of the adopted computational model, the tangential displacement may be represented in the form

$$u_{\alpha_z}(t,\alpha) = -\eta_x(t)\sin\alpha + \eta_y(t)\cos\alpha + \theta_z(t)e^tR_0, \tag{15.10}$$

where η_x, η_y, θ_z are the desired functions of one variable.

The thickness h_0 will be assumed independent of the coordinate α. Then, introducing (15.10) into (15.9), we obtain

$$\gamma\sin^2\beta\frac{1}{h_0}\frac{\partial}{\partial t}\left(h_1\frac{\partial u_{m_z}}{\partial t}\right) + \frac{\partial^2 u_{m_z}}{\partial \alpha^2} + \sin\beta e^t\left[\cos\alpha\frac{d}{dt}(e^{-t}\eta_x) +\right.$$
$$\left. + \sin\alpha\frac{d}{dt}(e^{-t}\eta_y)\right] = -\frac{R_0^2}{Gh_0\cos\beta}e^{2t}p_z. \tag{15.11}$$

Equation (15.11) contains four unknown functions.

Adding to (15.11) the equations of the cut-off equilibrium portion of the shell, written in displacements, we obtain a complete system of equations in $u_{m_z}(t,\alpha)$, $\eta_x(t)$, $\eta_y(t)$, $\theta_z(t)$.

For a circular conical shell, we have from (2.48)

$$\int_0^{2\pi} T_{m_z}\sin\beta\cos\alpha(1-\bar{Z})R_0d\alpha+\int_0^{2\pi} S_{\alpha_z m_z}\sin\alpha(1-\bar{Z})R_0d\alpha=-Q_x(\bar{Z}), \tag{15.12}$$

$$-\int_0^{2\pi} T_{m_z}\sin\beta\sin\alpha(1-\bar{Z})R_0d\alpha+\int_0^{2\pi} S_{\alpha_z m_z}\cos\alpha(1-\bar{Z})R_0d\alpha=Q_y(\bar{Z}), \tag{15.13}$$

$$\int_0^{2\pi} T_{m_z}\cos\beta(1-\bar{Z})R_0d\alpha=N_z(\bar{Z}), \tag{15.14}$$

$$\int_0^{2\pi} T_{m_z}\sin\beta\sin\alpha(1-\bar{Z})^2 l_0R_0d\alpha=M_x(\bar{Z}), \tag{15.15}$$

$$-\int_0^{2\pi} T_{m_z}\sin\beta\cos\alpha(1-\bar{Z})^2 l_0R_0d\alpha=M_y(\bar{Z}), \tag{15.16}$$

$$\int_0^{2\pi} S_{\alpha_z m_z}(1-\bar{Z})^2R_0^2d\alpha=M_z(\bar{Z}). \tag{15.17}$$

Expanding (15.12)-(15.17) with the aid of elasticity relations (15.5), (15.6) and formulas (15.7), (15.8), we obtain

$$-\gamma\sin^2\beta\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial t}\cos\alpha\frac{h_1}{h_0}d\alpha+\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial\alpha}\sin\alpha\,d\alpha+ \qquad (15.18)$$

$$+\pi e^t\sin\beta\frac{d}{dt}(e^{-t}\eta_x)=-\frac{1}{Gh_0}Q_x(t), \qquad (15.18)$$

$$\gamma\sin^2\beta\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial t}\sin\alpha\frac{h_1}{h_0}d\alpha+\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial\alpha}\cos\alpha\,d\alpha-$$

$$-\pi e^t\sin\beta\frac{d}{dt}(e^{-t}\eta_y)=\frac{1}{Gh_0}Q_y(t), \tag{15.19}$$

$$\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial t}\frac{h_1}{h_0}d\alpha=-\frac{N_z(t)}{Eh_0\sin\beta\cos\beta}, \tag{15.20}$$

$$\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial t}\frac{h_1}{h_0}\sin\alpha\,d\alpha=-\frac{M_x(t)e^{-t}}{Eh_0R_0\sin\beta\cos\beta}, \tag{15.21}$$

$$\int_0^{2\pi}\frac{\partial u_{m_z}}{\partial t}\frac{h_1}{h_0}\cos\alpha\,d\alpha=\frac{M_y(t)e^{-t}}{Eh_0R_0\sin\beta\cos\beta}, \tag{15.22}$$

$$\frac{d\theta_z}{dt}=-\frac{M_z(t)e^{-2t}}{2\pi R_0^2Gh_0\sin\beta}, \tag{15.23}$$

It is easy to show that the system of seven Eqs. (15.11), (15.18)-(15.23) in the four unknowns u_{m_z}, η_x, η_y, θ_z is simultaneous. We will integrate Eq. (15.2) over the coordinate α from zero to 2π, taking into consideration the periodicity of the tangential forces. Using the table on p. 453 and expression (p) of Part Three, we can obtain

$$\frac{d}{d\bar{Z}}\left[(1-\bar{Z})\int_0^{2\pi} T_{m_z}\cos\beta R_0 d\alpha - N_z\right]=0. \tag{15.24}$$

Multiplying Eq. (15.2) once by sin α, and again by cos α, we integrate it again over α from 0 to 2π. Taking the table on p. 653 and expression (p) of Part Three into account, we obtain

$$\frac{d}{d\bar{Z}}\left[\int_0^{2\pi} T_{m_z}\sin\alpha\sin\beta(1-\bar{Z})^2 l_0 R_0 d\alpha\right]+l_0\left[\int_0^{2\pi} T_{m_z}\sin\alpha\sin\beta(1-\bar{Z})R_0 d\alpha - \right.$$
$$\left. -\int_0^{2\pi} S_{n_z m_z}\cos\alpha(1-\bar{Z})R_0 d\alpha\right]+R_{0_z}=0, \tag{15.25}$$

$$\frac{d}{d\bar{Z}}\left[\int_0^{2\pi} T_{m_z}\cos\alpha\sin\beta(1-\bar{Z})^2 l_0 R_0 d\alpha\right]+l_0\left[\int_0^{2\pi} T_{m_z}\cos\alpha\sin\beta(1-\bar{Z})R_0 d\alpha + \right.$$
$$\left. +\int_0^{2\pi} S_{n_z m_z}\sin\alpha(1-\bar{Z})R_0 d\alpha\right]-R_{0_y}=0, \tag{15.26}$$

whence, using Eqs. (15.12), (15.13), we have

$$\frac{d}{d\bar{Z}}\left[\int_0^{2\pi} T_{m_z}\sin\alpha\sin\beta(1-\bar{Z})^2 l_0 R_0 d\alpha\right]=l_0 Q_y - R_{0_z}, \tag{15.27}$$

$$\frac{d}{d\bar{Z}}\left[\int_0^{2\pi} T_{m_z}\cos\alpha\sin\beta(1-\bar{Z})^2 l_0 R_0 d\alpha\right]=l_0 Q_z + R_{0_y}. \tag{15.28}$$

Since on the basis of expressions (p) of Part Three

$$l_0 Q_y - R_{0_z} = \frac{dM_z}{d\bar{Z}}, \tag{15.29}$$

$$l_0 Q_z - R_{0_y} = -\frac{dM_y}{d\bar{Z}}, \tag{15.30}$$

it follows from Equs. (15.29), (15.30) that

$$\frac{d}{d\bar{Z}}\left[\int_0^{2\pi} T_{m_z}\sin\alpha\sin\beta(1-\bar{Z})^2 l_0 R_0 d\alpha - M_z\right]=0, \tag{15.31}$$

$$\frac{d}{d\bar{Z}}\left[\int_0^{2\pi} T_{m_z}\cos\alpha\sin\beta(1-\bar{Z})^2 l_0 R_0 d\alpha + M_y\right]=0. \tag{15.32}$$

Comparing Eqs. (15.24), (15.31), (15.32) resulting from the system of Eqs. (15.2), (15.12), (15.13), with Eqs. (15.14)-(14.16), respectively, we conclude that the latter also result from system (15.2), (15.12), (15.13) if the solutions of this system are subordinated to the static boundary conditions. It follows that the system of seven Eqs. (15.2), (15.12)-(15.17) is simultaneous, and hence, the system of seven Eqs. (15.11), (15.18)-(15.23) in the four unknown functions $u_{m_z}, \eta_x, \eta_y, \vartheta_z$ is also simultaneous.

From Eqs. (15.12), (15.13), (15.15), (15.16) it is easy to obtain

$$\int_0^{2\pi} S_{a_z m_z} \cos\alpha\,(1-\bar{Z})^2 l_0 R_0 d\alpha = M_x(\bar{Z}) + (1-\bar{Z})\,l_0 Q_y(\bar{Z}) = M_{x_0}(\bar{Z}), \tag{15.33}$$

$$\int_0^{2\pi} S_{a_z m_z} \sin\alpha\,(1-\bar{Z})^2 l_0 R_0 d\alpha = M_y(\bar{Z}) - (1-\bar{Z})\,l_0 Q_x(\bar{Z}) = M_{y_0}(\bar{Z}). \tag{15.34}$$

Here M_{x_0}, M_{y_0} are the moments of the external forces, applied to the cut-off portion of the shell, about axes O_1x_0, O_1y_0, passing through the cone apex and parallel to the Ox, Oy axes.

Introducing the expanded expression for the force $S_{a_z m_z}$, into (15.33), (15.34) we obtain

$$\frac{1}{\pi}\int_0^{2\pi} \frac{\partial u_{m_z}}{\partial\alpha} \cos\alpha\, d\alpha - \sin\beta e^t \frac{d}{dt}(e^{-t}\eta_y) = \frac{M_{x_0}(t)\,e^{-t}}{\pi R_0 h_0 G \operatorname{ctg}\beta}, \tag{15.35}$$

$$\frac{1}{\pi}\int_0^{2\pi} \frac{\partial u_{m_z}}{\partial\alpha} \sin\alpha\, d\alpha + \sin\beta e^t \frac{d}{dt}(e^{-t}\eta_x) = \frac{M_{y_0}(t)\,e^{-t}}{\pi R_0 h_0 G \operatorname{ctg}\beta}. \tag{15.36}$$

We eliminate the unknowns η_x and η_y from (15.11) with the aid of expressions (15.35), (15.36). Transforming by parts the integrals entering into these expressions and allowing for the periodicity of the displacement u_{m_z}, we obtain

$$\gamma \sin^2\beta \frac{1}{h_0}\frac{\partial}{\partial t}\left(h_1 \frac{\partial u_{m_z}}{\partial t}\right) + \frac{\partial^2 u_{m_z}}{\partial\alpha^2} + \frac{1}{\pi}\int_0^{2\pi} u_{m_z}(t,\xi)\cos(\alpha-\xi)\,d\xi = \\ = \frac{e^{-t}}{\pi R_0 h_0 G \operatorname{ctg}\beta}\left[M_{x_0}(t)\sin\alpha - M_{y_0}(t)\cos\alpha\right] - \frac{R_0^2}{G h_0 \cos\beta} e^{2t} p_z. \tag{15.37}$$

From (15.35), (15.36), we also have

$$\eta_x(t)=\eta_x^0 e^t+\frac{e^t}{\pi\sin\beta}\int_0^t e^{-\xi}\int_0^{2\pi} u_{m_z}(\xi,\zeta)\cos\zeta\, d\zeta\, d\xi+ $$
$$+\frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{y_*}(\xi)e^{-2\xi}d\xi,$$
$$\eta_y(t)=\eta_y^0 e^t+\frac{e^t}{\pi\sin\beta}\int_0^t e^{-\xi}\int_0^{2\pi} u_{m_z}(\xi,\zeta)\sin\zeta\, d\zeta\, d\xi- \qquad (15.38)$$
$$-\frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{x_*}(\xi)e^{-2\xi}d\xi.$$

Integro-differential Eq. (15.37) in the unknown displacements $\bar{u}_{m_z}(t,\alpha)$ in the direction of the shell generatrices is the main resolvent of the problem. In particular, this equation reduces to the system of ordinary differential equations (Eqs. (m), (n) of Part Three) obtained by the variational method, if the desired displacements u_{m_z} are represented in the form of a unary series in the coordinate α.

The components $\eta_x(t)$, $\eta_y(t)$ of transitional displacement of the cross section of a shell as a solid for the displacement u_{m_1} obtained are determined by expressions (15.38).

The angle of rotation $\theta_z(t)$ of the cross section is determined independently of the displacement u_{m_z} and components η_x, η_y from Eq. (15.23). Hence, in torsion, the cross section remains planar, owing to the regular arrangement of the stringers.

The solution of system (15.37), (15.18)-(15.23) must satisfy the boundary conditions of the problem - the conditions of periodicity of the displacements and forces over the coordinate α and the boundary conditions on the ends of the shell when $Z=0$ $(t=0)$ and $Z=Z_1$ $(t=t_1)$.

The boundary conditions may be kinematic, static, or mixed. Moreover, in view of the adopted computational model, the static conditions in the longitudinal direction can be satisfied in detail, i.e., identically, and in the transverse direction, only integrally.

Let us now consider a shell with constant thickness $h = h_0$, reinforced along the generatrices $\alpha = \alpha_m = \frac{2\pi}{n} m$ $(m = 1, 2, \ldots, n)$ by a set of stringers with the same cross-sectional area $\Delta F = \Delta F(\bar{Z})$.

Proceeding from the obvious representation

$$F_1(\bar{Z}, \varphi_1, \varphi_2) = \int_{\varphi_1}^{\varphi_2} h_1(\bar{Z}, \alpha) R_0 (1 - \bar{Z}) d\alpha, \tag{15.39}$$

where $F_1(\bar{Z}, \varphi_1, \varphi_2)$ is the area of elements subjected to normal stresses, on the portion of arc of the crows section $\bar{Z}$ = const, bounded by arbitrary angles $\alpha = \varphi_1$, $\alpha = \varphi_2$, we will interpret the shell with the longitudinal structure as a shell of variable thickness

$$h_1(\bar{Z}, \alpha) = h_0 + \frac{\Delta F(\bar{Z})}{R_0(1 - \bar{Z})} \sum_{m=1}^{n} \delta(\alpha - \alpha_m), \tag{15.40}$$

where $\delta(\alpha - \alpha_m)$ is Dirac's delta function, concentrated at the point $\alpha = \alpha_m = \frac{2\pi}{n} m$ (see Section I.1 of Appendix I).

Introducing (15.40) into (15.39) and integrating, one can readily ascertain that the result corresponds to the stringer shell under consideration for any φ_1 and φ_2. This confirms the adopted concept.

Making the substitution of variables $1 - \bar{Z} = e^t$ in (15.40) and introducing the result into (15.37), we obtain the fundamental integro-differential equation of a stringer conical shell

$$\gamma \sin^2\beta \frac{\partial}{\partial t}\left\{\left[1+e^{-t}\frac{\Delta F(t)}{R_0 h_0}\sum_{m=1}^{n}\delta(\alpha-\alpha_m)\right]\frac{\partial u_{m_z}}{\partial t}\right\}+\frac{\partial^2 u_{m_z}}{\partial\alpha^2}+$$

$$+\frac{1}{\pi}\int_0^{2\pi} u_{m_z}(t,\xi)\cos(\alpha-\xi)\,d\xi=$$

$$=\frac{e^{-t}}{\pi R_0 h_0 G \operatorname{ctg}\beta}[M_{x_0}(t)\sin\alpha-M_{y_0}(t)\cos\alpha]-\frac{R_0^2}{Gh_0\cos\beta}e^{2t}p_z. \tag{15.41}$$

15.2. Honogenous Problem. Eigenvalues and Eigenfunctions

We will assume that the cross-sectional area of a stringer changes linearly along the length:

$$\Delta F=\Delta F_0(1-\bar{Z})\quad \Delta F_0 e^{t}. \tag{15.42}$$

In this case, we apply the Fourier method fo Eq. (15.41). Representing the partial solution of the corresponding homogeneous solution in the form

$$u_{m_z}(t,\alpha)=U(t)\Phi(\alpha), \tag{15.43}$$

we have

$$\gamma \sin^2\beta\left[1+\frac{\Delta F_0}{R_0 h_0}\sum_{m=1}^{n}\delta(\alpha-\alpha_m)\right]U''\Phi+U\Phi''+\frac{1}{\pi}U\int_0^{2\pi}\Phi(\xi)\cos(\alpha-\xi)\,d\xi=0. \tag{15.44}$$

Introducing (15.43) into equilibrium Eqs. (15.20)-(15.22), we find for the case of the homogeneous problem:

$$U'\int_0^{2\pi}\Phi k_1 d\alpha=0, \tag{15.45}$$

$$U'\int_0^{2\pi}\Phi\sin\alpha k_1 d\alpha=0, \tag{15.46}$$

$$U'\int_0^{2\pi}\Phi\cos\alpha k_1 d\alpha=0. \tag{15.47}$$

Dividing (15.44) by the product $U\Phi$, we obtain

$$\gamma \sin^2\beta\left[1+\frac{\Delta F_0}{R_0 h_0}\sum_{m=1}^{n}\delta(\alpha-\alpha_m)\right]\frac{U''}{U}=$$

$$=-\frac{\Phi''}{\Phi}-\frac{1}{\pi}\frac{1}{\Phi}\int_0^{2\pi}\Phi(\xi)\cos(\alpha-\xi)d\xi. \qquad (15.48)$$

The right-hand side of (15.48) depends only on the coordinate α, and hence

$$\frac{U''(t)}{U(t)}=\text{const}.$$

Denoting

$$\gamma \sin^2\beta\,\frac{U''}{U}=\lambda,$$

we obtain ordinary differential equations in U and Φ:

$$U''-\frac{\lambda}{\gamma\sin^2\beta}U=0, \qquad (15.49)$$

$$\Phi''+\lambda\left[1+\frac{\Delta F_0}{R_0 h_0}\sum_{m=1}^{n}\delta(\alpha-\alpha_m)\right]\Phi+\frac{1}{\pi}\int_0^{2\pi}\Phi(\xi)\cos(\alpha-\xi)d\xi=0, \qquad (15.50)$$

whose solutions must satisfy conditions (15.45)-(15.47).

The parameter λ must be determined from the periodicity conditions of the solution. From (15.43), and also (15.5)-(15.8) it follows that if the displacements u_{m_z} and internal forces N_{m_z}, $S_{n_z m}$ are periodic in α, then

$$\Phi(0)=\Phi(2\pi),$$
$$\Phi'(0)=\Phi'(2\pi). \qquad (15.51)$$

The general solution of Eq. (15.49) will be

when $\lambda\neq 0$

$$U(t)=C_1 e^{\sqrt{\frac{\lambda}{\gamma}}\frac{t}{\sin\beta}}+C_2 e^{-\sqrt{\frac{\lambda}{\gamma}}\frac{t}{\sin\beta}}, \qquad (15.52)$$

when $\lambda=0$

$$U(t)=C_0+C_1 t. \qquad (15.53)$$

Let us turn to Eq. (15.50).

For an arbitrary value of the parameter α, this equation, like any homogeneous equation, has a trivial solution $\Phi=0$. Nontrivial solutions of Eq. (15.50), satisfying conditions (15.51), are possible only for certain discrete values of the parameter λ. These values of the parameter and the corresponding solutions of Eq. (15.50) are called eigenvalues and eigenfunctions of the equation. Analyzing Eq. (15.50), one can easily establish that the eigenvalues of the parameter λ are real, and the eigenfunctions corresponding to the different eigenvalues are orthogonal with weight $1+\frac{\Delta F_0}{R_0 h_0}\sum_{m=1}^{n}\delta(\alpha-\alpha_m)$.

Let $\lambda = \lambda_s$ be some eigenvalue, and $\Phi(\alpha)=\Phi_s(\alpha)$ the corresponding eigenfunction. Then

$$\Phi_s'' + \lambda_s \varrho \Phi_s + \frac{1}{\pi}\left[\cos\alpha\int_0^{2\pi}\Phi_s(\alpha)\cos\alpha\,d\alpha + \sin\alpha\int_0^{2\pi}\Phi_s(\alpha)\sin\alpha\,d\alpha\right]=0, \tag{15.54}$$

where the weight function

$$\varrho(\alpha)=\frac{h_1(\alpha)}{h_0}=1+\frac{\Delta F_0}{R_0 h_0}\sum_{m=1}^{n}\delta(\alpha-\alpha_m), \tag{15.55}$$

and in addition,

$$\begin{aligned}\Phi_s(0)&=\Phi_s(2\pi),\\ \Phi_s'(0)&=\Phi_s'(2\pi).\end{aligned} \tag{15.56}$$

Multiplying (15.54) by $\Phi_s(\alpha)$, we integrate the result over α from 0 to 2π:

$$\int_0^{2\pi}\Phi_s''\Phi_s\,d\alpha + \lambda_s\int_0^{2\pi}\Phi_s^2\varrho\,d\alpha + \frac{1}{\pi}\left\{\left[\int_0^{2\pi}\Phi_s(\alpha)\cos\alpha\,d\alpha\right]^2 + \left[\int_0^{2\pi}\Phi_s(\alpha)\sin\alpha\,d\alpha\right]^2\right\}=0. \tag{15.57}$$

Integrating the first term in (15.57) by parts while considering (15.56), we obtain

$$\int_0^{2\pi}\Phi_s''\Phi_s\,d\alpha = -\int_0^{2\pi}\Phi_s'^2\,d\alpha. \tag{15.58}$$

Introducing (15.58) into (15.57), we find

$$\lambda_s=\frac{\int\limits_0^{2\pi}\Phi_s'^2 d\alpha-\frac{1}{\pi}\left\{\left[\int\limits_0^{2\pi}\Phi_s(\alpha)\cos\alpha d\alpha\right]^2+\left[\int\limits_0^{2\pi}\Phi_s(\alpha)\sin\alpha d\alpha\right]^2\right\}}{\int\limits_0^{2\pi}\Phi_s^2\varrho d\alpha}. \tag{15.59}$$

whence it follows that all the eigenvalues are real.

We will establish the lower limit of the eigenvalues. Integrating by parts while considering (15.56) in the second term of the numerator of (15.59), we have

$$\lambda_s=\frac{\int\limits_0^{2\pi}\Phi_s'^2 d\alpha-\frac{1}{\pi}\left\{\left[\int\limits_0^{2\pi}\Phi_s'(\alpha)\sin\alpha d\alpha\right]^2+\left[\int\limits_0^{2\pi}\Phi_s(\alpha)\sin\alpha d\alpha\right]^2\right\}}{\int\limits_0^{2\pi}\Phi_s^2\varrho d\alpha}. \tag{15.60}$$

Applying Bunjakowski's inequality to (15.60)

$$\int\limits_a^b\varphi\psi dx<\sqrt{\int\limits_a^b\varphi^2 dx}\sqrt{\int\limits_a^b\psi^2 dx},$$

we find

$$\lambda_s>-\frac{\int\limits_0^{2\pi}\Phi_s^2 d\alpha}{\int\limits_0^{2\pi}\Phi_s^2\varrho d\alpha}=-\frac{1}{1+\frac{\Delta F_0}{R_0h_0}\frac{\sum\limits_{m=1}^{n}\Phi_s^2(\alpha_m)}{\int\limits_0^{2\pi}\Phi_s^2 d\alpha}}\geqslant-1. \tag{15.61}$$

Estimate (15.61) does not give an exact lower boundary. Below, analyzing the solution of Eq. (15.54), we will show that $\lambda_s\geqslant 0$.

Integrating the right- and left-hand sides of Eq. (15.54) over α from 0 to 2π and allowing for periodicity $\Phi_s(\alpha)$ (15.56), we find

$$\lambda_s\int\limits_0^{2\pi}\Phi_s\varrho d\alpha=0,$$

whence, for eigenvalues different from zero

$$\int\limits_0^{2\pi}\Phi_s\varrho d\alpha=0,\quad \lambda_s\neq 0. \tag{15.62}$$

Successively multiplying (15.54) by sin α and cos α and integrating each time over α from 0 to 2π, taking (15.56) into account, we obtain

$$\lambda_s \int_0^{2\pi} \Phi_s(\alpha) \sin \alpha \varrho(\alpha) d\alpha = 0,$$

$$\lambda_s \int_0^{2\pi} \Phi_s(\alpha) \cos \alpha \varrho(\alpha) d\alpha = 0,$$

whence for eigenvalues different from zero

$$\int_0^{2\pi} \Phi_s(\alpha) \sin \alpha \varrho(\alpha) d\alpha = 0, \quad \lambda_s \neq 0,$$

$$\int_0^{2\pi} \Phi_s(\alpha) \cos \alpha \varrho(\alpha) d\alpha = 0, \quad \lambda_s \neq 0. \tag{15.63}$$

On the basis of (15.62), (15.63), considering (15.55), we conclude that relations (15.45)-(15.47) and hence, equilibrium Eqs. (15.20)-(15.22) are fulfilled identically when $\lambda_s \neq 0$.

Let $\lambda = \lambda_{s1}$, $\lambda = \lambda_{s2}$ be some eigenvalues and $\Phi(\alpha) = \Phi_{s1}(\alpha)$, $\Phi(\alpha) = \Phi_{s2}(\alpha)$ be the eigenfunctions. Then,

$$\Phi''_{s1} + \lambda_{s1} \varrho \Phi_{s1} + \frac{1}{\pi} \int_0^{2\pi} \Phi_{s1}(\xi) \cos(\alpha - \xi) d\xi = 0, \tag{15.64}$$

$$\Phi''_{s2} + \lambda_{s2} \varrho \Phi_{s2} + \frac{1}{\pi} \int_0^{2\pi} \Phi_{s2}(\xi) \cos(\alpha - \xi) d\xi = 0, \tag{15.65}$$

and in addition

$$\Phi_{s1}(0) = \Phi_{s1}(2\pi), \quad \Phi'_{s1}(0) = \Phi'_{s1}(2\pi),$$
$$\Phi_{s2}(0) = \Phi_{s2}(2\pi), \quad \Phi'_{s2}(0) = \Phi'_{s2}(2\pi). \tag{15.66}$$

Multiplying (15.64) by Φ_{s2} and (15.65) by Φ_{s1}, we integrate the left- and right-hand sides of the equations over α from 0 to 2π, taking (15.66) into account. We obtain

$$-\int_0^{2\pi} \Phi'_{s1} \Phi'_{s2} d\alpha + \lambda_{s1} \int_0^{2\pi} \Phi_{s1} \Phi_{s2} \varrho d\alpha + \frac{1}{\pi} \int_0^{2\pi}\int_0^{2\pi} \Phi_{s1}(\xi) \Phi_{s2}(\zeta) \cos(\zeta - \xi) d\xi d\zeta = 0, \tag{15.67}$$

$$-\int_0^{2\pi} \Phi'_{s2} \Phi'_{s1} d\alpha + \lambda_{s2} \int_0^{2\pi} \Phi_{s2} \Phi_{s1} \varrho d\alpha + \frac{1}{\pi} \int_0^{2\pi}\int_0^{2\pi} \Phi_{s2}(\xi) \Phi_{s1}(\zeta) \cos(\zeta - \xi) d\xi d\zeta = 0. \tag{15.68}$$

Subtracting (15.68) from (15.67) termwise, we find

$$(\lambda_{s1}-\lambda_{s2})\int_0^{2\pi}\Phi_{s1}\Phi_{s2}\varrho d\alpha=0,$$

whence it follows that the eigenfunctions corresponding to the different eigenvalues are orthogonal on the circle with weight $\varrho(\alpha)$ (15.55):

$$\int_0^{2\pi}\Phi_{s1}\Phi_{s2}\varrho d\alpha=0, \quad \lambda_{s1}\neq\lambda_{s2}. \tag{15.69}$$

In addition, introducing (15.68) into (15.67), we have for $\lambda_{s1}\neq\lambda_{s2}$

$$-\int_0^{2\pi}\Phi'_{s1}\Phi'_{s2}d\alpha+\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\Phi_{s1}(\zeta)\Phi_{s2}(\xi)\cos(\zeta-\xi)d\zeta d\xi=0. \tag{15.70}$$

It is also easy to show that the eigenfunctions are orthogonal with weight $\varrho(\alpha)$ with respect to their derivatives. Indeed, multiplying Eq. (15.54) by $\Phi'_s(\alpha)$ and integrating the left- and right-hard sides over α from 0 to 2π, we have

$$\int_0^{2\pi}\Phi'_s d(\Phi'_s)+\lambda_s\int_0^{2\pi}\Phi_s\Phi'_s\varrho d\alpha+\frac{1}{\pi}\left[\int_0^{2\pi}\Phi'_s\cos\alpha d\alpha\int_0^{2\pi}\Phi_s\cos\alpha d\alpha+\right.$$
$$\left.+\int_0^{2\pi}\Phi'_s\sin\alpha d\alpha\int_0^{2\pi}\Phi_s\sin\alpha d\alpha\right]=0,$$

whence, integrating by parts and considering periodicity conditions (15.56), we obtain

$$\int_0^{2\pi}\Phi_s\Phi'_s\varrho d\alpha=0, \quad \lambda_s\neq 0. \tag{15.71}$$

In accordance with estimate (15.61), it is also necessary to study the question of presence of zero eigenvalues in integro-differential Eq. (15.50).

Setting $\lambda = 0$ in (15.50), we obtain

$$\Phi''+\frac{1}{\pi}(A\cos\alpha+B\sin\alpha)=0, \tag{15.72}$$

where

$$A=\int_0^{2\pi}\Phi(\alpha)\cos\alpha d\alpha;\qquad B=\int_0^{2\pi}\Phi(\alpha)\sin\alpha d\alpha. \tag{15.73}$$

From (15.72) we find

$$\Phi(\alpha)=D_0+D_1\alpha+\frac{1}{\pi}(A\cos\alpha+B\sin\alpha), \tag{15.74}$$

where D_0, D_1 are arbitrary constants.

Satisfying periodicity conditions (15.56), we find

$$D_1=0. \tag{15.75}$$

Introducing (15.74) into (15.73) while considering (15.75), we obtain the identities

$$A = A,\ B = B,$$

whence it follows that the quantities A and B can be arbitrary, i.e., the solution of Eq. (15.50) satisfying periodicity condition (15.51) when $\lambda = 0$ will be

$$\Phi(\alpha)=D_0+D_1\sin\alpha+D_2\cos\alpha, \tag{15.76}$$

where D_0, D_1, D_2 are arbitrary constants.

Thus, $\lambda = 0$ constitutes a triply degenerate eigenvalue of Eq. (15.50). To within the multiplier, the corresponding eigenfunctions

$$\Phi_{00}=1;\quad \Phi_{01}=\sin\alpha;\quad \Phi_{02}=\cos\alpha. \tag{15.77}$$

Let us turn to relations (15.45)-(15.47). Since when $\lambda = 0$ the quadratures

$$\int_0^{2\pi}\Phi_{00}h_1 d\alpha,\quad \int_0^{2\pi}\Phi_{01}\sin\alpha h_1 d\alpha,\quad \int_0^{2\pi}\Phi_{02}\cos\alpha h_1 d\alpha$$

are known in advance to be different from zero, it is necessary to satisfy the conditions

$$\dot{U}'_{(a)}(t) = U'_{01}(t) = U'_{02}(t) = 0,$$

by setting in general solution (15.53), corresponding to a zero eigenvalue

$$C_1 = 0. \tag{15.78}$$

It follows that in the case of a homogeneous problem, functions (15.77) correspond to displacement of the shell as a solid. The remaining eigenfunctions correspond to different forms of warping of the cross section of a stringer shell acted on by a self-balanced boundary load.

Turning to expression (14.148), which constitutes the expansion of the displacement $u_{m_z}(t, \alpha)$ in special coordinate functions $\Phi_{1sr}(\alpha), \Phi_{2sr}(\alpha)$, we see that the first three terms of this expansion correspond to eigenfunctions (15.77) and functions (15.53) under condition (15.78). It will be shown below that all the eigenfunctions of Eq. (15.50) and corresponding functions U(t) (15.52) completely coincide with coordinate functions Φ_{1sr}, Φ_{2sr} and generalized displacements U_{1sr}, U_{2sr} of expansion (14.148). When the eigenfunctions corresponding to the degenerate eigenvalues are orthogonalized, this will permit us to construct a system of normal coordinate functions orthogonal on the cross sectional contour of the circular stringer shell.

Let us first consider a simpler problem by confining ourselves to the case in which the external load is such that the displacements $u_{n_z}(t, \alpha) \equiv 0$. This takes place if the external load reduces to a resultant directed along the axis of the shell, and also when the shell is loaded with a cyclic self-balanced load.

Setting $u_{n_z} = 0$ in (15.9), in view of (15.55), we have for the case of the homogeneous problem

$$\gamma \sin^2 \beta \varrho \frac{\partial^2 u_{m_z}}{\partial t^2} + \frac{\partial^2 u_{m_z}}{\partial \alpha^2} = 0. \tag{15.79}$$

Representing the partial solution of (15.79) in the form

$$u_{m_2}(t, \alpha) = U(t)\Phi(\alpha),$$

we obtain

$$U'' - \frac{\lambda}{\gamma \sin^2 \beta} U = 0, \tag{15.80}$$

$$\Phi'' + \lambda \left[1 + \mu \sum_{m=1}^{n} \delta(\alpha - \alpha_m)\right] \Phi = 0, \tag{15.81}$$

$$\begin{gathered} \Phi(0) = \Phi(2\pi), \\ \Phi'(0) = \Phi'(2\pi); \end{gathered} \tag{15.82}$$

where

$$\mu = \frac{\Delta F_0}{k h_0}. \tag{15.83}$$

The problem of finding the eigenvalues and eigenfunctions of Eq. (15.81) which satisfy conditions (15.82) is analogous to the classical Sturm-Liouville problem. The difference is that in the classical problem, the weight function $q(\alpha)$ is assumed to be continuous, while in the case under consideration $q(\alpha)$ has singularities represented by delta functions.

Problem (15.81)-(15.82) for eigenvalues is analyzed in Appendix IV. This problem is solved on the basis of the general method of integration of differential equations with singular coefficients, discussed in Appendix II. An orthonormalized spectrum of eigenvalues and eigenfunctions of problem (15.81)-(15.82) was obtained. The spectrum of eigenvalues forms a discrete sequence tending to infinity which contains one zero and an infinite number of positive eigenvalues. This spectrum consists of $\left[2n - 3E\left(\frac{n-1}{2}\right)\right]$ infinite chains of eigenvalues constituting the roots of $\left[2n - 3E\left(\frac{n-1}{2}\right)\right]$ different characteristic equations in the parameter $\lambda^* = \frac{\pi\sqrt{\lambda}}{n}$. With the exception of one equation, these equations completely coincide with the characteristic equations obtained in Section 14.2, where the problem was solved in trigonometric series.

The eigenfunctions of problem (15.81)-(15.82) corresponding to the coinciding roots and the coordinate functions obtained in Section 14.3 also coincide to within the multiplier.

The only exception are the coordinate functions Φ_{1s1} and Φ_{2s1}, which do not coincide with the analogous eigenfunctions of problem (15.81)-(15.82). The characteristic equations corresponding to them do not coincide either. This is explained by the fact that we obtained the coordinate functions Φ_{1s1}, Φ_{2s1} while solving infinite subsystems of differential equations corresponding to expansions in $\cos(qn\pm 1)\alpha$ and $\sin(qn\pm 1)\alpha$. Only the indicated subsystems contain the components η_x, η_y of the transverse displacement u_{n_z}, and hence, the coordinate functions Φ_{1sr}, Φ_{2sr} cannot be present in the spectrum of eigenfunctions of problem (15.81)-(15.82), since the problem was formulated under the condition $u_{n_z} = 0$.

Let us again return to general integro-differential equation (15.50). It was shown above that the first three terms of the expansion of the displacement $u_{m_r}(t, \alpha)$ in special coordinate functions correspond to eigenfunctions (15.77) of problem (15.50)-(15.51) when $\lambda = 0$. We will now show that the remaining special coordinate functions and the corresponding root of the characteristic equations are the eigenfunctions and eigenvalues of problem (15.50)-(15.51).

Turning to the initial representation of special coordinate functions in the form of trigonometric series, we can observe that none of the coordinate functions, with the exception of Φ_{1s1}, Φ_{2s1}, contain the first terms of the expansion sin α, cos α and hence, all are orthogonal with respect to these terms on the circle. Therefore, for any of the indicated coordinate functions, substitution into Eq. (15.50) yields

$$\Phi''+\lambda\varrho\Phi+\frac{1}{\pi}\left[\cos\alpha\int_0^{2\pi}\Phi\cos\alpha\,d\alpha+\sin\alpha\int_0^{2\pi}\Phi\sin\alpha\,d\alpha\right]=\Phi''+\lambda\varrho\Phi. \qquad (15.84)$$

We will specify the unknown parameter λ in the form

$$\lambda=\left(\frac{n}{\pi}\lambda^*\right)^2, \qquad (15.85)$$

where λ^* is the root of the characteristic equation obtained by solving the problem in trigonometric series, and corresponding to the coordinate function Φ in (15.84). However, as was shown above, the system of coordinate functions Φ is identical to the spectrum of eigenfunctions of the special Sturm-Liouville problem (see Section 3, Appendix IV). The only exceptions are the coordinate functions $\Phi_{1(1)}, \Phi_{2(1)}$ and eigenfunctions $Y_{I(1)}, Y_{II(1)}$. Therefore, taking into consideration the relationship between the parameter λ^* and eigenvalue λ of the special Sturm-Liouville problem (see Section 2, Appendix IV), we come to the conclusion that

$$\Phi''+\lambda\varrho\Phi=\Phi''+\lambda\varrho\Phi+\frac{1}{\pi}\left[\cos\alpha\int_0^{2\pi}\Phi\cos\alpha\,d\alpha+\sin\alpha\int_0^{2\pi}\Phi\sin\alpha\,d\alpha\right]=0 \qquad (15.86)$$

It follows that all the coordinate functions Φ with the exception of $\Phi_{1(1)}$, are the eigenfunctions of the general problem in eigenvalues (15.50)-(15.51). The corresponding eigenvalues λ are determined from the characteristic equations obtained in the special Sturm-Liouville problem (Section 2, Appendix IV), and also by solving the problem in trigonometric series (Section 14.2).

The functions $\Phi_{1(1)}$ and $\Phi_{2(1)}$ represented by expressions (14.143), (14.145), are also the eigenfunctions of problem (15.50)-(15.51), and the

λ values (15.85), where λ^* are the roots of characteristic Eq. (14.74), are its eigenvalues. This can be readily ascertained by directly substituting Φ_{1n} and Φ_{2n}, into Eq. (15.50). We will show how this is done.

The weight function $q(\alpha)$ (15.55) entering into Eq. (15.50) contains the delta functions $\delta(\alpha-\alpha_m)$. It is obvious, therefore, that the second derivative of the coordinate function Φ also contains delta-function type singularities, and the first derivative Φ undergoes discontinuities of the first kind at points $\alpha=\alpha_m$, i.e., in the usual sense, only one-sided derivatives exist at these points. Hence,

$$\Phi''(\alpha)=\Phi''(\alpha-0)+\sum_{m=1}^{n}[\Phi'(\alpha_m+0)-\Phi'(\alpha_m-0)]\,\delta(\alpha-\alpha_m) \tag{15.87}$$

where $\Phi'(\alpha_m+0)$, $\Phi''(\alpha_m-0)$ are one-sided first derivatives on the right and left at the point $\alpha=\alpha_m$, and $\Phi''(\alpha-0)$ is the second derivative on the left.

The coordinate functions Φ_{1n} and Φ_{2n}, in the form (14.143), (14.145) contain two variable quantities p and ξ related to α as follows:

$$\alpha=\frac{2\pi}{n}(p+\xi),$$

where

$$p=E\left(\frac{\alpha}{2\pi/n}\right),$$

$$0\leqslant\xi<1.$$

Therefore, in calculating $\Phi'(\alpha_m+0)$ it is necessary to assume

$$p=m,\ \xi=0, \tag{15.88}$$

and in calculating $\Phi'(\alpha_m-0)$

$$p=m-1,\ \xi\to 1. \tag{15.89}$$

The second derivative on the left $\Phi''(\alpha-0)$ should be calculated by assuming that

$$p = E\left(\frac{\alpha - 0}{2\pi/n}\right) \tag{15.90}$$

is independent of α.

From (14.143), (14.145) for values of p and ξ corresponding to (15.88), (15.89), we find

$$\begin{aligned} \Phi'_{1s1}(\alpha_m+0)-\Phi'_{1s1}(\alpha_m-0) &= -\frac{2n}{\pi}\lambda^{*2}_{s1}\sin\frac{2\pi}{n}m, \\ \Phi'_{2s1}(\alpha_m+0)-\Phi'_{2s1}(\alpha_m-0) &= -\frac{2n}{\pi}\lambda^{*2}_{s1}\cos\frac{2\pi}{n}m, \end{aligned} \tag{15.91}$$

and also

$$\begin{aligned} \Phi'_{1s1}(\alpha-0) &= -\frac{\sin\alpha}{\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1} - \frac{\lambda^*_{s1}\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2}{\cos 2\lambda^*_{s1}-\cos\frac{2\pi}{n}} \times \\ &\times\left[\sin\frac{2\pi}{n}p\sin 2\lambda^*_{s1}(1-\xi)+\sin\frac{2\pi}{n}(p+1)\sin 2\lambda^*_{s1}\xi\right], \\ \Phi'_{2s1}(\alpha-0) &= -\frac{\cos\alpha}{\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1} - \frac{\lambda^*_{s1}\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2}{\cos 2\lambda^*_{s1}-\cos\frac{2\pi}{n}} \times \\ &\times\left[\cos\frac{2\pi}{n}p\sin 2\lambda^*_{s1}(1-\xi)+\cos\frac{2\pi}{n}(p+1)\sin 2\lambda^*_{s1}\xi\right], \end{aligned} \tag{15.92}$$

where the p values are determined by relation (15.90).

Turning to the initial representation of Φ_{1s1}, Φ_{2s1} in the form of trigonometric series (14.132), (14.136), we find

$$\begin{aligned} &\int_0^{2\pi}\Phi_{1s1}\cos\alpha\,d\alpha=0, && \int_0^{2\pi}\Phi_{1s1}\sin\alpha\,d\alpha=-\pi, \\ &\int_0^{2\pi}\Phi_{2s1}\cos\alpha\,d\alpha=-\pi, && \int_0^{2\pi}\Phi_{2s1}\sin\alpha\,d\alpha=0. \end{aligned} \tag{15.93}$$

Now, expanding (15.87) with the aid of (15.91), (15.92) and introducing the result as well as (14.143), (14.145) into Eq. (15.50), we can obtain, taking (15.85), (15.93) into account,

$$
\begin{aligned}
&\Phi_{1s1}'' + \left(\frac{n}{\pi}\lambda_{s1}^{*}\right)^2 \varrho\Phi_{1s1} + \frac{1}{\pi}\left[\cos\alpha\int_0^{2\pi}\Phi_{1s1}\cos\alpha\, d\alpha + \sin\alpha\int_0^{2\pi}\Phi_{1s1}\sin\alpha\, d\alpha\right] = \\
&= \frac{2}{\pi}\,\frac{n}{\pi}\lambda_{s1}^{*2}\left[\frac{1}{\left(\frac{n}{\pi}\lambda_{s1}^{*}\right)^2 - 1} + \frac{\lambda_{s1}^{*}\sin 2\lambda_{s1}^{*}}{\cos 2\lambda_{s1}^{*} - \cos\frac{2\pi}{n}} - \pi\right] \times \\
&\times \sum_{m=1}^{n} \sin\alpha_m \delta(\alpha - \alpha_m), \\
&\Phi_{2s1}'' + \left(\frac{n}{\pi}\lambda_{s1}^{*}\right)^2 \varrho\Phi_{2s1} + \frac{1}{\pi}\left[\cos\alpha\int_0^{2\pi}\Phi_{2s1}\cos\alpha\, d\alpha + \sin\alpha\int_0^{2\pi}\Phi_{2s1}\sin\alpha\, d\alpha\right] = \\
&= \frac{2}{\pi}\,\frac{n}{\pi}\lambda_{s1}^{*2}\left[\frac{1}{\left(\frac{n}{\pi}\lambda_{s1}^{*}\right)^2 - 1} + \frac{\lambda_{s1}^{*}\sin 2\lambda_{s1}^{*}}{\cos 2\lambda_{s1}^{*} - \cos\frac{2\pi}{n}} - \pi\right] \times \\
&\times \sum_{m=1}^{n} \cos\alpha_m \delta(\alpha - \alpha_m).
\end{aligned}
\tag{15.94}
$$

Turning to characteristic Eq. (14.74) in λ_{s1}^{*}, we see that the expression in square brackets of (15.94) is equal to zero. Hence, the right-hand sides of (15.94) are identically equal to zero. This proves that the coordinate functions Φ_{1s1}, Φ_{2s1} are the eigenfunctions of problem (15.50)-(15.51), and the values $\lambda = \left(\frac{n}{\pi}\lambda_{s1}^{*}\right)^2$, where λ_{s1}^{*} are the roots of characteristic Eq. (14.74), are its eigenvalues.

Thus, the spectrum of eigenfunctions of general problem (15.50)-(15.51) is completely identical to the system of special coordinate functions of expansion (14.148). Therefore, hereinafter we will use the notation adopted in (14.148) for both.

The special Sturm-Liouville problem has a nonnegative spectrum of eigenvalues. Obviously, the spectrum of eigenvalues of general problem (15.50)-(15.51) will also be nonnegative if Eq. (14.74) has only real roots.

It follows from (15.59) that the value λ_{81} is real. Therefore, $\lambda_{s1}^{*} = \frac{\pi}{n}\sqrt{\lambda}$ can be either real or purely imaginary. We will show that λ_{s1}^{*}, satisfying Eqs. (14.74) can be only real. Let us assume the opposite:

$$\lambda_{s1}^{*} = \beta i, \tag{15.95}$$

where β is a real number. Introducing (15.95) into Eq. (14.74), we can readily reduce this equation to the form

$$-\frac{\beta \,\mathrm{sh}\, 2\beta}{\mathrm{ch}\, 2\beta - \cos\frac{2\pi}{n}} = \varkappa + \frac{1}{\left(\frac{n\beta}{\pi}\right)^2 \quad 1}.$$

We have reached a contradiction, since for any real β

$$\beta \,\mathrm{sh}\, 2\beta \geqslant 0,$$
$$\mathrm{ch}\, 2\beta - \cos\frac{2\pi}{n} > 0.$$

Hence λ_{s1}^{*} is real, and $\lambda_{s1}>0$.

Thus, we have obtained a spectrum of eigenvalues and eigenfunctions of general problem (15.50)-(15.51).

It was shown above that eigenfunctions corresponding to different eigenvalues were orthogonal with weight q(α). Eigenfunctions corresponding to the same eigenvalues (such eigenvalues are called degenerate) may also be nonorthogonal. However, by using them to construct the corresponding linear combination, one can arrive at orthogonal eigenfunctions in this case as well, as was done in Section 3, Appendix IV.

The spectrum of eigenfunctions of problem (15.50)-(15.51) contains the triply degenerate value λ = 0 and $E\left(\frac{n-1}{2}\right)$ of infinite chains of doubly degenerate eigenvalues $\lambda=\lambda_{sr}$ $\left(r=1,\ 2,\ldots\ \ldots,\ E\left(\frac{n-1}{2}\right);s=1,\ 2,\ldots,\ \infty\right)$. The eigenfunctions corresponding to the values $\lambda_{sr}\left[r=2,\ 3,\ldots,\ E\left(\frac{n-1}{2}\right);\ s=1,2,\ldots,\infty\right]$, coincide with the eigenfunctions $Y_{\mathrm{I}\,qr},\ Y_{\mathrm{II}\,qr}\left[r=2,\ 3,\ldots,\ E\left(\frac{n-1}{2}\right);\ q=1,\ 2,\ \ldots,\ \infty\right]$. orthogonalized in Section 3, Appendix IV. Hence, all that remains to be done is to examine the question of orthogonality of the eigenfunctions corresponding to λ = 0 and = λ_{s1}.

The eigenfunctions corresponding to $\lambda = 0$ are represented by expressions (15.77). Considering (14.3), (14.7), (14.8), (14.14) and (15.55), we have

$$\int_0^{2\pi} \Phi_{00}\Phi_{01}\varrho d\alpha = \frac{\Delta F_0}{R_0 h_0} \sum_{m=1}^{n} \sin\frac{2\pi}{n} m = 0,$$

$$\int_0^{2\pi} \Phi_{00}\Phi_{02}\varrho d\alpha = \frac{\Delta F_0^1}{R_0 h_0} \sum_{m=1}^{n} \cos\frac{2\pi}{n} m = 0,$$

$$\int_0^{2\pi} \Phi_{01}\Phi_{02}\varrho d\alpha = \frac{\Delta F_0}{R_0 h_0} \sum_{m=1}^{n} \sin\frac{2\pi}{n} m \cos\frac{2\pi}{n} m = 0, \tag{15.96}$$

i.e., the eigenfunctions $\Phi_{00}, \Phi_{01}, \Phi_{02}$, represented by expression (15.77) are orthogonal on the contour of a circular stringer shell.

The eigenfunctions $\Phi_{1\varepsilon 1}, \Phi_{2\varepsilon 1}$, corresponding to $\lambda = \lambda_{\varepsilon 1}$, are represented by expressions (14.143), (14.145). Let us turn to their initial representation in the form of trigonometric series (14.132), (14.136). It is evident that

$$\int_0^{2\pi} \Phi_{1\varepsilon 1}\Phi_{2\varepsilon 1} d\alpha = 0, \tag{15.97}$$

since (14.132) is a series in cos kα and (14.136), in sin kα. Therefore, considering (14.3), (14.14), (15.97), we have

$$\int_0^{2\pi} \Phi_{1\varepsilon 1}(\alpha)\Phi_{2\varepsilon 1}(\alpha)\varrho(\alpha) d\alpha = \frac{\Delta F_0}{R_0 h_0} \sum_{m=1}^{n} \Phi_{1\varepsilon 1}\left(\frac{2\pi}{n} m\right)\Phi_{2\varepsilon 1}\left(\frac{2\pi}{n} m\right) = 0, \tag{15.98}$$

i.e., the eigenfunctions $\Phi_{1\varepsilon 1}, \Phi_{2\varepsilon 1}$, represented by expression (14.143), (14.145), are orthogonal on the contour of a circular stringer shell.

In Section 3, Appendix IV, the eigenfunctions of the special Sturm-Liouville problem are normalized. Thus, all the eigenfunctions of problem (15.50)-(15.51) are also normalized, with the exception of the functions Φ_{01}, Φ_{02} and $\Phi_{1\varepsilon 1}, \Phi_{2\varepsilon 1}$. We will calculate for these functions the normalizing factors

$$\frac{1}{\sqrt{\int\limits_0^{2\pi} \Phi^2(\alpha)\,\varrho(\alpha)\,d\alpha}}.$$

On the basis of (15.55)

$$\int\limits_0^{2\pi} \Phi^2(\alpha)\varrho(\alpha)\,d\alpha = \int\limits_0^{2\pi} \Phi^2(\alpha)\,d\alpha + \frac{\Delta F_0}{R_0 h_0}\sum_{m=1}^{n}\Phi^2\left(\frac{2\pi}{n}m\right). \tag{15.99}$$

Expanding (15.99) and using (14.3), (14.10), (14.12), (14.13), we find for eigenfunctions (15.77)

$$\int\limits_0^{2\pi} \Phi_{01}^2\,\varrho(\alpha)\,d\alpha = \pi\left(1 + \frac{1}{\varkappa}\right),$$
$$\int\limits_0^{2\pi} \Phi_{02}^2\,\varrho(\alpha)\,d\alpha = \pi\left(1 + \frac{1}{\varkappa}\right). \tag{15.100}$$

On the basis of (15.59)

$$\int\limits_0^{2\pi} \Phi^2\varrho\,d\alpha = \frac{1}{\lambda_{s1}}\int\limits_0^{2\pi} \Phi'^2\,d\alpha - \frac{1}{\pi\lambda_{s1}}\left(\int\limits_0^{2\pi} \Phi\cos\alpha\,d\alpha\right)^2 - \frac{1}{\pi\lambda_{s1}}\left(\int\limits_0^{2\pi} \Phi\sin\alpha\,d\alpha\right)^2. \tag{15.101}$$

Turning to (14.132), (14.136), we find for the functions Φ_{1s1}, Φ_{2s1},

$$\int\limits_0^{2\pi} \Phi'^2\,d\alpha = \pi\left[1 + \bar{\lambda}_{s1}^4 n^2 \bar{S}(\bar{\lambda}_{s1})\right], \tag{15.102}$$

where

$$\bar{S}(\bar{\lambda}_{s1}) = \sum_{q_k=1}^{\infty}\left[\left(\frac{q_k + \frac{1}{n}}{\left(q_k + \frac{1}{n}\right)^2 - \bar{\lambda}_{s1}^2}\right)^2 + \left(\frac{q_k - \frac{1}{n}}{\left(q_k - \frac{1}{n}\right)^2 - \bar{\lambda}_{s1}^2}\right)^2\right]. \tag{15.103}$$

Sum (15.103) is easy to calculate by observing that

$$\bar{S}(\bar{\lambda}_{s1}) = \frac{1}{2\lambda}\frac{dS}{d\lambda}\bigg|_{\lambda=\bar{\lambda}_{s1}}, \tag{15.104}$$

where S is the sum determined by expression (14.72). From (14.73), after some simple transformations, we find

$$S = \frac{\lambda^* \sin 2\lambda^*}{\cos 2\lambda^* - \cos\frac{2\pi}{n}} + \frac{\left(\frac{n\lambda^*}{\pi}\right)^2}{\left(\frac{n\lambda^*}{\pi}\right)^2 - 1}. \tag{15.105}$$

Introducing (15.105) into (15.104), in view of (14.43), we find

$$\bar{S}=\left(\frac{\pi}{\lambda_{s1}^{*}}\right)^{2}\left[\varkappa\left(\varkappa+\frac{1}{2}\right)-\frac{2\varkappa+\frac{3}{2}}{\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^{2}-1}+\frac{\lambda_{s1}^{*2}\cos 2\lambda_{s1}^{*}}{\cos 2\lambda_{s1}^{*}-\cos\frac{2\pi}{n}}\right]. \tag{15.106}$$

Now from (15.101), using (15.93), (15.102), (15.100), we finally obtain

$$\int_{0}^{2\pi}\Phi_{1s1}^{2}\varrho\,d\alpha=\int_{0}^{2\pi}\Phi_{2s1}^{2}\varrho\,d\alpha=$$

$$=\pi\left[-\frac{\varkappa+1}{\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^{2}-1}+\lambda_{s1}^{*}\frac{\lambda_{s1}^{*}\cos 2\lambda_{s1}^{*}+\left(\varkappa+\frac{1}{2}\right)\sin 2\lambda_{s1}^{*}}{\cos 2\lambda_{s1}^{*}-\cos\frac{2\pi}{n}}\right]. \tag{15.107}$$

Thus, an orthonormalized spectrum of the eigenfunctions of general problem (15.50)-(15.51) has been obtained. These functions constitute normal coordinate functions, as natural for a stringer conical shell as the trigonometric functions sin $k\alpha$, cos $k\alpha$ for a smooth shell of revolution. It is easy to see that the spectrum of the eigenfunctions of general problem (15.50)-(15.51) in the case of a smooth shell $(\varkappa=\infty)$ is transformed into a system of trigonometric functions sin $k\alpha$, cos $k\alpha$.

In conclusion, we will give a complete summary of the orthonormalized eigenfunctions of problem (15.50)-(15.51) and their most important properties:

$$\Phi_{00}=\frac{1}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}},$$

$$\Phi_{01}(\alpha)=\frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\sin\alpha, \tag{15.108}$$

$$\Phi_{02}(\alpha)=\frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\cos\alpha,$$

$$\Phi_{1s0}(\alpha)=\frac{1}{\sqrt{\pi}}\sin 2\lambda_{1s0}^{*}\xi=\frac{1}{\sqrt{\pi}}\sin s\pi\alpha,$$

$$\Phi_{2s0}(\alpha)=-\frac{1}{B_{s0}\sqrt{2}}\,\frac{\cos 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)}{\cos\lambda_{2s0}^{*}}. \tag{15.109}$$

$$\Phi_{1s1}(\alpha)=\frac{1}{B_{s1}}\left\{\frac{\sin\alpha}{\varkappa\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right]-1}+\sin\frac{2\pi}{n}(p+1)\,\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+\right.$$

$$\left.+\sin\frac{2\pi}{n}p\,\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\},$$

$$\Phi_{2s1}(\alpha)=\frac{1}{B_{s1}}\left\{\frac{\cos\alpha}{\varkappa\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right]-1}+\cos\frac{2\pi}{n}(p+1)\,\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+\right. \tag{15.110}$$

$$\left.+\cos\frac{2\pi}{n}p\,\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\},$$

$$\Phi_{1sr}(\alpha)=\frac{1}{B_{sr}}\left\{\sin\frac{2\pi r}{n}(p+1)\,\frac{\sin 2\lambda^*_{sr}\xi}{\sin 2\lambda^*_{sr}}+\right.$$

$$\left.+\sin\frac{2\pi r}{n}p\,\frac{\sin 2\lambda^*_{sr}(1-\xi)}{\sin 2\lambda^*_{sr}}\right\},$$

$$\Phi_{2sr}(\alpha)=\frac{1}{B_{sr}}\left\{\cos\frac{2\pi r}{n}(p+1)\,\frac{\sin 2\lambda^*_{sr}\xi}{\sin 2\lambda^*_{sr}}+\right. \tag{15.111}$$

$$\left.+\cos\frac{2\pi r}{n}p\,\frac{\sin 2\lambda^*_{sr}(1-\xi)}{\sin 2\lambda^*_{sr}}\right\},$$

$$\Phi_{1s\frac{n}{2}}(\alpha)=\frac{1}{\sqrt{\pi}}(-1)^p\sin 2\lambda^*_{1s\frac{n}{2}}\xi=\frac{1}{\sqrt{\pi}}\sin\left(s\pi-\frac{\pi}{2}\right)\alpha,$$

$$\Phi_{2s\frac{n}{2}}(\alpha)=-\frac{1}{B_{s\frac{n}{2}}\sqrt{2}}(-1)^p\,\frac{\sin 2\lambda^*_{2s\frac{n}{2}}\left(\xi-\frac{1}{2}\right)}{\sin\lambda^*_{2s\frac{n}{2}}}. \tag{15.112}$$

Here $s=1,2,\ldots,\infty;\ r=2,3,\ldots,E\left(\frac{n-1}{2}\right)$; to the functions Φ_{00}, Φ_{01}, Φ_{02} there corresponds a zero value of the parameter λ^*, and for the remaining eigenfunctions, the values of the parameter are the roots of the characteristic equations:

$\{\lambda^*_{1s0}\}$ are the roots of the equation

$$\sin\lambda^*=0,\quad \lambda^*_{1s0}=\pi s; \tag{15.113}$$

$\{\lambda^*_{2s0}\}$ are the roots of the equation

$$\lambda^*\operatorname{ctg}\lambda^*=-\varkappa; \tag{15.114}$$

$\{\lambda^*_{s1}\}$ are the roots of the equation

$$\frac{\lambda^*\sin 2\lambda^*}{\cos 2\lambda^*-\cos\frac{2\pi}{n}}+\frac{1}{\left(\frac{n\lambda^*}{\pi}\right)^2-1}=\varkappa; \tag{15.115}$$

$\{\lambda^*_{sr}\}$ are the roots of the equations

$$\frac{\lambda^* \sin 2\lambda^*}{\cos 2\lambda^* - \cos \frac{2\pi r}{n}} = \varkappa \quad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right]; \tag{15.116}$$

$\left\{\lambda^*_{1s\frac{n}{2}}\right\}$ are the roots of the equation

$$\cos \lambda^* = 0, \quad \lambda^*_{1s\frac{n}{2}} = -\frac{\pi}{2} + \pi s; \tag{15.117}$$

$\left\{\lambda^*_{2s\frac{n}{2}}\right\}$ are the roots of the equation

$$\lambda^* \operatorname{tg} \lambda^* = \varkappa; \tag{15.118}$$

The normalizing factors are determined by the expressions

$$B^2_{sr} = \frac{\pi}{\sin^2 2\lambda^*_{sr}}\left(1 - \cos \frac{2\pi r}{n} \cos 2\lambda^*_s\right) + \frac{\pi}{2\varkappa} \tag{15.119}$$

$$\left[r = 0, 2, 3, \ldots, E\left(\frac{n}{2}\right)\right].$$

$$B^2_{s1} = \frac{\pi}{\left[\varkappa - \frac{1}{\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2 - 1}\right]^2} \bar{B}^2_s, \tag{15.120}$$

where

$$\bar{B}^2_s = -\frac{\varkappa + 1}{\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2 - 1} + \lambda^*_{s1} \frac{\lambda^*_{s1} \cos 2\lambda^*_{s1} + \left(\varkappa + \frac{1}{2}\right) \sin 2\lambda^*_{s1}}{\cos 2\lambda^*_{s1} - \cos \frac{2\pi}{n}}; \tag{15.121}$$

the variables p and ξ are related to α as follows:

$$p = E\left(\frac{\alpha}{2\pi/n}\right),$$
$$\xi = \frac{\alpha}{2\pi/n} - p, \quad 0 \leqslant \xi < 1, \tag{15.122}$$

where $E\left(\frac{\alpha}{2\pi/n}\right)$ is the integral part of the number $\frac{\alpha}{2\pi/n}$.

Eigenfunctions (15.108)-(15.112) are normalized so that

$$\int_0^{2\pi} \Phi^2(\alpha) \varrho(\alpha) d\alpha = 1, \tag{15.123}$$

$$\int_0^{2\pi} \Phi'^2 d\alpha - \frac{1}{\pi}\left\{\left[\int_0^{2\pi} \Phi(\alpha) \cos \alpha d\alpha\right]^2 + \left[\int_0^{2\pi} \Phi(\alpha) \sin \alpha d\alpha\right]^2\right\} = \left(\frac{n}{\pi} \lambda^*\right)^2, \tag{15.124}$$

and satisfy the following orthogonality conditions:

$$\int_0^{2\pi} \Phi_1(\alpha)\Phi_2(\alpha)\varrho(\alpha)\,d\alpha = 0, \tag{15.125}$$

$$\int_0^{2\pi} \Phi_1'\Phi_2'\,d\alpha - \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \Phi_1(\zeta)\Phi_2(\xi)\cos(\zeta-\xi)\,d\zeta\,d\xi = 0, \tag{15.126}$$

where Φ_1, Φ_2 is any pair of different eigenfunctions.

In addition,

$$\int_0^{2\pi} \Phi_{1sr}\cos\alpha\,d\alpha = \int_0^{2\pi} \Phi_{2sr}\sin\alpha\,d\alpha = 0,$$

$$\int_0^{2\pi} \Phi_{1sr}\sin\alpha\,d\alpha = \begin{cases} -\frac{\sqrt{\pi}}{B_s} & (r=1) \\ 0 & (r \neq 1) \end{cases}, \quad \int_0^{2\pi} \Phi_{2sr}\cos\alpha\,d\alpha = \begin{cases} -\frac{\sqrt{\pi}}{B_s} & (r=1) \\ 0 & (r \neq 1) \end{cases} \tag{15.127}$$

The arbitrary function can be made to correspond to the series in eigenfunctions (15.108)-(15.112), converging on the segment $[0, 2\pi]$:

$$f(\alpha) \sim a_{00}\Phi_{00} + a_{01}\Phi_{01}(\alpha) + a_{02}\Phi_{02}(\alpha) + \sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)} [a_{sr}\Phi_{2sr}(\alpha) + b_{sr}\Phi_{1sr}(\alpha)], \tag{15.128}$$

where

$$a_{00} = \Phi_{00}\int_0^{2\pi} f(\alpha)\varrho(\alpha)\,d\alpha,$$

$$a_{01} = \int_0^{2\pi} f(\alpha)\Phi_{01}(\alpha)\varrho(\alpha)\,d\alpha, \quad a_{02} = \int_0^{2\pi} f(\alpha)\Phi_{02}(\alpha)\varrho(\alpha)\,d\alpha, \tag{15.129}$$

$$a_{sr} = \int_0^{2\pi} f(\alpha)\Phi_{2sr}(\alpha)\varrho(\alpha)\,d\alpha, \quad b_{sr} = \int_0^{2\pi} f(\alpha)\Phi_{1sr}(\alpha)\varrho(\alpha)\,d\alpha,$$

provided that all the quadratures have meaning.

The Fourier series of eigenfunctions (15.109)-(15.112) in trigonometric functions have the form

$$\Phi_{2s0}(\alpha) = -\frac{1}{\sqrt{2\pi}B_{s0}} + \frac{\sqrt{2}}{\pi}\frac{\lambda_{2s0}^{*2}}{B_{s0}}\sum_{q_k=1}^{\infty}\frac{\cos q_k n\alpha}{n^2 q_k^2 - \lambda_{2s0}^{*2}},$$

$$\Phi_{1s1}(\alpha) = -\frac{\sin\alpha}{\sqrt{\pi}B_s} + \tag{15.130}$$

$$+\frac{\lambda_{s1}^{*2}}{\sqrt{\pi}B_s}\sum_{q_k=1}^{\infty}\left[\frac{\sin(q_k n+1)\alpha}{n^2\left(q_k+\frac{1}{n}\right)^2 - \lambda_{s1}^{*2}} - \frac{\sin(q_k n-1)\alpha}{n^2\left(q_k-\frac{1}{n}\right)^2 - \lambda_{s1}^{*2}}\right],$$

$$\Phi_{2s1}(\alpha) = -\frac{\cos\alpha}{\sqrt{\pi B_s}} +$$

$$+\frac{\lambda_{s1}^{*2}}{\pi B_s}\sum_{q_k=1}^{\infty}\left[\frac{\cos(q_k n+1)\alpha}{\pi^2\left(q_k+\frac{1}{n}\right)^2-\lambda_{s1}^{*2}}+\frac{\cos(q_k n-1)\alpha}{\pi^2\left(q_k-\frac{1}{n}\right)^2-\lambda_{s1}^{*2}}\right],$$

$$\Phi_{1sr}(\alpha) = \frac{\lambda_{sr}^{*2}}{\pi B_{sr}}\left\{\frac{\sin r\alpha}{\pi^2\left(\frac{r}{n}\right)^2-\lambda_{sr}^{*2}}+\right. \tag{15.130}$$

$$\left.+\sum_{q_k=1}^{\infty}\left[\frac{\sin(q_k n+r)\alpha}{\pi^2\left(q_k+\frac{r}{n}\right)^2-\lambda_{sr}^{*2}}-\frac{\sin(q_k n-r)\alpha}{\pi^2\left(q_k-\frac{r}{n}\right)^2-\lambda_{sr}^{*2}}\right]\right\},$$

$$\Phi_{2sr}(\alpha) = \frac{\lambda_{sr}^{*2}}{\pi B_{sr}}\left\{\frac{\cos r\alpha}{\pi^2\left(\frac{r}{n}\right)^2-\lambda_{sr}^{*2}}+\right.$$

$$\left.+\sum_{q_k=1}^{\infty}\left[\frac{\cos(q_k n+r)\alpha}{\pi^2\left(q_k+\frac{r}{n}\right)^2-\lambda_{sr}^{*2}}+\frac{\cos(q_k n-r)\alpha}{\pi^2\left(q_k-\frac{r}{n}\right)^2-\lambda_{sr}^{*2}}\right]\right\},$$

$$\Phi_{2s\frac{n}{2}}(\alpha) = \frac{\sqrt{2}}{\pi}\,\frac{\lambda_{2s\frac{n}{2}}^{*2}}{B_{s\frac{n}{2}}}\sum_{q_k=1}^{\infty}\frac{\cos\left(q_k-\frac{1}{2}\right)n\alpha}{\pi^2\left(q_k-\frac{1}{2}\right)^2-\lambda_{2s\frac{n}{2}}^{*2}}.$$

It is desirable to calculate the roots of the transcendental characteristic equations by the method of successive approximations.

We rewrite Eq. (15.114) in the form

$$\operatorname{ctg}\lambda^* = -\frac{\varkappa}{\lambda^*}, \tag{15.131}$$

whence it follows that the roots of Eq. (15.114) coincide with the abscissas of the points of intersection of the hyperbola $y = -\frac{\varkappa}{\lambda^*}$ and family of cotangent curves $y = \operatorname{ctg}\lambda^*$ (Fig. 15.1). It is evident that the infinite sequence of roots of Eq. (15.114) can be represented by the symptotic expressions

$$\lambda_{2s0}^{*} \approx \pi\left(s-\frac{1}{2}\right), \quad (15.132), \tag{15.132}$$

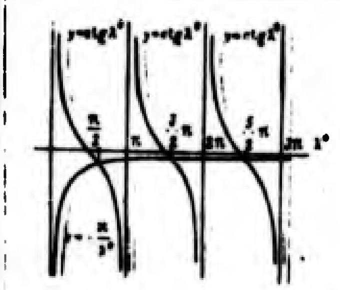

Fig. 15.1

Expression (15.132) can be refined by representing the roots of Eq. (15.114) in the form

$$\lambda^{*}_{2s0} = \pi\left(s-\frac{1}{2}\right)+\varepsilon_s.$$

Introducing this expression into Eq. (15.144) and linearizing (15.114) with respect to ε_s, we find

$$\lambda^{*}_{2s0} \approx \pi\left(s-\frac{1}{2}\right)+\frac{\varkappa}{\pi\left(s-\frac{1}{2}\right)}. \tag{15.133}$$

The following iteration relation follows from (15.131):

$$\lambda^{*(i+1)}_{2s0} = \operatorname{arcctg}\left(-\frac{\varkappa}{\lambda^{*(i)}_{2s0}}\right)+\pi(s-1). \tag{15.134}$$

Taking the asymptotic values (15.132) or (15.133) as the initial approximations of $\lambda^{*(0)}_{2s0}$, and using relation (15.134), one can compute the desired roots of Eq. (15.115) with any degree of accuracy. It should be noted that a symptotic formula (15.133) permits a significant reduction in the volume of computations. Thus, when x = 1, even for s = 36, the absolute error of asymptotic values (15.133) does not exceed 10^{-6}, whereas the same absolute error of asymptotic values (15.132) is reached only when $s \gg 1000$.

For Eq. (15.118), as in the case of Eq. (15.115), we obtain the asymptotic values of the roots

$$\lambda^{*}_{2s\frac{\pi}{2}} \approx \pi(s-1), \tag{15.135}$$

or more exactly

$$\lambda^{*}_{2s\frac{\pi}{2}} \approx \pi(s-1)+\frac{\varkappa}{\pi(s-1)} \quad (s=2,3,\ldots) \tag{15.136}$$

and the interation relation

$$\lambda^{*(i+1)}_{2s\frac{\pi}{2}} = \operatorname{arctg}\frac{\varkappa}{\lambda^{*(i)}_{2s\frac{\pi}{2}}}+\pi(s-1). \tag{15.137}$$

Let us turn to Eq. (15.155). Rewriting this equation in the form

$$\sin 2\lambda^* = \frac{1}{\lambda^*}\left(\cos 2\lambda^* - \cos\frac{2\pi}{n}\right)\left[\frac{1}{1-\left(\frac{n\lambda^*}{\pi}\right)^2} + \varkappa\right] \tag{15.138}$$

and revealing the indeterminacy when $\lambda^* \to \frac{\pi}{n}$, we arrive at an identity. However, as can be readily ascertained, for this value of λ^*, the functions $\Phi_{1n}(\alpha)$ and $\Psi_{2n}(\alpha)$, represented by expressions (15.110), identically become zero, and hence, $\lambda^* = \frac{\pi}{n}$ is not an eigenvalue of problem (15.50)-(15.51).

We will now show that the positive roots of Eq. (15.115) $\lambda^*_n > \frac{\pi}{n}$. To this end, we rewrite the equation in the form

$$\lambda^*\left[1-\left(\frac{n\lambda^*}{\pi}\right)^2\right] = \frac{\cos 2\lambda^* - \cos\frac{2\pi}{n}}{\sin 2\lambda^*}\left\{1+\left[1-\left(\frac{n\lambda^*}{\pi}\right)^2\right]\varkappa\right\}. \tag{15.139}$$

Let us analyze Eq. (15.139).

Figure 15.2 shows graphs of the functions

$$y_1(\lambda^*) = \lambda^*\left[1-\left(\frac{n\lambda^*}{\pi}\right)^2\right],$$
$$y_2(\lambda^*) = \frac{\cos 2\lambda^* - \cos\frac{2\pi}{n}}{\sin 2\lambda^*}. \tag{15.140}$$

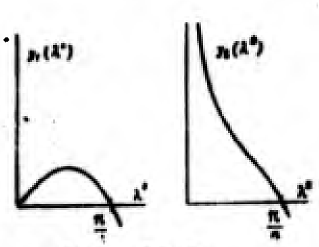

Fig. 15.2

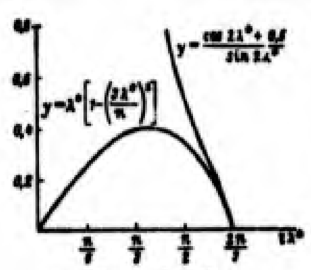

Fig. 15.3

It is evident that if the equation $y_1(\lambda^*)=y_2(\lambda^*)$ has no roots on $(0, \pi/n)$, then Eq. (15.139) certainly has no roots on this interval.

