AD-784 917

VIBRATIONS OF AXIALLY LOADED STIFFENED CYLINDRICAL SHELLS. PART I. THEORETICAL ANALYSIS

Josef Singer, et al

Technion - Israel Institute of Technology

Prepared for:

Air Force Office of Scientific Research

February 1974

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

ACCESSION for

NTIS	White Section	☐
DDC	Buff Section	☐
UNANNOUNCED		☐

JUSTIFICATION ....

....

BY ....

DISTRIBUTION/AVAILABILITY CODES

Dist.	AVAIL. and/or SPECIAL	
A		

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR - TR - 74 - 1362	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER: AD 784 917	
4. TITLE (and Subtitle): VIBRATIONS OF AXIALLY LOADED STIFFENED CYLINDRICAL SHELLS	5. TYPE OF REPORT & PERIOD COVERED: INTERIM
	6. PERFORMING ORG. REPORT NUMBER: TAE Report No 162
7. AUTHOR(s): J SINGER, A ROSEN	8. CONTRACT OR GRANT NUMBER(s): AFOSR-72-2394
9. PERFORMING ORGANIZATION NAME AND ADDRESS: TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, DEPARTMENT OF AERONAUTICAL ENGINEERING, HAIFA, ISRAEL	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 681307, 9782-01, 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS: AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA, 1400 WILSON BOULEVARD, ARLINGTON, VIRGINIA 22209	12. REPORT DATE: Feb 1974
	13. NUMBER OF PAGES: 99
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

VIBRATIONS OF SHELLS
STIFFENED CYLINCRICAL SHELLS
BUCKLING OF SHELLS

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A method of analysis and a computer program were developed for determination of the frequencies and mode shapes of the free vibrations of stiffened cylindrical shells. The shells are stiffened by closely spaced stringers and rings, have different boundary conditions, and are subjected to either a static axial load or a hydrostatic pressure. The stiffeners are considered "smeared" over the shell. The system of equations which was obtained is solved by a method known as the "exact method". The theoretical investigation includes a study of the variation in frequencies and mode shapes with increase in the static load until buckling. The results of the present analysis are compared to those of other mothods which appear in the literature, and good agreement is obtained.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AFOSR 72-2394

FEBRUARY, 1974

AFOSR - TR

VIBRATIONS OF AXIALLY LOADED STIFFENED CYLINDRICAL SHELLS

PART I - THEORETICAL ANALYSIS

by

AVIV ROSEN and JOSEF SINGER
Department of Aeronautical Engineering,
Technion - Israel Institute of Technology,
Haifa, Israel.

T.A.E. REPORT No. 162.

The research reported in this document has been sponsored by the Air Force Office of Scientific Research, through the European Office of Aerospace Research, United States Air Force under Grant 72-2394.

ia

ABSTRACT

A method of analysis and a computer program were developed for determination of the frequencies and mode shapes of the free vibrations of stiffened cylindrical shells. The shells are stiffened by closely spaced stringers and rings, have different boundary conditions, and are subjected to either a static axial load or a hydrostatic pressure. The stiffeners are considered "smeared" over the shell. The system of equations which was obtained is solved by a method known as the "exact method". The theoretical investigation includes a study of the variation in frequencies and mode shapes with increase in the static load until buckling.

The results of the present analysis are compared to those of other methods which appear in the literature, and good agreement is obtained.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONTINUED)

LIST OF SYMBOLS

A_ℓ	coefficients in displacements Eqs. (3-3) and (3-6)
$\bar{A}_\ell$	coefficients in characteristic equation (3-5)
$\bar{\bar{A}}_k$	stiffener cross-section [mm^2]
b_k	stiffener spacing (distance between centers of stiffeners) [mm]
B	coefficients in Appendix F
C_ℓ	coefficients in Eqs. (G.1)
c_k	stiffener width [mm]
D	$\frac{Eh^3}{12(1-\nu^2)}$ [kg - mm]
d_k	stiffener height [mm]
E	Young's modulus [$\frac{kg}{mm^2}$]
E_k	Young's modulus of stiffener [$\frac{kg}{mm^2}$]
$\bar{e}$	eccentricity of N_x, positive in z-direction [mm]
e_k	stiffener eccentricity (distance from shell middle surface to stiffener centroid [mm]
$f_{(x)}$	function in Appendix F
f	frequency [$\frac{1}{sec}$]
f_ℓ	coefficients in Eqs. (3-4) etc.
G	shear modulus [$\frac{kg}{mm^2}$]
G_k	shear modulus of stiffener [$\frac{kg}{mm^2}$]
g, h	coefficients in Eqs. (3-5)
g_ℓ, h_ℓ	coefficients in Eqs. (3-6)
$\bar{h}$	shell thickness [mm]
[I]	unit matrix
I_{ok}	moment of inertia of stiffener with respect to shell middle surface [mm^4)

J_k	torsional rigidity of stiffener [mm^4]
K	$\frac{R^2\bar{M}}{S}$ [Sec^2]
k	1 or 2 according to kind of stiffener
L	shell length [mm]
M_x, M_y, M_{xy}, M_{yx}	moment resultants [$\frac{kg}{mm}$]
$\bar{M}$	mass per unit area of stiffened shell [$\frac{kg\ sec^2}{mm^3}$]
m	number of half longitudinal waves
$[N]$	boundary-condition matrix of forces and moments, see Eq. (3-8)
N_x, N_y, N_{yx}	membrane force resultants [$\frac{kg}{mm}$]
$\bar{N}_x$	axial edge force per unit length of shell (positive when tension) [$\frac{kg}{mm}$]
$\bar{N}_{xy}$	tangential edge force on shell [$\frac{kg}{mm}$]
n	circumferential wave number
p	pressure of shell surface (positive in z direction) [$\frac{kg}{mm^2}$]
P	compressive load on shell [kg]
P_{cr}	axial buckling load on shell [kg]
R	radius of shell (middle surface) [mm]
S	$\frac{E\bar{h}}{(1-\nu^2)}$ [kg/mm]
T	$\frac{\bar{h}^2}{12R^2}$
t	time [sec]
U	strain energy [kg - mm]

u^*, v^*, w^*	dimensional displacements [mm]
u, v, w	non-dimensional displacements ($\frac{u^*}{R}$; $\frac{v^*}{R}$; $\frac{w^*}{R}$; respectively)
V	potential of external forces [kg-mm]
x^*, y^*, z^*	coordinates (see Fig. 1)
x, y, z	non-dimensional coordinates ($\frac{x^*}{R}$; $\frac{y^*}{R}$; $\frac{z^*}{R}$; respectively)
x_1, x_2, x_b	coordinates of shell edges
$[Y]$	geometrical boundary-condition matrix of shell
Z	value defined in Eq. (F.5)
ν	Poisson's ratio
μ_k	$\dfrac{(1-\nu^2)E_k\bar{\bar{A}}_k}{Eb_k\bar{h}}$
χ_k	$\dfrac{(1-\nu^2)E_k\bar{\bar{A}}_k e_k}{Eb_k\bar{h}R}$
η_{ok}	$\dfrac{E_k I_{ok}}{b_k D}$
ζ_k	$\dfrac{E_k\bar{\bar{A}}_k e_k R}{b_k D}$
η_{tk}	$\dfrac{G_k J_k}{b_k D}$
ω	angular velocity ($2\pi f$) [1/
λ	coefficient in Eq. (3.3)
$[\Phi], [\Psi]$	diagonal matrices, see Eq. (3.8)
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	strains

σ_x, σ_y	normal stresses in x, y directions respectively [kg/mm^2]
τ_{xy}	shear stress in x, y - plane [kg/mm^2]
ρ	mass density of shell [kg sec^2/mm]
ρ_k	mass density of stiffener [kg sec^2/mm]

INDICES

$(\ )_a$	refers to state a, see Eqs. (2.2)
$(\ )_b$	refers to state b, see Eqs. (2.2)
$(\ )_1$	refers to stringers
$(\ )_2$	refers to rings
$(\ )_{,r}$	differentiation with respect to r
$(\dot{\ })$	differentiation with respect to time
$\underline{(\ )}$	in sets b and c only in Eq. (3.1)
$\underline{\underline{(\ )}}$	in set c only in Eq. (3.1)
$(\ )_T$	to a point of the shell, not necessarily in the middle plane
$(\ )_{\ell n}$	matrix element, row ℓ, column n
$(\ )^o$	refers to case n = 0
k	1 or 2 according to kind of stiffener
ℓ	index ranging from 1 to 8
i	$\sqrt{-1}$
$(\ )''(\ )'$	see Appendix A

LIST OF TABLES

LIST OF FIGURES

FIGURE No.

1. INTRODUCTION

1.1. General

Free vibrations of stiffened shells subjected to static loads are studied in this report. The problem is of considerable engineering importance in the design of aircraft and missiles, since many of their structural components are stiffened cylindrical shells. Free vibrations are significant also as a step in the analysis of flutter, forced vibration and dynamic response.

Before discussion of the results of the present study, a detailed review of earlier studies is presented. This review is confined to free vibrations in vacuo. Vibration in a fluid, forced vibration, and dynamic response are not included.

1.2. Vibrations of Isotropic Cylindrical Shells

Vibrations of isotropic cylindrical shells are already mentioned in Rayleigh's book published in 1894 [1], but only for the extreme extensional cases, which obviously do not represent the general case. In 1927 Love [2] formulated the general equations for the latter but did not develop it further for different boundary conditions. In 1934 Flügge [3] also formulated the equations of motions for the cylindrical shell and solved the problem for the case of SS3 boundary conditions (simple support); he found that each mode of vibration is associated with three frequencies differing in the amplitude ratios of the three displacement components. The frequency of interest here is that mainly associated with radial motion of the shell, and it is by far the lowest of the three.

The work of Arnold and Warburton [4,5], first published in 1949 initiated the considerable research effort devoted universally to this field (not confined to cylindrical shells). Using the theory of shells as formulated by Love, they derived expressions for the elastic and kinetic energy of the shell and with the aid of Lagrange's principle - the appropriate vibration equations. They obtained an exact solution for the SS3 case, for which the exact mode of vibration is known. For the case of clamped B.C.'s they used Rayleigh's method, assuming the pattern of a clamped beam for the longitudinal mode. They carried out a comprehensive numerical analysis for both cases, for different shell geometries, and observed the so-called "cylindrical shell anomaly" (that in certain ranges transition to a more complicated mode results in a drop in the natural frequency), which they interpreted in energy terms. Their work included an experimental study which yielded very good agreement with theory. Reference [5] presents an interesting , though inaccurate, method for determination of the vibrations under intermediate boundary conditions between SS3 and clamping, which occur in many practical cases.

Several sets of equilibrium equations, are to be found in the theory of shells, and many investigators have discussed their relative merits. Naghdi and Bleich [6] compared Love's equations with those of Flügge and Donnell. In parallel, Baron and Bleich [7] presented a method for determination of the frequency and mode of vibration of infinite cylindrical shells, based on a correction to the pure membrane solution, and included tables for application of the procedure. Reissner [8], turned to another problem - subsequently dealt with by others - namely, whether in-plane inertia may be neglected relative to its radial counterpart. This aspect is of extreme importance, the neglect of in-plane inertia permits relatively easy solutions to otherwise very difficult equations.

Another significant early contribution was Yi-Yuan Yu's study [9], in which he derived a set of three general equilibrium equations (parallel to those obtained by Donnelll for the static case), and found an "exact solution" also used in the present study. He also dealt with the cases of SS3 (for which he compared Donnell's solution with more exact ones), clamping, and the combination of one edge clamped and the other SS3. In other studies [10, 11], he used Flügge's more exact theory and obtained a set of five equations, incorporating the effect of the transverse shear and rotational inertia. Warburton's comments [12] on Yi-Yuan Yu's first paper [9] were mainly a discussion, repeated in many other papers, of the accuracy of Donnell's equations.

In the last decade, the number of publications on the vibration of isotropic cylindrical shells increased rapidly. The main topics studied are now reviewed briefly.

First, an attempt was made to adapt the earlier results for use by engineers and designers. For example, Yamane [13] presented design curves for a wide range of geometries. Graphs and tables of this kind appeared in many other publications.

The advantage of the exact method, mentioned earlier, lies in the fact that it covers all boundary conditions and their combinations, without recourse to approximations beyond those of the theory of shells. Flügge already [3] outlined such a method in 1934, but no examples were presented because of the cumbersome numerical work involved. This difficulty was overcome to a certain extent by "logical" approximations [14] , or by a preliminary analysis (Refs. [15] to [17])which, in most cases, however, confines the method to certain ranges. The advent of the computer solved the problem of long complicated calculations, and hence in [18] the exact method is applied, without any assumptions or restrictions, to shells with different boundary conditions. With the introduction of computers, the method of finite differences was also applied extensively to the solution of the set of three differential equations.

Just as in the preceeding decade the emphasis had been on the choice of equations, the main problem in the sixties was the choice of method of solution. A comprehensive study on this subject was undertaken by Forsberg [19], who compared Flügge's and Donnell's equations on the one hand, and the exact solution, the finite differences method, and Arnold and Warburton's energy approach - on the other. In addition he also examined the effect of neglect of in-plane accelerations. In another study [20], Forsberg examined the effect of different boundary conditions with Flügge's equations and the exact method, for ten different cases which did not include, however, flexible supports. He found, for example, that the difference between SS3 and clamping is caused primarily not by the restriction on the gradient of the radial displacement, but rather by the additional restriction on the axial displacement, which is absent in the SS3 case. A similar effect was found by Sobel [21] for buckling under hydrostatic pressure. Nuckolls and Egle [22] studied a cylindrical shell with one edge SS3 and the other on springs. The spring model is very important, as it permits closer approximation of reality.

The computer also gave impetus to the finite-element approach. Although the simple isotropic cylindrical shell offers little scope for application of this method Webster [23], for example, used it for SS3 and clamped boundary conditions, with a view to comparing its accuracy with the known exact solutions.

The axisymmetric ($n = 0$) and beam-type ($n = 1$) vibrations are also of considerable engineering significance, though usually not related to the low-frequencies. Forsberg [24] studied both types and found that in certain cases they may be determined, with fair accuracy, by means of simpler equations. He showed that the axisymmetric case includes two types of vibration - one longitudinal, for which the shell may be represented by a beam model, and the other radial, corresponding to a ring model. The problem of beam-type vibration was also studied by Kornecki [25].

The above review is limited mainly to work reported in Western sources. The parallel studies published in the Russian literature, have not been included. Some important work was also published in Japan. Mizoguchi discussed the problem in two papers [26, 27] and derived equations for a shell with two types of clamping, one with restrained and one with unrestrained circumferential motion. Another team [28] studied shells clamped at one edge and free at the other, both theoretically and experimentally. One should also mention Yamaki [29] who presented curves and tables for determination of frequencies in cylindrical shells with a wide range of geometries, for eight boundary-condition variants.

The present review considers primarily methods that have widespread application, and does not dwell on special methods developed for specific problems (for example, Lin [30, 31]). The vibration of thick shells, for which the classical theories are inadequate and different approaches are called for are also excluded. However, it may be worth, in this context, to mention a comparison between exact theories (developed from theory of elasticity) and those of Flügge and Donnell, for the SS3 case, by Armenakas [32].

1.3. Vibrations of Anisotropic Cylindrical Shells

Anisotropic cylindrical shells, which include orthotropic and laminated shells (consisting of orthotropic layers), sandwich shells and heterogeneous shells, are widely used in aerospace structures. Das [33] solved the case of an orthotropic cylindrical shell by means of displacement functions, and presented examples of different boundary conditions for cylinders and cylindrical segments. White [34] analyzed a laminated cylindrical shell, consisting of alternate layers of two isotropic materials (for which he found the resultant elastic constants) and subsequently used Arnold and Warburton's method (Rayleigh's principle, with a beam type longitudinal mode) for the SS3 and clamped cases. Weingarten [35] also dealt with laminated shells and found good agreement between theory and experiment. Dong [36] used the exact method for laminated cylindrical shells comprising layers of different orthotropic materials and of different thickness.

Padovan and Koplik [37] studied cylindrical sandwich shells, and derived equations of motion which consider transverse shear in the core and face layers. These equations were solved for infinite and finite shells with SS3 boundary conditions, and the results were compared with those of the simpler theories. The authors emphasized the importance of inclusion of the face shear even in the case of very thin face layers.

Stavsky and Loewy [38] dealt with heterogeneous and orthotropic cylindrical shells, using a set of equations based on Love's approach, as well as a simpler set based on Donnell's equations. Their numerical analysis showed that heterogeneity may have a considerable effect, and also that in certain circumstances the two approaches may yield widely-differing results. Tasi [39] also studied the effect of heterogeneity on axisymmetric vibrations of cylindrical shells.