For the functions $y_1(\lambda^*), y_2(\lambda^*)$, we have

$$y_1\left(\frac{\pi}{n}\right)=y_2\left(\frac{\pi}{n}\right)=0,$$
$$y_1'(\lambda^*)\Big|_{\lambda^*=\frac{\pi}{n}}=y_2'(\lambda^*)\Big|_{\lambda^*=\frac{\pi}{n}}=-2, \tag{15.141}$$

whence it follows that the functions $y_1(\lambda^*), y_2(\lambda^*)$ when $\lambda^*=\frac{\pi}{n}$ have the common tangent

$$y(\lambda^*)=-2\left(\lambda^*-\frac{\pi}{n}\right). \tag{15.142}$$

Differentiating (15.140), we find on $(0, \pi/n)$

$$y_1''(\lambda^*)=-6\frac{n^2}{\pi^2}\lambda^*<0, \tag{15.143}$$
$$y_2''(\lambda^*)=\frac{4}{\sin^3 2\lambda^*}\left[\cos\frac{2\pi}{n}\sin^2 2\lambda^*+2\left(\cos 2\lambda^*-\cos\frac{2\pi}{n}\right)\right]>0 \text{ при } n>3. \tag{15.144}$$

On the basis of (15.141)-(15.144), we conclude that on $(0, \pi/n)$ when $n>3$

$$y_1(\lambda^*)<y(\lambda^*)<y_2(\lambda^*),$$

q.e.d.

When $n>3$, Eq. (15.115) is invalid. When $n = 3$, inequality (15.144) is not fulfilled anywhere on $(0, \pi/n)$ and hence, inequality $y(\lambda^*)<y_2(\lambda^*)$ is not fulfilled either on the entire interval $(0, \pi/n)$, but in this case the equation

$$\lambda^*\left[1-\left(\frac{3\lambda^*}{\pi}\right)^2\right]=\frac{\cos 2\lambda^*+0.5}{\sin 2\lambda^*}$$

has no roots on $(0, \pi/n)$, as can be readily ascertained by plotting the graphs shown in Fig. 15.3.

The process of successive approximations in the computation of the roots of Eq. (15.115) is easy to carry out by representing this equation in the form (15.138). A graphical interpretation of (15.138) is shown in Fig. 15.4. From (15.138), we have

$$\lambda_{s1}^{*[l+1]} = \frac{1}{2}\left\{(-1)^s \arcsin\left[\frac{1}{\lambda^{*[l]}}\left(\cos 2\lambda^{*[l]} - \cos\frac{2\pi}{n}\right)\times\right.\right.$$
$$\left.\left.\times\left(\frac{1}{1-\left(\frac{n\lambda^{*[l]}}{\pi}\right)^2}+\varkappa\right)\right]+\pi s\right\}. \tag{15.145}$$

As the initial approximations of $\lambda_{s1}^{*[0]}$ we can take the asymptotic values of the roots

$$\lambda_{s1}^{*} \approx \frac{\pi}{2} s. \tag{15.146}$$

Refining (15.146), we assume

$$\lambda_{s1}^{*} = \frac{\pi}{2} s + \varepsilon. \tag{15.147}$$

Introducing (15.147) into Eq. (15.138) and linearizing (15.138) with respect to ε, we find

$$\varepsilon \approx \frac{\pi s\left(1-\frac{n^2}{4}s^2\right)\left[1+\left(1-\frac{n^2}{4}s^2\right)\varkappa\right]}{\frac{\pi^2 s^2\left(1-\frac{n^2}{4}s^2\right)^2}{1-(-1)^n\cos\frac{2\pi}{n}}+2\left(1-\frac{n^2}{4}s^2\right)\left[1+\left(1-\frac{n^2}{4}s^2\right)\varkappa\right]-n^2 s^2}. \tag{15.148}$$

Finally, we turn to Eq. (15.116). We rewrite it in the form

$$\sin 2\lambda^{*} = \frac{\varkappa}{\lambda^{*}}\left(\cos 2\lambda^{*} - \cos\frac{2\pi r}{n}\right). \tag{15.149}$$

A graphical interpretation of (15.149) is shown in Fig. 15.5. From (15.149) we have

$$\lambda_{sr}^{*[l+1]} = \frac{1}{2}\left\{(-1)^{s-1}\arcsin\left[\frac{\varkappa}{\lambda^{*[l]}}\left(\cos 2\lambda^{*[l]} - \cos\frac{2\pi r}{n}\right)\right]+\pi(s-1)\right\}. \tag{15.150}$$

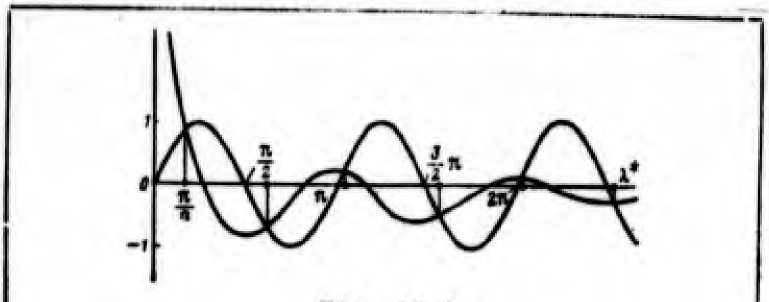

Fig. 15.4

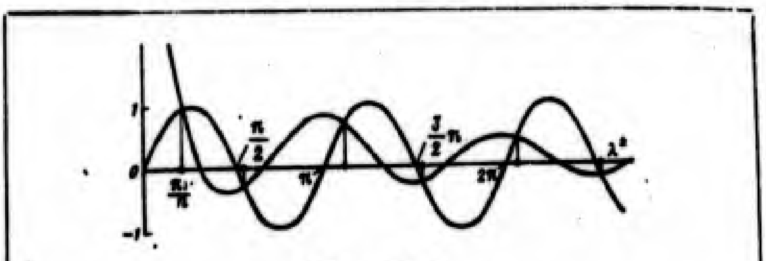

Fig. 15.5

As the initial approximations of $\lambda^{*(0)}_{sr}$ we can take the asymptotic value of the roots

$$\lambda^*_{sr} \approx \frac{\pi}{2}(s-1) \tag{15.151}$$

or more exactly

$$\lambda^*_{sr} \approx \frac{\pi}{2}(s-1) + \left[\frac{\pi}{n}\,\frac{s-1}{1-(-1)^{s-1}\cos\frac{2\pi r}{n}} + \frac{2}{\pi(s-1)}\right]^{-1}. \tag{15.152}$$

Iteration relation (15.150) is unsuitable for determining the first root λ^*_{1r}, since in this case the process of successive approximations diverges. It should be noted that for large values of the parameter $\varkappa$, the iteration process on the basis of relations (15.150) and (15.145) may prove to be divergent for several of the first roots. In such cases, a different iteration relation should be chosen for these roots, or other methods should be used.

As an illustration. Figs. 15.6-15.13 show the eigenfunctions of problem (15.50)-(15.51) corresponding to transcendental values λ^*_s

for s = 1, 2, 3 for the case $n=8,\ \dot{x}=1$.

15.3. Inhonogeneous Problem

The fundamental integro-differential Eq. (15.41) for an arbitrary external load in view of (15.42), (15.55) assumes the form

$$\gamma \sin^2\beta\, \rho(\alpha)\frac{\partial^2 u_{m_s}}{\partial t^2} + \frac{\partial^2 u_{m_s}}{\partial \alpha^2} + \frac{1}{\pi}\int_0^{2\pi} u_{m_s}(t,\xi)\cos(\alpha-\xi)\,d\xi = \frac{e^{-t}}{\pi R_0 h_0 G \operatorname{ctg}\beta}\left[M_{x_0}(t)\sin\alpha - M_{y_0}(t)\cos\alpha\right] + \frac{p_0^2}{G h_0 \cos\beta}\, e^{2t} p_z. \tag{15.153}$$

As was stated above, the solution of this equation must satisfy the periodicity conditions of the displacements and forces along the coordinate α and boundary conditions on the ends t = 0 and t = t_1 of the shell. Moreover, the solution satisfying the static boundary conditions will also satisfy Eqs. (15.20-(15.22).

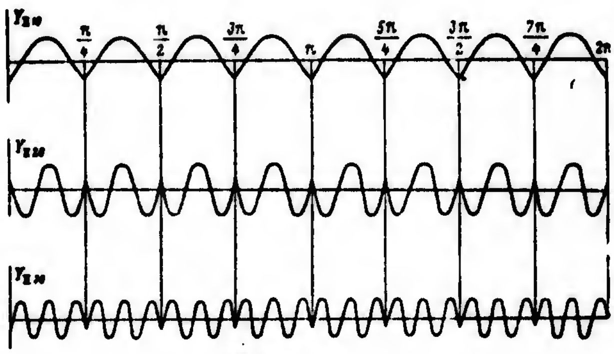

Fig. 15.6.

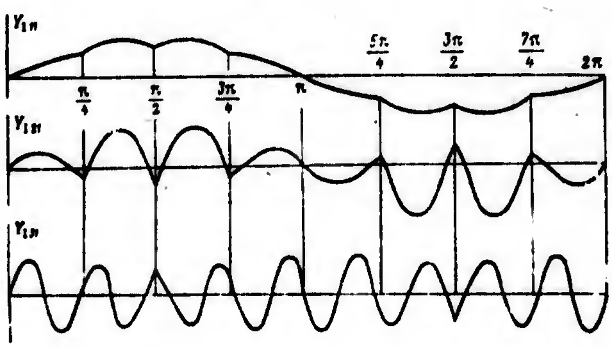

Fig. 15.7.

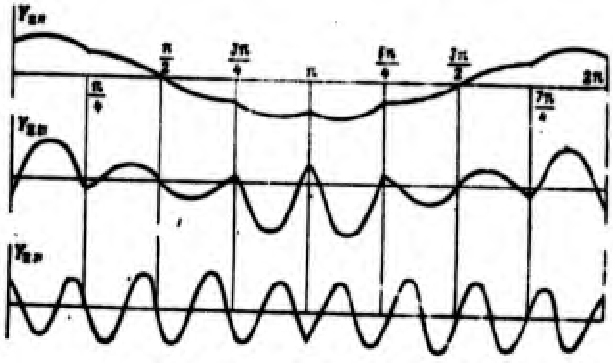

Fig. 15.8.

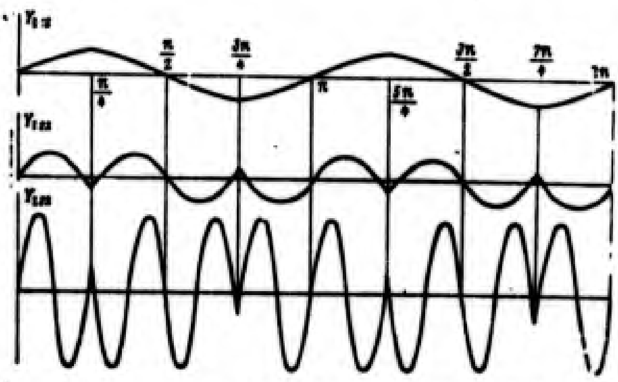

Fig. 15.9.

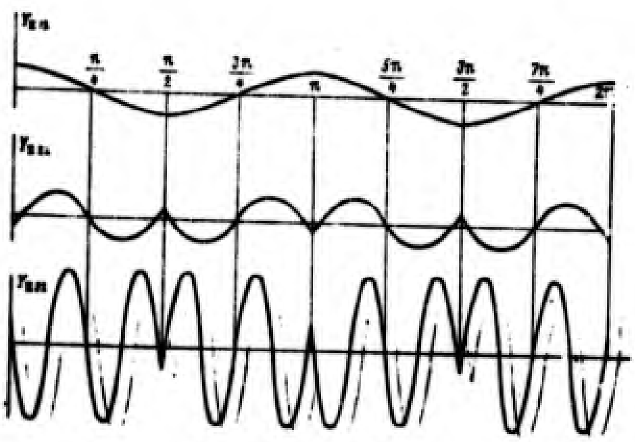

Fig. 15.10.

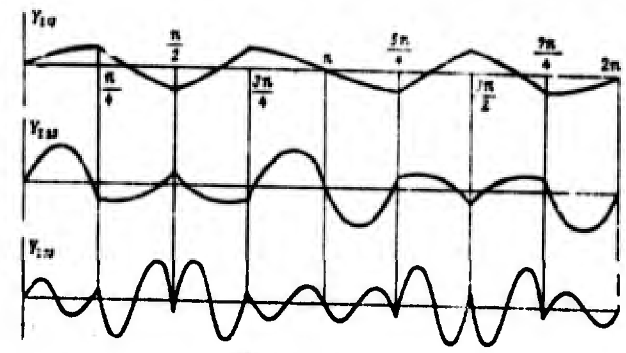

Fig. 15.11.

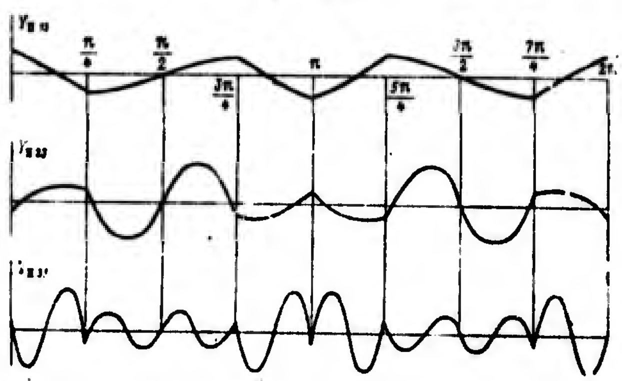

Fig. 15.12.

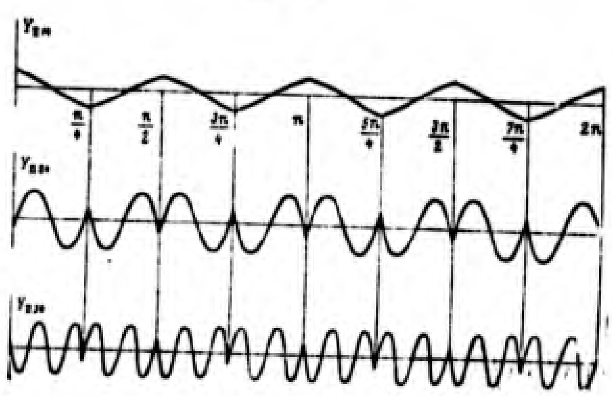

Fig. 15.13.

We will seek the solution of Eq. (15.153) in the form of a series in eigenfunctions (15.108)-(15.112) of problem (15.50)-(15.51):

$$u_{m_z}(t,\alpha)=U_{00}(t)\Phi_{00}+U_{01}(t)\Phi_{01}(\alpha)+U_{02}(t)\Phi_{02}(\alpha)+\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{s}{2}\right)}[U_{1sr}(t)\Phi_{1sr}(\alpha)+U_{2sr}(t)\Phi_{2sr}(\alpha)]. \tag{15.154}$$

Introducing (15.154) into (15.153), we obtain

$$\begin{aligned}&\gamma\sin^2\beta\dot U_{00}\rho\Phi_{00}+\gamma\sin^2\beta\dot U_{01}\rho\Phi_{01}+U_{01}\left[\Phi_{01}^{\cdot}+\frac{1}{\pi}\int_0^{2\pi}\Phi_{01}(\xi)\cos(\alpha-\xi)d\xi\right]+\\&+\gamma\sin^2\beta\dot U_{02}\rho\Phi_{02}+U_{02}\left[\Phi_{02}^{\cdot}+\frac{1}{\pi}\int_0^{2\pi}\Phi_{02}(\xi)\cos(\alpha-\xi)d\xi\right]+\\&+\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{s}{2}\right)}\left\{\gamma\sin^2\beta\dot U_{1sr}\rho\Phi_{1sr}+U_{1sr}\left[\Phi_{1sr}^{\cdot}+\frac{1}{\pi}\int_0^{2\pi}\Phi_{1sr}(\xi)\cos(\alpha-\xi)d\xi\right]+\right.\\&\left.+\gamma\sin^2\beta\dot U_{2sr}\rho\Phi_{2sr}+U_{2sr}\left[\Phi_{2sr}^{\cdot}+\frac{1}{\pi}\int_0^{2\pi}\Phi_{2sr}(\xi)\cos(\alpha-\xi)d\xi\right]\right\}=\\&=\frac{e^{-t}}{\pi R_0h_0G\operatorname{ctg}\beta}[M_{x_0}(t)\sin\alpha-M_{y_0}(t)\cos\alpha]-\frac{R_0^2}{Gh_0\cos\beta}e^{2t}p_z.\end{aligned} \tag{15.155}$$

Multiplying (15.155) successively by $\Phi_{00}, \Phi_{01}, \Phi_{02}$, and also by Φ_{1sr}, Φ_{2sr} with arbitrary but fixed values of s and r, and integrating each time over α from 0 to 2π while taking into consideration the conditions of orthogonality and normalization (15.123)-(15.127), we obtain

$$\dot U_{00}=-\frac{R_0^2}{Eh_0\sin^2\beta\cos\beta\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}}e^{2t}\int_0^{2\pi}p_z(t,\zeta)d\zeta, \tag{15.156}$$

$$\dot U_{01}=\frac{R_0^2}{Eh_0\sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\left\{\frac{1}{R_0^3}e^{-t}M_{x_0}(t)-\frac{1}{\sin\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\sin\zeta d\zeta\right\}, \tag{15.157}$$

$$\dot U_{02}=-\frac{R_0^2}{Eh_0\sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\left\{\frac{1}{R_0^3}e^{-t}M_{y_0}(t)+\frac{1}{\sin\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\cos\zeta d\zeta\right\}, \tag{15.158}$$

. . . .

$$U_{1s0}^{\cdot}-\left(\frac{ns}{\sqrt{\gamma}\sin\beta}\right)^2 U_{1s0}=-\frac{R_0^2}{\sqrt{\pi}Eh_0\sin^2\beta\cos\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\sin sn\zeta\,d\zeta$$

$$(s=1,\ 2,\ldots), \tag{15.159}$$

. . . .

$$U_{2s0}^{\cdot}-\left(\frac{n\lambda_{2s0}^{*}}{\pi\sqrt{\gamma}\sin\beta}\right)^2 U_{2s0}=-\frac{R_0^2}{Eh_0\sin^2\beta\cos\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\Phi_{2s0}(\zeta)d\zeta$$

$$(s=1,\ 2,\ldots), \tag{15.160}$$

. . . .

$$U_{1s1}^{\cdot}-\left(\frac{n\lambda_{s1}^{*}}{\pi\sqrt{\gamma}\cos\beta}\right)^2 U_{1s1}=-\frac{R_0^2}{Eh_0\sin\beta\cos\beta}\left\{\frac{e^{-t}}{\sqrt{\pi}\bar{B}_s R_0^3}M_{x_0}(t)+\right.$$

$$\left.+\frac{1}{\sin\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\Phi_{1s1}(\zeta)d\zeta\right\}\quad(s=1,\ 2,\ldots), \tag{15.161}$$

. . . .

$$U_{2s1}^{\cdot}-\left(\frac{n\lambda_{s1}^{*}}{\pi\sqrt{\gamma}\sin\beta}\right)^2 U_{2s1}=\frac{R_0^2}{Eh_0\sin\beta\cos\beta}\left\{\frac{e^{-t}}{\sqrt{\pi}\bar{B}_s R_0^3}M_{y_0}(t)-\right.$$

$$\left.-\frac{1}{\sin\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\Phi_{2s1}(\zeta)d\zeta\right\}\quad(s=1,\ 2,\ldots), \tag{15.162}$$

. . . .

$$U_{1sr}^{\cdot}-\left(\frac{n\lambda_{sr}^{*}}{\pi\sqrt{\gamma}\sin\beta}\right)^2 U_{1sr}=-\frac{R_0^2}{Eh_0\sin^2\beta\cos\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\Phi_{1sr}(\zeta)d\zeta$$

$$\left[s=1,\ 2,\ldots;\qquad r=2,\ 3,\ldots,\ E\left(\frac{n-1}{2}\right)\right], \tag{15.163}$$

. . . .

$$U_{2sr}^{\cdot}-\left(\frac{n\lambda_{sr}^{*}}{\pi\sqrt{\gamma}\sin\beta}\right)^2 U_{2sr}=-\frac{R_0}{Eh_0\sin^2\beta\cos\beta}e^{2t}\int_0^{2\pi}p_z(t,\zeta)\Phi_{2sr}(\zeta)d\zeta,$$

$$\left[s=1,\ 2,\ldots;\qquad r=2,\ 3,\ldots,\ E\left(\frac{n-1}{2}\right)\right], \tag{15.164}$$

. . . .

$$U_{1s\frac{n}{2}}^{\cdot}-\left[\frac{n\left(s-\frac{1}{2}\right)}{\sqrt{\gamma}\sin\beta}\right]^2 U_{1s\frac{n}{2}}=-\frac{R_0^2}{\sqrt{\pi}Eh_0\sin^2\beta\cos\beta}e^{2t}\times$$

$$\times\int_0^{2\pi}p_z(t,\zeta)\sin\left(s-\frac{1}{2}\right)n\zeta\,d\zeta\quad(s=1,\ 2,\ldots), \tag{15.165}$$

. . . .

$$U''_{2s\frac{n}{2}}-\left(\frac{n\lambda^{*}_{2s\frac{n}{2}}}{\pi\sqrt{\gamma}\sin\beta}\right)^{2}U_{2s\frac{n}{2}}=-\frac{r_0^2}{Eh_0\sin^2\beta\cos\beta}e^{2t}\times$$
$$\times\int_0^{2\pi}p_s(t,\ \zeta)\Phi_{2s\frac{n}{2}}(\zeta)d\zeta \qquad (s=1,\ 2,\ldots), \tag{15.166}$$

. . . .

We have thus obtained an infinite system of differential equations in the desired generalized displacements of a stringer conical shell under an arbitrary external load. In contrast to the coupled infinite system (14.22)-(14.37), obtained by solving the problem in trigonometric series, system (15.156)-(15.166), corresponding to a series expansion in normal coordinate functions, decomposed completely into independent second-order differential equations. Equations (15.156)-(15.158) correspond to the law of plane sections. Equations (15.159)-(166) determine the generalized warpings of the stringer shell.

Introducing (15.54) into (15.20)-(15.22), and considering the orthogonality and normalization conditions (15.123)-(15.127), we find

$$\begin{aligned}
U'_{00}(t)&=-\frac{N_z(t)}{Eh_0\sin\beta\cos\beta\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}},\\
U'_{01}(t)&=-\frac{M_z(t)e^{-t}}{Eh_0\sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}},\\
U'_{02}(t)&=\frac{M_x(t)e^{-t}}{Eh_0\sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}.
\end{aligned} \tag{15.167}$$

As already noted, Eqs. (15.20)-(15.22) must be fulfilled identically if the solution of the problem satisfies the static boundary conditions. For the time being, we will consider these conditions only in relation to the functions U_{00}, U_{01}, U_{02}, corresponding to the law of plane sections, since the functions U_{1sr}, U_{2sr} corresponding to warpings do not enter into relations (15.167).

We transform the right-hand side of Eqs. (15.156)-(15.158) by using the table of p. 453, and expressions (p) of Part Three, and also (15.26), (15.27) and (15.33), (15.34). We obtain

$$U_{00}^{\cdot} = -\frac{1}{Eh_0 \sin\beta\cos\beta\sqrt{2\pi\left(1+\frac{1}{x}\right)}} N_z^{\cdot}(t), \tag{15.168}$$

$$U_{01}^{\cdot} = -\frac{1}{Eh_0R_0 \sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{x}\right)}} (M_x e^{-t})', \tag{15.169}$$

$$U_{02}^{\cdot} = \frac{1}{Eh_0R_0 \sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{x}\right)}} (M_y e^{-t})'. \tag{15.170}$$

Integrating Eqs. (15.168)-(15.170) and introducing the result into relations (15.167), we find the desired values of the integration constants. As a result, the functions

$$U_{00}(t) = U_{00}(0) - \frac{1}{Eh_0 \sin\beta\cos\beta\sqrt{2\pi\left(1+\frac{1}{x}\right)}} \int_0^t N_z(\zeta)d\zeta, \tag{15.171}$$

$$U_{01}(t) = U_{01}(0) - \frac{1}{Eh_0R_0 \sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{x}\right)}} \int_0^t M_x(\zeta)e^{-\zeta}d\zeta, \tag{15.172}$$

$$U_{02}(t) = U_{02}(0) + \frac{1}{Eh_0R_0 \sin\beta\cos\beta\sqrt{\pi\left(1+\frac{1}{x}\right)}} \int_0^t M_y(\zeta)e^{-\zeta}d\zeta \tag{15.173}$$

satisfy the static boundary conditions.

Equations (15.159)-(15.167), which determine the generalized warpings, are of the same type. Applying the method of variation of arbitrary constants, we find for the second-order equation of the type $U''-k^2U=F(t)$

$$U(t) = Ce^{kt} + \bar{C}e^{-kt} - \frac{1}{k}\int_0^t \operatorname{sh} k(\zeta - t)F(\zeta)d\zeta. \tag{15.174}$$

Now, using the notation

$$\frac{n t^{*}}{\pi \sqrt{\gamma} \sin \beta} = t, \tag{15.175}$$

we expand (15.154) with the aid of (15.171)-(15.173) and of the general solution of the form (15.174) of Eqs. (15.159)-(15.166)

$$\begin{aligned}
u_{m_z}(t, \alpha) = & \frac{U_{00}(0)}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}} - \int_0^t \bar{N}_z(\zeta)\, d\zeta + \\
& + \left[\frac{U_{01}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} - \int_0^t \bar{M}_x(\zeta)\, e^{-\zeta} d\zeta\right] \sin\alpha + \\
& + \left[\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} + \int_0^t \bar{M}_y(\zeta)\, e^{-\zeta} d\zeta\right] \cos\alpha + \\
& + \sum_{s=1}^{\infty} \sum_{r=0}^{E\left(\frac{s}{2}\right)} \Bigg\{\left[C_{1sr} e^{\lambda_{1sr} t} + \bar{C}_{1sr} e^{-\lambda_{1sr} t} + \int_0^t \operatorname{sh} \lambda_{1sr}(\zeta - t)\, \bar{F}_{1sr}(\zeta)\, d\zeta\right] \Phi_{1sr}(\alpha) + \\
& + \left[C_{2sr} e^{\lambda_{2sr} t} + \bar{C}_{2sr} e^{-\lambda_{2sr} t} + \int_0^t \operatorname{sh} \lambda_{2sr}(\zeta - t)\, \bar{F}_{2sr}(\zeta)\, d\zeta\right] \Phi_{2sr}(\alpha)\Bigg\} + \\
& + \sum_{s=1}^{\infty} \int_0^t \left[\Phi_{1s1}(\alpha)\, \tilde{M}^s_{x_0}(\zeta) - \Phi_{2s1}(\alpha)\, \tilde{M}^s_{y_0}(\zeta)\right] \operatorname{sh} \lambda_{s1}(\zeta - t)\, e^{-\zeta} d\zeta.
\end{aligned} \tag{15.176}$$

Here

$$\bar{N}_z(\zeta) = \frac{N_z(\zeta)}{2\pi\left(1+\frac{1}{\varkappa}\right) E h_0 \sin\beta \cos\beta},$$

$$\bar{M}_x(\zeta) = \frac{M_x(\zeta)}{\pi\left(1+\frac{1}{\varkappa}\right) E h_0 R_0 \sin\beta \cos\beta}, \tag{15.177}$$

$$\bar{M}_y(\zeta) = \frac{M_y(\zeta)}{\pi\left(1+\frac{1}{\varkappa}\right) E h_0 R_0 \sin\beta \cos\beta},$$

$$\tilde{M}^s_{x_0}(\zeta) = -\frac{\sqrt{\pi\gamma}\, M_{x_0}(\zeta)}{n \lambda^{*}_{s1} \bar{B}_s E h_0 R_0 \cos\beta},$$

$$\tilde{M}^s_{y_0}(\zeta) = \frac{\sqrt{\pi\gamma}\, M_{y_0}(\zeta)}{n \lambda^{*}_{s1} \bar{B}_s E h_0 R_0 \cos\beta}, \tag{15.178}$$

$$\bar{F}_{1sr}(\zeta) = \frac{\pi \sqrt{\gamma} R_0^2}{n \lambda^{*}_{s1} E h_0 \sin\beta \cos\beta} e^{2\zeta} \int_0^{2\pi} p_z(\zeta, \eta)\, \Phi_{1sr}(\eta)\, d\eta,$$

$$\bar{F}_{2sr}(\zeta) = \frac{\pi \sqrt{\gamma} R_0^2}{n \lambda^{*}_{s1} E h_0 \sin\beta \cos\beta} e^{2\zeta} \int_0^{2\pi} p_z(\zeta, \eta)\, \Phi_{2sr}(\eta)\, d\eta. \tag{15.179}$$

$$\lambda_{1sr} = \lambda_{2sr} = \lambda_{sr} \quad \left(r \neq 0, \frac{s}{2}\right). \tag{15.180}$$

Expression (15.176) represents the general solution of the fundamental integro-differential Eq. (15.153) for an arbitrary external load.

In special cases, this expression is simplified. Let us consider, for example, the case of a surface load such that the component $p_z = p_{m_z} \cos\beta - p_{n_n} \sin\beta$ entering into (15.179) can be represented in the form

$$p_z(t, \alpha) = p_z^0(t) + p_z^1(t) \sin\alpha + p_z^2(t) \cos\alpha. \tag{15.181}$$

Expression (15.181) determines the distribution of the external surface load acting in the direction of the axis of revolution of the shell, a distribution corresponding to the law of the plane in the cross section of the shell and arbitrary in the direction of its axis. Such a distribution of the surface load is of considerable practical interest.

For the case of (15.181), considering the representations of coordinate functions in the form of trigonometric series (15.130), and also the table on p. 453 and expressions (p) of Part Three, it is easy to obtain

$$\begin{gathered}
\overline{F}_{1sr}(\zeta) = \begin{cases} -\dfrac{R_{0_x}(\zeta)\sqrt{\pi\gamma}}{Eh_0R_0 \cos\beta n\lambda^{\bullet}_{s1}\overline{B}_s} & (r = 1), \\ 0 & (r \neq 1), \end{cases} \\
\overline{F}_{2sr}(\zeta) = \begin{cases} \dfrac{R_{0_y}(\zeta)\sqrt{\pi\gamma}}{Eh_0R_0 \cos\beta n\lambda^{\bullet}_{s1}\overline{B}_s} & (r = 1), \\ 0 & (r \neq 1). \end{cases}
\end{gathered} \tag{15.182}$$

Introducing (15.182) into (15.176) and using (15.33), (15.34) and expression (p) of Part Three, we find

$$u_{m_z}(t, \alpha) = \frac{U_{00}(0)}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}} - \int_0^t \bar{N}_z(\zeta)\,d\zeta +$$

$$+\left[\frac{U_{01}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} - \int_0^t \bar{M}_x(\zeta)e^{-\zeta}d\zeta\right]\sin\alpha +$$

$$+\left[\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} + \int_0^t \bar{M}_y(\zeta)e^{-\zeta}d\zeta\right]\cos\alpha +$$

$$+\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left\{\left[C_{1sr}e^{\lambda_{1sr}t} + \bar{C}_{1sr}e^{-\lambda_{1sr}t}\right]\Phi_{1sr}(\alpha) +\right.$$

$$\left. + \left[C_{2sr}e^{\lambda_{2sr}t} + \bar{C}_{2sr}e^{-\lambda_{2sr}t}\right]\Phi_{2sr}(\alpha)\right\} -$$

$$-\sum_{s=1}^{\infty}\int_0^t \operatorname{sh}\lambda_{s1}(\zeta - t)\,\frac{\partial}{\partial\zeta}\left[e^{-\zeta}\left(\Phi_{1s1}(\alpha)\tilde{M}_x^s(\zeta) - \Phi_{2s1}(\alpha)\tilde{M}_y^s(\zeta)\right)\right]d\zeta, \tag{15.183}$$

где

$$\tilde{M}_x^s(\zeta) = \frac{\sqrt{\pi\gamma}}{\lambda_{s1}^{*}\bar{B}_s E h_0 R_0 n \cos\beta} M_x(\zeta),$$

$$\tilde{M}_y^s(\zeta) = \frac{\sqrt{\pi\gamma}}{\lambda_{s1}^{*}\bar{B}_s E h_0 R_0 n \cos\beta} M_y(\zeta). \tag{15.184}$$

The tangential displacement $u_{n_z}(t, \alpha)$ is represented by expression (15.10), where $\eta_x(t)$, $\eta_y(t)$ are determined by expressions (15.38), and $\theta_z(t)$ satisfies differential Eq. (15.23). Introducing general solution (15.176) into (15.38) and considering (15.127), we find

$$\eta_x(t) = \eta_x^0 e^t + \frac{1}{\sin\beta}\int_0^t e^{t-\tau}\left\{\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} + \int_0^{\xi} \bar{M}_y(\zeta)e^{-\zeta}d\zeta -\right.$$

$$-\sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}\,\bar{B}_s}\left[C_{2s1}e^{\lambda_{s1}\xi} + \bar{C}_{2s1}e^{-\lambda_{s1}\xi} +\right.$$

$$\left.\left. + \int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta - \xi)\left(\bar{F}_{2s1}(\zeta) - \tilde{M}_{y_0}^s(\zeta)e^{-\zeta}\right)d\zeta\right]\right\}d\xi +$$

$$+ \frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{y_0}(\xi)e^{-2\xi}d\xi, \tag{15.185}$$

$$\eta_y(t) = \eta_y^0 e^t + \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Bigg\{\frac{U_{01}(0)}{\sqrt{\varkappa\left(1+\frac{1}{\varkappa}\right)}} - \int_0^\xi \bar{M}_x(\zeta)e^{-\zeta}d\zeta -$$

$$-\sum_{s=1}^{\infty}\frac{1}{\sqrt{\varkappa}\,\bar{B}_s}\Big[C_{1s1}e^{\lambda_{s1}\xi} + \bar{C}_{1s1}e^{-\lambda_{s1}\xi} +$$

$$+\int_0^\xi \operatorname{sh}\lambda_{s1}(\zeta-\xi)\,(\bar{F}_{1s1}(\zeta) + \tilde{M}_{x_0}^t(\zeta)e^{-\zeta})\,d\zeta\Big]\Bigg\}d\xi -$$

$$-\frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{x_0}(\xi)e^{-2\xi}d\xi. \qquad (15.186)$$

Expressions (15.185), (15.186) correspond to the case of an arbitrary external load. If the component of the external load p_z can be represented in the form (15.181), then, considering (15.33), (15.34), (15.182) and expression (p) of Part Three, expressions (15.185), (15.186) can be reduced to the form

$$\eta_x(t) = \eta_x^0 e^t + \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Bigg\{\frac{U_{02}(0)}{\sqrt{\varkappa\left(1+\frac{1}{\varkappa}\right)}} + \int_0^\xi \bar{M}_y(\zeta)e^{-\zeta}d\zeta -$$

$$-\sum_{s=1}^{\infty}\frac{1}{\sqrt{\varkappa}\,\bar{B}_s}\Big[C_{2s1}e^{\lambda_{s1}\xi} + \bar{C}_{2s1}e^{-\lambda_{s1}\xi} +$$

$$+\int_0^\xi \operatorname{sh}\lambda_{s1}(\zeta-\xi)\,\frac{d}{d\zeta}(\tilde{M}_y^t(\zeta)e^{-\zeta})\,d\zeta\Big]\Bigg\}d\xi +$$

$$+\frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{y_0}(\xi)e^{-2\xi}d\xi, \qquad (15.187)$$

$$\eta_y(t) = \eta_y^0 e^t + \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Bigg\{\frac{U_{01}(0)}{\sqrt{\varkappa\left(1+\frac{1}{\varkappa}\right)}} - \int_0^\xi \bar{M}_x(\zeta)e^{-\zeta}d\zeta -$$

$$-\sum_{s=1}^{\infty}\frac{1}{\sqrt{\varkappa}\,\bar{B}_s}\Big[C_{1s1}e^{\lambda_{s1}\xi} + \bar{C}_{1s1}e^{-\lambda_{s1}\xi} -$$

$$-\int_0^\xi \operatorname{sh}\lambda_{s1}(\zeta-\xi)\,\frac{d}{d\zeta}(\tilde{M}_x^t(\zeta)e^{-\zeta})\,d\zeta\Big]\Bigg\}d\xi -$$

$$-\frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{x_0}(\xi)e^{-2\xi}d\xi. \qquad (15.188)$$

Integrating Eq. (15.23), we also obtain

$$\theta_z(t) = \theta_z^0 - \frac{1}{2\pi R_0^2 G h_0 \sin\beta} \int_0^t M_z(\xi) e^{-2\xi}\, d\xi. \tag{15.189}$$

Expressions (15.176), (15.185), (15.186) and (15.189) completely determine the displacements of a conical shell with a longitudinal structure under an arbitrary external load. Using these expressions, we will write the general expressions for the stresses.

Expanding (15.7) with the aid of general expression (15.176), we find

$$\begin{aligned}
\sigma_{m_z}(t, \alpha) = E\varepsilon_{m_z}(t, \alpha) = {} & \frac{N_z(t) e^{-t}}{2\pi R_0 h_0 \left(1 + \frac{1}{\varkappa}\right)\cos\beta} + \\
& + \frac{M_x(t)\sin\alpha - M_y(t)\cos\alpha}{\pi R_0^2 h_0 \left(1 + \frac{1}{\varkappa}\right)\cos\beta} e^{-2t} - \\
& - \frac{E\sin\beta}{R_0} e^{-t} \sum_{s=1}^{\infty} \sum_{r=0}^{E\left(\frac{n}{2}\right)} \Big\{ \lambda_{1sr} \Big[C_{1sr} e^{\lambda_{1sr} t} - \overline{C}_{1sr} e^{-\lambda_{1sr} t} - \\
& - \int_0^t \operatorname{ch} \lambda_{1sr}(\zeta - t)\, \overline{F}_{1sr}(\zeta)\, d\zeta \Big] \Phi_{1sr}(\alpha) + \\
& + \lambda_{2sr} \Big[C_{2sr} e^{\lambda_{2sr} t} - \overline{C}_{2sr} e^{-\lambda_{2sr} t} - \int_0^t \operatorname{ch} \lambda_{2sr}(\zeta - t)\, \overline{F}_{2sr}(\zeta)\, d\zeta \Big] \Phi_{2sr}(\alpha) \Big\} + \\
& + \frac{E\sin\beta}{R_0} e^{-t} \sum_{s=1}^{\infty} \lambda_{s1} \int_0^t \big[\Phi_{1s1}(\alpha) \tilde{M}_x^s(\zeta) - \Phi_{2s1}(\alpha) \tilde{M}_y^s(\zeta) \big] \operatorname{ch} \lambda_{s1}(\zeta - t)\, e^{-\zeta}\, d\zeta.
\end{aligned} \tag{15.190}$$

If the component p_z of the external load can be represented in the form (15.181), then, using (15.183), we obtain

$$\begin{aligned}
\sigma_{m_z}(t, \alpha) = {} & \frac{N_z(t) e^{-t}}{\cos\beta\,(F_{\text{обш}} + F_{\text{стр}})} + \frac{M_x(t)\sin\alpha - M_y(t)\cos\alpha}{\frac{R_0}{2}\cos\beta\,(F_{\text{обш}} + F_{\text{стр}})} e^{-2t} - \\
& - \frac{E\sin\beta}{R_0} e^{-t} \sum_{s=1}^{\infty} \sum_{r=0}^{E\left(\frac{n}{2}\right)} \Big\{ \lambda_{1sr} \Big[C_{1sr} e^{\lambda_{1sr} t} - \overline{C}_{1sr} e^{-\lambda_{1sr} t} \Big] \Phi_{1sr}(\alpha) + \\
& + \lambda_{2sr} \Big[C_{2sr} e^{\lambda_{2sr} t} - \overline{C}_{2sr} e^{-\lambda_{2sr} t} \Big] \Phi_{2sr}(\alpha) \Big\} - \\
& - \frac{E\sin\beta}{R_0} e^{-t} \sum_{s=1}^{\infty} \lambda_{s1} \int_0^t \operatorname{ch} \lambda_{s1}(\zeta - t) \times \\
& \times \frac{\partial}{\partial\zeta} \Big[\big(\Phi_{1s1}(\alpha) \tilde{M}_x^s(\zeta) - \Phi_{2s1}(\alpha) \tilde{M}_y^s(\zeta) \big) e^{-\zeta} \Big] d\zeta.
\end{aligned} \tag{15.191}$$

For tangential stresses, expanding (15.10) with the aid of (15.185), (15.186), (15.189), then introducing the results as well as (15.176) into (15.8), we find

$$\tau_{\alpha_z m_z}(t,\alpha)=G\gamma_{\alpha_z m_z}(t,\alpha)=\frac{M_{y_0}(t)\sin\alpha+M_{x_0}(t)\cos\alpha}{\pi R_0^2h_0\operatorname{ctg}\beta}e^{-2t}+$$
$$+\frac{M_z(t)}{2\pi R_0^2h_0}e^{-2t}+\frac{G}{R_0}e^{-t}\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{s}{2}\right)}\Bigg\{\Bigg[C_{1sr}e^{\lambda_{1sr}t}+\overline{C}_{1sr}e^{-\lambda_{1sr}t}+$$
$$+\int_0^t\operatorname{sh}\lambda_{1sr}(\zeta-t)\,F_{1sr}(\zeta)\,d\zeta\Bigg]\Phi'_{1sr}(\alpha)+$$
$$+\Bigg[C_{2sr}e^{\lambda_{2sr}t}+\overline{C}_{2sr}e^{-\lambda_{2sr}t}+\int_0^t\operatorname{sh}\lambda_{2sr}(\zeta-t)\,F_{2sr}(\zeta)\,d\zeta\Bigg]\Phi'_{2sr}(\alpha)\Bigg\}+$$
$$+\frac{G}{R_0}e^{-t}\sum_{s=1}^{\infty}\Bigg\{\frac{1}{\sqrt{\pi}\,B_s}\Big[(C_{1s1}\cos\alpha-C_{2s1}\sin\alpha)e^{\lambda_{s1}t}+$$
$$+(\overline{C}_{1s1}\cos\alpha-\overline{C}_{2s1}\sin\alpha)e^{-\lambda_{s1}t}\Big]+$$
$$+\int_0^t\operatorname{sh}\lambda_{s1}(\zeta-t)\Bigg[\tilde{M}^s_{x_0}(\zeta)\left(\Phi'_{1s1}(\alpha)+\frac{\cos\alpha}{\sqrt{\pi}\,B_s}\right)-$$
$$-\tilde{M}^s_{y_0}(\zeta)\left(\Phi'_{2s1}(\alpha)-\frac{\sin\alpha}{\sqrt{\pi}\,B_s}\right)+$$
$$+\frac{e^{\zeta}}{\sqrt{\pi}\,B_s}(\overline{F}_{1s1}(\zeta)\cos\alpha-\overline{F}_{2s1}(\zeta)\sin\alpha)\Bigg]e^{-\zeta}d\zeta\Bigg\}. \quad (15.192)$$

Expression (15.192) corresponds to the case of an arbitrary external load. If the component p_z of the external load is representable in the form (15.181), then, using (15.183) instead of (15.176) and (15.187), (15.188) instead of (15.185), (15.186), we find

$$\tau_{\alpha_z m_z}(t,\alpha)=\frac{M_{y_0}(t)\sin\alpha+M_{x_0}(t)\cos\alpha}{\pi R_0^2h_0\operatorname{ctg}\beta}e^{-2t}+\frac{M_z(t)}{2\pi R_0^2h_0}e^{-2t}+$$
$$+\frac{G}{R_0}e^{-t}\sum_{s=1}^{\infty}\Bigg\{\sum_{r=0}^{E\left(\frac{s}{2}\right)}\Big[(C_{1sr}e^{\lambda_{1sr}t}+\overline{C}_{1sr}e^{-\lambda_{1sr}t})\Phi'_{1sr}(\alpha)+ \quad (15.193)$$
$$+(C_{2sr}e^{\lambda_{2sr}t}+\overline{C}_{2sr}e^{-\lambda_{2sr}t})\Phi'_{2sr}(\alpha)\Big]+$$

$$+\frac{1}{\sqrt{\varkappa}\bar{B}_s}\left[(C_{1s1}\cos\alpha - C_{2s1}\sin\alpha)e^{\lambda_{s1}t} + (\bar{C}_{1s1}\cos\alpha - \bar{C}_{2s1}\sin\alpha)e^{\bar{\lambda}_{s1}t}\right] -$$

$$-\int_0^t \operatorname{sh}\lambda_{s1}(\zeta - t)\,\frac{\partial}{\partial\varkappa}\left[e^{-\zeta}\tilde{M}^s_{x_0}(\zeta)\left(\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\varkappa}\bar{B}_s}\right) -\right.$$

$$\left.-e^{-\zeta}\tilde{M}^s_{y_0}(\zeta)\left(\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\varkappa}\bar{B}_s}\right)\right] d\zeta\Bigg\}. \qquad (15.193)$$

We have thus obtained general expressions determining the stressed and strained state of a conical shell with a longitudinal structure, subjected to an arbitrary external load. These expressions contain the arbitrary derivatives $C_{1sr}, \bar{C}_{1sr}, C_{2sr}, \bar{C}_{2sr}$ $[s=1, 2, \ldots, \infty;\ r=0, 1, 2, \ldots; E\left(\frac{n}{2}\right)]$, which are to be determined from the conditions of loading and fixing of the shell ends.

The first terms of the expressions obtained correspond to the law of plane sections, and the infinite sums determine the warpings of the cross section and corresponding self-balanced normal and tangential stresses.

The external surface load in the general expressions for displacements and stresses is represented only by the longitudinal components p_z. Therefore, the detailed character of the distribution of radial and tangential components in the cross section of the shell is unimportant.

If the longitudinal component is distributed in the shell cross section according to the law of the plane, general expressions (15.176), (15.185), (15.186), (15.190), 15.192) correspondingly assume the form (15.183), (15.187), (15.188), (15.191), (15.193). Let us note that in this case, in the expressions for displacements as well as those for stresses, the external load is represented only by integral characteristics - bending moments $M_x(t)$, $M_y(t)$ and moments $M_{x_0}(t)$, $M_{y_0}(t)$. The latter are moments of the external load applied to the cut-off portion of the shell with respect to axes x_n, y_0 passing through the cone apex, and are determined by expressions (15.33), (15.34).

15.4. Boundary Conditions. Determination of Arbitrary Constants

Let the shell end $t = t^*$ $(t^* = 0, t_1)$ be loaded by external normal forces $\bar{T}(\alpha)$ directed along the generatrices, and tangential forces $\bar{S}(\alpha)$ acting along the tangent to the contour $t = t^*$. Referring these forces to a unit length of the end section contour, from the equilibrium conditions of external and internal forces, we have for a smooth shell

$$\sigma_{m_z}(t^*, \alpha)\, h(t^*, \alpha) = \bar{T}(\alpha),$$
$$\tau_{n_z m_z}(t^*, \alpha)\, h(t^*, \alpha) = \bar{S}(\alpha).$$

Extending these relations to the case of a shell of constant thickness with a longitudinal structure, whose section changes linearly along the length according to (15.42), we have

$$\sigma_{m_z}(t^*, \alpha)\, \rho(\alpha)\, h_0 = \bar{T}(\alpha), \tag{15.194}$$
$$\tau_{n_z m_z}(t^*, \alpha)\, h_0 = \bar{S}(\alpha), \tag{15.195}$$

where q(α) according to (15.55) is the ratio of equivalent thicknesses corresponding to elements subjected to normal and tangential stresses:

$$\rho(\alpha) = \frac{h_1(t^*, \alpha)}{h_0} = 1 + \frac{\Delta F_0}{R_0 h_0} \sum_{m=1}^{n} \delta(\alpha - \alpha_m).$$

Relation (15.194) containing delta functions should be understood as formal notation denoting the equality of the definite integrals of left- and right-hand sides over α within any limits. Relation (15.195) has the usual meaning. Let us recall that within the framework of the adopted model, while moving in its plane as a solid, the cross section of the shell has only three degrees of freedom in this plane. Therefore, condtiion (15.195) can be fulfilled only integrally in the form of equality of transverse forces Q_x, Q_y and twisting moment M_z to the corresponding forces and moment of the external load. It is evident that in the indicated sense, relation (15.195) is certain to be fulfilled regardless of the values of the arbitrary constants entering into the general expressions for tangential stresses, since these stresses identically satisfy the equilibrium conditions of the cut-off portion of the shell (15.12)-(15.17). Therefore, all arbitrary constants $C_{1sr}, \bar{C}_{1sr}, C_{2sr}, \bar{C}_{2sr}$ are determined from the static or kinematic boundary conditions imposed on the longitudinal forces and displacements, respectively, and only three integration constants, $\eta_{t_z}^0, \eta_{t_y}^0, \theta_z^0$, are determined from the conditions of equality of displacements of the end section in its plane as a solid to the specified displacements.

Satisfying condition (15.194), we multiply its right- and left-hand sides successively by $\Phi_{00}, \Phi_{01}, \Phi_{02}$, and also by Φ_{1sr}, Φ_{2sr} for arbitrary but fixed values of s and r, and integrate each time over α from 0 to 2π. At the same time, expanding $\sigma_{m_z}(l^*, \alpha)$ with the aid of general expression (15.190) and considering the conditions of orthogonality and normalization (15.123), (15.125), we obtain

$$
\begin{aligned}
N_z(l^*) &= \int_0^{2\pi} \overline{T}(\alpha) \cos\beta e^{l^*} R_0 d\alpha, \\
M_x(l^*) &= \int_0^{2\pi} \overline{T}(\alpha) \sin\alpha \cos\beta e^{2l^*} R_0^2 d\alpha, \\
M_y(l^*) &= -\int_0^{2\pi} \overline{T}(\alpha) \cos\alpha \cos\beta e^{2l^*} R_0^2 d\alpha,
\end{aligned}
\tag{15.196}
$$

. . . .

$$C_{1sr}e^{\lambda_{1sr}t^*} - \overline{C}_{1sr}e^{-\lambda_{1sr}t^*} = \int_0^{t^*} \operatorname{ch} \lambda_{1sr}(\zeta - t^*)\, \overline{F}_{1sr}(\zeta)\,d\zeta -$$

$$- \frac{R_0}{Eh_0 \sin \beta\lambda_{1sr}} e^{t^*} \int_0^{2\pi} \overline{T}(\alpha)\, \Phi_{1sr}(\alpha)\,d\alpha,$$

$$C_{2sr}e^{\lambda_{2sr}t^*} - \overline{C}_{2sr}e^{-\lambda_{2sr}t^*} = \int_0^{t^*} \operatorname{ch} \lambda_{2sr}(\zeta - t^*)\, \overline{F}_{2sr}(\zeta)\,d\zeta - \tag{15.197}$$

$$- \frac{R_0}{Eh_0 \sin \beta\lambda_{2sr}} e^{t^*} \int_0^{2\pi} \overline{T}(\alpha)\, \Phi_{2sr}(\alpha)\,d\alpha,$$

$$\left[r = 0, 2, 3, \ldots, E\left(\frac{n}{2}\right)\right], \quad (s = 1, 2, \ldots),$$

$$\left(\lambda^*_{1sr} = \lambda^*_{2sr} = \lambda^*_{sr}, \quad \text{при} \quad r \neq 0, \frac{n}{2}\right),$$

. . . .

$$C_{1s1}e^{\lambda_{s1}t^*} - \overline{C}_{1s1}e^{-\lambda_{s1}t^*} = \int_0^{t^*} \operatorname{ch} \lambda_{s1}(\zeta - t^*)\, (\overline{F}_{1s1}(\zeta) +$$

$$+ e^{-\zeta}\widetilde{M}^s_{x_1}(\zeta))\,d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{s1}} e^{t^*} \int_0^{2\pi} \overline{T}(\alpha)\, \Phi_{1s1}(\alpha)\,d\alpha, \tag{15.198}$$

$$C_{2s1}e^{\lambda_{s1}t^*} - \overline{C}_{2s1}e^{-\lambda_{s1}t^*} = \int_0^{t^*} \operatorname{ch} \lambda_{s1}(\zeta - t^*)\, (\overline{F}_{2s1}(\zeta) -$$

$$- e^{-\zeta}\widetilde{M}^c_{y_1}(\zeta))\,d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{s1}} e^{t^*} \int_0^{2\pi} \overline{T}(\alpha)\, \Phi_{2s1}(\alpha)\,d\alpha$$

$$(s = 1, 2, \ldots).$$

Relations (15.196) are identical because generalized displacements (15.171)-(15.173) corresponding to the law of plane already satisfy the static boundary conditions.

Relations (15.197), (15.198) represent an infinite decomposed system of algebraic equations, each of which contains only two arbitrary constants: C and $\overline{C}$.

Thus, the static boundary conditions at any of the shell ends reduce to Eqs. (15.197), (15.198). Let us examine the kinematic boundary conditions.

Let the following displacements be specified on the end $t = t^*$ (t^* 0,t_1)

$$u_{m_z}(t^*, \alpha) = \bar{u}(\alpha), \tag{15.199}$$

$$\gamma_{1x}(t^*) = \bar{\gamma}_{1x}, \quad \gamma_{1y}(t^*) = \bar{\gamma}_{1y}, \quad \theta_z(t^*) = \bar{\theta}_z. \tag{15.200}$$

Satisfying relation (15.199), we multiply its right- and left-hand sides by $\varrho\Phi_{00}$, $\varrho\Phi_{01}$, $\varrho\Phi_{02}$, successively and also by $\varrho\Phi_{1sr}$, $\varrho\Phi_{2sr}$ for arbitrary but fixed values of s and r, and integrate each time over α from 0 to 2π. Then expanding $u_{m_z}(t^*, \alpha)$ with the aid of general expression (15.176) and considering the conditions of orthogonality and normalization (15.123), (15.125), we obtain

$$U_{00}(0) = \frac{1}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}}\left(\frac{1}{Eh_0 \sin\beta\cos\beta}\int_0^{t^*} N_z(t)\,dt + \int_0^{2\pi}\bar{u}(\alpha)\rho(\alpha)\,d\alpha\right),$$

$$U_{01}(0) = \frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\left(\frac{1}{Eh_0R_0 \sin\beta\cos\beta}\int_0^{t^*} M_x(t)e^{-t}\,dt + \right.$$

$$\left. + \int_0^{2\pi}\bar{u}(\alpha)\sin\alpha\,\rho(\alpha)\,d\alpha\right), \tag{15.201}$$

$$U_{02}(0) = \frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\left(-\frac{1}{Eh_0R_0 \sin\beta\cos\beta}\int_0^{t^*} M_y(t)e^{-t}\,dt + \right.$$

$$\left. + \int_0^{2\pi}\bar{u}(\alpha)\cos\alpha\,\rho(\alpha)\,d\alpha\right),$$

. . . .

$$C_{1sr}e^{\lambda_{1sr}t^*} + \bar{C}_{1sr}e^{-\lambda_{1sr}t^*} = -\int_0^{t^*} \operatorname{sh}\lambda_{1sr}(\zeta - t^*)\,\bar{F}_{1sr}(\zeta)\,d\zeta +$$

$$+ \int_0^{2\pi}\bar{u}(\alpha)\Phi_{1sr}(\alpha)\rho(\alpha)\,d\alpha, \tag{15.202}$$

$$C_{2sr}e^{\lambda_{2sr}t^*} + \bar{C}_{2sr}e^{-\lambda_{2sr}t^*} = -\int_0^{t^*} \operatorname{sh}\lambda_{2sr}(\zeta - t^*)\,\bar{F}_{2sr}(\alpha)\,d\zeta +$$

$$+ \int_0^{2\pi}\bar{u}(\alpha)\Phi_{2sr}(\alpha)\rho(\alpha)\,d\alpha,$$

$$\left[r = 0,\ 2,\ 3, \ldots,\ E\left(\frac{n}{2}\right)\right], \quad (s = 1,\ 2, \ldots),$$

$$\left(\lambda_{1sr} = \lambda_{2sr} = \lambda_{sr} \quad \text{при} \quad r \neq 0,\ \frac{n}{2}\right).$$

$$C_{1s1}e^{\lambda_{s1}t^*}+\overline{C}_{1s1}e^{-\lambda_{s1}t^*}=-\int_0^{t^*}\operatorname{sh}\lambda_{s1}(\zeta-t^*)\,[\overline{F}_{1s1}+e^{-\zeta}\widetilde{M}^{s}_{x_0}(\zeta)]\,d\zeta+$$

$$+\int_0^{2\pi}\overline{u}(\alpha)\,\Phi_{1s1}(\alpha)\,\rho(\alpha)\,d\alpha,$$

$$C_{2s1}e^{\lambda_{s1}t^*}+\overline{C}_{2s1}e^{-\lambda_{s1}t^*}=-\int_0^{t^*}\operatorname{sh}\lambda_{s1}(\zeta-t^*)\,[\overline{F}_{2s1}-e^{-\zeta}\widetilde{M}^{s}_{y_0}(\zeta)]\,d\zeta+ \tag{15.203}$$

$$+\int_0^{2\pi}\overline{u}(\alpha)\,\Phi_{2s1}(\alpha)\,\rho(\alpha)\,d\alpha$$

$$(s=1,\ 2,\ldots).$$

Expanding (15.200) with the aid of general expressions (15.185), (15.186), we also obtain

$$\gamma_x^0=\overline{\gamma}_{1x}e^{-t^*}-\frac{1}{\sin\beta}\int_0^{t^*}e^{-\xi}\left\{\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{x}\right)}}+\int_0^{\xi}\overline{M}_{y}(\zeta)\,e^{-\zeta}d\zeta-\right.$$

$$-\sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}\,\overline{B}_s}\left[C_{2s1}e^{\lambda_{s1}\xi}+\overline{C}_{2s1}e^{-\lambda_{s1}\xi}+\int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta-\xi)\,(\overline{F}_{2s1}(\zeta)-\right.$$

$$\left.\left.-\widetilde{M}^{s}_{y_0}(\zeta)\,e^{-\zeta})\,d\zeta\right]\right\}d\xi-\frac{1}{\pi R_0h_0G\cos\beta}\int_0^{t^*}M_{y_0}(\xi)\,e^{-2\xi}d\xi. \tag{15.204}$$

$$\gamma_y^0=\overline{\gamma}_{1y}e^{-t^*}-\frac{1}{\sin\beta}\int_0^{t^*}e^{-\xi}\left\{\frac{U_{01}(0)}{\sqrt{\pi\left(1+\frac{1}{x}\right)}}-\int_0^{\xi}\overline{M}_{x}(\zeta)\,e^{-\zeta}d\zeta-\right.$$

$$-\sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}\,\overline{B}_s}\left[C_{1s1}e^{\lambda_{s1}\xi}+\overline{C}_{1s1}e^{-\lambda_{s1}\xi}+\right.$$

$$\left.\left.+\int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta-\xi)\,(\overline{F}_{1s1}(\zeta)+\widetilde{M}^{s}_{x_0}(\zeta)\,e^{-\zeta})\,d\zeta\right]\right\}d\xi-$$

$$-\frac{1}{\pi R_0h_0G\cos\beta}\int_0^{t^*}M_{x_0}(\xi)\,e^{-2\xi}d\xi. \tag{15.205}$$

$$\vartheta_z^0=\overline{\vartheta}_z+\frac{1}{2\pi R_0^2h_0\sin\beta}\int_0^{t^*}M_z(t)\,e^{-2t}dt. \tag{15.206}$$

Thus, we have obtained equations corresponding to the kinematic boundary conditions specified on either of the shell ends. Expressions (15.201) give the initial value of the desired generalized displacements U_{00}, U_{01}, U_{02} corresponding according to (15.176) to displacement of the end t = 0 as a solid. Using the table on p. 453 we have

$$\frac{U_{00}(0)}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}}=\eta_{1z}^{0}\cos\beta,$$

$$\frac{U_{01}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}=-\eta_{1y}^{0}\sin\beta+\theta_{x}^{0}R_{0}\cos\beta, \tag{15.207}$$

$$\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}=-\eta_{1x}^{0}\sin\beta-\theta_{y}^{0}R_{0}\cos\beta.$$

Relations (15.202), (15.203) represent an infinite decomposed system of algebraic equations, each of which contains only two arbitrary constants: C and $\bar{C}$. Finally, expressions (15.204)-(15.206) give the values of the integration constants $\eta_{1x}^{0}, \eta_{1y}^{0}, \theta_{z}^{0}$ for previously found values of arbitrary constants C, $\bar{C}$ and initial values of U_{00}, U_{01}, U_{02}.

Thus, when the problem is solved in normal coordinate functions, the infinite algebraic system in arbitrary constants C, $\bar{C}$ decomposes completely into independent pairs of equations in $C_{1sr}, \bar{C}_{1sr}$ and $C_{2sr}, \bar{C}_{2sr}$, respectively, whereas by solving the problem in trigonometric series, we would arrive at a coupled infinite system of algebraic equations.

Having formulated the corresponding boundary conditions on both shell ends, one can easily calculate the values of all the arbitrary constants, then, introducing them into general expressions (15.76), (15, 185), (15.186), (15.190), 15.192), find the displacements and stresses at any point of the shell.

For not too short shells, it is desirable, on the basis of St. Venant's principle, to neglect the mutual influence of the ends. In this case, the boundary conditions on the distant end should be replaced by certain requirements placed on the behavior of the solution far from the zone under consideration. For example, in studying the stressed and strained state outside the zone of the end $t = t_1$, it is permissible, without satisfying in detail the boundary conditions on this end, to replace the truncated shell by a nontruncated one and to find the constants $\bar{C}$ from the conditions of boundedness of the displacement at the shell apex.

Let us turn to general expression (15.176) for the displacement u_{m_z}. After some transformations, we obtain

$$u_{m_z}(t, \alpha) = u^0_{m_z}(t, \alpha) + \sum_{s=1}^{\infty} \sum_{r=0,2,4,\ldots}^{E\left(\frac{s}{2}\right)} \left\{ \left[C_{1sr} - \frac{1}{2} \int_0^t F_{1sr}(\zeta) e^{-\lambda_{1sr}\zeta} d\zeta \right] \times \right.$$
$$\times \Phi_{1sr}(\alpha) e^{\lambda_{1sr}t} + \left[\bar{C}_{1sr} + \frac{1}{2} \int_0^t F_{1sr}(\zeta) e^{\lambda_{1sr}\zeta} d\zeta \right] \Phi_{1sr}(\alpha) e^{-\lambda_{1sr}t} +$$
$$+ \left[C_{2sr} - \frac{1}{2} \int_0^t F_{2sr}(\zeta) e^{-\lambda_{2sr}\zeta} d\zeta \right] \Phi_{2sr}(\alpha) e^{\lambda_{2sr}t} +$$
$$\left. + \left[\bar{C}_{2sr} + \frac{1}{2} \int_0^t F_{2sr}(\zeta) e^{\lambda_{2sr}\zeta} d\zeta \right] \Phi_{2sr}(\alpha) e^{-\lambda_{2sr}t} \right\} +$$
$$+ \sum_{s=1}^{\infty} \left\{ \left[C_{1s1} - \frac{1}{2} \int_0^t (F_{1s1}(\zeta) + M'_{zs}(\zeta) e^{-\zeta}) e^{-\lambda_{1s1}\zeta} d\zeta \right] \Phi_{1s1}(\alpha) e^{\lambda_{1s1}t} + \right.$$
$$+ \left[\bar{C}_{1s1} + \frac{1}{2} \int_0^t (F_{1s1}(\zeta) + M'_{zs}(\zeta) e^{-\zeta}) e^{\lambda_{1s1}\zeta} d\zeta \right] \Phi_{1s1}(\alpha) e^{-\lambda_{1s1}t} +$$
$$+ \left[C_{2s1} - \frac{1}{2} \int_0^t (F_{2s1}(\zeta) - M'_{zs}(\zeta) e^{-\zeta}) e^{-\lambda_{2s1}\zeta} d\zeta \right] \Phi_{2s1}(\alpha) e^{\lambda_{2s1}t} +$$
$$\left. + \left[\bar{C}_{2s1} + \frac{1}{2} \int_0^t (F_{2s1}(\zeta) - M'_{zs}(\zeta) e^{-\zeta}) e^{\lambda_{2s1}\zeta} d\zeta \right] \Phi_{2s1}(\alpha) e^{-\lambda_{2s1}t} \right\}. \quad (15.208)$$

where

$$u^0_{m_z}(t, \alpha) = \frac{U_{00}(0)}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}} - \int_0^t \bar{N}_z(\zeta)\,d\zeta +$$

$$+\left[\frac{U_{01}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} - \int_0^t \bar{M}_x(\zeta)e^{-\zeta}d\zeta\right]\sin\alpha +$$

$$+\left[\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} + \int_0^t \bar{M}_y(\zeta)e^{-\zeta}d\zeta\right]\cos\alpha \tag{15.209}$$

is the displacement corresponding to the law of plane sections.

Satisfying the condition of boundedness of the displacements at the shell apex $(t=-\infty)$, we equate to zero the coefficients on the indefinitely increasing terms of the form $e^{-\lambda t}$.

Hence we find

$$\bar{C}_{1sr} = \frac{1}{2}\int_{-\infty}^{0} F_{1sr}(\zeta)e^{\lambda_{sr}\zeta}d\zeta,$$

$$\bar{C}_{2sr} = \frac{1}{2}\int_{-\infty}^{0} F_{2sr}(\zeta)e^{\lambda_{sr}\zeta}d\zeta$$

$$\left[r=0, 2, 3, \ldots, E\left(\frac{n}{2}\right)\right], \quad (s=1, 2, \ldots). \tag{15.210}$$

$$\bar{C}_{1s1} = \frac{1}{2}\int_{-\infty}^{0} (F_{1s1}(\zeta) + \tilde{M}'_{x_0}(\zeta)e^{-\zeta})e^{\lambda_{s1}\zeta}d\zeta \quad (s=1, 2, \ldots),$$

$$\bar{C}_{2s1} = \frac{1}{2}\int_{-\infty}^{0} (F_{2s1}(\zeta) - \tilde{M}'_{y_0}(\zeta)e^{-\zeta})e^{\lambda_{s1}\zeta}d\zeta \quad (s=1, 2, \ldots). \tag{15.211}$$

Relations (15.210), (15.211) represent the values of arbitrary constants $\bar{C}$ corresponding to an external load of arbitrary form. If however the load satisfies condition (15.181), we similarly find from (15.183)

$$\bar{C}_{1sr} = \bar{C}_{2sr} = 0 \quad \left[r=0, 2, 3, \ldots, E\left(\frac{n}{2}\right); \quad s=1, 2, \ldots\right]. \tag{15.212}$$

$$\overline{C}_{1s1} = -\frac{1}{2}\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\widetilde{M}_x^s(\zeta)\right]d\zeta \quad (s=1, 2, \ldots),$$

$$\overline{C}_{2s1} = \frac{1}{2}\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\widetilde{M}_y^s(\zeta)\right]d\zeta \quad (s=1, 2, \ldots). \tag{15.213}$$

Introducing the values of arbitrary constants $\bar{C}$ into the relations corresponding to the kinematic boundary conditions for $t = 0$, and also into the relations corresponding to static boundary conditions when $t = 0$, we can find the values of arbitrary constants C for both cases.

Let a system of external forces be given on the end $t = 0$, and the external load on the shell be arbitrary. Then, using (15.197), (15.198) and (15.210), (15.211), we find

$$C_{1sr} = \frac{1}{2}\int_{-\infty}^{0} \mathcal{F}_{1sr}(\zeta)e^{\lambda_{1sr}\zeta}d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{1sr}}\int_{0}^{2\pi}\overline{T}(\alpha)\Phi_{1sr}(\alpha)\,d\alpha,$$

$$C_{2sr} = \frac{1}{2}\int_{-\infty}^{0} \mathcal{F}_{2sr}(\zeta)e^{\lambda_{2sr}\zeta}d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{2sr}}\int_{0}^{2\pi}\overline{T}(\alpha)\Phi_{2sr}(\alpha)\,d\alpha \tag{15.214}$$

$$\left[r=0,\ 2,\ 3,\ldots,\ E\left(\frac{n}{2}\right);\qquad s=1,\ 2,\ \ldots\right],$$

$$C_{1s1} = \frac{1}{2}\int_{-\infty}^{0} \left(\mathcal{F}_{1s1}(\zeta) + \widetilde{M}_{x_0}^s(\zeta)e^{-\zeta}\right)e^{\lambda_{s1}\zeta}d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{s1}}\int_{0}^{2\pi}\overline{T}(\alpha)\Phi_{1s1}(\alpha)\,d\alpha,$$

$$C_{2s1} = \frac{1}{2}\int_{-\infty}^{0} \left(\mathcal{F}_{2s1}(\zeta) - \widetilde{M}_{y_0}^s(\zeta)e^{-\zeta}\right)e^{\lambda_{s1}\zeta}d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{s1}}\int_{0}^{2\pi}\overline{T}(\alpha)\Phi_{2s1}(\alpha)\,d\alpha \tag{15.215}$$

$$(s=1,\ 2,\ldots).$$

If the external satisfies condition (15.181), then by using (15.212), (15.213) instead of (15.210), (15.211), we find from (15.196), (15.197)

$$C_{1sr} = -\frac{R_0}{Eh_0 \sin \beta\lambda_{1sr}}\int_{0}^{2\pi}\overline{T}(\alpha)\Phi_{1sr}(\alpha)\,d\alpha,$$

$$C_{2sr} = -\frac{R_0}{Eh_0 \sin \beta\lambda_{2sr}}\int_{0}^{2\pi}\overline{T}(\alpha)\Phi_{2sr}(\alpha)\,d\alpha \tag{15.216}$$

$$\left[r=0,\ 2,\ 3,\ldots,\ E\left(\frac{n}{2}\right);\quad s=1,\ 2,\ldots\right],$$

$$C_{1s1}=-\frac{1}{2}\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}_x^s(\zeta)\right]d\zeta-\frac{R_0}{Eh_0\sin\beta\lambda_{s1}}\int_0^{2\pi}\overline{T}(\alpha)\Phi_{1s1}(\alpha)d\alpha,$$

$$C_{2s1}=\frac{1}{2}\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}_y^s(\zeta)\right]d\zeta-\frac{R_0}{Eh_0\sin\beta\lambda_{s1}}\int_0^{2\pi}\overline{T}(\alpha)\Phi_{2s1}(\alpha)d\alpha \tag{15.217}$$

$$(s=1,\ 2,\ldots).$$

Let the displacements on the end t = 0 be specified. Then, in the case of an arbitrary external load, from (15.202), (15.203) in view of (15.210), (15.211) we find

$$C_{1sr}=-\frac{1}{2}\int_{-\infty}^{0}\overline{F}_{1sr}(\zeta)e^{\lambda_{1sr}\zeta}d\zeta+\int_0^{2\pi}\overline{u}(\alpha)\Phi_{1sr}(\alpha)\rho(\alpha)d\alpha,$$

$$C_{2sr}=-\frac{1}{2}\int_{-\infty}^{0}\overline{F}_{2sr}(\zeta)e^{\lambda_{2sr}\zeta}d\zeta+\int_0^{2\pi}\overline{u}(\alpha)\Phi_{2sr}(\alpha)\rho(\alpha)d\alpha \tag{15.218}$$

$$\left[r=0,\ 2,\ 3,\ldots,\ E\left(\frac{n}{2}\right);\quad s=1,\ 2,\ldots\right],$$

$$C_{1s1}=-\frac{1}{2}\int_{-\infty}^{0}\left(\overline{F}_{1s1}(\zeta)+\tilde{M}_{x_0}^s(\zeta)e^{-\zeta}\right)e^{\lambda_{s1}\zeta}d\zeta+\int_0^{2\pi}\overline{u}(\alpha)\Phi_{1s1}(\alpha)\rho(\alpha)d\alpha,$$

$$C_{2s1}=-\frac{1}{2}\int_{-\infty}^{0}\left(\overline{F}_{2s1}(\zeta)-\tilde{M}_{y_0}^s(\zeta)e^{-\zeta}\right)e^{\lambda_{s1}\zeta}d\zeta+\int_0^{2\pi}\overline{u}(\alpha)\Phi_{2s1}(\alpha)\rho(\alpha)d\alpha \tag{15.219}$$

$$(s=1,\ 2,\ldots).$$

If the external load satisfies condition (15.181), then, using (15.212), (15.213) instead of (15.210), (15.211), we find from (15.202), (15.203)

$$C_{1sr}=\int_0^{2\pi}\overline{u}(\alpha)\Phi_{1sr}(\alpha)\rho(\alpha)d\alpha,\quad C_{2sr}=\int_0^{2\pi}\overline{u}(\alpha)\Phi_{2sr}(\alpha)\rho(\alpha)d\alpha \tag{15.220}$$

$$\left[r=0,\ 2,\ 3,\ldots,\ E\left(\frac{n}{2}\right);\quad s=1,\ 2,\ldots\right],$$

$$C_{1s1} = \frac{1}{2} \int_{-\infty}^{0} e^{\dot{\lambda}_{s1}\zeta} \frac{d}{d\zeta} \left[e^{-\zeta} \dot{\tilde{M}}_{v}^{s}(\zeta) \right] d\zeta + \int_{0}^{2\pi} \bar{u}(\alpha) \Phi_{1s1}(\alpha) \rho(\alpha) \, d\alpha,$$

$$C_{2s1} = -\frac{1}{2} \int_{-\infty}^{0} e^{\dot{\lambda}_{s1}\zeta} \frac{d}{d\zeta} \left[e^{-\zeta} \tilde{M}_{v}^{s}(\zeta) \right] d\zeta + \int_{0}^{2\pi} \bar{u}(\alpha) \Phi_{2s1}(\alpha) \rho(\alpha) \, d\alpha$$

$$(s = 1, 2, \ldots). \qquad (15.221)$$

Thus, we have obtained the values of all the arbitrary constants C and $\bar{C}$ corresponding to both static and kinematic boundary conditions formulated for t = 0. For a nontruncated shell $(t_1 = -\infty)$, these values are exact. For a truncated shell (t_1 is finite), the values obtained for the arbitrary constants are approximate, but they can be used to study the stressed and strained state everywhere with the exception of the zone adjoining the end $t = t_1$. Moreover, for not too short shells, the error will be practically imperceptible.