A problem faced by many authors, both in isotropic and anisotropic systems, was the accuracy of the shell theories for the dynamic case. This motivated Mirsky's comparison [40] of the results of shell theory for orthotoropic shells with those of three dimensional theory of elasticity for the axisymmetric case.

The effect of orthotropy,as well as that of boundary conditions and eccentricity on vibrations of cylindrical shells, are studied in [41], where orthotropy represents stringer and ring stiffening. Stiffened shells, which are a special case of anisotropic shells are, however, discussed in more detail below.

1.4. Vibrations of Stiffened Cylindrical Shells

Work on stiffened shells began in the fifties. Early contributions include those of Junger [42] and Baron [43],which were theoretical studies of special cases of vibrations of infinite stiffened cylindrical shells, and that of Galletly [61], which is discussed later. Hoppmann [44,45] considered the stiffened shell as an orthotropic shell, whose elastic constants he determined experimentally, and found that the "isotropic anomaly" exists also in stiffened shells. His experiments with stringer- and ring-stiffened shells showed good agreement with theory. In 1967, Penzes [46] used Hoppmann's theoretical considerations to determine the effect of boundary conditions on the vibrations of stiffened cylindrical shells. Miller [47] analyzed stringer-ring-stiffened cylindrical shells with SS3 boundary conditions using a discrete approach, but did not progress beyond the formulation of equations and gave no numerical examples. By contrast, Bleich [48] presented a practical (though inaccurate) method for a ring-stiffened cylindrical shell.

The main approaches used in the analysis of stiffened shells are now discussed in detail: One approach sets out from the concept that if the stiffeners are numerous, closely-spaced and uniformly distributed, their effect may also be regarded as uniformly distributed, and the shell becomes an equivalent orthotropic system in which the eccentricity of the stiffeners is allowed for. This approach is known accordingly as "smearing". A simple "smeared" theory, developed by Baruch and Singer [49,50], is utilized in buckling and vibration analyses of shells by many authors, including

Mikulas, McElman and Stein [51,52], who extended the equations developed in [49] and solved them numerically for the SS3 case. The numerical analysis showed that the eccentricity of the stiffeners is important in vibrations, as shown earlier in buckling. For example, external stringers were found to cause higher frequencies than internal ones. Resnik and Dugundji [41] whose study was already mentioned, also used the "smeared" approach for theoretical and experimental vibration analysis of cylindrical shells under different boundary conditions. They obtained good agreement between experiment and theory, and found (as expected) that the boundary conditions have a considerable influence on the frequencies and modes. Their study included also the effect of in-plane inertia. (In [51] and [52], only radial inertia was taken into account). Sewall and Naumann [53] included in their analysis also the effect of rotational inertia and carried out an experimental study of vibrations under different boundary conditions. They found that the effect of rotational inertia is insignificant, unlike that of the boundary conditions and eccentricity.

In all these studies, based on the "smeared" approach, the equilibrium equations were derived (variationally, by the Lagrange principle, or by Rayleigh's method) from an expression for the total potential of the stiffened shell. This method may also be used in a "discrete" approach, whereby each stiffener is treated separately in terms of location, geometry and material. This approach is suitable for shells with a small number of heavy stiffeners, differing in dimensions, material and spacing. In the discrete approach, the analysis of vibration was again preceeded by extensive work on buckling (for example [54, 55, 56]). The approach was outlined for vibrations already in [41] without detailed derivations or numerical work. It was also employed in the theoretical and experimental work of Parthan and Johns [57 - 59]. They studied mainly the SS3 case, examining the effect of different factors and comparing theories, and concluded that, even if the sitffeners are fairly widely spaced, both approaches yield close results. Parallel work on buckling of stiffened shells yielded similar conclusions. The only significant difference in the case of vibrations is in the spectrum(the curve presenting frequency versus circumferential wave number), being smooth in the "smeared" approach and wavy in the "discrete" approach. These investigators also found that while in-plane inertia cannot be neglected, rotational inertia has only a negligible effect. A similar "discrete" approach was used by Soder [60].

One of the earliest studies of the vibrations of stiffened cylindrical shells is that of Galletly [61], who dealt with ring-stiffened shells. It is outstanding also in that it takes into account inter-ring displacements in addition to those of the shell as a whole. The study considered the case of SS3 B.C's and found that inter-ring motion may sometimes have a considerable effect on the frequency and mode shape. Also in subsequent investigations [62 - 64] the equilibrium equations were derived from the total potential expression. Since the approach is "discrete", each stiffener was considered separately, and the displacements are also valid for inter-ring motion. This approach was employed by Eagle and Sewall in [62]. Reference [63] provided experimental confirmation to the theory. It showed in tests, for example, the double-resonance effect predicted for widely spaced large stiffeners. Another team [64] also found, in a theoretical and experimental study, that the discrete treatment is of importance for widely spaced and relatively large stiffeners.

The studies discussed above were based at least initially on an energy approach. There are also "discrete" approaches of a different type. One series of studies was carried out by a team headed by Wah [65]-[68]. In them, the ring-stiffened shell was subdivided into cylindrical segments and rings. The cylindrical segments were analysed by shell theory and the rings by more exact methods derived from the theory of elasticity, with an appropriate continuity requirement for forces and displacements at the point of contact and appropriate boundary conditions at the edges. The resulting set of difference equations could be solved without difficulty. In [65], a solution was presented for the symmetric case of uniformly-spaced rings, in [66] the analysis was extended to non-symmetric vibrations, and in [67] the eccentricity of the rings was also taken into consideration. Reference [68] compared the results obtained by this method, with those found by others. These studies showed that for large circumferential wave numbers, inter-ring motion becomes important. In these cases some doubt is cast on the accuracy of the "smeared" theory, which neglects this motion. Warburton and Al-Najafi [69] also used this method and extended it. After comparing its results to those of other methods, including Forsberg's [77] exact analysis to be discused later, they found the method to be very accurate.

A similar method was used by McDonald [70] for a stringer-stiffened shell with SS3 B.C's. He likewise subdivided the stiffened shell into cylindrical segments and beams, and enforced appropriate continuity requirements. The method yielded good results and again emphasized the importance of eccentricity of stiffeners. Schnell and Heinrichsbauer [71] applied a similar method to stringer-stiffened shells.

Other discrete methods are discussed in [72]-[74]. Reference [72] used the discrete-mass method and presented a solution for certain cases. In [73] the stiffeners were replaced by an equivalent system of forces which depend on the displacement at each point. In [74] Harari and Baron again dealt separately with shell segments and rings, enforcing appropriate continuity requirements. Comparison of their results with those obtained by "smeared" theory showed again that the "smeared" approach yields very good results for weak and closely-spaced stiffeners.

Special approaches were also developed for particular types of shells. Basdekas [75] presented a method, based on the variational principle, for vibrations of shells of variable thickness, stiffened with rings of variable shape (also in circumferential direction) and spacing. His method permits inclusion of lumped masses attached to the shell or rings. The effect of lumped masses on the vibrations of ring-stiffened cylindrical shells, which has obvious engineering significance, was also studied in [76].

To-day, computer programs are available which cover even the vibrations of anisotropic shells of complex geometry with arbitrarily located stiffeners. These programs are usually complicated and lengthy, and even the simplest ones apply to shells of revolution, of which the cylinder is only a particular case. The programs utilize either finite difference or finite elements.

The wealth and variety of methods necessitated assessment of their relative advantages. Forsberg's study [77] referred to above, contains such an assessment. The ring-stiffened shell is again subdivided into cylindrical segments (analyzed by Flügge's equations) and rings, with the appropriate continuity requirements and eight boundary conditions. The rings are analyzed by an exact method and affect the shell segment via the boundary conditions. Comparison of these "exact" results with those obtained by finite elements, finite differences, or other approximate methods showed fairly close agreement. The paper also surveys the influence of different stiffener-and shell parameters.

1.5. Vibrations of Cylindrical Shells under Initial Stress.

Whereas the general field of vibrations of cylindrical shells has been thoroughly investigated, the particular case of shells under initial stress has drawn much less attention, though it is of considerable practical importance. One may note, that in their outstanding contribution [5], the authors (who confined their study to an unloaded cylindrical shell) wrote: " The results are also true for a cylinder subjected to uniform stress; thus internal fluid pressure in a container will not affect the frequency", and a similar erroneous conclusion appeared also in their first paper [4]. Soon, however, the error in this statement was realized and fairly extensive work, motivated by engineering requirements, was undertaken on the vibrations of cylindrical shells under external or internal pressure. Apparently the earliest investigation in this field was that of Fung, Sechler and Kaplan [78]. They linearized Timoshenko's shell equations, assuming small displacement, studied the cylindrical shell with SS3 boundary conditions as a particular case, and discussed different approximations to the frequency equation. In order to verify the theory, tests were carried out on cylindrical shells under internal pressure. It was found that the internal pressure has a significant effect on the frequency of the different modes (the frequency rises with increase in pressure), and changes the mode with the lowest frequency. Koval [79], in a comment on the above paper, showed that one of the simpler approximate vibration equations is identical with that obtained from Donnell's equations, if the effect of radial acceleration is added. He also showed how Yu's method [9] can be applied to a clamped cylindrical shell under internal pressure. Tests on vibration of cylindrical shells were also conducted by Gottenberg [80] with internal pressure, and yielded good agreement between theory and experiment.

Recently new efforts were applied. The effects of prestress due to uniform pressure on the vibrations of spherical caps and conical shells was studied by Ebner [81] with the aid of a large computer program. Dym, in [82], studied the vibrations of orthotropic cylindrical membranes under external pressure, and in [83] extended the discussion to orthotropic cylindrical shells. His conclusion was that if the in-plane inertia is neglected and only the radial inertia taken into consideration, a very close approximation is obtained. He also found that a change in the circumferential stiffness causes considerable changes in the frequencies, while internal pressure increases the frequency appreciably and can hence offset the changes in circumferential stiffness. The effect of an attached mass on the natural vibrations of a pressurized stiffened cylindrical shell was studied in [84]. It was found that any weight addition even if it is very light evokes marked local effects which may lower the fundamental frequency considerably and offset the stiffening due to pressurization. In another study [85] Greenspon used Flügge's theory to derive a set of equations for a stiffened cylindrical shell under pressure (valid also for sandwich shells), which he solved for SS3 boundary conditions. He showed that the resulting frequency equation may be simplified by various assumptions and approximations, to yield a convenient relationship between the natural frequency of a stressed and an unstressed shell.

Besides internal and external pressure, other preloads were also studied. Koval and Cranch [86] considered cylindrical shells under an initial torque. They employed three methods of analysis: an exact solution (assuming a large circumferential wave number and thereby reducing the order of the characteristic equation from eight to four), an approximate solution by the Galerkin method, and an approximate solution using Lagrange's equations. The authors solved several cases with different boundary conditions and found good agreement between the three methods, which were also confirmed experimentally up to buckling. Both theory and experiment showed that the natural frequency decreases with increasing torque. Weingarten [87] studied the vibrations of a cylindrical shell under an initial bending moment and internal pressure, using Donnell-type equations with only radial inertia taken into account. The equations were solved by Galerkin's method and results verified experimentally. He found that the frequency decreases with increasing moment for certain modes and increases for others,apparently due to the fact that bending produces both a zone of compression (which reduces stability) and a zone of tension (which increases it). Seggelke [88] used Flügge's equations and obtained an "exact" solution for an isotropic shell under axial load or radial pressure for different boundary conditions.

In the studies cited above, linear theory was used, with the preload state considered as one of membrane stress. A more exact treatment is given in a series of studies lead by Herrmann. In [89], Herrmann and Armenakas, presented some linear theories, based on the three-dimensional theory of elasticity, for the vibrations of cylindrical shells under initial stress. These theories are more exact than the earlier ones in that they take into consideration not only a membrane state of intial stress, but also initial moments and transverse shears, as well as the change due to deformation in the direction and magnitude of the forces. In [90], the same authors applied their general bending theory to a study of the effect of uniform circumferential stress, bending moment and radial shear on the vibrations of an infinite cylindrical shell. Later Armenakas [91] extended the investigation to a finite cylindrical shell, with SS3 boundary conditions, subjected to uniform circumferential and axial stress. The results were compared with those obtained by simpler methods, and the approximate methods were found to be satisfactory in most cases. Herrmann and Shaw [92] added an experimental part to the investigation in which the vibrations of cylindrical shells under axial load and external pressure were tested. Herrmann's exact method was extended also to anisotropic shells by Baker and Herrmann [93] in a study of the vibrations of sandwich shells under an initial stress.

In another approach, Cooper [94] used Saunders non-linear equations for shells of revolution for investigating the vibrations of prestressed shells. He took into account some factors, like initial deformation due to preload, not included in the linear theory which considers only an initial membrane-state. He developed a numerical method based on finite differences and obtained a closed solution for the SS3 case. In another report [95], Cooper used linearized equations to study the effect of axial compression or tension, and internal or external pressure on the vibration of cylindrical shells with SS3 boundary conditions. Recently the vibrations of anisotropic cylindrical shells subjected to nonuniform lateral prestress were also studied by Padovan [96], with Hoppmann's equations [44], [45].

2. DIFFERENTIAL EQUATIONS, DISPLACEMENTS AND BOUNDARY CONDITIONS

2.1. Assumptions and Basic Equations

The approach presented here is based on "smearing" of the stiffeners. This approach has been discussed in detail by Singer, Baruch and Harari [49] and [97]. It was also employed by Mikulas, McElman and Stein [51], [52] and in other investigations mentioned in the previous Chapter. Its accuracy, compared with the more exact "discrete" methods, is also discussed in many studies cited above. From References [57,58,59,74] it can be seen that the "smeared" approach yields accurate results, provided the stiffeners are uniformly spaced, fairly close and not too heavy. Since here the stiffeners obey these criteria, the use of "smeared" theory is justified. The rotational inertia of the shell sections is neglected, in accordance with [53],[57-59], where it was shown that its effect is indeed negligible in stiffened shells. Use is made of Donnell's and Flügge's theories [3]. Equilibrium equations and natural boundary conditions are derived by application of the first variational principle to the total potential of the shell. The derivations according to Flügge and Donnell are given in Appendices A and B respectively. The non-linear expressions used subsequently in this Chapter also appear in Appendix B.