In studying the zone adjoining the end $t = t_1$, we can neglect the influence on this zone of the boundary conditions on the end t = 0. The corresponding values of arbitrary constants for not too short shells can be found as was done above.

In analyzing short shells, for which the mutual influence of the ends is appreciable, the arbitrary constants $\dot{C}_{1sr}, \bar{C}_{1sr}$ and $\dot{C}_{2sr}, \bar{C}_{2sr}$ for fixed s and r should be determined by solving a system of two algebraic equations with two unknowns, which for specific boundary conditions can be obtained from (15.197), (15.198) and (15.202), (15.203). It is thus easy to establish the limits of applicability of approximate solutions corresponding to arbitrary constants found in the present section.

15.5. Analysis of a Stringer Shell for a Given Boundary Load.

We examined above a conical shell with longitudinal structure loaded by an arbitrary system of external forces. Both static and kinematic boundary conditions were formulated. In the present section, on the basis of the general relations obtained in Section 3 for not too short shells, we will examine certain important special cases of loading of a shell by different forms of a boundary load.

1. Cyclically symmetric normal load

The external load $T(\alpha)$ will be referred to as cyclically symmetric if its distribution between a pair of neighboring stringers is the same over the entire contour. In this case

$$\bar{T}\left(\alpha+\frac{2\pi}{n}p\right)=\bar{T}(\alpha)$$

for any α and integral values of p, whence, considering (14.138), we have

$$\bar{T}(\alpha)=\bar{T}(\xi). \tag{15.222}$$

In calculating the arbitrary constants, we will need, according to (15.214), (15.215), the Fourier coefficients of the external load $\bar{T}(\alpha)$ in normal coordinate functions $\Phi(\alpha)$. Considering (14.138) and (15.222), we have

$$\int_0^{2\pi}\bar{T}(\alpha)\Phi(\alpha)\,d\alpha=\sum_{p=0}^{n-1}\int_{\frac{2\pi}{n}p}^{\frac{2\pi}{n}(p+1)}\bar{T}(\alpha)\Phi(\alpha)\,d\alpha=$$
$$=\frac{2\pi}{n}\sum_{p=0}^{n-1}\int_0^1\bar{T}(\xi)\Phi(\xi,\,p)\,d\xi=\frac{2\pi}{n}\int_0^1\bar{T}(\xi)\sum_{p=0}^{n-1}\Phi(\xi,\,p)\,d\xi. \tag{15.223}$$

We first calculate the values of the following sums·

$$S_{1r}=\sum_{p=0}^{n-1}\sin\frac{2\pi r}{n}p,\quad S_{2r}=\sum_{p=0}^{n-1}\cos\frac{2\pi r}{n}p,$$

$$\left[r=1,\ 2,\dots,\ E\left(\frac{n-1}{2}\right)\right]. \tag{15.224}$$

On the basis of (14.7), (14.8)

$$S_{1r}=S_{2r}=0. \tag{15.225}$$

Using (15.225), we also find

$$S_{3r}=\sum_{p=0}^{n-1}\sin\frac{2\pi r}{n}(p+1)=0, \tag{15.226}$$

$$S_{4r}=\sum_{p=0}^{n-1}\cos\frac{2\pi r}{n}(p+1)=0.$$

Now, for the normal coordinate functions represented by expressions (15.108)-(15.112) and using (15.225), (15.226), we obtain

$$\sum_{p=0}^{n-1}\Phi_{01}(\xi,\ p)=\sum_{p=0}^{n-1}\Phi_{02}(\xi,\ p)=\sum_{p=0}^{n-1}\Phi_{1s1}(\xi,\ p)=\sum_{p=0}^{n-1}\Phi_{2s1}(\xi,\ p)=$$

$$=\dots=\sum_{p=0}^{n-1}\Phi_{1sr}(\xi,\ p)=\sum_{p=0}^{n-1}\Phi_{2sr}(\xi,\ p)=\dots=\sum_{p=0}^{n-1}\Phi_{1s\frac{n}{2}}(\xi,\ p)=$$

$$=\sum_{p=0}^{n-1}\Phi_{2s\frac{n}{2}}(\xi,\ p)=0\quad\left[r=2,\ 3,\dots,\ E\left(\frac{n-1}{2}\right)\right]. \tag{15.227}$$

$$\sum_{p=0}^{n-1}\Phi_{00}=\frac{n}{\sqrt{2\pi\left(1+\frac{1}{\pi}\right)}}. \tag{15.228}$$

$$\sum_{p=0}^{n-1}\Phi_{1s0}(\xi)=\frac{n}{\sqrt{\pi}}\sin 2\lambda_{2s0}^{*}\xi, \tag{15.229}$$

$$\sum_{p=0}^{n-1}\Phi_{2s0}(\xi)=-\frac{n}{B_{s0}\sqrt{2}}\,\frac{\cos 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)}{\cos\lambda_{2s0}^{*}}. \tag{15.230}$$

Expanding (15.223) with the aid of (15.227)-(15.230), we find the values of arbitrary constants (15.214), (15.215) and (15.210), (15.211). Since for the case under consideration, the external surface load is absent, and the end cyclic load is directed along the generatrices, the moments $\bar{M}^0_{z_s}(\xi)$, $\bar{M}^0_{y_s}(\xi)$, entering into (15.215) are absent. Taking this into account, we have

$$\begin{gathered}
\bar{C}_{1sr}=\bar{C}_{2sr}=0\quad\left[r=0,\ 1,\ 2,\ldots,\ E\left(\frac{n}{2}\right)\right],\\
C_{1sr}=C_{2sr}=0\quad\left[r=1,\ 2,\ldots,\ E\left(\frac{n}{2}\right)\right],\\
C_{1s0}=-\frac{2\pi^2R_0\sqrt{\frac{\ }{1}}}{Eh_0n\lambda^{\circ}_{1s0}}\int_0^1\bar{T}(\xi)\,\Phi_{1s0}(\xi)\,d\xi,\\
C_{2s0}=-\frac{2\pi^2R_0\sqrt{\frac{\ }{1}}}{Eh_0n\lambda^{\circ}_{2s0}}\int_0^1\bar{T}(\xi)\,\Phi_{2s0}(\xi)\,d\xi\\
(s=1,\ 2,\ldots).
\end{gathered}\tag{15.231}$$

Introducing the values of arbitrary constants (15.231) into (15.176) and (15.190), (15.192), we write the expressions for displacements u_{m_z} and stresses σ_{m_z}, $\tau_{\alpha_z m_z}$ for the case under consideration.

We have

$$\begin{gathered}
u_{m_z}(t,\ \alpha)=u^0_{m_z}(0,\ \alpha)-\frac{N_z t}{2\pi Eh_0\sin\beta\cos\beta\left(1+\frac{1}{\varkappa}\right)}-\\
-\frac{2\pi^2R_0\sqrt{\frac{\ }{1}}}{Eh_0n}\sum_{s=1}^{\infty}\left[e^{\lambda_{1s0}t}\Phi_{1s0}(\alpha)\int_0^1\bar{T}(\xi)\,\Phi_{1s0}(\xi)\,d\xi+\right.\\
\left.+e^{\lambda_{2s0}t}\Phi_{2s0}(\alpha)\int_0^1\bar{T}(\xi)\,\Phi_{2s0}(\xi)\,d\xi\right].
\end{gathered}\tag{15.232}$$

Here

$$\begin{gathered}
u^0_{m_z}(0,\ \alpha)=\frac{U_{00}(0)}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}}+\frac{U_{01}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\sin\alpha+\\
+\frac{U_{02}(0)}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\cos\alpha
\end{gathered}\tag{15.233}$$

is the displacement of the end t = 0, corresponding to the law of plane sections. On the basis of (15.207)

$$\begin{gathered}
u^0_{m_z}(0,\ \alpha)=\gamma^0_{z}\cos\beta+(-\gamma^0_{y}\sin\beta+\theta^0_{z}R_0\cos\beta)\sin\alpha-\\
-(\gamma^0_{x}\sin\beta+\theta^0_{y}R_0\cos\beta)\cos\alpha.
\end{gathered}\tag{15.234}$$

For normal and tangential stresses, considering that the resultant of the end cyclic load is directed along the Oz axis, we obtain

$$\sigma_{m_z}(t,\alpha)=\frac{N_z e^{t}}{2\pi R_0 h_0\cos\beta\left(1+\frac{1}{\varkappa}\right)}-\frac{2\pi}{h_0}e^{-t}\sum_{s=1}^{\infty}\left[e^{\lambda_{1s0}t}\Phi_{1s0}(\alpha)\int_0^1\bar{T}(\xi)\Phi_{1s0}(\xi)d\xi+\right.$$

$$\left.+e^{\lambda_{2s0}t}\Phi_{2s0}(\alpha)\int_0^1\bar{T}(\xi)\Phi_{2s0}(\xi)d\xi\right], \qquad (15.235)$$

$$\tau_{s_z m_z}(t,\alpha)=-\frac{2\pi^2}{n\sqrt{\bar{\gamma}}h_0}e^{-t}\sum_{s=1}^{\infty}\left[\frac{1}{\lambda^{\circ}_{1s0}}e^{\lambda_{1s0}t}\Phi'_{1s0}(\alpha)\int_0^1\bar{T}(\xi)\Phi_{1s0}(\xi)d\xi+\right.$$

$$\left.+\frac{1}{\lambda^{\circ}_{2s0}}e^{\lambda_{2s0}t}\Phi'_{2s0}(\alpha)\int_0^1\bar{T}(\xi)\Phi_{2s0}(\xi)d\xi\right]. \qquad (15.236)$$

Introducing expressions (15.109) into (15.232) and (15.235), (15.236), we rewrite them in expanded form.

We obtain

$$u_{m_z}(t,\alpha)=u^0_{m_z}(0,\alpha)-\frac{N_z t}{2\pi E h_0\sin\beta\cos\beta\left(1+\frac{1}{\varkappa}\right)}-$$

$$-\frac{2\pi R_0\sqrt{\bar{\gamma}}}{E h_0 n}\sum_{s=1}^{\infty}\left\{\frac{1}{\pi s}e^{\frac{\pi s}{\sqrt{\bar{\gamma}}\sin\beta}t}\sin 2\pi s\xi\int_0^1\bar{T}(\xi)\sin 2\pi s\xi\,d\xi+\right.$$

$$\left.+\frac{e^{\lambda_{2s0}t}}{\lambda^{\circ}_{2s0}\left(1+\frac{\cos^2\lambda^{\circ}_{2s0}}{\varkappa}\right)}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right)\int_0^1\bar{T}(\xi)\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right)d\xi\right\}, \qquad (15.237)$$

$$\sigma_{m_z}(t,\alpha)=\frac{N_z e^{-t}}{2\pi R_0 h_0\cos\beta\left(1+\frac{1}{\varkappa}\right)}+$$

$$+\frac{2e^{-t}}{h_0}\sum_{s=1}^{\infty}\left\{\int_0^1\bar{T}(\xi)\sin 2\pi s\xi\,d\xi\, e^{\frac{\pi s}{\sqrt{\bar{\gamma}}\sin\beta}t}\sin 2\pi s\xi+\right.$$

$$\left.+\frac{\int_0^1\bar{T}(\xi)\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right)d\xi}{\left(1+\frac{\cos^2\lambda^{\circ}_{2s0}}{\varkappa}\right)}e^{\lambda_{2s0}t}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right)\right\}. \qquad (15.238)$$

$$\tau_{s_z m_z}(t,\alpha)=-\frac{2e^{-t}}{\sqrt{\bar{\gamma}}h_0}\sum_{s=1}^{\infty}\left\{\int_0^1\bar{T}(\xi)\sin 2\pi s\xi\,d\xi\, e^{\frac{\pi s}{\sqrt{\bar{\gamma}}\sin\beta}t}\cos 2\pi s\xi-\right.$$

$$\left.-\frac{\int_0^1\bar{T}(\xi)\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right)d\xi}{1+\frac{\cos^2\lambda^{\circ}_{2s0}}{\varkappa}}e^{\lambda_{2s0}t}\sin 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right)\right\}. \qquad (15.239)$$

In expressions (15.237), (15.238), from the equilibrium conditions of the cut-off portion of the shell as a solid, the axial force

$$N_z = 2\pi R_0 \cos\beta \int_0^1 \overline{T}(\xi)\, d\xi. \tag{15.240}$$

The parameter λ^{*}_{2s0} is the root of the transcendental characteristic equation

$$\lambda^{*} \operatorname{ctg} \lambda^{*} = -\varkappa = -\frac{F_{обш}}{F_{стр}} = -\frac{2\pi R_0 h_0}{n\Delta F_0}.$$

Relations (15.237)-(15.239) can be easily extended to the case of a cylindrical shell by passing to the limit when $\beta \to 0$.

Introducing the relative coordinate $\overline{Z} = \frac{Z}{R} = (1 - e^{t}) \operatorname{ctg} \beta$, we have when $\beta \to 0$:

$$\begin{gathered} e^{t} = 1 - \overline{Z}\beta, \\ e^{-t} = 1 + \overline{Z}\beta + (\overline{Z}\beta)^2 + (\overline{Z}\beta)^3 + \ldots, \\ t = -\overline{Z}\beta - \frac{(\overline{Z}\beta)^2}{2} - \frac{(\overline{Z}\beta)^3}{3} - \ldots \end{gathered} \tag{15.241}$$

Considering (15.241), we find

$$\begin{aligned} u_{m_z}(\overline{Z}, \alpha) = u^{0}_{m_z}(0, \alpha) + \frac{N_z \overline{Z}}{2\pi E h_0 \left(1 + \frac{1}{\varkappa}\right)} - \\ - \frac{2R_0 \sqrt{\overline{1}}}{E h_0 \pi} \sum_{s=1}^{\infty} \left\{ \frac{1}{s} \int_0^1 \overline{T}(\xi) \sin 2\pi s\xi\, d\xi\; e^{-\frac{\pi s}{\sqrt{\overline{1}}} \overline{Z}} \sin 2\pi s \xi + \right. \\ \left. + \frac{\pi \int_0^1 \overline{T}(\xi) \cos 2\lambda^{*}_{2s0} \left(\xi - \frac{1}{2}\right) d\xi}{\lambda^{*}_{2s0} \left(1 + \frac{\cos^2 \lambda^{*}_{2s0}}{\varkappa}\right)} e^{-\lambda_{2s0} \overline{Z}} \cos 2\lambda^{*}_{2s0} \left(\xi - \frac{1}{2}\right) \right\}, \end{aligned} \tag{15.242}$$

$$\sigma_{m_z}(\overline{Z}, \alpha) = \frac{N_z}{2\pi R_0 h_0 \left(1 + \frac{1}{\varkappa}\right)} +$$

$$+\frac{2}{h_0}\sum_{s=1}^{\infty}\left\{\int_0^1 \bar{T}(\xi)\sin 2\pi s\xi\,d\xi\, e^{-\frac{\pi s}{\sqrt{\gamma}}\bar{Z}}\sin 2\pi s\xi+\right.$$

$$\left.+\frac{\int_0^1 \bar{T}(\xi)\cos 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)d\xi}{1+\frac{\cos^2\lambda_{2s0}^{*}}{\varkappa}}e^{-\lambda_{2s0}\bar{Z}}\cos 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)\right\}, \tag{15.243}$$

$$\tau_{n_2 m_2}(\bar{Z},\alpha)=-\frac{2}{\sqrt{\gamma}\,h_0}\sum_{s=1}^{\infty}\left\{\int_0^1 \bar{T}(\xi)\sin 2\pi s\xi\,d\xi\, e^{-\frac{\pi s}{\sqrt{\gamma}}\bar{Z}}\cos 2\pi s\xi-\right.$$

$$\left.-\frac{\int_0^1 \bar{T}(\xi)\cos 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)d\xi}{1+\frac{\cos^2\lambda_{2s0}^{*}}{\varkappa}}e^{-\lambda_{2s0}\bar{Z}}\sin 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)\right\}. \tag{15.244}$$

Here and below, for a cylindrical shell

$$\lambda=\frac{n}{\pi\sqrt{\gamma}}\lambda^{*},$$

for a conical shell, as before,

$$\lambda=\frac{n}{\pi\sqrt{\gamma}\sin\beta}\lambda^{*}.$$

2. Cyclic System of Concentrated Forces Directed Along the Generatrices

Let the external load in the section t = 0 be represented by a system of arbitrary concentrated forces $P_m (m=1, 2, \ldots, n)$, applied to the stringers in the direction of the generatrices. The corresponding linear load $\bar{T}(\alpha)$ referred to a unit length of the cross sectional contour t = 0 will be represented in the form

$$\bar{T}(\alpha)=\frac{1}{R_0}\sum_{m=1}^{n}P_m\delta(\alpha-\alpha_m), \tag{15.245}$$

(15.245) being understood as formal notation denoting the equality of the definite integrals of the left- and right-hand sides over α within any limits.

The Fourier coefficients of external load (15.245) in normal coordinate functions Φ(α) will be

$$\int_0^{2\pi} \bar{T}(\alpha)\Phi(\alpha)\,d\alpha = \frac{1}{R_0}\sum_{m=1}^{n} P_m \Phi\left(\frac{2\pi}{n} m\right). \tag{15.246}$$

Setting in (15.108)-(15.112)

$$p \quad m, \quad 0.$$

we find

$$\begin{aligned}
&\Phi_{00}\left(\frac{2\pi}{n} m\right) = \frac{1}{\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}},\\
&\Phi_{01}\left(\frac{2\pi}{n} m\right) = \frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} \sin\frac{2\pi}{n} m,\\
&\Phi_{02}\left(\frac{2\pi}{n} m\right) = \frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}} \cos\frac{2\pi}{n} m,\\
&\Phi_{1s0}\left(\frac{2\pi}{n} m\right) = 0,\\
&\Phi_{2s0}\left(\frac{2\pi}{n} m\right) = -\frac{1}{B_{s0}\sqrt{2}},\\
&\Phi_{1s1}\left(\frac{2\pi}{n} m\right) = \frac{\varkappa}{\sqrt{\pi}\,B_s} \sin\frac{2\pi}{n} m,\\
&\Phi_{2s1}\left(\frac{2\pi}{n} m\right) = \frac{\varkappa}{\sqrt{\pi}\,B_s} \cos\frac{2\pi}{n} m,\\
&\Phi_{1sr}\left(\frac{2\pi}{n} m\right) = \frac{1}{B_{sr}} \sin\frac{2\pi r}{n} m \qquad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right],\\
&\Phi_{2sr}\left(\frac{2\pi}{n} m\right) = \frac{1}{B_{sr}} \cos\frac{2\pi r}{n} m \qquad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right],\\
&\Phi_{1s\frac{n}{2}}\left(\frac{2\pi}{n} m\right) = 0,\\
&\Phi_{2s\frac{n}{2}}\left(\frac{2\pi}{n} m\right) = \frac{1}{\sqrt{2}\,B_{s\frac{n}{2}}}(-1)^m.
\end{aligned} \tag{15.247}$$

Let us represent the system of concentrated forces P_m in the form of interpolation polynomials (13.78), (13.79):

for an odd number n of stringers

$$P_m = A_0 + \sum_{r=1}^{\frac{n-1}{2}} \left(A_r \cos\frac{2\pi m}{n} r + B_r \sin\frac{2\pi m}{n} r\right), \tag{15.248}$$

for an even number n

$$P_m = A_0 + \sum_{r=1}^{\frac{n}{2}-1} \left(A_r \cos\frac{2\pi m}{n} r + B_r \sin\frac{2\pi m}{n} r\right) + (-1)^m A_{\frac{n}{2}}.$$

Then, expanding (15.246) with the aid of (15.247) and considering (13.80), we find

$$
\begin{aligned}
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{00}(\alpha)\,d\alpha = \frac{n}{R_0\sqrt{2\pi\left(1+\frac{1}{\varkappa}\right)}}\,A_0,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{01}(\alpha)\,d\alpha = \frac{n}{2R_0\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\,B_1,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{02}(\alpha)\,d\alpha = \frac{n}{2R_0\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\,A_1,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{1s0}(\alpha)\,d\alpha = 0,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{2s0}(\alpha)\,d\alpha = -\frac{n}{\sqrt{2}R_0B_{s0}}\,A_0,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{1s1}(\alpha)\,d\alpha = \frac{n\varkappa}{2\sqrt{\pi}R_0\bar{B}_s}\,B_1,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{2s1}(\alpha)\,d\alpha = \frac{n\varkappa}{2\sqrt{\pi}R_0\bar{B}_s}\,A_1,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{1sr}(\alpha)\,d\alpha = \frac{n}{2R_0B_{sr}}\,B_r\quad \left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right],\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{2sr}(\alpha)\,d\alpha = \frac{n}{2R_0B_{sr}}\,A_r\quad \left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right],\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{1s\frac{n}{2}}(\alpha)\,d\alpha = 0,\\
&\int_0^{2\pi} \bar{T}(\alpha)\Phi_{2s\frac{n}{2}}(\alpha)\,d\alpha = \frac{n}{\sqrt{2}R_0B_{s\frac{n}{2}}}\,A_{\frac{n}{2}}.
\end{aligned}
\tag{15.249}
$$

Now, introducing Fourier coefficients (15.249) into expressions (15.214), (15.215) and using the coefficients of interpolation polynomial (15.248), we can find the values of arbitrary constants for any distribution of concentrated forces P_m. Let us consider certain special cases.

A. Constant forces

Let the forces $P_m = P =$ const.

Then the coefficients of interpolation polynomial (15.248) will be according to (13.80):

$$A_0 = P,$$
$$A_r = B_r = 0 \quad \left[r = 1, 2, \ldots, E\left(\frac{n}{2}\right)\right]. \tag{15.250}$$

Introducing (15.250) into (15.249), we then find the values of arbitrary constants (15.214), (15.215) and (15.210), (15.211). In the absence of an external surface load, considering that in the case at hand the moments $\mathfrak{M}^{*}_{z_r}(\zeta)$, $\mathfrak{M}^{*}_{y_r}(\zeta)$ are also absent, we find

$$\bar{C}_{1sr} = \bar{C}_{2sr} = 0 \quad \left[r = 0, 1, 2, \ldots, E\left(\frac{n}{2}\right);\ s = 1, 2, \ldots\right], \tag{15.251}$$
$$C_{1sr} = C_{2sr} = 0 \quad \left[r = 1, 2, \ldots, E\left(\frac{n}{2}\right);\ s = 1, 2, \ldots\right],$$
$$C_{1s0} = 0 \qquad (s = 1, 2, \ldots),$$
$$C_{2s0} = \frac{\pi\sqrt{\gamma}}{\sqrt{2Eh_0}\,\lambda^{*}_{2s_0}B_{s0}}P \quad (s = 1, 2, \ldots).$$

Introducing the values of arbitrary constants (15.251) into (15.176) and (15.190), (15.192), we write the expressions for displacements u_{m_z} and σ_{m_z}, $\tau_{\alpha_z m_z}$. We have

$$u_{m_z}(t, \alpha) = u^{0}_{m_z}(0, \alpha) - \frac{N_z t}{2\pi E h_0 \sin\beta\cos\beta\left(1 + \frac{1}{\kappa}\right)} - \frac{\pi\sqrt{\gamma}\,P}{\sqrt{2Eh_0}}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{\lambda^{*}_{2s0}B_{s0}}\Phi_{2s0}(\alpha), \tag{15.252}$$

$$\sigma_{m_z}(t, \alpha) = \frac{N_z e^{-t}}{2\pi R_0 h_0 \cos\beta\left(1 + \frac{1}{\kappa}\right)} - \frac{\pi P}{R_0 h_0}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{B_{s0}}\Phi_{2s0}(\alpha), \tag{15.253}$$

$$\tau_{\alpha_z m_z}(t, \alpha) = \frac{\pi P}{\sqrt{2h_0}\sqrt{\gamma}}e^{-t}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{\lambda^{*}_{2s0}B_{s0}}\Phi'_{2s0}(\alpha). \tag{15.254}$$

where from the equilibrium conditions

$$N_z = Pn\cos\beta.$$

Introducing into (15.252)-(15.254) expression (15.109) for the function Φ_{3s0}, we have

$$u_{m_z}(t,\alpha) = u^0_{m_z}(0,\alpha) - \frac{N_z t}{2\pi E h_0 \sin\beta\cos\beta\left(1+\frac{1}{\varkappa}\right)} -$$

$$-\frac{\sqrt{\gamma}P}{Eh_0}\sum_{s=1}^{\infty}\frac{1}{\lambda^{\circ}_{2s0}\left(\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}\right)}e^{\lambda_{2s0}t}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right), \tag{15.255}$$

$$\sigma_{m_z}(t,\alpha) = \frac{N_z e^{-t}}{2\pi R_0 h_0\left(1+\frac{1}{\varkappa}\right)\cos\beta} +$$

$$+\frac{Pn}{\pi R_0 h_0}e^{-t}\sum_{s=1}^{\infty}\frac{1}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}e^{\lambda_{2s0}t}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right), \tag{15.256}$$

$$\tau_{n_z m_z}(t,\alpha) = \frac{Pn}{\sqrt{\gamma}\pi R_0 h_0}e^{-t}\sum_{s=1}^{\infty}\frac{1}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}\times$$

$$\times e^{\lambda_{2s0}t}\sin 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right). \tag{15.257}$$

For a cylindrical shell, using (15.241), we find

$$u_{m_z}(Z,\alpha) = u^0_{m_z}(0,\alpha) + \frac{PnZ}{2\pi h_0 E\left(1+\frac{1}{\varkappa}\right)} -$$

$$-\frac{\sqrt{\gamma}P}{Eh_0}\sum_{s=1}^{\infty}\frac{1}{\lambda^{\circ}_{2s0}\left(\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}\right)}e^{-\lambda_{2s0}Z}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right), \tag{15.258}$$

$$\sigma_{m_z}(Z,\alpha) = \frac{Pn}{2\pi R_0 h_0\left(1+\frac{1}{\varkappa}\right)} +$$

$$+\frac{Pn}{\pi R_0 h_0}\sum_{s=1}^{\infty}\frac{1}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}e^{-\lambda_{2s0}Z}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right), \tag{15.259}$$

$$\tau_{n_s m_s}(\bar{Z}, a) = \frac{Pn}{\sqrt{\bar{\gamma}}\pi R_0 h_0} \sum_{s=1}^{\infty} \frac{1}{\frac{1}{\cos \lambda^{\circ}_{2s0}} + \frac{\cos \lambda^{\circ}_{2s0}}{\varkappa}} \times$$

$$\times e^{-\lambda_{2s0}^{\circ 2} \bar{t}} \sin 2\lambda^{\circ}_{2s0}\left(\xi - \frac{1}{2}\right). \qquad (15.260)$$

In expressions (15.252)-(15.260), the infinite sums determine the warpings of the cross section of a reinforced shell and the corresponding self-balanced normal and tangential stresses. Changing to the dimensionless form, we write

$$\bar{\sigma}_{m_s} = 1 + \frac{\sigma^{\infty}_{m_s}}{\sigma^{0}_{m_s}}, \quad \bar{\tau}_{n_s m_s} = \frac{\tau^{\infty}_{n_s m_s}}{\sigma^{0}_{m_s}}, \qquad (15.261)$$

where $\sigma^0_{m_s}$ are stresses corresponding to the law of plane sections: $\sigma^{\infty}_{m_s}, \tau^{\infty}_{m_s}$ are self-balanced stresses.

Expanding (15.261), we obtain

$$\bar{\sigma}_{m_s} = 1 + 2(1+\varkappa) \sum_{s=1}^{\infty} \frac{e^{\lambda^{\circ}_{2s0}\bar{t}} \cos 2\lambda^{\circ}_{2s0}\left(\xi - \frac{1}{2}\right)}{\cos \lambda^{\circ}_{2s0} - \frac{\lambda^{\circ}_{2s0}}{\sin \lambda^{\circ}_{2s0}}}, \qquad (15.262)$$

$$\bar{\tau}_{n_s m_s} = -\frac{2}{\sqrt{\bar{\gamma}}}(1+\varkappa) \sum_{s=1}^{\infty} \frac{e^{\lambda^{\circ}_{2s0}\bar{t}} \sin 2\lambda^{\circ}_{2s0}\left(\xi - \frac{1}{2}\right)}{\cos \lambda^{\circ}_{2s0} - \frac{\lambda^{\circ}_{2s0}}{\sin \lambda^{\circ}_{2s0}}}, \qquad (15.263)$$

where the relative coordinate

$$\bar{t} = \begin{cases} \dfrac{n}{\pi\sqrt{\bar{\gamma}} \sin\beta}\, t = \dfrac{n}{\pi\sqrt{\bar{\gamma}} \sin\beta} \ln\left(1 - \operatorname{tg}\beta \dfrac{z}{R_0}\right) & \text{- for conical shell} \\ -\dfrac{n}{\pi\sqrt{\bar{\gamma}}} \bar{Z} = -\dfrac{n}{\pi\sqrt{\bar{\gamma}}} \dfrac{Z}{R_0} & \text{- for cylindrical shell} \end{cases} \qquad (15.264)$$

and λ°_{2s0} is the root of the equation $\lambda^{\circ} \operatorname{ctg} \lambda^{\circ} = -\varkappa$.

Dimensionless expressions (15.262), (15.263) contain only one geometric parameter

$$\varkappa = \frac{2\pi R_0 h_0}{n \Delta F_0}$$

and are therefore very convenient for analysis.

Figures 15.14-15.17 show graphs of $\bar{\sigma}_{m_s}=\bar{\sigma}_{m_s}(\alpha)$ and $\bar{\tau}_{n_s m_s}=\bar{\tau}_{n_s m_s}(\alpha)$ for different values of the dimensionless coordinate $\bar{t}$ and geometric parameter $\varkappa$. The calculation was carried out with a computer. The roots of the transcendental characteristic equation λ^* cotan $\lambda^* = -\varkappa$ were determined (see Fig. 15.1) on the basis of the iteration relation

$$\lambda_s^{(i+1)} = -\operatorname{arcctg}\frac{\varkappa}{\lambda_s^{(i)}} + \pi(s-1)$$

for

$$\lambda_s^{(0)} = \frac{\pi}{2} + \pi(s-1). \tag{15.265}$$

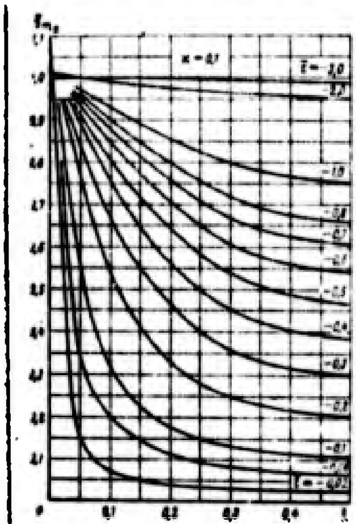

Fig. 15.14.

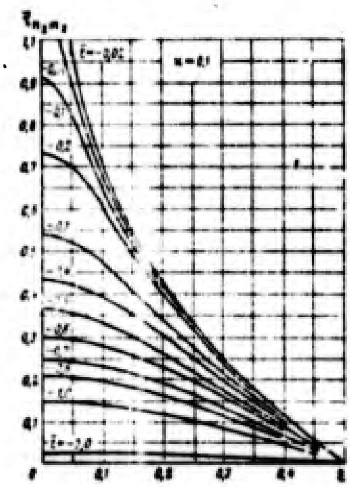

Fig. 15.15.

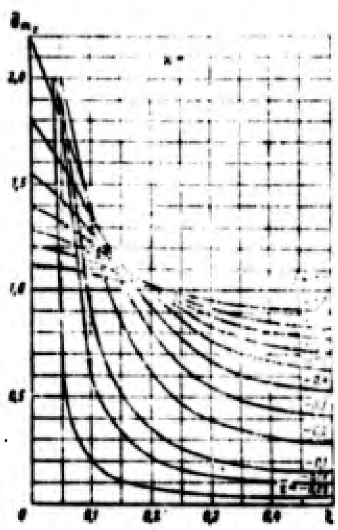

Fig. 15.16.

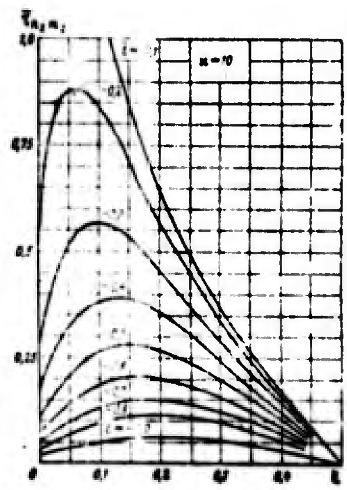

Fig. 15.17.

B. **Forces distributed according to the law of the plane**

Let the forces

$$P_m = A_0 + A_1 \cos \frac{2\pi m}{n} + B_1 \sin \frac{2\pi m}{n}, \qquad (15.266)$$

i.e., be distributed according to the law of the plane.

The first term in (15.266) corresponds to a uniform distribution of the forces. Since this case has already been discussed, we will assume below for the sake of brevity that $A_0 = 0$, since when $A_0 \neq 0$, the corresponding solution can be obtained on the basis of the superposition principle.

Setting in (15.249)

$$A_0 = A_r = B_r = 0 \left[r = 1, 2, 3, \ldots, E\left(\frac{n}{2}\right) \right],$$

we will find the values of arbitrary constants (15.214), (15.215) and also (15.210), (15.211). Since the forces P_m are directed along the generatrices of a cone, then on the basis of the equilibrium condition of the cut-off portion of the shell, the moments $\bar{M}^{\prime}_{z_0}, \bar{M}^{\prime}_{y_0}$ (15.178), entering into (15.211), (15.215) will be as before be equal to zero. Hence, in the absence of an external surface load, we will have

$$\begin{aligned} &\bar{C}_{1sr} = \bar{C}_{2sr} = 0 \quad \left[r = 0, 1, 2, \ldots, E\left(\frac{n}{2}\right);\; s = 1, 2, \ldots \right], \\ &C_{1sr} = C_{2sr} = 0 \quad \left[r = 0, 2, 3, \ldots, E\left(\frac{n}{2}\right);\; s = 1, 2, \ldots \right], \\ &C_{1s1} = -\frac{\sqrt{\pi}\sqrt{\gamma\varkappa}}{2Eh_0 B_s \lambda^{*}_{s1}} B_1 \quad (s = 1, 2, \ldots), \\ &C_{2s1} = -\frac{\sqrt{\pi}\sqrt{\gamma\varkappa}}{2Eh_0 B_s \lambda^{*}_{s1}} A_1 \quad (s = 1, 2, \ldots). \end{aligned} \qquad (15.267)$$

For the values of arbitrary constants obtained, we will write out the displacements u_{m_z}, η_z, η_y and stresses $\sigma_{m_z}, \tau_{n_z m_z}$.

From the equilibrium condition of the cut-off portion of the shell

$$M_x(t) = \sum_{m=1}^{n} P_m R_0 e^t \cos\beta \sin\frac{2\pi}{n} m,$$

$$M_y(t) = -\sum_{m=1}^{n} P_m R_0 e^t \cos\beta \cos\frac{2\pi}{n} m. \tag{15.268}$$

Expanding (15.268) with the aid of (15.266) and considering (14.3) and (14.12)-(14.14), we obtain

$$M_x(t) = B_1 \frac{n}{2} R_0 \cos\beta e^t,$$

$$M_y(t) = -A_1 \frac{n}{2} R_0 \cos\beta e^t. \tag{15.269}$$

Introducing (15.267), into general expressions (15.176) and (15.185), (15.186), we find

$$u_{m_z}(t, \alpha) = u_{m_z}(0, \alpha) - \frac{nt}{2\pi\left(1 + \frac{1}{\varkappa}\right) E h_0 \sin\beta} (A_1 \cos\alpha + B_1 \sin\alpha) -$$

$$-\frac{\sqrt{\pi\gamma}\varkappa}{2Eh_0} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1}^{*} B_s} \left[A_1 \Phi_{s1}(\alpha) + B_1 \Phi_{1s1}(\alpha)\right], \tag{15.270}$$

$$\eta_x = \eta_x^0 e^t - \frac{\bar{U}_{02}(0)}{\sin\beta}(1 - e^t) - \frac{A_1}{2Eh_0 \sin\beta} \left[\frac{n}{\pi\left(1 + \frac{1}{\varkappa}\right)\sin\beta} (e^t - 1 - t) + \right.$$

$$\left. + \varkappa\sqrt{\gamma} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1}^{*} B_s^2 (\lambda_{s1} - 1)} \right], \tag{15.271}$$

$$\eta_y = \eta_y^0 e^t - \frac{\bar{U}_{01}(0)}{\sin\beta}(1 - e^t) - \frac{B_1}{2Eh_0 \sin\beta} \left[\frac{n}{\pi\left(1 + \frac{1}{\varkappa}\right)\sin\beta} (e^t - 1 - t) + \right.$$

$$\left. + \varkappa\sqrt{\gamma} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1}^{*} B_s^2 (\lambda_{s1} - 1)} \right]. \tag{15.272}$$

Here

$$\bar{U}_{01}(0) = \frac{U_{01}(0)}{\sqrt{\pi\left(1 + \frac{1}{\varkappa}\right)}}; \quad \bar{U}_{02}(0) = \frac{U_{02}(0)}{\sqrt{\pi\left(1 + \frac{1}{\varkappa}\right)}} \tag{15.273}$$

are arbitrary constants. According to (15.207),

(15.274)

For normal and tangential stresses, we obtain from (15.190), (15.192),

$$\sigma_{m_s}(t, \alpha) = \frac{n e^{-t}}{2\pi R_0 h_0\left(1+\frac{1}{\varkappa}\right)}(A_1 \cos\alpha + B_1 \sin\alpha) +$$

$$+\frac{n\varkappa e^{-t}}{2\sqrt{\pi} R_0 h_0}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s}[A_1\Psi_{s1}(\alpha) + B_1\Phi_{s1}(\alpha)], \tag{15.275}$$

$$\tau_{\alpha_s m_s}(t, \alpha) = -\frac{\varkappa\sqrt{\pi}}{2R_0 h_0\sqrt{\gamma}}e^{-t}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s\lambda^{\circ}_{s1}}\left[A_1\left(\Psi'_{s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\gamma}B_s}\right) +\right.$$

$$\left.+ B_1\left(\Phi'_{s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\gamma}B_s}\right)\right]. \tag{15.276}$$

In expanded form, in view of (15.110), expressions (15.270), (15.275) take the form

$$u_{m_s}(t, \alpha) = u^0_{m_s}(0, \alpha) - \frac{nt}{2\pi\left(1+\frac{1}{\varkappa}\right)Eh_0\sin\beta}(A_1\cos\alpha + B_1\sin\alpha) -$$

$$-\frac{\varkappa\sqrt{\gamma}}{2Eh_0}\left\{(A_1\cos\alpha + B_1\sin\alpha)\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\lambda^{\circ}_{s1}B_s^2\left[\left(\frac{n\lambda^{\circ}_{s1}}{\varkappa}\right)^2 - 1\right]} +\right.$$

$$+P_{p+1}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\lambda^{\circ}_{s1}B_{s1}B_s}\frac{\sin 2\lambda^{\circ}_{s1}\xi}{\sin 2\lambda^{\circ}_{s1}} +$$

$$\left.+P_p\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\lambda^{\circ}_{s1}B_{s1}B_s}\frac{\sin 2\lambda^{\circ}_{s1}(1-\xi)}{\sin 2\lambda^{\circ}_{s1}}\right\}, \tag{15.277}$$

$$\sigma_{m_s}(t, \alpha) = \frac{n}{2\pi R_0 h_0\left(1+\frac{1}{\varkappa}\right)}e^{-t}(A_1\cos\alpha + B_1\sin\alpha) +$$

$$+\frac{n\varkappa}{2\pi R_0 h_0}e^{-t}\left\{(A_1\cos\alpha + B_1\sin\alpha)\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s^2\left[\left(\frac{n\lambda^{\circ}_{s1}}{\varkappa}\right)^2 - 1\right]} +\right.$$

$$+P_{p+1}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s B_{s1}}\frac{\sin 2\lambda^{\circ}_{s1}\xi}{\sin 2\lambda^{\circ}_{s1}} +$$

$$\left.+P_p\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s B_{s1}}\frac{\sin 2\lambda^{\circ}_{s1}(1-\xi)}{\sin 2\lambda^{\circ}_{s1}}\right\}, \tag{15.278}$$

$$\tau_{\alpha_s m_s}(t, \alpha) = -\frac{\varkappa n}{2\pi R_0 h_0\sqrt{\gamma}}e^{-t}\left\{(B_1\cos\alpha - A_1\sin\alpha)\frac{n}{\varkappa}\times\right.$$

$$\times\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s^2}\frac{\lambda^{\circ}_{s1}}{\left(\frac{n\lambda^{\circ}_{s1}}{\varkappa}\right)^2 - 1} + P_{p+1}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s B_{s1}}\frac{\cos 2\lambda^{\circ}_{s1}\xi}{\sin 2\lambda^{\circ}_{s1}} -$$

$$\left.-P_p\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_s B_{s1}}\frac{\cos 2\lambda^{\circ}_{s1}(1-\xi)}{\sin 2\lambda^{\circ}_{s1}}\right\}. \tag{15.279}$$

Here P_p, P_{p+1} are forces P_m (15.266) at points corresponding to

$$m = p = E\left(\frac{n}{2\pi}\alpha\right)$$

and

$$m = p+1 = E\left(\frac{n}{2\pi}\alpha\right)+1.$$

For a cylindrical shell, using (15.241) and (15.274), we find

$$u_{m_z}(\bar{Z},\alpha) = u^0_{m_z}(0,\alpha) + \frac{n\bar{Z}}{2\pi\left(1+\frac{1}{\varkappa}\right)Eh_0}(A_1\cos\alpha + B_1\sin\alpha) -$$

$$-\frac{\varkappa\sqrt{\gamma}}{2Eh_0}\left\{(A_1\cos\alpha + B_1\sin\alpha)\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda^*_{s1}\bar{B}^2_s\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right]}+\right.$$

$$+P_{p+1}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda^*_{s1}B_{s1}\bar{B}_s}e^{-\lambda_{s1}\bar{Z}}\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+$$

$$\left.+P_p\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda^*_{s1}B_{s1}\bar{B}_s}e^{-\lambda_{s1}\bar{Z}}\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\}, \qquad (15.280)$$

$$\eta_x = \eta^0_x + \theta^0_y R_0\bar{Z} - \frac{A_1}{2Eh_0}\left[\frac{n}{2\pi\left(1+\frac{1}{\varkappa}\right)}\bar{Z}^2 + \frac{\pi\varkappa\gamma}{n}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda^{*3}_{s1}\bar{B}^2_s}\right], \qquad (15.281)$$

$$\eta_y = \eta^0_y - \theta^0_x R_0\bar{Z} - \frac{B_1}{2Eh_0}\left[\frac{n}{2\pi\left(1+\frac{1}{\varkappa}\right)}\bar{Z}^2 + \frac{\pi\varkappa\gamma}{n}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda^{*3}_{s1}\bar{B}^2_s}\right], \qquad (15.282)$$

$$\sigma_{m_z}(\bar{Z},\alpha) = \frac{n}{2\pi R_0 h_0\left(1+\frac{1}{\varkappa}\right)}(A_1\cos\alpha + B_1\sin\alpha) + \frac{n\varkappa}{2\pi R_0 h_0}\times$$

$$\times\left\{(A_1\cos\alpha + B_1\sin\alpha)\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}^2_s\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right]}+P_{p+1}\sqrt{\pi}\times\right.$$

$$\left.\times\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s B_{s1}}\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+P_p\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s B_{s1}}\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\}, \qquad (15.283)$$

$$\tau_{n_z m_z}(\bar{z},\alpha)=-\frac{\varkappa n}{2\pi R_0 h_0 \sqrt{\bar{\gamma}}}\Big\{(B_1\cos\alpha-A_1\sin\alpha)\frac{n}{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{z}}}{B_s^2}\times$$

$$\times\frac{\lambda_{s1}^{*}}{\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^2-1}+P_{p+1}\sqrt{\bar{\gamma}}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{z}}}{\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{*}\xi}{\sin 2\lambda_{s1}^{*}}-$$

$$-P_p\sqrt{\bar{\gamma}}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{z}}}{\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{*}(1-\xi)}{\sin 2\lambda_{s1}^{*}}\Big\}. \qquad (15.284)$$

In expressions (15.270)-(15.284), the infinite sums determine the warpings of the cross section and the corresponding self-balanced normal and tangential stresses. Changing to the dimensionless form, we write

$$\sigma_{m_z}=\sigma_{1m_z}^0\bar{\sigma}_{1m_z}+\sigma_{2m_z}^0\bar{\sigma}_{2m_z};\quad \tau_{n_z m_z}=\sigma_{1m_z}^0\bar{\tau}_{1n_z m_z}+\sigma_{2m_z}^0\bar{\tau}_{2n_z m_z}. \qquad (15.285)$$

Here $\sigma_{1m_z}^0$, $\sigma_{2m_z}^0$ are stresses distributed according to the law of plane sections, due to the system of forces $P_m=P_{1m}=B_1\sin\frac{2\pi}{n}m$, and correspondingly $P_m=P_{2m}=A_1\cos\frac{2\pi}{n}m$;

$\bar{\sigma}_{1m_z}$, $\bar{\sigma}_{2m_z}$, $\bar{\tau}_{1n_z m_z}$, $\bar{\tau}_{2n_z m_z}$ are relative stresses:

$$\bar{\sigma}_{1m_z}=1+\frac{\sigma_{1m_z}^{\infty}}{\sigma_{1m_z}^0},\quad \bar{\sigma}_{2m_z}=1+\frac{\sigma_{2m_z}^{\infty}}{\sigma_{2m_z}^0},$$

$$\bar{\tau}_{1n_z m_z}=\frac{\tau_{1n_z m_z}^{\infty}}{\sigma_{1m_z}^0},\quad \bar{\tau}_{2n_z m_z}=\frac{\tau_{2n_z m_z}^{\infty}}{\sigma_{2m_z}^0}, \qquad (15.286)$$

where $\sigma_{1m_z}^{\infty}$, $\tau_{1n_z m_z}^{\infty}$ и $\sigma_{2m_z}^{\infty}$, $\tau_{2n_z m_z}^{\infty}$ are the self-balanced stresses due to systems of forces P_{1m} and P_{2m}, respectively.

Expanding (15.286), we have

$$\bar{\sigma}_{1m_z}=1+(1+\varkappa)\left[S_0^{\circ}(\bar{z})+\frac{\sin\frac{2\pi}{n}(p+1)\,S^{\circ}(\bar{z},\xi)+\sin\frac{2\pi}{n}p\,S^{\circ}[\bar{z},(1-\xi)]}{\sin\alpha}\right],$$

$$\bar{\sigma}_{2m_z}=1+(1+\varkappa)\left[S_0^{\circ}(\bar{z})+\frac{\cos\frac{2\pi}{n}(p+1)\,S^{\circ}(\bar{z},\xi)+\cos\frac{2\pi}{n}p\,S^{\circ}[\bar{z},(1-\xi)]}{\cos\alpha}\right],$$

$$\bar{\tau}_{1n_z m_z}=-\frac{1+\varkappa}{\sqrt{\bar{\gamma}}}\times \qquad (15.287)$$

$$\times\left[\operatorname{ctg}\alpha\, S_0^{\circ}(\bar{z})+\frac{\sin\frac{2\pi}{n}(p+1)\,S^{\circ}(\bar{z},\xi)-\sin\frac{2\pi}{n}p\,S^{\circ}[\bar{z},(1-\xi)]}{\sin\alpha}\right],$$

$$\bar{\tau}_{n_s m_s} = -\frac{1+\varkappa}{\sqrt{\pi}} \times$$

$$\times\left[-\operatorname{tg}\alpha S_0^{\circ}(\bar{t}) + \frac{\cos\frac{2\pi}{n}(p+1)S^{\circ}(\bar{t},\xi) - \cos\frac{2\pi}{n}p\,S^{\circ}[\bar{t},(1-\xi)]}{\cos\alpha}\right]. \qquad (15.287)$$

Here Здесь

$$S_0^{\circ}(\bar{t}) = \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1}^{\circ}\bar{t}}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1\right]},$$

$$S_0^{\circ}(\bar{t}) = \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1}^{\circ}\bar{t}}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1\right]} \frac{n\lambda_{s1}^{\circ}}{\pi},$$

$$S^{\circ}(\bar{t},\xi) = \sqrt{\pi}\sum_{s=1}^{\infty} \frac{e^{\lambda_{s1}^{\circ}\bar{t}}}{\bar{B}_s B_{s1}} \frac{\sin 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}}, \qquad (15.288)$$

$$S^{\circ}(\bar{t},\xi) = \sqrt{\pi}\sum_{s=1}^{\infty} \frac{e^{\lambda_{s1}^{\circ}\bar{t}}}{\bar{B}_s B_{s1}} \frac{\cos 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}},$$

where $\bar{t}$ is the relative longitudinal coordinate determined by relations (15.264).

Dimensionless stresses (15.287) contain only two independent geometric parameters: the relative area $\varkappa = \frac{2\pi R_0 h_0}{n\Delta F_0}$ and numbers of stringers n. For this reason, expressions (15.287) are very convenient for the analysis of the stressed state of a shell reinforced with a regular set of stringers, for different values of geometric parameters R_0, β, n, ΔF_0, h_0.

C. Alternating forces

Let a shell with an even number of stringers be loaded by a system of forces of the same magnitude but alternating in sign:

$$P_n = (-1)^m P.$$

Then the coefficients of the interpolation polynomial

$$A_0=A_r=B_r=0 \quad \left(r=1,2,\ldots,\frac{n}{2}-1\right),$$
$$A_{\frac{n}{2}}=P. \tag{15.289}$$

Introducing (15.289) into (15.249), we find the values of arbitrary constants (15.214), (15.215) and (15.210), (15.211). Since in the case under consideration the moments $\tilde{M}^t_{x_s}, \tilde{M}^t_{y_s}$ are equal to zero, in the absence of an external surface load, we obtain

$$\begin{gathered}
\bar{C}_{1sr}=\bar{C}_{2sr}=0 \quad \left(r=0,1,2,\ldots,\frac{n}{2};\ \ s=1,2,\ldots\right),\\
C_{1sr}=C_{2sr}=0 \quad \left(r=0,1,2,\ldots,\frac{n}{2}-1;\ \ s=1,2,\ldots\right),\\
C_{1s\frac{n}{2}}=0 \quad (s=1,2,\ldots),\\
C_{2s\frac{n}{2}}=-\frac{\pi\sqrt{\gamma}P}{\sqrt{2}Eh_0\lambda^0_{2s\frac{n}{2}}B_{s\frac{n}{2}}} \quad (s=1,2,\ldots).
\end{gathered} \tag{15.290}$$

Introducing the values of arbitrary constants (15.290) and (15.176) and (15.190), (15.192), we write the expanded expressions:

$$u_{m_s}(t,\alpha)=u^0_{m_s}(0,\alpha)+\frac{\sqrt{\gamma}P}{Eh_0}(-1)^p\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s\frac{n}{2}}t}}{\lambda^0_{2s\frac{n}{2}}\left(\frac{1}{\sin\lambda^0_{2s\frac{n}{2}}}+\frac{\sin\lambda^0_{2s\frac{n}{2}}}{\pi}\right)}\times$$
$$\times\sin 2\lambda^0_{2s\frac{n}{2}}\left(\xi-\frac{1}{2}\right), \tag{15.291}$$

$$\sigma_{m_s}(t,\alpha)=-\frac{nP}{\pi R_0h_0}(-1)^pe^{-t}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s\frac{n}{2}}t}}{\frac{1}{\sin\lambda^0_{2s\frac{n}{2}}}+\frac{\sin\lambda^0_{2s\frac{n}{2}}}{\pi}}\times$$
$$\times\sin 2\lambda^0_{2s\frac{n}{2}}\left(\xi-\frac{1}{2}\right), \tag{15.292}$$

$$\tau_{n_sm_s}(t,\alpha)=\frac{nP(-1)^p}{\sqrt{\gamma}\pi R_0h_0}e^{-t}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s\frac{n}{2}}t}}{\frac{1}{\sin\lambda^0_{2s\frac{n}{2}}}+\frac{\sin\lambda^0_{2s\frac{n}{2}}}{\pi}}\times$$
$$\times\cos 2\lambda^0_{2s\frac{n}{2}}\left(\xi-\frac{1}{2}\right). \tag{15.293}$$

For a cylindrical shell, using (15.241), we find

$$u_{\alpha_p}(Z,\alpha)=u^0_{\alpha_p}(0,\alpha)+\frac{\sqrt{\gamma}P}{Eh_0}(-1)^p\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s\frac{n}{2}}Z}}{\lambda^*_{2s\frac{n}{2}}\left(\dfrac{1}{\sin\lambda^*_{2s\frac{n}{2}}}+\dfrac{\sin\lambda^*_{2s\frac{n}{2}}}{\nu}\right)}\times\sin 2\lambda^*_{2s\frac{n}{2}}\left(t-\frac{1}{2}\right). \tag{15.294}$$

$$\sigma_{\alpha_p}(Z,\alpha)=-\frac{nP}{\pi R_0h_0}(-1)^p\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s\frac{n}{2}}Z}}{\dfrac{1}{\sin\lambda^*_{2s\frac{n}{2}}}+\dfrac{\sin\lambda^*_{2s\frac{n}{2}}}{\nu}}\times\sin 2\lambda^*_{2s\frac{n}{2}}\left(t-\frac{1}{2}\right). \tag{15.295}$$

$$\tau_{\alpha_p\alpha_p}(Z,\alpha)=\frac{nP(-1)^p}{\sqrt{\gamma}\pi R_0h_0}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s\frac{n}{2}}Z}}{\dfrac{1}{\sin\lambda^*_{2s\frac{n}{2}}}+\dfrac{\sin\lambda^*_{2s\frac{n}{2}}}{\nu}}\times\cos 2\lambda^*_{2s\frac{n}{2}}\left(t-\frac{1}{2}\right). \tag{15.296}$$

3. Regular End Load

Let the fluxes of normal and tangential forces

$$\bar{T}=\bar{T}(\alpha),\ \bar{S}=\bar{S}(\alpha)$$

contain no delta function type singularities, in contrast to (15.245). In this case, it is evident that the external end load is applied to the skin only, and the stringers at $t = 0$ are unloaded.

Indeed, for any regular function $f(\alpha)$

$$\lim_{\varepsilon\to 0}\int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\xi)\,d\xi=0$$

and hence, the longitudinal and tangential forces applied to the stringer $\alpha=\alpha_m=\frac{2\pi}{n}m$ at $t = 0$ are equal to zero:

$$(P_{n,t})_m = \lim_{\epsilon \to 0} \int_{\alpha_m-\epsilon}^{\alpha_m+\epsilon} \bar{T}(\xi)d\xi = 0,$$

$$(P_{n,s})_m = \lim_{\epsilon \to 0} \int_{\alpha_m-\epsilon}^{\alpha_m+\epsilon} \bar{S}(\xi)d\xi = 0.$$

Thus, the problem consists in studying the inclusion of a stringer in a combined operation with the shell proper, whereas in subsection 2 of the present section, the problem of inclusion of the skin was considered.

Various types of regular end load are examined below.

A. Uniform normal load

Let $\bar{T}(\alpha) = \bar{T} = \text{const}, \bar{S}(\alpha) = 0$, and there be no external surface load. The solution for this simplest case can be obtained directly from the solution for the general case of a cyclically symmetric load, examined in Subsection 1 of the present section.

Let us turn to expression (15.232), (15.235), (15.236). Setting $\bar{T}(\xi)$ = const, we calculate the quadratures:

$$\begin{gathered} \int_0^1 \bar{T} \sin 2\pi s\xi \, d\xi = 0, \\ \int_0^1 \bar{T} \cos 2\lambda^*_{1s0}\left(\xi - \frac{1}{2}\right) d\xi = \bar{T} \frac{\sin \lambda^*_{1s0}}{\lambda^*_{1s0}}. \end{gathered} \tag{15.297}$$

Using characteristic Eq. (15.114), we obtain

$$\int_0^1 \bar{T} \cos 2\lambda^*_{1s0} \left(\xi - \frac{1}{2}\right) d\xi = -\bar{T} \frac{\cos \lambda^*_{1s0}}{\varkappa}. \tag{15.298}$$

Introducing the values of the quadratures in accordance with (15.297), (15.298) into general expressions (15.232), (15.235), (15.236), we obtain

$$u_{m_s}(t,\alpha)=u^0_{m_s}(0,\alpha)-\frac{R_0\bar{T}t}{Eh_0\sin\beta\left(1+\frac{1}{\varkappa}\right)}+\frac{2\pi R_0\bar{T}\sqrt{\gamma}}{Eh_0\pi\varkappa}\times$$

$$\times\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{\lambda^{\circ}_{2s0}\left(\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}\right)}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right), \qquad (15.299)$$

$$\sigma_{m_s}(t,\alpha)=\frac{\bar{T}e^{-t}}{h_0\left(1+\frac{1}{\varkappa}\right)}-\frac{2\bar{T}}{\varkappa h_0}e^{-t}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}\times$$

$$\times\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right), \qquad (15.300)$$

$$\tau_{\alpha_s m_s}(t,\alpha)=-\frac{2\bar{T}}{\sqrt{\gamma}h_0\varkappa}e^{-t}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}\times$$

$$\times\sin 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right). \qquad (15.301)$$

For a cylindrical shell, using (15.241), we find

$$u_{m_s}(\bar{Z},\alpha)=u^0_{m_s}(0,\alpha)-\frac{R_0\bar{T}\bar{Z}}{Eh_0\left(1+\frac{1}{\varkappa}\right)}+\frac{2R_0\bar{T}}{Eh_0\varkappa}\times$$

$$\times\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s0}\bar{Z}}}{\lambda_{2s0}\left(\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}\right)}\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right). \qquad (15.302)$$

$$\sigma_{m_s}(\bar{Z},\alpha)=\frac{\bar{T}}{h_0\left(1+\frac{1}{\varkappa}\right)}-\frac{2\bar{T}}{\varkappa h_0}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s0}\bar{Z}}}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}\times$$

$$\times\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right). \qquad (15.303)$$

$$\tau_{\alpha_s m_s}(\bar{Z},\alpha)=\frac{2\bar{T}}{\sqrt{\gamma}h_0\varkappa}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s0}\bar{Z}}}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}\sin 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right). \qquad (15.304)$$

Changing the dimensionless form, we find

$$\bar{\sigma}_{m_s}=\frac{\sigma_{m_s}}{\sigma^0_{m_s}}=1-2\frac{1+\varkappa}{\varkappa^2}\sum_{s=1}^{\infty}\frac{e^{\lambda^{\circ}_{2s0}\bar{T}}}{\frac{1}{\cos\lambda^{\circ}_{2s0}}+\frac{\cos\lambda^{\circ}_{2s0}}{\varkappa}}\times$$

$$\times\cos 2\lambda^{\circ}_{2s0}\left(\xi-\frac{1}{2}\right). \qquad (15.305)$$

$$\bar{\tau}_{a_s m_s} = \frac{\tau_{a_s m_s}}{\sigma^0_{m_s}} = -\frac{2}{\sqrt{\gamma}} \frac{1+\varkappa}{\varkappa^2} \sum_{s=1}^{\infty} \frac{e^{\lambda^*_{2s0}\bar{t}}}{\frac{1}{\cos\lambda^*_{2s0}} + \frac{\cos\lambda^*_{2s0}}{\varkappa}} \times$$

$$\times \sin 2\lambda^*_{2s0}\left(\xi - \frac{1}{2}\right). \quad (15.306)$$

It is of interest to compare the results of the problem under consideration with the general expressions obtained in Subsection 3.A of the present section in the case of loading of stringers with identical forces $P_m = P$. It is evident that the infinite sums in the expressions for displacements and stresses are the same in both cases, but of opposite signs. Therefore, on the basis of the superposition principle, a relation can be found between the forces P and the linear load $\bar{T}$ for which the total warpings and self-balanced stresses disappear when one state is superposed on the other. It is easy to see that this will occur when

$$\frac{\bar{T}}{h_0} = \frac{P}{\Delta F},$$

which corresponds to a uniform distribution of the external load over the entire section t = 0. In the study of the stressed and strained state, this fact makes it possible in the case at hand to use the graphs of dimensionless stresses shown in Fig. 15.14-15.17 by means of a simple conversion.

B. Normal load distributed according to the law of the plane

Let the longitudinal forces be distributed in the end section according to the law of the plane, and the tangential forces be absent:

$$\begin{aligned} \bar{T}(\alpha) &= a_0 + a_1\cos\alpha + b_1\sin\alpha, \\ \bar{S}(\alpha) &= 0. \end{aligned} \quad (15.307)$$

The first term in (15.307), corresponding to a uniform distribution of the forces, will for brevity be assumed equal to zero, since this case has already been discussed above.

Turning to expressions (15.214), (15.215), we first find the values of the Fourier coefficients of load (15.307) in normal coordinate functions. On the basis of (15.127), only

$$\int_0^{2\pi} \overline{T}(\alpha)\Phi_{1s1}(\alpha)d\alpha = -b_1\frac{\sqrt{\pi}}{\overline{B}_s}.$$
$$\int_0^{2\pi} \overline{T}(\alpha)\Psi_{1s1}(\alpha)d\alpha = -a_1\frac{\sqrt{\pi}}{\overline{B}_s}. \qquad (15.308)$$

are different from zero.

In the absence of an external surface load, in view of (15.308), we find from (15.210), (15.211) and (15.214), (15.215)

$$\begin{gathered}\overline{C}_{1sr}=\overline{C}_{2sr}=0 \quad \left[r=0,1,2,\ldots,E\left(\frac{n}{2}\right);\quad s=1,2,\ldots\right],\\ C_{1sr}=C_{2sr}=0 \quad \left[r=0,2,3,\ldots,E\left(\frac{n}{2}\right);\quad s=1,2,\ldots\right],\\ C_{1s1}=\frac{\pi\sqrt{\pi}R_0\sqrt{\gamma}}{Enh_0\lambda^{\circ}_{s1}\overline{B}_s}b_1 \quad (s=1,2,\ldots),\\ C_{2s1}=\frac{\pi\sqrt{\pi}R_0\sqrt{\gamma}}{Enh_0\lambda^{\circ}_{s1}\overline{B}_s}a_1 \quad (s=1,2,\ldots)\end{gathered} \qquad (15.309)$$

For the values of arbitrary constants obtained, we will write out the expressions for displacements u_m, v_t, w_v and stresses $\sigma_{m_s}, \tau_{s_s m_s}$.

From the equilibrium condition of the cut-off portion of the shell

$$M_x(t)=\int_0^{2\pi}\overline{T}(\alpha)\cos\beta\sin\alpha\, R_0^2e^t d\alpha=\pi R_0^2\cos\beta b_1e^t,$$
$$M_y(t)=-\int_0^{2\pi}\overline{T}(\alpha)\cos\beta\cos\alpha\, R_0^2e^t d\alpha=-\pi R_0^2\cos\beta a_1e^t. \qquad (15.310)$$

Introducing (15.309), (15.310) and into general expressions (15.176), (15.185), (15.186), in view of (15.207), we find

$$u_{m_z}(t, \alpha) = u^0_{m_z}(0, \alpha) - \frac{R_0 \bar{T}(\alpha)\, t}{Eh_0\left(1 + \frac{1}{\varkappa}\right)\sin\beta} +$$

$$+ \frac{\sqrt{\pi} R_0}{Eh_0 \sin\beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}_s} \left[a_1 \Phi_{2s1}(\alpha) + b_1 \Phi_{1s1}(\alpha)\right], \tag{15.311}$$

$$\eta_x(t) = \eta^0_x + \theta^0_y R_0 \operatorname{ctg}\beta (1 - e^t) - \frac{a_1}{2Eh_0 \sin^2\beta} \left[\frac{2R_0}{1 + \frac{1}{\varkappa}}(e^t - 1 - t) - \right.$$

$$\left. - 2R_0 \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1} \bar{B}^2_s (\lambda_{s1} - 1)}\right]. \tag{15.312}$$

$$\eta_y(t) = \eta^0_y - \theta^0_x R_0 \operatorname{ctg}\beta (1 - e^t) - \frac{b_1}{2Eh_0 \sin^2\beta} \left[\frac{2R_0}{1 + \frac{1}{\varkappa}}(e^t - 1 - t) - \right.$$

$$\left. - 2R_0 \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1} \bar{B}^2_s (\lambda_{s1} - 1)}\right] \tag{15.313}$$

Using (15.190), (15.192), we obtain

$$\sigma_{m_z}(t, \alpha) = \frac{\bar{T}(\alpha) e^{-t}}{h_0\left(1 + \frac{1}{\varkappa}\right)} - \frac{\sqrt{\pi}}{h_0} e^{-t} \sum_{s=1}^{\infty} e^{\lambda_{s1} t} \left[a_1 \Phi_{2s1}(\alpha) + b_1 \Phi_{1s1}(\alpha)\right], \tag{15.314}$$

$$\tau_{\alpha_z m_z}(t, \alpha) = \frac{\sqrt{\pi} e^{-t}}{\gamma h_0 \sin\beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}_s} \left[a_1\left(\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\pi} b_s}\right) + \right.$$

$$\left. + b_1\left(\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\pi} b_s}\right)\right]. \tag{15.315}$$

Expanding $\Phi_{1s1}(\alpha)$ and $\Phi_{2s1}(\alpha)$ with the aid of (15.110), we reduce expressions (15.311), (15.314), (15.315) to the form

$$u_{m_z}(t, \alpha) = u^0_{m_z}(0, \alpha) - \frac{R_0 \bar{T}(\alpha)\, t}{\left(1 + \frac{1}{\varkappa}\right) Eh_0 \sin\beta} +$$

$$+ \frac{R_0}{Eh_0 \sin\beta} \left\{\bar{T}(\alpha) \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}^2_s \left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2 - 1\right]} + \right.$$

$$+ \bar{T}\left[\frac{2\pi}{n}(p+1)\right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} B_{s1} \bar{B}_s} \frac{\sin 2\lambda^*_{s1} \xi}{\sin 2\lambda^*_{s1}} +$$

$$\left. + T\left(\frac{2\pi}{n} p\right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} B_{s1} \bar{B}_s} \frac{\sin 2\lambda^*_{s1} (1 - \xi)}{\sin 2\lambda^*_{s1}}\right\}. \tag{15.316}$$

$$\sigma_{m_s}(t,\alpha)=\frac{\bar{T}(\alpha)e^{-t}}{h_0\left(1+\frac{1}{\varkappa}\right)}-\frac{e^{-t}}{h_0}\left\{\bar{T}(\alpha)\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2-1\right]}+\right.$$

$$+\bar{T}\left[\frac{2\pi}{n}(p+1)\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_{s1}\bar{B}_s}\frac{\sin 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}}+$$

$$\left.+\bar{T}\left(\frac{2\pi}{n}p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_{s1}\bar{B}_s}\frac{\sin 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}, \qquad (15.317)$$

$$\tau_{\alpha_s m_s}(t,\alpha)=\frac{1}{\sqrt{\gamma}h_0}e^{-t}\left\{\frac{d\bar{T}(\alpha)}{d\alpha}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_s^2}\frac{\frac{n\lambda_{s1}^{\circ}}{\pi}}{\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2-1}+\right.$$

$$+\bar{T}\left[\frac{2\pi}{n}(p+1)\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_sB_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}}-$$

$$\left.-\bar{T}\left(\frac{2\pi}{n}p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_sB_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}. \qquad (15.318)$$

For a cylindrical shell, using (15.241), we find

$$u_{m_s}(\bar{Z},\alpha)=u_{m_s}^{0}(0,\alpha)+\frac{R_0\bar{Z}}{\left(1+\frac{1}{\varkappa}\right)Eh_0}\bar{T}(\alpha)+$$

$$+\frac{\pi R_0\sqrt{\gamma}}{nEh_0}\left\{\bar{T}(\alpha)\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda_{s1}^{\circ}\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)-1\right]}+\right.$$

$$+\bar{T}\left[\frac{2\pi}{n}(p+1)\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{\circ}B_{s1}\bar{B}_s}e^{-\lambda_{s1}\bar{Z}}\frac{\sin 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}}+$$

$$\left.+\bar{T}\left(\frac{2\pi}{n}p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{\circ}B_{s1}\bar{B}_s}e^{-\lambda_{s1}\bar{Z}}\frac{\sin 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}, \qquad (15.319)$$

$$\eta_z(\bar{Z})=\eta_z^0+\theta_y^0R_0\bar{Z}-\frac{a_1}{2Eh_0}\left[\frac{R_0}{1+\frac{1}{\varkappa}}\bar{Z}^2-\frac{2\pi^2\gamma R_0}{n^2}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}\bar{Z}}}{\lambda_{s1}^{\circ}\bar{B}_s^2}\right], \qquad (15.320)$$

$$\eta_y(\bar{Z})=\eta_y^0-\theta_z^0R_0\bar{Z}-\frac{b_1}{2Eh_0}\left[\frac{R_0}{1+\frac{1}{\varkappa}}\bar{Z}^2-\frac{2\pi^2\gamma R_0}{n}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}\bar{Z}}}{\lambda_{s1}^{\circ}\bar{B}_s^2}\right]. \qquad (15.321)$$

$$\sigma_{m_s}(\bar{Z},\alpha)=\frac{\bar{T}(u)}{h_0\left(1+\frac{1}{\varkappa}\right)}-\frac{1}{h_0}\left\{\bar{T}(u)\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}t}}{B_s\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2-1\right]}+\right.$$

$$+\bar{T}\left[\frac{2\pi}{n}(p+1)\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}2}}{B_sB_{s1}}\frac{\sin 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}}+$$

$$\left.+\bar{T}\left(\frac{2\pi}{n}p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}2}}{B_sB_{s1}}\frac{\sin 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}; \qquad (15.322)$$

$$\tau_{n_s m_s}(\bar{Z},\alpha)=\frac{1}{\sqrt{\gamma}h_0}\left\{\frac{d\bar{T}(u)}{du}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}2}}{B_s}\frac{\frac{n\lambda_{s1}^{\circ}}{\pi}}{\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2-1}+\right.$$

$$+\bar{T}\left[\frac{2\pi}{n}(p+1)\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}2}}{B_sB_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}}-$$

$$\left.-\bar{T}\left(\frac{2\pi}{n}p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}^{\circ}2}}{B_sB_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}. \qquad (15.323)$$

Changing to the dimensionless form as in (15.285), (15.286), we write

$$\bar{\sigma}_{1m_s}=1-\left(1+\frac{1}{\varkappa}\right)\left[S_0^{\circ}(\bar{t})+\right.$$

$$\left.+\frac{\sin\frac{2\pi}{n}(p+1)S^{\circ}(\bar{t},\xi)+\sin\frac{2\pi}{n}p\,S^{\circ}(\bar{t},(1-\xi))}{\sin\alpha}\right],$$

$$\bar{\sigma}_{2m_s}=1-\left(1+\frac{1}{\varkappa}\right)\left[S_0^{\circ}(\bar{t})+\right. \qquad (15.324)$$

$$\left.+\frac{\cos\frac{2\pi}{n}(p+1)S^{\circ}(\bar{t},\xi)+\cos\frac{2\pi}{n}p\,S^{\circ}(\bar{t},(1-\xi))}{\cos\alpha}\right],$$

$$\bar{\tau}_{1n_sm_s}=\frac{1}{\sqrt{\gamma}}\left(1+\frac{1}{\varkappa}\right)\left[\operatorname{ctg}\alpha\,S_0^{*}(\bar{t})+\right.$$

$$\left.+\frac{\sin\frac{2\pi}{n}(p+1)S^{*}(\bar{t},\xi)-\sin\frac{2\pi}{n}p\,S^{*}(\bar{t},(1-\xi))}{\sin\alpha}\right],$$

$$\bar{\tau}_{2n_sm_s}\frac{1}{\sqrt{\gamma}}\left(1+\frac{1}{\varkappa}\right)\left[-\operatorname{tg}\alpha\,S_0^{*}(\bar{t})+\right.$$

$$\left.+\frac{\cos\frac{2\pi}{n}(p+1)S^{*}(\bar{t},\xi)-\cos\frac{2\pi}{n}p\,S^{*}(\bar{t},(1-\xi))}{\cos\alpha}\right].$$

Here $\bar{\sigma}_{1m_z}$, $\bar{\tau}_{1m_z}$ и $\bar{\sigma}_{2m_z}$, $\bar{\tau}_{1m_z}$ are the relative normal and tangential stresses corresponding to the loads $\bar{T}_1(\alpha)=b_1\sin\alpha$ and $\bar{T}_2(\alpha)=a_1\cos\alpha$; $S_0^{\circ}(\bar{t})$, $S_0^{\circ}(\bar{t})$, $S^{\circ}(\bar{t},t)$, $\bar{S}^{\circ}(\bar{t},t)$ are infinite series determined by expressions (15.288).

Comparing (15.324) and (15.287), we can easily ascertain that for a certain relation between the coefficients A_1, B_1, determining the forces P_m (15.266) and coefficients a_1, b_1 determining the linear load $\bar{T}(\alpha)$ (15.307), the total warpings and self-balanced stresses cancel each other out when one state is superposed on the other. We see that this will take place when

$$a_1=\frac{m}{2\pi R_0}A_1;\ b_1=\frac{m}{2\pi R_0}B_1,$$

which corresponds to the distribution of stresses σ_{m_z} over the entire section $t = 0$ according to the unified law of the plane. Thus, an interrelation is established between the problem of inclusion of the stringers in the combined operation with a shell and the opposite problem of inclusion of the shell proper in the combined operation with stringers, provided that in both cases the external load is distributed in the section $t = 0$ according to the law of the plane.

C. Tangential load

Let the end section $t = 0$ be loaded with a flux of tangential forces, and longitudinal forces be absent. By virtue of the adopted computational model, the detailed character of the flux $\bar{S}(\alpha)$ is not important, and therefore it is sufficient to specify it in the form

$$\bar{S}(\alpha)=a_0+a_1\cos\alpha+b_1\sin\alpha. \quad (15.325)$$

The first term in (15.325), corresponding to a uniform distribution of tangential forces, determines the twisting moment $M_z=2\pi R_0^2 a_0$.

From (15.189), in the absence of an external surface load

$$\theta_z(t)=\theta_z^0+\frac{a_0}{2Gh_0\sin\beta}(e^{-2t}-1). \tag{15.236}$$

From (15.8), in view of (15.10) and (15.326), we also find

$$\tau_{n_zm_z}(t,\alpha)=G\gamma_{n_zm_z}=\frac{a_0}{h_0}e^{-2t}. \tag{15.327}$$

Since Eq. (15.23) in the angle of twist θ_z separates from the general system of resolvents, for simplicity we will hereinafter assume that $a_0 = 0$, which does not reduce the generality of the problem.

Let us turn to expressions (15.212), (15.213) and (15.216), (15.217) in arbitrary constants $\bar{C}$ and C.

From the equilibrium condition of the cut-off portion of the shell, we find

$$\begin{aligned} M_x(t)&=\int_0^{2\pi}\bar{S}(\alpha)\cos\alpha R_0l_0(1-e^t)d\alpha=P_yl_0(1-e^t),\\ M_y(t)&=\int_0^{2\pi}\bar{S}(\alpha)\sin\alpha R_0l_0(1-e^t)d\alpha=-P_xl_0(1-e^t), \end{aligned} \tag{15.328}$$

where

$$P_y=\int_0^{2\pi}\bar{S}(\alpha)\cos\alpha R_0d\alpha=\pi R_0a_1$$

and

$$P_x=-\int_0^{2\pi}\bar{S}(\alpha)\sin\alpha R_0d\alpha=-\pi R_0b_1 \tag{15.329}$$

are the components of the resultant of external tangential forces along the Ox, Oy axes.

Considering (15.328) and (15.184), we obtain

$$\begin{aligned} &\bar{C}_{1sr}=C_{1sr}=\bar{C}_{2sr}=C_{2sr}=0\\ &\qquad\left[r=0,\,2,\,3,\dots,\,E\left(\frac{n}{2}\right);\ s=1,\,2,\dots\right],\\ &\bar{C}_{1s1}=C_{1s1}=\frac{\sqrt{\pi\gamma}}{2Eh_0\pi\sin\beta\,\lambda_{s1}^{\circ}\bar{B}_s}\,\frac{P_y}{(\lambda_{s1}-1)}\quad(s=1,\,2,\dots),\\ &\bar{C}_{2s1}=C_{2s1}=\frac{\sqrt{\pi\gamma}}{2Eh_0\pi\sin\beta\,\lambda_{s1}^{\circ}\bar{B}_s}\,\frac{P_x}{(\lambda_{s1}-1)}\quad(s=1,\,2,\dots). \end{aligned} \tag{15.330}$$

For the values of arbitrary constants obtained, we will write out the displacements u_{m_z}, η_x, η_y and stresses σ_{m_z}, $\tau_{n_z m_z}$.

Introducing (15.328), (15.330) into general expressions (15.183) and (15.187), (15.188), in the absence of an external surface load we obtain

$$u_{m_z}(t, \alpha) = u^0_{m_z}(0, \alpha) + \frac{(1 - e^{-t} - t) R_0}{E h_0 \sin^2\beta \left(1 + \frac{1}{\varkappa}\right)} \frac{d\bar{S}(\alpha)}{d\alpha} +$$

$$+ \frac{R_0 \sqrt{\pi}}{E h_0 \sin^2\beta} \sum_{s=1}^{\infty} \frac{\lambda_{s1} e^{-t} + e^{\lambda_{s1} t}}{\lambda_{s1} (\lambda_{s1}^2 - 1) \bar{B}_s} [a_1 \Phi_{1s1}(\alpha) - b_1 \Phi_{2s1}(\alpha)], \tag{15.331}$$

$$\eta_x(t) = \eta_x^0 + \theta_y^0 R_0 \operatorname{ctg}\beta (1 - e^t) + \frac{b_1 R_0}{E h_0 \sin^3\beta} \left[\left(\gamma \sin^2\beta - \frac{\varkappa}{1+\varkappa}\right) \operatorname{sh} t + \frac{\varkappa}{1+\varkappa} t\right] +$$

$$+ \frac{\pi \sqrt{\gamma} b_1 R_0}{n E h_0 \sin^2\beta} \sum_{s=1}^{\infty} \frac{1}{\lambda_{s1}^2 (\lambda_{s1}^2 - 1) \bar{B}_s^2} \left(\frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1} - 1} + \lambda_{s1} \operatorname{sh} t\right), \tag{15.332}$$

$$\eta_y(t) = \eta_y^0 - \theta_x^0 R_0 \operatorname{ctg}\beta (1 - e^t) - \frac{a_1 R_0}{E h_0 \sin^3\beta} \left[\left(\gamma \sin^2\beta - \frac{\varkappa}{1+\varkappa}\right) \operatorname{sh} t + \frac{\varkappa}{1+\varkappa} t\right] -$$

$$- \frac{\pi \sqrt{\gamma} a_1 R_0}{n E h_0 \sin^2\beta} \sum_{s=1}^{\infty} \frac{1}{\lambda_{s1}^2 (\lambda_{s1}^2 - 1) \bar{B}_s^2} \left(\frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1} - 1} + \lambda_{s1} \operatorname{sh} t\right). \tag{15.333}$$

On the basis of (15.191), (15.193), the stresses

$$\sigma_{m_z}(t, \alpha) = - \frac{(1 - e^t) e^{-2t}}{h_0 \sin\beta \left(1 + \frac{1}{\varkappa}\right)} \frac{d\bar{S}(\alpha)}{d\alpha} -$$

$$- \frac{e^{-t} \sqrt{\pi}}{h_0 \sin\beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} - e^{-t}}{\bar{B}_s (\lambda_{s1}^2 - 1)} [a_1 \Phi_{1s1}(\alpha) - b_1 \Phi_{2s1}(\alpha)], \tag{15.334}$$

$$\tau_{n_z m_z}(t, \alpha) = \frac{e^{-2t}}{h_0} \bar{S}(\alpha) - \frac{e^{-t} \sqrt{\pi}}{h_0 \gamma \sin^2\beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} + \lambda_{s1} e^{-t}}{\lambda_{s1} (\lambda_{s1}^2 - 1) \bar{B}_s} \left\{a_1 \left[\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\pi} \bar{B}_s}\right] - \right.$$

$$\left. - b_1 \left[\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\pi} B_s}\right]\right\}. \tag{15.335}$$

Expanding Φ_{1s1} and Φ_{2s1} with the aid of (15.110), from expressions (15.331), (15.334), (15.335) we obtain:

$$u_{m_z}(t, \alpha)=u^0_{m_z}(0, \alpha)+\frac{(1-e^{-t}-t)R_0}{Eh_0\sin^2\beta\left(1+\frac{1}{\varkappa}\right)}\frac{d\bar S(\alpha)}{d\alpha}-$$

$$-\frac{R_0}{Eh_0\sin^2\beta}\left\{\frac{d\bar S(\alpha)}{d\alpha}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\lambda_{s1}\bar B_s^2(\lambda_{s1}^2-1)\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right]}+\right.$$

$$+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}(p+1)}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\lambda_{s1}B_s^2(\lambda_{s1}^2-1)}\,\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+$$

$$\left.+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}p}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\lambda_{s1}B_s^2(\lambda_{s1}^2-1)}\,\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\}, \qquad (15.336)$$

$$\sigma_{m_z}(t, \alpha)=-\frac{(1-e^{t})e^{-2t}}{h_0\sin\beta\left(1+\frac{1}{\varkappa}\right)}\frac{d\bar S(\alpha)}{d\alpha}+$$

$$+\frac{e^{-t}}{h_0\sin\beta}\left\{\frac{d\bar S(\alpha)}{d\alpha}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}-e^{-t}}{\bar B_s^2\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right](\lambda_{s1}^2-1)}+\right.$$

$$+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}(p+1)}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{\left(e^{\lambda_{s1}t}-e^{-t}\right)}{B_{s1}\bar B_s(\lambda_{s1}^2-1)}\,\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+$$

$$\left.+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}p}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{\left(e^{\lambda_{s1}t}-e^{-t}\right)}{B_{s1}\bar B_s(\lambda_{s1}^2-1)}\,\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\}, \qquad (15.337)$$

$$\tau_{n_z m_z}(t, \alpha)=\frac{e^{-2t}}{h_0}\bar S(\alpha)+\frac{e^{-t}}{\sqrt{\gamma}h_0\sin\beta}\times$$

$$\times\left\{\bar S(\alpha)\sum_{s=1}^{\infty}\frac{1}{\bar B_s^2}\,\frac{\frac{n\lambda^*_{s1}}{\pi}}{\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1}\,\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\lambda_{s1}^2-1}+\right.$$

$$+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}(p+1)}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\bar B_s B_{s1}}\,\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\lambda_{s1}^2-1}\,\frac{\cos 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}-$$

$$\left.-\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}p}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\bar B_s B_{s1}}\,\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\lambda_{s1}^2-1}\,\frac{\cos 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\}. \qquad (15.338)$$

For a cylindrical shell, using (15.241), we find

$$u_{m_z}(\bar Z, \alpha)=u^0_{m_z}(0, \alpha)+\frac{R_0\bar Z^2}{2Eh_0\left(1+\frac{1}{\varkappa}\right)}\frac{d\bar S(\alpha)}{d\alpha}-$$

$$-\frac{R_0}{Eh_0}\left\{\frac{d\bar S(\alpha)}{d\alpha}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^2\bar B_s^2\left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2-1\right]}+\right.$$

$$+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}(p+1)}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^2\bar B_s B_{s1}}\,\frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}}+$$

$$\left.+\left.\frac{d\bar S(\alpha)}{d\alpha}\right|_{\alpha=\frac{2\pi}{n}p}\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^3\bar B_s B_{s1}}\,\frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}}\right\}. \qquad (15.339)$$

$$\eta_x(\bar{Z}) = \eta_x^0 + \theta_y^0 R_0 \bar{Z} - \frac{a_1 R_0}{E h_0}\left[\gamma \bar{Z}\left(1 + \frac{n^2}{\pi^2}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{\circ 2}\bar{B}_s^2}\right) - \frac{\varkappa}{1+\varkappa}\frac{\bar{Z}^3}{6}\right], \qquad (15.340)$$

$$\eta_y(\bar{Z}) = \eta_y^0 - \theta_x^0 R_0 \bar{Z} + \frac{a_1 R_0}{E h_0}\left[\gamma \bar{Z}\left(1 + \frac{n^2}{\pi^2}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{\circ 2}\bar{B}_s^2}\right) - \frac{\varkappa}{1+\varkappa}\frac{\bar{Z}^3}{6}\right], \qquad (15.341)$$

$$\sigma_{m_z}(\bar{Z}, \alpha) = -\frac{\bar{Z}}{h_0\left(1 + \frac{1}{\varkappa}\right)}\frac{d\bar{S}(\alpha)}{d\alpha}, \qquad (15.342)$$

$$\tau_{z_s m_z}(\bar{Z}, \alpha) = -\frac{1}{h_0}\bar{S}(\alpha) + \frac{1}{h_0}\left\{\bar{S}(\alpha)\sum_{s=1}^{\infty}\frac{1}{\bar{B}_s^2}\frac{1}{\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1} + \right.$$

$$+ \left.\frac{d\bar{S}(\alpha)}{d\alpha}\right|_{\alpha = \frac{2\pi}{n}(p+1)}\frac{\pi\sqrt{\pi}}{n}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{\circ}\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}} -$$

$$\left. - \left.\frac{d\bar{S}(\alpha)}{d\alpha}\right|_{\alpha = \frac{2\pi}{n}p}\frac{\pi\sqrt{\pi}}{n}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{\circ}\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}. \qquad (15.343)$$

Thus, for the case under consideration, the normal stresses σ_{m_z} in a cylindrical shell obey the law of plane sections, and the corresponding tangential stresses $\tau_{z_s m_z}$ are independent of the longitudinal coordinate. This does not occur in a conical shell; in addition to the beam terms, expressions (15.334), (15.335) also contain self-balanced terms.

15.6. Stringer Conical Shell under Cyclic Fixing Conditions

1. Cantilever Shell

Let the end t = 0 of a shell be completely fixed. Then, setting

$$\eta_x^0 = \eta_y^0 = \eta_z^0 = \theta_x^0 = \theta_y^0 = \theta_z^0 = 0$$

and

$$\bar{u}(\alpha) = 0,$$

we can write expanded expressions for displacements $u_{m_z}, \eta_x, \eta_y, \theta_z$ and stresses $\sigma_{m_z}, \tau_{z_s m_z}$ for an arbitrary external load. To keep the treatment relatively simple, we will confine it to the case in which the longitudinal component P_z of the external surface load can be represented in the form (15.181). In this case, from (15.212), (15.213), (15.220), (15.221), we have

$$\overline{C}_{1sr} = \overline{C}_{2sr} - C_{1sr} - C_{2sr} = 0 \qquad \left[r = 0, 2, 3, \ldots, E\left(\frac{n}{2}\right)\right],$$

$$C_{1s1} = -\overline{C}_{1s1} = \frac{1}{2}\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}_x^0(\zeta)\right]d\zeta, \tag{15.344}$$

$$C_{2s1} = -\overline{C}_{2s1} = -\frac{1}{2}\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}_y^0(\zeta)\right]d\zeta$$

$$(s = 1, 2, \ldots).$$

Introducing (15.344), we obtain

$$\begin{aligned} u_{m_2}(t, \alpha) = {} & -\int_0^t \overline{N}_z(\zeta)d\zeta - \sin\alpha\int_0^t \overline{M}_x(\zeta)e^{-\zeta}d\zeta + \cos\alpha\int_0^t \overline{M}_y(\zeta)e^{-\zeta}d\zeta + \\ & + \sum_{s=1}^{\infty}\Big\{\Phi_{1s1}(\alpha)\Big[\operatorname{sh}\lambda_{s1}t\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_x^0(\zeta)\right)d\zeta - \\ & - \int_0^t \operatorname{sh}\lambda_{s1}(\zeta - t)\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_x^0(\zeta)\right)d\zeta\Big] - \\ & - \Phi_{2s1}(\alpha)\Big[\operatorname{sh}\lambda_{s1}t\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_y^0(\zeta)\right)d\zeta - \\ & - \int_0^t \operatorname{sh}\lambda_{s1}(\zeta - t)\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_y^0(\zeta)\right)d\zeta\Big]\Big\}, \end{aligned} \tag{15.345}$$

$$\begin{aligned} \eta_x(t) = {} & \frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{y_0}(\xi)e^{-2\xi}d\xi + \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Big\{\int_0^{\xi}\overline{M}_y(\zeta)e^{-\zeta}d\zeta + \\ & + \sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}B_s}\Big[\operatorname{sh}\lambda_{s1}\xi\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_y^0(\zeta)\right)d\zeta - \\ & - \int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta - \xi)\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_y^0(\zeta)d\zeta\right)\Big]\Big\}d\xi. \end{aligned} \tag{15.346}$$

$$\begin{aligned} \eta_y(t) = {} & -\frac{e^t}{\pi R_0 h_0 G\cos\beta}\int_0^t M_{x_0}(\xi)e^{-2\xi}d\xi - \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Big\{\int_0^{\xi}\overline{M}_x(\zeta)e^{-\zeta}d\zeta + \\ & + \sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}B_s}\Big[\operatorname{sh}\lambda_{s1}\xi\int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_x^0(\zeta)\right)d\zeta - \\ & - \int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta - \xi)\frac{d}{d\zeta}\left(e^{-\zeta}\tilde{M}_x^0(\zeta)d\zeta\right)\Big]\Big\}d\xi. \end{aligned} \tag{15.347}$$

$$
\begin{aligned}
\sigma_{m_s}(t, \alpha) = {} & \frac{N_z(t)\, e^{-t}}{2\pi R_0 h_0 \cos\beta\left(1 + \frac{1}{x}\right)} + \frac{M_x(t)\sin\alpha - M_y(t)\cos\alpha}{\pi R_0^2 h_0 \cos\beta\left(1 + \frac{1}{x}\right)} e^{-2t} - \\
& - \frac{En}{\pi R_0 \sqrt{\gamma}} e^{-t} \sum_{s=1}^{\infty} \lambda_{s1}^{\circ} \Bigg\{ \Phi_{1s1}(\alpha) \Bigg[\operatorname{ch} \lambda_{s1} t \int_{-\infty}^{0} e^{\lambda_{s1}\zeta} \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_x^{\circ}(\zeta)\right) d\zeta + \\
& + \int_{0}^{t} \operatorname{ch} \lambda_{s1}(\zeta - t) \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_x^{\circ}(\zeta)\right) d\zeta \Bigg] - \\
& - \Phi_{2s1}(\alpha) \Bigg[\operatorname{ch} \lambda_{s1} t \int_{-\infty}^{0} e^{\lambda_{s1}\zeta} \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_y^{\circ}(\zeta)\right) d\zeta - \\
& - \int_{0}^{t} \operatorname{ch} \lambda_{s1}(\zeta - t) \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_y^{\circ}(\zeta)\right) d\zeta \Bigg]\Bigg\}.
\end{aligned} \tag{15.348}
$$

$$
\begin{aligned}
\tau_{n_s m_s}(t, \alpha) = {} & \frac{M_{y_0}(t)\sin\alpha + M_{x_0}(t)\cos\alpha}{\pi R_0^2 h_0 \operatorname{ctg}\beta} e^{-2t} + \frac{M_z(t)}{2\pi R_0^2 h_0} e^{-2t} + \\
& + \frac{G}{R_0} e^{-t} \sum_{s=1}^{\infty} \Bigg\{ \left(\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\pi \bar{B}_s}}\right) \Bigg[\operatorname{sh} \lambda_{s1} t \int_{-\infty}^{0} e^{\lambda_{s1}\zeta} \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_x^{\circ}(\zeta)\right) d\zeta - \\
& - \int_{0}^{t} \operatorname{sh} \lambda_{s1}(\zeta - t) \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_x^{\circ}(\zeta)\right) d\zeta \Bigg] - \\
& - \left(\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\pi \bar{B}_s}}\right) \Bigg[\operatorname{sh} \lambda_{s1} t \int_{-\infty}^{0} e^{\lambda_{s1}\zeta} \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_y^{\circ}(\zeta)\right) d\zeta - \\
& - \int_{0}^{t} \operatorname{sh} \lambda_{s1}(\zeta - t) \frac{d}{d\zeta} \left(e^{-\zeta} \tilde{M}_y^{\circ}(\zeta)\right) d\zeta \Bigg]\Bigg\}.
\end{aligned} \tag{15.349}
$$

Thus, all expressions (15.345)-(15.349) contain infinite series in the normal coordinate functions Φ_{1s1}, Φ_{2s1}. This means that even for a completely fixed end, the stressed and strained state of a cantilever conical shell with a longitudinal structure, in contrast to a smooth shell under the indicated load, does not follow the law of plane sections. As can be shown by passing to the limit, this is also true of a cylindrical shell.

Let us consider the case in which there is no external surface load on the shell. Let P_x, P_y, P_z, be the components of the resultant external load applied to the end $t = t_1$, and M_{x_0}, M_{y_0}, M_{z_0} be its moments about the axes passing through the cone apex. Then, from the equilibrium conditions of the cut-off portion of the shell

$$
\begin{gathered}
Q_x(t)=P_x, \quad Q_y(t)=P_y, \quad N_z(t)=P_z, \\
M_x(t)=M_{x_0}-e^t l_0 P_y, \quad M_y(t)=M_{y_0}+e^t l_0 P_x, \quad M_z(t)=M_{z_0}.
\end{gathered} \tag{15.350}
$$

Introducing (15.350) into general expression (15.345), we can easily obtain

$$
\begin{aligned}
&u_{m_3}(t, \alpha)= \\
&=-\frac{\left[\frac{1}{2} P_z-(P_y \sin \alpha+P_x \cos \alpha) \operatorname{ctg} \beta\right] R_0 t+\left(M_{x_0} \sin \alpha-M_{y_0} \cos \alpha\right)\left(1-e^{-t}\right)}{\pi R_0 h_0 E\left(1+\frac{1}{\varkappa}\right) \sin \beta \cos \beta}- \\
&-\frac{1}{\sqrt{\pi E h_0 R_0} \sin \beta \cos \beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}-e^{-t}}{\bar{B}_s\left[(\lambda_{s1})^2-1\right]}\left[M_{x_0} \Phi_{1s1}(\alpha)-M_{y_0} \Phi_{2s1}(\alpha)\right].
\end{aligned} \tag{15.351}
$$

Expanding (15.351) with the aid of (15.110), we get

$$
\begin{aligned}
u_{m_3}(t, \alpha)= & -\frac{\left[\frac{1}{2} P_z-P_0 \operatorname{ctg} \beta \cos \left(\alpha-\alpha_p\right)\right] R_0 t+M_0\left(1-e^{-t}\right) \sin \left(\alpha-\alpha_{M_0}\right)}{\pi R_0 h_0 E\left(1+\frac{1}{\varkappa}\right) \sin \beta \cos \beta}- \\
& -\frac{M_0}{\pi E h_0 R_0 \sin \beta \cos \beta}\left\{\sin \left(\alpha-\alpha_{M_0}\right) \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}-e^{-t}}{\bar{B}_s^2\left[\left(\frac{n \lambda_{s1}^{\circ}}{\pi}\right)^2-1\right]\left(\lambda_{s1}^2-1\right)}+\right. \\
& +\sin \left[\frac{2 \pi}{n}(p+1)-\alpha_{M_0}\right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{1}{B_{s1} \bar{B}_s} \frac{e^{\lambda_{s1} t}-e^{-t}}{\lambda_{s1}^2-1} \frac{\sin 2 \lambda_{s1}^{\circ} \xi}{\sin 2 \lambda_{s1}^{\circ}}+ \\
& \left.+\sin \left(\frac{2 \pi}{n} p-\alpha_{M_0}\right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{1}{B_{s1} \bar{B}_s} \frac{e^{\lambda_{s1} t}-e^{-t}}{\lambda_{s1}^2-1} \frac{\sin 2 \lambda_{s1}^{\circ}(1-\xi)}{\sin 2 \lambda_{s1}^{\circ}}\right\},
\end{aligned} \tag{15.352}
$$

where

$$
\begin{aligned}
& P_0=\sqrt{P_x^2+P_y^2}, \quad \alpha_p=\operatorname{arctg} \frac{P_y}{P_x}, \\
& M_0=\sqrt{M_{x_0}^2+M_{y_0}^2}, \quad \alpha_{M_0}=\operatorname{arctg} \frac{M_y}{M_x}.
\end{aligned} \tag{15.353}
$$

Expanding (15.346), (15.347), we also obtain

$$
\begin{aligned}
\eta_x(t)= & \frac{M_{y_0}}{\pi R_0 h_0 E \cos \beta \sin^2 \beta}\left[\left(\gamma \sin^2 \beta-\frac{\varkappa}{1+\varkappa}\right) \operatorname{sh} t+\frac{\varkappa}{1+\varkappa}\left(e^t-1\right)\right]- \\
& -\frac{P_x\left(t+1-e^t\right)}{\pi E h_0\left(1+\frac{1}{\varkappa}\right) \sin^2 \beta}- \\
& -\frac{M_{y_0}}{\pi E h_0 R_0 \cos \beta \sin^2 \beta} \sum_{s=1}^{\infty} \frac{1}{B_s^2\left(\lambda_{s1}^2-1\right)}\left(\frac{e^{\lambda_{s1} t}-e^t}{\lambda_{s1}-1}-+\operatorname{sh} t\right),
\end{aligned} \tag{15.354}
$$

$$
\begin{aligned}
\eta_0(t) = & -\frac{M_{z_0}}{\pi R_0 h_0 E \cos\beta \sin^2\beta}\left[\left(\gamma \sin^2\beta - \frac{\varkappa}{1+\varkappa}\right) \operatorname{sh} t + \frac{\varkappa}{1+\varkappa}(e^t - 1)\right] - \\
& - \frac{P_y(t + 1 - e^t)}{\pi E h_0 \left(1 + \frac{1}{\varkappa}\right) \sin^2\beta} + \\
& + \frac{M_{z_0}}{\pi E h_0 R_0 \cos\beta \sin^2\beta} \sum_{s=1}^{\infty} \frac{1}{\bar{B}_s^2(\lambda_{s1}^2 - 1)}\left(\frac{e^{\lambda_{s1} t} - e^t}{\lambda_{s1} - 1} + \operatorname{sh} t\right).
\end{aligned} \tag{15.355}
$$

For normal and tangential stresses, from (15.348), (15.349) we get

$$
\begin{aligned}
& \sigma_{m_z}(t, \alpha) = \\
& = \frac{\left[\frac{1}{2} P_z - (P_y \sin\alpha + P_x \cos\alpha) \operatorname{ctg}\beta\right] R_0 e^{-t} + (M_{x_0} \sin\alpha - M_{y_0} \cos\alpha) e^{-2t}}{\pi R_0^2 h_0 \cos\beta \left(1 + \frac{1}{\varkappa}\right)} + \\
& + \frac{e^{-t}}{\sqrt{\pi} h_0 R_0^2 \cos\beta} \sum_{s=1}^{\infty} \frac{\lambda_{s1} e^{\lambda_{s1} t} + e^{-t}}{\bar{B}_s(\lambda_{s1}^2 - 1)} [\Phi_{1s1}(\alpha) M_{x_0} - \Phi_{2s1}(\alpha) M_{y_0}],
\end{aligned} \tag{15.356}
$$

$$
\begin{aligned}
\tau_{n_z m_z}(t, \alpha) = & \frac{M_{y_0} \sin\alpha + M_{x_0} \cos\alpha}{\pi R_0^2 h_0 \operatorname{ctg}\beta} e^{-2t} + \frac{M_{z_0}}{2\pi R_0^2 h_0} e^{-2t} - \\
& - \frac{e^{-t}}{\sqrt{\pi} \gamma E h_0 R_0^2 \sin\beta \cos\beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} - e^{-t}}{\bar{B}_s(\lambda_{s1}^2 - 1)} \times \\
& \times \left[M_{x_0}\left(\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\pi}\bar{B}_s}\right) - M_{y_0}\left(\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\pi}\bar{B}_s}\right)\right].
\end{aligned} \tag{15.357}
$$

Expanding (15.356), (15.357) with the aid of (15.110) and using (15.353), we have

$$
\begin{aligned}
\sigma_{m_z}(t, \alpha) = & \frac{\left[\frac{1}{2} P_z - P_0 \operatorname{ctg}\beta \cos(\alpha - \alpha_p)\right] R_0 e^{-t} + M_0 e^{-2t} \sin(\alpha - \alpha_{M_0})}{\pi R_0^2 h_0 \cos\beta \left(1 + \frac{1}{\varkappa}\right)} + \\
& + \frac{M_0 e^{-t}}{\pi R_0^2 h_0 \cos\beta}\left\{\sin(\alpha - \alpha_{M_0}) \sum_{s=1}^{\infty} \frac{\lambda_{s1} e^{\lambda_{s1} t} + e^{-t}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1\right](\lambda_{s1}^2 - 1)} + \right. \\
& + \sin\left[\frac{2\pi}{n}(p+1) - \alpha_{M_0}\right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{\lambda_{s1} e^{\lambda_{s1} t} + e^{-t}}{B_{s1} \bar{B}_s (\lambda_{s1}^2 - 1)} \frac{\sin 2\lambda_{s1}^{\circ} \xi}{\sin 2\lambda_{s1}^{\circ}} + \\
& \left. + \sin\left(\frac{2\pi}{n} p - \alpha_{M_0}\right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{\lambda_{s1} e^{\lambda_{s1} t} + e^{-t}}{B_{s1} B_s (\lambda_{s1}^2 - 1)} \frac{\sin 2\lambda_{s1}^{\circ}(1 - \xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}.
\end{aligned} \tag{15.358}
$$

$$\tau_{n_z m_z}(t, \alpha) = \frac{M_0 \operatorname{tg} \beta \cos(\alpha - \alpha_{M_0}) + \frac{1}{2} M_{z_0}}{\pi R_0^2 h_0} e^{-2t} -$$

$$- \frac{n e^{-t} M_0}{\pi^2 \gamma E h_0 R_0^2 \sin\beta \cos\beta} \left\{ \cos(\alpha - \alpha_{M_0}) \frac{n}{\pi} \sum_{s=1}^{\infty} \lambda_{s1}^{\circ 2} \frac{e^{\lambda_{s1} t} - e^{-t}}{\bar{B}_s^2 \left| \left(\frac{n \lambda_{s1}^{\circ}}{\pi} \right)^2 - 1 \right| (\lambda_{s1}^2 - 1)} + \right.$$

$$+ \sin\left[\frac{2\pi}{n} (p+1) - \alpha_{M_0} \right] \sqrt{\pi} \sum_{s=1}^{\infty} \lambda_{s1}^{\circ} \frac{e^{\lambda_{s1} t} - e^{-t}}{\bar{B}_s B_{s1} (\lambda_{s1}^2 - 1)} \frac{\cos 2\lambda_{s1}^{\circ} \xi}{\sin 2\lambda_{s1}^{\circ}} -$$

$$\left. - \sin\left(\frac{2\pi}{n} p - \alpha_{M_0} \right) \sqrt{\pi} \sum_{s=1}^{\infty} \lambda_{s1}^{\circ} \frac{e^{\lambda_{s1} t} - e^{-t}}{B_s B_{s1} (\lambda_{s1}^2 - 1)} \frac{\cos 2\lambda_{s1}^{\circ} (1 - \xi)}{\sin 2\lambda_{s1}^{\circ}} \right\}. \qquad (15.359)$$

The expressions obtained can, as before, be extended to the case of a cylindrical shell by passing to the limit when $\beta \to 0$.

From (13.350), we have

$$M_{x_0} = M_{x_1} + e^{t_1} l_0 P_y, \quad M_{y_0} = M_{y_1} - e^{t_1} l_0 P_x, \qquad (15.360)$$

where M_{x_1}, M_{y_1} are specified moments of external forces on the end $t = t_1$.

Introducing (15.360) into (15.35?)-(15.355) and (15.358), (15.359) and using (15.241), we obtain for a cylindrical shell

$$u_{m_z}(\bar{Z}, \alpha) = \frac{\frac{1}{2} P_z R_0 + P_0 R_0 \left(\frac{\bar{Z}}{2} - \bar{Z}_1 \right) \cos(\alpha - \alpha_p) + M_1 \sin(\alpha - \alpha_{M_1})}{\pi R_0 E h_0 \left(1 + \frac{1}{\varkappa} \right)} \bar{Z} -$$

$$- \frac{P_0}{\pi E h_0} \left\{ \cos(\alpha - \alpha_p) \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1} \bar{Z}} - 1}{\bar{B}_s^2 \lambda_{s1}^2 (\gamma \lambda_{s1}^2 - 1)} + \right.$$

$$+ \cos\left[\frac{2\pi}{n} (p+1) - \alpha_p \right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1} \bar{Z}} - 1}{B_{s1} \bar{B}_s \lambda_{s1}^2} \frac{\sin 2\lambda_{s1}^{\circ} \xi}{\sin 2\lambda_{s1}^{\circ}} +$$

$$\left. + \cos\left(\frac{2\pi}{n} p - \alpha_p \right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1} \bar{Z}} - 1}{B_{s1} B_s \lambda_{s1}^2} \frac{\sin 2\lambda_{s1}^{\circ} (1 - \xi)}{\sin 2\lambda_{s1}} \right\}. \qquad (15.361)$$

$$\eta_x(\bar{Z}) = \frac{\varkappa}{1+\varkappa} \frac{M_{y_1}}{2\pi R_0 E h_0} \bar{Z}^2 + \frac{P_x}{\pi E h_0} \left[\gamma \bar{Z} + \frac{\varkappa}{1+\varkappa} \left(\frac{1}{2} \bar{Z}_1 \bar{Z}^2 - \frac{1}{6} \bar{Z}^3 \right) + \right.$$

$$\left. + \sum_{s=1}^{\infty} \frac{1}{\lambda_{s1}^3 \bar{B}_s^2} \left(e^{-\lambda_{s1} \bar{Z}} - \lambda_{s1} \bar{Z} - 1 \right) \right]. \qquad (15.362)$$

$$\eta_y(\bar{Z}) = -\frac{\varkappa}{1+\varkappa} \frac{M_{x_1}}{2\pi R_0 E h_0} \bar{Z}^2 + \frac{P_y}{\pi E h_0} \left[\gamma \bar{Z} + \frac{\varkappa}{1+\varkappa} \left(\frac{1}{2} \bar{Z}_1 \bar{Z}^2 - \frac{1}{6} \bar{Z}^3 \right) + \right.$$

$$\left. + \sum_{s=1}^{\infty} \frac{1}{\lambda_{s1}^3 \bar{B}_s^2} \left(e^{-\lambda_{s1} \bar{Z}} - \lambda_{s1} \bar{Z} - 1 \right) \right]. \qquad (15.363)$$

$$\sigma_{m_s}(\bar{Z}, \alpha)=\frac{P_z\frac{R_0}{2}-P_0R_0(\bar{Z}_1-\bar{Z})\cos(\alpha-\alpha_p)+M_1\sin(\alpha-\alpha_{M_1})}{\pi R_0^2h_0\left(1+\frac{1}{\varkappa}\right)}+$$

$$+\frac{P_0}{\pi R_0h_0}\left\{\cos(\alpha-\alpha_p)\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s^2\lambda_{s1}(\gamma\lambda_{s1}^2-1)}+\right.$$

$$+\cos\left[\frac{2\pi}{n}(p+1)-\alpha_p\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{B_{s1}\bar{B}_s\lambda_{s1}}\frac{\sin 2\lambda_{s1}^*\xi}{\sin 2\lambda_{s1}^*}+$$

$$\left.+\cos\left(\frac{2\pi}{n}p-\alpha_p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{B_{s1}\bar{B}_s\lambda_{s1}}\frac{\sin 2\lambda_{s1}^*(1-\xi)}{\sin 2\lambda_{s1}^*}\right\}, \tag{15.364}$$

$$\tau_{\alpha_s m_s}(\bar{Z}, \alpha)=-\frac{P_0\sin(\alpha-\alpha_p)}{\pi R_0h_0}+\frac{M_{z_0}}{2\pi R_0^2h_0}+$$

$$+\frac{P_0}{\pi Eh_0R_0}\left\{\sin(\alpha-\alpha_p)\frac{n}{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}-1}{\bar{B}^2(\gamma\lambda_{s1}^2-1)}-\right.$$

$$-\cos\left[\frac{2\pi}{n}(p+1)-\alpha_p\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}-1}{B_{s1}\bar{B}_s\lambda_{s1}}\frac{\cos 2\lambda_{s1}^*\xi}{\sin 2\lambda_{s1}^*}+$$

$$\left.+\cos\left(\frac{2\pi}{n}p-\alpha_p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}-1}{B_{s1}\bar{B}_s\lambda_{s1}}\frac{\cos 2\lambda_{s1}^*(1-\xi)}{\sin 2\lambda_{s1}^*}\right\}, \tag{15.365}$$

where

$$M_1=\sqrt{M_{x_0}^2+M_{y_0}^2};\quad \alpha_{M_1}=\operatorname{arctg}\frac{M_{y_0}}{M_{x_0}}. \tag{15.366}$$

Thus, the stressed and strained state of a shell with a longitudinal structure, even in the absence of a surface load, does not follow the law of plane sections; as we have ascertained, this applies not only to a conical shell but also to a cylindrical one. It should also be noted that for a cylindrical shell, deviations from the law of plane sections have the character of the edge effect, which damps out rapidly as the distance from the fixed end increases. Therefore, for a cylindrical shell, calculation based on beam theory outside the edge zone is justified, but this theory is generally inapplicable to a conical shell. Infinite series describing the deviations from the law of plane sections for a conical shell contain the exponent e^{-t} in addition to the rapidly damping exponential terms of the form $e^{\lambda_n t}$. This term corresponds to the effect of internal constraint, characteristic of conical shells, and must usually be taken into consideration.

2. Shell with Discretely Fixed End

Let the shell end t = 0 be completely fixed against tangential displacements, and in the direction of the generatrices, only at the points where the stringers $\alpha=\frac{2\pi}{n}m\ (m=1, 2, \ldots, n)$ are located. Fictitiously discarding the supports which prevent the displacements u_{m_z}, we will replace their actions by unknown support reactions. Then the problem reduces to the analysis of the shell for an end load of intensity

$$\bar{T}(\alpha)=\frac{1}{R_0}\sum_{m=1}^{n}P_m\delta\left(\alpha-\frac{2\pi}{n}m\right), \tag{15.367}$$

where P_m are unknown reactions applied to the stringers in the longitudinal direction, with

$$\eta_t(0)=\eta_\psi(0)=\theta_z(0)=0. \tag{15.368}$$

Satisfying conditions (15.368), we find from (15.185), (15.186), (15.189)

$$\eta_t^0=\eta_\psi^0=\theta_z^0=0. \tag{15.369}$$

As was shown during the formulation of the boundary conditions in Section 13.1, the arbitrary constants C, $\bar{C}$ are subject to determination from the kinematic or static conditions imposed, respectively, on the displacements or longitudinal forces in the end sections of the shell. In the case at hand, the conditions with respect to the longitudinal components on the end t = 0 are mixed. Therefore, first satisfying the static boundary conditions with the aid of (15.367) to within the unknown forces P_m, one must then find these forces from the kinematic conditions

$$u_{m_z}\left(0, \frac{2\pi}{n} m\right) = 0 \qquad (m = 1, 2, \ldots, n). \tag{15.370}$$

We will confine ourselves here to the case in which the external load is such that its longitudinal component p_z can be represented in the form (15.181), and the mutual influence of the ends can be neglected. These limitations, which significantly simplify the operations and final formulas, are not fundamental in character. We will introduce into consideration the interpolational polynomial (13.78), (13.79) corresponding to the desire reactions of the supports:

for an odd number n of stringers

$$P_m = A_0 + \sum_{r=1}^{\frac{n-1}{2}} \left(A_r \cos\frac{2\pi m}{n} + B_r \sin\frac{2\pi m}{n}\right), \tag{15.371}$$

for an even number n

$$P_m = A_0 + \sum_{r=1}^{\frac{n}{2}-1} \left(A_r \cos\frac{2\pi m}{n} + B_r \sin\frac{2\pi m}{n}\right) + (-1)^m A_{\frac{n}{2}}.$$

Introducing into equilibrium conditions (15.196) the linear load $\bar{T}(\alpha)$ represented by expression (15.367), for $t^* = 0$ we obtain

$$\begin{gathered} \cos\beta \sum_{m=1}^{n} P_m = N_z(0), \\ R_0 \cos\beta \sum_{m=1}^{n} P_m \sin\frac{2\pi}{n} m = M_x(0), \\ R_0 \cos\beta \sum_{m=1}^{n} P_m \cos\frac{2\pi}{n} m = -M_y(0), \end{gathered} \tag{15.372}$$

whence, using (13.80), we find

$$
\begin{aligned}
A_0 &= \frac{N_z(0)}{n \sin \beta}, \\
A_1 &= -\frac{M_y(0)}{n \frac{R_0}{2} \cos \beta}, \\
B_1 &= \frac{M_x(0)}{n \frac{R_0}{2} \cos \beta}.
\end{aligned}
\tag{15.373}
$$

The Fourier coefficients of an arbitrary system of forces P_m specified by interpolation polynomial (15.371) in normal coordinate functions $\Phi(\alpha)$ are represented by expressions (15.249).

Introducing (15.249) into (15.216), (15.217), we find

$$
\begin{aligned}
& C_{1s0} = 0, \\
& C_{2s0} = \frac{\pi \sqrt{\gamma}}{\sqrt{2} E h_0 \lambda_{2s0}^{\circ} B_{s0}} A_0, \\
& C_{1s1} = -\frac{1}{2} \int_{-\infty}^{0} e^{\lambda_{s1} \zeta} \frac{d}{d\zeta} \left[e^{-\zeta} \tilde{M}_x^{\circ}(\zeta) \right] d\zeta - \frac{\sqrt{\pi \gamma} \pi}{2 E h_0 \lambda_{s1}^{\circ} B_s} B_1, \\
& C_{2s1} = \frac{1}{2} \int_{-\infty}^{0} e^{\lambda_{s1} \zeta} \frac{d}{d\zeta} \left[e^{-\zeta} \tilde{M}_y^{\circ}(\zeta) \right] d\zeta - \frac{\sqrt{\pi \gamma} \pi}{2 E h_0 \lambda_{s1}^{\circ} B_s} A_1, \\
& C_{1sr} = -\frac{\pi \sqrt{\gamma}}{2 E h_0 \lambda_{sr}^{\circ} B_{sr}} B_r \qquad \left[r = 1, 2, \ldots, E\left(\frac{n-1}{2}\right) \right], \\
& C_{2sr} = -\frac{\pi \sqrt{\gamma}}{2 E h_0 \lambda_s^{\circ} B_{sr}} A_r \qquad \left[r = 1, 2, \ldots, E\left(\frac{n-1}{2}\right) \right], \\
& C_{1s\frac{n}{2}} = 0 \\
& C_{2s\frac{n}{2}} = -\frac{\pi \sqrt{\gamma}}{\sqrt{2} E h_0 \lambda_{2s\frac{n}{2}}^{\circ} B_{s\frac{n}{2}}} A_{\frac{n}{2}}.
\end{aligned}
\tag{15.374}
$$

From (15.212), (15.213), we also have

$$
\begin{aligned}
& \bar{C}_{1sr} = \bar{C}_{2sr} = 0 \qquad \left[r = 0, 2, 3, \ldots, E\left(\frac{n}{2}\right) \right], \\
& \bar{C}_{1s1} = -\frac{1}{2} \int_{-\infty}^{0} e^{\lambda_{s1} \zeta} \frac{d}{d\zeta} \left[e^{-\zeta} \tilde{M}_x^{\circ}(\zeta) \right] d\zeta, \\
& \bar{C}_{2s1} = \frac{1}{2} \int_{-\infty}^{0} e^{\lambda_{s1} \zeta} \frac{d}{d\zeta} \left[e^{-\zeta} \tilde{M}_y^{\circ}(\zeta) \right] d\zeta.
\end{aligned}
\tag{15.375}
$$

From (15.183), we obtain

$$u_{m_z}(0, \alpha)=\frac{1}{\sqrt{\pi\left(1+\frac{1}{\varkappa}\right)}}\left[\frac{1}{\sqrt{2}}U_{r0}(0)+U_{01}(0)\sin\alpha+U_{02}(0)\cos\alpha\right]+$$

$$+\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left[(C_{1sr}+\bar{C}_{1sr})\Phi_{1sr}(\alpha)+(C_{2sr}+\bar{C}_{2sr})\Phi_{2sr}(\alpha)\right]. \qquad (15.376)$$

Introducing arbitrary constants (15.374), (15.375) into (15.376), in view of (15.207) and (15.369), when t = 0, we find

$$u_{m_z}(0, \alpha)=\eta_z^0\cos\beta+(\theta_x^0\sin\alpha-\theta_y^0\cos\alpha)R_0\cos\beta+$$

$$+\frac{\pi\sqrt{\gamma}}{\sqrt{2}Eh_0}\sum_{s=1}^{\infty}\left[\frac{A_0}{\lambda_{2s0}^{*}B_{s0}}\Phi_{2s0}(\alpha)-\frac{A_{\frac{n}{2}}}{\lambda_{2s\frac{n}{2}}^{*}B_{s\frac{n}{2}}}\Phi_{2s\frac{n}{2}}(\alpha)\right]-$$

$$-\frac{\varkappa\sqrt{\pi\gamma}}{2Eh_0}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{*}B_s}(A_1\Phi_{2s1}+B_1\Phi_{1s1})-$$

$$-\frac{\pi\sqrt{\gamma}}{2Eh_0}\sum_{s=1}^{\infty}\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\frac{1}{\lambda_{sr}^{*}B_{sr}}(A_r\Phi_{2sr}+B_r\Phi_{1sr})- \qquad (15.377)$$

$$-\sum_{s=1}^{\infty}\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left\{e^{-\zeta}\left[\Phi_{1s1}(\alpha)\tilde{M}_x^s(\zeta)-\Phi_{2s1}(\alpha)\tilde{M}_y^s(\zeta)\right]\right\}d\zeta.$$

Satisfying fixing conditions (15.370), in view of (15.185), (15.247) and (15.373), we obtain from (15.377)

$$\eta_z^0\cos\beta-N_z(0)\frac{\pi\sqrt{\gamma}}{2Eh_0 n\cos\beta}\sum_{s=1}^{\infty}\frac{1}{\lambda_{2s0}^{*}B_{s0}^{2}}+$$

$$+\left\{-\theta_y^0R_0\cos\beta+\frac{\sqrt{\gamma}\varkappa}{Eh_0R_0 n\cos\beta}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{*}B_s^{2}}\left[\varkappa M_y(0)+\right.\right.$$

$$\left.\left.+\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}(e^{-\zeta}M_x(\zeta))d\zeta\right]\right\}\cos\frac{2\pi}{n}m+\left\{\theta_x^0R_0\cos\beta-\right.$$

$$\left.-\frac{\sqrt{\gamma}\varkappa}{Eh_0R_0 n\cos\beta}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{*}B_s^{2}}\left[\varkappa M_x(0)+\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}(e^{-\zeta}M_x(\zeta))d\zeta\right]\right\}\sin\frac{2\pi}{n}m-$$

$$-\frac{\pi\sqrt{\gamma}}{2Eh_0}\sum_{r=1}^{E\left(\frac{n-1}{2}\right)}\left(A_r\cos\frac{2\pi r}{n}m+B_r\sin\frac{2\pi r}{n}m\right)\sum_{s=1}^{\infty}\frac{1}{\lambda_{sr}^{*}B_{sr}^{2}}- \qquad (15.378)$$

$$-\frac{\pi\sqrt{\gamma}}{2Eh_0}A_{\frac{n}{2}}(-1)^m\sum_{s=1}^{\infty}\frac{1}{\lambda_{2s\frac{n}{2}}^{*}B_{s\frac{n}{2}}^{2}}=0 \quad (m=1, 2, \ldots, n).$$

Expressions (15.378) represent a system of n equations in the three integration constants $\eta_x^0, \theta_x^0, \theta_y^0$ и $(n-3)$, and (n-3) coefficients of the interpolation polynomial A_r, B_r $\left[r=2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right]$ and $A_{\frac{n}{2}}$.
Let us note that system (15.378) was written, strictly speaking, only for an even number n of stringers. For odd n, $E\left(\frac{n-1}{2}\right)=\frac{n-1}{2}$, and the term containing $A_{\frac{n}{2}}(-1)^m$ is absent. This should also be kept in mind below, but, as can be ascertained, the final results apply to any n.

The solution of system (15.378) is obvious. It is easy to see that this system can be made to correspond to an interpolation polynomial assuming zero values at the points $\alpha=\frac{2\pi}{n}m(m=1, 2, \ldots, n)$. However, in this case, on the basis of (13.70), all the coefficients of this polynomial must be equal to zero. In view of these considerations, by equating to zero in (15.378) the free term and the coefficients on $\sin\frac{2\pi r}{n}m, \cos\frac{2\pi r}{n}m$, $\left[r=1, 2, \ldots, E\left(\frac{n-1}{2}\right)\right]$, and also on $(-1)^m$, we find

$$\eta_x^0=N_x(0)\frac{\pi\sqrt{\gamma}}{2Eh_0 n\cos^2\beta}\sum_{s=1}^{\infty}\frac{1}{\lambda_{2s0}^{*}B_{s0}^2},$$

$$\theta_x^0=-\frac{\pi\sqrt{\gamma}}{Bh_0R_0^2 n\cos^2\beta}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{*}\overline{B}_s^2}\left[\varkappa M_x(0)+\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left(e^{-\zeta}M_x(\zeta)\right)d\zeta\right],\qquad(15.379)$$

$$\theta_y^0=-\frac{\pi\sqrt{\gamma}}{Eh_0R_0^2 n\cos^2\beta}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^{*}\overline{B}_s^2}\left[\varkappa M_y(0)+\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left(e^{-\zeta}M_y(\zeta)\right)d\zeta\right]$$

$$A_r=B_r=0\quad\left[r=2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right],\qquad(15.380)$$

$$A_{\frac{n}{2}}=0.$$

Thus, if the longitudinal component of the external surface load contains no self-balanced components, then for discrete cyclic fixing of a not too short stringer conical shell, the reactions of the supports are distributed according to the law of the plane:

$$P_m = \frac{1}{n\cos\beta} N_z(0) - \frac{2}{nR_0\cos\beta}\left[M_y(0)\cos\frac{2\pi}{n}m - M_x(0)\sin\frac{2\pi}{n}m\right]. \quad (15.381)$$

Introducing the values of arbitrary constants (15.374), (15.375) for the obtained coefficients of the interpolation polynomial A_0, A_1, B_1 (15.373) into the general expressions for displacements (15.183), (15.187), (15.188) and stresses (15.191), (15.193), we obtain

$$\begin{aligned} u_{m_z}(t,\alpha) = {} & u^0_{m_z}(0,\alpha) - \int_0^t \overline{N}_z(\zeta)\,d\zeta - \sin\alpha \int_0^t \overline{M}_x(\zeta)e^{-\zeta}d\zeta + \\ & + \cos\alpha \int_0^t \overline{M}_y(\zeta)e^{-\zeta}d\zeta - \sum_{s=1}^{\infty}\Big\{ -\overline{N}^s_z(0)e^{\lambda_{s0}t}\Phi_{s0}(\alpha) + \\ & + \operatorname{ch}\lambda_{s1}t \int_{-\infty}^{0} e^{\lambda_{s1}\zeta}\frac{\partial}{\partial\zeta}\left[e^{-\zeta}(\Phi_{1s1}(\alpha)\widetilde{M}^s_x(\zeta) - \Phi_{2s1}(\alpha)\widetilde{M}^s_y(\zeta))\right]d\zeta + \\ & + xe^{\lambda_{s1}t}\left[\Phi_{1s1}(\alpha)\widetilde{M}^s_x(0) - \Phi_{2s1}(\alpha)\widetilde{M}^s_y(0)\right] + \\ & + \int_0^t \operatorname{sh}\lambda_{s1}(\zeta - t)\frac{\partial}{\partial\zeta}\left[e^{-\zeta}(\Phi_{1s1}(\alpha)\widetilde{M}^s_x(\zeta) - \Phi_{2s1}(\alpha)\widetilde{M}^s_y(\zeta))\right]d\zeta, \end{aligned} \quad (15.382)$$

$$\begin{aligned} \eta_x(t) = {} & \theta^0_y R_0 \operatorname{ctg}\beta(1 - e^t) + \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Big\{\int_0^{\xi}\overline{M}_y(\zeta)e^{-\zeta}d\zeta - \\ & - \sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}B_s}\Big[\operatorname{ch}\lambda_{s1}\xi\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}(e^{-\zeta}\widetilde{M}^s_y(\zeta))\,d\zeta + x\widetilde{M}^s_y(0)e^{\lambda_{s1}\xi} + \\ & + \int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta - \xi)\frac{d}{d\zeta}(e^{-\zeta}\widetilde{M}^s_y(\zeta))\,d\zeta\Big]\Big\}d\xi + \frac{e^t}{R_0 h_0 G\cos\beta}\int_0^t M_{y_\kappa}(\zeta)e^{-\zeta}d\zeta, \end{aligned}$$

(15.383)

$$\begin{aligned} \eta_y(t) = {} & -\theta^0_x R_0 \operatorname{ctg}\beta(1 - e^t) - \frac{1}{\sin\beta}\int_0^t e^{t-\xi}\Big\{\int_0^{\xi}\overline{M}_x(\zeta)e^{-\zeta}d\zeta - \\ & - \sum_{s=1}^{\infty}\frac{1}{\sqrt{\pi}B_s}\Big[\operatorname{ch}\lambda_{s1}\xi\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}(e^{-\zeta}\widetilde{M}^s_x(\zeta))\,d\zeta + x\widetilde{M}^s_x(0)e^{\lambda_{s1}\xi} + \\ & + \int_0^{\xi}\operatorname{sh}\lambda_{s1}(\zeta - \xi)\frac{d}{d\zeta}(e^{-\zeta}\widetilde{M}^s_x(\zeta))\,d\zeta\Big]\Big\}d\xi - \frac{e^t}{R_0 h_0 G\cos\beta}\int_0^t M_{x_\kappa}(\zeta)e^{-\zeta}d\zeta, \end{aligned}$$

(15.384)

$$\sigma_{n_z}(t,\alpha)=\frac{N_z(t)e^{-t}}{2\pi R_0h_0\left(1+\frac{1}{\varkappa}\right)\cos\beta}-\frac{M_x(t)\sin\alpha-M_y(t)\cos\alpha}{\pi R_0^2h_0\left(1+\frac{1}{\varkappa}\right)\cos\beta}e^{-2t}+$$

$$+\frac{En}{\pi R_0\sqrt{\gamma}}e^{-t}\sum_{s=1}^{\infty}\lambda_{s1}^{\circ}\Big\{-\frac{\lambda_{2s0}^{\circ}}{\lambda_{s1}^{\circ}}\bar{N}_z^{\circ}(0)e^{\lambda_{2s0}t}\Phi_{2s0}(\alpha)+$$

$$+\operatorname{sh}\lambda_{s1}t\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{\partial}{\partial\zeta}\left[e^{-\zeta}\left(\Phi_{1s1}(\alpha)\bar{M}_x^{\circ}(\zeta)-\right.\right.$$

$$\left.\left.-\Phi_{2s1}(\alpha)\bar{M}_y^{\circ}(\zeta)\right)\right]d\zeta+\varkappa e^{\lambda_{s1}t}\left[\Phi_{1s1}(\alpha)\bar{M}_x^{\circ}(0)-\Phi_{2s1}(\alpha)\bar{M}_y^{\circ}(0)\right]-$$

$$-\int_{0}^{t}\operatorname{ch}\lambda_{s1}(\zeta-t)\,\frac{\partial}{\partial\zeta}\left[e^{-\zeta}\left(\Phi_{1s1}(\alpha)\bar{M}_x^{\circ}(\zeta)-\Phi_{2s1}(\alpha)\bar{M}_y^{\circ}(\zeta)\right)\right]d\zeta\Big\}. \qquad (15.385)$$

$$\tau_{n_zm_z}(t,\alpha)=\frac{M_z(t)}{2\pi R_0^2h_0}e^{-2t}+\frac{M_{y_0}(t)\sin\alpha+M_{x_0}(t)\cos\alpha}{\pi R_0^2h_0\operatorname{ctg}\beta}e^{-2t}-$$

$$-\frac{G}{R_0}e^{-t}\sum_{s=1}^{\infty}\Big\{-\bar{N}_z^{\circ}(0)e^{\lambda_{2s0}t}\Phi'_{2s0}(\alpha)+\operatorname{ch}\lambda_{s1}t\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\times$$

$$\times\frac{\partial}{\partial\zeta}\Big\{e^{-\zeta}\Big[\left(\Phi'_{1s1}(\alpha)+\frac{\cos\alpha}{\sqrt{\pi}\bar{B}_s}\right)\bar{M}_x^{\circ}(\zeta)-\left(\Phi'_{2s1}(\alpha)-\frac{\sin\alpha}{\sqrt{\pi}\bar{B}_s}\right)\bar{M}_y^{\circ}(\zeta)\Big]\Big\}d\zeta+$$

$$+\varkappa e^{\lambda_{s1}t}\Big[\left(\Phi'_{1s1}(\alpha)+\frac{\cos\alpha}{\sqrt{\pi}\bar{B}_s}\right)\bar{M}_x^{\circ}(0)-\left(\Phi'_{2s1}(\alpha)-\frac{\sin\alpha}{\sqrt{\pi}\bar{B}_s}\right)\bar{M}_y^{\circ}(0)\Big]+$$

$$+\int_{0}^{t}\operatorname{sh}\lambda_{s1}(\zeta-t)\frac{\partial}{\partial\zeta}\Big\{e^{-\zeta}\Big[\left(\Phi'_{1s1}(\alpha)+\frac{\cos\alpha}{\sqrt{\pi}\bar{B}_s}\right)\bar{M}_x^{\circ}(\zeta)-$$

$$-\left(\Phi'_{2s1}(\alpha)-\frac{\sin\alpha}{\sqrt{\pi}\bar{B}_s}\right)\bar{M}_y^{\circ}(\zeta)\Big]\Big\}d\zeta\Big\}. \qquad (15.386)$$

Here

$$u_{n_z}^{0}(0,\alpha)=\eta_z^{0}\cos\beta+R_0\cos\beta\,(\theta_x^{0}\sin\alpha-\theta_y^{0}\cos\alpha), \qquad (15.387)$$

where $\eta_z^0, \theta_x^0, \theta_y^0$ are determined by relations (15.379);

$$\bar{N}_z^{\circ}=\frac{\pi\sqrt{\gamma}}{\sqrt{2}Eh_0\pi\cos\beta\lambda_{2s0}^{\circ}B_{s0}}N_z. \qquad (15.388)$$

Let us consider the case in which there is no external surface load on the shell. Let P_x, P_y, P_z be the components of the resultant of the external load, applied to the end t = t_1, and $M_{x_0}, M_{y_0}, M_{z_0}$, it moments about the axes passing through the cone apex. In this case, the transverse forces and moments in the current section of the shell are determined by relations (15.350). Introducing (15.350) into expressions (15.382)-(15.386), we can obtain

$$
\begin{aligned}
u_{m_z}(t,\alpha) = {} & u^{c}_{m_z}(t,\alpha) - \\
& - \frac{\left[\frac{1}{2}P_z - (P_y \sin\alpha + P_x \cos\alpha)\operatorname{ctg}\beta\right] R_0 t + (M_{x_0}\sin\alpha - M_{y_0}\cos\alpha)(1 - e^{-t})}{\pi R_0^2 h_0 E\left(1 + \frac{1}{\varkappa}\right)\sin\beta\cos\beta} + \\
& + \frac{1}{\sqrt{\pi} E h_0 R_0 \sin\beta\cos\beta}\sum_{s=1}^{\infty}\left\{\frac{\sqrt{\pi}R_0 P_z}{\sqrt{2}\lambda_{2s0}B_{s0}} e^{\lambda_{2s0}t}\Phi_{2s0}(\alpha) - \right. \\
& - \frac{\varkappa e^{\lambda_{s1}t}}{\lambda_{s1}\bar{B}_s}\left[\Phi_{1s1}(\alpha)(M_{x_0} - l_0 P_y) - \Phi_{2s1}(\alpha)(M_{y_0} + l_0 P_x)\right] + \\
& \left. + \frac{e^{\lambda_{s1}t} + \lambda_{s1}e^{-t}}{\lambda_{s1}(\lambda_{s1}^2 - 1)\bar{B}_s}\left[\Phi_{1s1}(\alpha)M_{x_0} - \Phi_{2s1}(\alpha)M_{y_0}\right]\right\}, \qquad (15.389)
\end{aligned}
$$

$$
\begin{aligned}
\eta_x(t) = {} & \theta_y^0 R_0 \operatorname{ctg}\beta(1 - e^t) + \frac{M_{y_0}}{\pi R_0 h_0 E\cos\beta\sin^2\beta}\left[\left(\gamma\sin^2\beta - \frac{\varkappa}{1+\varkappa}\right)\operatorname{sh} t + \right. \\
& \left. + \frac{\varkappa}{1+\varkappa}(e^t - 1)\right] - \frac{P_x(t + 1 - e^t)}{\pi E h_0\left(1 + \frac{1}{\varkappa}\right)\sin^2\beta} - \frac{1}{\pi E h_0 R_0\cos\beta\sin^2\beta}\sum_{s=1}^{\infty}\left\{(M_{y_0} + \right. \\
& \left. + l_0 P_x)\varkappa\frac{e^{\lambda_{s1}t} - e^t}{\lambda_{s1}(\lambda_{s1} - 1)\bar{B}_s^2} - \frac{M_{y_0}}{\lambda_{s1}(\lambda_{s1}^2 - 1)\bar{B}_s^2}\left(\lambda_{s1}\operatorname{sh} t + \frac{e^{\lambda_{s1}t} - e^t}{\lambda_{s1} - 1}\right)\right\}, \qquad (15.390)
\end{aligned}
$$

$$
\begin{aligned}
\eta_y(t) = {} & -\theta_x^0 R_0 \operatorname{ctg}\beta(1 - e^t) - \frac{M_{x_0}}{\pi R_0 h_0 E\cos\beta\sin^2\beta}\left[\left(\gamma\sin^2\beta - \frac{\varkappa}{1+\varkappa}\right)\operatorname{sh} t + \right. \\
& \left. + \frac{\varkappa}{1+\varkappa}(e^t - 1)\right] - \frac{P_y(t + 1 - e^t)}{\pi E h_0\left(1 + \frac{1}{\varkappa}\right)\sin^2\beta} + \\
& + \frac{1}{\pi E h_0 R_0\cos\beta\sin^2\beta}\sum_{s=1}^{\infty}\left\{(M_{x_0} - l_0 P_y)\varkappa\frac{e^{\lambda_{s1}t} - e^t}{\lambda_{s1}(\lambda_{s1} - 1)\bar{B}_s^2} - \right. \\
& \left. - \frac{M_{x_0}}{\lambda_{s1}(\lambda_{s1}^2 - 1)\bar{B}_s^2}\left(\lambda_{s1}\operatorname{sh} t + \frac{e^{\lambda_{s1}t} - e^t}{\lambda_{s1} - 1}\right)\right\}, \qquad (15.391)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{m_z}(t,\alpha) = {} & \frac{\left[\frac{1}{2}P_z - (P_y\sin\alpha + P_x\cos\alpha)\operatorname{ctg}\beta\right] R_0 e^{-t} + (M_{x_0}\sin\alpha - M_{y_0}\cos\alpha)e^{-2t}}{\pi R_0^2 h_0\cos\beta\left(1 + \frac{1}{\varkappa}\right)} + \\
& + \frac{e^{-t}}{\sqrt{\pi}R_0^2 h_0\cos\beta}\sum_{s=1}^{\infty}\left\{-\frac{\sqrt{\pi}R_0 P_z}{\sqrt{2}B_{s0}} e^{\lambda_{2s0}t}\Phi_{2s0}(\alpha) + \right. \\
& + \frac{\varkappa e^{\lambda_{s1}t}}{\bar{B}_s}\left[\Phi_{1s1}(\alpha)(M_{x_0} - l_0 P_y) - \Phi_{2s1}(\alpha)(M_{y_0} + l_0 P_x)\right] - \\
& \left. - \frac{e^{\lambda_{s1}t} - e^{-t}}{(\lambda_{s1}^2 - 1)\bar{B}_s}\left[\Phi_{1s1}(\alpha)M_{x_0} - \Phi_{2s1}(\alpha)M_{y_0}\right]\right\}. \qquad (15.392)
\end{aligned}
$$

$$\tau_{n_z m_z}(t,\alpha) = \frac{M_{z_0}}{2\pi R_0^2 h_0} e^{-2t} + \frac{M_{y_0} \sin\alpha + M_{x_0} \cos\alpha}{\pi R_0^2 h_0 \operatorname{ctg}\beta} e^{-2t} -$$

$$- \frac{e^{-t}}{\sqrt{\pi}\eta h_0 R_0^2 \sin\beta\cos\beta} \sum_{s=1}^{\infty} \Bigg\{ - \frac{\sqrt{\pi} R_0 P_z}{\sqrt{2}\lambda_{2s0} B_{s0}} e^{\lambda_{2s0} t} \Phi'_{2s0}(\alpha) +$$

$$+ \frac{n e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}_s} \left[\left(\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\pi}\bar{B}_s} \right) (M_{x_0} - l_0 P_y) - \left(\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\pi}\bar{B}_s} \right) \times \right.$$

$$\left. \times (M_{y_0} + l_0 P_x) \right] - \frac{e^{\lambda_{s1} t} + \lambda_{s1} e^{-t}}{\lambda_{s1}(\lambda_{s1}^2 - 1)\bar{B}_s} \left[\left(\Phi'_{1s1}(\alpha) + \frac{\cos\alpha}{\sqrt{\pi}\bar{B}_s} \right) M_{x_0} - \right.$$

$$\left. - \left(\Phi'_{2s1}(\alpha) - \frac{\sin\alpha}{\sqrt{\pi}\bar{B}_s} \right) M_{y_0} \right] \Bigg\}. \qquad (15.393)$$

Introducing into (15.389), (15.392), (15.393) expanded expressions (15.109), (15.11)) for the normal coordinate functions Φ'_{2s0}, Φ_{1s1}, Φ_{2s1} and considering (15.353), (15.366), we finally obtain:

$$u_{m_z}(t,\alpha) = u^0_{m_z}(0,\alpha) -$$

$$- \frac{\left[\frac{1}{2} P_z - P_0 \operatorname{ctg}\beta \cos(\alpha - \alpha_p) \right] R_0 t + M_0 (1 - e^{-t}) \sin(\alpha - \alpha_{M_0})}{\pi R_0 h_0 E \left(1 + \frac{1}{n} \right) \sin\beta\cos\beta} -$$

$$- \frac{P_z}{2 E h_0 \sin\beta\cos\beta} \sum_{s=1}^{\infty} \frac{e^{\lambda_{2s0} t}}{\lambda_{2s0} B_{s0}^2} \frac{\cos 2\lambda^{\circ}_{2s0} \left(\xi - \frac{1}{2} \right)}{\cos \lambda^{\circ}_{2s0}} -$$

$$- \frac{M_0 n}{\pi E h_0 R_0 \sin\beta\cos\beta} \Bigg\{ \sin(\alpha - \alpha_{M_0}) \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}_s^2 \left[\left(\frac{n \lambda^{\circ}_{s1}}{\pi} \right)^2 - 1 \right]} +$$

$$+ \sin\left[\frac{2\pi}{n}(p+1) - \alpha_{M_0} \right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}_s B_{s1}} \frac{\sin 2\lambda^{\circ}_{s1} \xi}{\sin 2\lambda^{\circ}_{s1}} +$$

$$+ \sin\left(\frac{2\pi}{n} p - \alpha_{M_0} \right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t}}{\lambda_{s1} \bar{B}_s B_{s1}} \frac{\sin 2\lambda^{\circ}_{s1} (1 - \xi)}{\sin 2\lambda^{\circ}_{s1}} \Bigg\} +$$

$$+ \frac{M_0}{\pi E h_0 R_0 \sin\beta\cos\beta} \Bigg\{ \sin(\alpha - \alpha_{M_0}) \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} + \lambda_{s1} e^{-t}}{\lambda_{s1} (\lambda_{s1}^2 - 1) \left[\left(\frac{n \lambda^{\circ}_{s1}}{\pi} \right)^2 - 1 \right] \bar{B}_s^2} +$$

$$+ \sin\left[\frac{2\pi}{n}(p+1) - \alpha_{M_0} \right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} + \lambda_{s1} e^{-t}}{\lambda_{s1} (\lambda_{s1}^2 - 1) \bar{B}_s B_{s1}} \frac{\sin 2\lambda^{\circ}_{s1} \xi}{\sin 2\lambda^{\circ}_{s1}} + \qquad (15.394)$$

$$+ \sin\left(\frac{2\pi}{n} p - \alpha_{M_0} \right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{\lambda_{s1} t} + \lambda_{s1} e^{-t}}{\lambda_{s1} (\lambda_{s1}^2 - 1) \bar{B}_s B_{s1}} \frac{\sin 2\lambda^{\circ}_{s1} (1 - \xi)}{\sin 2\lambda^{\circ}_{s1}} \Bigg\},$$

$$
\begin{aligned}
\sigma_{m_z}(t,\alpha) = {} & \frac{\left[\frac{1}{2}P_z - P_0 \operatorname{ctg}\beta \cos(\alpha-\alpha_p)\right] R_0 e^{-t} + M_0 e^{-2t}\sin(\alpha-\alpha_{M_0})}{\pi R_0^2 h_0 \cos\beta\left(1+\frac{1}{\varkappa}\right)} + \\
& + \frac{P_z e^{-t}}{2R_0 h_0 \cos\beta}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{B_{s0}^2}\,\frac{\cos 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)}{\cos\lambda_{2s0}^{*}} + \\
& + \frac{M_z \varkappa e^{-t}}{\pi R_0^2 h_0 \cos\beta}\left\{\sin(\alpha-\alpha_{M_0})\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^2-1\right]} + \right. \\
& + \sin\left[\frac{2\pi}{n}(p+1)-\alpha_{M_0}\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_{s1}\bar{B}_s}\,\frac{\sin 2\lambda_{s1}^{*}\xi}{\sin 2\lambda_{s1}^{*}} + \\
& \left. + \sin\left(\frac{2\pi}{n}p-\alpha_{M_0}\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{B_{s1}\bar{B}_s}\,\frac{\sin 2\lambda_{s1}^{*}(1-\xi)}{\sin 2\lambda_{s1}}\right\} - \\
& - \frac{M_0 e^{-t}}{\pi R_0^2 h_0 \cos\beta}\left\{\sin(\alpha-\alpha_{M_0})\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}-e^{-t}}{\bar{B}_s^2(\lambda_{s1}^2-1)\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^2-1\right]} + \right. \\
& + \sin\left[\frac{2\pi}{n}(p+1)-\alpha_{M_0}\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}-e^{-t}}{(\lambda_{s1}^2-1)\bar{B}_s B_{s1}}\,\frac{\sin 2\lambda_{s1}^{*}\xi}{\sin 2\lambda_{s1}^{*}} + \\
& \left. + \sin\left(\frac{2\pi}{n}p-\alpha_{M_0}\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}-e^{-t}}{(\lambda_{s1}^2-1)\bar{B}_s B_{s1}}\,\frac{\sin 2\lambda_{s1}^{*}(1-\xi)}{\sin 2\lambda_{s1}}\right\}.
\end{aligned}
\qquad (15.395)
$$

$$
\begin{aligned}
\tau_{n_z m_z}(t,\alpha) = {} & \frac{M_0 \operatorname{tg}\beta\cos(\alpha-\alpha_{M_0}) + \frac{1}{2}M_{z_0}}{\pi R_0^2 h_0}e^{-2t} + \\
& + \frac{P_z e^{-t}}{2\sqrt{\gamma}\, h_0 R_0 \cos\beta}\sum_{s=1}^{\infty}\frac{e^{\lambda_{2s0}t}}{B_{s0}^2}\,\frac{\sin 2\lambda_{2s0}^{*}\left(\xi-\frac{1}{2}\right)}{\cos\lambda_{2s0}^{*}} - \\
& - \frac{\varkappa M e^{-t}}{\pi\sqrt{\gamma}\, R_0^2 h_0 \cos\beta}\left\{\cos(\alpha-\alpha_{M_0})\frac{n}{\pi}\sum_{s=1}^{\infty}\lambda_{s1}^{*}\frac{e^{\lambda_{s1}t}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^2-1\right]} + \right. \\
& + \sin\left[\frac{2\pi}{n}(p+1)-\alpha_{M_0}\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_s B_{s1}}\,\frac{\cos 2\lambda_{s1}^{*}\xi}{\sin 2\lambda_{s1}^{*}} - \\
& \left. - \sin\left(\frac{2\pi}{n}p-\alpha_{M_0}\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}}{\bar{B}_s B_{s1}}\,\frac{\cos 2\lambda_{s1}^{*}(\xi-1)}{\sin 2\lambda_{s1}^{*}}\right\} + \\
& + \frac{M_0 e^{-t}}{\pi\sqrt{\gamma}\, R_0^2 h_0 \cos\beta}\left\{\cos(\alpha-\alpha_{M_0})\frac{n}{\pi}\sum_{s=1}^{\infty}\lambda_{s1}^{*}\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\bar{B}_s^2(\lambda_{s1}^2-1)\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^2-1\right]} + \right. \\
& + \sin\left[\frac{2\pi}{n}(p+1)-\alpha_{M_0}\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\bar{B}_s B_{s1}(\lambda_{s1}^2-1)}\,\frac{\cos 2\lambda_{s1}^{*}\xi}{\sin 2\lambda_{s1}^{*}} - \\
& \left. - \sin\left(\frac{2\pi}{n}p-\alpha_{M_0}\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{\lambda_{s1}t}+\lambda_{s1}e^{-t}}{\bar{B}_s B_{s1}(\lambda_{s1}^2-1)}\,\frac{\cos 2\lambda_{s1}^{*}(1-\xi)}{\sin 2\lambda_{s1}^{*}}\right\}.
\end{aligned}
\qquad (15.396)
$$