2.2. Equilibrium Equations and Boundary Conditions for the Vibrations of Prestressed Stiffened Cylindrical Shells.

The vibrating prestressed shell is a case of small deformation superposed on finite deformations. This involves distinction between three states of the elastic body, First, the zero state with the body free of stresses. When subjected to a given force, the body assumes state "a", a state of finite deformation under a given state of stress. When a small deformation is added to state "a" the body assumes state "b". The equilibrium equations are now obtained for state "b" by a linear approach, which is based on the assumption that in the system of coordinates there is no difference between the three states. Hence, for example, the system of cylindrical coordinates used here for all three states is exact only in the zero state. The displacement components of the shell in state "a" are u_a, v_a, w_a, and in state "b" u, v, w, where the following relations apply:

$$\begin{aligned} u &= u_a + u_b \\ v &= v_a + v_b \\ w &= w_a + w_b \end{aligned} \qquad (2.1)$$

u_b, v_b, w_b being the components of the small increment characterizing the change from state "a" to state "b". Here the "b" displacements are components of the dynamic displacements caused by the vibrations of the shell. In the same manner one can write:

$$\begin{aligned} N_x &= N_{x_a} + N_{x_b} \\ N_y &= N_{y_a} + N_{y_b} \\ N_{xy} &= N_{xy_a} + N_{xy_b} \\ M_x &= M_{x_a} + M_{x_b} \\ M_y &= M_{y_a} + M_{y_b} \\ M_{xy} &= M_{xy_a} + M_{xy_b} \\ M_{yx} &= M_{yx_a} + M_{yx_b} \end{aligned} \qquad (2.2)$$

The expression for state "a" are obtained by substitution of u_a, v_a, w_a for u, v, w, in Eqs. (B.5), of Appendix B, yielding:

$$
\begin{aligned}
N_{x_a} &= S\,[(1+\mu_1)(u_{a,x} + \tfrac{1}{2} w_{a,x}^2) + \nu(v_{a,y} - w_a + \tfrac{1}{2} w_{a,y}^2) - \chi_1 w_{a,x}] \\
N_{y_a} &= S[(1+\mu_2)(v_{a,y} - w_a + \tfrac{1}{2} w_{a,y}^2) + \nu(u_{a,y} + \tfrac{1}{2} w_{a,x}^2) - \chi_2 w_{a,yy}] \\
N_{xy_a} &= N_{yx_a} = S\,\frac{1-\nu}{2}\,[u_{a,y} + v_{a,x} + w_{a,x}w_{a,y}] \\
M_{x_a} &= -\frac{D}{R^2}\,[(1+\eta_{o1})w_{a,xx} + \nu w_{a,yy} - \zeta_1(u_{a,x} + \tfrac{1}{2} w_{a,x}^2)] \qquad (2.3) \\
M_{y_a} &= -\frac{D}{R^2}\,[(1+\eta_{o2})w_{a,yy} + \nu w_{a,xx} - \zeta_2(v_{a,y} - w_a + \tfrac{1}{2} w_{a,y}^2)] \\
M_{xy_a} &= -\frac{D}{R^2}\,[1-\nu+\eta_{t1}]w_{a,xy} \\
M_{yx_a} &= -\frac{D}{R^2}\,[1-\nu+\eta_{t2}]w_{a,xy}
\end{aligned}
$$

If Eqs. (2.1) are substituted in Eqs. (B.5) and Eqs. (2.3) are subtracted, where only linear terms in u_b, v_b, w_b are retained (in view of their smallness), one obtains:

$$
\begin{aligned}
N_{x_b} &= S[(1+\mu_1)(u_{b,x} + w_{a,x}w_{b,x}) + \nu(v_{b,y} - w_b + w_{a,y}w_{b,y}) - \chi_1 w_{b,xx}] \\
N_{y_b} &= S[(1+\mu_2)(v_{b,y} - w_b + w_{a,y}w_{b,y}) + \nu(u_{b,x} + w_{a,x}w_{b,x}) - \chi_2 w_{b,yy}] \\
N_{xy_b} &= N_{yx_b} = S\,\frac{1-\nu}{2}\,[u_{b,y} + v_{b,x} + w_{a,x}w_{b,y} + w_{a,y}w_{b,x}] \\
M_{x_b} &= -\frac{D}{R^2}[(1+\eta_{o1})w_{b,xx} + \nu w_{b,yy} - \zeta_1(u_{b,x} + w_{a,x}w_{b,x})] \qquad (2.4) \\
M_{y_b} &= -\frac{D}{R^2}[(1+\eta_{o2})w_{b,yy} + \nu w_{b,xx} - \zeta_2(v_{b,y} - w_b + w_{a,y}w_{b,y})] \\
M_{xy_b} &= -\frac{D}{R^2}\,[1-\nu+\eta_{t1}]w_{b,xy} \\
M_{yx_b} &= -\frac{D}{R^2}\,[1-\nu+\eta_{t2}]w_{b,xy}
\end{aligned}
$$

State "a" is a state of static equilibrium, in which a pressure p acts lateraly ,edge forces $\bar{N}_x$ act on the shell at a distance $z^* = \bar{e}$ from its middle surface and edge forces $\bar{N}_{xy}$ act at the middle surface. Eqs. (B-3) then yields the equilibrium equations for state "a":

$$N_{x_{a,x}} + N_{xy_{a,y}} = 0$$

$$N_{y_{a,y}} + N_{xy_{a,x}} = 0$$

$$M_{x_{a,xx}} + M_{y_{a,yy}} + M_{xy_{a,xy}} + M_{yx_{a,xy}} + N_{y_a} w_{a,xx} + N_{y_a} w_{a,yy} + 2N_{xy_a} w_{a,xy} +$$

$$+ N_{x_{a,x}} w_{a,x} + N_{y_{a,y}} w_{a,y} + N_{xy_{a,x}} w_{a,y} + N_{xy_{a,y}} w_{a,x} + Rp = 0$$

(2.5)

The boundary conditions for state "a" are obtained from Eqs. (B.4):

(i) $u_a = 0$, or $N_{x_a} - \bar{N}_x = 0$

(ii) $v_a = 0$, or $N_{xy_a} - \bar{N}_{xy} = 0$

(iii) $w_a = 0$, or $N_{x_a} w_{a,x} + N_{xy_a} w_{a,y} + M_{x_{a,x}} + M_{xy_{a,y}} + M_{yx_{a,y}} = 0$

(iv) $w_{a,x} = 0$, or $M_{x_a} - \bar{N}_x \frac{\bar{e}}{R} = 0$

(2.6)

State "b" is also an equilibrium state. The transition from "a" to "b" involves no increment to the edge forces and pressure, but only a dynamic displacement. Substituting Eqs. (2.2) into (B.3) and subtracting Eqs. (2.5), one obtains the equlibrium equations for "b".

$$N_{x_{b,x}} + N_{xy_{b,y}} = R^2\bar{M}\ddot{u}_b$$

$$N_{y_{b,y}} + N_{xy_{b,x}} = R^2\bar{M}\ddot{v}_b$$

$$M_{x_{b,xx}} + M_{y_{b,yy}} + M_{xy_{b,xy}} + M_{yx_{b,xy}} + N_{y_b} + N_{x_a} w_{b,xx} + N_{x_b} w_{a,xx} +$$
$$+ N_{y_a} w_{b,yy} + N_{y_b} w_{a,yy} + 2N_{xy_a} w_{b,xy} + 2N_{xy_b} w_{a,xy} + N_{x_{a,x}} w_{b,x} +$$
$$+ N_{x_{b,x}} w_{a,x} + N_{y_{a,y}} w_{b,y} + N_{y_{b,y}} w_{a,y} + N_{xy_{a,x}} w_{b,y} + N_{xy_{b,x}} w_{a,y} +$$
$$+ N_{xy_{a,y}} w_{b,x} + N_{xy_{b,y}} w_{a,x} = R^2\bar{M}\ddot{w}_b \qquad (2.7)$$

Substitution of Eqs. (2.2) into (B.4) yields similarly the boundary conditions for "b". Upon recalling that they are the same for both states, one may subtract Eqs. (2.6) from the result and finally obtain the boundary conditions for "b", in the following form:

(i) $u_b = 0$ or $N_{x_b} = 0$

(ii) $v_b = 0$ or $N_{xy_b} = 0$

(iii) $w_b = 0$ or $N_{x_a} w_{b,x} + N_{x_b} w_{a,x} + N_{xy_a} w_{b,y} + N_{xy_b} w_{a,y} + M_{x_{b,x}} +$
$$+ M_{xy_{b,y}} + M_{yx_{b,y}} = 0$$

(iv) $w_{b,x} = 0$ or $M_{x_b} = 0$

(2.8)

2.3. Axisymmetric Initial Static State of Stress

If state "a" is axisymmetric

$$N_{xy_a} = M_{xy_a} = M_{yx_a} = 0$$

$$v_a = 0$$

$$(\ _a)_{,y} = 0 \qquad (2.9)$$

The last condition in Eq. (2.9) signifies that in state "a" a derivative of any parameter with respect to y vanishes, i.e. the parameters are invariant in y. With conditions (2.9) Eqs. (2.3) reduce to:

$$\begin{aligned}
N_{x_a} &= S[(1+\mu_1)(u_{a,x} + \tfrac{1}{2} w_{a,x}^2) - \nu w_a - \chi_1 w_{a,xx}] \\
N_{y_a} &= S[-(1+\mu_2)w_a + \nu(u_{a,x} + \tfrac{1}{2} w_{a,x}^2)] \\
M_{x_a} &= -\frac{D}{R^2}[(1+\eta_{o1})w_{a,xx} - \zeta_1(u_{a,x} + \tfrac{1}{2} w_{a,x}^2)] \\
M_{y_a} &= -\frac{D}{R^2}[\nu w_{a,xx} + \zeta_2 w_a]
\end{aligned} \tag{2.10}$$

The equilibrium Equations (2.5) then reduce to:

$$\begin{aligned}
&N_{x_{a,x}} = 0 \\
&M_{x_{a,xx}} + N_{y_a} + N_{x_a} w_{a,xx} + N_{x_{a,x}} w_{a,x} + Rp = 0
\end{aligned} \tag{2.11}$$

and the boundary conditions Eqs. (2.6) become for an axisymmetric state "a"

$$\begin{aligned}
&\text{(i)}\quad u_a = 0 &&\text{or } N_{x_a} - \bar{N}_x = 0 \\
&\text{(ii)}\quad w_a = 0 &&\text{or } N_{x_a} w_{a,x} + M_{x_{a,x}} = 0 \\
&\text{(iii)}\quad w_{a,x} = 0 &&\text{or } M_{x_a} - \bar{N}_x \frac{\bar{e}}{R} = 0
\end{aligned} \tag{2.12}$$

Equations (2.11) further simplify to:

$$\begin{aligned}
&N_{x_a} = \text{Const} \\
&M_{x_{a,xx}} + N_{y_a} + N_{x_a} w_{a,xx} + Rp = 0
\end{aligned} \tag{2.13}$$

and Eqs. (2.4) become for the axisymmetric case:

$$N_{x_b} = S[(1+\mu_1)(u_{b,x} + w_{a,x}w_{b,x}) + \nu(v_{b,y} - w_b) - \chi_1 w_{b,xx}]$$

$$N_{y_b} = S[(1+\mu_2)(v_{b,y} - w_b) + \nu(u_{b,x} + w_{a,x}w_{b,x}) - \chi_2 w_{b,yy}]$$

$$N_{xy_b} = N_{yx_b} = S\,\frac{1-\nu}{2}\,[u_{b,y} + v_{b,x} + w_{a,x}w_{b,y}]$$

$$M_{x_b} = -\frac{D}{R^2}\,[(1+\eta_{o1})w_{b,xx} + \nu w_{b,yy} - \zeta_1(u_{b,x} + w_{a,x}w_{b,x})] \qquad (2.14)$$

$$M_{y_b} = -\frac{D}{R^2}\,[(1+\eta_{o2})w_{b,yy} + \nu w_{b,xx} - \zeta_2(v_{b,y} - w_b)]$$

$$M_{xy_b} = -\frac{D}{R^2}\,[1-\nu+\eta_{t1}]w_{b,xy}$$

$$M_{yx_b} = -\frac{D}{R^2}\,[1-\nu+\eta_{t2}]w_{b,xy}$$

The equilibrium Equations for state "b" then simplify from Eqs. (2.7) to:

$$N_{x_{b,x}} + N_{xy_{b,y}} = R^2\,\bar{M}\,\ddot{u}_b$$

$$N_{y_{b,y}} + N_{xy_{b,x}} = R^2\,\bar{M}\,\ddot{v}_b \qquad (2.15)$$

$$M_{x_{b,xx}} + M_{y_{b,yy}} + M_{xy_{b,xy}} + M_{yx_{b,xy}} + N_{y_b} + N_{x_a}w_{b,xx} + N_{x_b}w_{a,xx} +$$

$$+ N_{y_a}w_{b,yy} + N_{x_{b,x}}w_{a,x} + N_{xy_{b,y}}w_{a,x} = R^2\,\bar{M}\,\ddot{w}_b$$

and the corresponding boundary conditions to:

(i) $u_b = 0$ or $N_{x_b} = 0$

(ii) $v_b = 0$ or $N_{xy_b} = 0$

(iii) $w_b = 0$ or $N_{x_a} w_{b,x} + N_{x_b} w_{a,x} + M_{x_{b,x}} + M_{xy_{b,y}} + M_{yx_{b,y}} = 0$

(iv) $w_{b,x} = 0$ or $M_{x_b} = 0$

(2.16)

2.4. "Membrane" Initial Axisymmetric Static State of Stress

When the initial state of stress is a "membrane" axisymmetric state the following hold:

$$N_{x_a} = \bar{N}_x$$

$$N_{y_a} = -pR \qquad (2.17)$$

$$w_a = \text{const.}$$

From Eqs. (2.14) the equations for state "b" then become (Note that the subscript "b" is dropped from now on for convenience)

$$N_x = S[(1+\mu_1)u_{,x} + \nu(v_{,y} - w) - \chi_1 w_{,xx}]$$

$$N_y = S[(1+\mu_2)(v_{,y} - w) + \nu u_{,x} - \chi_2 w_{,yy}]$$

$$N_{xy} = S\frac{1-\nu}{2}[u_{,y} + v_{,x}]$$

$$M_x = -\frac{D}{R^2}[(1+\eta_{o1})w_{,xx} + \nu w_{,yy} - \zeta_1 u_{,x}]$$

$$M_y = -\frac{D}{R^2}[(1+\eta_{o2})w_{,yy} + \nu w_{,xx} - \zeta_2(v_{,y} - w)] \qquad (2.18)$$

$$M_{xy} = -\frac{D}{R^2}[1-\nu+\eta_{t1}]w_{,xy}$$

$$M_{yx} = -\frac{D}{R^2}[1-\nu+\eta_{t2}]w_{,xy}$$

The equilibrium equations are then, according to Eqs. (2.15):

$$N_{x,x} + N_{xy,y} = R^2 \bar{M} \ddot{u}$$

$$N_{y,y} + N_{xy,x} = R^2 \bar{M} \ddot{v}$$

$$M_{x,xx} + M_{y,yy} + M_{xy,xy} + M_{yx,xy} + N_y + \bar{N}_x w_{,xx} - pRw_{,yy} = R^2 \bar{M} \ddot{w} \tag{2.19}$$

and the corresponding boundary conditions according to Eqs. (2.16):

$$\begin{array}{llll} \text{(i)} & u = 0 & \text{or} & N_x = 0 \\ \text{(ii)} & v = 0 & \text{or} & N_{xy} = 0 \\ \text{(iii)} & w = 0 & \text{or} & \bar{N}_x w_{,x} + M_{x,x} + M_{xy,y} + M_{yx,y} = 0 \\ \text{(iv)} & w_{,x} = 0 & \text{or} & M_x = 0 \end{array} \tag{2.20}$$

2.5. <u>Equilibrium Equations and Boundary Conditions for Vibrations of Stiffened Cylindrical Shells Subjected to a "Membrane" Axi-symmetric Static State of Stress, with the Initial Stress State Considered According to Donnell's Theory and the Dynamic Displacements according to Flügge's Theory</u>.

Equations (2.17) are still valid. The resultant forces and moments in this case are obtained according to Eqs. (A.31) of Appendix A and are not repeated here. The equilibrium equations are obtained from Eqs. (A.29), with addition of the contribution of the initial stress according to Eqs. (2.19). Hence

$$N_{x,x} + N_{yx,y} = R^2 \bar{M} \ddot{u}$$

$$N_{y,y} - M_{y,y} + N_{xy,x} - M_{xy,x} = R^2 \bar{M} \ddot{v} \tag{2.21}$$

$$N_y + M_{x,xx} + M_{xy,xy} + M_{yx,xy} + M_{y,yy} + \bar{N}_x w_{,xx} - pRw_{,yy} = R^2 \bar{M} \ddot{w}$$

The boundary conditions are obtained from Eqs. (A.30) with addition of the contribution of the initial stress to the third equation:

$$\begin{aligned}
&\text{(i)} \quad u = 0 \quad \text{or} \quad N_x = 0 \\
&\text{(ii)} \quad v = 0 \quad \text{or} \quad N_{xy} - M_{xy} = 0 \\
&\text{(iii)} \quad w = 0 \quad \text{or} \quad \bar{N}_x w_{,x} + M_{x,x} + M_{xy,y} + M_{yx,y} = 0 \\
&\text{(iv)} \quad w_{,x} = 0 \quad \text{or} \quad M_x = 0
\end{aligned} \tag{2.22}$$

One may point out again that the above equations represent an intermediate formulation between the Donnell and Flügge theories. The contribution of the small displacements is taken according to Flügge and that of the initial stress according to Donnell. Below, the latter is also taken according to Flügge, and the complete set by Flügge's theory is obtained. The intermediate set permits assessment of the influence of more accuracy in each contribution on the final result.

2.6. Equilibrium Equations and Boundary Conditions for Vibrations of Stiffened Cylindrical Shells Subjected to a "Membrane" Axisymmetric Static State of Stress According to Flügge's Theory.