Here the bending moment in section t = 0

$$M_2 = \sqrt{(M_{x_0} - l_0 P_y)^2 + (M_{y_0} + l_0 P_x)^2},$$

$$\alpha_{M_0} = \operatorname{arctg} \frac{M_{y_0} + l_0 P_x}{M_{x_0} - l_0 P_y}. \qquad (15.397)$$

For a cylindrical shell, passing to the limit when $\beta \to 0$, in view of (15.241) and (15.360, we obtain

$$u_{m_z}(\bar{Z}, \alpha) = u^0_{m_z}(0, \alpha) +$$

$$+ \frac{\frac{1}{2} P_z P_0 + P_0 R_0 \left(\frac{\bar{Z}}{2} - \bar{Z}_1\right) \cos(\alpha - \alpha_p) + M_1 \sin(\alpha - \alpha_{M_1})}{\pi R_0 E h_0 \left(1 + \frac{1}{\varkappa}\right)} \bar{Z} -$$

$$- \frac{P_z}{2Eh_0} \sum_{s=1}^{\infty} \frac{e^{-\lambda_{2s0}\bar{Z}}}{\lambda_{2s0} B^2_{s0}} \frac{\cos 2\lambda^*_{2s0}\left(\xi - \frac{1}{2}\right)}{\cos \lambda^*_{2s0}} -$$

$$- \frac{M_1 n}{\pi E h_0 F_0} \left\{ \sin(\alpha - \alpha_{M_0}) \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda_{s1} \bar{B}^2_s \left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2 - 1\right]} + \right.$$

$$+ \sin\left[\frac{2\pi}{n}(p+1) - \alpha_{M_0}\right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda_{s1} \bar{B}_s B_{s1}} \frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}} +$$

$$\left. + \sin\left(\frac{2\pi}{n} p - \alpha_{M_0}\right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1}\bar{Z}}}{\lambda_{s1} \bar{B}_s B_{s1}} \frac{\sin 2\lambda_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}} \right\} +$$

$$+ \frac{P_0}{\pi E h_0 R_0^2} \left\{ \cos(\alpha - \alpha_p) \sum_{s=1}^{\infty} \frac{1}{\lambda^2_{s1} \left[\left(\frac{n\lambda^*_{s1}}{\pi}\right)^2 - 1\right] \bar{B}^2_s} + \right.$$

$$+ \cos\left[\frac{2\pi}{n}(p+1) - \alpha_p\right] \sqrt{\pi} \sum_{s=1}^{\infty} \frac{1}{\lambda^2_{s1} \bar{B}_s B_{s1}} \frac{\sin 2\lambda^*_{s1}\xi}{\sin 2\lambda^*_{s1}} +$$

$$\left. + \cos\left(\frac{2\pi}{n} p - \alpha_p\right) \sqrt{\pi} \sum_{s=1}^{\infty} \frac{1}{\lambda^2_{s1} \bar{B}_s B_{s1}} \frac{\sin 2\lambda^*_{s1}(1-\xi)}{\sin 2\lambda^*_{s1}} \right\}, \qquad (15.398)$$

$$\eta_x(\bar{Z}) = \vartheta^0_y R_0 \bar{Z} + \frac{\varkappa}{1+\varkappa} \frac{M_{y_1}}{2\pi R_0 E h_0} \bar{Z}^2 + \frac{P_x}{\pi E h_0} \left[\gamma \bar{Z} + \frac{\varkappa}{1+\varkappa}\left(\frac{1}{2}\bar{Z}_1 \bar{Z}^2 - \frac{1}{\beta}\bar{Z}^3\right)\right] -$$

$$- \frac{M_{y_1} - P_x \bar{Z}_1 R_0}{\pi E h_0 R_0} \varkappa \sum_{s=1}^{\infty} \frac{e^{-\lambda_{s1}\bar{Z}} - 1}{\lambda^2_{s1} \bar{B}^2_s} + \frac{P_x}{\pi E h_0} \sum_{s=1}^{\infty} \frac{1}{\lambda^3_{s1} \bar{B}^2_s}. \qquad (15.399)$$

$$\eta_r(\bar{Z}) = -\theta_x^0 R_0 \bar{Z} - \frac{\varkappa}{1+\varkappa}\frac{M_{x_1}}{2\pi R_0 E h_0}\bar{Z}^2 + \frac{P_v}{\pi E h_0}\left[\gamma\bar{Z} + \frac{\varkappa}{1+\varkappa}\left(\frac{1}{2}\bar{Z}_1\bar{Z}^2 - \frac{1}{6}\bar{Z}^3\right)\right] +$$

$$+ \frac{M_{x_1} + P_v\bar{Z}_1 R_0}{\pi E h_0 R_0}\varkappa\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}} - 1}{\lambda_{s1}^2\bar{B}_s^2} + \frac{P_v}{\pi E h_0}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}^2\bar{B}_s^2}, \tag{15.400}$$

$$\sigma_{m_z}(\bar{Z}, \alpha) = \frac{P_z\frac{R_0}{2} - P_0 R_0(\bar{Z}_1 - \bar{Z})\cos(\alpha - \alpha_p) + M_1\sin(\alpha - \alpha_{M_1})}{\pi R_0^2 h_0\left(1 + \frac{1}{\varkappa}\right)} +$$

$$+ \frac{P_z}{2R_0 h_0}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s0}\bar{Z}}}{B_{s0}^2}\frac{\cos 2\lambda_{2s0}^{\circ}\left(\xi - \frac{1}{2}\right)}{\cos\lambda_{2s0}^{\circ}} +$$

$$+ \frac{M_0\varkappa}{\pi R_0^2 h_0}\left\{\sin(\alpha - \alpha_{M_0})\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1\right]} +\right.$$

$$+ \sin\left[\frac{2\pi}{n}(p+1) - \alpha_{M_0}\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{B_{s1}\bar{B}_s}\frac{\sin 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}} +$$

$$\left. + \sin\left(\frac{2\pi}{n}p - \alpha_{M_0}\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{B_{s1}\bar{B}_s}\frac{\sin 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}. \tag{15.401}$$

$$\tau_{\alpha_z m_z}(\bar{Z}, \alpha) = -\frac{P_0\sin(\alpha - \alpha_p)}{\pi R_0 h_0} + \frac{M_{z_0}}{2\pi R_0^2 h_0} +$$

$$+ \frac{P_z}{2\sqrt{\gamma}h_0 R_0}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{2s0}\bar{Z}}}{B_{s0}^2}\frac{\sin 2\lambda_{2s0}^{\circ}\left(\xi - \frac{1}{2}\right)}{\cos\lambda_{2s0}^{\circ}} -$$

$$- \frac{\varkappa M_2}{\pi\sqrt{\gamma}R_0^2 h_0}\left\{\cos(\alpha - \alpha_{M_0})\frac{n}{\pi}\sum_{s=1}^{\infty}\lambda_{s1}^{\circ}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1\right]} +\right.$$

$$+ \sin\left[\frac{2\pi}{n}(p+1) - \alpha_{M_0}\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}} -$$

$$\left. - \sin\left(\frac{2\pi}{n}p - \alpha_{M_0}\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{e^{-\lambda_{s1}\bar{Z}}}{\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\} +$$

$$+ \frac{P_0}{\pi\sqrt{\gamma}R_0 h_0}\left\{-\sin(\alpha - \alpha_p)\sqrt{\gamma}\sum_{s=1}^{\infty}\frac{1}{\bar{B}_s^2\left[\left(\frac{n\lambda_{s1}^{\circ}}{\pi}\right)^2 - 1\right]} +\right.$$

$$+ \cos\left[\frac{2\pi}{n}(p+1) - \alpha_p\right]\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}\xi}{\sin 2\lambda_{s1}^{\circ}} -$$

$$\left. - \cos\left(\frac{2\pi}{n}p - \alpha_p\right)\sqrt{\pi}\sum_{s=1}^{\infty}\frac{1}{\lambda_{s1}\bar{B}_s B_{s1}}\frac{\cos 2\lambda_{s1}^{\circ}(1-\xi)}{\sin 2\lambda_{s1}^{\circ}}\right\}. \tag{15.402}$$

Here M_2 is the bending moment in the section $\bar{Z} = $). From (15.397) when $\beta \to 0$ in view of (15.360), we have

$$M_z = \sqrt{(M_{x_1} + P_y \bar{Z}_1 l_0)^2 + (M_{y_1} - P_x \bar{Z}_1 l_0)^2},$$
$$\alpha_{M_z} = \operatorname{arctg} \frac{M_{y_1} - P_x \bar{Z}_1 l_0}{M_{x_1} + P_y \bar{Z}_1 l_0}. \qquad (15.403)$$

Thus, the stressed and strained state of a stringer shell for discrete cyclic fixing has been determined for the case where there is no surface load on the shell, and the end $t = t_1$ is loaded by an arbitrary system of external forces. One can similarly write the solution in the presence of a surface load. Moreover, if its longitudinal component p_z has the form (15.181), the solution is represented by expressions (15.382)-(15.386). However, it should be noted that in studying the concentration of stresses caused by discrete fixing, it is permissible to replace the distributed load by its resultant, and, normalizing the latter to the end $t = t_1$, to use expressions (15.390), (15.391), (15.394)-(15.396) for a conical shell and (15.398)-(15.402) for a cylindrical shell. Outside the zone of concentration of stresses caused by discrete fixing, it is desirable to use the solution corresponding to continuous fixing of the end $t = 0$. For the case where the component p_z of the surface load has the form (15.181), this solution is represented by expressions (15.345)-(15.349).

15.7 Stringer Conical Shell with an End Ring

Let a stringer shell be reinforced in the section $\bar{Z} = 0$ by an end ring of constant cross section. The problem of reinforcement of a smooth shell with an end ring is discussed in Section 13.6. The problem discussed below does not differ from the other problem fundamentally, i.e., the arbitrary constants must also be determined from the strain compatibility condition of the shell end ring. However, while in the former case the stressed and strained state of both the ring and the shell was represented by trigonometric series, so that the arbitrary constants for each of the harmonics were determined independently of one another, in the case at hand this will not occur. The strained state of the ring must as before be represented in the form of a trigonometric series, whereas the displacements and stresses of a stringer shell were obtained above in the form of series in normal coordinate functions. Since these functions, generally speaking, are not orthogonal to trigonometric functions, we will arrive at infinite systems of algebraic equations in arbitrary constants, which will complicate the problem and generally will not permit us to obtain a solution in closed form.

In addition, if a smooth shell is joined to the ring along its entire contour, the character of the connection between the stringer shell and the ring may be different. Two cases will be considered in the present section: a) connection along the entire contour, b) connection between the ring and the stringers only, without contact with the shell.

As above, we will confine the treatment to the case in which the longitudinal component p_z of the external surface load on the shell contains no self-balanced components, and the mutual influence of the ends is negligible. As already noted, these limitations are not fundamental in character, but substantially simplify the operations and final results.

1. Connection Along the Entire Contour

Let $T=T^*(\alpha)$, $S=S^*(\alpha)$ be the unknown forces of the interaction of the ring with the shell, referred to a unit length of the contour $\bar{Z} = 0$. These forces are subject to determination from the equilibrium conditions and strain compatibility conditions of the shell and ring:

$$[u^*_{m_z}(0, \alpha)]_{\text{оболочки}} = [u^*_{m_z}(\alpha)]_{\text{шпангоута}}, \tag{15.404}$$

where $u^*_{m_z}$ are the displacements in the direction of the generatrices, corresponding to warpings of the contour $\bar{Z} = 0$.

For a stringer shell, we have from (15.176)

$$u^*_{m_z}(0, \alpha) = \sum_{s=1}^{\infty} \sum_{r=0}^{E\left(\frac{n}{2}\right)} [(C_{1sr}+\bar{C}_{1sr})\Phi_{1sr}(\alpha)+(C_{2sr}+\bar{C}_{2sr})\Phi_{2sr}(\alpha)], \tag{15.405}$$

where according to (15.212), (15.213), (15.216), (15.217)

$$\begin{gathered}
C_{1sr}+\bar{C}_{1sr} = -\frac{R_0}{Eh_0 \sin \beta\lambda_{1sr}} \int_0^{2\pi} T^*(\alpha)\Phi_{1sr}(\alpha)\,d\alpha, \\
C_{2sr}+\bar{C}_{2sr} = -\frac{R_0}{Eh_0 \sin \beta\lambda_{2sr}} \int_0^{2\pi} T^*(\alpha)\Phi_{2sr}(\alpha)\,d\alpha \\
\left[r=0, 2, 3, \ldots, E\left(\frac{n}{2}\right)\right], \quad (s=1, 2, \ldots), \\
C_{1s1}+\bar{C}_{1s1} = -\int_{-\infty}^{0} e^{\lambda_{s1}\zeta} \frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}^s_x(\zeta)\right] d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{s1}} \int_0^{2\pi} T^*(\alpha)\Phi_{1s1}(\alpha)\,d\alpha, \\
C_{2s1}+\bar{C}_{2s1} = \int_{-\infty}^{0} e^{\lambda_{s1}\zeta} \frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}^s_y(\zeta)\right] d\zeta - \frac{R_0}{Eh_0 \sin \beta\lambda_{s1}} \int_0^{2\pi} T^*(\alpha)\Phi_{2s1}(\alpha)\,d\alpha \\
(s=1, 2, \ldots).
\end{gathered} \tag{15.406}$$

The corresponding displacements of the ring will be represented in the form

$$u^*_{m_z}(\alpha) = u^{T*}_{m_z}(\alpha) + u^{S*}_{m_z}(\alpha), \tag{15.407}$$

e

where $u^{0*}_{m_s}$ - are the displacements of the ring due to the given external load to within its displacements as a solid;

$u^{1*}_{m_s}$ - are the displacements of the ring due to the self-balanced components of the contact forces.

Expanding the contact forces $T^*(\alpha)$, $S^*(\alpha)$ in trigonometric series, on the basis of the superposition principle, in view of (13.193), (13.194), we obtain

$$u^*_{m_s}(\alpha)=u^{0*}_{m_s}(\alpha)+\frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\left[\frac{1}{d_k^T}\int_0^{2\pi}T^*(\alpha)\cos k\alpha d\alpha+\frac{1}{d_k^S}\int_0^{2\pi}S^*(\alpha)\sin k\alpha d\alpha\right]\times\right.$$
$$\left.\times\cos k\alpha+\left[\frac{1}{d_k^S}\int_0^{2\pi}S^*(\alpha)\cos k\alpha d\alpha+\frac{1}{d_k^T}\int_0^{2\pi}T^*(\alpha)\sin k\alpha d\alpha\right]\sin k\alpha\right\}, \tag{15.408}$$

where d_k^T, d_k^S are some generalized rigidities of the ring.

Introducing expressions (15.405), (15.408), constituting the displacements of the shell and ring along the line of contact, into strain compatibility conditions (15.404), we can establish a relation between the Fourier coefficients of the unknown contact load $T^*(\alpha)$, $S^*(\alpha)$ in trigonometric functions $\sin k\alpha$, $\cos k\alpha$ and the quadratures

$$\int_0^{2\pi}T^*(\alpha)\Phi_{1rr}(\alpha)\,d\alpha,\quad \int_0^{2\pi}T^*(\alpha)\Phi_{2rr}(\alpha)\,d\alpha. \tag{15.409}$$

It should be considered at the same time that within the framework of the computation model adopted, the constraint between the shell and ring with respect to tangential displacements is realized without considering the deformations of the line of contact in its plane. Hence, the tangential contact forces $S^*(\alpha)$ are not independent and can be determined from elasticity relations.

When t = 0, from (15.193) we have

$$
S^*(\alpha) = \frac{M_{\xi_0}(0)\sin\alpha + M_{\eta_0}(0)\cos\alpha}{\pi R_0^2 \operatorname{ctg}\beta} + \frac{M_\zeta(0)}{2\pi R_0^2} +
$$

$$
+ \frac{Gh_0}{R_0}\sum_{s=1}^{\infty}\left\{\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left[(C_{1sr}+\bar{C}_{1sr})\Phi'_{1sr}(\alpha)+(C_{2sr}+\bar{C}_{2sr})\Phi'_{2sr}(\alpha)\right]+ \right. \tag{15.410}
$$

$$
\left. + \frac{1}{\sqrt{\pi}\,\bar{B}_s}\left[(C_{1s1}+\bar{C}_{1s1})\cos\alpha-(C_{2s1}+\bar{C}_{2s1})\sin\alpha\right]\right\}.
$$

Now, introducing (15.410) into (15.408) and considering expansions (15.130) of normal coordinate functions in trigonometric series, after some transformations we obtain

$$
u^*_{m_z}(\alpha) = u^{0*}_{m_z}(\alpha) + \frac{1}{\pi}\sum_{k=2}^{\infty}\left\{\frac{1}{d_k^T}\left[\sin k\alpha\int_0^{2\pi}T^*(\alpha)\sin k\alpha d\alpha + \right.\right.
$$

$$
\left. + \cos k\alpha\int_0^{2\pi}T^*(\alpha)\cos k\alpha d\alpha\right] + \frac{kGh_0}{d_k^S R_0}\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left[(C_{1sr}+\bar{C}_{1sr})\times\right.
$$

$$
\left.\left. \times\sin k\alpha\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha d\alpha - (C_{2sr}+\bar{C}_{2sr})\cos k\alpha\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha d\alpha\right]\right\}. \tag{15.411}
$$

We expand strain compatibility conditions (15.404). Using (15.405), (15.411), we obtain

$$
\sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left\{(C_{1sr}+\bar{C}_{1sr})\left[\Phi_{1sr}(\alpha) - \frac{Gh_0}{\pi R_0}\sum_{k=2}^{\infty}\frac{k}{d_k^S}\sin k\alpha\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha d\alpha\right]+\right.
$$

$$
\left. +(C_{2sr}+\bar{C}_{2sr})\left[\Phi_{2sr}(\alpha)+\frac{Gh_0}{\pi R_0}\sum_{k=2}^{\infty}\frac{k}{d_k^S}\cos k\alpha\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha d\alpha\right]\right\} - \tag{15.412}
$$

$$
-\frac{1}{\pi}\sum_{k=2}^{\infty}\frac{1}{d_k^T}\left[\sin k\alpha\int_0^{2\pi}T^*(\alpha)\sin k\alpha d\alpha + \right.
$$

$$
\left. + \cos k\alpha\int_0^{2\pi}T^*(\alpha)\cos k\alpha d\alpha\right] - u^{0*}_{m_z}(\alpha) = 0.
$$

Introducing into (15.412) the values of the arbitrary constants according to (15.406), we finally obtain

$$
\begin{gathered}
\frac{\pi\lambda\bar{\gamma}P_0}{l h \pi}\sum_{s=1}^{\infty}\sum_{r=1}^{r\left(\frac{s}{2}\right)}\left\{\frac{1}{\lambda_{1sr}^{c}}\int_0^{2\pi}T^*(\alpha)\Phi_{1sr}(\alpha)d\alpha\left[\Phi_{1sr}(\alpha)-\right.\right. \\
\left.-\frac{Gh_0}{\pi R_0}\sum_{k=2}^{\infty}\frac{k}{d_k^S}\sin k\alpha\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha d\alpha\right]+\frac{1}{\lambda_{2sr}^{c}}\int_0^{2\pi}T^*(\alpha)\Phi_{2sr}(\alpha)d\alpha\times \\
\left.\times\left[\Phi_{2sr}(\alpha)+\frac{Gh_0}{\pi R_0}\sum_{k=2}^{\infty}\frac{k}{d_k^S}\cos k\alpha\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha d\alpha\right]\right\}+ \\
+\frac{1}{\pi}\sum_{k=2}^{\infty}\frac{1}{d_k^T}\left[\sin k\alpha\int_0^{2\pi}T^*(\alpha)\sin k\alpha d\alpha+\cos k\alpha\int_0^{2\pi}T^*(\alpha)\cos k\alpha d\alpha\right]+ \\
+u_{m_z}^{0*}(\alpha)+\sum_{s=1}^{\infty}\left\{C_x^s\left[\Phi_{1s1}(\alpha)-\frac{Gh_0}{\pi R_0}\sum_{k=2}^{\infty}\frac{k}{d_k^S}\sin k\alpha\int_0^{2\pi}\Phi_{1s1}(\alpha)\sin k\alpha d\alpha\right]-\right. \\
\left.-C_y^s\left[\Phi_{2s1}(\alpha)+\frac{Gh_0}{\pi R_0}\sum_{k=2}^{\infty}\frac{k}{d_k^S}\cos k\alpha\int_0^{2\pi}\Phi_{2s1}(\alpha)\cos k\alpha d\alpha\right]\right\}=0,
\end{gathered}
\qquad (15.413)
$$

where

$$
\begin{gathered}
C_x^s=-\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}_x^s(\zeta)\right]d\zeta, \\
C_y^s=\int_{-\infty}^{0}e^{\lambda_{s1}\zeta}\frac{d}{d\zeta}\left[e^{-\zeta}\tilde{M}_y^s(\zeta)\right]d\zeta.
\end{gathered}
\qquad (15.414)
$$

Expression (15.413) represents the fundamental integral resolvent in contact load T*(α). This equation reduces to an infinite system of algebraic equations in Fourier coefficients of the desired contact load $T^*(\alpha)$ in $\sin k\alpha, \cos k\alpha$ and quadratures (15.409). Such a system can be obtained by orthogonalizing (15.413) to any system of functions that is complete on the circle. In the case at hand, it is natural to select for this either the trigonometric functions $\sin k\alpha, \cos k\alpha$, or the normal coordinate functions Φ_{1sr}, Φ_{2sr}. Moreover, in selecting the unknown Fourier series coefficients of the contact load t* in trigonometric functions as the fundamental unknowns, it is necessary to express quadratures (15.409) in terms of them. This can be done by using expansions of normal coordinate functions in trigonometric series. Conversely, after choosing quadratures (15.409) as the fundamental unknowns, it is necessary to use them to express the Fourier series coefficients of the contact load in trigonometric functions, having first expanded them as series in normal coordinate functions.

Below, we will need the following orthogonality conditions, which can be obtained by using expansions of normal coordinate functions in trigonometric series (15.130):

$$\int_0^{2\pi} \Phi_{1sr} \cos k\alpha d\alpha = \int_0^{2\pi} \Phi_{2sr} \sin k\alpha d\alpha = 0 \; (k, s, r - \text{ any}$$

$$\int_0^{2\pi} \Phi_{1sr}(\alpha) \sin q_k n\alpha d\alpha = \begin{cases} 0 & (r \neq 0), \\ 0 & (r = 0;\; s \neq q_k), \\ \sqrt{\pi} & (r = 0;\; s = q_k), \end{cases}$$

$$\int_0^{2\pi} \Phi_{2sr}(\alpha) \cos q_k n\alpha d\alpha = \begin{cases} 0 & (r \neq 0), \\ \frac{\pi\sqrt{2}}{\varkappa B_{s0}} \frac{\lambda_{2s0}^{*2}}{\pi^2 q_k^2 - \lambda_{2s0}^{*2}} & (r = 0;\; q_k \neq 0), \\ -\frac{\pi\sqrt{2}}{\varkappa B_{s0}} & (r = 0;\; q_k = 0), \end{cases}$$

$$\int_0^{2\pi} \Phi_{1sr}(\alpha) \sin (q_k n \pm 1)\alpha d\alpha = \begin{cases} 0 & (r \neq 1), \\ \pm \frac{\sqrt{\pi}}{B_s} \frac{\lambda_{s1}^{*2}}{\pi^2\left(q_k \pm \frac{1}{n}\right)^2 - \lambda_{s1}^{*2}} & (r = 1;\; q_k \neq 0), \\ \mp \frac{\sqrt{\pi}}{B_s} & (r = 1;\; q_k \neq 0), \end{cases}$$

$$\int_0^{2\pi} \Phi_{2sr}(\alpha) \cos (q_k n \pm 1)\alpha d\alpha = \begin{cases} 0 & (r \neq 1), \\ \frac{\sqrt{\pi}}{B_s} \frac{\lambda_{s1}^{*2}}{\pi^2\left(q_k \pm \frac{1}{n}\right)^2 - \lambda_{s1}^{*2}} & (r = 1;\; q_k \neq 0), \\ -\frac{\sqrt{\pi}}{B} & (r = 1;\; q_k = 0), \end{cases}$$

$$\int_0^{2\pi} \Phi_{1sr}(\alpha) \sin (q_k n \pm \bar{r})\alpha d\alpha = \begin{cases} 0 & (r \neq \bar{r}), \\ \pm \frac{\pi}{\varkappa B_{s\bar{r}}} \frac{\lambda_{s\bar{r}}^{*2}}{\pi^2\left(q_k \pm \frac{\bar{r}}{n}\right)^2 - \lambda_{s\bar{r}}^{*2}} & (r = \bar{r}), \end{cases}$$
$$\left[\bar{r} = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right] \qquad (15.415)$$

$$
\int_0^{2\pi} \Psi_{2sr}(\alpha)\cos(q_s n \pm \bar{r})\alpha d\alpha = \begin{cases} 0 & (r \neq \bar{r}), \\ \dfrac{n}{\pi B_{\bar{s}\bar{r}}} \dfrac{\lambda^{*2}_{s\bar{r}}}{n^2\left(q_s \pm \dfrac{\bar{r}}{n}\right)^2 - \lambda^{*2}_{s\bar{r}}} & (r = \bar{r}), \end{cases}
$$

$$
\left[\bar{r} = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right]
$$

$$
\int_0^{2\pi} \Phi_{1sr}(\alpha)\sin\left(q_s - \frac{1}{2}\right)n\alpha d\alpha = \begin{cases} 0 & \left(r \neq \dfrac{n}{2}\right), \\ 0 & \left(r = \dfrac{n}{2};\ s \neq q_s\right); \\ \sqrt{\pi} & \left(r = \dfrac{n}{2};\ s = q_s\right) \end{cases}
$$

$$
\int_0^{2\pi} \Phi_{2sr}(\alpha)\cos\left(q_s - \frac{1}{2}\right)n\alpha d\alpha = \begin{cases} 0 & \left(r \neq \dfrac{n}{2}\right), \\ \dfrac{n\sqrt{2}}{\pi B_{s\frac{n}{2}}} \dfrac{\lambda^{*2}_{s\frac{n}{2}}}{n^2\left(q_s - \dfrac{1}{2}\right)^2 - \lambda^{*2}_{s\frac{n}{2}}} & \left(r = \dfrac{n}{2}\right). \end{cases} \qquad (15.415)
$$

To within the multiplier $1/\pi$, quadratures (15.415) constitute Fourier series coefficients of normal coordinate functions in trigonometric functions. However, in this case we will be interested in Fourier series coefficients of trigonometric functions in normal coordinate functions:

$$
\int_0^{2\pi} \sin k\alpha \Phi(\alpha) \rho(\alpha) d\alpha, \qquad \int_0^{2\pi} \cos k\alpha \Phi(\alpha) \rho(\alpha) d\alpha.
$$

To calculate these coefficients, we will use the following method. We multiply fundamental integro-differential Eq. (15.54) successively by $\sin k\alpha$, $\cos k\alpha$ and integrate both times over α from 0 to 2π. Considering periodicity conditions (15.56), we can obtain

$$
\begin{aligned}
&\int_0^{2\pi} \sin k\alpha \Phi(\alpha) \rho(\alpha) d\alpha = \left(\frac{nk}{n\lambda^*}\right)^2 \int_0^{2\pi} \Phi(\alpha) \sin k\alpha d\alpha \quad (k \neq 1), \\
&\int_0^{2\pi} \cos k\alpha \Phi(\alpha) \rho(\alpha) d\alpha = \left(\frac{nk}{n\lambda^*}\right)^2 \int_0^{2\pi} \Phi(\alpha) \cos k\alpha d\alpha \quad (k \neq 1),
\end{aligned} \qquad (15.416)
$$

where $\Phi(\alpha)$ and λ^* are some eigenfunction and its corresponding eigenvalues. When $k = 1$, we obtain the obvious result

$$
\int_0^{2\pi} \cos \alpha \Phi(\alpha) \rho(\alpha) d\alpha = \int_0^{2\pi} \sin \alpha \Phi(\alpha) \rho(\alpha) d\alpha = 0. \qquad (15.417)
$$

since sin α, cos α are the eigenfunctions of problem (15.54), (15.56) corresponding to zero eigenvalue.

Formulas (15.416) have established a relationship between Fourier series coefficients of trigonometric functions in normal coordinate functions and Fourier series coefficients of normal coordinate functions and trigonometric functions, represented by formulas (15.415). It follows from (15.415) and (15.416) that the expansions of trigonometric functions sin $(q_k n \pm \bar{r})\alpha$, $\cos(q_k n \pm \bar{r})\alpha$ in normal coordinate functions Ψ_{1sr}, Ψ_{2sr} contain no terms corresponding to $r \neq \bar{r}$. Therefore, expansions IV.70 of trigonometric functions sin $(q_k n \pm \bar{r})\alpha$, $\cos(q_k n \pm \bar{r})\alpha$ in eigenfunctions of the special Sturm-Liouville problem, obtained in Section 5, Appendix IV, coincide completely for $\bar{r} \neq 1$ with the corresponding expansions in normal coordinate functions. According to the notation adopted in the present chapter, we have

$$\cos q_k n\alpha = \frac{1}{1+\varkappa} + \frac{\sqrt{2}\pi^3}{\varkappa} \sum_{s=1}^{\infty} \frac{1}{B_{s0}} \frac{q_k^2}{\pi^2 q_k^2 - \lambda_{2s0}^{*2}} \Phi_{2s0}(\alpha),$$

$$\cos(q_k n \pm r) = \frac{\pi^3}{\varkappa} \sum_{s=1}^{\infty} \frac{1}{B_{sr}} \frac{\left(q_k \pm \frac{r}{n}\right)^2}{\pi^2\left(q_k \pm \frac{r}{n}\right)^2 - \lambda_{sr}^{*2}} \Phi_{2sr}(\alpha) \qquad (15.418)$$

$$\left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right],$$

$$\cos\left(q_k n - \frac{n}{2}\right)\alpha = \frac{\sqrt{2}\pi^3}{\varkappa} \sum_{s=1}^{\infty} \frac{1}{B_{s\frac{n}{2}}} \frac{\left(q_k - \frac{1}{2}\right)^2}{\pi^2\left(q_k - \frac{1}{2}\right)^2 - \lambda_{2s\frac{n}{2}}^{*2}} \Phi_{2s\frac{n}{2}}(\alpha).$$

$$\sin(q_k n \pm r)\alpha = \pm \frac{\pi^3}{\varkappa} \sum_{s=1}^{\infty} \frac{1}{B_{sr}} \frac{\left(q_k \pm \frac{r}{n}\right)^2}{\pi^2\left(q_k \pm \frac{r}{n}\right)^2 - \lambda_{sr}^{*2}} \Phi_{1sr}(\alpha)$$

$$\left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right].$$

Using (15.415), (15.416), we also find

$$\cos(q_k n \pm 1)\alpha = \pi^2 \sqrt{\pi} \sum_{s=1}^{\infty} \frac{1}{\bar{B}_s} \frac{\left(q_k \pm \frac{1}{n}\right)^2}{\pi^2\left(q_k \pm \frac{1}{n}\right)^2 - \lambda_{s1}^{*2}} \Phi_{2s1}(\alpha),$$

$$\sin(q_k n \pm 1)\alpha = \pm \pi^2 \sqrt{\pi} \sum_{s=1}^{\infty} \frac{1}{\bar{B}_s} \frac{\left(q_k \pm \frac{1}{n}\right)^2}{\pi^2\left(q_k \pm \frac{1}{n}\right)^2 - \lambda_{s1}^{*2}} \Phi_{1s1}(\alpha). \quad (15.419)$$

Let us turn to Eq. (15.413). Analysis of this equation shows that it is most desirable to select quadratures (15.409), as the fundamental unknowns, then orthogonalize (15.413) to a system of normal coordinate functions. This method of solution is most effective. Also important is the fact that the arbitrary constants C, $\bar{C}$ represented by expressions (15.406) explicitly depend on quadratures (15.409).

We will first eliminate the Fourier coefficients of the desired contact load t* from (15.413). Introducing the notation

$$\int_0^{2\pi} T^*(\alpha)\sin k\alpha d\alpha = X_{1,k}; \quad \int_0^{2\pi} T^*(\alpha)\cos k\alpha d\alpha = X_{2,k},$$

$$\int_0^{2\pi} T^*(\alpha)\Phi_{1sr}(\alpha)d\alpha = Y_{1sr}; \quad \int_0^{2\pi} T^*(\alpha)\Phi_{2sr}(\alpha)d\alpha = Y_{2sr} \quad (15.420)$$

and expanding the trigonometric function as series in normal coordinate functions, we have

$$X_{1,k} = \sum_{s=1}^{\infty} \sum_{r=0}^{E\left(\frac{n}{2}\right)} Y_{1sr} \int_0^{2\pi} \sin k\alpha \Phi_{1sr}(\alpha)\rho(\alpha)d\alpha,$$

$$X_{2,k} = \int_0^{2\pi} T^*(\alpha)d\alpha \int_0^{2\pi} \cos k\alpha \rho(\alpha)d\alpha + \sum_{s=1}^{\infty} \sum_{r=0}^{E\left(\frac{n}{2}\right)} Y_{2sr} \int_0^{2\pi} \cos k\alpha \Phi_{2sr}(\alpha)\rho(\alpha)d\alpha, \quad (15.421)$$

whence, considering relations (15.416) as well as (15.196), we obtain

$$X_{1,k}=-\left(\frac{nk}{n}\right)^2\sum_{s=1}^{\infty}\sum_{r=1}^{E\left(\frac{s}{2}\right)}\frac{1}{\lambda_{1sr}^{*2}}Y_{1sr}\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha d\alpha,$$

$$X_{2,k}=\frac{N_z(0)}{R_0\cos\beta}\int_0^{2\pi}\cos k\alpha\rho(\alpha)\,d\alpha+$$

$$+\left(\frac{nk}{n^*}\right)^2\sum_{s=1}^{\infty}\sum_{r=1}^{E\left(\frac{s}{2}\right)}\frac{1}{\lambda_{2sr}^{*2}}Y_{2sr}\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha d\alpha. \tag{15.422}$$

Now, introducing (15.422) into Eq. (15.413), in view of (15.418), we finally obtain

$$\frac{\pi}{n}\sum_{s=1}^{\infty}\sum_{r=1}^{E\left(\frac{s}{2}\right)}\Bigg\{\frac{1}{\lambda_{1sr}^{*}}Y_{1sr}\Bigg[\frac{\sqrt{\gamma}R_0}{Eh_0}\Phi_{1sr}(\alpha)+$$
$$+\frac{1}{\pi}\sum_{k=1}^{\infty}k\sin k\alpha\left(\frac{\pi}{nk\lambda_{1sr}^{*}}\frac{k}{d_k^2}-\frac{1}{\sqrt{\gamma}d_k^2}\right)\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha d\alpha\Bigg]+$$
$$+\frac{1}{\lambda_{2sr}^{*}}Y_{2sr}\Bigg[\frac{\sqrt{\gamma}R_0}{Eh_0}\Phi_{2sr}(\alpha)+\frac{1}{\pi}\sum_{k=1}^{\infty}k\cos k\alpha\left(\frac{\pi}{nk\lambda_{2sr}^{*}}\frac{k}{d_k^2}+\frac{1}{\sqrt{\gamma}d_k^2}\right)\times$$
$$\times\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha d\alpha\Bigg]\Bigg\}+\alpha_{m,z}^{T}(\alpha)+\frac{2}{\pi R_0\cos\beta}N_z(0)\sum_{q_k=1}^{\infty}\frac{\cos q_k n\alpha}{d_{q_k n}^{T}}+$$
$$+\sum_{s=1}^{\infty}\Bigg\{C_s^{*}\Bigg[\Phi_{1s1}(\alpha)-\frac{Gh_0}{nR_0}\sum_{k=1}^{\infty}\frac{k}{d_k^2}\sin k\alpha\int_0^{2\pi}\Phi_{1s1}(\alpha)\sin k\alpha d\alpha\Bigg]-$$
$$-C_s^{*}\Bigg[\Phi_{2s1}(\alpha)+\frac{Gh_0}{nR_0}\sum_{k=1}^{\infty}\frac{k}{d_k^2}\cos k\alpha\int_0^{2\pi}\Phi_{2s1}(\alpha)\cos k\alpha d\alpha\Bigg]\Bigg\}=0. \tag{15.423}$$

We will orthogonalize (15.423) with respect to a system of normal coordinate functions. For this purpose, successively multiplying (15.423) by $\varrho\Phi_{1\bar{s}0},\ \varrho\Phi_{1\bar{s}1},\ \varrho\Phi_{2\bar{s}1},\ \varrho\Phi_{1\bar{s}\bar{r}},\ \varrho\Phi_{2\bar{s}\bar{r}},\ \varrho\Phi_{2\bar{s}\frac{\bar{s}}{2}}$ where $\bar{s},\bar{r}$ are some fixed values of s and r, we integrate the result each time over α from 0 to 2π. Then, considering expansions (15.418), (15.419) as well as orthogonality conditions (15.415) and denoting

$$\frac{\pi}{n}\frac{k}{d_k^2}-\frac{\lambda_{1sr}^{*}}{\sqrt{\gamma}d_k^2}=\frac{1}{D_{k,1sr}},\quad\frac{\pi}{n}\frac{k}{d_k^2}+\frac{\lambda_{2sr}^{*}}{\sqrt{\gamma}d_k^2}=\frac{1}{D_{k,2sr}},$$
$$\frac{\left(\frac{k}{n}\right)^2}{\left[\left(\frac{nk}{n}\right)^2-\lambda_{1sr}^{*2}\right]\left[\left(\frac{nk}{n}\right)^2-\lambda_{2sr}^{*2}\right]}=A_{k,sr}^{\bar{}}, \tag{15.424}$$

we obtain

$$\frac{\sqrt{\gamma}R_0}{Eh_0}\frac{\pi}{n\lambda^{\circ}_{2\bar{s}0}}Y^{\circ}_{2\bar{s}0}+\frac{2\pi^4}{\varkappa^2 B_{\bar{s}0}}\sum_{s=1}^{\infty}\frac{1}{B_{s0}}Y_{2s0}\sum_{q_k=1}^{\infty}\frac{A^{\bar{s}}_{q_k n, s0}}{D_{q_k n, 2s0}}+$$

$$+\int_0^{2\pi}u^{0^*}_{m_z}(\alpha)\Phi_{2\bar{s}0}(\alpha)\rho(\alpha)d\alpha+\frac{2\sqrt{2}\pi^3 N_z(0)}{\varkappa^2 R_0\cos\beta B_{\bar{s}0}}\sum_{q_k=1}^{\infty}\frac{1}{d^{T}_{q_k n}}\frac{q_k^2}{\pi^2 q_k^2-\lambda^{\circ 2}_{2\bar{s}0}}=0$$

$$(\bar{s}=1,\ 2,\dots). \tag{15.425}$$

$$\frac{\sqrt{\gamma}R_0}{Eh_0}\frac{\pi}{n\lambda^{\circ}_{\bar{s}1}}Y_{1\bar{s}1}+\frac{\pi^3}{\bar{B}_{\bar{s}}}\sum_{s=1}^{\infty}\frac{1}{\bar{B}_s}Y_{1s1}\sum_{q_k=1}^{\infty}\left[\frac{A^{\bar{s}}_{q_k n-1, s1}}{D_{q_k n-1, 1s1}}+\frac{A^{\bar{s}}_{q_k n+1, s1}}{D_{q_k n+1, 1s1}}\right]+$$

$$+\int_0^{2\pi}u^{0^*}_{m_z}(\alpha)\Phi_{1\bar{s}1}(\alpha)\rho(\alpha)d\alpha+C^{\bar{s}}_x-$$

$$-\frac{\pi^2 G h_0 n}{R_0 B_{\bar{s}}}\sum_{s=1}^{\infty}C^s_x\frac{\lambda^{\circ 2}_{s1}}{\bar{B}_s}\sum_{q_k=1}^{\infty}\left[\frac{A^{\bar{s}}_{q_k n-1, s1}}{d^S_{q_k n-1}}+\frac{A^{\bar{s}}_{q_k n+1, s1}}{d^S_{q_k n+1}}\right]=0\quad(\bar{s}=1,\ 2,\dots). \tag{15.426}$$

$$\frac{\sqrt{\gamma}R_0}{Eh_0}\frac{\pi}{n\lambda}Y_{2\bar{s}1}+\frac{\pi^3}{\bar{B}_{\bar{s}}}\sum_{s=1}^{\infty}\frac{1}{\bar{B}_s}Y_{2s1}\sum_{q_k=1}^{\infty}\left[\frac{A^{\bar{s}}_{q_k n-1, s1}}{D_{q_k n-1, 2s1}}+\frac{A^{\bar{s}}_{q_k n+1, s1}}{D_{q_k n+1, 2s1}}\right]+$$

$$+\int_0^{2\pi}u^{0^*}_{m_z}(\alpha)\Phi_{2\bar{s}1}(\alpha)\rho(\alpha)d\alpha-C^{\bar{s}}_v-$$

$$-\frac{\pi^2 G h_0 n}{B_0 B_{\bar{s}}}\sum_{s=1}^{\infty}C^s_v\frac{\lambda^{\circ 2}_{s1}}{\bar{B}_s}\sum_{q_k=1}^{\infty}\left[\frac{A^{\bar{s}}_{q_k n-1, s1}}{d^S_{q_k n-1}}+\frac{A^{\bar{s}}_{q_k n+1, s1}}{d^S_{q_k n+1}}\right]=0\quad(\bar{s}=1,\ 2,\dots). \tag{15.427}$$

$$\frac{\sqrt{\gamma}R_0}{Eh_0}\frac{\pi}{n\lambda^{\circ}_{\bar{s}\bar{r}}}Y_{1\bar{s}\bar{r}}+\frac{\pi^4}{\varkappa^2 B_{\bar{s}\bar{r}}}\sum_{s=1}^{\infty}\frac{1}{B_{s\bar{r}}}Y_{1s\bar{r}}\left\{\frac{A^{\bar{s}}_{\bar{r}, s\bar{r}}}{D_{\bar{r}, 1s\bar{r}}}+\right.$$

$$\left.+\sum_{q_k=1}^{\infty}\left[\frac{A^{\bar{s}}_{q_k n-\bar{r}, s\bar{r}}}{D_{q_k n-\bar{r}, 1s\bar{r}}}+\frac{A^{\bar{s}}_{q_k n+\bar{r}, s\bar{r}}}{D_{q_k n+\bar{r}, 1s\bar{r}}}\right]\right\}+\int_0^{2\pi}u^{0^*}_{m_z}(\alpha)\Phi_{1\bar{s}\bar{r}}(\alpha)\rho(\alpha)d\alpha=0$$

$$\left[\bar{s}=1,\ 2,\dots;\qquad \bar{r}=2,\ 3,\dots,\ E\left(\frac{n-1}{2}\right)\right]. \tag{15.428}$$

$$\frac{\sqrt{\gamma}R_0}{Eh_0}\frac{\pi}{n\lambda^{\circ}_{\bar{s}\bar{r}}}Y_{2\bar{s}\bar{r}}+\frac{\pi^4}{\varkappa^2 B_{\bar{s}\bar{r}}}\sum_{s=1}^{\infty}\frac{1}{B_{s\bar{r}}}Y_{2s\bar{r}}\left\{\frac{A^{\bar{s}}_{\bar{r}, s\bar{r}}}{D_{\bar{r}, 2s\bar{r}}}+\right.$$

$$\left.+\sum_{q_k=1}^{\infty}\left[\frac{A^{\bar{s}}_{q_k n-\bar{r}, s\bar{r}}}{D_{q_k n-\bar{r}, 2s\bar{r}}}+\frac{A^{\bar{s}}_{q_k n+\bar{r}, s\bar{r}}}{D_{q_k n+\bar{r}, 2s\bar{r}}}\right]\right\}+\int_0^{2\pi}u^{0^*}_{m_z}(\alpha)\Phi_{2\bar{s}\bar{r}}(\alpha)\rho(\alpha)d\alpha=0$$

$$\left[\bar{s}=1,\ 2,\dots;\qquad \bar{r}=2,\ 3,\dots,\ E\left(\frac{n-1}{2}\right)\right]. \tag{15.429}$$

$$\frac{\sqrt{\gamma}R_0}{Eh_0}\frac{\pi}{n\lambda^{\circ}_{\bar{s}\frac{n}{2}}}Y_{2\bar{s}\frac{n}{2}}+\frac{2\pi^4}{\varkappa^2 B_{\bar{s}\frac{n}{2}}}\sum_{s=1}^{\infty}\frac{1}{B_{s\frac{n}{2}}}Y_{2s\frac{n}{2}}\sum_{q_k=1}^{\infty}\frac{A^{\bar{s}}_{q_k n-\frac{n}{2}, s\frac{n}{2}}}{D_{q_k n-\frac{n}{2}, 2s\frac{n}{2}}}+$$

$$+\int_0^{2\pi}u^{0^*}_{m_z}(\alpha)\Phi_{2\bar{s}\frac{n}{2}}(\alpha)\rho(\alpha)d\alpha=0 \tag{15.430}$$

$$(\bar{s}=1,\ 2,\ \dots).$$

Expressions (15.425)-(15.430) represent n subsystems of algebraic equations in unknown quadratures $Y_{1ir}\left[r=1, 2, \ldots, E\left(\frac{n-1}{2}\right)\right]$, $Y_{2ir}\left[r=0, 1, 2, \ldots, E\left(\frac{n}{2}\right)\right]$, determined by expressions (15.420). Each of these subsystems, corresponding to a fixed $r = \bar{r}$, constitutes an infinite system of algebraic equations whose coefficients are expressed by infinite series, and therefore the solution of these equations in closed form is impossible. However, a numerical solution with any degree of accuracy is not difficult. It is most desirable to use the method of iterations, writing each of subsystems (15.425)-(15.430) in the form

$$Y = \sum_{q_k=1}^{\infty} B_{q_k} Y + C, \tag{15.431}$$

where Y, C are infinite column matrices of the unknowns and right-hand parts of any of the subsystems;

$\sum_{q_k=1}^{\infty} B_{q_k}$ is an infinite coefficient matrix, represented as a sum of infinite matrices whose elements contain the generalized rigidities of the ring d_k^r, d_k^s.

Taking as the initial approximation

$$Y^{(0)} = C, \tag{15.432}$$

we should carry out the process of successive approximations on the basis of an iteration relation determining an infinite but completely decomposed system

$$Y^{(l+1)} = \sum_{q_k=1}^{\infty} B_{q_k} Y^{(l)} + C. \tag{15.433}$$

The interation process described permits a clear interpretation. As we can see, initial approximation (15.432) corresponds to an absolutely rigid ring, since when $d_k^r = d_k^s = \infty$, all infinite matrices B_{q_k} become zero.

The contact load $V^{(q)}$ found in this manner is applied to an elastic ring and is successively refined. A satisfactory convergence is thus obtained rather quickly.

It should also be noted that the rigidities d_k^T, d_k^S increase rapidly with increasing k. Thus, according to (13.233), for a frame type ring

$$d_k^T = \frac{1}{R^4} \frac{k^2(k^2-1)^2}{\frac{1}{GI_{kp}} + k^2 \frac{1}{EI}}.$$

Therefore, infinite series of the type $\sum_{qk}$ entering into the coefficients of the equations of subsystems (15.425)-(15.430), converge very quickly. For practical purposes, when $n > 8-10$, it is permissible to neglect terms containing the generalized rigidities d_k^T, d_k^S corresponding to the numbers $k > \frac{n}{2}$, and in most cases, even to assume $d_k^S = \infty$ for any k. Series of the type $\sum_{q_k}$ disappear, and the infinite systems of equations corresponding to subsystems (15.428), (15.429) are substantially simplified, and subsystems (15.425)-(15.427) and (15.430) decompose completely.

All that remains to be done is to obtain equations in Y_{1s0} and $Y_{1s\frac{n}{2}}$. For this purpose, multiplying (15.413) by Φ_{1s0} and $\Psi_{1s\frac{n}{2}}$ successively, we integrate the result both times over α from 0 to 2π. Considering (15.109), (15.112), we find

$$Y_{1s0} = \frac{-\frac{1}{\sqrt{\pi}} \int\limits_0^{2\pi} u_{m_z}^{0^*}(\alpha) \sin sn\alpha d\alpha}{\frac{\sqrt{\gamma} R_0}{Eh_0 n} \frac{1}{s} + \frac{1}{d_{sn}^T} - \frac{1}{\sqrt{\gamma} d_{sn}^S}}, \qquad (15.435)$$

$$Y_{1s\frac{n}{2}} = \frac{-\frac{1}{\sqrt{\pi}} \int\limits_0^{2\pi} u_{m_z}^{0^*}(\alpha) \sin \left(s - \frac{1}{2}\right) n\alpha d\alpha}{\frac{\sqrt{\gamma} R_0}{Eh_0 n} \frac{1}{s - \frac{1}{2}} + \frac{1}{d^T_{\left(s-\frac{1}{2}\right)n}} - \frac{1}{\sqrt{\gamma} d^S_{\left(s-\frac{1}{2}\right)n}}} \qquad (15.435)$$

$$(s = 1, 2, \ldots, \infty).$$

Thus, the calculation of a stringer shell reinforced with an end ring, during the action of an arbitrary load, reduces to the solution of (n+2) independent subsystems of algebraic equations, n of which, (15.425) -(15.430), constitute coupled infinite systems, and two (15.434), (15.435), decompose completely.

Let us note that subsystems (15.430), (15.435) pertain only to the case where the number of stringers is even. For odd n, one should omit these systems, and in (15.428), (15.429) take $\bar{r} = 2, 3, \ldots, \frac{n-1}{2}$.

Having determined the coefficients of the desired contact load, then calculated the values of arbitrary constants C, $\bar{C}$ from formulas (15.212), (15.213), (15.406), by using general expressions (15.183), (15.187), (15.188), (15.191, (15.193), one can find the displacements and stresses at any point of the shell.

Subsystems (15.425)-(15.43) and (15.434), (15.435) correspond to an external load of arbitrary type, satisfying only conditions (15.181). In special cases, the problem is considerably simplified. Thus, for example, if the external load applied to the ring satisfies the conditions of cyclic symmetry, i.e., can be represented as a function of only one coordinate $\xi = \frac{\alpha}{2\pi/n} - E\left(\frac{\alpha}{2\pi/n}\right)$, all the coefficients $Y_{1\sigma r}, Y_{2\sigma r}$ with the exception of $Y_{1\sigma 0}, Y_{2\sigma 0}$ turn out to be equal to zero, and hence, in this case it is necessary to find the solution of only one infinite subsystem (15.425).

Let us note in conclusion that in a definite range of rigidity relations of the ring and shell, for example, in the case of a relatively weak ring, the numerical procedure of determination of the contact load is more

The contact load $Y^{(0)}$ found in this manner is applied to an elastic ring and is successively refined. A satisfactory convergence is thus obtained rather quickly.

It should also be noted that the rigidities d_k^T, d_k^S increase rapidly with increasing k. Thus, according to (13.233), for a frame type ring

$$d_k^T=\frac{1}{R^4}\,\frac{k^2(k^2-1)^2}{\frac{1}{GI_{kp}}+k^2\frac{1}{EI}}.$$

Therefore, infinite series of the type Σ_{qk} entering into the coefficients of the equations of subsystems (15.425)-(15.430), converge very quickly. For practical purposes, when n>8-10, it is permissible to neglect terms containing the generalized rigidities d_k^T, d_k^S, corresponding to the numbers $k>\frac{n}{2}$, and in most cases, even to assume $d_k^S=\infty$ for any k. Series of the type Σ_{q_k} disappear, and the infinite systems of equations corresponding to subsystems (15.428), (15.429) are substantially simplified, and subsystems (15.425)-(15.427) and (15.430) decompose completely.

All that remains to be done is to obtain equations in Y_{1s0} and $Y_{1s\frac{n}{2}}$. For this purpose, multiplying (15.413) by Φ_{1s0} and $\Phi_{1s\frac{n}{2}}$, successively, we integrate the result both times over α from 0 to 2π. Considering (15.109), (15.112), we find

$$Y_{1s0}=\frac{-\frac{1}{\sqrt{\pi}}\int\limits_0^{2\pi}u_{m_z}^{0*}(\alpha)\sin snad\alpha}{\frac{\sqrt{\gamma}R_0}{Eh_0n}\,\frac{1}{s}+\frac{1}{d_{sn}^T}-\frac{1}{\sqrt{\gamma}d_{sn}^S}}, \tag{15.435}$$

$$Y_{1s\frac{n}{2}}=\frac{-\frac{1}{\sqrt{\pi}}\int\limits_0^{2\pi}u_{m_z}^{0*}(\alpha)\sin\left(s-\frac{1}{2}\right)nad\alpha}{\frac{\sqrt{\gamma}R_0}{Eh_0n}\,\frac{1}{s-\frac{1}{2}}+\frac{1}{d^T_{\left(s-\frac{1}{2}\right)n}}-\frac{1}{\sqrt{\gamma}d^S_{\left(s-\frac{1}{2}\right)n}}} \tag{15.435}$$

$$(s=1,\ 2,\ldots,\ \infty).$$

effective if in strain compatibility Eq. (15.404) the unknown taken is not the contact load itself, but the resultant self-balanced load on the ring (the algebraic sum of the unknown contact load applied to the ring and specified external load). Then the solution for the shell constitutes a superposition of two solutions: that due to the specified external load and that due to the unknown self-balanced load, both applied directly to the shell. The first of these solutions corresponds to a shell without the ring, and the second determines the correction introduced by the ring.

A system of equations in self-balanced load can be obtained directly from Eqs. (15.425)-(15.430), (15.434), (15.435). However, it is simpler to reconstruct the system by using strain compatibility Eq. (15.404).

2. Connection with Stringers without Contact with the Shell

In this case, the longitudinal contact load reduces to a system of concentrated forces $P_m^*(m=1, 2, \ldots, n)$, set up as a result of the interaction of the ring with the stringers. The linear load corresponding to these forces, referred to a unit length of the contour $\bar{Z} = 0$

$$T^*(\alpha)=\frac{1}{R_0}\sum_{m=1}^{n}P_m^*\delta(\alpha-\alpha_m). \qquad (15.436)$$

As above, the tangential contact forces are determined by expressions (15.410).

The unknown forces P_m^* are subject to determination from the equilibrium conditions and strain compatibility conditions of the shell and ring.

As in continuous fastening of the end ring to the shell, the compatibility conditions are formulated in the form of equality of the warping displacements of the ring and shell. However, in the case at hand, such equality should be fulfilled only at the points $\alpha=\frac{2\pi}{n}m$. It should be noted that it would be incorrect in this case to require the fulfillment of condition (15.404) at these points, since the values of the warping displacements $u^{*}_{m_z}(0,\alpha)$, of the shell and $u^{*}_{m_z}(\alpha)$ of the ring at the points $\alpha=\frac{2\pi}{n}m$ can, generally speaking, be distributed according to the law of the plane.

In order to formulate the strain compatibility conditions for the case at hand, we will represent the displacements $u^{*}_{m_z}(0,\alpha)$ and $u^{*}_{m_z}(\alpha)$ at the points $\alpha=\frac{2\pi}{n}m$ by interpolation polynomials of the form (13.78), (13.79):

for even n

$$A_0+\sum_{l=1}^{\frac{n}{2}-1}\left(A_l\cos\frac{2\pi m}{n}l+B_l\sin\frac{2\pi m}{n}l\right)+(-1)^m A_{\frac{n}{2}};$$

for odd n

$$A_0+\sum_{l=1}^{\frac{n-1}{2}}\left(A_l\cos\frac{2\pi m}{n}l+B_l\sin\frac{2\pi m}{n}l\right). \tag{15.437}$$

The first three terms in (15.437) with coefficients A_0, A_1, B_1 represent the distribution of the corresponding displacements according to the law of the plane. Therefore, formulating the strain compatibility conditions for the case at hand to within the displacements of the line of contact $\bar{z}=0$ as a solid, it is necessary to assume

$$\left[u^{*}_{m_z}(0,\alpha)-u^{*}_{m_z 0}(0,\alpha)\right]_{\alpha=\frac{2\pi}{n}m}=\left[u^{*}_{m_z}(\alpha)-u^{*}_{m_z 0}(\alpha)\right]_{\alpha=\frac{2\pi}{n}m} \quad (m=1,2,\ldots,n), \tag{15.438}$$

where $u^{*}_{m_z 0}$ are the components of the displacements $u^{*}_{m_z}$ of the form

$$A_0+A_1\cos\frac{2\pi}{n}m+B_1\sin\frac{2\pi}{n}m.$$

We will represent the unknown concentrated forces P_m^* also in the form of interpolation polynomial (13.78), (13.79):

for an odd n of stringers

$$P_m^* = A_0 + \sum_{l=1}^{\frac{n-1}{2}} \left(A_l \cos \frac{2\pi m}{n} l + B_l \sin \frac{2\pi m}{n} l \right);$$

for an even number n

$$P_m^* = A_0 + \sum_{l=1}^{\frac{n}{2}-1} \left(A_l \cos \frac{2\pi m}{n} l + B_l \sin \frac{2\pi m}{n} l \right) + (-1)^m A_{\frac{n}{2}}. \qquad (15.439)$$

Introducing (15.436) into (15.420) and considering (15.439), for even n we have

$$Y = \frac{1}{R_0} \left\{ A_0 \sum_{m=1}^{n} \Phi\left(\frac{2\pi}{n} m\right) + \sum_{l=1}^{E\left(\frac{n-1}{2}\right)} \left[A_l \sum_{m=1}^{n} \cos \frac{2\pi l}{n} m \, \Phi\left(\frac{2\pi}{n} m\right) + \right. \right.$$
$$\left. \left. + B_l \sum_{m=1}^{n} \sin \frac{2\pi l}{n} m \, \Phi\left(\frac{2\pi}{n} m\right) \right] + A_{\frac{n}{2}} \sum_{m=1}^{n} (-1)^m \Phi\left(\frac{2\pi}{n} m\right) \right\}. \qquad (15.440)$$

Hereinafter, for odd n, terms containing $A_{\frac{n}{2}}$, must be omitted. Expanding (15.440) with the aid of expressions (15.247), then using formulas (14.3), (14.12)-(14.14), we find

$$\begin{aligned}
&Y_{1s0} = 0, && Y_{2s0} = -\frac{n}{\sqrt{2} R_0 B_{s0}} A_0, \\
&Y_{1s1} = \frac{n\pi}{2\sqrt{\pi} R_0 B_{s1}} B_1, && Y_{2s1} = \frac{n\pi}{2\sqrt{\pi} R_0 B_{s1}} A_1, \\
&Y_{1sr} = \frac{n}{2R_0 B_{sr}} B_r, && Y_{2sr} = \frac{n}{2R_0 B_{sr}} A_r \quad \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right) \right], \\
&Y_{1s\frac{n}{2}} = 0, && Y_{2s\frac{n}{2}} = \frac{n}{\sqrt{2} R_0 B_{s\frac{n}{2}}} A_{\frac{n}{2}}.
\end{aligned} \qquad (15.441)$$

Expressions (15.441) permit one to reduce the problem of determination of arbitrary constants C and $\bar{C}$, related to coefficients Y by relations (15.406), to the determination of n coefficients A and B of the interpolation polynomial. The coefficients A_0, A_1, B_1 corresponding to the distribution of the contact forces P_m^* according to the law of the plane are determined from the equilibrium conditions. Expanding (15.436) with the aid of (15.439) and introducing the result into (15.196), in view of (14.3), (14.12)-(14.14), we find

$$A_0 = \frac{N_x(0)}{n\cos\beta},$$
$$A_1 = -\frac{M_y(0)}{\frac{n}{2}R_0\cos\beta}; \quad B_1 = \frac{M_x(0)}{\frac{n}{2}R_0\cos\beta}. \tag{15.442}$$

The remaining coefficients of the interpolation polynomial, corresponding to the self-balanced distribution of the contact forces P_m^*, are subject to determination from strain compatibility conditions (15.438).

From (15.405) when $\alpha = \frac{2\pi}{n}m$, we have for the shell

$$\overset{\bullet}{u}_{m_s}\left(0, \frac{2\pi}{n}m\right) = \sum_{s=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left[(C_{1sr}+\bar{C}_{1sr})\,\Phi_{1sr}\left(\frac{2\pi}{n}m\right) + (C_{2sr}+\bar{C}_{2sr})\,\Psi_{2sr}\left(\frac{2\pi}{n}m\right)\right].$$

On the basis of (15.247), omitting the terms corresponding to the distribution of displacements $\overset{\bullet}{u}_{m_s}\left(0, \frac{2\pi}{n}m\right)$ according to the law of the plane, we find

$$\left[\overset{\bullet}{u}_{m_s}(0,\alpha) - \overset{\bullet}{u}_{m_s0}(0,\alpha)\right]_{\alpha=\frac{2\pi}{n}m} = \sum_{r=1}^{E\left(\frac{n-1}{2}\right)}\left[\sin\frac{2\pi r}{n}m\sum_{s=1}^{\infty}\frac{1}{B_{sr}}(C_{1sr}+\bar{C}_{1sr}) +\right.$$
$$\left. + \cos\frac{2\pi r}{n}m\sum_{s=1}^{\infty}\frac{1}{B_{sr}}(C_{2sr}+\bar{C}_{2sr})\right] + \frac{(-1)^m}{\sqrt{2}}\sum_{s=1}^{\infty}\frac{1}{B_{s\frac{n}{2}}}\left(C_{2s\frac{n}{2}}+\bar{C}_{2s\frac{n}{2}}\right). \tag{15.443}$$

Introducing into (15.443) the values of arbitrary constants according to (15.406), in view of (15.420) and (15.441) we finally obtain

$$\left[\overset{\bullet}{u}_{m_s}(0,\alpha) - \overset{\bullet}{u}_{m_s0}(0,\alpha)\right]_{\alpha=\frac{2\pi}{n}m} = -\frac{\pi\sqrt{2}}{2Eh_0}\left\{\sum_{r=1}^{E\left(\frac{n-1}{2}\right)}\left[B_r\sin\frac{2\pi r}{n}m +\right.\right.$$
$$\left.\left. + A_r\cos\frac{2\pi r}{n}m\right]\sum_{s=1}^{\infty}\frac{1}{\lambda^{\bullet}_{sr}B^2_{sr}} + A_{\frac{n}{2}}(-1)^m\sum_{s=1}^{\infty}\frac{1}{\lambda^{\bullet}_{2s\frac{n}{2}}B^2_{s\frac{n}{2}}}\right\}. \tag{15.444}$$

From (15.411) in view of (15.422) when $\alpha=\frac{2\pi}{n}m$, we have for the ring

$$
\begin{aligned}
u^{\bullet}_{m_s}\left(\frac{2\pi}{n}m\right)=u^{0\bullet}_{m_s}\left(\frac{2\pi}{n}m\right)+\frac{1}{\pi}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\sum_{s=1}^{\infty}\Bigg\{\sum_{k=2}^{\infty}\sin\frac{2\pi k}{n}m\times \\
\times\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha\,d\alpha\left[\frac{1}{d^T_k}\left(\frac{\pi k}{n}\right)^2\frac{Y_{1sr}}{\lambda^{*2}_{1sr}}+\frac{kGh_0}{d^S_k R_0}\left(C_{1sr}+\bar{C}_{1sr}\right)\right]+ \\
+\sum_{k=2}^{\infty}\cos\frac{2\pi k}{n}m\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha\,d\alpha\left[\frac{1}{d^T_k}\left(\frac{\pi k}{n}\right)^2\frac{Y_{2sr}}{\lambda^{*2}_{2sr}}+\frac{kGh_0}{d^S_k R_0}\left(C_{2sr}+\bar{C}_{2sr}\right)\right]\Bigg\}+ \\
+\frac{N_z(0)}{\pi R_0\cos\beta}\sum_{k=2}^{\infty}\frac{1}{d^T_k}\int_0^{2\pi}\cos k\alpha\,\varrho(\alpha)\,d\alpha.
\end{aligned}
\tag{15.445}
$$

We establish a correspondence between the displacements $u^{0\bullet}_{m_s}\left(\frac{2\pi}{n}m\right)$ and the interpolation polynomial

$$
u^{0\bullet}_{m_s}\left(\frac{2\pi}{n}m\right)=A^{\bullet}_0+\sum_{l=1}^{E\left(\frac{n-1}{2}\right)}\left(A^{\bullet}_l\cos\frac{2\pi m}{n}l+B^{\bullet}_l\sin\frac{2\pi m}{n}l\right)+(-1)^m A^{\bullet}_{\frac{n}{2}}. \tag{15.446}
$$

Introducing (15.446) into expression (15.445) and in addition, considering (15.130) and (15.418), we get

$$
\begin{aligned}
u^{\bullet}_{m_s}\left(\frac{2\pi}{n}m\right)=A^{\bullet}_0+\frac{N_z(0)}{\pi R_0\cos\beta}\sum_{q_k=1}^{\infty}\frac{1}{d^T_{q_k n}}\int_0^{2\pi}\cos q_k n\alpha\,\varrho(\alpha)\,d\alpha+ \\
+\sum_{s=1}^{\infty}\sum_{q_k=1}^{\infty}a^{q_k n}_{2s0}+\sin\frac{2\pi}{n}m\left[B^{\bullet}_1+\sum_{s=1}^{\infty}\sum_{q_k=1}^{\infty}\left(a^{q_k n-1}_{1s1}+a^{q_k n+1}_{1s1}\right)\right]+ \\
+\cos\frac{2\pi}{n}m\left[A^{\bullet}_1+\sum_{s=1}^{\infty}\sum_{q_k=1}^{\infty}\left(a^{q_k n-1}_{2s1}+a^{q_k n+1}_{2s1}\right)\right]+ \\
+\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\Bigg(\sin\frac{2\pi r}{n}m\left\{B^{\bullet}_r+\sum_{s=1}^{\infty}\left[a^{r}_{1sr}+\sum_{q_k=1}^{\infty}\left(a^{q_k n-r}_{1sr}+a^{q_k n+r}_{1sr}\right)\right]\right\}+ \\
+\cos\frac{2\pi r}{n}m\left\{A^{\bullet}_r+\sum_{s=1}^{\infty}\left[a^{r}_{2sr}+\sum_{q_k=1}^{\infty}\left(a^{q_k n-r}_{2sr}+a^{q_k n+r}_{2sr}\right)\right]\right\}\Bigg)+ \\
+(-1)^m\left[A^{\bullet}_{\frac{n}{2}}+\sum_{s=1}^{\infty}\sum_{q_k=1}^{\infty}a^{q_k n-\frac{n}{2}}_{2s\frac{n}{2}}\right].
\end{aligned}
\tag{15.447}
$$

where symbols of abbreviation are assumed:

$$a_{1sr}^{k}=\frac{1}{\pi}\int_{0}^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha\,d\alpha\left[\frac{1}{d_k^7}\left(\frac{\pi k}{n}\right)^2\frac{Y_{1sr}}{\lambda_{1sr}^{\circ 2}}+\frac{kGh_0}{d_k^3R_0}(C_{1sr}+\overline{C}_{1sr})\right],$$

$$a_{2sr}^{k}=\frac{1}{\pi}\int_{0}^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha\,d\alpha\left[\frac{1}{d_k^7}\left(\frac{\pi k}{n}\right)^2\frac{Y_{2sr}}{\lambda_{2sr}^{\circ 2}}-\frac{kGh_0}{d_k^3R_0}(C_{2sr}+\overline{C}_{2sr})\right]. \quad (15.448)$$

Now, omitting in (15.447) the terms corresponding to the distribution of displacements of the ring $u_{m_s}^{\circ}\left(\frac{2\pi}{n}m\right)$ according to the law of the plane, in view of (15.448), (15.406), (15.420) and (15.441), we finally obtain

$$[u_{m_s}^{\circ}(\alpha)-u_{m_s0}^{\circ}(\alpha)]_{\alpha=\frac{2\pi}{n}m}=$$

$$=\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\left(\sin\frac{2\pi r}{n}m\left\{B_r^{\circ}+B_r\sum_{s=1}^{\infty}\left[b_{1sr}'+\sum_{q_k=1}^{\infty}\left(b_{1sr}^{q_kn-r}+b_{1sr}^{q_kn+r}\right)\right]\right\}+\right.$$

$$\left.+\cos\frac{2\pi r}{n}m\left\{A_r^{\circ}+A_r\sum_{s=1}^{\infty}\left[b_{2sr}'+\sum_{q_k=1}^{\infty}\left(b_{2sr}^{q_kn-r}+b_{2sr}^{q_kn+r}\right)\right]\right\}\right)+$$

$$+(-1)^m\left[A_{\frac{n}{2}}^{\circ}+\sqrt{2}A_{\frac{n}{2}}\sum_{s=1}^{\infty}\sum_{q_k=1}^{\infty}b_{2s\frac{n}{2}}^{q_kn-\frac{n}{2}}\right], \quad (15.449)$$

where

$$b_{1sr}^{k}=\frac{k}{2R_0B_{sr}\lambda_{1sr}^{\circ 2}}\int_{0}^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha\,d\alpha\left[\frac{1}{d_k^7}\frac{\pi k}{n}-\frac{\lambda_{1sr}^{\circ}}{\sqrt{\gamma}d_k^3}\right],$$

$$b_{2sr}^{k}=\frac{k}{2R_0B_{sr}\lambda_{2sr}^{\circ 2}}\int_{0}^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha\,d\alpha\left[\frac{1}{d_k^7}\frac{\pi k}{n}+\frac{\lambda_{2sr}^{\circ}}{\sqrt{\gamma}d_k^3}\right]. \quad (15.450)$$

Introducing (15.444), (15.449) into strain compatibility conditions (15.438), we obtain

$$\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\left(\sin\frac{2\pi r}{n}m\left\{B_r^{\circ}+B_r\sum_{s=1}^{\infty}\left[\frac{\pi\sqrt{\gamma}}{2Eh_0}\frac{1}{\lambda_{sr}^{\circ}B_{sr}^2}+b_{1sr}'+\right.\right.\right.$$

$$\left.\left.+\sum_{q_k=1}^{\infty}\left(b_{1sr}^{q_kn-r}+b_{1sr}^{q_kn+r}\right)\right]\right\}+\cos\frac{2\pi r}{n}m\left\{A_r^{\circ}+A_r\sum_{s=1}^{\infty}\left[\frac{\pi\sqrt{\gamma}}{2Eh_0}\frac{1}{\lambda_{sr}^{\circ}B_{sr}^2}+\right.\right.$$

$$\left.\left.\left.+b_{2sr}'+\sum_{q_k=1}^{\infty}\left(b_{2sr}^{q_kn-r}+b_{2sr}^{q_kn+r}\right)\right]\right\}\right)+ \quad (15.451)$$

$$+(-1)^m\left[A_{\frac{n}{2}}^{\circ}+A_{\frac{n}{2}}\sum_{s=1}^{\infty}\left(\frac{\pi\sqrt{\gamma}}{2Eh_0}\frac{1}{\lambda_{2s\frac{n}{2}}^{\circ}B_{s\frac{n}{2}}^2}+\sqrt{2}\sum_{q_k=1}^{\infty}b_{2s\frac{n}{2}}^{q_kn-\frac{n}{2}}\right)\right]=0$$

$$(m=1,\ 2,\ \ldots,\ n).$$

Expressions (15.451) represent a system of n equations in (n-3) unknowns $A_r, B_r, \left[r=2, 3, \ldots, E\left(\frac{n-1}{2}\right)\right]$ and $A_{\frac{n}{2}}$. This system is simultaneous.

Obviously, as before, a correspondence can be established between system (15.451) and the interpolation polynomial, assuming zero values at the points $\alpha=\frac{2\pi}{n}m$. Therefore, equating the multipliers on $\sin\frac{2\pi r}{n}m$, $\cos\frac{2\pi r}{n}m$ and $(-1)^m$ to zero, we find

$$B_r=\frac{-B_r^*}{\sum\limits_{s=1}^{\infty}\left[\frac{\pi\sqrt{i}}{2Eh_0}\frac{1}{\lambda_{sr}^*B_{sr}^2}+b_{1sr}^r+\sum\limits_{q_k=1}^{\infty}\left(b_{1sr}^{q_kn-r}+b_{1sr}^{q_kn+r}\right)\right]},$$

$$A_r=\frac{-A_r^*}{\sum\limits_{s=1}^{\infty}\left[\frac{\pi\sqrt{i}}{2Eh_0}\frac{1}{\lambda_{sr}^*B_{sr}^2}+b_{2sr}^r+\sum\limits_{q_k=1}^{\infty}\left(b_{2sr}^{q_kn-r}+b_{2sr}^{q_kn+r}\right)\right]}, \tag{15.452}$$

$$A_{\frac{n}{2}}=\frac{-A_{\frac{n}{2}}^*}{\sum\limits_{s=1}^{\infty}\left(\frac{\pi\sqrt{i}}{2Eh_0}\frac{1}{\lambda_{s\frac{n}{2}}^*B_{s\frac{n}{2}}^2}+\sqrt{2}\sum\limits_{q_k=1}^{\infty}b_{2s\frac{n}{2}}^{q_kn-\frac{n}{2}}\right)}.$$

The problem has thus been solved. Introducing the obtained coefficients of the interpolation polynomial of the contact load, represented by expressions (15.442), and (15.452), into relations (15.441), then determining the arbitrary constants C and $\bar{C}$ from formulas (15.212), (15.213), (15.406), by using general expressions (15.183), (15.187), (15.188), (15.191), (15.193) we can then calculate the stresses and displacements at any point of the shell.

In conclusion, we will write out the expanded expressions for the coefficients of the contact load interpolation polynomial.

Using formulas (13.78), expansions (15.130), and writing

$$\frac{k}{\left(\frac{\pi k}{n}\right)^2-\lambda_{sr}^{*2}}=L_{k,sr}, \tag{14.453}$$

we can obtain

$$A_r=\frac{-\frac{4R_0\varkappa}{\pi n}\sum_{m=1}^{n}u^{0r}_{m_z}\left(\frac{2\pi}{n}m\right)\cos\frac{2\pi r}{n}m}{\sum_{s=1}^{\infty}\frac{1}{B^2_{sr}}\left[\frac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{*}_{sr}}+\frac{L_{r,sr}}{D_{r,3sr}}+\sum_{q_k=1}^{\infty}\left(\frac{L_{q_kn+r,sr}}{D_{q_kn+r,3sr}}+\frac{L_{q_kn-r,sr}}{D_{q_kn-r,3sr}}\right)\right]},$$

$$B_r=\frac{-\frac{4R_0\varkappa}{\pi n}\sum_{m=1}^{n}u^{0r}_{m_z}\left(\frac{2\pi}{n}m\right)\sin\frac{2\pi r}{n}m}{\sum_{s=1}^{\infty}\frac{1}{B^2_{sr}}\left[\frac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{*}_{sr}}+\frac{L_{r,sr}}{D_{r,3sr}}+\sum_{q_k=1}^{\infty}\left(\frac{L_{q_kn+r,sr}}{D_{q_kn+r,1sr}}-\frac{L_{q_kn-r,sr}}{D_{q_kn-r,1sr}}\right)\right]}, \qquad (15.454)$$

$$A_{\frac{n}{2}}=\frac{-\frac{R_0\varkappa}{\pi n}\sum_{m=1}^{n}(-1)^m u^{0r}_{m_z}\left(\frac{2\pi}{n}m\right)}{\sum_{s=1}^{\infty}\frac{1}{B^2_{s\frac{n}{2}}}\left[\frac{\varkappa\sqrt{\gamma}R_0}{2Eh_0\lambda^{*}_{2s\frac{n}{2}}}+\sum_{q_k=1}^{\infty}\frac{L_{q_kn-\frac{n}{2},s\frac{n}{2}}}{D_{q_kn-\frac{n}{2},3s\frac{n}{2}}}\right]},$$

where the generalized rigidities $D_{k,1sr}$, $D_{k,3sr}$ are determined by expressions (15.424).

As in the previous problem, when n 8-10, to simplify the computational formulas it is permissible in this problem to assume

$$d^T_k=\infty,\quad k>\frac{n}{2},$$

$$d^s_k=\infty,\quad k-\text{ any}$$

We can then obtain from (15.454)

$$A_r=\frac{-\frac{4R_0\varkappa}{\pi n}\sum_{m=1}^{n}u^{0r}_{m_z}\left(\frac{2\pi}{n}m\right)\cos\frac{2\pi r}{n}m}{\sum_{s=1}^{\infty}\frac{1}{B^2_{sr}}\left[\frac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{*}_{sr}}+\frac{\lambda_{r,sr}}{D_{r,3sr}}\right]},$$

$$B_r=\frac{-\frac{4R_0\varkappa}{\pi n}\sum_{m=1}^{n}u^{0r}_{m_z}\left(\frac{2\pi}{n}m\right)\sin\frac{2\pi r}{n}m}{\sum_{s=1}^{\infty}\frac{1}{B^2_{sr}}\left[\frac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{*}_{sr}}+\frac{L_{r,sr}}{D_{r,1sr}}\right]}, \qquad (15.455)$$

$$A_{\frac{n}{2}}=\frac{-\frac{R_0\varkappa}{\pi n}\sum_{m=1}^{n}(-1)^m u^{0r}_{m_z}\left(\frac{2\pi}{n}m\right)}{\sum_{s=1}^{\infty}\frac{1}{B^2_{s\frac{n}{2}}}\left[\frac{\varkappa\sqrt{\gamma}R_0}{2Eh_0\lambda^{*}_{2s\frac{n}{2}}}+\frac{L_{\frac{n}{2},s\frac{n}{2}}}{D_{\frac{n}{2},2s\frac{n}{2}}}\right]},$$

3. Joint Along the Entire Contour. Approximate Solution

The results obtained in Subsection 2 for a sufficiently large number n of stringers may be treated as an approximate solution for the case of continuous fastening of the end ring to the shell. For small values of the parameter $\varkappa$, corresponding to heavy stringers and relatively weak skin, the accuracy of this solution is completely satisfactory. Below, for the case of continuous fixing of the end ring to the shell, another approximate solution with wider applicability limits is discussed.

We will seek the contact load in the form

$$T^{*}(\alpha)=\frac{1}{R_0}\sum_{m=1}^{n}P_m^{*}\left[\delta(\alpha-\alpha_m)+\frac{R_0 h_0}{\Delta F_0}\varphi_m(\alpha)\right], \qquad (15.456)$$

where P_m^* are the unknown contact forces in the stringers;

$\varphi_m(\alpha)$ are some regular functions satisfying the conditions

$$\varphi_m\left(\frac{2\pi}{n}m\right)=1,$$

$$\varphi_m(\alpha)=0 \quad \text{при} \quad \alpha\leqslant\frac{2\pi}{n}(m-1) \text{ и } \alpha\geqslant\frac{2\pi}{n}(m+1). \qquad (15.457)$$

The linear contact load T^*, represented by expression (15.456), corresponds to the continuous distribution of normal stresses on the contour $\bar{Z} = 0$. The functions φ_m should be preselected on the basis of intuitive representations or experimental data. For a sufficiently large number of stringers, the stress distribution on the portions between the stringers can be considered linear. In this case, on the segment $\frac{2\pi}{n}(m-1)\leqslant\alpha\leqslant\frac{2\pi}{n}(m+1)$, the function

$$\varphi_m(\alpha)=(p+1-m)(1-\xi)-(p-m)\xi, \qquad (15.458)$$

where, as before,

$$\rho=E\left(\frac{\alpha}{2\pi/n}\right);\ \xi=\frac{\alpha}{2\pi/n}-\rho.$$

For functions φ_m specified in some manner, contact forces (15.456) contain only n unknown forces P_m^* in the stringers, which must be determined from equilibrium conditions (15.196) and strain compatibility conditions (15.404) of the shell and ring. For continuous fastening of the end ring to the shell, relation (15.404) can be fulfilled identically, but in view of the fact that contact load (15.456) contains only n unknowns, the strain compatibility conditions can be satisfied only approximately. This can be done in different ways. In determining the forces P_m^*, the simplest way is to equate the displacements of the ring and shell at the points $\alpha=\frac{2\pi}{n}m$. In this case, the strain compatibility conditions are disturbed and the spans between the stringers, but if the system of functions φ_m has been successfully chosen, the divergence of the displacements is insignificant, and hence, the error of the approximate solution will be slight.

As above, the unknown concentrated forces P_m^* will be represented in the form of interpolation polynomials (15.439).

We expand (15.420) with the aid of (15.456), and obtain

$$Y=\frac{1}{R_0}\sum_{m=1}^{n}P_m^*\left[\Phi\left(\frac{2\pi}{n}m\right)+\frac{R_0h_2}{\Delta F_0}\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha\right]. \quad (15.459)$$

Introducing into (15.459) the expression for P_m^* in the form of interpolation polynomial (15.439), we will have for even n

$$Y=\frac{1}{R_0}\left\{A_0\sum_{m=1}^{n}\left(\Phi\left(\frac{2\pi}{n}m\right)+\frac{R_0h_0}{\Delta F_0}\int_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha\right)+\right.$$

$$+\sum_{l=1}^{E\left(\frac{n-1}{2}\right)}\left[A_l\sum_{m=1}^{n}\cos\frac{2\pi l}{n}m\left(\Phi\left(\frac{2\pi}{n}m\right)+\frac{R_0h_0}{\Delta F_0}\int_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha\right)+\right.$$

$$\left.+B_l\sum_{m=1}^{n}\sin\frac{2\pi l}{n}m\left(\Phi\left(\frac{2\pi}{n}m\right)+\frac{R_0h_0}{\Delta F_0}\int_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha\right)\right]+$$

$$\left.+A_{\frac{n}{2}}\sum_{m=1}^{n}(-1)^m\left(\Phi\left(\frac{2\pi}{n}m\right)+\frac{R_0h_0}{\Delta F_0}\int_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha\right)\right\}. \quad (15.460)$$

Using expanded expressions (15.109)-(15.112) for normal coordinate functions, for the chosen functions φ_m we can calculate all the quadratures of the form

$$\int_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha. \quad (15.461)$$

Let $\varphi_{\bar{m}}$ be some completely arbitrary function corresponding to a fixed value $m = \bar{m}$, a function satisfying conditions (15.457) only. Our treatment will be limited to the case in which all the remaining functions φ_m can be obtained by shifting $\varphi_{\bar{m}}$ by an angle $\alpha_0=\frac{2\pi}{n}(m-\bar{m})$:

$$\varphi_m(\alpha)=\varphi_{\bar{m}}\left[\alpha-\frac{2\pi}{n}(m-\bar{m})\right]\ (m=1,2,\ldots,n). \quad (15.462)$$

The arbitrary function $\varphi_{\bar{m}}$ on the portion $\frac{2\pi}{n}(\bar{m}-1)\leqslant\alpha\leqslant\frac{2\pi}{n}(\bar{m}+1)$ can be represented in the form

$$\varphi_{\bar{m}}(\alpha)=\varphi_{\bar{m}}(p,\xi)=(\bar{m}-p)\chi_-(\xi)+(\bar{m}-p-1)[\chi_+(\xi)-1], \quad (15.463)$$

where $\chi_-(\xi),\ \chi_+(\xi)$ are functions satisfying only the conditions

$$\chi_-(0)=\chi_+(0)=0,$$
$$\chi_-(1)=\chi_+(1)=1. \quad (15.464)$$

Obviously, the function $\varphi_{\bar{m}}(\alpha)$ on the portion $\frac{2\pi}{n}(\bar{m}-1)<\alpha<\frac{2\pi}{n}\bar{m}$ is represented by the function $\chi_-(\xi)$, and on the portion $\frac{2\pi}{n}\bar{m}<\alpha<\frac{2\pi}{n}(\bar{m}+1)$, by the function $\chi_+(\xi)$. Therefore, by selecting the functions $\chi_-(\xi)$ and $\chi_+(\xi)$ in some manner, in accordance with (15.463), we completely determine the functions $\varphi_{\bar{m}}(\alpha)$, and along with it, according to (15.462), all the remaining functions $\varphi_m(\alpha)$ as well. We take $\bar{m} = 0$. Then, introducing (15.463) into (15.462), we finally obtain

$$\varphi_m(\alpha)=(m-p)\chi_-(\xi)+(m-p-1)[\chi_+(\xi)-1]. \tag{15.465}$$

For the functions represented by expression (15.465), we have

$$\begin{gathered}\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi(\alpha)d\alpha=\frac{2\pi}{n}\left\{\int\limits_0^1\chi_-(\xi)\Phi(m-1,\xi)d\xi+\int\limits_0^1[1-\chi_+(\xi)]\times\right.\\ \left.\times\Phi(m,\xi)d\xi\right\}=\frac{2\pi}{n}\int\limits_0^1\{\chi_-(\xi)\Phi(m-1,\xi)+[1-\chi_+(\xi)]\Phi(m,\xi)\}d\xi.\end{gathered} \tag{15.466}$$

Using (15.466), with the aid of general expressions (15.109)-(15.112), we obtain

$$\begin{gathered}\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi_{1\nu_0}(\alpha)d\alpha=a_{1\nu_0},\\ \int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi_{2\nu_0}(\alpha)d\alpha=a_{2\nu_0},\\ \int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\Phi_{1\nu r}(\alpha)d\alpha=a_{3\nu r}\sin\frac{2\pi r}{n}m+a_{1\nu r}\cos\frac{2\pi r}{n}m\\ \left[r=1,2,\ldots,E\left(\frac{n-1}{2}\right)\right].\end{gathered} \tag{15.467}$$

$$\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)} \varphi_m(\alpha)\,\Phi_{2sr}(\alpha)\,d\alpha = -a_{2sr}\sin\frac{2\pi r}{n}m + a_{2sr}\cos\frac{2\pi r}{n}m$$

$$\left[r = 1, 2, \ldots, E\left(\frac{n-1}{2}\right)\right],$$

$$\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)} \varphi_m(\alpha)\,\Phi_{1s\frac{n}{2}}(\alpha)\,d\alpha = a_{1s\frac{n}{2}}(-1)^m,$$

$$\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)} \varphi_m(\alpha)\,\Phi_{2s\frac{n}{2}}(\alpha)\,d\alpha = a_{2s\frac{n}{2}}(-1)^m, \qquad (15.467)$$

where

$$a_{1s0} = \frac{2\pi}{n}\int_0^1 [\chi_-(\xi) - \chi_+(\xi)]\,\Phi_{1s0}(\xi)\,d\xi,$$

$$a_{2s0} = \frac{2\pi}{n}\left\{\frac{1}{\sqrt{2B_{s0}^2}} + \int_0^1 [\chi_-(\xi) - \chi_+(\xi)]\,\Phi_{2s0}(\xi)\,d\xi\right\},$$

$$a_{1sr} = \frac{2\pi}{n}\int_0^1 \left\{[1 - \chi_+(\xi)]\,\Phi_{1sr}(0,\xi) - \chi_-(\xi)\,\Phi_{1sr}(0, 1-\xi)\right\}d\xi, \qquad (15.468)$$

$$a_{2sr} = \frac{2\pi}{n}\int_0^1 \left\{[1 - \chi_+(\xi)]\,\Phi_{2sr}(0,\xi) + \chi_-(\xi)\,\Phi_{2sr}(0, 1-\xi)\right\}d\xi,$$

$$a_{1s\frac{n}{2}} = \frac{2\pi}{n}\left\{\frac{1}{\sqrt{\pi}\lambda^{*}_{1s\frac{n}{2}}} - \int_0^1 [\chi_-(\xi) + \chi_+(\xi)]\,\Phi_{1s\frac{n}{2}}(0,\xi)\,d\xi\right\},$$

$$a_{2s\frac{n}{2}} = -\frac{2\pi}{n}\int_0^1 [\chi_-(\xi) + \chi_+(\xi)]\,\Phi_{2s\frac{n}{2}}(0,\xi)\,d\xi.$$

Thus, having calculated coefficients (15.468) for functions χ_-, χ_+ chosen in some manner, by using (15.467), we can find the values of all the quadratures entering into (15.460).

Expressions (15.468) correspond to the arbitrary functions χ_-, χ_+. These expressions are simplified in special cases. Let $\chi_- = \chi_+ = \chi$. Then from (15.468) we obtain in expanded form

$$a_{1s0}=0,$$

$$a_{2s0}=\frac{\sqrt{2\pi}}{n\pi B_{s0}},$$

$$a_{1s1}=\frac{\pi}{n}\,\frac{\sin\frac{2\pi}{n}}{B_{s1}}\left\{\frac{\frac{n}{\pi}\operatorname{tg}\frac{\pi}{n}}{\pi\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^{2}-1\right]-1}+\frac{\operatorname{tg}\lambda_{s1}^{*}}{\lambda_{s1}^{*}}-\right.$$

$$\left.-2\int_{0}^{1}\chi(\xi)\left[\frac{1}{\pi\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^{2}-1\right]}\,\frac{\cos\frac{2\pi}{n}\left(\xi-\frac{1}{2}\right)}{\cos\frac{\pi}{n}}+\frac{\cos 2\lambda_{s1}^{*}\left(\xi-\frac{1}{2}\right)}{\cos\lambda_{s1}^{*}}\right]d\xi\right\},$$

$$a_{2s1}=\frac{2\pi}{n}\,\frac{\cos^{2}\frac{\pi}{n}}{B_{s1}}\left\{\frac{\frac{n}{\pi}\operatorname{tg}\frac{\pi}{n}}{\pi\left[\left(\frac{n\lambda_{s1}}{\pi}\right)^{2}-1\right]-1}+\frac{\operatorname{tg}\lambda_{s1}^{*}}{\lambda_{s1}^{*}}+\right.$$

$$+2\operatorname{tg}^{2}\frac{\pi}{n}\int_{0}^{1}\chi(\xi)\left[\frac{1}{\pi\left[\left(\frac{n\lambda_{s1}^{*}}{\pi}\right)^{2}-1\right]-1}\,\frac{\sin\frac{2\pi}{n}\left(\xi-\frac{1}{2}\right)}{\sin\frac{\pi}{n}}+\right.$$

$$\left.\left.+\frac{\sin 2\lambda_{s1}^{*}\left(\xi-\frac{1}{2}\right)}{\sin\lambda_{s1}^{*}}\right]d\xi\right\},$$

$$a_{1sr}=\frac{\pi}{n}\,\frac{\sin\frac{2\pi r}{n}}{B_{sr}}\left\{\frac{\operatorname{tg}\lambda_{sr}^{*}}{\lambda_{sr}^{*}}-\frac{2}{\cos\lambda_{sr}^{*}}\int_{0}^{1}\chi(\xi)\cos 2\lambda_{sr}^{*}\left(\xi-\frac{1}{2}\right)d\xi\right\}$$

$$\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right],$$

$$a_{2sr}=\frac{2\pi}{n}\,\frac{\cos^{2}\frac{\pi r}{n}}{B_{sr}}\left\{\frac{\operatorname{tg}\lambda_{sr}^{*}}{\lambda_{sr}^{*}}+\frac{2\operatorname{tg}^{2}\frac{\pi}{n}}{\sin\lambda_{sr}^{*}}\int_{0}^{1}\chi(\xi)\sin 2\lambda_{sr}^{*}\left(\xi-\frac{1}{2}\right)d\xi\right\}$$

$$\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right],$$

$$a_{1s\frac{n}{2}}=\frac{2\sqrt{\pi}}{n}\left[\frac{1}{\lambda_{1s\frac{n}{2}}^{*}}-2\int_{0}^{1}\chi(\xi)\sin 2\lambda_{1s\frac{n}{2}}^{*}\xi\,d\xi\right],$$

$$a_{2s\frac{n}{2}}=\frac{2\sqrt{2\pi}}{n}\,\frac{1}{B_{s\frac{n}{2}}}\int_{0}^{1}\chi(\xi)\,\frac{\sin 2\lambda_{2s\frac{n}{2}}^{*}\left(\xi-\frac{1}{2}\right)}{\sin\lambda_{2s\frac{n}{2}}^{*}}\,d\xi. \qquad (15.469)$$

As already noted, for a sufficiently large number n of stringers, the function m may be assumed to be piecewise-linear (15.458). In this case $\chi(\xi)=\xi$, and from (15.459) we obtain

$$a_{1s0}=0;\quad a_{2s0}=\frac{\sqrt{2}\pi}{n\varkappa B_{s0}},$$

$$a_{1s1}=0;\quad a_{2s1}=\frac{2\sqrt{\pi}}{nB_s}\left[\frac{\sin^2\frac{\pi}{n}}{\lambda_{s1}^{*2}}(\varkappa+1)-1\right],$$

$$a_{1sr}=0;\quad a_{2sr}=\frac{2\pi}{nB_{sr}}\left(\frac{\sin^2\frac{\pi r}{n}}{\lambda_{sr}^{*2}}-\frac{1}{\varkappa}\right), \tag{15.470}$$

$$a_{1s\frac{n}{2}}=0;\quad a_{2s\frac{n}{2}}=\frac{\pi\sqrt{2}}{nB_{s\frac{n}{2}}}\left(\frac{1}{\lambda_{2s\frac{n}{2}}^{*2}}-\frac{1}{\varkappa}\right).$$

Let us now return to expression (15.460). Introducing into (15.460) the values of the normal coordinate functions at the points $\alpha=\frac{2\pi}{n}m$ and the values of quadratures of the form (15.461) according to (15.247), (15.248), by using (14.3), (14.12)-(14.14), we can find

$$\begin{gathered}
Y_{1s0}=A_0b'_{s0},\qquad Y_{2s0}=A_0b_{s0},\\
Y_{1sr}=A_rb'_{sr}+B_rb_{sr},\qquad Y_{2sr}=A_rb_{sr}-B_rb'_{sr}\\
\left[r=1,2,\ldots,E\left(\frac{n-1}{2}\right)\right],\\
Y_{1s\frac{n}{2}}=A_{\frac{n}{2}}b'_{s\frac{n}{2}},\qquad Y_{2s\frac{n}{2}}=A_{\frac{n}{2}}b_{s\frac{n}{2}},
\end{gathered} \tag{15.471}$$

where

$$b_{sr}=\begin{cases}
\left(-\frac{1}{\sqrt{2}R_0B_{s0}}+\frac{h_0}{\Delta F_0}a_{2s0}\right)n & (r=0),\\
\left(\frac{\varkappa}{\sqrt{\pi}R_0B_s}+\frac{h_0}{\Delta F_0}a_{2s1}\right)\frac{n}{2} & (r=1),\\
\left(\frac{1}{R_0B_{sr}}+\frac{h_0}{\Delta F_0}a_{2sr}\right)\frac{n}{2} & \left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right],\\
\left(\frac{1}{\sqrt{2}R_0B_{s\frac{n}{2}}}+\frac{h_0}{\Delta F_0}a_{2s\frac{n}{2}}\right)n & \left(r=\frac{n}{2}\right),
\end{cases}$$

$$b'_{sr}=\begin{cases}
\frac{nh_0}{\Delta F_0}a_{1sr} & \left(r=0,\frac{n}{2}\right),\\
\frac{nh_0}{2\Delta F_0}a_{1sr} & \left[r=1,2,\ldots,E\left(\frac{n-1}{2}\right)\right].
\end{cases} \tag{15.472}$$

By means of expressions (15.471), the problem of determination of the arbitrary constants C, $\bar{C}$, related to the coefficients Y by relations (15.406), reduces to the determination of n coefficients A and B of the intrpolation polynomial of contact forces P_m^*.

The coefficients $A_0, A_1, B_1,$ corresponding to the distribution of forces P_m^* according to the law of the plane are determined from equilibrium conditions (15.196). Introducing contact forces (15.456) into (15.196), we have

$$N_z(0)=\cos\beta\sum_{m=1}^{n}P_m^*\left[1+\frac{R_0k_0}{\Delta F_0}\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\,d\alpha\right],$$

$$M_x(0)=R_0\cos\beta\sum_{m=1}^{n}P_m^*\left[\sin\frac{2\pi}{n}m+\frac{R_0k_0}{\Delta F_0}\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\sin\alpha\,d\alpha\right], \tag{15.473}$$

$$M_y(0)=-R_0\cos\beta\sum_{m=1}^{n}P_m^*\left[\cos\frac{2\pi}{n}m+\frac{R_0k_0}{\Delta F_0}\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\cos\alpha\,d\alpha\right].$$

For φ_m specified in the form (15.465), we can obtain

$$\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\,d\alpha=a_{00},$$

$$\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\sin\alpha\,d\alpha=a_{02}\sin\frac{2\pi}{n}m+a_{01}\cos\frac{2\pi}{n}m,$$

$$\int\limits_{\frac{2\pi}{n}(m-1)}^{\frac{2\pi}{n}(m+1)}\varphi_m(\alpha)\cos\alpha\,d\alpha=-a_{01}\sin\frac{2\pi}{n}m+a_{02}\cos\frac{2\pi}{n}m, \tag{15.474}$$

where

$$a_{00}=\frac{2\pi}{n}\left\{1+\int\limits_0^1[\chi_-(\xi)-\chi_+(\xi)]\,d\xi\right\},$$

$$a_{01}=1-\cos\frac{2\pi}{n}-\frac{2\pi}{n}\int\limits_0^1\left[\chi_-(\xi)\sin\frac{2\pi}{n}(1-\xi)+\chi_+(\xi)\sin\frac{2\pi}{n}\xi\right]d\xi,$$

$$a_{02}=\sin\frac{2\pi}{n}+\frac{2\pi}{n}\int\limits_0^1\left[\chi_-(\xi)\cos\frac{2\pi}{n}(1-\xi)-\chi_+(\xi)\cos\frac{2\pi}{n}\xi\right]d\xi. \tag{15.475}$$

Expressions (15.475) correspond to the arbitrary functions χ_-, χ_+. Let $\chi_-=\chi_+=\chi$, then

$$a_{00}=\frac{2\pi}{n},$$

$$a_{01}=2\sin\frac{\pi}{n}\left[\sin\frac{\pi}{n}-\frac{2\pi}{n}\int\limits_0^1\chi(\xi)\cos\frac{2\pi}{n}\left(\xi-\frac{1}{2}\right)d\xi\right],$$

$$a_{02}=2\sin\frac{\pi}{n}\left[\cos\frac{\pi}{n}+\frac{2\pi}{n}\int\limits_0^1\chi(\xi)\sin\frac{2\pi}{n}\left(\xi-\frac{1}{2}\right)d\xi\right]. \tag{15.476}$$

For the piecewise-linear function φ_m, setting $\chi(\xi)=\xi$, we find

$$a_{00}=\frac{2\pi}{n};\; a_{01}=0;\; a_{02}=\frac{2n}{\pi}\sin^2\frac{\pi}{n}. \tag{15.477}$$

Now, expanding (15.473) with the aid of (15.474), we have

$$\begin{aligned}
N_z(0)&=\cos\beta\left(1+\frac{R_0h_0}{\Delta F_0}a_{00}\right)\sum_{m=1}^{n}P_m^*,\\
M_x(0)&=R_0\cos\beta\left[\left(1+\frac{R_0h_0}{\Delta F_0}a_{02}\right)\sum_{m=1}^{n}P_m^*\sin\frac{2\pi}{n}m+\right.\\
&\left.+\frac{R_0h_0}{\Delta F_0}a_{01}\sum_{m=1}^{n}P_m^*\cos\frac{2\pi m}{n}\right],\\
M_y(0)&=-R_0\cos\beta\left[\left(1+\frac{R_0h_0}{\Delta F_0}a_{02}\right)\sum_{m=1}^{n}P_m^*\cos\frac{2\pi}{n}m-\right.\\
&\left.-\frac{R_0h_0}{\Delta F_0}a_{01}\sum_{m=1}^{n}P_m^*\sin\frac{2\pi m}{n}\right],
\end{aligned} \tag{15.478}$$

whence on the basis of formula (13.68), we obtain a system of equations in the coefficients A_0, A_1, B_1 of the interpolation polynomial:

$$\begin{aligned}
N_z(0)&=n\cos\beta\left(1+\frac{R_0h_0}{\Delta F_0}a_{00}\right)A_0,\\
M_x(0)&=\frac{1}{2}nR_0\cos\beta\left[\left(1+\frac{R_0h_0}{\Delta F_0}a_{02}\right)B_1+\frac{R_0h_0}{\Delta F_0}a_{01}A_1\right],\\
M_y(0)&=-\frac{1}{2}nR_0\cos\beta\left[\left(1+\frac{R_0h_0}{\Delta F_0}a_{02}\right)A_1-\frac{R_0h_0}{\Delta F_0}a_{01}B_1\right].
\end{aligned} \tag{15.479}$$

From (15.479) we obtain

$$A_0=\frac{N_z(0)}{n\cos\beta\left(1+\frac{R_0h_0}{\Delta F_0}a_{00}\right)}, \tag{15.480}$$

$$A_1 = \frac{2\Delta F_0}{nR_0^2h_0\cos\beta}\,\frac{M_x(0)\,a_{01} - M_y(0)\left(\frac{\Delta F_0}{R_0h_0} + a_{02}\right)}{\left(\frac{\Delta F_0}{R_0h_0} + a_{02}\right)^2 + a_{01}^2},$$

$$B_1 = \frac{2\Delta F_0}{nR_0^2h_0\cos\beta}\,\frac{M_x(0)\left(\frac{\Delta F_0}{R_0h_0} + a_{02}\right) + M_y(0)\,a_{01}}{\left(\frac{\Delta F_0}{R_0h_0} + a_{02}\right)^2 + a_{01}^2}. \tag{15.480}$$

For piecewise-linear functions ψ_m, in view of (15.477), we obtain

$$A_0 = \frac{N_x(0)}{n\cos\beta\,(1+\varkappa)}, \tag{15.481}$$

$$A_1 = -\frac{2}{nR_0\cos\beta}\,\frac{M_y(0)}{1+\varkappa\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^2},$$

$$B_1 = \frac{2}{nR_0\cos\beta}\,\frac{M_x(0)}{1+\varkappa\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^2}.$$

The remaining coefficients of the contact load interpolation polynomial, which correspond to the self-balanced distribution of forces P_m^*, must be determined from the strain compatibility conditions. As noted above, these conditions are satisfied approximately and are formulated in the form of equality of the warping displacements of the shell and ring at the points $\alpha = \frac{2\pi}{n}m$. For a contact load of arbitrary type, the warping displacements of the shell are represented by relation (15.443).

Introducing into (15.443) the values of the arbitrary constants according to (15.406), and considering (15.420) and (15.471), we obtain

$$\left[u^*_{m_x}(0,\alpha) - u^*_{m_x0}(0,\alpha)\right]_{\alpha=\frac{2\pi}{n}m} =$$

$$= -\frac{\pi\sqrt{\gamma}R_0}{Eh_0 n}\left\{\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\left[\sin\frac{2\pi r}{n}m\left(A_r\sum_{s=1}^{\infty}\frac{b'_{sr}}{\lambda^*_{sr}B_{sr}} - B_r\sum_{s=1}^{\infty}\frac{b_{sr}}{\lambda^*_{sr}B_{sr}}\right)+\right.\right. \tag{15.482}$$

$$\left.+\cos\frac{2\pi r}{n}m\left(A_r\sum_{s=1}^{\infty}\frac{b_{sr}}{\lambda^*_{sr}B_{sr}} - B_r\sum_{s=1}^{\infty}\frac{b'_{sr}}{\lambda^*_{sr}B_{sr}}\right)\right]+$$

$$\left.+\frac{1}{\sqrt{2}}(-1)^m A_{\frac{n}{2}}\sum_{s=1}^{\infty}\frac{b_{s\frac{n}{2}}}{\lambda^*_{s\frac{n}{2}}B_{s\frac{n}{2}}}\right\}.$$

The displacements of the ring at the points $\alpha=\frac{2\pi}{n}m$ for an arbitrary contact load are represented by expression (15.447). Omitting in (15.447) the terms corresponding to the distribution of ring displacements $u^{\circ}_{m_z}\left(\frac{2\pi}{n}m\right)$ according to the law of the plane and taking (15.448), (15.406), (15.420) and (15.471) into account, we obtain

$$\begin{aligned}
\left[u^{\circ}_{m_z}(\alpha)-u^{\circ}_{m_z0}(\alpha)\right]_{\alpha=\frac{2\pi}{n}m} = \sum_{r=2}^{E\left(\frac{n-1}{2}\right)} \Bigg(& \sin\frac{2\pi r}{n}m\Bigg\{B^{\circ}_r+B_r\sum_{s=1}^{\infty}\Bigg[g^{r}_{1sr}+ \\
& +\sum_{q_k=1}^{\infty}\left(g^{q_kn-r}_{1sr}+g^{q_kn+r}_{1sr}\right)\Bigg]+A_r\sum_{s=1}^{\infty}\Bigg[\bar{g}^{r}_{1sr}+\sum_{q_k=1}^{\infty}\left(\bar{g}^{q_kn-r}_{1sr}+\bar{g}^{q_kn+r}_{1sr}\right)\Bigg]\Bigg\}+ \\
& +\cos\frac{2\pi r}{n}m\Bigg\{A^{\circ}_r+A_r\sum_{s=1}^{\infty}\Bigg[g^{r}_{2sr}+\sum_{q_k=1}^{\infty}\left(g^{q_kn-r}_{2sr}+g^{q_kn+r}_{2sr}\right)\Bigg]- \\
& -B_r\sum_{s=1}^{\infty}\Bigg[\bar{g}^{r}_{2sr}+\sum_{q_k=1}^{\infty}\left(\bar{g}^{q_kn-r}_{2sr}+\bar{g}^{q_kn+r}_{2sr}\right)\Bigg]\Bigg\}\Bigg)+ \\
& +(-1)^m\Bigg[A^{\circ}_{\frac{n}{2}}+A_{\frac{n}{2}}\sum_{s=1}^{\infty}\sum_{q_k=1}^{\infty}g^{q_kn-\frac{n}{2}}_{2s\frac{n}{2}}\Bigg].
\end{aligned} \tag{15.483}$$

where the following notation is used:

$$\begin{aligned}
g^{k}_{1sr} &= \frac{kb_{sr}}{n\lambda^{\circ 2}_{1sr}}\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha\, d\alpha\left[\frac{1}{d^T_k}\,\frac{\pi k}{n}-\frac{\lambda^{\circ}_{1sr}}{\sqrt{\gamma}d^S_k}\right], \\
g^{k}_{2sr} &= \frac{kb_{sr}}{n\lambda^{\circ 2}_{2sr}}\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha\, d\alpha\left[\frac{1}{d^T_k}\,\frac{\pi k}{n}+\frac{\lambda^{\circ}_{2sr}}{\sqrt{\gamma}d^S_k}\right], \\
\bar{g}^{k}_{1sr} &= \frac{kb'_{sr}}{n\lambda^{\circ 2}_{1sr}}\int_0^{2\pi}\Phi_{1sr}(\alpha)\sin k\alpha\, d\alpha\left[\frac{1}{d^T_k}\,\frac{\pi k}{n}-\frac{\lambda^{\circ}_{1sr}}{\sqrt{\gamma}d^S_k}\right], \\
\bar{g}^{k}_{2sr} &= \frac{kb'_{sr}}{n\lambda^{\circ 2}_{2sr}}\int_0^{2\pi}\Phi_{2sr}(\alpha)\cos k\alpha\, d\alpha\left[\frac{1}{d^T_k}\,\frac{\pi k}{n}+\frac{\lambda^{\circ}_{2sr}}{\sqrt{\gamma}d^S_k}\right].
\end{aligned} \tag{15.484}$$

Now, introducing (15.482) and (15.483) into strain compatibility conditions (15.438), we obtain a simultaneous system of n equations in (n-3) unknown coefficients $A_r, B_r\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right]$ and $A_{\frac{n}{2}}$ of the interpolation polynomial of the contact load P^{*}_{m}:

$$\sum_{r=2}^{E\left(\frac{n-1}{2}\right)}\Bigg(\sin\frac{2\pi r}{n}m\Bigg\{B_r^\circ+B_r\sum_{s=1}^{\infty}\Bigg[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b_{sr}}{\lambda_{sr}^\circ B_{sr}}+g'_{1sr}+\sum_{q_k=1}^{\infty}\left(g_{1sr}^{q_kn-r}+\right.$$

$$\left.+g_{1sr}^{q_kn+r}\right)\Bigg]+A_r\sum_{s=1}^{\infty}\Bigg[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b'_{sr}}{\lambda_{sr}^\circ B_{sr}}+\bar g'_{1sr}+\sum_{q_k=1}^{\infty}\left(\bar g_{1sr}^{q_kn-r}+\bar g_{1sr}^{q_kn+r}\right)\Bigg]\Bigg\}+$$

$$+\cos\frac{2\pi r}{n}m\Bigg\{A_r^\circ+A_r\sum_{s=1}^{\infty}\Bigg[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b_{sr}}{\lambda_{sr}^\circ B_{sr}}+g'_{2sr}+\sum_{q_k=1}^{\infty}\left(g_{2sr}^{q_kn-r}+\right.\qquad(15.485)$$

$$\left.+g_{2sr}^{q_kn+r}\right)\Bigg]-B_r\sum_{s=1}^{\infty}\Bigg[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b'_{sr}}{\lambda_{sr}^\circ B_{sr}}+\bar g'_{2sr}+\sum_{q_k=1}^{\infty}\left(\bar g_{2sr}^{q_kn-r}+\bar g_{2sr}^{q_kn+r}\right)\Bigg]\Bigg\}\Bigg)+$$

$$+(-1)^m\left[A^\circ_{\frac{n}{2}}+A_{\frac{n}{2}}\sum_{s=1}^{\infty}\left(\frac{\pi\sqrt{\gamma}R_0}{Eh_0n\sqrt{2}}\,\frac{b_{s\frac{n}{2}}}{\lambda^\circ_{2s\frac{n}{2}}B_{s\frac{n}{2}}}+\sum_{q_k=1}^{\infty}g_{2s\frac{n}{2}}^{q_kn-\frac{n}{2}}\right)\right]=0$$

$$(m=1,2,\ldots,n).$$

Equating to zero in (15.485) the multipliers on $\sin\frac{2\pi r}{n}m,\ \cos\frac{2\pi r}{n}m,\ (-1)^m$, we obtain

$$A_rS'_{1r}+B_rS_{1r}+B_r^\circ=0,\quad A_rS_{2r}-B_rS'_{2r}+A_r^\circ=0$$
$$\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right].\qquad(15.486)$$

$$A_{\frac{n}{2}}=-\frac{A^\circ_{\frac{n}{2}}}{S_{2\frac{n}{2}}},\qquad(15.487)$$

where

$$S_{1r}=\sum_{s=1}^{\infty}\left[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b_{sr}}{\lambda_{sr}^\circ B_{sr}}+g'_{1sr}+\sum_{q_k=1}^{\infty}\left(g_{1sr}^{q_kn-r}+g_{1sr}^{q_kn+r}\right)\right],$$

$$S_{2r}=\sum_{s=1}^{\infty}\left[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b_{sr}}{\lambda_{sr}^\circ B_{sr}}+g'_{2sr}+\sum_{q_k=1}^{\infty}\left(g_{2sr}^{q_kn-r}+g_{2sr}^{q_kn+r}\right)\right],$$

$$S'_{1r}=\sum_{s=1}^{\infty}\left[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b'_{sr}}{\lambda_{sr}^\circ B_{sr}}+\bar g'_{1sr}+\sum_{q_k=1}^{\infty}\left(\bar g_{1sr}^{q_kn-r}+\bar g_{1sr}^{q_kn+r}\right)\right],\qquad(15.488)$$

$$S'_{2r}=\sum_{s=1}^{\infty}\left[\frac{\pi\sqrt{\gamma}R_0}{Eh_0n}\,\frac{b'_{sr}}{\lambda_{sr}^\circ B_{sr}}+\bar g'_{2sr}+\sum_{q_k=1}^{\infty}\left(\bar g_{2sr}^{q_kn-r}+\bar g_{2sr}^{q_kn+r}\right)\right]$$
$$\left[r=2,3,\ldots,E\left(\frac{n-1}{2}\right)\right].$$

$$S_{2\frac{n}{2}} = \sum_{s=1}^{\infty} \left(\frac{\pi\sqrt{\gamma}R_0}{Eh_0 n\sqrt{2}} \, \frac{b_{s\frac{n}{2}}}{\lambda^{\circ}_{2s\frac{n}{2}} B_{s\frac{n}{2}}} + \sum_{s_k=1}^{\infty} K_{2s\frac{n}{2}}^{s_k n - \frac{n}{2}} \right). \tag{15.489}$$

Solving system (15.486) for A_r, B_r, we finally find

$$A_r = -\frac{S'_{2r}B^{\circ}_r + S_{1r}A^{\circ}_r}{S'_{1r}S'_{2r} + S_{1r}S_{2r}}; \quad B_r = \frac{S'_{1r}A^{\circ}_r - S_{2r}B^{\circ}_r}{S'_{1r}S'_{2r} + S_{1r}S_{2r}}. \tag{15.490}$$

The problem has thus been solved. Introducing the obtained coefficients of the contact load interpolation polynomial (15.481), (15.487), (15.490) into relations (15.471), then determining the arbitrary constants C, $\bar{C}$ from formulas (15.212), (15.213), (15.406), by using general expressions (15.183), (15.187), (15.188), (15.191), one can calculate the stresses and displacements at any point of the shell.

The solutions obtained pertains to the case in which the approximating functions φ_m are determined by the arbitrary functions $X-$ and $X+$. For a sufficiently large number n of stringers, when φ_m can be assumed to be piecewise-linear (15.458), all the expressions are considerably simplified. The coefficients of the contact load interpolation polynomial A_0, A_1, B_1 for this case are represented by expression (15.481). For the coefficients $A_r, B_r \left[r = 2, 3, \ldots, E\left(\frac{n-1}{2}\right) \right]$ and $A_{\frac{n}{2}}$, in view of (15.470), (15.472), (15.484), (15.488) and notations (15.424) and (15.453), we can obtain from (15.487), (15.490)

$$A_r=\frac{-\dfrac{4R_0}{\pi n\sin^2\dfrac{\pi r}{n}}\sum_{m=1}^{n}u^{0^\circ}_{m_s}\left(\dfrac{2\pi}{n}m\right)\cos\dfrac{2\pi r_m}{n}}{\sum\limits_{s=1}^{\infty}\dfrac{1}{\lambda^{02}_{sr}B_{sr}}\left[\dfrac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{\circ}_{sr}}+\dfrac{L_{r,sr}}{D_{r,2sr}}+\sum\limits_{q_k=1}^{\infty}\left(\dfrac{L_{q_kn+r,sr}}{D_{q_kn+r,2sr}}+\dfrac{L_{q_kn-r,sr}}{D_{q_kn-r,2sr}}\right)\right]},$$

$$B_r=\frac{-\dfrac{4R_0}{\pi n\sin^2\dfrac{\pi r}{n}}\sum_{m=1}^{n}u^{0^\circ}_{m_s}\left(\dfrac{2\pi}{n}m\right)\sin\dfrac{2\pi r}{n}m}{\sum\limits_{s=1}^{\infty}\dfrac{1}{\lambda^{02}_{sr}B_{sr}}\left[\dfrac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{\circ}_{sr}}+\dfrac{L_{r,sr}}{D_{r,1sr}}+\sum\limits_{q_k=1}^{\infty}\left(\dfrac{L_{q_kn+r,sr}}{D_{q_kn+r,1sr}}-\dfrac{L_{q_kn-r,sr}}{D_{q_kn-r,1sr}}\right)\right]},$$

$$A_{\frac{n}{2}}=\frac{-\dfrac{R_0}{\pi n}\sum_{m=1}^{n}(-1)^m u^{0^\circ}_{m_s}\left(\dfrac{2\pi}{n}m\right)}{\sum\limits_{s=1}^{\infty}\dfrac{1}{\lambda^{02}_{2s\frac{n}{2}}B_{s\frac{n}{2}}}\left[\dfrac{\varkappa\sqrt{\gamma}R_0}{2Eh_0\lambda^{\circ}_{2s\frac{n}{2}}}+\sum\limits_{q_k=1}^{\infty}\dfrac{L_{q_kn-\frac{n}{2},s\frac{n}{2}}}{D_{q_kn-\frac{n}{2},2s\frac{n}{2}}}\right]}. \qquad (15.491)$$

In this problem, as in the previous one, to simplify the computational formulas it is desirable to assume for n 8-10:

$$d^{T}_{k}=\infty \quad \left(k>\frac{n}{2}\right),$$
$$d^{s}_{k}=\infty \quad (k \text{ — любые}).$$

Then, from (15.491), we obtain

$$A_r=\frac{-\dfrac{4R_0}{\pi n\sin^2\dfrac{\pi r}{n}}\sum_{m=1}^{n}u^{0^\circ}_{m_s}\left(\dfrac{2\pi}{n}m\right)\cos\dfrac{2\pi r}{n}m}{\sum\limits_{s=1}^{\infty}\dfrac{1}{\lambda^{02}_{sr}B_{sr}}\left[\dfrac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{\circ}_{sr}}+\dfrac{L_{r,sr}}{D_{r,2sr}}\right]},$$

$$B_r=\frac{-\dfrac{4R_0}{\pi n\sin^2\dfrac{\pi r}{n}}\sum_{m=1}^{n}u^{0^\circ}_{m_s}\left(\dfrac{2\pi}{n}m\right)\sin\dfrac{2\pi r}{n}m}{\sum\limits_{s=1}^{\infty}\dfrac{1}{\lambda^{02}_{sr}B_{sr}}\left[\dfrac{\varkappa\sqrt{\gamma}R_0}{Eh_0\lambda^{\circ}_{sr}}+\dfrac{L_{r,sr}}{D_{r,1sr}}\right]}, \qquad (15.492)$$

$$A_{\frac{n}{2}}=\frac{-\dfrac{R_0}{\pi n}\sum_{m=1}^{n}(-1)^m u^{0^\circ}_{m_s}\left(\dfrac{2\pi}{n}m\right)}{\sum\limits_{s=1}^{\infty}\dfrac{1}{\lambda^{02}_{2s\frac{n}{2}}B_{s\frac{n}{2}}}\left[\dfrac{\varkappa\sqrt{\gamma}R_0}{2Eh_0\lambda^{\circ}_{2s\frac{n}{2}}}+\dfrac{L_{\frac{n}{2},s\frac{n}{2}}}{D_{\frac{n}{2},2s\frac{n}{2}}}\right]}.$$

APPENDICES

Appendix 1. THE DELTA FUNCTION

In the construction of idealized models of real processes and objects, abstract interpretations of the continuous and discrete are reflected in such concepts as distributed and concentrated quantities. The mathematical interpretation of these concepts in the frame work of classical representations varies.

Quantitative relationships between distributed quantities are formulated mostly by using the language of classical theory of functions. The basis for this is the representation of the intensity of distributed quantities as a point function, a representation that permits an effective use of the powerful device of classical analysis. For concentrated quantities, the concept of intensity is deprived of meaning in the framework of classical interpretations. Therefore, quantitative relationships between them occur only in the form of finite relations.

The indicated discontinuity in the mathematical interpretation of distributed and concentrated quantities in cases where the idealized model is based on a synthesis of these concepts substantially complicates the research. However, this discontinuity can be eliminated, and the difficulties produced by it circumvented by interpreting a concentrated quantity as the limiting form of the corresponding distributed quantity. This view makes it possible to achieve the necessary logical unity between the continuous and the discrete, formulated mathematically in the language of generalized functions.

The foundations of modern theory of generalized functions were laid by S. L. Sobolev* and L. Schwartz.** The simplest generalized function is the delta function, or, as it is also known, the symbolic impulse function.

The delta function is usually associated with the name of P. Dirac, who used it systematically in his quantum mechanical studies.*** The delta function also occurred earlier in theoretical work in mathematics.

A set of generalized functions is an extension of a set of ordinary functions, just as a set of radial numbers extends to a set of real numbers. While the introduction of the concept of real numbers is aimed at making certain operations feasible, for example, the computation of roots or logarithms, the introduction of generalized functions always permits one to make the differentiation operation feasible, which, as we know, is not always possible, even for continuous functions.

Generalized functions are certain limiting forms, i.e., "ideal elements." They are not functions in the usual sense of the word, but

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* S. L. Sobolev. Methode nouvelle a resoudre le probleme de Cauchy pour les equations lineaires hyperboliques normales. Matematicheskiy Sbornik, 1936, No. 1 (43).

** L. Schwartz. Theorie des distributions, t. I, 195C, t, II, 1951, Paris.

***P. A.-M. Dirac. The Physical Interpretation of the Quantum Dynamics. Proceedings of the Royal Society of London, Series A, 1927, vol. CXIII. No. 765. p. 621-641.

in a certain intuitive sense can be approximated by ordinary functions. For example, the delta function is a mathematical form of such an idealized concept as the intensity of a quantity concentrated at a point. In one- and two-dimensional problems, this may be a linear load and the pressure on some surface, corresponding to a concentrated force. In the space of three measurements, one can speak of the density generated by a material point, of the intensity of a point heat source, etc. It is obvious that by using mathematically permissible functions and the operation of passage to the limit, one can dispense with the introduction of the delta function. However, by defining instead a set of formal operations on the delta function as a special kind of object, one can circumvent considerable difficulties involved in classical analysis, and in many cases, obtain new results.*

I.1. Fundamental Concepts and Definitions

Mathematical Interpretation

The delta function is one of a number of singular functions that cannot be correctly defined in the framework of classical theory of functions. According to the meaning given to it, delta function $\delta(x-x_0)$ should always be equal to zero, with the exception of the point $x = x_0$,

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*G. G. Onanov. Equations with Singular Coefficients of the Type of The Delta Function and Its Derivatives in Problems of Structural Mechanics. MTT, 1971, No. 5; G. G. Onanov. Normal Coordinate Functions of a Shell of Revolution with a Discrete Longitudinal Structure. Uchenyye Zapiski TsAGI, 1972, Vol. 3, No. 1; V. I. Narinskiy and G. G. Onanov. Normal Coordinate Functions of a Shell of Revolution with a Biperiodic Discrete Longitudinal Structure. Uchenyye Zapiski TsAGI, 1972, Vol. 3, No. 4.

at which it is "equal to infinity," and this in such a way that

$$\int \delta(x-x_0)\,dx=1\,^{*} \tag{I.1}$$

i.e., the delta function is the intensity of a unit action "applied" at the point $x = x_0$.

It is obvious that this definition is incompatible with classical definitions of a function and integral, but it can be analyzed so as to discover the real mathematical content of the delta function.

To this end, let us consider for example a material point of unit mass placed at the origin. It is "physically" evident that the distribution of masses, i.e., the density produced by this material point, will be a delta function.

Let us "smear out" the preliminary mass of the point under consideration uniformly inside a sphere S_ϵ. We will thus obtain an average density

$$f_\epsilon(x)=\begin{cases}\dfrac{3}{4\pi\epsilon^3}, & |x|<\epsilon,\\ 0, & |x|>\epsilon.\end{cases} \tag{I.2}$$

Initially, we will take as the density $\delta(x)$ the pointwise limit of the average densities $f_\epsilon(x)$ when $\epsilon\to 0$. Then

$$\delta(x)=\lim_{\epsilon\to 0} f_\epsilon(x)=\begin{cases}\infty, & x=0,\\ 0, & x\neq 0.\end{cases} \tag{I.3}$$

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*Hereinafter $x=(x_1, x_2, \ldots, x_n)$ denotes a point of n-dimensional space R_n. For simplicity, one can visualize a point on a straight line. Integrating over a limited region of the space R_n, we will write $\int_{R_n} \ldots dx$. When the integration is carried out over the entire space, the symbol R_n under the integration sign is omitted.

It is natural to require that the integral of the density $\delta(x)$ over any volume G give the mass of this volume. Therefore, in our case it is necessary that

$$\int_G \delta(x)\,dx = \begin{cases} 1, & 0 \in G, \\ 0, & 0 \notin G. \end{cases} \qquad \text{(I.4)}$$

However, it is easy to see by considering (I.3) that the integral on the left side of (I.4) is always equal to zero if it is taken to mean an improper integral:

$$\int_G \delta(x)\,dx = \lim_{\varepsilon\to 0}\left[\int_{G-S_\varepsilon} \delta(x)\,dx\right] = 0.$$

We have thus arrived at a contradiction, which shows that the pointwise limit of the sequence $\delta_\varepsilon(x)$, $\varepsilon\to 0$ cannot be adopted as the density $\delta(x)$.

However, in addition to the pointwise limit, other forms of passage to the limit are possible. Depending on the mathematical content of a given problem, the treatment of the convergence of a sequence of functions may be different.

A sequence of functions

$$\{u_k(x)\} = u_1(x),\ u_2(x), \ldots, u_k(x), \ldots \qquad \text{(I.5)}$$

is said to converge uniformly in some region of space R_n if for any $\varepsilon>0$ one can indicate an N such that when k, l>N for any x from R_n, the following condition will be fulfilled:

$$|u_k(x) - u_l(x)| < \varepsilon \text{ при } k,\ l > N.$$

Sequence (I.5) is said to converge in the mean in R_n if for any $\varepsilon>0$ one can indicate an N such that for k, l>N

$$\int_{R_n} |u_k(x) - u_l(x)|^2\,dx < \varepsilon.$$

Finally, sequence (I.5) is said to converge weakly in R_n with respect to functions $\varphi(x)$ from a certain class (ϕ), if for any $\varphi(x)$ there exists

$$\lim_{n\to\infty}\int_{R_n}\varphi(x)u_n(x)dx.$$

In discussing a convergent sequence, the set of its elements is usually supplemented with a limiting element. The limiting element of a weakly convergent sequence $u_k(x)$ must obviously be taken to mean a certain element u introduced by the axiomatic relation

$$\int_{R_n} u\varphi(x)dx = \lim_{n\to\infty}\int_{R_n} u_n(x)\varphi(x)dx. \qquad \text{(I.6)}$$

The limiting element of a uniformly convergent sequence of continuous functions always belongs to the class of continuous functions, but generally speaking this may fail to occur for sequences converging in the mean, and all the more for sequences converging weakly. If the limiting element does not belong to the class of functions under consideration, this class is expanded so as to include the limiting elements as well. As was noted above, the concept of expansion is encountered in real number theory, when irrational numbers are introduced as the limiting elements, defined by a class of equivalent numerical sequences. Let us recall that the numerical sequences $\{a_n\}$ and $\{b_n\}$ are considered to be equivalent if the sequence $\{a_n - b_n\}$ converges to zero. Similarly, two weakly convergent sequences $\{u_n\}$ and $\{v_n\}$ are said to have the same limiting elements, i.e., are equivalent, if, whatever the continuous function $\varphi(x)$.

$$\lim_{n\to\infty}\int_{R_n}[u_n(x) - v_n(x)]\varphi(x)dx = 0.$$

The principle of weak convergence ("weak definition") underlies the representation of generalized functions as an expansion of a class of ordinary functions.

Instead of requiring from some functions f(x) a definite value at every point, one can describe it just as successfully with the aid of a set of "scalar products" (f, φ) with all functions :

$$(f, \varphi) = \int f(x)\varphi(x)\,dx. \qquad (I.7)$$

For continuous functions, this definition is equivalent to the usual one, but in addition, it contains the possibility of generalizations, since by specifying the scalar products (f, φ), in some way, we give an exhaustive characterization to the symbol of function f. For example, if in defining generalized functions one proceeds from the representation of limiting elements of a class of equivalent sequences, one must assume

$$(f, \varphi) = \lim_{n \to \infty} \int f_n(x)\varphi(x)\,dx. \qquad (I.8)$$

We will compute the weak limit of the sequence of functions $\delta_\varepsilon(x)$, i.e., find for any continuous function the limit of the numerical sequence

$$\int \delta_\varepsilon(x)\varphi(x)\,dx \text{ при } \varepsilon \to 0.$$

Because of the continuity of the functions $\varphi(x)$ for any $\eta > 0$ as small as desired, there exists such $\varepsilon_0 > 0$, that when $|x| < \varepsilon_0$, we have $|\varphi(x) - \varphi_0| < \eta$

Hence, for all $\varepsilon \leq \varepsilon_0$, we obtain

$$\left|\int \delta_\varepsilon(x)\varphi(x)\,dx - \varphi(0)\right| = \frac{3}{4\pi\varepsilon^3}\left|\int_{|x|<\varepsilon} [\varphi(x) - \varphi(0)]\,dx\right| <$$
$$< \frac{3}{4\pi\varepsilon^3}\int_{|x|<\varepsilon} |\varphi(x) - \varphi(0)|\,dx < \eta\frac{3}{4\pi\varepsilon^3}\int_{|x|<\varepsilon} dx = \eta.$$

Hence,

$$\lim_{\varepsilon \to 0} \int \delta_\varepsilon(x)\varphi(x)\,dx = \varphi(0), \qquad (I.9)$$

i.e., sequence (I.2) is weakly convergent.

We set $\varphi(x) = 1$. Then from (I.9) it follows that the limiting element of sequence (I.2) satisfies conditions (I.3), which determine the density $\delta(x)$. Therefore, it is natural to define the delta function as the limiting element of a class of equivalent weakly convergent sequences $\delta_n(x)$:

$$(\delta, \varphi) = \int \delta\varphi(x)\,dx = \overline{\lim_{n\to\infty}} \int \delta_n(x)\varphi(x)\,dx = \varphi(0). \tag{I.10}$$

Such a definition is completely rigorous and makes it possible, by analyzing a class of equivalent sequences of the indicated type, to establish the essential properties of the delta function.

It should be noted that generalized functions can also be introduced in a different way. It has now become common practice in specialized literature* to define generalized functions, not on the basis of the representation of classes of equivalent sequences, but on the basis of an axiomatic introduction of the scalar product (f, φ) as a linear continuous functional $\Lambda[\varphi]$ defined on a certain space K of "good functions" $\varphi(x)$:

$$(f, \varphi) = \Lambda[\varphi]. \tag{I.11}$$

Now, to each given functional $\Lambda[\varphi]$ there will correspond a definite generalized function f. Thus, for the delta function

$$(\delta, \varphi) = \int \delta\varphi(x)\,dx = \varphi(0). \tag{I.12}$$

Definition (I.11) will be exhaustive provided that a class of "fairly good" functions is indicated on which the functional is defined. Ordinary integrable functions also fit this scheme; for such functions one can compute the integral from their product by a "good function."

Two generalized functions are considered to be identical if the values of the corresponding functionals coincide on any of the "good functions" $\varphi(x) = \varphi_0(x)$, and different if for at least one function $\varphi(x)$ these values

---

*See for example I. M. Gel'fand and G. Ye. Shilov. Generalized Functions. Moscow, Fizmatgiz, 1959.

are different.

It is also evident that a generalized function must be considered equal to zero if on any of the "good functions" the corresponding functional is equal to zero.

Local Properties

The representation of limiting elements of weakly convergent sequence does not, strictly speaking, provide the basis for speaking of the values of generalized functions at individual points. However, having tried to identify a generalized function with an ordinary continuous function, one can partly remove this indeterminacy.

Let f be some generalized function

$$\int f\varphi(x)\,dx = \lim_{n\to\infty}\int f_n(x)\varphi(x)\,dx,$$

and in some limited region of space G

$$\lim_{n\to\infty}\int_G f_n(x)\varphi(x)\,dx = 0.$$

Then for any $\varphi(x)$ which becomes zero outside the region G, one can write

$$\int_G f\varphi(x)\,dx = 0.$$

Hence, identifying the generalized function with the ordinary continuous function f = f(x), on the basis of the known lemma of variational calculus, we have

$$f(x) \equiv 0 \quad \text{for} \quad x \in G.$$

If the generalized function f is equal to zero in region G, the additional set R_n-G is said to be the carrier of f. The generalized function f is also said to be concentrated on the set R_n-G. A generalized function with a limited carrier is said to be <u>finite</u>. It is easy to show, for example, that the delta function is concentrated at the point x = 0. Indeed, by definition of the delta function, for any (x)

$$\int \delta\varphi(x)\,dx = \varphi(0).$$

Setting (0) = 0, we can write

$$\int \delta\varphi(x)\,dx = \int_{R_n} \delta\varphi(x)\,dx = 0,$$

where the integration extends to any region R_n not containing the point x = 0. In such a region, (x) is completely arbitrary. Therefore, identifying the delta function with an ordinary continuous function, we have

$$\delta = \delta(x) = 0 \text{ for } x \neq 0.$$

Hereinafter, the delta function will always be denoted by $\delta(x)$, keeping in mind that when x = 0, it is not identifiable with an ordinary function.

Delta-Shaped Sequences

Sequences whose limiting element is the delta function are said to be delta-shaped. Let us consider the sequence of nonnegative functions $\{\delta_{\varepsilon_n}(x)\}$ ($\varepsilon_n \to 0$ when $n \to \infty$), satisfying the conditions $\delta_{\varepsilon_n}(x) = 0$, if $|x| \geq \varepsilon_n$ and $\int \delta_{\varepsilon_n}(x)\,dx = 1$.

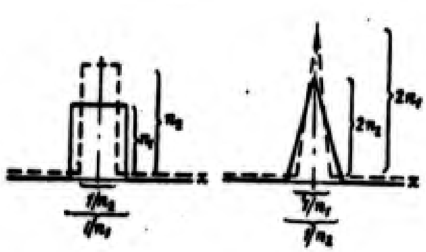


Fig. I.1. Normalized local sequences.

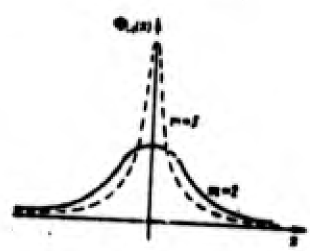


Fig. I.2. Elements of delta-shaped sequence.

Such a sequence is said to be a normalized local sequence of the point $x = 0$. Examples of such sequences for the case of one variable are shown in Fig. I.1.

We will compute the weak limit:

$$\lim_{n\to\infty}\int \delta_{\epsilon_n}(x)\varphi(x)\,dx=\lim_{n\to\infty}\int_{|x|<\epsilon_n}\delta_{\epsilon_n}(x)\varphi(x)\,dx=$$
$$=\lim_{n\to\infty}\varphi(x^*)\int_{|x|<\epsilon_n}\delta_{\epsilon_n}(x)\,dx=\lim_{n\to\infty}\varphi(x^*)=\varphi(0),$$

since $|x^*|<\epsilon_n$ and $\varphi(x)$ is continuous.

Hence, any normalized sequence will be delta-shaped.

Let us consider a sequence of continuous functions

$$\Phi_m(x)=\frac{m}{\pi}\,\frac{1}{1+(mx)^2} \tag{I.13}$$

of one independent variable.

Sequence (I.13) is not local. The elements of this sequence for different values of m are shown in Fig. I.2. It is obvious that

$$\lim_{m\to\infty}\Phi_m=\begin{cases}0 \text{ при } x\neq 0,\\ \infty \text{ при } x=0,\end{cases} \tag{I.14}$$

and in addition, the integral

$$\int_{-\infty}^{\infty}\Phi_m(x)\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\,(mx)}{1+(mx)^2}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}=1. \tag{I.15}$$

The question arises as to whether sequence (I.13) delta-shaped, i.e., whether the condition is fulfilled for this sequence whatever the continuous functions $\varphi(x)$. This question is informal, despite the fact that when $x \neq 0$, sequence (I.13) converges uniformly to zero with respect to x, and in addition, $\int_{-\infty}^{\infty}\Phi_m dx=1$ for any m.

Indeed, setting $\varphi(x)=e^{x}$, for example, we arrive at the following improper integral divergent for any values of the parameter m:

$$\int_{-\infty}^{\infty} \frac{e^{\frac{t}{m}}}{1+t^2}\,dt,$$

so that sequence (I.13) will also diverge. However, on the other hand, having imposed certain requirements on the behavior of the continuous functions $\varphi(x)$ when $|x|\to\infty$, one can obviously see to it that the improper integral $\int\limits_{-\infty}^{\infty}\Phi_m(x)\varphi(x)dx$ converges and that, in addition, there exists

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\Phi_m(x)\varphi(x)dx=\varphi(0).$$

We have formally

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\Phi_m(x)\varphi(x)dx=\frac{1}{\pi}\lim_{m\to\infty}\int_{-\infty}^{\infty}\frac{\varphi(x)}{1+(mx)^2}\,d(mx)=$$
$$=\frac{1}{\pi}\lim_{m\to\infty}\int_{-\infty}^{\infty}\frac{\varphi(t/m)}{1+t^2}\,dt=\frac{1}{\pi}\int_{-\infty}^{\infty}\lim_{m\to\infty}\frac{\varphi(t/m)}{1+t^2}\,dt= \tag{I.16}$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(0)\frac{dt}{1+t^2}=\varphi(0)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}=\varphi(0)$$

Of course, the result obtained applies only when the conditions ensuring the validity of passage to the limit under the sign of the improper integral are fulfilled.

For example, let $\varphi(x)$ be a limited continuous function. Then $\varphi(t/m)\leqslant M$ for any t, and hence, the improper integral $\int\limits_{-\infty}^{\infty}\frac{\varphi(t/m)}{1+t^2}$ converges uniformly with respect to any m.

We will also show that when $m\to\infty$, the function $\varphi(t/m)\to\varphi(0)$ is uniform with respect to t in the interval
[-A, A] for any A.

In fact, because of the continuity of $\varphi(x)$ with respect to its argument for any $\varepsilon > 0$ as small as desired, there exists such $\eta > 0$ that $|\varphi(t/m) - \varphi(0)| < \varepsilon$ as soon as $\frac{|t|}{m} < \eta$. However, for any A one can then choose $m = m_0$ from the condition $\frac{A}{m_0} < \eta$, when $|\varphi(t/m) - \varphi(0)| < \varepsilon$ for all t from the interval [-A, A], as soon as $m = m_0$.

From the uniform convergence of the improper integral with respect to the parameter m, and uniform convergence, with respect to t, of the integrand in the limiting function when $m \to \infty$ in any interval [-A, A], and also from the integrability in the strict sense of the function $\frac{\varphi(t/m)}{1+t^2}$ on this interval, it follows that the passage to the limit is permissible under the sign of the improper integral in (I.16). Therefore, with respect to any continuous function $\varphi(x)$, sequence (I.14) is delta-shaped.

The above discussed example shows the necessity of first qualifying the set of "sufficiently good" functions with respect to which we will discuss the weak convergence of given sequences (or, in other words, on which the corresponding functionals will be determined).

We take as such a set a collection of all real functions $\varphi(x)$ infinitely differentiable and finite, i.e., identically equal to zero outside a certain bounded region [arbitrary for each of the functions $\varphi(x)$]. These functions are known as <u>base functions</u>, and their entire collection, as the <u>base space K</u>.

The base functions can be added up, multiplied by real numbers, or differentiated any number of times, producing new base functions.

A set of base functions form the region of determination of generalized functions. For finite generalized functions, because of their local properties, the region of determination is expanded on the basis of the following lemma, which is cited without proof.

Lemma. For any bounded closed set G and region F containing it, there exists a base function h(x) equal to unity on G, to zero outside F, and always included between 0 and 1.

It is evidence that arbitrary changes of the base function outside the carrier of a finite generalized function do not affect the magnitude of the corresponding functional. Hence, the region of determination of a finite generalized function can, in addition to the base functions, be made to include any infinitely differentiable function $\varphi^*(x)$ in the vicinity of the carrier by constructing the base function $\varphi(x)$ coinciding with $\varphi^*(x)$ in the vicinity of the carrier. This can be done, by setting, for example, $\varphi(x)=\varphi^*(x)h(x)$.

It is clear from the above that the region of determination of the delta function generally includes all functions infinitely differentials in the vicinity of zero. Let us also note that in specific problems, this region may be even wider.

Derivatives of Delta Functions

The interpretation of a generalized function as the limiting element of a weakly convergent sequence of certain continuous functions suggests a natural representation of the derivative of a generalized function as the limiting element of a sequence of derivatives of these functions.

Let f be a generalized function, defined in a one-dimensional region. We have

$$\int_{-\infty}^{\infty} f\varphi(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\varphi(x)\,dx,$$

where the functions $f_n(x)$ are assumed to be infinitely differentiable.

Then, for the derivative f' of this generalized function, we have

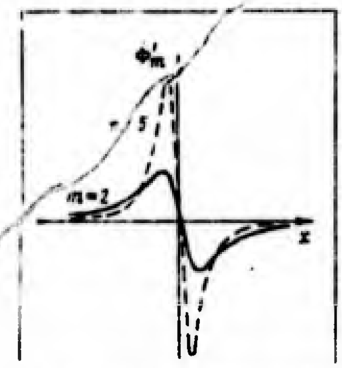


Fig. I.3. Derivatives of elements of a delta-shaped sequence.

$$\begin{aligned}\int_{-\infty}^{\infty} f'\varphi(x)\,dx &= \lim_{n\to\infty}\int_{-\infty}^{\infty} f'_n(x)\varphi(x)\,dx = \\ &= \lim_{n\to\infty}\left[f_n(x)\varphi(x)\Big|_{-\infty}^{\infty}\right] - \\ &- \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\varphi'(x)\,dx = \\ &= -\int_{-\infty}^{\infty} f\varphi'(x)\,dx, \end{aligned} \tag{I.17}$$

since in view of the finiteness of the base function $f_n(x)\varphi(x)\Big|_{-\infty}^{\infty}=0$ and in addition, $\varphi(x)$ belongs to the base space K.

Hence, according to (I.17), the following relation should be set down as the basis for the definition of the derivative of a generalized function:

$$\int f'\varphi(x)\,dx = -\int f\varphi'(x)\,dx. \tag{I.18}$$

Thus, every generalized function has a derivative which in turn will also be a generalized function. It is easy to check that the ordinary rules of differentiation are fulfilled: the derivative of the sum is equal to the sum of the derivatives, the constant multiplier is factored out of the sign of the derivative, etc.

For example, we will show that the derivative of the product of an infinitely differentiable function a(x) by a generalized function f obeys the classic differentiation formula. On the basis of the idea of the limiting element of a sequence, we write

$$\int [a(x)f]\varphi(x)dx = \lim_{n\to\infty}\int [a(x)f_n(x)]\varphi(x)dx =$$
$$= \lim_{n\to\infty}\int f_n(x)[a(x)\varphi(x)]dx = \int f[a(x)\varphi(x)]dx,$$

i.e., the product of the generalized function by the "good function" is determined by the relation

$$\int [a(x)f]\varphi(x)dx = \int f[a(x)\varphi(x)]dx. \tag{I.19}$$

Now, using (I.18) and (I.19), we can obtain

$$\int [af]'\varphi dx = -\int (af)\varphi' dx = -\int f(a\varphi')dx = -\int f[(a\varphi)' - a'\varphi]dx =$$
$$= \int [f'(a\varphi) + f(a'\varphi)]dx = \int (af' + a'f)\varphi dx, \tag{I.20}$$

as was stated above.