In order to derive these equations by the energy method, one would have to include non-linear terms in the expressions for the deformations in Appendix A, and follow the procedure outlined in Appendix B and Sections 2.2 - 2.4. Since the derivations here are more cumbersome, a different procedure was adopted. From Appendix A it is apparent that the equilibrium equations and boundary conditions in terms of the resultant forces and moments are identical with those presented by Flügge for an isotropic shell [3], except that the forces and moments here include also the effect of the stiffeners. Hence, Flügge's original equations may be used for stiffened shells, provided the expressions for the resultant forces and moments are modified appropriately. In the Chapter on Buckling in Flügge's book the equilibrium equations are presented with the contributions of the initial stresses. For the axisymmetric initial stresses the equilibrium equations become

$$\begin{aligned}
&N_{x,x} + N_{yx,y} - pR(u_{,yy} + w_{,x}) + \bar{N}_x u_{,xx} = R\bar{M}\ddot{u} \\
&N_{y,y} + N_{xy,x} - M_{y,y} - M_{xy,x} - pR(v_{,yy} - w_{,y}) + \bar{N}_x v_{,xx} = R\bar{M}\ddot{v} \\
&M_{y,yy} + M_{xy,xy} + M_{yx,xy} + M_{x,xx} + N_y + pR(u_{,x} - v_{,y} - w_{,yy}) + \\
&\qquad + \bar{N}_x w_{,xx} = R\bar{M}\ddot{w}
\end{aligned} \tag{2.23}$$

The boundary conditions are not formulated explicitly in Flügge's book, but are according to his approach

$$\begin{aligned}
&\text{(i)} \quad u = 0 \quad \text{or} \quad N_x = 0 \\
&\text{(ii)} \quad v = 0 \quad \text{or} \quad N_{xy} - M_{xy} + \bar{N}_x u_{,y} = 0 \\
&\text{(iii)} \quad w = 0 \quad \text{or} \quad \bar{N}_x w_{,x} + M_{x,x} + M_{xy,y} + M_{yx,y} = 0 \\
&\text{(iv)} \quad w_{,x} = 0 \quad \text{or} \quad M_x = 0
\end{aligned} \tag{2.24}$$

The appropriate formulae for the resultant forces and moments which include the effect of the stiffeners, are Eqs. (A.31) in Appendix A.

3. METHOD OF SOLUTION

Three sets of equilibrium equations and boundary conditions for an axisymmetric membrane initial state of stress have been derived. The first set (a) is based on the Donnell theory; the second set (b) considers the small displacements according to Flügge, and the contribution of the initial stresses according to Donnell; the third set (c) is based completely on the Flügge theory. Now, as a first step towards solution, the equilibrium equations and boundary conditions are rewritten in terms of the displacements,

$$(1 + \mu_1 + \frac{\bar{N}_x}{\underline{\underline{S}}})u_{,xx} + [\frac{1-\nu}{2}(1 + \underline{T}) - \frac{pR}{\underline{\underline{S}}}]u_{,yy} + \frac{1+\nu}{2}v_{,xy} - (\nu + \frac{pR}{\underline{\underline{S}}})w_{,x}$$
$$- (\chi_1 - \underline{T})w_{,xxx} - \underline{T\,\frac{1-\nu}{2}w_{,xyy}} = \frac{R^2\bar{M}}{S}\ddot{u}$$

$$\frac{1+\nu}{2}u_{,xy} + (1 + \mu_2 - \frac{pR}{\underline{\underline{S}}})v_{,yy} + [\frac{1-\nu}{2}(1 + \underline{3T}) + \underline{\eta_{t1}T} + \frac{N_x}{\underline{\underline{S}}}]v_{,xx}$$
$$- (1 + \mu_2 + \underline{\chi_2} + \underline{T\eta_{o2}} - \frac{pR}{\underline{\underline{S}}})w_{,y} - (\chi_2 + \underline{T\eta_{o2}})w_{,yyy} + \underline{T(\frac{3-\nu}{2} + \eta_{tI})w_{,xxy}} =$$
$$\frac{R^2\bar{M}}{S}\ddot{v}$$

$$(\nu + \frac{pR}{\underline{\underline{S}}})u_{,x} + (\chi_1 - \underline{T})u_{,xxx} + \underline{T\,\frac{1-\nu}{2}u_{,xyy}} + (1 + \mu_2 + \underline{\chi_2 + T\eta_{o2}} - \frac{pR}{\underline{\underline{S}}})v_{,y}$$
$$+ (\chi_2 + \underline{T\eta_{o2}})v_{,yyy} - \underline{T(\frac{3-\nu}{2} + \eta_{t1})v_{,xxy}} - (1 + \mu_2 + \underline{2\chi_2})w$$
$$- (2\chi_2 + \frac{pR}{S})w_{,yy} + \frac{\bar{N}_x}{S}w_{,xx} - \underline{T[(1 + 3\eta_{o2})w} + \underline{2w_{,yy}(1 + 2\eta_{o2})}$$
$$+ (1 + \eta_{o1})w_{,xxxx} + (2 + \eta_{t1} + \eta_{t2})w_{,xxyy} + (1 + \eta_{o2})w_{,yyyy}] = \frac{R^2\bar{M}}{S}\ddot{w}$$

(3.1)

where the terms underlined twice are those which appear in set c only, those underlined once - relate to sets b and c but not to set a.

The boundary conditions, are given with the same marking for the different sets:

$$\text{(i)}\quad u = 0 \quad \text{or} \quad (1 + \underline{\mu_1})u_{,x} + \nu(v_{,y} - w) - \chi_1 w_{,xx} + \underline{Tw_{,xx}} = 0$$

$$\text{(ii)}\quad v = 0 \quad \text{or} \quad \left(\frac{1-\nu}{2} + \frac{\bar{N}_x}{\underline{\underline{S}}}\right)u_{,y} + \frac{1-\nu}{2}(1+\underline{3T})v_{,x} + \underline{T\eta_{t1}v_{,x}} + {}$$

$$+ \underline{3T\frac{1-\nu}{2}w_{,xy}} + \underline{T\eta_{t1}w_{,xy}} = 0$$

$$\text{(iii)}\quad w = 0 \quad \text{or} \quad (\underline{1} - \zeta_1)u_{,xx} - \underline{\frac{1-\nu}{2}u_{,yy}} + \underline{\left(\frac{3-\nu}{2} + \eta_{t1}\right)v_{,xy}} + {}$$

$$+ (1+\eta_{o1})w_{,xxx} + (2-\nu+\eta_{t1}+\eta_{t2})w_{,xyy} - \frac{\bar{N}_x R^2}{D}w_{,x} = 0$$

$$\text{(iv)}\quad w_{,x} = 0 \quad \text{or} \quad (\underline{1} - \zeta_1)u_{,x} + \nu\underline{v_{,y}} + \nu w_{,yy} + (1+\eta_{o1})w_{,xx} = 0$$

(3.2)

The following displacements are now substituted in Eqs. (3.1)

$$u = Ahe^{\lambda x}\sin(ny)\,e^{i\omega t}$$
$$v = Age^{\lambda x}\cos(ny)\,e^{i\omega t} \qquad (3.3)$$
$$w = Ae^{\lambda x}\sin(ny)\,e^{i\omega t}$$

This is the usual substitution in the "exact solution" method discussed in Chapter 1.

The substitution of the displacements (3.3) in the equilibrium equations (3.1) yields:

$$\begin{bmatrix} [f_1\lambda^2 + f_2] & [f_3\lambda] & [f_4\lambda^3 + f_5\lambda] \\ [f_3\lambda] & [f_6\lambda^2 + f_7] & [f_8\lambda^2 + f_9] \\ [f_4\lambda^3 + f_5\lambda] & [f_8\lambda^2 + f_9] & [f_{10}\lambda^4 + f_{11}\lambda^2 + f_{12}] \end{bmatrix} \begin{bmatrix} Ah \\ Ag \\ A \end{bmatrix} = 0 \tag{3.4}$$

where the coefficients f defined in Eqs. (C.1) in Appendix C. The conditions for a non-trivial solution is the vanishing of the determinant of the coefficients, namely:

$$\bar{A}_5\lambda^8 + \bar{A}_4\lambda^6 + \bar{A}_3\lambda^4 + \bar{A}_2\lambda^2 + \bar{A}_1 = 0 \tag{3.5}$$

where the coefficients $\bar{A}$ are defined by Eqs. (C.2) in Appendix C. Eq. (3.5) has eight roots (determined here by the Newton-Raphson method), and hence the displacement components are:

$$\begin{aligned} u &= \sin (ny)\, e^{i\omega t} \sum_{j=1}^{8} A_j h_j e^{\lambda_j x} \\ v &= \cos (ny)\, e^{i\omega t} \sum_{j=1}^{8} A_j g_j e^{\lambda_j x} \\ w &= \sin (ny)\, e^{i\omega t} \sum_{j=1}^{8} A_j e^{\lambda_j x} \end{aligned} \tag{3.6}$$

where h_j and g_j are obtained from Eq. (3.4):

$$\begin{aligned} h_j &= \frac{-\lambda_j [\lambda_j^4 f_4 f_6 + \lambda_j^2 (f_4 f_7 + f_5 f_6 - f_3 f_8) - f_3 f_9 + f_5 f_7]}{f_1 f_6 \lambda_j^4 + \lambda_j^2 (f_2 f_6 + f_1 f_7 - f_3^2) + f_2 f_7} \\ g_j &= \frac{\lambda_j^4 (f_3 f_4 - f_1 f_8) + \lambda_j^2 (f_3 f_5 - f_2 f_8 - f_1 f_9) - f_2 f_9}{f_1 f_6 \lambda_j^4 + \lambda_j^2 (f_2 f_6 + f_1 f_7 - f_3^2) + f_2 f_7} \end{aligned} \tag{3.7}$$

Substitution of the displacement components Eqs. (3.6) in the boundary conditions Eqs. (3.2) yields:

$$\text{a)}\quad \sum_{j=1}^{8} A_j h_j e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{8} [(1+\mu_1)\lambda_j h_j - \nu n g_j - \nu - \chi_1 \lambda_j^2 + \underline{T\lambda_j^2}] e^{\lambda_j x_b} A_j = 0$$

$$\text{b)}\quad \sum_{j=1}^{8} A_j g_j e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{8} \{(\frac{1-\nu}{2} + \frac{\bar{N}_x}{\underline{\underline{S}}}) n h_j + [\frac{1-\nu}{2}(1+\underline{3T}) + T\eta_{t1}]\lambda_j g_j + \underline{T(3\frac{1-\nu}{2} + \eta_{t1}) n\lambda_j}\} e^{\lambda_j x_b} A_j = 0$$

$$\text{c)}\quad \sum_{j=1}^{8} A_j e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{8} [(\underline{1}-\zeta_1) h_j \lambda_j^2 + \underline{\frac{1-\nu}{2} n^2 h_j} - \underline{(\frac{3-\nu}{2} + \eta_{t1})\lambda_j n g_j} + \lambda_j^3 (1+\eta_{o1}) - (2-\nu+\eta_{t1}+\eta_{t_2}) n^2 \lambda_j - \frac{\bar{N}_x R^2}{D} \lambda_j] e^{\lambda_j x_b} A_j = 0$$

$$\text{d)}\quad \sum_{j=1}^{8} A_j \lambda_j e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{8} [(\underline{1}-\zeta_1)\lambda_j h_j - \underline{\nu n g_j} - \nu n^2 + (1+\eta_{o1})\lambda_j^2] e^{\lambda_j x_b} A_j = 0$$

(3.7)*

where x_b is the end coordinate, and here $x_b = \pm L/2R$ (see Fig. 1). In abbreviated form, the boundary conditions read:

$$[[\phi][N] + [\psi][Y]]\{A\} = 0 \qquad (3.8)$$

where [Y], [N], [ϕ] and [ψ] are 8x8 square matrices and {A} a column matrix of eight elements. [Y] describes the geometrical boundary conditions, and [N] the natural ones (of the forces and moments). [ϕ] and [ψ] are diagonal matrices, whose diagonals consist of ones and zeroes, that determine which of the two possible boundary conditions is valid. The elements of all the matrices are defined by Eqs. (C.3) to (C.6) in Appendix C. Clearly, always:

$$[\phi] + [\psi] = [I] \tag{3.9}$$

where $[I]$ is a unit matrix.

For Eq. (3.8) to have a non-trivial solution, the determinant of its coefficients must vanish, namely:

$$\left| [\phi][N] + [\psi][Y] \right| = 0 \tag{3.10}$$

Since the aim is to find the frequency and mode shape of the free vibrations of a shell with a given geometry, load, boundary conditions and circumferential wave number, the solution proceeds as follows:

(1) A certain initial frequency is chosen.

(2) The initial frequency is substituted in Eq. (3.5) and the roots of the equations are found.

(3) h_j and g_j are found from Eqs. (3.7).

(4) Equation (3.10) is checked for compliance.

(5) If it is not satisfied, the frequency is increased and steps (2)-(4) are repeated.

(6) If Eq. (3.10) is satisfied the natural frequency has been found.

(7) To obtain the corresponding mode shape, the relationship between the coefficients A_j are found from Eq. (3.8) with the aid of Cramer's rule, and substituted in Eqs. (3.6).

(8) Substitution of the displacements, Eqs. (3.6), in Eq. (C.7) of Appendix C then yields the resultant forces and moments.

Having found the first frequency, the others are determined in a similar manner. The calculation is carried out for a certain n (circumferential wave number). n represents therefore a known parameter in the calculations. It should be noted that determination of the frequency at which Eq. (3.10) vanishes is quite difficult, and hence in practice one looks for the frequency at which the determinant changes its sign. The required frequency is then known to be between this last frequency and the one preceeding it, and is obtained by iteration to the desired accuracy. There are, however, additional computational difficulties. The roots of Eq. (3.5) and hence the determinant in Eq. (3.10) are generally complex. This raises the question which part of the determinant, the real or the imaginary one, must be checked for change of sign. A discussion of this point is given in Appendix D.

The case of a shell under a uniform compressive load is of special interest. The total load on the shell is P. and therefore:

$$\bar{N}_x = -\frac{P}{2\pi R} \tag{3.11}$$

The variation of the natural frequencies and mode shapes with increasing P are now studied. At a certain value of P the shell buckles. The definition of buckling depends also on the point of view. Dynamically, it may be regarded as the state of vanishing frequency, at which the shell departs from the state of equilibrium and does not return to it. Accordingly, the value of P is sought for which the natural frequency is zero. The procedure is as follows:

(1) All terms containing ω in Eqs. (3.5), (3.7) and (3.10) are made to vanish.

(2) An initial load P is chosen.

(3) The initial load is substituted in Eq. (3.5) and the roots of the equation are found.

(4) h_j and g_j are found from Eq. (3.7)

(5) Eq. (3.10) is checked for compliance.

(6) If it is not satisfied, P is increased and steps (3)-(5) are repeated.

(7) If Eq. (3.10) is satisfied, the buckling load has been found.

(8) If the pattern is sought, the relationships between the coefficients A_j are found from Eq. (3.8) and substituted in Eqs. (3.6).

Obviously, the result is the buckling load for a given n, the actual buckling load being the lowest one for any possible value of n.

The axisymmetric case $(n = 0)$ mentioned in Chapter 1,differs from the others and is discussed in Appendix E. The case of pure torsion is discussed in Appendix F. The case of SS3 boundary conditions is often discussed in the literature, since its exact displacement functions are relatively simple. Here it is discussed in Appendix G.

The computations by the above procedure are generally lengthy and cumbersome. A computer program was prepared based largely on the program proposed by Cooper [98], who employed the "exact" approach for isotropic shells of revolution with axial curvature.

4. VERIFICATION OF THE METHOD OF ANALYSIS

The first step in the verification of the method developed, is a comparison with results obtained by other investigators with different approaches and procedures. In the absence of a suitable reference for comparison of the variation of the natural frequencies under intial static load (axial or external pressure) and with different boundary conditions, the comparison is carried out for no-load frequencies and buckling loads.

The no-load frequencies are compared with the results of Sewall and Naumann [53],(whose theory assumes "smeared" stiffeners, is based on the Novozhilov strain expressions and takes into account also the rotational inertia of the shell cross-section). Their solution employed the Rayleigh-Ritz principle, and represents the longitudinal vibration modes by an arbitrary number of beam modes with suitable boundary conditions. The calculations by the present method are based on set c of the equilibrium equations and B.C.'s, which for zero load is identical with set b. Figures 2-5 show the comparisons for externally stringer-stiffened cylindrical shells with 4 different sets of boundary conditions. The dimensions and properties of the shells are given in Table 1. Agreement between the frequencies computed by the two methods is generally good, bearing in mind that the curves reproduced from Ref. [53] have been scaled up, which reduces their accuracy. The largest discrepancies appear for the free-free B.C.'s (Fig. 5) and even there, the largest difference is less than 9%,being generally much smaller. Sewall and Naumann's Fig. 7 compares their results with those of Ref. [51] for the SS3 case, (in [51] "smeared" stiffener theory is used as in the present study, except that there only Donnell's theory, set a is considered). The differences there are very small, and of the same nature as in Fig. 2 of the present study. Sewall' and Naumann's results are lower at small wave numbers and higher at large ones.