In space R_n, the derivatives of the generalized functions are introduced as in (I.18) by the relation

$$\int \frac{\partial f}{\partial x_i}\varphi(x)dx = -\int f\frac{\partial\varphi}{\partial x_i}dx. \tag{I.21}$$

It can also be shown that mixed derivatives of generalized functions are independent of the order of differentiation.

For higher-order derivatives we have the obvious relation

$$\int f^{(n)}\varphi(x)dx = (-1)^n\int f\varphi^{(n)}(x)dx. \tag{I.22}$$

From (I.22) it is also evident that the carriers of derivatives of a generalized function coincide with the carrier of the function itself.

For the delta function, on the basis of (I.10), (I.18), we have

$$\int \delta'(x)\varphi(x)dx = -\int \delta(x)\varphi'(x)dx = -\varphi'(0), \tag{I.23}$$

$$\int \delta^{(n)}(x)\varphi(x)dx = (-1)^n \int \delta(x)\varphi^{(n)}(x)dx = (-1)^n \varphi^{(n)}(0). \tag{I.24}$$

As was shown above, the functions

$$\Phi_m(x) = \frac{1}{\pi}\frac{m}{1+(mx)^2}$$

when $m \to \infty$ form a delta-shaped sequence. The derivatives of these functions

$$\Phi'_m(x) = -\frac{2}{\pi}\frac{m^3 x}{[1+(mx)^2]^2}$$

form a sequence which by definition converges to the derivative of the delta function when . Figure I.3 shows graphs of the functions $\Phi'_m(x)$ for different values of m.

The derivatives of the delta function are concentrated at the point x = 0. When x ≠ 0, the derivatives $\delta^{(n)}(x) \equiv 0$.

Unit Function

The function of one variable

(I.25)

is known as the Heaviside unit function (Fig. I.4). The unit function is not a singular function, but at the point x = 0 it is not differentiable in the ordinary sense, and can be shown to be related to the delta function as follows:

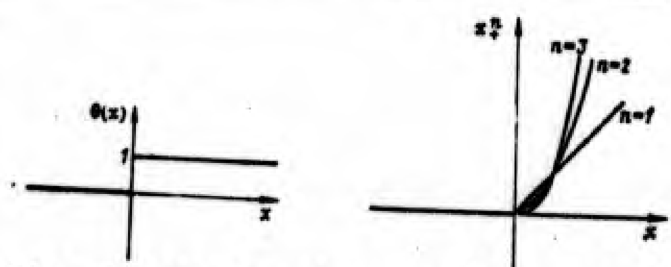
$$\frac{d\theta(x)}{dx} = \delta(x), \tag{I.26}$$

where the derivative is understood in the "generalized sense," i.e., as the derivative of a generalized function.

Indeed,

$$\int \theta'(x)\varphi(x)dx = -\int \theta(x)\varphi'(x)dx = -\int_0^{\infty} \varphi'(x)dx = \varphi(0)$$

and hence, according to basic definition (I.10), relation (I.26) is valid.



Diag. 1-single function Diag. 2-function x_+^n

It follows from (I.26) that

$$\theta(x) = \int_{-\infty}^{x} \delta(\xi)d\xi. \tag{I.27}$$

<u>The Function x_+^n</u>

The following function is related to the unit function $\theta(x)$

$$x_+ = \begin{cases} x \text{ при } x \geqslant 0, \\ 0 \text{ при } x \leqslant 0. \end{cases} \tag{I.28}$$

It is obvious that

$$x_+ = \int_{-\infty}^{x} \theta(\xi)d\xi. \tag{I.29}$$

Integrating by parts, we have

$$x_+ = x\theta(x) - \int_{-\infty}^{x} x\delta(x)dx = x\theta(x). \tag{I.30}$$

The following function is simiarly introduced (Fig. I.5)

$$x_+^n = \begin{cases} x^n \text{ when } & x \geqslant 0, \\ 0 \text{ when } & x \leqslant 0. \end{cases} \tag{I.31}$$

Obviously,

and $$x_+^n = x^n\theta(x) \tag{I.32}$$

$$\frac{d^k x_+^n}{dx^k} = \frac{n!}{(n-k)!}x_+^{n-k}. \tag{I.33}$$

1.2 Special Properties of the Delta Function

In the previous section, we analyzed the mathematical content of the delta function as some function which "is equal to zero everywhere with the exception of the point x = 0; at this point, it is equal to ∞, and in such a way that $\int_{-\infty}^{\infty}\delta(x)dx = 1$".

We can now define a collection of formal operations on the delta function as a special kind of object, so that hereinafter we can deal with it as with an "almost ordinary" function without having to reflect on its mathematical nature.

We will proceed from the fact that when x ≠ 0, the delta function can be identified with the ordinary function $\delta(x) = 0$; the point x = 0 is the carrier of the generalized function

$$\int_G \delta(x)\varphi(x)dx = \varphi(0),$$

if region G contains the point x = 0.

On the basis of this representation, we will summarize the basic operations involving the delta function.

1. $$f(x)\delta(x) = f(0)\delta(x). \tag{I.34}$$

Identity (I.34) should be understood literally (in the general sense) when x ≠ 0, and also in the sense of equality of the generalized functions with the carrier concentrated at the point x = 0.

$$\int_G [f(x)\delta(x)]\varphi(x)dx = \int_G [f(0)\delta(x)]\varphi(x)dx. \tag{I.35}$$

Here

G is any region;

$\varphi(x)$ is any function continuous in the vicinity of the point x = 0.

If $f(0)=0$, then

$$f(x)\delta(x)=0. \tag{I.36}$$

On the basis of (I.34)

$$f(x)\delta(x-t)=f(t)\delta(x-t), \tag{I.37}$$

where $\delta(x-t)$ is a "mixed" delta function with a carrier concentrated at the point x = t.

If $f(t)=0$, then

$$f(x)\delta(x-t)=0. \tag{I.38}$$

The delta function is even:

$$\delta(-x)=\delta(x), \tag{I.39}$$

therefore

$$\int \delta(x)dx=\int \delta(|a|x)d(|a|\,x)=|a|\int \delta(ax)dx,$$

whence

$$\delta(ax)=\frac{1}{|a|}\delta(x). \tag{I.40}$$

2. The delta function in space R_n may be represented in the form of the product

$$\delta(x)=\delta(x_1)\delta(x_2)\ldots\delta(x_i)\ldots\delta(x_n), \tag{I.41}$$

where $\delta(x_i)$ is a one-dimensional delta function.

3. For a one-dimensional delta function, we have

$$\int_{-\infty}^{x}\delta(\xi-t)d\xi=\theta(x-t)=\begin{cases}0 & \text{when} \quad x<t,\\ 1 & \text{when} \quad x>t,\end{cases} \tag{I.42}$$

$$\int_{-\infty}^{x} f(\xi)\delta(\xi-t)d\xi = \int_{-\infty}^{x} f(t)\delta(\xi-t)d\xi = f(t)\theta(x-t). \quad (I.43)$$

4. The n-th order derivative of a one-dimensional delta function when x ≠ 0 can be identified with an ordinary function $\delta^{(n)}(x)=0$; the point x = 0 is the carrier of the generalized function:

$$\int_{-\infty}^{\infty} \delta^{(n)}(x)\varphi(x)dx = (-1)^n\varphi^n(0), \quad (I.44)$$

where $\alpha \neq 0$.

The derivative $\delta'(x)$ is odd:

$$\delta'(-x) = -\delta'(x), \quad (I.45)$$
$$\delta^n(-x) = (-1)^n\delta^n(x).$$

5. If x is a point on a straight line, then differentiating (I.34), we obtain

$$f'(x)\delta(x) + f(x)\delta'(x) = f(0)\delta'(x),$$

whence

$$f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x). \quad (I.46)$$

Identity (I.46) expresses the local properties of the derivatives $\delta'(x)$. When x ≠ 0, the identity should be understood in the usual sense, and also in the sense of equality of the generalized functions to the carrier concentrated at the point x = 0.

We set $f(x)=x$ in (I.46). Then, differentiating, we obtain

$$\delta'(x) = -\frac{1}{x}\delta(x). \quad (I.47)$$

The following formula applies to higher-order derivatives:

$$f(x)\frac{d^n}{dx^n}\psi(x) = \sum_{i=0}^{n}(-1)^i C_n^i \frac{d^{n-i}}{dx^{n-i}}\left[\psi(x)\frac{d^i}{dx^i}f(x)\right], \quad (I.48)$$

where C_n^i is the number of combinations of n elements with respect to i.

Hence, setting $\psi(x)=\delta(x)$ and considering (I.34), we have

$$f(x)\delta^{(n)}(x)=\sum_{l=0}^{n}(-1)^{l}C_n^l f^{(l)}(0)\,\delta^{(n-l)}(x). \tag{I.49}$$

Formula (I.48) is easily proven by using the method of mathematical induction. We assume that it is valid for a certain n. Then

$$f(x)\frac{d^{n+1}}{dx^{n+1}}\psi(x)=\frac{d}{dx}\left[f(x)\frac{d^n}{dx^n}\psi(x)\right]-\frac{df}{dx}\frac{d^n}{dx^n}\psi(x).$$

Using (I.48), we have

$$\begin{aligned} f(x)\frac{d^{n+1}}{dx^{n+1}}\psi(x)=&\sum_{l=0}^{n}(-1)^{l}C_n^l\frac{d^{n+1-l}}{dx^{n+1-l}}\left[\psi(x)\frac{d^l}{dx^l}f(x)\right]-\\ &-\sum_{l=0}^{n}(-1)^{l}C_n^l\frac{d^{n-l}}{dx^{n-l}}\left[\psi(x)\frac{d^{l+1}}{dx^{l+1}}f(x)\right]. \end{aligned} \tag{I.50}$$

The second sum on the right-hand side of (I.50) can be transformed to

$$-\sum_{l=1}^{n+1}(-1)^{l}C_n^{l-1}\frac{d^{n+1-l}}{dx^{n+1-l}}\left[\psi(x)\frac{d^l}{dx^l}f(x)\right]. \tag{I.51}$$

Using (I.51), we combine the two sums in (I.50). Then, considering that

$$C_n^l+C_n^{l-1}=\frac{n!}{l!\,(n-l)!}+\frac{n!}{(l-1)!\,(n-l+1)!}=\frac{(n+1)!}{l!\,(n+1-l)!}=C_{n+1}^l,$$

we obtain

$$f(x)\frac{d^{n+1}}{dx^{n+1}}\psi(x)=\sum_{l=0}^{n+1}(-1)^{l}C_{n+1}^l\frac{d^{n-l+1}}{dx^{n-l+1}}\left[\psi(x)\frac{d^l}{dx^l}f(x)\right], \tag{I.52}$$

i.e., formula (I.48) is also valid for n+1. Since for n = 1 formula (I.48) is obvious, it holds for any n.

We assume that $f(x)=x^n$. Then, considering (I.36), we obtain from formula (I.49)

$$\frac{d^n\delta(x)}{dx^n}=(-1)^n\frac{n!}{x^n}\delta(x). \tag{I.53}$$

6. Let x be a point of n-dimensional space R_n, and $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ be an integral vector with nonnegative components α_j. We will denote the derivative of the function F(x) of order $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_n$ by

$$D^{\alpha}F(x)=\frac{\partial^{|\alpha|}F(x_1,x_2,\ldots,x_n)}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2},\ldots,\partial x_n^{\alpha_n}}.$$

Then, generalizing formula (I.48), we can obtain

$$f(x)D^{\alpha}\psi(x)=\sum_{i=0}^{\alpha}(-1)^{|i|}\prod_{j=1}^{n}C_{\alpha_j}^{i_j}D^{\alpha-i}[\psi(x)D^{i}f(x)]. \tag{I.54}$$

Where $i=(i_1, i_2,\ldots, i_n)$ is an integral vector whose components i_j assume the values $0,1,\ldots,\alpha_j$; n is the multiple sum

$$\sum_{i=0}^{\alpha}=\sum_{i_1,\ldots,i_n=0}^{i_1=\alpha_1,\ldots,i_n=\alpha_n}=\sum_{i_1=0}^{\alpha_1}\cdots\sum_{i_n=0}^{\alpha_n}.$$

From expression (I.54), setting $\psi(x)=\delta(x)$, we obtain

$$f(x)D^{\alpha}\delta(x)=\sum_{i=0}^{\alpha}(-1)^{|i|}\prod_{j=1}^{n}C_{\alpha_j}^{i_j}[D^{i}f(x)]_{x=0}D^{\alpha-i}\delta(x). \tag{I.55}$$

7. Let $\{\varphi_m(x)\}$ be a system of functions, complete orthogonal and normalized in region G. We will show that the delta function can be expanded as a Fourier series in these functions:

$$\delta(x)=\sum_{m=1}^{\infty}c_m\varphi_m(x),$$

where

$$c_m=\int_G\delta(x)\varphi_m(x)dx=\varphi_m(0)$$

in the sense that the sequence $\delta_n(x)=\sum_{m=1}^{n}c_m\varphi_m(x)$ will be delta-shaped.

Indeed, computing the weak limit of such a sequence, for any continuous g(x) we will have

$$\begin{aligned}\lim_{n\to\infty}\int_G\delta_n(x)g(x)dx&=\lim_{n\to\infty}\int_G\sum_{m=1}^{n}\varphi_m(0)\varphi_m(x)g(x)dx=\\&=\lim_{n\to\infty}\sum_{m=1}^{n}\varphi_m(0)\int_G\varphi_m(x)g(x)dx=\sum_{m=1}^{\infty}\varphi_m(0)\int_G\varphi_m(x)g(x)dx=g(0),\end{aligned} \tag{I.56}$$

since $\int_0 \varphi_m(x)g(x)dx$ is the coefficient of expansion of the function g(x) as a Fourier series in functions $\varphi(x)$.

Appendix II. EQUATIONS WITH IMPULSE TYPE CHARACTERISTICS

II.1. Equations with Singular Coefficients of the Type of the Delta Function and Its Derivatives*

Partly Degenerate Equations

Let T_1, T_2 be operators with regions of definition $\mathfrak{J}_1$ and $\mathfrak{J}_2$. The equation

$$Tu = F \tag{II.1}$$

where $T = T_1+T_2$, will be referred to as partly degenerate if the operator T_1^{-1} reciprocal to T_1 exists, and operator T_2 is degenerate:

$$T_2u = \Phi[x, (f_1u), (f_2u), \ldots, (f_ku)], \tag{II.2}$$

where Φ is some function of point x of n-dimensional space and k parameters (f_l, u) $(l = 1, 2, \ldots, k)$, constituting the values of certain functionals f_1 on u(x). We will denote by $\mathfrak{R}$ the region of values of the operator T, i.e., a set of elements of the type $\{T\varphi\}$, where φ ranges over $\mathfrak{J} = \mathfrak{J}_1 \cap \mathfrak{J}_2$. Obviously, if $F \in \mathfrak{R}$, the solution $u = u^*$ of Eq. (II.2) exists, and $u \in \mathfrak{J}$.

Since $T_1u^* = F - T_2u^*$, then, considering (II.2) and setting $(f_l, u^*) = \mu_l^*$, we have

$$u^* = T_1^{-1}[F - \Phi(x, \mu_1^*, \mu_2^*, \ldots, \mu_k^*)], \tag{II.3}$$

and

$$\mu_l^* = (f_l, T_1^{-1})[F - \Phi(x, \mu_1^*, \mu_2^*, \ldots, \mu_k^*)] \quad (l = 1, 2, \ldots, k). \tag{II.4}$$

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* G. G. Onanov. Equations with Singular Coefficients of the Type of the Delta Function and Its Derivatives. DAN SSSR, 1970, vol. 191, No. 5.

Expression (II.3) containing k parameters μ_1^* and a system of k equations (II.4) in these parameters represents the formal solution of partly degenerate Eq. (II.1) in "mixed form." In order to obtain the solution of this equation in explicit form, it is necessary to find the solution of system (II.4) satisfying the equations

$$T_1^{-1}[F-\Phi(x,\mu_1^*,\mu_2^*,\ldots,\mu_k^*)]\in\mathfrak{I}_1, \qquad \text{(II.5)}$$
$$F-\Phi(x,\mu_1^*,\mu_2^*,\ldots,\mu_k^*)=F^*\in\mathfrak{R}_1,$$

where $\mathfrak{R}_1$ is the region of values of the operator T_1. If such a solution of system (II.4) exists, then for the obtained values of the parameters μ_1^*, expression (II.3) gives the solution of Eq. (II.1).

Examples of partly degenerate equations are integral and integro-differential equations with a degenerate kernel.

Equations with Singular Coefficients

Let

$$T_2u=\sum_{|\alpha|<r}\sum_{k=1}^{m}D^{\alpha}\delta(x-x_k)\cdot T_{2\alpha k}u, \qquad \text{(II.6)}$$

where $D^{\alpha}\delta(x-x_k)$ is a partial derivative of order $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_n$ of the delta function concentrated at point x_k; $T_{2\alpha k}\cdot$ are some operators. On the basis of (I.55)

$$D^{\alpha}\delta(x-x_k)\cdot T_{2\alpha k}u=\sum_{l=0}^{\alpha}(-1)^{|l|}\prod_{j=1}^{n}C_{\alpha_j}^{l_j}D^{l}T_{2\alpha k}u\Big|_{x=x_k}D^{\alpha-l}\delta(x-x_k), \qquad \text{(II.7)}$$

whence it follows that operator (II.6) is degenerate, since the quantities $D^{l}T_{2\alpha k}u|_{x=x_k}$ can be treated as values of certain singular functionals $f_{\alpha lk}$ on u(x). Thus, the equation with singular coefficients $(T_1+T_2)u=F$ is partly degenerate:

$$T_1u+\sum_{|\alpha|<r}\sum_{k=1}^{m}\sum_{l=0}^{\alpha}C_{\alpha l}(f_{\alpha lk},u)D^{\alpha-l}\delta(x-x_k)=F, \qquad \text{(II.8)}$$

where

$$C_{\alpha l}=(-1)^{|l|}\prod_{j=1}^{n} C_{\alpha_j}^{l_j}.$$

According to (II.3), (II.4), the formal solution of Eq. (II.8) will be

$$u=T_1^{-1}\left[F-\sum_{|\alpha|\leqslant r}\sum_{k=1}^{m}\sum_{l=0}^{\alpha} C_{\alpha l}\mu_{\alpha l k} D^{\alpha-l}\delta(x-x_k)\right], \tag{II.9}$$

$$\mu_{\tilde{\alpha}\tilde{l}\tilde{k}}=D^{\tilde{l}}T_{2\tilde{\alpha}}T_1^{-1}\left[F-\sum_{|\alpha|\leqslant r}\sum_{k=1}^{m}\sum_{l=0}^{\alpha} C_{\alpha l}\mu_{\alpha l k} D^{\alpha-l}\delta(x-x_k)\right]\Bigg|_{x=x_{\tilde{k}}} \tag{II.10}$$

$$(|\tilde{\alpha}|\leqslant r;\ \tilde{k}=1,2,\ldots,m;\ \tilde{l}_j\leqslant\tilde{\alpha}_j).$$

If the solution $\bar{\mu}=\bar{\mu}^*$ of system (II.10), which also satisfies the conditions $T_1^{-1}F^*\in\mathfrak{Z}_2$ and $F^*\in\mathfrak{M}_1$, where $F^*=F-\sum_{|\alpha|\leqslant r}\sum_{k=1}^{m}\sum_{l=0}^{\alpha} C_{\alpha l}\mu^*_{\alpha l k} D^{\alpha-l}\delta(x-x_k)$, exists, then for $\bar{\mu}=\bar{\mu}^*$ expression (II.9) gives the solution of Eq. (II.8).

Let x be a point on a straight line. We will consider the simplest examples.

<u>Example 1</u>. $\ddot{y}''+\delta(x)y=f''(x)$ f(0) exists.

Passing to the partially degenerate form, we obtain

$$\dot{y}''+\dot{y}(0)\delta(x)=\dot{f}''(x).$$

In mixed form, the solution will be

$$y(x)=C_1x+C_2-y(0)x_++f(x),$$
$$y(0)=C_2+f(0),$$

where

$$x_+=\begin{cases} x & \text{when } x\geqslant 0,\\ 0 & \text{when } x\leqslant 0.\end{cases}$$

Eliminating y(0), we finally obtain

$$y(x)=C_1x+C_2(1-x_+)+f(x)-f(0)x_+.$$

<u>Example 2</u>. $y'+\delta(x)y=f'(x)$ f(0) exists.

Solution in mixed form:

$$y(x)=C_1-y(0)\theta(x)+f(x),$$
$$y(0)=C_1-y(0)\theta(0)+f(0).$$

Let us note that, generally speaking, the functions $\theta(x)=\begin{cases} 0 & \text{when } x<0 \\ 1 & \text{when } x>0. \end{cases}$ at the point x = 0 is not defined. However, $\theta(x)$ is not a singular function, and therefore at the point x = 0, its definition can be supplemented by any finite value.

Eliminating y(0) we have

$$y(x)=C[\theta(x)-\theta(0)-1]+f(x)-f(0),$$

where the new arbitrary constant

$$C=-\frac{C_1+f(0)}{1+\theta(0)}.$$

<u>Example 3</u>. $y'+\delta'(x)y=f'(x)$ f'(0) exists).

The product $\delta'(x)y(x)$ has meaning if y'(0) exists. In this case

$$y'(x)=-y(0)\delta'(x)+y'(0)\delta(x)+f'(x),$$
$$y(x)=C-y(0)\delta(x)+y'(0)\theta(x)+f(x).$$

Setting x = 0 in these expressions, we will obviously arrive at an absurdity if at least one of the quantities y(0) and y'(0) is different from zero.

Therefore $y(0)=y'(0)=0$, so that the solution in mixed form has the form

$$y(x)=C+f(x),$$
$$0=C+f(0),$$
$$0=f'(0).$$

It follows that when $f'(0)\neq 0$, the equation under consideration has no solution, and when $f'(0)=0$, the following degenerate solution exists:

$$y(x)=f(x)-f(0).$$

II.2. Impulse Type Characteristics on the Righ-Hand Side of Ordinary Linear Differential Equations

Let us consider the ordinary differential equation

$$L^*[y] = V^*, \tag{II.11}$$

where the linear differential operator

$$L^* = D^n + X_1^* D^{n-1} + \ldots + X_{n-1}^* D^1 + X_n^*.$$

In the general case, any of the coefficients $X^*(x)$ of Eq. (II.11), as well as the right-hand side $V^*(x)$, may contain singularities of the type of the delta function and its derivatives, and in addition, finite discontinuities:

$$X^*(x) = X(x) + \sum_i \left[X_i(x)\theta(x - x_i) + \sum_{m=0}^{r} a_{im}\delta^{(m)}(x - x_i) \right],$$

$$V^*(x) = V(x) + \sum_i \left[V_i(x)\theta(x - x_i) + \sum_{m=0}^{r} b_{im}\delta^{(m)}(x - x_i) \right].$$

Here X, X_1, V, V_1 are specified continuous functions

a_{im}, b_{im} are specified coefficients.

Taking relation (I.49) into consideration, we arrive at the ordinary differential equation

$$L[y] = V^{**} \tag{II.12}$$

with regular coefficients of the type

$$X(x) + \sum_i X_i(x)\theta(x - x_i) \quad \text{and the right-hand side}$$

$$V^{**}(x) = V(x) + \sum_i \left[V_i(x)\theta(x - x_i) + \sum_{m=0}^{r} C_{im}\delta^{(m)}(x - x_i) \right], \tag{II.13}$$

where the coefficients C_{im} constitute linear combinations of the values of certain singular functionals on the solution $y = y^*(x)$ of Eq. (II.11)

Let $y_1(x)$, $y_2(x), \ldots, y_j(x), \ldots, y_n(x)$ be the fundamental system of solutions of the homogeneous equation $L[y] = 0$. Representing the partial solution $y = y^{**}$ of Eq. (II.12) in the form

$$y^{**}(x)=\sum_{j=1}^{n} C_j(x)\,y_j(x),\tag{II.14}$$

where $C_j(x)$ are the desired functions, using the method of variation of arbitrary constants, we find

$$\sum_{j=1}^{n} C_j'(x)\,y_j^{(s)}(x)=0 \quad (s=0,\,1,\,2,\,\ldots,\,n-2),$$

$$\sum_{j=1}^{n} C_j'(x)\,y_j^{(n-1)}(x)=V^{**}(x),$$

whence откуда

$$C_j'(x)=(-1)^{n+j}\frac{W_{nj}(x)}{W(x)}V^{**}(x),\tag{II.15}$$

where $W_{nj}(x)$ is the minor of the element $y_j^{(n-1)}(x)$ of the Wronskian $W(x)$.

Integrating (II.15) while considering (II.13) and relations (I.49), and introducing the result into (II.14), we obtain

$$\begin{aligned}
y^{**}(x)={}&\sum_{j=1}^{n}(-1)^{n+j}y_j(x)\int\Big[V(x)+\sum_{i}V_i'(x)\theta(x-x_i)\Big]\frac{W_{nj}(x)}{W(x)}\,dx+\\
&+\sum_{i}\sum_{m=0}^{r}C_{im}\Big\{\sum_{k=0}^{m-1}(-1)^k C_m^k\frac{d^{m-k-1}}{dx^{m-k-1}}\delta(x-x_i)\sum_{j=1}^{n}(-1)^{n+j}y_j(x)\times\\
&\times\Big[\frac{d^k}{dx^k}\Big(\frac{W_{nj}(x)}{W(x)}\Big)\Big]_{x=x_i}+(-1)^m\theta(x-x_i)\sum_{j=1}^{n}(-1)^{n+j}y_j(x)\times\\
&\times\Big[\frac{d^m}{dx^m}\Big(\frac{W_{nj}(x)}{W(x)}\Big)\Big]_{x=x_i}\Big\}.
\end{aligned}\tag{II.16}$$

Partial solution (II.16) contains both step and generalized functions concentrated at the points $x=x_i$ $(i=1,\,2,\,\ldots,\,l)$. We will study the behavior of this solution in the vicinity of the indicated points.

Using relations (I.48), (I.49), we can obtain

$$\begin{aligned}
&\frac{d^{m-k-1}}{dx^{m-k-1}}\delta(x-x_i)\sum_{j=1}^{n}(-1)^{n+j}y_j(x)\Big[\frac{d^k}{dx^k}\Big(\frac{W_{nj}(x)}{W(x)}\Big)\Big]_{x=x_i}=\\
&=\sum_{l=0}^{m-k-1}(-1)^l C_{m-k-1}^l\frac{d^{m-k-1-l}}{dx^{m-k-1-l}}\delta(x-x_i)\sum_{j=1}^{n}(-1)^{n+j}\Big[\frac{d^l}{dx^l}y_j(x)\times\\
&\times\frac{d^k}{dx^k}\Big(\frac{W_{nj}(x)}{W(x)}\Big)\Big]_{x=x_i}=\sum_{l=0}^{m-k-1}(-1)^l C_{m-k-1}^l\frac{d^{m-k-1-l}}{dx^{m-k-1-l}}\delta(x-x_i)\times\\
&\times\sum_{j=1}^{n}(-1)^{n+j}\Big\{\sum_{q=0}^{k}(-1)^q C_k^q\frac{d^{k-q}}{dx^{k-q}}\Big[\Big(\frac{d^{l+q}}{dx^{l+q}}y_j(x)\Big)\frac{W_{nj}(x)}{W(x)}\Big]\Big\}_{x=x_i}=
\end{aligned}$$

$$= \sum_{t=0}^{m-k-1} (-1)^t C^t_{m-k-1} \frac{d^{m-k-1-t}}{dx^{m-k-1-t}} \delta(x-x_i) \times$$

$$\times \sum_{q=0}^{k} (-1)^q C^q_k \left\{ \frac{d^{k-q}}{dx^{k-q}} \left[\frac{1}{W(x)} \sum_{j=1}^{n} (-1)^{n+j} W_{nj}(x) \frac{d^{t+q}}{dx^{t+q}} y_j(x) \right] \right\} \Bigg|_{x=x_i} =$$

$$= \sum_{t=0}^{m-k-1} (-1)^t C^t_{m-k-1} \frac{d^{m-k-1-t}}{dx^{m-k-1-t}} \delta(x-x_i) \times$$

$$\times \sum_{q=0}^{k} (-1)^q C^q_k \left[\frac{d^{k-q}}{dx^{k-q}} \left(\frac{W_{t+q}(x)}{W(x)} \right) \right|_{x=x_i}, \qquad \text{(II.17)}$$

where

$$W_{t+q}(x) = \sum_{j=1}^{n} (-1)^{n+j} \left(\frac{d^{t+q}}{dx^{t+q}} y_j(x) \right) W_{nj}(x)$$

is a functional determinant obtained from the Wronskian by replacing its last row by the row of derivatives of order t+q. Obviously, the determinant $W_{t+q}(x) \equiv 0$, if $t+q<n-1$, since in this case its last row coincides with the (t+q)th row. However, as is evident from (II.17), t and q are independent summation indices. Consequently,

$$(t+q)_{max} = t_{max} + q_{max} = (m-k-1)+k = m-1;$$

where m is also a summation index. As is evident from (II.16), the maximum value m = r, whence $(t+q)_{max,\,max} = r-1$. The latter fact leads to the conclusion that when

$$r < n \qquad \text{(II.18)}$$

the determinant $W_{t+q}(x)$ for all values of t and k, and hence, expressions (II.17) for all k and m are identically equal to zero.

With the aid of formula (I.48), we can also obtain

$$\sum_{j=1}^{n} (-1)^{n+j} y_j(x_i) \left[\frac{d^m}{dx^m} \left(\frac{W_{nj}(x)}{W(x)} \right) \right] \Bigg|_{x=x_i} =$$

$$= \sum_{j=1}^{n} (-1)^{n+j} \left\{ \sum_{t=0}^{m} (-1)^t C^t_m \frac{d^{m-t}}{dx^{m-t}} \left[\frac{W_{nj}(x)}{W(x)} \frac{d^t}{dx^t} y_j(x) \right] \right\} \Bigg|_{x=x_i} =$$

$$= \sum_{t=0}^{m} (-1)^t C^t_m \left\{ \frac{d^{m-t}}{dx^{m-t}} \left[\frac{1}{W(x)} \sum_{j=1}^{n} (-1)^{n+j} W_{nj}(x) \frac{d^t}{dx^t} y_j(x) \right] \right\} \Bigg|_{x=x_i} =$$

$$= \sum_{t=0}^{m} (-1)^t C^t_m \left[\frac{d^{m-t}}{dx^{m-t}} \left(\frac{W_t(x)}{W(x)} \right) \right|_{x=x_i}. \qquad \text{(II.19)}$$

where the functional determinant $W_t(x) = W_{t+q}(x)|_{q=0}$.

The determinant $W_t(x) \equiv 0$, if $t < n-1$. However, as is evident from (II.19) and (II.16), $t_{max} = m$, and $m_{max} = r$, whence $t_{max,\ max} = r$. This leads to the conclusion that when

$$r < n-1 \tag{II.20}$$

the determinant $W_t(x)$ is identically equal to zero for all values of t, and hence, expressions (II.19) are also equal to zero for all values of m. Expressions of the form (II.17) constitute the coefficients of the terms of partial solution (II.16) of Eq. (II.12), which contain singularities of the type of the delta function and its derivatives; expressions (II.19) are the coefficients of terms with finite discontinuity type singularities. Comparing estimates (II.18) and (II.20) one can easily conclude that when

$$r < n-1 \tag{II.21}$$

the general solution of Eq. (II.12) will be continuous. When

$$r = n-1 \tag{II.22}$$

the solution in the general case will contain finite discontinuity type singularities. Finally, when

$$r > n-1 \tag{II.23}$$

the solution in the general case also contains singularities of the type of the delta function and its derivatives up to order (r-n) inclusive.

Let $r < n-1$. Then the general solution of Eq. (II.12) may be represented in the form

$$y(x) = \sum_{j=1}^{n} \Big\{ C_j + (-1)^{n+j} \Big[\int_{a}^{x} V(\xi) \frac{W_{nj}(\xi)}{W(\xi)} d\xi + \sum_{i} \theta(x - x_i) \Big(\int_{x_i}^{x} V_i(\xi) \frac{W_{nj}(\xi)}{W(\xi)} d\xi + \sum_{m=0}^{r} (-1)^m C_{im} \frac{d^m}{dx^m} \frac{W_{nj}(x)}{W(x)} \Big|_{x=x_i} \Big) \Big] \Big\} y_j(x). \tag{II.24}$$

where C_j are arbitrary constants. At the points $x=x_l\ (l=1,2,\ldots)$, expression (II.24) assumes the form

$$y(x_l)=\sum_{j=1}^{n}\left\{C_j+(-1)^{n+j}\left[\int_{x_0}^{x_l}V(\xi)\frac{W_{nj}(\xi)}{W(\xi)}d\xi+\sum_{i=1}^{l-1}\left(\int_{x_i}^{x_l}V_i(\xi)\frac{W_{nj}(\xi)}{W(\xi)}d\xi+\right.\right.\right.$$

$$\left.\left.\left.+\sum_{m=0}^{r}(-1)^m C_{im}\frac{d^m}{dx^m}\frac{W_{nj}(x)}{W(x)}\bigg|_{x=x_i}\right)\right]\right\}y_j(x_l). \tag{II.25}$$

Generalizing estimate (II.21), we can readily see that the derivative of order p will be continuous if

$$r<n-1-p. \tag{II.26}$$

It follows that all the derivatives up to order n-r-2 inclusive will be continuous. For the pth derivative $(p<n-r-1)$, we have

$$\frac{d^p}{dx^p}y(x)=\sum_{j=1}^{n}\left\{C_j+(-1)^{n+j}\left[\int_{x_0}^{x}V(\xi)\frac{W_{nj}(\xi)}{W(\xi)}d\xi+\right.\right.$$

$$+\sum_{i}\theta(x-x_i)\left(\int_{x_i}^{x}V_i(\xi)\frac{W_{nj}(\xi)}{W(\xi)}d\xi+\right.$$

$$\left.\left.\left.+\sum_{m=0}^{r}(-1)^m C_{im}\frac{d^m}{dx^m}\frac{W_{nj}(x)}{W(x)}\bigg|_{x=x_i}\right)\right]\right\}\frac{d^p}{dx^p}y_j(x), \tag{II.27}$$

причем

$$\frac{d^p}{dx^p}y(x)\bigg|_{x=x_l}=\sum_{j=1}^{n}\left\{C_j+(-1)^{n+j}\left[\int_{x_0}^{x_l}V(\xi)\frac{W_{nj}(\xi)}{W(\xi)}d\xi+\right.\right.$$

$$+\sum_{i=1}^{l-1}\left(\int_{x_i}^{x_l}V_i(\xi)\frac{W_{nj}(\xi)}{W(\xi)}d\xi+\right. \tag{II.28}$$

$$\left.\left.\left.+\sum_{m=0}^{r}(-1)^m C_{im}\frac{d^m}{dx^m}\frac{W_{nj}(x)}{W(x)}\bigg|_{x=x_i}\right)\right]\right\}\frac{d^p}{dx^p}y_j(x)\bigg|_{x=x_l}.$$

II.3. Differential Equations with Step Function Type Coefficients

We will indicate one method that permits one to derive the solution of differential equations with variable step-function type coefficients by reducing these equations to partly degenerate ones. As an example illustrating this method, let us consider the equation

$$y'' - k^2\left[1 + \sum_{i=1}^{n} a_i\theta(x - x_i)\right] y = 0,$$

where a_i are constant coefficients, and $\theta(x-x_i)$ are asymmetric Heaviside functions $[\theta(0) = 0; \dot{1}]$.

We multiply the equation by the function $\theta(x-x_j)$. Considering that for asymmetric functions the product $\theta(x-x_i)\theta(x-x_j) = \theta(x-x_{\max(i,j)})$ for any i, j, we obtain

$$Y_j'' - \sum_{i=1}^{n} A_i Y_i = y'(x_j)\theta(x-x_j) + y(x_j)\theta'(x-x_j) \qquad (j = 1, 2, \ldots, n),$$

where

$$Y_j(x) = y(x)\theta(x - x_j),$$

$$A_i = \begin{cases} k^2\left(1 + \sum_{i=1}^{j} a_i\right) & \text{when } i = j, \\ k^2 a_i & \text{when } i > j. \end{cases}$$

We thus arrive at a system of n equations with constant coefficients in n unknown functions Y_j. As we know, a system of ordinary linear differential equations can always be reduced to a single equation in any of the unknown functions, and also in any linear combination of these functions. In the case at hand, we are interested in $Z(x) = k^2 \sum_{i=1}^{n} a_i Y_i$. The general solution of the system in this linear combination will contain 2n arbitrary constants. Considering the boundedness of the solution y(x) of the initial equation, theseconstants must be determined from the condition $Z(x < x_{i\,\min}) \equiv 0$ Introducing the partial solution Z = Z* thus obtained into the initial equation, we finally obtain a partly degenerate equation with constant coefficients

$$y'' - k^2 y = Z^* [x;\ y(x_1), \ldots, y(x_n);\ y'(x_1), \ldots, y'(x_n)].$$

Let n = 1. Then the equation in Z will assume the form

$$Z'' - k_1^2 Z = (k_1^2 - k^2)\, y(2x_1 - x)\, \theta'(x - x_1),$$

where

$$k_1^2 = (1 + a_1) k^2.$$

Using (II.24), we find the general solution

$$Z = C_1 e^{k_1 x} + C_2 e^{-k_1 x} + \frac{k_1^2 - k^2}{k_1} \frac{d}{d\xi} [y(2x_1 - \xi) \operatorname{sh} k_1 (\xi - x)] \Big|_{\xi = x_1} \theta(x - x_1),$$

on the basis of which

$$y'' - k^2 y = \frac{k_1^2 - k^2}{k_1} \frac{d}{d\xi} [y(2x_1 - \xi) \operatorname{sh} k_1 (\xi - x)] \Big|_{\xi = x_1} \theta(x - x_1).$$

Integrating this equation with the aid of (II.24), we obtain the general solution of the initial equation. In explicit form

$$y(x) = y^*(x)\, \theta(x_1 - x) + \frac{1}{k_1} \frac{d}{d\xi} [y^*(2x_1 - \xi) \operatorname{sh} k_1 (\xi - x)] \Big|_{\xi = x_1} \theta(x - x_1),$$

where

$$y^*(x) = C_1 \operatorname{ch} kx + C_2 \operatorname{sh} kx,$$
$$\theta(x_1 - x) = 1 - \theta(x - x_1).$$

Appendix III. EQUATIONS WITH IMPULSE TYPE SINGULARITIES IN PROBLEMS OF STRUCTURAL MECHANICS

As an illustration, we will examine certain problems of structural mechanics, reducing to ordinary differential equations and partial differential equations with impulse type singularities. These singularities may be present on the right-hand side of the equations, and also in the form of variable coefficients.

III.1. Some Equations with Singularities on the Right-Hand Side Beam on a Regular-Type Elastic Base

The equation of bending of a beam on an elastic base (Fig. III.1) in the presence of a longitudinal compressive force has the form

$$(EIy'')''+(Ny')'+ky=q. \qquad \text{(III.1)}$$

Here E(x)I(x) is the rigidity coefficient of the beam; N(x) is the longitudinal compressive force; k(x) is the rigidity coefficient of the elastic base; q(x) is the transverse load; y(x) is the deflection of the beam.

The elastic base will be said to be regular if the rigidity coefficient K(x) contains no signularities of the type of the delta function and its derivatives. The remaining coefficients of Eq. (III.1) are also ordinary functions, i.e., are piecewise-continuous in the general case.

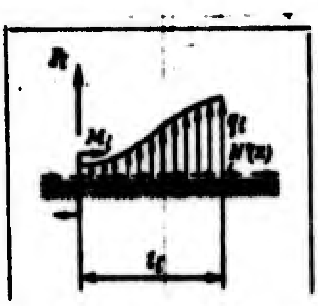

Fig. III.1 Beam on regular type elastic vase.

As we know, axisymmetric problems for shells of revolution also reduce to Eq. (III.1).

The transverse load will be represented in the form

$$q(x)=\sum_i \{q_i(x)[\theta(x-x_i)-\theta(x-x_i-l_i)]+P_i\delta(x-x_i)+M_i\delta'(x-x_i)\}, \tag{III.2}$$

where $q_i(x)[\theta(x-x_i)-\theta(x-x_i-l_i)]$ is a linear load of intensity $q_i(x)$, distributed over the segment $x_i<x<x_i+l_i$; $P_i\delta(x-x_i)+M_i\delta'(x-x_i)$ is a distributed load, corresponding to the force P_i and moment M_i, concentrated in the section $x = x_i$. Expression (III.2) constitutes the most general form of notation of the transverse load. The summation is carried out over all the points of discontinuity of the load. In the absence of any force factor at the point $x = x_i$, it should be set equal to zero.

The general solution of Eq. (III.1) with the right-hand side of (III.2) may be written by using expression (II.24).

On the basis of the Ostrogadskiy-Liouville formula,

$$W(x)=x(0)\left[\frac{E(0)I(0)}{E(x)I(x)}\right]^2. \tag{III.3}$$

In view of (III.3), we obtain

$$\begin{aligned} y(x)=\frac{1}{E(0)I(0)}\sum_{j=1}^{4} y_j(x)\Big\{C_j+(-1)^j\sum_i\Big[\theta(x-x_i)\int_{x_i}^{x}\bar{E}\bar{I}q_i\bar{W}_j d\xi-\\ -\theta(x-x_i-l_i)\int_{x_i+l_i}^{x}\bar{E}\bar{I}q_i\bar{W}_j d\xi+\bar{E}\bar{I}\bar{W}_j\Big|_{x=x_i}P_i\theta(x-x_i)-\\ -(\bar{E}\bar{I}\bar{W}_j)'\Big|_{x=x_i}M_i\theta(x-x_i)\Big]\Big\}, \end{aligned} \tag{III.4}$$

where

$$\bar{E}(x)=\frac{E(x)}{E(0)};\quad \bar{I}(x)=\frac{I(x)}{I(0)};\quad \bar{W}_j(x)=\frac{W_{4j}(x)}{W(0)}.$$

Expression (III.4) represents the universal equation of a beam on an elastic base in longitudinal-transverse bending. By determining the integration constants in accordance with the boundary conditions, from (III.4) one can obtain the solution for any special cases.

Universal Equation of the Elastic Line of a Beam

Let us consider as an example a beam of constant cross section in the absence of an elastic base and of a longitudinal compressive force. We assume that

$$y_1 = 1, \quad y_2 = x, \quad y_3 = x^2, \quad y_4 = x^3.$$

Then

$$W = 12, \quad \overline{W}_1 = \frac{x^3}{6}, \quad \overline{W}_2 = \frac{x^2}{2}, \quad \overline{W}_3 = \frac{x}{2}, \quad \overline{W}_4 = \frac{1}{6}.$$

Expression (III.4) thus readily reduces to the form

$$\begin{aligned} y(x) = {} & y(0) + y'(0)\,x + \frac{M(0)}{2EI}x^2 + \frac{Q(0)}{6EI}x^3 + \\ & + \frac{1}{EI}\sum_i \Bigg\{ \theta(x - x_i)\int_{x_i}^{x}\frac{(x-\xi)^3}{6}q_i(\xi)\,d\xi - \theta(x - x_i - l_i)\times \\ & \times \int_{x_i + l_i}^{x}\frac{(x-\xi)^3}{6}q_i(\xi)\,d\xi + \theta(x - x_i)\frac{(x - x_i)^3}{6}P_i + \theta(x - x_i)\frac{(x - x_i)^2}{2}M_i \Bigg\}. \end{aligned} \qquad \text{(III.5)}$$

Expression (III.5) represents the well-known universal equation of an elastic axis of a beam in bending.

III.2. Some Differential Equations with Singular Coefficients Beam on a Singular Type Elastic Base

Let us again consider a beam on a regular elastic base with rigidity coefficient k(x). Let the mobility of the individual sections of the beam be bounded by concentrated constraints restricting the deflection $y(x_i)$ and the angle of rotation $y'(x_i)$ in these sections (Fig. III.2). The action of such constraints on the beam is equivalent to the forces $P_i = k_{1i}y(x_i)$ and moments $M_i = k_{2i}y'(x_i)$ concentrated in the section $x = x_i$ of the beam.

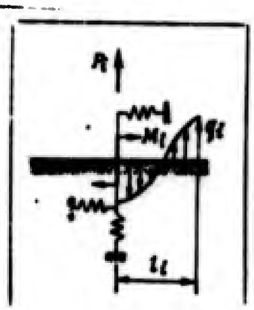

Fig. III.2. Beam on singular type elastic base.

The distributed load corresponding to these force factors may be represented in the form

$$\sum_i [-k_{1i}y(x_i)\delta(x-x_i)+$$
$$+k_{2i}y'(x_i)\delta'(x-x_i)],$$

or, on the basis of the properties of the delta function, in the form

$$\sum_i \{-k_{1i}y(x)\delta(x-x_i)+k_{2i}[y'(x)\delta(x-x_i)]'\}. \tag{III.6}$$

Introducing additional transverse load (III.6) into Eq. (III.1), we have

$$(EIy'')''+(N^*y')'+k^*y=q, \tag{III.7}$$

where

$$N^*(x)=N(x)-\sum_i k_{2i}\delta(x-x_i), \quad k^*(x)=k(x)+\sum_i k_{1i}\delta(x-x_i) \tag{III.8}$$

are variable coefficients containing delta function type singularities, and q, as before, is transverse load (III.2) of the most general type.

Equation (III.7) has a very general character. This equation can be used as the basis of studies of a wide range of questions pertaining to the analysis of beams with concentrated constraints and also to the study of axisymmetric eformations of shells of revolution reinforced with a transverse structure.

Let us write the general solution of Eq. (III.7) in "mixed form." Reducing (III.7) to a partly degenerate form, we obtain

$$(EIy'')''+(Ny')'+ky=\sum_i [-k_{1i}y(x_i)\delta(x-x_i)+k_{2i}y'(x_i)\delta'(x-x_i)]+q. \tag{III.9}$$

Using general expression (II.24), we can obtain

$$y(x)=\sum_{j=1}^{4} y_j(x)\left\{C_j-\sum_i \theta(x-x_i)[\alpha_{ji}y(x_i)+\beta_{ji}y'(x_i)]\right\}+y_q(x), \tag{III.10}$$

where $y_j(x)$ $(j=1, 2, 3, 4)$ is the fundamental system of solutions of the homogeneous equation corresponding to (III.1);

$y_q(x)$ is the partial solution of Eq. (III.1) due to and arbitrary external load;

$$\alpha_{jl}=(-1)^j \frac{k_{1l}}{E(0)I(0)} \bar{E}(x_l)\bar{I}(x_l)\bar{W}_j(x_l);$$
$$\beta_{jl}=(-1)^j \frac{k_{2l}}{E(0)I(0)} [\bar{E}(x)\bar{I}(x)\bar{W}_j(x)]'\Big|_{x=x_l}.$$

Expression (III.10) may be said to be a universal equation of an elastic axis of a beam on a singular elastic base. To this expression one should add a system of algebraic equations in $y(x_i)$ and $y'(x_i)$. Using general expressions (II.25), (II.28), one can obtain

$$y(x_l)+\sum_{i=1}^{l-1}\left[y(x_i)\sum_{j=1}^{4}\alpha_{ji}y_j(x_l)+y'(x_i)\sum_{j=1}^{4}\beta_{ji}y_j(x_l)\right]=\sum_{j=1}^{4}C_j y_j(x_l)+y_0(x_l)$$
$$(l=1,2,\ldots), \tag{III.11}$$

$$y'(x_l)+\sum_{i=1}^{l-1}\left[y(x_i)\sum_{j=1}^{4}\alpha_{ji}y_j'(x_l)+y'(x_i)\sum_{j=1}^{4}\beta_{ji}y_j'(x_l)\right]=\sum_{j=1}^{4}C_j y_j'(x_l)+y_0'(x_l)$$
$$(l=1,2,\ldots). \tag{III.12}$$

Expression (III.10) and system of algebraic equations (III.11), (III.12) represent the general solution of Eq. (III.7) in mixed form. In order to obtain the explicit form of the general solution, it is necessary to solve system (III.11), (III.12) for the unknowns $y(x_l),\ y'(x_l)\ (l=1,2,\ldots,n)$ and eliminate them from expression (III.10).

System (III.11), (III.12) in the matrix formulation may be represented in the form

$$\mathbf{AZ}=\mathbf{B}_0+\sum_{j=1}^{4}C_j\mathbf{B}_j, \tag{III.13}$$

where

$$\mathbf{A}=\begin{pmatrix} \mathbf{E} & & \mathbf{0} \\ & \mathbf{E} & \\ \mathbf{M}_{li} & & \mathbf{E} \end{pmatrix}$$

is a nondegenerate triangular matrix, where $\mathbf{E}$ is a 2 x 2 unit matrix $\mathbf{M}_{li}=\sum_{j=1}^{4}\mathbf{M}_{lij}$ is a second-order square matrix, where

$$\mathbf{M}_{lij}=\begin{pmatrix} \alpha_{ji}y_j(x_l) & \beta_{ji}y_j(x_l) \\ \alpha_{ji}y_j'(x_l) & \beta_{ji}y_j'(x_l) \end{pmatrix};$$

the matrix column

$$Z = [Z_i],$$

where

$$Z_i = \begin{pmatrix} y(x_i) \\ y'(x_i) \end{pmatrix};$$

the matrix columns

$$B_0 = [B_{0i}], \quad B_j = [B_{ji}],$$

where

$$B_{0i} = \begin{pmatrix} y_0(x_i) \\ y_0'(x_i) \end{pmatrix}, \quad B_{ji} = \begin{pmatrix} y_j(x_i) \\ y_j'(x_i) \end{pmatrix}.$$

From (III.13) we have

$$Z = Z^0 + \sum_{j=1}^{4} C_j Z^j, \tag{III.14}$$

where

$$Z^j = A^{-1} B_j \qquad (j = 0, 1, 2, 3, 4).$$

The solution in the form (III.14) contains integration constants in explicit form. Eliminating the unknowns $y(x_i)$, $y'(x_i)$ from expression (III.10) with the aid of (III.14), we arrive at the explicit form of a general solution with four arbitrary constants. It should be emphasized that because of the triangularity of the matrix A, the process of finding the vectors Z^j is trivial independently of the number of elastic supports.

Torsion of an Elastically Clamped Rod

For clarity, we will consider a simpler example. Let the rod be connected with some elastic medium resisting torsion. In this case

$$GI_{кр}\varphi'' - k\varphi = m(x),$$

where $\varphi(x)$ is the twist angle of the cross section of the rod; $k(x)$ is the rigidity coefficient of the medium. If the rod is elastically clamped only in individual sections, then we should set

$$k(x) = \sum_i k_i \delta(x - x_i).$$

In this case the solution in mixed form will be are the linear mass and moment of inertia, m and μ being ordinary functions, and M_i and J_i being the mass and moment of inertia of the load lumped in the section $x = x_i$.

Applying the Fourier method to Eq. (III.16), we arrive at the equation

$$(EIY'')'' + [(N - k_2)Y']' + k_1 Y - \lambda[m^* Y - (\mu^* Y')'] = 0 \tag{III.17}$$

in normal mode Y(x).

On the basis of general expressions (II.24)-(II.28), the solution of Eq. (III.17) has the form

$$Y(x) = \sum_{j=1}^{4}\left\{C_j + \sum_{i}\theta(x - x_i)[\alpha_{ji}Y(x_i) + \beta_{ji}Y'(x_i)]\right\}Y_j(x), \tag{III.18}$$

$$\begin{gathered} Y(x_l) - \sum_{i=1}^{l-1}\left[Y(x_i)\sum_{j=1}^{4}\alpha_{ji}Y_j(x_l) + Y'(x_i)\sum_{j=1}^{4}\beta_{ji}Y_j(x_l)\right] = \sum_{j=1}^{4}C_jY_j(x_l) \\ (l = 1,\ 2, \ldots), \\ Y'(x_l) - \sum_{i=1}^{l-1}\left[Y(x_i)\sum_{j=1}^{4}\alpha_{ji}Y'_j(x_l) + Y'(x_i)\sum_{j=1}^{4}\beta_{ji}Y'_j(x_l)\right] = \sum_{j=1}^{4}C_jY'_j(x_l) \\ (l = 1,\ 2, \ldots), \end{gathered} \tag{III.19}$$

where

$$\begin{gathered} \alpha_{ji} = (-1)^j\frac{\lambda M_i - k_{1i}}{E(0)I(0)}\bar{E}(x_i)\bar{I}(x_i)\bar{W}_j(x_i), \\ \beta_{ji} = (-1)^j\frac{\lambda J_i - k_{2i}}{E(0)I(0)}\left[\bar{E}(x)\bar{I}(x)\bar{W}_j(x)\right]'\Big|_{x = x_i}. \end{gathered} \tag{III.20}$$

Y_1, Y_2, Y_3, Y_4 is the fundamental system of solutions of the equation

$$(EIY'')'' + (NY')' + kY - \lambda[mY - (\mu Y')'] = 0.$$

Solving system (III.19) for the unknowns $Y(x_l),\ Y'(x_l)$ $(l = 1, 2, \ldots)$ and eliminating these unknowns from expression (III.18), we arrive at a general solution of Eq. (III.17) containing four arbitrary constants. Then, satisfying the boundary conditions on the ends of the beam and adding the condition of nontriviality of the solution to them, we can obtain the spectrum of natural frequencies and the corresponding normal modes of the beam.

We represent system (III.19) in matrix form

$$A(\lambda)Z(\lambda)=\tilde{B}(\lambda)C. \tag{III.21}$$

Here A and Z are a nondegenerate triangular matrix and a column matrix, respectively, similar in structure to the corresponding matrices in Eq. (III.13); $C=[C_j]$ is a column matrix of arbitrary constants; $\tilde{B}$ is a matrix consisting of four columns which are analogous in structure to the column matrices B_j in Eq. (III.13).

The solution of Eq. (III.21)

$$Z(\lambda)=A^{-1}(\lambda)B(\lambda)C. \tag{III.22}$$

Thus, as above, the problem reduces to finding the vectors $A^{-1}B_j$, but in contrast to (III.14), the elements of the matrix A and vectors B_j are not numbers, but some functions of the parameter λ, which complicates the problem if the number of concentrated masses and moments of inertia is large. Nevertheless, even in this case the application of numerical methods is possible, although the corresponding algorithms are fairly cumbersome. However, an analytical solution of Eq. (III.21), which is of considerable interest, involves major mathematical difficulties and is possible, generally speaking, only in individual cases.

Let us consider one method which in certain important cases makes it possible to construct an analytical solution.

Let $\gamma(\lambda)=[\gamma_{ij}(\lambda)]$ be a nondegenerate fourth-order square matrix whose elements are some functions of the parameter λ. Let us consider the matrix equation

$$A(\lambda)\Xi(\lambda)=B(\lambda)\gamma(\lambda), \tag{III.23}$$

where $\Xi(\lambda)$ is the desired four-column matrix.

The solution of Eq. (III.23)

$$\Xi(\lambda)=A^{-1}(\lambda)B(\lambda)\gamma(\lambda), \tag{III.24}$$

whence

$$A^{-1}(\lambda)B(\lambda)=\Xi(\lambda)\gamma^{-1}(\lambda). \tag{III.25}$$

Introducing (III.25) into (III.22), we obtain

$$\Xi(\lambda)=\Xi(\lambda)\gamma^{-1}(\lambda)C. \tag{III.26}$$

In certain cases, the matrix $\gamma(\lambda)$ can be chosen so that the corresponding matrix $\Xi(\lambda)$ can be found by artificial procedures. Then expression (III.26) gives an analytical solution of Eq. (III.21) for arbitrary C and λ.

The above-described approach is used in constructing the solution of the special Sturm-Liouville problem (Appendix IV). It can of course also be applied to any other eigenvalue problem.

Let us consider separately the important case in which the beam proper is weightless, i.e., when the number of degrees of freedom of the system is determined by the total number of concentrated masses and moments of inertia of the loads. In this case, Y_1, Y_2, Y_3, Y_4 in (III.18), (III.19) is the fundamental system of solutions of the equation $(EIY'')''+(NY')'+kY=0$, so that the parameter λ will be contained explicitly only in the coefficients α_{ji}, β_{ji} of (III.20). As can be readily ascertained, the solution of system (III.19) can therefore be represented in the form of polynomials in powers of the parameter λ:

$$Y(x_l)=\sum_{s=0}^{l-1}\frac{1}{s!}\lambda^s\sum_{j=1}^{4}a_{lsj}C_j,\quad Y'(x_l)=\sum_{s=0}^{l-1}\frac{1}{s!}\lambda^s\sum_{j=1}^{4}b_{lsj}C_j \quad (l=1,2,\ldots), \tag{III.27}$$

where a_{lsj}, b_{lsj} are some coefficients. Since

$$a_{lsk}=\frac{\partial^s}{\partial\lambda^s}Y(x_l)\Big|_{c_j=\delta_{j,k}}^{\lambda=0};\quad b_{lsk}=\frac{\partial^s}{\partial\lambda^s}Y'(x_l)\Big|_{c_j=\delta_{j,k}}^{\lambda=0}\ (k=1,\ 2,\ 3,\ 4),$$

then from (III.19), (III.20) we have

$$\begin{aligned}
&a_{lsk}=\frac{1}{E(0)I(0)}\Big[-\sum_{i=s+1}^{l-1}(k_{1i}\alpha^{*}_{li}a_{isk}+k_{2i}\beta^{*}_{li}b_{isk})+\\
&+s\sum_{i=s}^{l-1}(M_i\alpha^{*}_{li}a_{i,s-1,k}+J_i\beta^{*}_{li}b_{i,s-1,k})\Big]\quad (l,\ s=1,\ 2,\ldots),\\
&a_{l0k}=Y_k(x_l),\\
&b_{lsk}=\frac{1}{E(0)I(0)}\Big[-\sum_{i=s+1}^{l-1}(k_{1i}\alpha^{**}_{li}a_{isk}+k_{2i}\beta^{**}_{li}b_{isk})+\\
&+s\sum_{i=s}^{l-1}(M_i\alpha^{**}_{li}a_{i,s-1,k}+J_i\beta^{**}_{li}b_{i,s-1,k})\Big]\quad (l,\ s=1,\ 2,\ldots),\\
&b_{l0k}=Y'_k(x_l),
\end{aligned}\qquad\text{(III.28)}$$

where

$$\alpha^{*}_{li}=\bar{E}(x_i)\bar{I}(x_i)\sum_{j=1}^{4}(-1)^j Y_j(x_l)\bar{W}_j(x_i),$$

$$\alpha^{**}_{li}=\bar{E}(x_i)\bar{I}(x_i)\sum_{j=1}^{4}(-1)^j Y'_j(x_l)\bar{W}_j(x_i),$$

$$\beta^{*}_{li}=\sum_{j=1}^{4}(-1)^j Y_j(x_l)[\bar{E}(x)\bar{I}(x)\bar{W}_j(x)]'\Big|_{x=x_i},$$

$$\beta^{**}_{li}=\sum_{j=1}^{4}(-1)^j Y'_j(x_l)[\bar{E}(x)\bar{I}(x)\bar{W}_j(x)]'\Big|_{x=x_i}.$$

Expressions (III.28) constitute recursion relations that permit one successively to find all the coefficients of polynomials (III.27). This procedure is carried out numerically and is therefore elementary.

Eliminating the unknowns $Y(x_i), Y'(x_i)$ from (III.18) with the aid of (III.27), we have

$$Y(x)=\sum_{j=1}^{4}C_jY_j(x)\Big[1+\sum_{i}\theta(x-x_i)\sum_{s=0}^{l-1}\frac{1}{s!}\lambda^s(\alpha_{ji}a_{isj}+\beta_{ji}b_{isj})\Big].\qquad\text{(III.29)}$$

Expression (III.29) constitutes an explicit form of the general solution of Eq. (III.17) under the condition $m(x)=\mu(x)=0$, containing four arbitrary constants. Now, satisfying the boundary conditions on the beam

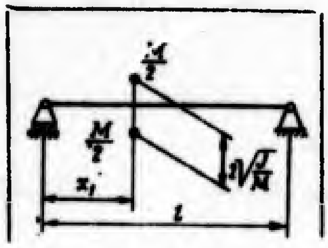

Fig. III.4. Weightless beam with concentrated load.

ends and adding the condition of nontriviality of the solution to them, we can readily obtain the spectrum of frequencies and normal modes. A very important fact is that the frequency equation for any number of concentrated masses and moments of inertia is the condition of equality to zero of the second-order determinant, whose elements are polynomials in powers of λ.

As the simplest example, we will consider a weightless beam of constant cross section hinged at the ends, with a load of mass M and moment of inertia J, concentrated in the section $x = x_1$ (Fig. III.4). In this case, Eq. (III.17) assumes the form

$$EIY^{IV} - \lambda\{M\delta(x-x_1)Y(x) - J[\delta(x-x_1)Y'(x)]'\} = 0,$$

whence откуда

$$Y(x) = \frac{\lambda}{EI}\left[MY(x_1)\frac{(x-x_1)_+^3}{6} - JY'(x_1)\frac{(x-x_1)_+^2}{2}\right] + C_1x^3 + C_2x^2 + C_3x + C_4.$$

Excluding $Y(x_1)$ and $Y'(x_1)$, we find

$$\begin{aligned} Y(x) = {} & C_1\left[\frac{M}{EI}\lambda x_1^3\frac{(x-x_1)_+^3}{6} - \frac{J}{EI}\lambda 3x_1^2\frac{(x-x_1)_+^2}{2} + x^3\right] + \\ & + C_2\left[\frac{M}{EI}\lambda x_1^2\frac{(x-x_1)_+^3}{6} - \frac{J}{EI}\lambda 2x_1\frac{(x-x_1)_+^2}{2} + x^2\right] + \\ & + C_3\left[\frac{M}{EI}\lambda x_1\frac{(x-x_1)_+^3}{6} - \frac{J}{EI}\lambda\frac{(x-x_1)_+^2}{2} + x\right] + \\ & + C_4\left[\frac{M}{EI}\lambda\frac{(x-x_1)_+^3}{6} + 1\right]. \end{aligned} \qquad \text{(III.30)}$$

Satisfying the boundary conditions $Y(0) = Y''(0) = Y(l) = Y''(l) = 0,$

$$C_2=0,$$
$$C_4=0,$$
$$C_1\left\{\frac{\lambda}{EI}\left[Mx_1^3\frac{(l-x_1)^3}{6}-3Jx_1^2\frac{(l-x_1)^2}{2}\right]+l^3\right\}+$$
$$+C_3\left\{\frac{\lambda}{EI}\left[Mx_1\frac{(l-x_1)^3}{6}-J\frac{(l-x_1)^2}{2}\right]+l\right\}=0,$$
$$C_1\left\{\frac{\lambda}{EI}\left[Mx_1^3(l-x_1)-3Jx_1^2\right]+6l\right\}+C_3\left\{\frac{\lambda}{EI}\left[Mx_1(l-x_1)-J\right]\right\}=0. \qquad \text{(III.31)}$$

Equating the determinant of this system to zero, we arrive at the quadratic equation

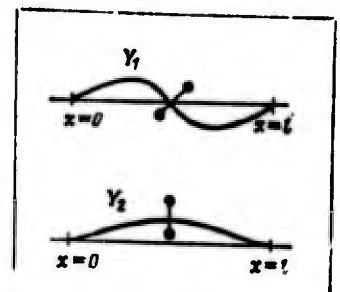

Fig. III.5. Normal modes of beam with concentrated load.

$$a\lambda^2+b\lambda+c=0, \qquad \text{(III.32)}$$

where

$$a=\frac{2}{3}MJx_1^3(l-x_1)^3,$$
$$b=-2EIl\{Mx_1^3(l-x_1)^3+$$
$$+J[l^2-3x_1(l-x_1)]\},$$
$$c=6(EIl)^2.$$

The roots $\lambda_{1,2}$ of Eq. (III.32) determine two frequencies $\omega_{1,2}=\sqrt{\lambda_{1,2}}$ of natural vibrations. The corresponding modes are determined by expression (III.30) for values of the constants satisfying Eqs. (III.31). For example, let $x_1=\frac{l}{2}$. Then from (III.32) and (III.33) we find

$$\lambda_1=\frac{EI}{J}\,\frac{12}{l}, \quad \lambda_2=\frac{EI}{M}\,\frac{48}{l^3}.$$

Let us turn to system (III.31). When $\lambda=\lambda_{1,2}$, we correspondingly have

$$(C_1l^2+4C_3)(Ml^2-4J)=0,$$
$$\left(C_1l^2+\frac{4}{3}C_3\right)(Ml^2-4J)=0,$$

whence, if $\frac{J}{M}\neq\frac{l^2}{4}$, we correspondingly find

$$\frac{C_3}{C_1}=-\frac{l^2}{4}, \quad \frac{C_3}{C_1}=-\frac{3}{4}l^2. \qquad \text{(III.33)}$$

Introducing (III.33) into (III.30), we obtain to within the multiplier:

$$Y_1(x)=x\left(x^2-\frac{l^2}{4}\right)-3l\left(x-\frac{l}{2}\right)_+^2,$$

$$Y_2(x)=x\left(x^2-\frac{3l^2}{4}\right)-2\left(x-\frac{l}{2}\right)_+^3. \qquad \text{(III.34)}$$

If $\frac{J}{M}=\frac{l^2}{4}$, then $\lambda_1=\lambda_2$, i.e., the frequencies merge. In this case, we obtain from (III.30)

$$Y_{1,2}(x)=AY_1(x)+BY_2(x),$$

where A, B are arbitrary constants, and $Y_1(x)$, $Y_2(x)$ are determined by expressions (III.34) (Fig. III.5).

Cross-Bending of Reinforced Plates

The equation of bending of plates of constant thickness $D\nabla^4 w=q$ is easily extended to the case of a plate with a reinforcing structure if one resorts to generalized functions. For instance, let the plate be reinforced by ribs oriented along the lines $x=x_i$ $(i=1,2,\ldots)$ symmetrically with respect to the middle plane. In this case, the elasticity relations establishing the relationship between linear bending and twisting moments in the cross sections of the plate and the deflection function should obviously be taken in the form

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)-D_{yy}^{\circ}\frac{\partial^2 w}{\partial y^2},$$

$$H_x=-(1-\mu)D\frac{\partial^2 w}{\partial x\partial y},\quad H_y=-(1-\mu)D\frac{\partial^2 w}{\partial y\partial x}-D_{yx}^{\circ}\frac{\partial^2 w}{\partial y\partial x},$$

where

$$D=\frac{Eh^3}{12(1-\mu^2)},$$

$$D_{yy}^{\circ}=\sum_i E_i(y)I_i(y)\delta(x-x_i),\qquad D_{yx}^{\circ}=\sum_i G_i(y)I_{kp\,i}(y)\delta(x-x_i).$$

The plate deflection equation, constituting the condition of equilibrium of an element of the plate with respect to normal displacements, then becomes

$$D\nabla^4 w+\frac{\partial}{\partial x}\left(D_{yx}^{\circ}\frac{\partial^3 w}{\partial y^2\partial x}\right)+D_{yy}^{\circ}\frac{\partial^4 w}{\partial y^4}=q. \qquad \text{(III.35)}$$

We will assume for simplicity that each of the ribs has flexural and torsional rigidities constant along the length. The plate edges y = 0 and y = b will be assumed freely supported:

$$w\Big|_{y=0}=\frac{\partial^2 w}{\partial y^2}\Big|_{y=0}=w\Big|_{y=b}=\frac{\partial^2 w}{\partial y^2}\Big|_{y=b}=0.$$

The solution of Eq. (III.35) may be sought in this case in the form of the unary trigonometric series

$$w(x, y)=\sum_{k=1}^{\infty} X_k(x) \sin\frac{k\pi y}{b}. \tag{III.36}$$

Introducing (III.36) into Eq. (III.35), we have

$$X_k^{\mathrm{IV}}-\left(\frac{k\pi}{b}\right)^2\left[\left(2+\frac{D^*_{yx}}{D}\right)X'_k\right]'+\left(\frac{k\pi}{b}\right)^4\left(1+\frac{D^*_{yy}}{D}\right)X_k=\\=\frac{2}{bD}\int_0^b q(x, y)\sin\frac{k\pi y}{b}\,dy. \tag{III.37}$$

Expression (III.37) constitutes and ordinary differential equation whose coefficients contain signularities of the type of the delta function and its derivative. The general solution of this equation can be easily written by using general expression (II.24). As in the case of Eq. (III.7), it is easy to change from the mixed form of the general solution to the explicit form, containing four arbitrary constants. These constants are determined from the conditions at the plate edges x = 0 and x = a. Such a solution obviously constitutes an extension of M. Levi's well-known solution to the case of a plate reinforced with a unidirectional structure.

The problems discussed in the present appendix illustrate the method, presented in Appendix II, of constructing the solution of equations whose coefficients contain singularities of the type of the delta function and its derivatives, and form the basis of certain very important conclusions of general character.

Let us note first of all that in studying any linear problems reducible to ordinary differential equations of the type under consideration, systems of equations in indefinite values of the corresponding singular functionals in mixed form of the solution always have a triangular nondegenerate matrix, so that the procedure of elimination of these values is trivial independently of their number (i.e., of the number of singularities: ribs, concentrated constraints, etc.) and can be carried out numerically. The only exceptions are problems in eigenvalues, to which the study of questions of stability and vibrations is reduced. In this case, the procedure of elimination of the values of singular functionals from the solution in mixed form is complicated by the fact that the triangular system contains the parameter of the initial equation. However, if only this parameter enters into the degenerate operator, the solution of the triangular system may be represented in the form of finite series in powers of the parameter, and this permits one to construct the general solution of the equation with the parameter in explicit form. The coefficients of these series are determined by recursion relations, and the corresponding numerical procedure is therefore trivial. In the presence of the general solution, the eigenvalues are determined from the conditions of nontriviality of the solution satisfying the boundary conditions. The corresponding equation then constitutes the condition of equality to zero of a determinant of the same order as for the "smooth" problem, the elements of this determinant being polynomials in powers of the parameter. This fact considerably simplifies the algorithmic aspect of the problem (for example, in studying the vibrations of a system with a finite number of degrees of freedom).

Let us note another important point.

The above solution to the problem of cross-bending of a reinforced plate constitutes an extension of M. Levi's classical solution for a smooth plate. In completely analogous fashion, any solutions in unary trigonometric series for smooth plates and shells extend to the case of a unidirectional structure (or weakening) of any type. Moreover, in the direction perpendicular to the structure, we arrive at ordinary differential equations with variable coefficients of the type of the delta function and its derivative (if the torsional rigidity of the rib is considered). However, as in the case of absence of the structure, the boundary conditions at the edges parallel to this direction are assumed to be hinged, and for rings of closed shells of revolution reinforced with this structure, these conditions are replaced by the conditions of periodicity of the solution along the circumferential coordinate.

It may be concluded from the above that the problem of cross-bending of plates illustrates the general method which can be used as the basis for the most diverse studies of the operations of shells of revolution reinforced with stringers or rings, and of the operation of stringer plates.

We have confined our discussion to a fairly narrow range of comparatively simple problems. It is obvious, however, that equations with impulse type singularities can serve as a general mathematical basis for the study of the most diverse objects combining the elements of discreteness and continuity. In structural mechanics, such objects include primarily bars, plates and shells with masses and moments of inertia concentrated at points, on lines and individual surfaces, with a reinforcing structure, supporting layers of zero thickness, etc., and also with a gradual change in geometric and physical characteristics. Such objects are described by differential equations with singular coefficients of the type of the delta function and its derivatives, and also of the type of step functions. In theoretical studies dealing with generalized functions, such equations are not considered, but they do occur in certain applied studies published comparatively recently. Representing the desired solution in the form of binary or unary infinite series, the authors of these studies arrived with the aid of the Bubnov-Galerkin procedure at coupled infinite systems of algebraic or ordinary differential equations which, however, could be easily obtained from the steady-state condition of the energy functional even without resorting to the device of generalized functions. Therefore, the use of this device is purely formal in character, since the absence of the procedure of sewing of solutions in individual portions is essentially determined, not by generalized functions, but by the representation of the desired solution in the form of a series. We should also mention certain other applied studies whose authors, using the operation method, constructed the solutions of individual equations of the type under consideration in mixed form, but then, instead of eliminating from this solution the undetermined values of the corresponding singular functionals on the desired solution of the equation and obtaining the general solution in explicit form, considered an expanded triangular system in place of the triangular one by adding to the triangular system additional equations resulting from the boundary conditions.