Figures 6 and 7 present a comparison for externally ring-stiffened shells, and the agreement is again satisfactory.

The comparison of buckling loads is carried out for different circumferential wave numbers (see[101]) with results obtained with the BOSOR 3 program [99],[100], which is based on the method of finite-differences. The agreement for the buckling loads is also good.

The results of these comparisons increases the confidence in the soundness of the method of analysis developed here.

5. CONCLUSION

A method has been developed for analysis of the vibrations of pre-loaded stiffened cylindrical shells with different boundary condition. "Smeared" stiffener theory is employed. Three sets of equilibrium equations and boundary conditions, based on different theoretical approaches, have been obtained. The differences between results obtained from the three sets of equations have, however, been shown in [101] to be very small. A computer program, utilizing the "exact" solution, has been developed for solution of the three sets of equations and to obtain the frequencies, buckling loads and mode-shapes. This method presents a very easy solution for any combination of the natural boundary conditions. The method has been verified by comparison of the results with those obtained by other methods. The good agreement observed lends confidence to the proposed method of analysis.

Verification of the theory with experiments has also been performed. Again, good agreement is obtained. Details are given in the parallel report [101].

APPENDIX A - DERIVATION OF EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS FOR STIFFENED SHELLS ACCORDING TO FLÜGGE'S THEORY.

(A.1) INTRODUCTION

As shown in Fig. 1, two coordinate systems are used: the dimensional one (x^*, y^*, z^*) and the non-dimensional system (x, y, z), with:

$$x = \frac{x^*}{R} \; ; \quad y = \frac{y^*}{R} \; ; \; z = \frac{z^*}{R} \tag{A.1}$$

The shell boundaries are at x_1^*, x_2^*, (or x_1, x_2). The derivation proceeds as follows: The expressions for the strain energy of the shell and the stiffeners are found and a variation is performed on every term. The sum of the variations, plus that of the potential of external forces. represents the first variation of the total potential of the shell. When made to vanish, it yields the equilibrium equations and natural boundary conditions.

Now, u_T^*, v_T^*, w_T^* are the displacement components in the x^*, y^*, z^* - directions respectively. According to Flügge [3],

$$\begin{aligned} u_T^* &= u^* - z^* w^*_{,x^*} \\ v_T^* &= v^* \frac{R - z^*}{R} - z^* w^*_{,y^*} \\ w_T^* &= w^* \end{aligned} \tag{A.2}$$

where u^*, v^*, w^* are the displacements at the middle surface of the shell, where $z^* = 0$. The strains are, according to Flügge's theory:

$$\begin{aligned} \varepsilon_{xT} &= u^*_{T,x^*} \\ \varepsilon_{yT} &= \frac{R}{R - z^*} (v^*_{T,y^*} - \frac{w^*}{R}) \\ \gamma_{xyT} &= (v^*_{T,x^*} + \frac{R}{R - z^*} u^*_{T,y^*}) \end{aligned} \tag{A.3}$$

Substitution of Eqs. (A.2) in Eqs. (A.3) yields

$$\varepsilon_{xT} = u^*_{,x^*} - z^* w^*_{,x^*x^*}$$

$$\varepsilon_{yT} = v^*_{,y^*} - \frac{Rz^*}{R - z^*} w^*_{,y^*y^*} - \frac{w^*}{R - z^*} \qquad \text{(A.4)}$$

$$\gamma_{xyT} = \frac{R - z^*}{R} v^*_{,x^*} + \frac{R}{R - z^*} u^*_{,y^*} - \frac{Rz}{R - z^*} w^*_{,x^*y^*} - z^* w^*_{,x^*y^*}$$

By Hooke's law,

$$\sigma_{xT} = \frac{E}{1 - \nu^2} (\varepsilon_{xT} + \nu\varepsilon_{yT})$$

$$\sigma_{yT} = \frac{E}{1 - \nu^2} (\varepsilon_{yT} + \nu\varepsilon_{xT}) \qquad \text{(A.5)}$$

$$\tau_{xyT} = \frac{E}{2(1 + \nu)} \gamma_{xyT}$$

With these expressions for strains and stresses the strain energy of the shell can now be computed.

(A.2) Strain Energy of the Shell

To avoid lengthy calculations, the expressions for the strain energy of an isotropic cylindrical shell obtained by Bleich and Dimaggio [102] from Flugge's theory are employed after some modification of the coordinate system. The strain energy of the shell may hence be written

$$U_{sh} = \frac{S}{2} \iint [u^{*2}_{,x^*} + (v^*_{,y^*} - \frac{w^*}{R})^2 + 2\nu u^*_{,x^*}(v^*_{,y^*} - \frac{w^*}{R})$$

$$+ \frac{1 - \nu}{2}(u^*_{,y^*} + v^*_{,x^*})^2] dx^* dy^* + \frac{D}{2} \iint [w^{*2}_{,x^*x^*} + (w^*_{,y^*y^*} + \frac{w^*}{R})^2$$

$$+ \frac{1 - \nu}{2}(w^*_{,x^*y^*} - \frac{1}{R} u^*_{,y^*})^2 + \frac{3(1 - \nu)}{2}(\frac{v^*_{,x^*}}{R} + w^*_{,x^*y^*})^2 +$$

$$+ 2\nu w^*_{,x^*x^*}(w^*_{,y^*y^*} + \frac{v^*_{,y^*}}{R}) + \frac{2}{R} u^*_{,x^*} w^*_{,x^*x^*}] dx^* dy^* \qquad \text{(A.6)}$$

Rewriting the above in non-dimensional displacements and coordinates, and performing the variation, one obtains

$$\delta U_{sh} = R^2 \iint \{ [N_x'\delta u_{,x} + N_{yx}'\delta u_{,y}] + [(N_y' - M_y')\delta v_{,y} + (N_{xy}' - M_{xy}')\delta v_{,x}] -$$

$$- (N_y'\delta w + M_x'\delta w_{,xx} + M_y'\delta w_{,yy} + (M_{xy}' + M_{yx}')\delta w_{,xy}]\} \, dx \, dy \qquad \text{(A.7)}$$

where

$$N_x' = S[u_{,x} + \nu(v_{,y} - w) + Tw_{,xx}]$$

$$N_y' = S[v_{,y} - w + \nu u_{,x} - T(w_{,yy} + w)]$$

$$N_{xy}' = S\frac{1-\nu}{2}[u_{,y} + v_{,x} + T(v_{,x} + w_{,xy})]$$

$$N_{yx}' = S\frac{1-\nu}{2}[u_{,y} + v_{,x} + T(u_{,y} - w_{,xy})]$$

$$M_x' = -\frac{D}{R^2}[w_{,xx} + \nu(w_{,yy} + v_{,y}) + u_{,x}] \qquad \text{(A.8)}$$

$$M_y' = -\frac{D}{R^2}[w_{,yy} + w + \nu w_{,xx}]$$

$$M_{xy}' = -\frac{D}{R^2}(1-\nu)[w_{,xy} + v_{,x}]$$

$$M_{yx}' = -\frac{D}{R^2}(1-\nu)[w_{,xy} - \frac{1}{2}u_{,y} + \frac{1}{2}v_{,x}]$$

Integration of Eq. (A.7) by parts, bearing in mind that the integration path along y is closed and that all variables are continuous in it, yields

$$\delta U_{sh} = -R^2\iint\{[N_{x,x}' + N_{yx,y}']\delta u + [N_{y,y}' - M_{y,y}' + N_{xy,x}' - M_{xy,x}']\delta v +$$

$$+ [N_y' + M_{x,xx}' + M_{y,yy}' + M_{xy,xy}' + M_{yx,xy}']\delta w\} dx \, dy$$

$$+ R^2\int [N_x'\delta u + (N_{xy}' - M_{xy}')\delta v + (M_{x,x}' + M_{xy,y}' + M_{yx,x}')\delta w -$$

$$- M_x'\delta w_{,x}]_{x=x_1}^{x=x_2} dy$$

$$\text{(A.9)}$$

One may note that the quantities in Eqs. (A.8) are identical to the ones in Flügge's theory, except that the moments are divided by R.

(A.3) Strain Energy of the Stringers

The strain energy of the stringers is contributed by bending and torsion. The flexural strain energy is

$$U_{stb} = \frac{c_1}{2b_1} \iiint_{\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}+d_1} E_1 \varepsilon_{xT}^2 dz^* dx^* dy^* \tag{A.10}$$

and its variation

$$\delta U_{stb} = \frac{c_1}{b_1} \iiint_{\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}+d_1} E_1 \varepsilon_{xT} \delta\varepsilon_{xT} dz^* dx^* dy^* \tag{A.11}$$

Substitution of the strains, Eqs. (A.3), and integration by parts with respect to z^* yields

$$\delta U_{stb} = - R^2 \iint [N''_{x,x} \delta u + M''_{x,xx} \delta w] dx\, dy \;+$$

$$+ R^2 \int [N''_x \delta u + M''_{x,x} \delta w - M''_x \delta w_{,x}]_{x=x_1}^{x=x_2} dy \tag{A.12}$$

where

$$\begin{aligned} N''_x &= S[\mu_1 u_{,x} - \chi_1 w_{,xx}] \\ M''_x &= - \frac{D}{R^2} [\eta_{o1} w_{,xx} - \zeta_1 u_{,x}] \end{aligned} \tag{A.13}$$

The torsional strain energy is

$$U_{stt} = \frac{1}{2b_1} \iint G_1 J_1 (w^*_{,x^*y^*} + \frac{v^*_{,x^*}}{R})^2 dx^* dy^* \tag{A.14}$$

and its variation:

$$\delta U_{stt} = \frac{G_1 J_1}{b_1} \iint (w^*_{,x^*y^*} + \frac{v^*_{,x^*}}{R}) (\delta w^*_{,x^*y^*} + \frac{\delta v^*_{,x^*}}{R}) dx^* dy^* \tag{A.15}$$

Rewriting Eq. (A.15) in non-dimensional form and integrating by parts, one obtains

$$\delta U_{stt} = D\eta_{t1} \iint [-(w_{,xxy} + v_{,xx})\delta v + (w_{,xxyy} + v_{,xxy})\delta w] dxdy -$$
$$- D\eta_{t1} \int [-(w_{,xy} + v_{,x})\delta v + (w_{,xyy} + v_{,xy})\delta w]_{x=x_1}^{x=x_2} dy \qquad (A.16)$$

(A.4) <u>Strain Energy of the Rings</u>

Similarly, the flexural strain energy of the rings is

$$U_{rb} = \frac{c_2}{2b_2} \iiint_{\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}+d_2} E_2 \varepsilon_{yT}^2 \, dz^* dx^* dy^* \qquad (A.17)$$

and its variation

$$\delta U_{rb} = \frac{c_2}{b_2} \iiint_{\frac{\bar{h}}{2}}^{\frac{\bar{h}}{2}+d_2} E_2 \varepsilon_{yT} \delta\varepsilon_{yT} \, dz^* dx^* dy^* \qquad (A.18)$$

Substitution of the strains Eqs. (A.4) and integration by parts, yields

$$\delta U_{rb} = R^2 \iint [(-N''_{y,y} + M''_{y,y})\delta v - (N''_y + M''_{y,yy})\delta w] dxdy \qquad (A.19)$$

where

$$N''_y = S[\mu_2(v_{,y} - w) + \chi_2(v_{,y} - 2w - w_{,yy}) - T\eta_{o2}(2w_{,yy} + 3w - v_{,y})]$$
$$M''_y = -\frac{D}{R^2}\eta_{o2}(w_{,yy} + 2w - v_{,y}) + S\chi_2(v_{,y} - w) \qquad (A.20)$$

The torsional strain energy of the rings is

$$U_{rt} = \frac{1}{2b_2} \iint G_2 J_2 w^{*2}_{,x^* y^*} dx^* dy^* \qquad (A.21)$$

$$\delta U_{rt} = \frac{G_2 J_2}{b_2} \iint w^*_{,x^* y^*} \, \delta w^*_{,x^* y^*} dx^* dy^* \qquad (A.22)$$

Rewriting again Eq. (A.22) in non-dimensional form and integrating by parts, one obtains

$$\delta U_{rt} = D\eta_{t2}\iint w_{,xxyy}\delta w\,dxdy - D\eta_{t2}\int [w_{,xyy}\delta w]_{x=x_1}^{x=x_2} dy \tag{A.23}$$

(A.5) The Potential of the External Forces

The external forces include the edge and surface loads ($\bar{N}_x$ and $\bar{N}_{xy}$, p) and inertia forces due to motion of the shell ($-\bar{M}\ddot{u}$, $-\bar{M}\ddot{v}$, and $-\bar{M}\ddot{w}$). $\bar{N}_x$ acts at a distance $z^* = \bar{e}$ from the middle surface of the shell, and $\bar{N}_{xy}$ - at the middle surface. $\bar{M}$ is the mass per unit area given by:

$$\bar{M} = \rho\bar{h} + \frac{\rho_1\bar{\bar{A}}_1}{b_1} + \frac{\rho_2\bar{\bar{A}}_2}{b_2} \tag{A.24}$$

Bearing in mind that the rotational inertia of the shell cross sections is neglected, the variation of the potential becomes

$$\delta V = -\iint p\delta w^* dx^* dy^* - \int [\bar{N}_x \delta u_T^*\Big|_{z^*=\bar{e}}]_{x^*=x_1}^{x^*=x_2} dy^* - \int [\bar{N}_{xy}\delta v^*]_{x^*=x_1}^{x^*=x_2} dy^* +$$

$$+ \bar{M}\iint [\ddot{u}^*\delta u^* + \ddot{v}^*\delta v^* + \ddot{w}^*\delta w^*]\,dx^* dy^* \tag{A.25}$$

Substituting for $\delta u_T^*\Big|_{z^*=e^*}$ its value given by Eq. (A.2) and non-dimensionalizing one obtains

$$\delta V = R^2\iint [\bar{M}\ddot{u}\delta u + \bar{M}\ddot{v}\delta v + (\bar{M}\ddot{w} - \frac{p}{R})\delta w]\,dx\,dy +$$

$$+ R^2\int [-\bar{N}_x\delta u - \bar{N}_{xy}\delta v + \bar{N}_x\frac{\bar{e}}{R}\delta w_{,x}]_{x=x_1}^{x=x_2} dy \tag{A.26}$$

(A.6) Derivation of Equilibrium Equations and Natural Boundary Conditions

The equilibrium equations are obtained from the conditions

$$\delta V + \delta U_{sh} + \delta U_{stb} + \delta U_{stt} + \delta U_{rb} + \delta U_{rt} = 0 \tag{A.27}$$

After substitution of the appropriate values, Eq. (A.27) becomes

$$- \iint [N'_{x,x} + N'_{yx,y} + N''_{x,x} - R^2 \bar{M} \ddot{u}] \delta u \, dx \, dy$$

$$- \iint [N'_{y,y} - M'_{y,y} + N'_{xy,x} - M'_{xy,x} + N''_{y,y} - M''_{y,y} + \frac{\eta_{t1} D}{R^2} (w_{,xxy} + v_{,xx}) -$$
$$- \bar{M} R^2 \ddot{v}] \delta v \, dx \, dy$$

$$- \iint [N'_y + M'_{x,xx} + M'_{y,yy} + M'_{xy,xy} + M'_{yx,xy} + M''_{x,xx} + N''_y + M''_{y,yy} -$$
$$- \frac{D}{R^2} (\eta_{t1} + \eta_{t2}) w_{,xxyy} - \frac{D}{R^2} \eta_{t1} v_{,xxy} - \bar{M} R^2 \ddot{w} + Rp] \delta w \, dx \, dy +$$
$$+ \int [N'_x + N''_x - \bar{N}_x) \delta u]_{x=x_1}^{x=x_2} dy$$

$$+ \int \{[N'_{xy} - M'_{xy} + \frac{D}{R^2} \eta_{t1} (w_{,xy} + v_{,x}) - \bar{N}_{xy}] \delta v\}_{x=x_1}^{x=x_2} dy$$

$$+ \int \{[M'_{x,x} + M'_{xy,y} + M'_{yx,y} + M''_{x,x} - \frac{D}{R^2} \eta_{t2} w_{,xyy} - \frac{D}{R^2} \eta_{t1} (w_{,xyy} + v_{,xy})] \delta w\}_{x=x_1}^{x=x_2} dy$$

$$+ \int [(- M'_x - M''_x + \bar{N}_x \frac{\bar{e}}{R}) \delta w_{,x}]_{x=x_1}^{x=x_2} dy = 0 \qquad \text{(A.28)}$$

which in turn yields the equations of equilibrium

$$N_{x,x} + N_{yx,y} = R^2 \bar{M} \ddot{u}$$
$$N_{y,y} - M_{y,y} + N_{xy,x} - M_{xy,x} = R^2 \bar{M} \ddot{v} \qquad \text{(A.29)}$$
$$N_y + M_{x,xx} + M_{xy,xy} + M_{yx,xy} + M_{y,yy} + Rp = \bar{M} R^2 \ddot{w}$$

and the four boundary conditions, to be satisfied at $x = x_1$ and $x = x_2$

a) $u = 0$ or $N_x - \bar{N}_x = 0$

b) $v = 0$ or $N_{xy} - M_{xy} - \bar{N}_{xy} = 0$ (A.30)

c) $w = 0$ or $M_{x,x} + M_{xy,y} + M_{yx,y} = 0$

d) $w_{,x} = 0$ or $M_x - \bar{N}_x \frac{\bar{e}}{R} = 0$

where

$$N_x = N_x' + N_x'' = S[(1+\mu_1)u_{,x} + \nu(v_{,y} - w) - \chi_1 w_{,xx} + Tw_{,xx}]$$

$$N_y = N_y' + N_y'' = S\{(1+\mu_2)(v_{,y} - w) + \nu u_{,x} + \chi_2(v_{,y} - 2w - w_{,yy}) - $$
$$- T[w_{,yy} + w + \eta_{o2}(2w_{,yy} + 3w - v_{,y})]\}$$

$$N_{xy} = N_{xy}' = S\,\frac{1-\nu}{2}[u_{,y} + v_{,x} + T(v_{,x} + w_{,xy})]$$

$$N_{yx} = N_{yx}' = S\frac{1-\nu}{2}[u_{,y} + v_{,x} + T(u_{,y} - w_{,xy})]$$

$$M_x = M_x' + M_x'' = -\frac{D}{R^2}[(1 + \eta_{o1})w_{,xx} + \nu(w_{,yy} + v_{,y}) + (1 - \zeta_1)u_{,x}]$$

$$M_y = M_y' + M_y'' = -\frac{D}{R^2}[(1 + \eta_{o2})(w_{,yy} + w) + \nu w_{,xx} + (\eta_{o2} + \zeta_2)(w - v_{,y})]$$

$$M_{xy} = M_{xy}' - \frac{\eta_{t1}D}{R^2}(w_{,xy} + v_{,x}) = -\frac{D}{R^2}[(1-\nu)(w_{,xy} + v_{,x}) + \eta_{t1}(w_{,xy} + v_{,x})]$$

$$M_{yx} = M_{yx}' - \frac{D}{R^2}\,\eta_{t2}w_{,xy} = -\frac{D}{R^2}\{(1-\nu)[w_{,xy} + \frac{1}{2}(v_{,x} - u_{,y})] + \eta_{t2}w_{,xy}\}$$

(A.31)

Substitution of Eqs. (A.31) into Eqs. (A.29), yields the equilibrium equations in terms of the displacements

$$(1+\mu_1)u_{,xx} + \frac{1-\nu}{2}(1+T)u_{,yy} + \frac{1+\nu}{2}v_{,xy} - \nu w_{,x} - (\chi_1 - T)w_{,xxx} - T\,\frac{1-\nu}{2}w_{,xyy} = \frac{R^2\bar{M}}{S}\ddot{u}$$

$$\frac{1+\nu}{2}u_{,xy} + (1+\mu_2)v_{,yy} + [\frac{1-\nu}{2}(1+3T) + \eta_{t1}T]v_{,xx} - (1+\mu_2+\chi_2+T\eta_{o2})w_{,y} - $$
$$-(\chi_2+T\eta_{o2})w_{,yyy} + T(\frac{3-\nu}{2} + \eta_{t1})w_{,xxy} = \frac{R^2\bar{M}}{S}\ddot{v}$$

$$\nu u_{,x} + (\chi_1 - T)u_{,xxx} + T\,\frac{1-\nu}{2}u_{,xyy} + (1+\mu_2+\chi_2+T\eta_{o2})v_{,y} + (\chi_2+T\eta_{o2})v_{,yyy} - $$
$$- T(\frac{3-\nu}{2} + \eta_{t1})v_{,xxy} - (1+\mu_2+2\chi_2)w - 2\chi_2 w_{,yy} - T[(1+3\eta_{o2})w + 2w_{,yy}(1+2\eta_{o2}) + $$
$$+ (1+\eta_{o1})w_{,xxxx} + (2+\eta_{t1}+\eta_{t2})w_{,xxyy} + (1+\eta_{o2})w_{,yyyy}] + \frac{Rp}{S} = \frac{R^2\bar{M}}{S}\ddot{w}$$

(A.32)

and similarly the boundary conditions in terms of the displacements are

a) $u = 0$ or $S[(1+\mu_1)u_{,x} + \nu(v_{,y} - w) - \chi_1 w_{,xx} + Tw_{,xx}] = \bar{N}_x$

b) $v = 0$ or $S\frac{1-\nu}{2}[u_{,y} + v_{,x} + 3T(v_{,x} + w_{,xy})] + \frac{D}{R^2}\eta_{t1}(w_{,xy} + v_{,x}) = \bar{N}_{xy}$

c) $w = 0$ or $(1-\zeta_1)u_{,xx} - \frac{1-\nu}{2}u_{,yy} + (\frac{3-\nu}{2} + \eta_{t1})v_{,xy} + (1+\eta_{o1})w_{,xxx} +$

$$+(2 - \nu + \eta_{t1} + \eta_{t2})w_{,xyy} = 0$$

d) $w_{,x} = 0$ or $-\frac{D}{R^2}[(1-\zeta_1)u_{,x} + \nu v_{,y} + \nu w_{,yy} + (1+\eta_{o1})w_{,xx}] = \bar{N}_x \frac{\bar{e}}{R}$

(A.33)

The above equations reduce to those given by Flügge [3], when the contribution of the stiffeners vanishes.

APPENDIX B - DERIVATION OF EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS FOR STIFFENED SHELLS ACCORDING TO DONNELL'S THEORY.

The derivation and the coordinate system are similar to Appendix A, and hence only differences will be pointed out

According to Donnell

$$
\begin{aligned}
u_T^* &= u^* - z\, w^*_{,x^*} \\
v_T^* &= v^* - z\, w^*_{,y^*} \\
w_T^* &= w^*
\end{aligned}
\qquad \text{(B.1)}
$$

After incorporation of non-linear terms for the three strains, they become

$$
\begin{aligned}
\varepsilon_{xT} &= u^*_{T,x^*} + \frac{1}{2} w^{*2}_{,x^*} = u^*_{,x^*} + \frac{1}{2} w^{*2}_{,x^*} - z\, w^*_{,x^*x^*} \\
\varepsilon_{yT} &= v^*_{T,y^*} - \frac{w^*}{R} + \frac{1}{2} w^{*2}_{,y^*} = v^*_{,y^*} - \frac{w^*}{R} + \frac{1}{2} w^{*2}_{,y^*} - z\, w^*_{,y^*y^*} \\
\gamma_{xyT} &= v^*_{T,x^*} + u^*_{T,y^*} + w^*_{,x^*} w^*_{,y^*} = u^*_{,y^*} + v^*_{,x^*} + w^*_{,x^*} w^*_{,y^*} - 2z\, w^*_{,x^*y^*}
\end{aligned}
$$

(B.2)

Application of Hooke's law and the first variational principle yields the equilibrium equations

$$
\begin{aligned}
& N_{x,x} + N_{xy,y} = R^2 \bar{M}\, \ddot{u} \\
& N_{y,y} + N_{xy,x} = R^2 \bar{M}\, \ddot{v} \\
& M_{x,xx} + M_{y,yy} + M_{xy,xy} + M_{yx,xy} + N_y + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} \\
& N_{x,x} w_{,x} + N_{y,y} w_{,y} + N_{xy,x} w_{,y} + N_{xy,y} w_{,x} + Rp = R^2 \bar{M}\, \ddot{w}
\end{aligned}
$$

(B.3)

and the natural boundary conditions

$$
\begin{aligned}
&\text{a)} \quad u = 0 \quad \text{or} \quad N_x = \bar{N}_x \\
&\text{b)} \quad v = 0 \quad \text{or} \quad N_{xy} = \bar{N}_{xy} \\
&\text{c)} \quad w = 0 \quad \text{or} \quad N_x w_{,x} + N_{xy} w_{,y} + M_{x,x} + M_{xy,y} + M_{yx,y} = 0 \\
&\text{d)} \quad w_{,x} = 0 \quad \text{or} \quad M_x = \bar{N}_x \frac{\bar{e}}{R}
\end{aligned}
\tag{B.4}
$$

where

$$
\begin{aligned}
N_x &= \frac{Eh}{(1-\nu^2)} [(1+\mu_1)(u_{,x} + \tfrac{1}{2} w_{,x}^2) + \nu(v_{,y} - w + \tfrac{1}{2} w_{,y}^2) - \chi_1 w_{,xx}] \\
N_y &= \frac{Eh}{(1-\nu^2)} [(1+\mu_2)(v_{,y} - w + \tfrac{1}{2} w_{,y}^2) + \nu(u_{,x} + \tfrac{1}{2} w_{,x}^2) - \chi_2 w_{,yy}] \\
N_{xy} = N_{yx} &= \frac{Eh}{2(1+\nu)} [u_{,y} + v_{,x} + w_{,x} w_{,y}] \\
M_x &= -\frac{D}{R^2} [(1+\eta_{o1}) w_{,xx} + \nu w_{,yy} - \zeta_1 (u_{,x} + \tfrac{1}{2} w_{,x}^2)] \\
M_y &= -\frac{D}{R^2} [(1+\eta_{o2}) w_{,yy} + \nu w_{,xx} - \zeta_2 (v_{,y} - w + \tfrac{1}{2} w_{,y}^2)] \\
M_{xy} &= -\frac{D}{R^2} [(1-\nu) + \eta_{t1}] w_{,xy} \\
M_{yx} &= -\frac{D}{R^2} [(1-\nu) + \eta_{t2}] w_{,xy}
\end{aligned}
\tag{B.5}
$$

Substituting Eqs. (B.5) in Eqs. (B.3) and (B.4) one obtains the equilibrium equations and boundary conditions in terms of the displacements. To stress the differences between the two theories, the equations can be written in the following form

$$(1+\mu_1)u_{,xx} + \frac{1-\nu}{2} u_{,yy} + \boxed{T\frac{1-\nu}{2} u_{,yy}} + \frac{1+\nu}{2} v_{,xy} - \nu w_{,x} - \chi_1 w_{,xxx}$$

$$+ \boxed{T(w_{,xxx} - \frac{1-\nu}{2} w_{,xyy})} = \frac{R^2\bar{M}}{S} \ddot{u}$$

$$\frac{1+\nu}{2} u_{,xy} + (1+\mu_2)v_{,yy} + \frac{1-\nu}{2} v_{,xx} + \boxed{3\frac{1-\nu}{2} Tv_{,xx} + T\eta_{t1}v_{,xx}} - \chi_2 w_{,yyy}$$

$$- (1+\mu_2)w_{,y} \boxed{-\chi_2 w_{,y} + T[(\frac{3-\nu}{2} + \eta_{t1})w_{,xxy} - \eta_{o2}(w_{,yyy} - w_{,y})]} = \frac{R^2\bar{M}}{S} \ddot{v}$$

$$\nu u_{,x} + \chi_1 u_{,xxx} - \boxed{T(u_{,xxx} - \frac{1-\nu}{2} u_{,xyy})} + (1+\mu_2)v_{,y} + \chi_2 v_{,yyy}$$

$$+ \boxed{\chi_2 v_{,y} - T[(\frac{3-\nu}{2} + \eta_{t1})v_{,xxy} - \eta_{o2}(v_{,y} + v_{,yyy})]} - w(1+\mu_2)$$

$$\boxed{-2\chi_2 w} - 2\chi_2 w_{,yy} - T[\boxed{(1+3\eta_{o2})w + 2w_{,yy}(1+2\eta_{o2})} + (1+\eta_{o1})w_{,xxxx}$$

$$+ w_{,xxyy}(2 + \eta_{t1} + \eta_{t2}) + (1 + \eta_{o2})w_{,yyyy}] + \frac{Rp}{S} = \frac{R^2\bar{M}}{S} \ddot{w}$$

(B.6)

and the boundary conditions, similarly

(a) $u = 0$ or $S[(1+\mu_1)u_{,x} + \nu(v_{,y} - w) - \chi_1 w_{,xx} + \boxed{Tw_{,xx}}] = \bar{N}_x$

(b) $v = 0$ or $S\frac{1-\nu}{2}[u_{,y}+v_{,x}+\boxed{3T(v_{,x}+w_{,xy})}] + \boxed{\frac{D}{R^2}\eta_{t1}(w_{,xy}+v_{,x})} = \bar{N}_{xy}$

(c) $w = 0$ or $-\zeta_1 u_{,xx} + \boxed{u_{,xx} - \frac{1-\nu}{2} u_{,yy} + (\frac{3-\nu}{2} + \eta_{t1})v_{,xy}} +$

$+ (1+\eta_{o1})w_{,xxx} + (2 - \nu + \eta_{t1} + \eta_{t2})w_{,xyy} = 0$

(d) $w_{,x} = 0$ or $-\frac{D}{R^2}[-\zeta_1 u_{,x} + \boxed{u_{,x}+\nu v_{,y}} + \nu w_{,yy} + (1+\eta_{o1})w_{,xx}] = \bar{N}_x \frac{\bar{e}}{R}$

(B.7)

where the "boxed" expressions appear only in Flügge's equations and not in Donnell's. Note that the above equations contain no non-linear terms.

APPENDIX C - SUPPLEMENT TO CHAPTER 3

The coefficients in Eq. (3.4) are

$$\begin{aligned}
f_1 &= 1 + \mu_1 + \underline{\underline{\frac{\bar{N}_x}{S}}} \\
f_2 &= -n^2\left[\frac{1-\nu}{2}(1 + \underline{T}) - \underline{\underline{\frac{pR}{S}}}\right] + K\omega^2 \\
f_3 &= -n\,\frac{1+\nu}{2} \\
f_4 &= \underline{T} - \chi_1 \\
f_5 &= -\left(\nu + \underline{\underline{\frac{pR}{S}}}\right) + \underline{T\frac{1-\nu}{2}\,n^2} \\
f_6 &= -\frac{1-\nu}{2}(1 + \underline{3T}) - \underline{\eta_{t1}T} - \underline{\underline{\frac{\bar{N}_x}{S}}} \\
f_7 &= n^2\left(1 + \mu_2 - \underline{\underline{\frac{pR}{S}}}\right) - K\omega^2 \\
f_8 &= \underline{-nT\left(\frac{3-\nu}{2} + \eta_{t1}\right)} \\
f_9 &= n\left(1 + \mu_2 + \underline{\chi_2 + T\eta_{o2}} - \underline{\underline{\frac{pR}{S}}}\right) - n^3(\chi_2 + \underline{T\eta_{o2}}) \\
f_{10} &= T(1 + \eta_{o1}) \\
f_{11} &= -Tn^2(2 + \eta_{t1} + \eta_{t2}) - \frac{\bar{N}_x}{S} \\
f_{12} &= 1 + \mu_2 + \underline{2\chi_2} + \underline{T(1 + 3\eta_{o2})} - n^2\left[2\chi_2 + \frac{pR}{S} + \underline{2T(1 + 2\eta_{o2})}\right] + \\
&\quad + Tn^4(1 + \eta_{o2}) - K\omega^2
\end{aligned} \tag{C.1}$$

The terms underlined twice appear in set c only; those underlined once do not appear in a.