Such an approach yields only a partial solution of the problem, corresponding to a specific load and specific boundary conditions. It should be considered numerical, since every time the load or the boundary conditions change, it is necessary to resort to a cumbersome numerical procedure of construction of the solution of the expanded system.

In contrast to such an approach, the methods presented above yield the general solution of the problem. The numerical procedure must be carried out only once in constructing the general integral of the homogeneous equation, and, very importantly, as applied to the triangular system. Let us also note that by using the general formula obtained in Appendix II, one can, in contrast to the operation method, write the solution of the equation with both constant and any, including, singular, variable coefficients.

Having the fundamental system of solutions of a homogeneous equation with singular coefficients, one can then, by using the method of variation of arbitrary constants, construct the general solution of the inhomogeneous equation corresponding to an arbitrary external load, and hence, satisfying the boundary conditions as is done in "smooth" problems obtain the solution corresponding to an arbitrary external load and any fixing conditions.

The above discussion makes it possible to speak, not formally, but essentially about a single methodological base of the study of both "discrete-continuous" and "smooth" objects with the aid of the device of generalized functions.

Appendix IV. THE STURM-LIOUVILLE SPECIAL PROBLEM

IV.1. General Solution

We will consider the ordinary differential equation

$$y''+\lambda^2 \varrho y=0, \tag{IV.1}$$

where the weight function

$$\varrho(x)=1+\mu\sum_{m=1}^{n}\delta(x-x_m); \tag{IV.2}$$

μ is a constant

$$x_m=\frac{2\pi}{n}\,m\ (m=1,\ 2,\ldots,\ n).$$

We will seek nontrivial solutions of Eq. (IV.1) satisfying the periodicity conditions

$$y(0)=y(2\pi),\quad y'(0)=y'(2\pi). \tag{IV.3}$$

We reduce Eq. (IV.1) to the partly degenerate form

$$y''+\lambda^2 y=-\mu\sum_{m=1}^{n}y(x_m)\delta(x-x_m). \tag{IV.4}$$

The general solution of Eq. (IV.4) in mixed form will be, on the basis of (II.24), (II.25)

$$y(x)=C_1\sin\lambda x+C_2\cos\lambda x-\mu\lambda\sum_{m=1}^{n}y(x_m)\sin\lambda(x-x_m)\theta(x-x_m), \tag{IV.5}$$

$$y(x_l)=C_1\sin\lambda x_l+C_2\cos\lambda x_l-\mu\lambda\sum_{m=1}^{l-1}y(x_m)\sin\lambda(x_l-x_m) \tag{IV.6}$$

$$(l=1,\ 2,\ldots,\ n).$$

In order to obtain the solution of Eq. (IV.1) in closed form, it is necessary, having solved system (IV.6) with arbitrary C_1 and C_2 for the parameters $y(x_1)$, to eliminate the latter from expression (IV.5).

We will treat (IV.6) not as an algebraic system, but as a single finite difference equation with variable coefficients of a special type with respect to the function of the discrete argument $y(x_1)$, writing this equation in the form

$$y_l+\bar{\mu}\bar{\lambda}\sum_{m=1}^{l-1} y_m \sin\bar{\lambda}(l-m)=C_1^* e^{i\bar{\lambda}l}+C_2^* e^{-i\bar{\lambda}l}, \tag{IV.7}$$

where

$$y_l=y(x_l),\quad \bar{\lambda}=\frac{2\pi}{n}\lambda,\quad \bar{\mu}=\frac{n}{2\pi}\mu,$$

$$C_1^*=\frac{1}{2}(C_1-iC_2),\quad C_2^*=\frac{1}{2}(C_1+iC_2). \tag{IV.8}$$

The solution of Eq. (IV.7) will be sought in the form

$$y_l=e^{isl}. \tag{IV.9}$$

Introducing (IV.9) into (IV.7) and carrying out the summation, we obtain

$$\left(1+\frac{\bar{\mu}\bar{\lambda}}{2}\frac{\sin\bar{\lambda}}{\cos s-\cos\bar{\lambda}}\right)e^{isl}-$$
$$-\left(C_1^*-\frac{\bar{\mu}\bar{\lambda}}{2i}\frac{1}{e^{i(\bar{\lambda}-s)}-1}\right)e^{i\bar{\lambda}l}-\left(C_2^*-\frac{\bar{\mu}\bar{\lambda}}{2i}\frac{1}{e^{-i(\bar{\lambda}-s)}-1}\right)e^{-i\bar{\lambda}l}=0. \tag{IV.10}$$

Let us note that expression (IV.10), written for arbitrary C_1* and C_2*, becomes an identity in 1 for particular values of the constants

$$C_1^*=C_{10}^*=\frac{\bar{\mu}\bar{\lambda}}{2i}\left[e^{i(\bar{\lambda}-s)}-1\right]^{-1},\quad C_2^*=C_{20}^*=-\frac{\bar{\mu}\bar{\lambda}}{2i}\left[e^{-i(\bar{\lambda}-s)}-1\right]^{-1}, \tag{IV.11}$$

if the parameter s is related to $\bar{\lambda}$ by the relation

$$1+\frac{\bar{\mu}\bar{\lambda}}{2}\frac{\sin\bar{\lambda}}{\cos s-\cos\bar{\lambda}}=0. \tag{IV.12}$$

In this case, as a result of the parity of cos s, Eq. (IV.7) will be satisfied by

$$y_l=y_l^+=e^{isl},\quad y_l=y_l^-=e^{-isl}. \tag{IV.13}$$

Thus, expressions (IV.13), where s is related to $\bar{\lambda}$ by relation (IV.12), represent two particular solutions of Eq. (IV.7), corresponding to values of the constants determined by expressions (IV.11). These solutions will be linearly independent if

$$\begin{vmatrix} y_i^+ & y_i^- \\ y_{i+1}^+ & y_{i+1}^- \end{vmatrix} = -2i \sin s \neq 0, \text{ i.e. for } s \neq \pi k.$$

The solution of Eq. (IV.7), corresponding to arbitrary C_1^* and C_2^* (and hence, to arbitrary C_1 and C_2), can now be constructed for $s \neq \pi k$ by a linear combination of particular solutions (IV.13):

$$y(x_i) = \alpha y_i^+ + \beta y_i^-, \tag{IV.14}$$

where the multipliers α and β, as a result of the linearity of difference Eq. (IV.7), are determined from the system

$$\begin{aligned} \alpha C_{10}^+ + \beta C_{10}^- &= C_1, \\ \alpha C_{20}^+ + \beta C_{20}^- &= C_2. \end{aligned} \tag{IV.15}$$

Here C^+ and C^- are constants (IV.11) of s and -s, respectively.

We transform (IV.15) with the aid of (IV.11). Making use of (IV.8), (IV.12), we obtain

$$\begin{aligned} e^{is}\alpha + e^{-is}\beta &= C_1 \sin \bar{\lambda} + C_2 \cos \bar{\lambda}, \\ \alpha + \beta &= C_2. \end{aligned} \tag{IV.16}$$

The condition of solvability of system (IV.16) is readily seen to coincide with the condition of linear independence of particular solutions (IV.13). Solving (IV.16) for α and β and introducing the solutions in (IV.14), we obtain

$$y(x_i) = C_1 \frac{\sin \bar{\lambda}}{\sin s} \sin is + C_2 \left(\frac{\mu \bar{\lambda}}{2} \frac{\sin \bar{\lambda}}{\sin s} \sin is + \cos is \right), \tag{IV.17}$$

ore, replacing $C_1 + \frac{\mu \bar{\lambda}}{2} C_2$ by the new constant C_1,

$$y(x_i) = C_1 \frac{\sin \bar{\lambda}}{\sin s} \sin is + C_2 \cos is. \tag{IV.18}$$

Let us note that relation (IV.12) between the parameters s and $\bar{\lambda}$ loses meaning when $\bar{\lambda}=\pi k$. We rewrite (IV.12) in the form

$$\cos s=\cos\bar{\lambda}-\frac{\mu\bar{\lambda}}{2}\sin\bar{\lambda}. \tag{IV.19}$$

For values $\bar{\lambda}=\bar{\lambda}_{kq}$, to which the values $s=s_k=\pi k$, will correspond according to (IV.19), solution (IV.17) of Eq. (IV.7) also loses meaning. However, the values $\bar{\lambda}=\bar{\lambda}_{kq}$ form a discrete sequence of points in whose vicinity, as small as desired, solution (IV.17), where s and $\bar{\lambda}$ are now related by relation (IV.19), obviously satisfies Eq. (IV.7) identically for any C_1 and C_2. It follows, as can be readily ascertained, that when $\bar{\lambda}=\bar{\lambda}_{kq}$, the solution of Eq. (IV.7) is the limit of solution (IV.17), (IV.19) when $\bar{\lambda}\to\bar{\lambda}_{kq}$. Indeed, introducing (IV.17) into (IV.7), we have

$$\lim_{\bar{\lambda}\to\bar{\lambda}_{kq}}\left[y_l+\bar{\mu}\bar{\lambda}\sum_{m=1}^{l-1}y_m\sin\bar{\lambda}(l-m)-C_1\sin\bar{\lambda}l-C_2\cos\bar{\lambda}l\right]=0,$$

whence откуда

$$\lim_{\bar{\lambda}\to\bar{\lambda}_{kq}}y_l+\bar{\mu}\bar{\lambda}_{kq}\sum_{m=1}^{l-1}\lim_{\bar{\lambda}\to\bar{\lambda}_{kq}}y_m\sin\bar{\lambda}_{kq}(l-m)=C_1\sin\bar{\lambda}_{kq}l+C_2\cos\bar{\lambda}_{kq}l.$$

Thus, we conclude that expressions (IV.18), (IV.19) are the solution of Eq. (IV.7) for arbitrary C_1, C_2 and any values of λ, and the passage to the limit when $s = \pi k$ is hereinafter legitimate.

Introducing the solution obtained (IV.18) of Eq. (IV.7) and therefore of system (IV.6) into expression (IV.5), we obtain the closed form of the general solution of Eq. (IV.1).

The sum in expression (IV.5) can be transformed if the coordinate is represented in the form

$$x=\frac{2\pi}{n}(p+\xi), \tag{IV.20}$$

where

$p = E\left(\frac{x}{2\pi/n}\right)$ is the integral part of the number $\frac{x}{2\pi/n}$; (IV.21)

$$\xi = x - \frac{2\pi}{n} p. \tag{IV.22}$$

It is obvious that $p = 0, 1, \ldots, n;\ 0 \leq \xi < 1.$

In view of (IV.20)-(IV.22)

$$\sum_{m=1}^{n} y(x_m) \sin\lambda(x - x_m)\theta(x - x_m) = \sum_{m=1}^{p} y(x_m) \sin\lambda(x - x_m). \tag{IV.23}$$

Expanding $y(x_m)$ in (IV.5) with the aid of (IV.18), transforming the sum according to (IV.23), carrying out the summation and as above replacing $C_1 + \frac{\mu\bar{\lambda}}{2} C_2$ by the new constant C_1, we obtain

$$\begin{aligned} y(x) = {} & C_1 \left[\sin ps \sin\bar{\lambda}(1-\xi) + \sin(p+1)s \sin\bar{\lambda}\xi\right] \frac{1}{\sin s} + \\ & + C_2 \left[\cos ps \sin\bar{\lambda}(1-\xi) + \cos(p+1)s \sin\bar{\lambda}\xi\right] \frac{1}{\sin\bar{\lambda}}. \end{aligned} \tag{IV.24}$$

Expression (IV.24) constitutes the general solution of Eq. (IV.1). It is apparent that this solution is continuous, but its derivative generally undergoes finite discontinuities at the points $x = \frac{2\pi}{n} l$ $(l = 1, 2, \ldots, n)$. Let us note that in computing the derivative of the function y(x), written in the variables y(x), one of which is discrete, the differentiation should be carried out only with respect to the coordinate ξ in the usual sense: $\frac{d}{dx} = \frac{n}{2\pi} \frac{d}{d\xi}$ In so doing, at the points $x_l (p = l, \xi = 0)$ we will obtain the expressions for the derivatives on the right. The derivatives on the left are computed as the limit when $p = l - 1, \xi \to 1$. The generalized derivative can be written by differentiating expression (IV.5) in the generalized sense. Then the expression for the first derivative will contain $\theta(x - x_m)$, and for the second, also the delta function.

IV.2. Eigenvalues and Eigenfunctions

Having general solution (IV.24) of Eq. (IV.1) at one's disposal, one can construct particular solutions of this equation which satisfy periodicity conditions (IV.3).

We will subordinate (IV.24) to conditions (IV.3), setting according to (IV.20)-(IV.22) $p = 0$, $\xi = 0$ when $x = 0$ and $p = n$, $\xi = 0$ when $x = 2\pi$. Then, considering (IV.19), we obtain

$$C_1 \sin \bar{\lambda} \frac{\sin ns}{\sin s} - C_2 (1 - \cos ns) = 0,$$

$$\lambda \left\{ C_1 \left(1 - \cos ns + \frac{\bar{\mu}\bar{\lambda}}{2} \sin \bar{\lambda} \frac{\sin ns}{\sin s}\right) - C_2 \left[\frac{\bar{\mu}\bar{\lambda}}{2}(1 - \cos ns) - \frac{\sin ns \sin s}{\sin \bar{\lambda}}\right]\right\} = 0. \quad \text{(IV.25)}$$

Since we are interested only in the nontrivial solutions of Eq. (IV.1), and hence, of system (IV.25), we equate its determinant to zero. We then find for $\lambda \neq 0$

$$\cos ns = 1. \quad \text{(IV.26)}$$

In the case $\lambda = 0$, Eq. (IV.1) has the following obvious solution satisfying (IV.3):

$$y(x) = C. \quad \text{(IV.27)}$$

To determine the eigenvalues, we eliminate the parameter s from (IV.19) with the aid of (IV.26), and obtain

$$\cos \bar{\lambda} - \frac{\bar{\mu}\bar{\lambda}}{2} \sin \bar{\lambda} = \cos \frac{2\pi r}{n} \; (r = 0, 1, 2, \ldots). \quad \text{(IV.28)}$$

Obviously, as a result of the periodicity and parity of $\cos \frac{2\pi r}{n}$, it is sufficient to consider only the values $r = 0, 1, \ldots, E\left(\frac{n}{2}\right)$, where $E\left(\frac{n}{2}\right)$ is the integral part of $\frac{n}{2}$. For these values of r, expression (IV.28) determines the following change of eigenvalues:

when r = 0, i.e., s = 0,

$$\bar{\lambda} = 2\pi q \qquad (q = 1, 2, \ldots) \tag{IV.29}$$

and

$$\bar{\lambda} \operatorname{ctg} \frac{\bar{\lambda}}{2} = -\frac{2}{\bar{\mu}};$$

when r = $1, 2, \ldots, E\left(\frac{n-1}{2}\right)$, т. е. $s = \frac{2\pi r}{n}$, (IV.30)

$$\cos \bar{\lambda} - \frac{\bar{\mu}\bar{\lambda}}{2} \sin \bar{\lambda} = \cos \frac{2\pi r}{n}; \tag{IV.31}$$

when r = $\frac{n}{2}$ (for even n), i.e., $s = \pi$,

$$\bar{\lambda} = \pi(2q - 1) \qquad (q = 1, 2, \ldots) \tag{IV.32}$$

and

$$\bar{\lambda} \operatorname{tg} \frac{\bar{\lambda}}{2} = \frac{2}{\bar{\mu}}. \tag{IV.33}$$

To each of the chains of eigenvalues (IV.29)-(IV.33) there correspond chains of eigenvalues of the problem.

Let us consider system (IV.25) for values of $\bar{\lambda}$ and s determined by expressions (IV.29)-(IV.33). In the cases s = 0, π, terms of the form $\frac{\sin ns}{\sin s}$, $\frac{\sin ns \sin s}{\sin \bar{\lambda}}$ lose meaning, and, as was indicated above, they should be replaced by the corresponding limiting values when $s \to 0, \pi$. According to l'Hopital's rule

$$\lim \frac{\sin ns}{\sin s} = n \frac{\cos ns}{\cos s}, \tag{IV.34}$$

$$\lim \frac{\sin ns \sin s}{\sin \bar{\lambda}} = \lim \frac{(n \cos ns \sin s + \sin ns \cos s) \frac{ds}{d\bar{\lambda}}}{\cos \bar{\lambda}}. \tag{IV.35}$$

The value of $\frac{ds}{d\bar{\lambda}}$ is found by using relation (IV.19), whence

$$\frac{ds}{d\bar{\lambda}} = -\frac{\frac{\bar{\mu}}{2}(\sin \bar{\lambda} + \bar{\lambda} \cos \bar{\lambda}) + \sin \bar{\lambda}}{\sin s}. \tag{IV.36}$$

Introducing (IV.36) into (IV.35), we have

$$\lim \frac{\sin ns \sin s}{\sin \bar{\lambda}} = -\overline{\bar{\mu} n \bar{\lambda}}.$$

From system (IV.25) it follows that

when $s = 0$ and $\bar{\lambda} = 2\pi q$

$$C_2 = 0,\ C_1 - \text{any}; \qquad \text{(IV.37)}$$

when $s = 0$ and $\bar{\lambda}$ satisfying (IV.30)

$$C_1 = 0,\ C_2 - \text{any}; \qquad \text{(IV.38)}$$

when $s = \frac{2\pi r}{n}\left[r = 1, 2, \ldots, E\left(\frac{n-1}{2}\right)\right]$ and $\bar{\lambda}$ satisfying (IV.31),

$$C_1,\ C_2 - \text{any}; \qquad \text{(IV.39)}$$

when $s = \pi$ and $\bar{\lambda} = \pi(2q-1)$ (for even n)

$$C_2 = 0,\ C_1 - \text{any}; \qquad \text{(IV.40)}$$

when $s = \pi$ and $\bar{\lambda}$ satisfying (IV.33) (for even n)

$$C_1 = 0,\ C_2 - \text{any}. \qquad \text{(IV.41)}$$

Turning to general solution (IV.24), for values of $\bar{\lambda}$ and s determined by (IV.29)-(IV.33), and the corresponding constants (IV.37)-(IV.41), we obtain the expressions for the eigenfunctions of the problem:

$$y_{Iq0}(x) = \sin 2\pi q\xi = \sin qnx \quad (q = 1, 2, \ldots), \qquad \text{(IV.42)}$$

$$y_{IIq0}(x) = \cos\bar{\lambda}_{IIq0}\left(\xi - \frac{1}{2}\right), \qquad \text{(IV.43)}$$

$$y_{Iqr}(x) = \sin\frac{2\pi r}{n}p\sin\bar{\lambda}_{qr}(1-\xi) + \sin\frac{2\pi r}{n}(p+1)\sin\bar{\lambda}_{qr}\xi, \qquad \text{(IV.44)}$$

$$y_{IIqr}(x) = \cos\frac{2\pi r}{n}p\sin\bar{\lambda}_{qr}(1-\xi) + \cos\frac{2\pi r}{n}(p+1)\sin\bar{\lambda}_{qr}\xi, \qquad \text{(IV.45)}$$

$$y_{Iq\frac{n}{2}}(x) = (-1)^p\sin\pi(2q-1)\xi = \sin\frac{n}{2}(2q-1)x \quad (q = 1, 2, \ldots), \qquad \text{(IV.46)}$$

$$y_{IIq\frac{n}{2}}(x) = (-1)^p\sin\bar{\lambda}_{IIq\frac{n}{2}}\left(\xi - \frac{1}{2}\right), \qquad \text{(IV.47)}$$

where $\bar{\lambda}_{IIq0}$ are the roots of Eq. (IV.30); $\bar{\lambda}_{qr}$ are the roots of Eqs. (IV.31); $\bar{\lambda}_{IIq\frac{n}{2}}$ are the roots of Eq. (IV.33).

In addition, there is another eigenfunction

$$y(x) = 1, \qquad \text{(IV.48)}$$

corresponding to the eigenvalue $\lambda = 0$.

IV.3. Orthonormalized System of Eigenfunctions

Let us turn to Eq. (IV.1). For any pair of eigenfunctions $y_i(x), y_j(x)$, we have

$$y_i'' + \lambda_i^2 q y_i = 0,$$
$$y_j'' + \lambda_j^2 q y_j = 0. \tag{IV.49}$$

Multiplying the first of Eqs. (IV.49) by $y_j(x)$ and the second by $y_i(x)$, integrating over x from 0 to 2π while considering periodicity conditions (IV.3), then subtracting one from the other, we have

$$(\lambda_i^2 - \lambda_j^2) \int_0^{2\pi} y_i(x) y_j(x) q(x) dx = 0, \tag{IV.50}$$

whence all the eigenfunctions corresponding to different λ are orthogonal on $[0, 2\pi]$ with weight q(x).

In addition, multiplying the first Eq. (IV.49) by $y_j(x)$ and integrating over x from 0 to 2π, considering (IV.3) and (IV.50), we obtain

$$\lambda_i^2 \int_0^{2\pi} y_i(x) y_j(x) q(x) dx = \int_0^{2\pi} y_i'(x) y_j'(x) dx = 0 \quad (i \neq j), \tag{IV.51}$$

i.e., the derivatives of any eigenfunctions corresponding to different λ are orthogonal on $[0,2\pi]$.

Let us consider the question of orthogonality of eigenfunctions corresponding to the twice degenerate values $\lambda = \lambda_{qr}$. To this end, we set up the linear combination

$$\bar{y}_{II\,qr} = A y_{I\,qr} + y_{II\,qr}, \tag{IV.52}$$

requiring the orthogonality of the functions $y_{I\,qr}$ and $\bar{y}_{II\,qr}$ with weight q(x) on $[0.2\pi]$. We obtain

$$A = -\frac{\int_0^{2\pi} y_{I\,qr} y_{II\,qr} q dx}{\int_0^{2\pi} y_{I\,qr}^2 q dx} \tag{IV.53}$$

or, considering (IV.51),

$$A = -\frac{1}{\lambda^2}\frac{\int\limits_0^{2\pi} y_{\mathrm{I}qr} y'_{\mathrm{II}qr} dx}{\int\limits_0^{2\pi} y^2_{\mathrm{I}qr} q\, dx}. \tag{IV.54}$$

We find the derivatives $y'_{\mathrm{I}qr}$, $y'_{\mathrm{II}qr}$ by differentiating (IV.44), (IV.45):

$$\begin{aligned} y'_{\mathrm{I}qr} &= \lambda\left[-\sin\frac{2\pi r}{n}p\cos\bar{\lambda}(1-\xi)+\sin\frac{2\pi r}{n}(p+1)\cos\bar{\lambda}\xi\right], \\ y'_{\mathrm{II}qr} &= \lambda\left[-\cos\frac{2\pi r}{n}p\cos\bar{\lambda}(1-\xi)+\cos\frac{2\pi r}{n}(p+1)\cos\bar{\lambda}\xi\right]. \end{aligned} \tag{IV.55}$$

The integral in the numerator of (IV.54) will be computed in the form of a sum of integrals over sections:

$$\begin{aligned} \int\limits_0^{2\pi} y'_{\mathrm{I}qr}(x)\, y'_{\mathrm{II}qr}(x)\, dx &= \lambda^2\frac{n}{2\pi}\sum_{p=0}^{n-1}\int\limits_0^1 y'_{\mathrm{I}qr}(p,\xi)\, y'_{\mathrm{II}qr}(p,\xi)\, d\xi = \\ &= \lambda^2\frac{n}{2\pi}\left[\int\limits_0^1\cos^2\bar{\lambda}(1-\xi)\, d\xi\sum_{p=0}^{n-1}\sin\frac{2\pi r}{n}p\cos\frac{2\pi r}{n}p+\right. \\ &+\int\limits_0^1\cos^2\bar{\lambda}\xi\, d\xi\sum_{p=0}^{n-1}\sin\frac{2\pi r}{n}(p+1)\cos\frac{2\pi r}{n}(p+1)- \\ &\left.-\int\limits_0^1\cos\bar{\lambda}(1-\xi)\cos\bar{\lambda}\xi\, d\xi\sum_{p=0}^{n-1}\sin\frac{2\pi r}{n}(2p+1)\right]. \end{aligned} \tag{IV.56}$$

Using summation formulas (14.3), (14.12)-(14.14), one can easily ascertain that all the sums entering into (IV.56) are equal to zero. Hence

$$\int\limits_0^{2\pi} y'_{\mathrm{I}qr}(x)\, y'_{\mathrm{II}qr}(x)\, dx = 0, \tag{IV.57}$$

and from (IV.54) we obtain A = 0, i.e., all the eigenfunctions corresponding to twice degenerate values of λ are also orthogonal with the weight q(x) on $[0, 2\pi]$, and on the basis of (IV.51), their derivatives are also orthogonal.

We will normalize all the eigenfunctions of the problem so that

$$\int_0^{2\pi} \left[\frac{y_i(x)}{N_i}\right]^2 \varrho(x)\,dx = 1. \tag{IV.58}$$

Then for the normalizing factors N_i, on the basis of (IV.58), (IV.51), we obtain

$$N_i^2 = \frac{1}{\lambda_i^2} \int_0^{2\pi} [y_i'(x)]^2\,dx. \tag{IV.59}$$

Computing the normalizing factor for functions (IV.42)-(IV.47) in accordance with (IV.59) we obtain

$$\begin{aligned}
&N^2_{I\varphi 0} = N^2_{I\varphi \frac{\pi}{2}} = \pi, \\
&N^2_{II\varphi 0} = \pi\left(1 + \bar{\mu}\cos^2\frac{\bar{\lambda}_{II\varphi 0}}{2}\right), \\
&N^2_{I\varphi r} = N^2_{II\varphi r} = \pi\left(1 - \cos\frac{2\pi r}{n}\cos\bar{\lambda}_{\varphi r} + \frac{\bar{\mu}}{2}\sin^2\bar{\lambda}_{\varphi r}\right), \\
&N^2_{I\varphi \frac{\pi}{2}} = \pi\left(1 + \bar{\mu}\sin^2\frac{\bar{\lambda}_{II\varphi \frac{\pi}{2}}}{2}\right).
\end{aligned} \tag{IV.60}$$

In addition, for the function $y_0(x) = 1$, we have

$$N_0^2 = 2\pi(1 + \bar{\mu}).$$

IV.4. Expansion of the Eigenfunctions of the Problem in Infinite Trigonometric Series

Any of the orthonormalized eigenfunctions $Y_i(x) = \frac{y_i(x)}{N_i}$ can be made to correspond to the Fourier series

$$Y_i(x) = \frac{a_{i,0}}{2} + \sum_{k=1}^{\infty} (a_{i,k}\cos kx + b_{i,k}\sin kx), \tag{IV.61}$$

where

$$\begin{aligned}
a_{i,k} &= \frac{1}{\pi}\int_0^{2\pi} Y_i(x)\cos kx\,dx, \\
b_{i,k} &= \frac{1}{\pi}\int_0^{2\pi} Y_i(x)\sin kx\,dx.
\end{aligned} \tag{IV.62}$$

The expansions for the function $Y_{1q0}, Y_{1q\frac{n}{2}}$ are obvious and contain only one term each.

Let us turn to the eigenfunctions corresponding to the transcendental values of λ. To compute the coefficients $a_{l,k}, b_{l,k}$, we will use an artificial method. We multiply Eq. (IV.1) successively by cox kx, sin kx and integrate over x from 0 to 2π, taking (IV.2), (IV.3) into account. We obtain

$$(k^2-\lambda_l^2)\int_0^{2\pi} Y_l(x)\cos kx\,dx=\lambda_l^2\mu\sum_{m=1}^{n} Y_l(x_m)\cos kx_m,$$

$$(k^2-\lambda_l^2)\int_0^{2\pi} Y_l(x)\sin kx\,dx=\lambda_l^2\mu\sum_{m=1}^{n} Y_l(x_m)\sin kx_m,$$

Hence, since $k^2\neq\lambda^2$,

$$a_{l,k}=\frac{\mu}{\pi}\,\frac{\lambda_l^2}{k^2-\lambda_l^2}\sum_{m=1}^{n} Y_l(x_m)\cos\frac{2\pi k}{n}m,$$

$$b_{l,k}=\frac{\mu}{\pi}\,\frac{\lambda_l^2}{k^2-\lambda_l^2}\sum_{m=1}^{n} Y_l(x_m)\sin\frac{2\pi k}{n}m. \tag{IV.63}$$

The expressions for $Y_l(x_m)=Y_l(p=m,\ \xi=0)$ have the form:

$$Y_{11q0}(x_m)=\frac{1}{N_{11q0}}\cos\frac{\bar{\lambda}_{11q0}}{2},$$

$$Y_{11q\frac{n}{2}}(x_m)=-\frac{(-1)^m}{N_{11q\frac{n}{2}}}\sin\frac{\bar{\lambda}_{11q\frac{n}{2}}}{2},$$

$$Y_{1qr}(x_m)=\frac{1}{N_{qr}}\sin\bar{\lambda}_{qr}\sin\frac{2\pi r}{n}m, \tag{IV.64}$$

$$Y_{11qr}(x_m)=\frac{1}{N_{qr}}\sin\bar{\lambda}_{qr}\cos\frac{2\pi r}{n}m.$$

Introducing (IV.64) into (IV.63), and carrying out the summation by means of formulas (14.3), (14.12)-(14.14), we obtain

$$a_{11q0,k}=\begin{cases}0, & k\neq nt,\\ \dfrac{\mu n}{\pi}\dfrac{1}{N_{11q0}}\cos\dfrac{\bar{\lambda}_{11q0}}{2}\dfrac{\lambda^2_{11q0}}{k^2-\lambda^2_{11q0}} & (k=nt),\end{cases}$$

$$a_{11q\frac{n}{2},k}=\begin{cases}0, & k\neq\dfrac{n}{2}(2t-1),\\ -\dfrac{\mu n}{\pi}\dfrac{1}{N_{11q\frac{n}{2}}}\sin\dfrac{\bar{\lambda}_{11q\frac{n}{2}}}{2}\dfrac{\lambda^2_{11q\frac{n}{2}}}{k^2-\lambda^2_{11q\frac{n}{2}}} & \left[k=\dfrac{n}{2}(2t-1)\right],\end{cases}$$

$$a_{11qr,k}=\pm b_{1qr,k}=\begin{cases}0, & k\neq tn\pm r,\\ \dfrac{\mu n}{2\pi}\dfrac{1}{N_{qr}}\sin\bar{\lambda}_{qr}\dfrac{\lambda^2_{qr}}{k^2-\lambda^2_{qr}} & (k=tn\pm r),\end{cases} \tag{IV.65}$$

$$b_{11q0,k}=b_{11q\frac{n}{2},k}=a_{1qr,k}=b_{11qr,k}=0,$$

where $t=1, 2, \ldots$.

Introducing the expressions for the Fourier coefficients (IV.65) into (IV.61), we finally obtain

$$Y_{11q0}(x)=\frac{\mu n}{\pi}\frac{\cos\dfrac{\bar{\lambda}_{11q0}}{2}}{N_{11q0}}\left[-\frac{1}{2}+\lambda^2_{11q0}\sum_{t=1}^{\infty}\frac{\cos tnx}{(tn)^2-\lambda^2_{11q0}}\right],$$

$$Y_{11q\frac{n}{2}}(x)=-\frac{\mu n}{\pi}\frac{\sin\dfrac{\bar{\lambda}_{11q\frac{n}{2}}}{2}}{N_{11q\frac{n}{2}}}\lambda^2_{11q\frac{n}{2}}\sum_{t=1}^{\infty}\frac{\cos\left(tn-\frac{n}{2}\right)x}{\left(tn-\frac{n}{2}\right)^2-\lambda^2_{11q\frac{n}{2}}}, \tag{IV.66}$$

$$Y_{1qr}(x)=\frac{\mu n}{2\pi}\frac{\sin\bar{\lambda}_{qr}}{N_{qr}}\lambda^2_{qr}\left\{\frac{\sin rx}{r^2-\lambda^2_{qr}}-\right.$$

$$\left.-\sum_{t=1}^{\infty}\left[\frac{\sin(tn-r)x}{(tn-r)^2-\lambda^2_{qr}}-\frac{\sin(tn+r)x}{(tn+r)^2-\lambda^2_{qr}}\right]\right\},$$

$$Y_{11qr}(x)=\frac{\mu n}{2\pi}\frac{\sin\bar{\lambda}_{qr}}{N_{qr}}\lambda^2_{qr}\left\{\frac{\cos rx}{r^2-\lambda^2_{qr}}+\right.$$

$$\left.+\sum_{t=1}^{\infty}\left[\frac{\cos(tn-r)x}{(tn-r)^2-\lambda^2_{qr}}+\frac{\cos(tn+r)x}{(tn+r)^2-\lambda^2_{qr}}\right]\right\}.$$

IV.5. Fourier Series in Eigenfunctions of the Problem

An arbitrary function f(x) specified on [0,2π] can be made to correspond to the series

$$f(x)\sim F(x)=\bar{a}_{00}Y_0+\sum_{q=1}^{\infty}\sum_{r=0}^{E\left(\frac{n}{2}\right)}\left[\bar{a}_{qr}Y_{IIqr}(x)+\bar{b}_{qr}Y_{Iqr}(x)\right]. \qquad \text{(IV.67)}$$

The coefficients of this series are determined by the Euler-Fourier method. Let us assume that f(x) = F(x). Then, multiplying (IV.67) successively by all $Y_i(x)\varrho(x)$ and integrating over x from 0 to 2π, taking the orthonormalization of the system of eigenfunctions $Y_i(x)$, we obtain

$$\bar{a}_{00}=Y_0\int_0^{2\pi} f(x)\varrho(x)dx,$$

$$\bar{a}_{qr}=\int_0^{2\pi} f(x)Y_{IIqr}(x)\varrho(x)dx,\quad \bar{b}_{qr}=\int_0^{2\pi} f(x)Y_{Iqr}(x)\varrho(x)dx \qquad \text{(IV.68)}$$

$$\left[q=1,2,\ldots;\ r=0,1,\ldots,E\left(\frac{n}{2}\right)\right].$$

We construct the expansions in the eigenfunctions of the problem for the trigonometric functions sin kx, cos kx. Instead of directly computing the coefficients from formulas (IV.68), we establish a relationship between coefficients (IV.68) and the expressions obtained above for coefficients (IV.62). Multiplying Eq. (IV.1) successively by sin kx, cos kx and integrating over x from 0 to 2π, in view of expressions (IV.62), (IV.68) when $\lambda \neq 0$ we obtain

$$\pi k^2 b_{Iqr,k}=\lambda_{Iqr}^2\bar{b}_{qr,k},\quad \pi k^2 a_{IIqr,k}=\lambda_{IIqr}^2\bar{a}_{qr,k}$$

$$\left[r=0,1,\ldots,E\left(\frac{n}{2}\right)\right]. \qquad \text{(IV.69)}$$

Relations (IV.69) make it possible to write the expansions of the trigonometric functions in the eigenfunctions of the problem by using expansions (IV.66). We have:

$$\cos tnx = \frac{\mu n}{2\pi + \mu n} + \mu n (tn)^2 \sum_{q=1}^{\infty} \frac{\cos \frac{\bar{\lambda}_{11q0}}{2}}{N_{11q0}} \frac{Y_{11q0}}{(tn)^2 - \lambda_{11q0}^2},$$

$$\cos\left(tn - \frac{n}{2}\right)x = -\mu n \left(tn - \frac{n}{2}\right)^2 \sum_{q=1}^{\infty} \frac{\sin \frac{\bar{\lambda}_{11q\frac{n}{2}}}{2}}{N_{11q\frac{n}{2}}} \frac{Y_{11q\frac{n}{2}}}{\left(tn - \frac{n}{2}\right)^2 - \lambda_{11q\frac{n}{2}}^2}, \qquad \text{(IV.70)}$$

$$\cos(tn \pm l)x = \frac{\mu n}{2}(tn \pm l)^2 \sum_{q=1}^{\infty} \frac{\sin \bar{\lambda}_{qr}}{N_{qr}} \frac{Y_{11qr}}{(tn \pm l)^2 - \lambda_{qr}^2},$$

$$\sin(tn \pm l)x = \pm \frac{\mu n}{2}(tn \pm l)^2 \sum_{q=1}^{\infty} \frac{\sin \bar{\lambda}_{qr}}{N_{qr}} \frac{Y_{1qr}}{(tn \pm l)^2 - \lambda_{qr}^2}$$

$$\left[t = 1, 2, \ldots; \; l = 1, 2, \ldots, E\left(\frac{n-1}{2}\right)\right].$$

The expansions for $\sin \frac{n}{2} tx$ contain only one term each.

$$\sin tnx = \sqrt{\pi}\, Y_{1t0}(x), \quad \sin\left(tn - \frac{n}{2}\right)x = \sqrt{\pi}\, Y_{1t\frac{n}{2}}(x).$$

Finally, on the basis of expansion (IV.67), we can raise the question of the series expansion of the function of the discrete argument

$$f(x_l) = f\left(\frac{2\pi}{n} l\right) \qquad (l = 1, 2, \ldots, n).$$

Such a formulation of the question is adequate to the expansion of the function f(x) in a series convergent on the point set $\{x_m\}$. In this case, we will be interested in the values of the sum of series (IV.67) only at the points $x = x_l$:

$$f(x_l) \sim F(x_l) = \bar{a}_{00} Y_0 + \sum_{q=1}^{\infty} \sum_{r=0}^{E\left(\frac{n}{2}\right)} \left[\bar{a}_{qr} Y_{11qr}(x_l) + \bar{b}_{qr} Y_{1qr}(x_l)\right]. \qquad \text{(IV.71)}$$

Considering expressions (IV.64), we reduce series (IV.71) to the form

$$f(x_l) \sim F(x_l) = A_0 + \sum_{r=1}^{E\left(\frac{n}{2}-1\right)} \left(A_r \cos \frac{2\pi r}{n} l + B_r \sin \frac{2\pi r}{n} l\right) + A_{\frac{n}{2}} (-1)^l, \qquad \text{(IV.72)}$$

where

$$A_0 = \bar{a}_{00} Y_0 + \sum_{\bar{q}=1}^{\infty} \frac{\bar{a}_{\bar{q}0}}{N_{11\bar{q}0}} \cos\frac{\bar{\lambda}_{11\bar{q}0}}{2},$$

$$A_{\bar{r}} = \sum_{\bar{q}=1}^{\infty} \frac{\bar{a}_{\bar{q}\bar{r}}}{N_{\bar{q}\bar{r}}} \sin\bar{\lambda}_{\bar{q}\bar{r}}, \qquad \text{(IV.73}$$

$$B_{\bar{r}} = \sum_{\bar{q}=1}^{\infty} \frac{\bar{b}_{\bar{q}\bar{r}}}{N_{\bar{q}\bar{r}}} \sin\bar{\lambda}_{\bar{q}\bar{r}},$$

$$A_{\frac{n}{2}} = -\sum_{\bar{q}=1}^{\infty} \frac{\bar{a}_{\bar{q}\frac{n}{2}}}{N_{11\bar{q}\frac{n}{2}}} \sin\frac{\bar{\lambda}_{11\bar{q}\frac{n}{2}}}{2}.$$

The coefficients of expansion (IV.72) will also be determined by the Euler-Fourier method, assuming that $f(x_1) = F(x_1)$. In this case, successively multiplying (IV.72) by $Y_0, Y_{1\bar{q}\bar{r}}, Y_{11\bar{q}\bar{r}} [\bar{q}=1,2,\ldots; \bar{r}=0,1,\ldots,E\left(\frac{n}{2}\right)]$, and integrating each time in the sense of Lebesgue-Stieltjes over the region of convergence (i.e., over the point set $\{x_m\}$), we will obtain an infinite system of algebraic equations in n unknown coefficients (IV.72). However, it is apparent that for fixed values of $\bar{r}$, equations corresponding to different $\bar{q}$ will be proportional. Therefore, the infinite system actually constitutes a system of n equations:

$$\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right) = A_0 n + \sum_{r=1}^{E\left(\frac{n-1}{2}\right)} \left(A_r \sum_{m=1}^{n} \cos\frac{2\pi r}{n}m + B_r \sum_{m=1}^{n} \sin\frac{2\pi r}{n}m\right) + A_{\frac{n}{2}} \sum_{m=1}^{n} (-1)^m,$$

$$\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right)\cos\frac{2\pi\bar{r}}{n}m = A_0 \sum_{m=1}^{n} \cos\frac{2\pi\bar{r}}{n}m + \sum_{r=1}^{E\left(\frac{n-1}{2}\right)} \left(A_r \sum_{m=1}^{n} \cos\frac{2\pi r}{n}m\cos\frac{2\pi\bar{r}}{n}m + B_r \sum_{m=1}^{n} \sin\frac{2\pi r}{n}m\cos\frac{2\pi\bar{r}}{n}m\right) + A_{\frac{n}{2}} \sum_{m=1}^{n} (-1)^m \cos\frac{2\pi\bar{r}}{n}m,$$

$$\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right)\sin\frac{2\pi\bar{r}}{n}m = A_0 \sum_{m=1}^{n} \sin\frac{2\pi\bar{r}}{n}m + \sum_{r=1}^{E\left(\frac{n-1}{2}\right)} \left(A_r \sum_{m=1}^{n} \cos\frac{2\pi r}{n}m\sin\frac{2\pi\bar{r}}{n}m + B_r \sum_{m=1}^{n} \sin\frac{2\pi r}{n}m\sin\frac{2\pi\bar{r}}{n}m\right) +$$

$$+A_{\frac{n}{2}}\sum_{m=1}^{n}(-1)^m \sin\frac{2\pi r}{n}m,$$

$$\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right)(-1)^m = A_0\sum_{m=1}^{n}(-1)^m +$$

$$+\sum_{r=1}^{E\left(\frac{n-1}{2}\right)}\left[A_r\sum_{m=1}^{n}\cos\frac{2\pi r}{n}m(-1)^m + B_r\sum_{m=1}^{n}\sin\frac{2\pi r}{n}m(-1)^m\right] + A_{\frac{n}{2}}n. \qquad \text{(IV.74)}$$

On the basis of summation formulas (14.3), (14.12)-(14.14), and considering that for odd n it is necessary to set $A_{\frac{n}{2}}=0$, and for even n $\sum_{m=1}^{n}(-1)^m=0$, we can easily observe that the system of n equations (IV.74) decomposes completely. We have

$$A_0=\frac{1}{n}\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right),$$

$$A_r=\frac{2}{n}\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right)\cos\frac{2\pi r}{n}m, \qquad \text{(IV.75)}$$

$$B_r=\frac{2}{n}\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right)\sin\frac{2\pi r}{n}m,$$

$$A_{\frac{n}{2}}=\frac{1}{n}\sum_{m=1}^{n} f\left(\frac{2\pi}{n}m\right)(-1)^m.$$

Formulas (IV.75) together with expression (IV.72) determine the series expansion in the eigenfunctions of problem (IV.1)-(IV.3) of an arbitrary function of a discrete argument $f(x_l)=f\left(\frac{2\pi}{n}l\right)$ and coincide with the polynomial given previously (13.67)/(13.69)/ and formulas (13.68)/(13.70)/, confirming the convergence of series (IV.72) to the function $f(x_1)$.

Appendix V. GENERAL ALGORITHM OF THE ANALYSIS OF OBLIQUE THIN-WALLED SYSTEMS AND ITS REALIZATION ON A COMPUTER

V.1. Differential Resolvents of a Conical Shell of Arbitrary Outline

It is most convenient to proceed from the mixed form of differential resolvents, which in the general case, neglecting the bending strain of the middle surface, can be represented in the form

$$
\begin{gathered}
P_j^{*\prime} = \left(1 - \frac{z}{l_0}\right) \sum_{i=1}^{N} \tilde{a}_{ji} V_i' + \sum_{i=1}^{N} \tilde{b}_{ji} V_i - P_j^{\prime\prime}, \\
P_j^{\prime\prime} - \sum_{i=1}^{N} \tilde{b}_{ij} V_i' - \frac{1}{1 - \frac{z}{l_0}} \sum_{i=1}^{N} (\tilde{c}_{ji} + \tilde{c}_{nji}) V_i = -R_j - R_j' \qquad \text{(V.1)} \\
(j = 1, 2, \ldots, N),
\end{gathered}
$$

where

$$
P_j^{*} = \frac{1}{\lambda_j G} P_j \qquad \text{(V.2)}
$$

are reduced generalized forces.

The coefficients of Eqs. (V.1) are determined by the expressions

$$
\begin{gathered}
\tilde{a}_{ji} = \frac{2}{1-\nu} \left\{ \oint \left[\psi_{jm_s} \psi_{im_s} + \nu (\psi_{jn_s} \psi_{im_s} + \psi_{jm_s} \psi_{in_s}) + \psi_{jn_s} \psi_{in_s} \right] \frac{l_s}{l_0} \sin \chi h dS + \right. \\
\left. + (1 - \nu^2) \sum_k \psi_{jm_s}(S_k) \psi_{im_s}(S_k) \left(\frac{l_s}{l_0} \right)_k \Delta F_k \right\} + \oint \psi_{jn_s m_s} \psi_{in_s m_s} \frac{l_0}{l_s} \sin \chi h dS, \\
\tilde{b}_{ji} = \frac{2}{1-\nu} \left\{ \oint \left[\psi_{jm_s} \bar{\psi}_{im_s} + \nu (\psi_{jn_s} \bar{\psi}_{im_s} + \psi_{jm_s} \bar{\psi}_{in_s}) + \psi_{jn_s} \bar{\psi}_{in_s} \right] \frac{l_s}{l_0} \sin \chi h dS \right\} + \\
+ \oint \psi_{jn_s m_s} \bar{\psi}_{in_s m_s} \frac{l_s}{l_0} \sin \chi h dS, \qquad \text{(V.3)} \\
\tilde{c}_{ji} = \frac{2}{1-\nu} \left\{ \oint \left[\bar{\psi}_{jm_s} \bar{\psi}_{im_s} + \nu (\bar{\psi}_{jn_s} \bar{\psi}_{im_s} + \bar{\psi}_{jm_s} \bar{\psi}_{in_s}) + \bar{\psi}_{jn_s} \bar{\psi}_{in_s} \right] \frac{l_s}{l_0} \sin \chi h dS \right\} + \\
+ \oint \bar{\psi}_{jn_s m_s} \bar{\psi}_{in_s m_s} \frac{l_s}{l_0} \sin \chi h dS, \\
\tilde{c}_{nji} = 2(1+\nu) \oint \bar{\psi}_{jm_s} \bar{\psi}_{im_s} \Delta \bar{F}_n dS_n + \left(1 - \frac{z}{l_0}\right) F_{nn} \tilde{h}_{nk} \Delta_{ji} \\
[k = j - (6 + n) = 1, 2, \ldots, p],
\end{gathered}
$$

where

$$
\begin{gathered}
\psi_{im_s} = \frac{l_0}{l_s} \varphi_{im_s}; \quad \psi_{in_s} = -\frac{l_0}{l_s} \operatorname{ctg} \chi \varphi_{in_s}, \qquad \text{(V.4)} \\
\psi_{in_s m_s} = \frac{l_0}{l_s} (-\operatorname{ctg} \chi \varphi_{im_s} + \varphi_{in_s}).
\end{gathered}
$$

$$\bar{\psi}_{lm_s}=0;\ \psi_{ln_s}=\frac{1}{\sin\chi}\varphi'_{ln_s}-\frac{1}{l_s}\varphi_{lm_s}-\frac{1}{R_0}\varphi_{ln_n}, \tag{V.5}$$

$$\bar{\psi}_{ln_s m_s}=\frac{1}{\sin\chi}\varphi'_{lm_s}+\frac{1}{l_s}\varphi_{ln_s},$$

$$\bar{\varphi}_{lm_s}=\varphi'_{lm_s}-\frac{1-\frac{1}{2}(l_s^0)'}{l_s\sin\chi}\varphi_{ln_s}-\frac{\sin^2\chi}{R_0}\varphi_{ln_n}. \tag{V.6}$$

On the right-hand side of the equations

$$\begin{aligned}
\bar{R}_j&=\frac{1}{G}\left(1-\frac{Z}{l_0}\right)\oint \mathbf{p}\cdot\boldsymbol{\varphi}_j\frac{l_s}{l_0}\sin\chi dS,\\
\bar{R}^t_j&=2\frac{1+\nu}{1-\nu}\alpha\oint t(Z,S)\bar{\psi}_{jn_s}\frac{l_s}{l_0}\sin\chi h dS,\\
\sigma^t_j&=2\frac{1+\nu}{1-\nu}\alpha\left(1-\frac{Z}{l_0}\right)\oint t(Z,S)(\psi_{jm_s}+\psi_{jn_s})\frac{l_s}{l_0}\sin\chi h dS+\\
&+2(1+\nu)\alpha\sum_h t(Z,S_h)\psi_{jm_s}(S_h)\left(\frac{l_s}{l_0}\right)_h\Delta F_h
\end{aligned} \tag{V.7}$$

are the known functions of the Z coordinate, which allow for the surface load on the shell and for the action of a temperature field.

The external surface load and the temperature field are conveniently represented in the form of the expansions

$$\mathbf{p}(Z,S)=\sum_k p_k(Z)\mathbf{p}^0_k(S), \tag{V.8}$$

$$t(Z,S)=\sum_k t_k(Z)t^0_k(S). \tag{V.9}$$

In this case, the right-hand sides of the resolvents may also be represented in the form of the expansions

$$\begin{aligned}
\bar{R}_j(Z)&=\frac{1}{G}\left(1-\frac{Z}{l_0}\right)\sum_k p_k(Z)\bar{R}_{jk},\\
\bar{R}^t_j(Z)&=2\frac{1+\nu}{1-\nu}\alpha\sum_k t_k(Z)\bar{R}^t_{jk},\\
P^{\sigma t}_j(Z)&=2\frac{1+\nu}{1-\nu}\alpha\left(1-\frac{Z}{l_0}\right)\sum_k t_k(Z)P^{\sigma t}_{jk},
\end{aligned} \tag{V.10}$$

where

$$\begin{aligned}
\bar{R}_{jk}&=\oint \mathbf{p}^0_k\cdot\boldsymbol{\varphi}_j\frac{l_s}{l_0}\sin\chi dS,\\
\bar{R}^t_{jk}&=\oint t^0_k\bar{\psi}_{jn_s}\frac{l_s}{l_0}\sin\chi h dS,\\
P^{\sigma t}_{jk}&=\oint t^0_k(\psi_{jm_s}+\psi_{jn_s})\frac{l_s}{l_0}\sin\chi h dS+\\
&+\frac{1-\nu}{1-\frac{Z}{l_0}}\sum_r t^0_k(S_r)\psi_{jm_s}(S_r)\left(\frac{l_s}{l_0}\right)_r\Delta F_r.
\end{aligned} \tag{V.11}$$

In practical calculations, it is usually sufficient to represent the dependence of the surface load and temperature field on the Z coordinate in the form of a certain parabola. In this case, it is convenient to assume that

$$p_k(Z)=t_k(Z)=\left(1-\frac{Z}{l_0}\right)^k. \tag{V.12}$$

The shell thickness may also be represented in the form of the expansion

$$h(Z,S)=\sum_k h_k(Z)h_k^*(S), \tag{V.13}$$

and in many cases, it is sufficient to confine onself to only one of its terms, assuming that

$$h(Z,S)=h(Z)h^0(S), \tag{V.14}$$

where h)Z) is some piecewise-linear or even step function (h(0) = 1). Representation (V.14) determines, for the entire wing, the same character of distribution of the skin thickness in the sections Z = const as in the root section Z = 0.

Equations (V.1) and expressions (V.2)-(V.7) are most general in character. Specifying the number of degrees of freedom N of the contour Z - const and choosing the coordinate functions φ_i corresponding to them, by integrating system (V.1), one can determine the generalized displacements $U_i=\frac{1}{\lambda_i}V_i$, then find the displacements, strains, and stresses at any point of a conical shell of arbitrary outline.

In the general case, the number of degrees of freedom

$$N=6+n+p+n_1, \tag{V.15}$$

where

n is the number of degrees of freedom corresponding to warpings of the contour out of its plane;

$p \pm n_1$ is the number of degrees of freedom corresponding to strains of the contour in its plane (p corresponds to shear strains, and n_1, to bending strains of the ribs in their plane).

Assuming in a special case that N = 6+n, we will have a model without considering the rib elasticity. An appropriate selection of the coordinate functions $\varphi_k\ (k=i-6)$ will make it possible to obtain both a model with the contour Z = const nondeformable in its plane, and a fundamental static-geometric model.

The elastic displacement vector is determined by the expansion

$$\mathrm{U}(Z,S)=\sum_{i=1}^{N} U_i(Z)\lambda_i(Z)\varphi_i(S), \tag{V.16}$$

where

$$\lambda_i(Z)=\begin{cases} 1 & (i=1,2,3), \\ 1-\dfrac{Z}{l_0} & (i=4,5,6), \\ 1 & (i=7,8,\ldots,n), \\ 1-\dfrac{Z}{l_0} & (i=n+1,n+2,\ldots,n+p), \\ 1 & (i=n+p+1,n+p+2,\ldots,N). \end{cases} \tag{V.17}$$

The internal forces are determined in the general case by elasticity relations (5.6). The linear and angular strains of the middle surface are determined by expansions of the type

$$\sum_{i=1}^{N}\left(V_i'\eta_i+\frac{1}{1-\frac{Z}{l_0}}V_i\bar{\eta}_i\right). \tag{V.18}$$

Representing system (V.1) in matrix form, we will solve it for the derivatives

$$\begin{aligned} &\left(1-\frac{Z}{l_0}\right)\frac{d}{dZ}\mathrm{V}=-\mathrm{A}^{-1}\mathrm{B}\mathrm{V}+\mathrm{A}^{-1}\mathrm{P}^{*}+\mathrm{A}^{-1}\mathrm{P}^{*\prime},\\ &\left(1-\frac{Z}{l_0}\right)\frac{d}{dZ}\mathrm{P}^{*}=-(\mathrm{B}'\mathrm{A}^{-1}\mathrm{B}-\mathrm{C})\mathrm{V}+\mathrm{B}'\mathrm{A}^{-1}\mathrm{P}^{*}+\\ &\qquad+\mathrm{B}'\mathrm{A}^{-1}\mathrm{P}^{*\prime}-\tilde{\mathrm{R}}-\tilde{\mathrm{R}}'. \end{aligned} \tag{V.19}$$

Here V, P_1^* are the column matrices of the unknowns V_i and P^*_i consisting of N elements each; $\tilde{R}, \tilde{R}^i, P^{*i}$ are the column matrices of the terms $\dot{R}_i, \dot{R}_i^i, \dot{P}_i^{*i}$, which determine the external action on the shell, consisting of N elements each; $A, B, \ddot{C}$ are square matrices of the coefficients $\ddot{a}_{ij}, \bar{b}_{ji}, \bar{c}_{ji}+\bar{c}_{*ji}$, calculated in the section Z = 0, of Nth order each, the matrices A and C being symmetric; A^{-1} is the matrix reciprocal to A; B' is a matrix obtained by transposing the matrix B.

The solution of system (V.19) must satisfy the boundary conditions when Z = 0 and $Z = Z_1$. In the general case, these conditions may be represented in the form

$$M_{0V} V(0) + M_{0P} P^*(0) = N_0, \tag{V.20}$$

$$M_{1V} V(Z_1) + M_{1P} P^*(Z_1) = N_1, \tag{V.21}$$

where

$M_{0V}, M_{1V}, M_{0P}, M_{1P}$ are some specified square matrices of Nth order; N_0, N_1 are specified matrix columns consisting of N elements each.

Expressions (V.20), (V.21) represent boundary conditions of general form; geometric, static, mixed, and also boundary conditions corresponding to elastic fixing of the shell. For example, if on the end Z = 0 the displacements are specified, and on the end $Z = Z_1$ the forces are specified, the matrices $\bar{M}_{0V}, \bar{M}_{1P}$ will be unit matrices, and $\bar{M}_{1V}, M_{0P}$, null matrices. Then N_0 will constitute a column of specified values $V_i(0)$, and N_1, a column of specified values $P_i^*(Z_1)$.

The algorithm of the calculation and its realization on a computer include two main steps, i.e., formation of matrices of coefficients (V.3) and right-hand sides (V.7) of system (V.19), and numerical integration of the system under boundary conditions (V.20), (V.21). The chief problem here is to minimize the total volume of initial information on the geometric parameters and shell material, and on the external action, fixing conditions and collection of the retained coordinate functions.

V.2. Algorithm of Calculation of the Coefficients and Right-Hand Sides of Differential Resolvents

The geometry of the middle surface of a conical shell of arbitrary outline is completely determined by the position of the apex and outline of the contour of the generatrix. The position of the apex will be specified by the parameter r and magnitude of the angle x_0, and the outline of the generatrix, by the coordinates $(r=0, 1, 2, \ldots)$ of the nodal points, chosen on its contour. Information on the outline of the contour of the generatrix is introduced into the working storage of the computer in blocks corresponding to the decomposition of this contour into several characteristic arcs (Fig. V.1).

We will approximate each of the characteristic arcs of the contour of the generatrix by the Lagrange interpolation polynomial

$$x_0[S(\xi)] \cong L_{xq}(\xi), \quad y_0[S(\xi)] \cong L_{yq}(\xi), \tag{V.22}$$

where ξ is some coordinate measured along this arc, and q is the number of interpolation nodes. Taking for the coordinate of the rth nodal point

$$\xi_r = r, \tag{V.23}$$

we have

$$L_{xq}(\xi) = \Pi_q(\xi) \sum_{r=0}^{q-1} \frac{x_{0r}}{(\xi - r)\Pi_q'(r)},$$

$$L_{yq}(S) = \Pi_q(\xi) \sum_{r=0}^{q-1} \frac{y_{0r}}{(\xi - r)\Pi_r'(r)}, \tag{V.24}$$

where

$$\Pi_q(\xi)=\xi(\xi-1)(\xi-2)\dots(\xi-q+1),$$
$$\Pi'_q(r)=r(r-1)(r-2)\dots 2\cdot 1\cdot(-1)(-2)\dots(r-q+1). \qquad (V.25)$$

At the nodal points of the contour of the generatrix, the derivatives

$$\left.\frac{dx_0}{d\xi}\right|_{\xi=r}=\Pi'_q(r)\sum_{\substack{p=0\\ p\neq r}}^{q-1}\frac{x_{0p}}{(r-p)\,\Pi'_q(p)}+\sum_{\substack{p=0\\ p\neq r}}^{q-1}\frac{x_{0r}}{r-p},$$
$$\left.\frac{dy_0}{d\xi}\right|_{\xi=r}=\Pi'_q(r)\sum_{\substack{p=0\\ p\neq r}}^{q-1}\frac{y_{0p}}{(r-p)\,\Pi'_q(p)}+\sum_{\substack{p=0\\ p\neq r}}^{q-1}\frac{y_{0r}}{r-p}. \qquad (V.26)$$

The derivative of the arc length of the contour of the generatrix

$$\frac{dS}{d\xi}=\sqrt{\frac{1}{\sin^2\chi_0}\left(\frac{dx_0}{d\xi}\right)^2+\left(\frac{dy_0}{d\xi}\right)^2}. \qquad (V.27)$$

Introducing into (V.27) the values of the derivatives $\frac{dx_0}{d\xi}$, $\frac{dy_0}{d\xi}$ at the nodal points, determined by expressions (V.26), we obtain the values of $\frac{dS}{d\xi}$ at these points. Now, using the interpolation polynomial, we have

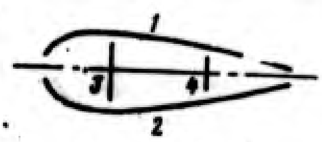

Fig. V.1.

$$\frac{dS}{d\xi}=L_{sq}(\xi)=\Pi_q(\xi)\sum_{r=0}^{q-1}\frac{\left.\frac{dS}{d\xi}\right|_{\xi=r}}{(\xi-r)\,\Pi'_q(r)}, \qquad (V.28)$$

whence, integrating (V.28) with the aid of Simpson's formula, we find

$$S(\xi)=\int_0^\xi L_{sq}(\zeta)\,d\zeta. \qquad (V.29)$$

Using (V.26), (V.27), we also obtain the values at the nodal points of the derivatives

$$x'_0(r)=\left.\frac{dx_0}{dS}\right|_{\xi=r}=\frac{\left.\frac{dx_0}{d\xi}\right|_{\xi=r}}{\left.\frac{dS}{d\xi}\right|_{\xi=r}}, \quad y'_0(r)=\left.\frac{dy_0}{dS}\right|_{\xi=r}=\frac{\left.\frac{dy_0}{d\xi}\right|_{\xi=r}}{\left.\frac{dS}{d\xi}\right|_{\xi=r}} \qquad (V.30)$$

then, approximating the derivatives $\frac{dx_0}{d\xi}, \frac{dy_0}{d\xi}$ by the Lagrange interpolation polynomial, we find

$$x_0'(\xi)=\Pi_q(\xi)\sum_{r=0}^{q-1}\frac{x_0'(r)}{(\xi-r)\,\Pi_q'(r)},$$

$$y_0'(\xi)=\Pi_q(\xi)\sum_{r=0}^{q-1}\frac{y_0'(r)}{(\xi-r)\,\Pi_q'(r)}. \tag{V.31}$$

The values of the second derivatives $x_0''(r), y_0''(r)$, then $x_0''(\xi), y_0''(\xi)$ are now determined in completely analogous fashion.

Knowing the values of $x_0, y_0, x_0', y_0', x_0'', y_0''$ at the nodal points, as well as the values of the coordinate functions p_k^*, t_k^* of the external load and temperature field at the nodal points, by interpolation we can find the values of the integrands in the expressions for the elements of matrices A, B, C and columns $\bar{R}, \bar{R}^t, \bar{P}^{*t}$ of Eqs. (V.19) at any point of the contour of the generatrix. Then, calculating the quadratures with the aid of Simpson's formula, we find the elements of the indicated matrices and columns with any degree of accuracy.

V.3. Numerical Integration of the Boundary Value Problem

The algorithm of the numerical integration is based on the reduction of the linear boundary value problem to a series of Cauchy problems.

We represent the boundary conditions for Z = 0 (V.20) in the form

$$M_{0Y}\,Y=N_0-M_{0X}\,X, \tag{V.32}$$

where X, Y are column matrices of N elements each, satisfying the conditions

$$\{X_j\}\cup\{Y_j\}=\{V_j(0)\}\cup\{P_j(0)\}; \tag{V.33}$$

M_{0X}, M_{0Y} are the corresponding square matrices of Nth order.

It is evident that if boundary conditions (V.20) are correct, the column matrix Y can always be chosen so that the matrix M_{0Y} turns out to be nondegenerate. In this case, we have from (V.32)

$$Y = M_{0Y}^{-1} N_0 - M_{0Y}^{-1} M_{0X} X. \quad \text{(V.34)}$$

Let

$$X = \sum_{i=1}^{N} X_i X_{0i}, \quad \text{(V.35)}$$

where

$$X_{0i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_{0i} = 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{(V.36)}$$

is a column matrix of N elements.

Introducing (V.35) into (V.34), we have

$$Y = Y_0 + \sum_{i=1}^{N} X_i Y_{0i}, \quad \text{(V.37)}$$

where

$$Y_0 = M_{0Y}^{-1} N_0,$$

$$Y_{0i} = -M_{0Y}^{-1} M_{0X} X_{0i}. \quad \text{(V.38)}$$

Now, the solution of the system of differential resolvents (V.19), satisfying conditions (V.20) when Z = 0, may be represented in the form

$$\begin{pmatrix} V(Z) \\ P^*(Z) \end{pmatrix} = \begin{pmatrix} V_0(Z) \\ P_0^*(Z) \end{pmatrix} + \sum_{i=1}^{n} X_i \begin{pmatrix} V_{0i}(Z) \\ P_{0i}^*(Z) \end{pmatrix}, \quad \text{(V.39)}$$

where

$\begin{pmatrix} V_0(Z) \\ P_0^*(Z) \end{pmatrix}$ is the solution of system (V.19) for the initial conditions

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ Y_0 \end{pmatrix}; \quad \text{(V.40)}$$

$\begin{pmatrix} V_N(Z) \\ P_N^{\circ}(Z) \end{pmatrix}$ is the solution of a homogeneous system of differential resolvents for initial conditions

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X_{0i} \\ Y_{0i} \end{pmatrix}. \tag{V.41}$$

Solution (V.39) contains n arbitrary elements X_i which are determined from boundary conditions (V.21) when $Z = Z_1$. For the X_i values obtained, expression (V.39) determines the solution of the system of differential resolvents (V.19), satisfying boundary conditions (V.20), (V.21). This solution was obtained by a linear combination of solutions of N+1 Cauchy problems for initial conditions (V.40), (V.41).

In practice, so as not to overburden the working storage of the computer, the matrix column X have been determined, it is desirable to find the matrix Y by using expression (V.37); then, without resorting to expression (V.39), to find the solution of the conclusive Cauchy problem by integrating Eqs. (V.19) for the initial conditions obtained.

The algorithm described, which makes it possible to reduce the solution of the boundary value problem to the solution of N+1 Cauchy problems, can be transposed with minor changes to the case of special boundary conditions, when either of the shell ends is fixed discretely against displacements, i.e., only at a certain number k of points.

For example, let special boundary conditions be given on the end $Z = 0$, and the conditions for $Z = Z_1$ be of ordinary form. In this case, the desired generalized displacements should satisfy 3k relations

$$\sum_{i=1}^{N} V_i(0)\varphi_i(S_l)=0 \quad (l=1,2,\ldots,k), \tag{V.42}$$

if the indicated points of the end $Z = 0$ are completely fixed against displacements, and k relations

$$\sum_{i=1}^{N} V_i(0)\varphi_{ii_l}(S_l)=0 \ (l=1,2,\ldots,k), \tag{V.43}$$

if the indicated points are fixed against displacements only in certain directions $t = t_1$. Also possible is the case in which only some of the indicated points are completely fixed, the remaining ones being fixed only partially. In this case, we will have m relations of the type (V.42), (V.43), $k\leq m\leq 3k$. The above-described three cases in the sense of construction of an algorithm for reducing a boundary value problem to Cauchy problems do not differ from one another, and for this reason we will not differentiate between them hereinafter, keeping in mind that in all cases $m < N$.

Fictitiously discarding k discrete supports, we will replace them by the unknown support reactions. In the general case, we will have m support reactions with respect to the number of discarded constraints. At the same time, we will have m relations of type (V.42), (V.43). We will present these relations in the form

$$M_{0Y}\,Y + M_{0X}\,X = 0, \tag{V.44}$$

where

X, Y are column matrices of N-m and m elements, respectively, and

$$\{X_j\} \cup \{Y_j\} = \{V_j(0)\}; \tag{V.45}$$

M_{0X}, M_{0Y} are rectangular matrices of order $(N-m)\times m$ and $m\times m$ corresponding to them.

Obviously, if the conditions of point fixing are specified correctly, then the column matrix Y can always be chosen so that the matrix M_{0Y} turns out to be nondegenerate. In this case, from (V.44) we have

$$Y = -M_{0Y}^{-1} M_{0X}\,X. \tag{V.46}$$

We assume that

$$X = \sum_{j=1}^{N-m} X_j X_{0j}, \tag{V.47}$$

where

$$X_{0j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_{0j} = 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{V.48}$$

is a column matrix of N-m elements.

Introducing (V.47) into (V.46), we have

$$Y = \sum_{j=1}^{N-m} X_j Y_{0j}, \tag{V.49}$$

where

$$Y_{0j} = -M_{0Y}^{-1} M_{0X} X_{0j}. \tag{V.50}$$

We now represent the column matrix of generalized forces P* in the section Z = 0 in the form

$$P^*(0) = \sum_{l=1}^{m} R_l P^*_{0l}, \tag{V.51}$$

where

R_l are the unknown support reactions;

P^*_{01} are the column matrices of the generalized forces, corresponding to the load

$$R_i = \begin{cases} 1 & (i = l), \\ 0 & (i \neq l). \end{cases} \tag{V.52}$$

Each of the matrices P^*_{0l} consists of n elements

$$P^*_{0lj} = \frac{1}{\lambda_j G} P_{0lj}, \tag{V.53}$$

where the generalized forces P_{01j} constitute the work of the end load

(V.52) on displacements $\lambda_i\varphi_i$. Therefore

$$P^{\bullet}_{0ij}=\frac{1}{G}t_i\cdot\varphi_j(S_i). \qquad (V.54)$$

where t_i is a unit vector oriented in the direction of the reaction R_i.

The solution of the system of differential resolvents (V.19), satisfying the conditions of discrete fixing, may now be represented in the form

$$\begin{pmatrix}V(Z)\\ P^{\bullet}(Z)\end{pmatrix}=\begin{pmatrix}V_0(Z)\\ P^{\bullet}_0(Z)\end{pmatrix}+\sum_{i=1}^{N-m}X_i\begin{pmatrix}V_{0i}(Z)\\ P^{\bullet}_{0i}(Z)\end{pmatrix}+\sum_{i=1}^{m}R_i\begin{pmatrix}V_{0i}(Z)\\ P^{\bullet}_{0i}(Z)\end{pmatrix}, \qquad (V.55)$$

where $\begin{pmatrix}V_0(Z)\\ P^{\bullet}_0(Z)\end{pmatrix}$ is the solution of system (V.19) for initial conditions

$$\begin{pmatrix}X\\ Y\\ P^{\bullet}_0(0)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}; \qquad (V.56)$$

$\begin{pmatrix}V_{0i}(Z)\\ P^{\bullet}_{0i}(Z)\end{pmatrix}$ is the solution of the honogeneous system of differential resolvents for initial conditions

$$\begin{pmatrix}X\\ Y\\ P^{\bullet}_0(0)\end{pmatrix}=\begin{pmatrix}X_{0i}\\ Y_{0i}\\ 0\end{pmatrix}; \qquad (V.57)$$

$\begin{pmatrix}V_{0i}(Z)\\ P^{\bullet}_{0i}(Z)\end{pmatrix}$. is the solution of the homogeneous system of differential resolvents for initial conditions

$$\begin{pmatrix}X\\ Y\\ P^{\bullet}(0)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ P^{\bullet}_{0i}\end{pmatrix}. \qquad (V.58)$$

Solution (V.55) contains N-m arbitrary quantities X_i and m unknown support reactions R_i, which are determined from boundary conditions (V.21) when $Z = Z_1$. For the X_i and R_i values obtained, expression (V.55) determines the solution of the differential resolvents which satisfies the conditions of discrete fixing when Z = 0, and boundary conditions of ordinary type when $Z = Z_1$. Practically, in order to avoid encumbering the working storage of the computer, it is desirable, as above, to determine the matrices X, $P'(0)$, find the matrix Y, using expressions (V.49); then, without resorting to expression (V.55), to integrate Eqs. (V.19) for the initial conditions obtained.

Boundary conditions (V.42), (V.43) are homogeneous and correspond to absolutely rigid point fixing of the end. Also possible are more complex cases, for example, when the end is fixed pointwise only against longitudinal displacements, and completely in its plane. In the latter case, the conditions of point fixing, in contrast to (V.42), (V.43), contain the desired support reactions. Nevertheless, the boundary value problem can always be reduced to a series of Cauchy problems as discussed above.

The algorithm described was used as the basis for a unified general program of analysis of a conical shell of arbitrary outline, to be used with the M-20 computer. In the presence of appropriate data, this program makes it possible to study the most diverse shells of aeronautical type acted on by a static load as well as a temperature field. All the results presented in Part Two of calculations involving straight and swept wings as well as low-aspect wings, both with and without consideration of rib elasticity, were obtained by using this program. Despite its generality, the program is very compact. By using only the working storage of the computer and one magnetic drum (MD), one can retain in the expansion of the elastic displacement vector up to 20 coordinate functions ($N = 20$), integrating a system of order forty. The numerical integration of the Cauchy problems was performed by using the Runge-Kutta method. It should be noted that in some cases, the process of numerical integration is associated with instability phenomena: the solution of the total Cauchy problem does not satisfy with sufficient accuracy the boundary conditions on the end $Z = Z_1$ with respect to the self-balanced components of the stressed state. This is explained by an insignificant mutual influence of the shell ends with respect to the self-balanced components, especially those with a high degree of variability. This phenomenon can be avoided by using special integration methods.* However, the weak mutual influence of the ends usually makes it possible to take a simpler approach, i.e., to study the zone of each of the ends, ignoring the incomplete fulfillment of the boundary conditions on the remote end. Moreover, having first evaluated the size of the zone of the edge effect, one can consider short shells instead of long ones. In this case, with a rational selection of the coordinate functions, the process of numerical integration is usually stable. Let us note that for local coordinate functions, the phenomenon of instability of the numerical solution takes place with a greater number of retained functions than in the case of power coordinate functions.

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*See, for example, S. N. Godunov. Numerical Solution of Boundary Value Problems for Systems of Linear Ordinary Differential Equations. UMN, 1961, issue 3, Vol. XVI.