The coefficients in Eq. (3.5) are

$$\bar{A}_5 = f_1 f_6 f_{10} - f_6 f_4^2$$

$$\bar{A}_4 = f_2 f_6 f_{10} + f_1 f_7 f_{10} + f_1 f_6 f_{11} - f_1 f_8^2 - f_3^2 f_{10} - f_4^2 f_7 + 2 f_3 f_4 f_8 - 2 f_4 f_5 f_6$$

$$\bar{A}_3 = f_2 f_7 f_{10} + f_2 f_6 f_{11} + f_1 f_7 f_{11} + f_1 f_6 f_{12} - f_5^2 f_6 - 2 f_4 f_5 f_7 - f_2 f_8^2 -$$
$$- 2 f_1 f_8 f_9 - f_3^2 f_{11} + 2 f_3 f_5 f_8 + 2 f_3 f_4 f_9$$

$$\bar{A}_2 = f_2 f_7 f_{11} + f_2 f_6 f_{12} + f_1 f_7 f_{12} - 2 f_2 f_8 f_9 - f_1 f_9^2 - f_3^2 f_{12} + 2 f_3 f_5 f_9 - f_5^2 f_7$$

$$\bar{A}_1 = f_2 f_7 f_{12} - f_2 f_9^2$$

(C.2)

Now the elements of matrix [Y], defining the geometric boundary conditions in Eq. (3.8) are

$$Y_{1j} = h_j e^{-\frac{\lambda_j L}{2R}} \qquad Y_{5j} = h_j e^{\frac{\lambda_j L}{2R}}$$

$$Y_{2j} = g_j e^{-\frac{\lambda_j L}{2R}} \qquad Y_{6j} = g_j e^{\frac{\lambda_j L}{2R}}$$

$$Y_{3j} = e^{-\frac{\lambda_j L}{2R}} \qquad Y_{7j} = e^{\frac{\lambda_j L}{2R}}$$

$$Y_{4j} = \lambda_j e^{-\frac{\lambda_j L}{2R}} \qquad Y_{8j} = \lambda_j e^{\frac{\lambda_j L}{2R}}$$

(C.3)

and those of matrix [N], defining the natural boundary conditions, are

$$N_{1j} = [(1+\mu_1)\lambda_j h_j - \nu n g_j - \nu - \chi_1 \lambda_j^2 + T\lambda_j^2] e^{-\frac{\lambda_j L}{2R}}$$

$$N_{2j} = \{(\frac{1-\nu}{2} + \frac{\bar{N}_x}{S}) n h_j + [\frac{1-\nu}{2}(1+3T) + T\eta_{t1}]\lambda_j g_j + T[3\frac{1-\nu}{2} + \eta_{t1}] n\lambda_j\} e^{-\frac{\lambda_j L}{2R}}$$

$$N_{3j} = [(1-\zeta_1) h_j \lambda_j^2 + \frac{1-\nu}{2} n^2 h_j - (\frac{3-\nu}{2} + \eta_{t1})\lambda_j n g_j + \lambda_j^3 (1+\eta_{o1}) -$$
$$- (2-\nu+\eta_{t1}+\eta_{t2}) n^2 \lambda_j - \frac{\bar{N}_x R^2}{D}\lambda_j] e^{-\frac{\lambda_j L}{2R}}$$

$$N_{4j} = [(\underline{1} - \zeta_1)\lambda_j h_j - \underline{\nu n g_j} - \nu n^2 + (1 + \eta_{o1})\lambda_j^2]e^{-\frac{\lambda_j L}{2R}}$$

$$N_{5j} = [(1+\mu_1)\lambda_j h_j - \nu n g_j - \nu - \chi_1\lambda_j^2 + \underline{T\lambda_j^2}]e^{\frac{\lambda_j L}{2R}}$$

$$N_{6j} = \{(\frac{1-\nu}{2} + \underline{\underline{\frac{\bar{N}_x}{S}}})nh_j + [\frac{1-\nu}{2}(1+\underline{3T}) + \underline{T\eta_{t1}}]\lambda_j g_j + \underline{T[3\frac{1-\nu}{2} + \eta_{t1}]n\lambda_j}\}e^{\frac{\lambda_j L}{2R}}$$

$$N_{7j} = [(\underline{1}-\zeta_1)h_j\lambda_j^2 + \underline{\frac{1-\nu}{2}n^2 h_j} - (\frac{3-\nu}{2} + \eta_{t1})\lambda_j n g_j + \lambda_j^3(1+\eta_{o1}) -$$

$$- (2 - \nu + \eta_{t1} + \eta_{t2})n^2\lambda_j - \frac{\bar{N}_x R^2}{D}\lambda_j]e^{\frac{\lambda_j L}{2R}}$$

$$N_{8j} = [(\underline{1}-\zeta_1)\lambda_j h_j - \underline{\nu n g_j} - \nu n^2 + (1 + \eta_{o1})\lambda_j^2]e^{\frac{\lambda_j L}{2R}} \qquad (C.4)$$

In both matrices, the first four rows represent the boundary conditions at $x = - L/2R$ and the last four at $x = L/2R$. In $[Y]$, the first and fifth rows represent $u = 0$, the second and sixth $v = 0$, and the third and seventh $w = 0$, and the fourth and eighth $w_{,x} = 0$.

In $[N]$, the first and fifth rows represent $N_x = 0$, the second and sixth $N_{xy} - \underline{M_{xy}} + \underline{\underline{\bar{N}_x u_{,y}}} = 0$, and the third and seventh $\bar{N}_x w_{,x} + M_{x,x} + M_{xy,y} + M_{yx,y} = 0$, the fourth and eighth $M_x = 0$.

For the SS3 case ($N_x = 0$, $v = 0$, $w = 0$, $M_x = 0$ at the edges) the matrices $[\psi]$ and $[\phi]$ become

$$[\psi] \quad = \quad \begin{bmatrix} 0 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & 0 & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & 0 & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 0 \end{bmatrix}$$

$$[\phi] = [I] - [\psi] = \begin{bmatrix} 1 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & & & & 0 & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & 0 & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 1 \end{bmatrix} \tag{C.5}$$

For clamped edges, $u = v = w = w_{,x} = 0$, these matrices become

$$\psi = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & 0 & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & 0 & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \tag{C.6}$$

$$\phi = I - \psi = \quad [0]$$

The resultant forces and moments are obtained by substitution of Esq. (3.6) in Eqs. (A.31):

$$N_x = S \cdot \sin(ny) \cdot e^{i\omega t} \sum_{j=1}^{8} [(1+\mu_1)h_j\lambda_j - \nu n g_j - \nu - \chi_1\lambda_j^2 + \underline{T\lambda_j^2}]e^{\lambda_j x} A_j$$

$$N_y = S \cdot \sin(ny) \cdot e^{i\omega t} \sum_{j=1}^{8} [\nu\lambda_j h_j - n g_j(1 + \mu_2 + \underline{\chi_2 + T\eta_{o2}}) -$$

$$- (1 + \mu_2 \underline{-2\chi_2 - 3T\eta_{o2}} - \underline{T}) + n^2(\chi_2 + \underline{T + 2}T\eta_{o2})]e^{\lambda_j x} A_j$$

$$N_{xy} = S\,\frac{1-\nu}{2}\cos(ny) \cdot e^{i\omega t} \sum_{j=1}^{8} [n h_j + \lambda_j g_j(1 + \underline{T}) + T\lambda_j n]e^{\lambda_j x} A_j$$

$$N_{yx} = S\,\frac{1-\nu}{2}\cos(ny)\cdot e^{i\omega t}\sum_{j=1}^{8}[(1+\underline{T})nh_j + \lambda_j g_j - \underline{T\lambda_j}n]e^{\lambda_j x}A_j$$

$$M_x = -\frac{D}{R^2}\sin(ny)\cdot e^{i\omega t}\sum_{j=1}^{8}[(\underline{1} - \zeta_1)\lambda_j h_j + \underline{\nu n g_j} + (1+\eta_{o1})\lambda_j^2 - \nu n^2]e^{\lambda_j x}A_j$$

$$M_y = -\frac{D}{R^2}\sin(ny)\cdot e^{i\omega t}\sum_{j=1}^{8}[(\underline{\eta_{o2}} + \zeta_2)ng_j + \underline{1+2\eta_{o2}} + \zeta_2 -$$

$$- n^2(1+\eta_{o2}) + \nu\lambda_j^2]e^{\lambda_j x}A_j$$

$$M_{xy} = -\frac{D}{R^2}\cos(ny)\cdot e^{i\omega t}\sum_{j=1}^{8}[(1-\nu+\eta_{t1})(\underline{\lambda_j g_j} + n\lambda_j)]e^{\lambda_j x}A_j$$

$$M_{yx} = -\frac{D}{R^2}\cos(ny)\cdot e^{i\omega t}\sum_{j=1}^{8}[\underline{\frac{1-\nu}{2}nh_j} + \underline{\frac{1-\nu}{2}\lambda_j g_j} + (1-\nu+\eta_{t2})n\lambda_j]e^{\lambda_j x}A_j$$

(C.7)

APPENDIX D - DISCUSSION OF DETERMINANT (3.10)

Before consideration of the determinant itself, the roots of Eq. (3.5) are checked. One observes that the characteristic equation yields four solutions for λ^2, which in turn, on extraction of the square root, yield the eight roots of the equation. It must be borne in mind that the conjugate of a complex root is also a root of the equation. Hence the following sets of combinations are possible:

1. 4 full complex
   4 pure real
2. 4 full complex
   2 pure real
   2 pure imaginary
3. 4 full complex
   4 pure imaginary
4. 8 full complex
5. 8 pure real
6. 2 pure imaginary
   6 pure real
7. 4 pure imaginary
   4 pure real
8. 6 pure imaginary
   2 pure real
9. 8 pure imaginary

Now, as can be seen from Eqs. (C.3) and (C.4), the columns of the determinant consist of functions of λ_j, the function being the same throughout each row. Thus two columns representing a pair of conjugate roots are themselves a conjugate pair. A determinant changes sign when two columns are interchanged. On interchanging all columns with their conjugates, the new value of the determinant is the old one multiplied by $(-1)^k$ (2k being the number of full-complex and pure-imaginary roots of the characteristic equation), or in effect the conjugate of the latter (as is the case when the variable in a complex function is replaced by its own conjugate). For the above to be satisfied, the determinant must be real for k = 0,2,4 and imaginary for k = 1,3. Of the nine combinations listed, (1), (3), (4), (5), (7) and (9) correspond to k = 2,4, and (2), (6) and (8) to k = 1,3. In other words, the real part of the determinant must be considered in the first case, and its imaginary part - in the second case. In practice, however, the value obtained is not pure real or pure imaginary, due to the cumulative error involved in the computation.

APPENDIX E - AXISYMMETRIC CASE

In the axisymmetric case $v = 0$ and all parameters are invariant in y. Hence the equilibrium equations Eqs. (3.1) reduce to:

$$\left(1 + \mu_1 + \underline{\underline{\frac{\bar{N}_x}{S}}}\right)u^o_{,xx} - \left(\nu + \underline{\underline{\frac{pR}{S}}}\right)w^o_{,x} - (\chi_1 - \underline{T})w^o_{,xxx} = \frac{R^2\bar{M}}{S}\ddot{u}^o$$

$$\left(\nu + \underline{\underline{\frac{pR}{S}}}\right)u^o_{,x} + (\chi_1 - \underline{T})u^o_{,xxx} - (1+\mu_2+\underline{2\chi_2})w^o + \frac{\bar{N}_x}{S}w^o_{,xx} - \qquad \text{(E.1)}$$

$$- T[\underline{(1 + 3\eta_{o2})w^o} + (1 + \eta_{o1})w^o_{,xxxx}] = \frac{R^2\bar{M}}{S}\ddot{w}^o$$

and the boundary conditions to:

a) $u^o = 0$ or $(1+\mu_1)u^o_{,x} - \nu w^o - \chi_1 w^o_{,xx} + \underline{Tw^o_{,xx}} = 0$

b) $w^o = 0$ or $(\underline{1}-\zeta_1)u^o_{,xx} + (1+\eta_{o1})w^o_{,xxx} - \dfrac{\bar{N}_x R^2}{D} w^o_{,x} = 0$ (E.2)

c) $w^o_{,x} = 0$ or $(\underline{1}-\zeta_1)u^o_{,x} + (1+\eta_{o1})w^o_{,xx} = 0$

Substituting in Eqs. (E.1) the displacements

$$u^o = A^o h^o e^{\lambda x} e^{i\omega t}$$
$$w^o = A^o e^{\lambda x} e^{i\omega t} \qquad \text{(E.3)}$$

one obtains

$$\begin{bmatrix} [f_1\lambda^2 + f_2] & [f_3\lambda^3 + f_4\lambda] \\ [f_3\lambda^3 + f_4\lambda] & [f_5\lambda^4 + f_6\lambda^2 + f_7] \end{bmatrix} \begin{bmatrix} A^o h^o \\ A^o \end{bmatrix} = 0 \qquad \text{(E.4)}$$

where

$$f_1 = 1 + \mu_1 + \underline{\underline{\frac{\bar{N}_x}{S}}} \qquad f_5 = T(1+\eta_{o1})$$

$$f_2 = K\omega^2 \qquad f_6 = -\frac{\bar{N}_x}{S} \qquad \text{(E.5)}$$

$$f_3 = \underline{T} - \chi_1 \qquad f_7 = 1+\mu_2+\underline{2\chi_2} + \underline{T(1+3\eta_{o2})}-K\omega^2$$

$$f_4 = -\left(\nu + \underline{\underline{\frac{pR}{S}}}\right)$$

For a non-trivial solution the determinant of the coefficients in Eq. (E.4) must vanish. Hence

$$\bar{A}_4^o\lambda^6 + \bar{A}_3^o\lambda^4 + \bar{A}_2^o\lambda^2 + \bar{A}_1^o = 0 \tag{E.6}$$

where

$$\begin{aligned}
\bar{A}_4^o &= f_1f_5 - f_3^2 \\
\bar{A}_3^o &= f_2f_5 + f_1f_6 - 2f_3f_4 \\
\bar{A}_2^o &= f_2f_6 \\
\bar{A}_1^o &= f_2f_7
\end{aligned} \tag{E.7}$$

From Eq. (E.4) one also obtains

$$h_j^o = \frac{-\lambda(\lambda^2 f_3 + f_4)}{f_1\lambda^2 + f_2} \tag{E.8}$$

Eq. (E.6) yields six roots and hence

$$\begin{aligned}
u^o &= e^{i\omega t} \sum_{j=1}^{6} e^{\lambda_j x} h_j^o A_j^o \\
w^o &= e^{i\omega t} \sum_{j=1}^{6} e^{\lambda_j x} A_j^o
\end{aligned} \tag{E.9}$$

Substitution of the displacement components Eqs. (E.9) in Eqs. (E.2) yields:

$$\sum_{j=1}^{6} A_j^o h_j^o e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{6} [(1+\mu_1)\lambda_j h_j^o - \nu + (\underline{T}-\chi_1)\lambda_j^2] e^{\lambda_j x_b} A_j^o = 0$$

$$\sum_{j=1}^{6} A_j^o e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{6} [(\underline{1}-\zeta_1)\lambda_j^2 h_j^o + (1+\eta_{o1})\lambda_j^3 - \frac{\bar{N}_x R^2}{D}\lambda_j] e^{\lambda_j x_b} A_j^o = 0$$

$$\sum_{j=1}^{6} A_j^o \lambda_j e^{\lambda_j x_b} = 0 \quad \text{or} \quad \sum_{j=1}^{6} [(\underline{1}-\zeta_1)\lambda_j h_j^o + (1+\eta_{o1})\lambda_j^2] e^{\lambda_j x_b} A_j^o = 0$$

(E.10)

The boundary conditions may be formulated as follows

$$[[\phi^o][N^o] + [\psi^o][Y^o]]\{A^o\} = 0 \tag{E.11}$$

where $[Y^o],[\psi^o],[N^o],[\phi^o]$ are 6×6 square matrices.

$[\phi]$ and $[\psi]$ are diagonal matrices whose diagonal consists of ones and zeroes, and they determine which of the two alternative boundary conditions is satisfied. Obviously,

$$[\phi] + [\psi] = [I] \tag{E.12}$$

$[Y]$ is again the matrix of the geometric boundary conditions, with elements

$$\begin{aligned}
Y^o_{1j} &= h^o_j e^{-\frac{\lambda_j L}{2R}} & \qquad Y^o_{4j} &= h^o_j e^{\frac{\lambda_j L}{2R}} \\
Y^o_{2j} &= e^{-\frac{\lambda_j L}{2R}} & \qquad Y^o_{5j} &= e^{\frac{\lambda_j L}{2R}} \\
Y_{3j} &= \lambda_j e^{-\frac{\lambda_j L}{2R}} & \qquad Y^o_{6j} &= \lambda_j e^{\frac{\lambda_j L}{2R}}
\end{aligned} \tag{E.13}$$

whose first three rows represent the boundary conditions at $x = -L/2R$ and the last three - those at $x = L/2R$; the first and fourth rows give $u = 0$, the second and fifth $w = 0$, and third and sixth $w_{,x} = 0$.

$[N]$ is the matrix of the boundary conditions of the resultant forces and moments, with elements

$$N^o_{1j} = [(1 + \mu_1)\lambda_j h^o_j - \nu - \chi_1\lambda_j^2 + \underline{T\lambda_j^2}]e^{-\frac{\lambda_j L}{2R}}$$

$$N^o_{2j} = [(\underline{1} - \zeta_1)\lambda_j^2 h^o_j + (1 + \eta_{o1})\lambda_j^3 - \frac{\bar{N}_x R^2}{D}\lambda_j]e^{-\frac{\lambda_j L}{2R}}$$

$$N^o_{3j} = [(\underline{1} - \zeta_1)\lambda_j h^o_j + (1 + \eta_{o1})\lambda_j^2]e^{-\frac{\lambda_j L}{2R}}$$

$$N^o_{4j} = [(1 + \mu_1)\lambda_j h^o_j - \nu - \chi_1\lambda_j^2 + \underline{T\lambda_j^2}]e^{\frac{\lambda_j L}{2R}}$$

$$N^o_{5j} = [(\underline{1} - \zeta_1)\lambda_j^2 h_j^o + (1 + \eta_{o1})\lambda_j^3 - \frac{\bar{N}_x R^2}{D}\lambda_j]e^{\frac{\lambda_j L}{2R}}$$

$$N^o_{6j} = [(\underline{1} - \zeta_1)\lambda_j h_j^o + (1 + \eta_{o1})\lambda_j^2]e^{\frac{\lambda_j L}{2R}}$$

$$\text{(E.14)}$$

where as in [Y], the first three rows represent the boundary conditions at $x = -L/2R$ and the last three - those at $x = L/2R$, the first and fourth rows gives $N_x = 0$, the second and fifth $\bar{N}_x w_{,x} + M_{x,x} = 0$ and the third and sixth $M_x = 0$.

For simple supports SS3,

$$[\psi^o] = \begin{bmatrix} 0 & & & & & \\ & 1 & & & 0 & \\ & & 0 & & & \\ & & & 0 & & \\ & 0 & & & 1 & \\ & & & & & 0 \end{bmatrix} ; \quad [\phi^o] = [I] - [\psi^o] = \begin{bmatrix} 1 & & & & & \\ & 0 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 0 & \\ & & & & & 1 \end{bmatrix}$$

$$\text{(E.15)}$$

and for clamped edges

$$[\psi^o] = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix} ; \quad [\phi^o] = [I] - [\psi^o] = [0]$$

For Eq. (E.11) to have a non-trivial solution, the determinant of the coefficients must vanish

$$\left| [\phi^{o}][N^{o}] + [\psi^{o}][Y^{o}] \right| = 0 \tag{E.16}$$

The vibration and buckling problems are then solved in the same manner as in Chapter 3.

APPENDIX F - SHELL IN PURE TORSION

In the case of pure torsion

$$v = v(x,t) \; ; \quad u = w = 0 \tag{F.1}$$

i.e. the cross-sections remain circular.

There remains therefore only one equilibrium equation

$$\left[\frac{1-\nu}{2}(1 + \underline{3T}) + \underline{\eta_{t1}T} + \underline{\underline{\frac{\bar{N}_x}{S}}}\right] v_{,xx} = \frac{R^2\bar{M}}{S}\ddot{v} \tag{F.2}$$

Assuming a solution in the form

$$v = f(x)\,e^{i\omega t} \tag{F.3}$$

and substituting Eq. (F.3) in Eq. (F.2), one obtains

$$f(x)_{,xx} + \omega^2 z^2 f(x) = 0 \tag{F.4}$$

where

$$z^2 = \frac{R^2\bar{M}}{S\left[\frac{1-\nu}{2}(1+\underline{3T}) + \underline{\eta_{t1}T} + \underline{\underline{\frac{\bar{N}_x}{S}}}\right]} \tag{F.5}$$

From Eqs. (3.2) for the boundary conditions one has here

$$f(x)_{,x} = 0 \quad \text{or} \quad f(x) = 0 \tag{F.6}$$

The boundaries are $x = \pm L/2R$, and there are three cases:

a) $f(x) = 0$ for both edges, in which case

$$f(x) = B\cos\frac{\pi x R}{L} \tag{F.7}$$

and the frequency is

$$\omega^2 = \frac{\left(\frac{\pi R}{L}\right)^2}{z^2} \tag{F.8}$$

(b) $f(x)_{,x} = 0$ for both edges, in which case

$$f(x) = B \sin \frac{\pi R x}{L} \tag{F.9}$$

and the frequency is the same as above.

(c) $f(x) = 0$ for one edge and $f(x)_{,x} = 0$ for the other, in which case

$$f(x) = B\sin\left(\frac{\pi R x}{2L} + \frac{\pi}{4}\right) \tag{F.10}$$

and the frequency is

$$\omega^2 = \frac{\left(\frac{\pi R}{2L}\right)^2}{Z^2} \tag{F.11}$$

APPENDIX G - THE CASE OF SS3 BOUNDARY CONDITIONS

If the boundary conditions are SS3 for both edges, the displacement components are given by:

$$
\begin{aligned}
u &= C_1 \cos(ny) \cos m\pi\left(\frac{Rx}{L} + \frac{1}{2}\right) e^{i\omega t} \\
v &= C_2 \sin(ny) \sin m\pi\left(\frac{Rx}{L} + \frac{1}{2}\right) e^{i\omega t} \\
w &= C_3 \cos(ny) \sin m\pi\left(\frac{Rx}{L} + \frac{1}{2}\right) e^{i\omega t}
\end{aligned}
\tag{G.1}
$$

Substitution of Eqs. (G.1) in Eqs. (3.1) yields

$$
\begin{bmatrix}
(f_1 + K\omega^2) & (f_2) & (f_3) \\
(f_2) & (f_4 + K\omega^2) & (f_5) \\
(f_3) & (f_5) & (f_6 + K\omega^2)
\end{bmatrix}
\cdot
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0
\tag{G.2}
$$

where

$$
\begin{aligned}
f_1 &= -\left(1+\mu_1 + \underline{\underline{\frac{\bar{N}_x}{S}}}\right)\left(\frac{m\pi R}{L}\right)^2 - n^2\left[\frac{1-\nu}{2}(1 + \underline{T}) - \underline{\underline{\frac{pR}{S}}}\right] \\
f_2 &= \frac{1+\nu}{2}\, n \left(\frac{m\pi R}{L}\right) \\
f_3 &= -\left(\nu + \underline{\underline{\frac{pR}{S}}}\right)\left(\frac{m\pi R}{L}\right) + (\chi_1 - \underline{T})\left(\frac{m\pi R}{L}\right)^3 + \underline{Tn^2\,\frac{1-\nu}{2}\left(\frac{m\pi R}{L}\right)} \\
f_4 &= -n^2\left(1 + \mu_2 - \underline{\underline{\frac{pR}{S}}}\right) - \left(\frac{m\pi R}{L}\right)^2\left[\frac{1-\nu}{2}(1 + \underline{3T}) + \underline{\eta_{t1}T} + \underline{\underline{\frac{\bar{N}_x}{S}}}\right] \\
f_5 &= n\left(1 + \mu_2 + \underline{\chi_2} + \underline{T\eta_{o2}} - \underline{\underline{\frac{pR}{S}}}\right) - n^3(\chi_2 + \underline{T\eta_{o2}}) - \underline{Tn\left(\frac{m\pi R}{L}\right)^2\left(\frac{3-\nu}{2} + \eta_{t1}\right)} \\
f_6 &= -(1 + \mu_2 + \underline{2\chi_2}) + n^2\left(2\chi_2 + \frac{pR}{S}\right) - \frac{\bar{N}_x}{S}\left(\frac{m\pi R}{L}\right)^2 - T\left[(1 + 3\eta_{o2}) - 2n^2(1 + 2\eta_{o2})\right. \\
&\quad \left. + (1 + \eta_{o1})\left(\frac{m\pi R}{L}\right)^2 + (2 + \eta_{t1} + \eta_{t2})\left(\frac{m\pi R}{L}\right)^2 n^2 + (1 + \eta_{o2})n^4\right]
\end{aligned}
\tag{G.3}
$$

For a non-trivial solution, the determinant of the coefficients must vanish. Hence

$$K^3\omega^6 + \bar{A}_3K^2\omega^4 + \bar{A}_2K\omega^2 + \bar{A}_1 = 0 \tag{G.4}$$

where

$$\begin{aligned} \bar{A}_3 &= f_1 + f_4 + f_6 \\ \bar{A}_2 &= f_1f_6 + f_1f_4 + f_4f_6 - f_2^2 - f_3^2 - f_5^2 \\ \bar{A}_3 &= f_1f_4f_6 + 2f_2f_3f_5 - f_2^2f_6 - f_3^2f_4 - f_1f_5^2 \end{aligned} \tag{G.5}$$

Equation (G.4) yields the natural frequencies. As stated in Chapter 1, there are three different frequencies of which the lowest (mainly associated with the radial motion) is of prime interest; the other two are much higher and are mainly associated with in-plane motion.

ACKNOWLEDGEMENT

The authors would like to thank Professor M. Baruch for useful advice and comments. They also wish to thank Mr. A. Greenwald and Mr. A. Sudak for preparation of the figures and Mrs. D. Zirkin for typing the report.

REFERENCES

1. Lord Rayleigh, The Theory of Sound, 2nd Ed. Dover Publications, New York 1945, pp.395-432, Vol. I.
2. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity 4th Ed. Dover Publications, New York, 1945, pp. 538-552.
3. Flügge, W., Statik und Dynamik der Schalen, 1st Ed. Springer Berlin, 1934. Also, Stresses in Shells, 2nd Ed. Springer, Berlin 1973.
4. Arnold, R.N. and Warburton, G.B., "Flexural Vibrations of the Walls of Thin Cylindrical Shells Having Freely Supported Ends," Proc. Roy. Soc. A, Vol. 197, 1949, pp. 238-256.
5. Arnold, R.N. and Warburton, G.B., "The Flexural Vibrations of Thin Cylinders," Proc. (A) Inst. Mech. Eng. (London), Vol. 167, 1953, pp.62-80.
6. Naghdi, P.M. and Berry J.G., "On the Equation of Motion of Cylindrical Shells," Journal of Appl. Mech., Trans. ASME, Vol. 76, 1954, pp.160-166.
7. Baron, M.L. and Bleich, H.H., "Tables for Frequencies and Modes of Free Vibration of Infinitely Long Thin Cylindrical Shells," Journal of Appl. Mech. Trans. ASME, Vol. 76, 1954, pp.178-184.
8. Reissner, E., "On the Transverse Vibrations of Thin, Shallow Elastic Shells," Quart. of Appl. Math. Vol. 13, 1955, pp. 169-176.
9. Yi-Yuan Yu, "Free Vibrations of Thin Cylindrical Shells Having Finite Lengths with Freely Supported and Clamped Edges," Journal of Appl. Mech. Trans ASME, Vol. 77, 1955, pp.547-552.

10. Yi-Yuan Yu, "Dynamic Equations of Donnell's Type for Cylindrical Shells with Application to Vibration Problems," 9th Congress Intern. Me'can Appl. Univ., Bruxelles, 1957, pp. 261-278.

11. Yi-Yuan Yu,"Vibrations of Thin Cylindrical Shells Analyzed by Means of of Donnell-type Equations,"Journal of Aero/Space Sci. Vol. 25, 1958, pp. 699-715.

12. Warburton, G.B., "Free Vibrations of Thin Cylindrical Shells Having Finite Lengths with Freely Supported and Clamped Edges, Discussion," Journal of Appl. Mech., Trans. ASME, Vol. 78, 1956, pp. 484-486.

13. Yamane, J.R.,"Natural Frequency Curves for Simply Supported Cylindrical Shells," AIAA Journal Vol. 3, 1965, pp. 180-181.

14. Weingarten, V.I., "Free Vibration of Thin Cylindrical Shells," AIAA Journal Vol. 2, 1964, pp. 717-722.

15. Warburton, G.B., " Vibration of Thin Cylindrical Shells," Journal Mech. Eng. Sci., Vol. 7, 1965, pp.399-407.

16. Smith, B.L. and Haft E.E.," Natural Frequencies of Clamped Cylindrical Shells," AIAA Journal Vol. 6, 1968, pp.720-721.

17. Warburton, G.B. and Higgs, J., "Natural Frequencies of Thin Cantilever Cylindrical Shells," Journal Sound Vib. Vol. 11, 1970, pp.335-338.

18. Vronay, D.F. and Smith, B.L.,"Free Vibration of Circular Cylindrical Shells of Finite Length," AIAA Journal Vol. 8, 1970, pp.601-603.

19. Forsberg, K., "A Review of Analytical Methods Used to Determine the Modal Characteristics of Cylindrical Shells," NASA Contractor Report, NASA CR-613, Sept. 1966, 67 pp.

20. Forsberg, K., "Influence of Boundary Conditions on the Modal Characteristics of Thin Cylindrical Shells," AIAA Journal Vol.2, 1964, pp.2150-2156.

21. Sobel, H., "Effects of Boundary Conditions on the Stability of Cylinders Subject to Lateral and Axial Pressures," AIAA Journal Vol. 2,1964, p.1437.

22. Nuckolls, C.E. and Egle, D.M., " The Influence of Elastic Boundary Conditions on the Resonant Response of Thin Cylindrical Shells," AIAA/ASME 12th Structures, Structural Dynamics and Materials Conf. Anaheim, Calif. April 19-21, 1971, AIAA Paper No. 71-335.

23. Webster, J.J., " The Accuracy of Finite-Element Solution for the Modal Characteristics of Shells of Revolution," Int. Journal Mech. Sci. Vol. 12, 1970, pp.157-168.

24. Forsberg, K.,"Axisymmetric and Beam-Type Vibrations of Thin Cylindrical Shells," AIAA Journal Vol. 7, 1969, pp.221-227.

25. Kornecki, A., "A Note on Beam Type Vibrations of Circular Cylindrical Shells," Journal Sound.Vib. Vol.14, 1971, pp.1-6.

26. Mizoguchi, K., "On Fundamental Differential Equations of Vibrations of Cylindrical Shells," Bulletin of JSME, Vol. 5, No. 17, 1962, pp.43-49.

27. Mizoguchi, K.,"Vibration of a Cylindrical Shell Clamped at Both Ends," Bulletin of University of Osaka, Perfecture Series (A), 1968, pp.1-20.

28. Takahashi, S., Hirano, Y., Suzuki, K., and Nakachi, I., "The Vibration of a Cylindrical Shell with Built-in-Free Ends," Bulletin of the JSME, Vol. 13, No. 57, 1970, pp.341-347.

29. Yamaki, N., "Flexural Vibrations of Thin Cylindrical Shells," Reports of the Institute of High Speed Mechanics. Tohoku University, Vol. 22, 1970, pp.109-136.

30. Lin C.W., and Bell, F.L., "On the Non-Symmetric Vibrations of Thin Cylindrical Shells with Clamped-Clamped Edges", Nuclear Eng. and Design, Vol. 7, 1968, pp.194-200.

31. Lin, C.W., "Nonsymmetric Vibrations of Thin Cylindrical Shells," Journal of the Struc. Div. Proc. ASCE, Vol. 96, 1970,pp.2175-2187.

32. Armenakas, A.E., "On the Accuracy of Some Dynamic Shells Theories," Journal of the Eng. Mech. Div. Proc. ASCE, Vol.93, 1967, pp.95-109.

33. Das, Y.C., "Vibrations of Orthotropic Cylindrical Shells", Appl. Sci. Res. (A), Vol. 12, 1964, pp.317-326.

34. White, J.C., "The Flexural Vibrations of Thin Laminated Cylinders," Journal of Eng. for Industry, Trans. of ASME,Vol.83, 1961,pp.397-402.

35. Weingarten, V.I., "Free Vibrations of Multilayered Cylindrical Shells," Exp. Mech., Vol. 4, 1964, pp.200-205.

36. Dong, S.B., "Free Vibration of Laminated Orthotropic Cylindrical Shells," Journal of Acous. Soc. Am.Vol.44, No.6, 1968, pp. 1628-1635.

37. Padovan, J., and Koplik, B., "Vibrations of Closed and Open Sandwich Cylindrical Shells Using Refined Theory," Journal of Acous. Soc. Am. Vol. 47, No. 3, 1970, pp.862-869.

38. Stavsky, Y., and Loewy, R., "On Vibrations of Heterogeneous Orthotropic cylindrical shells," Journal of Sound and Vibration,Vol.15,No.2, 1971, pp.235-256.

39. Tasi, J., "Effect of Heterogeneity on the Axisymmetric Vibrations of Cylindrical Shells," Journal of Sound and Vibration,Vol.14,No.3, 1971, pp.325-338.

40. Mirsky, I., "Three-Dimensional and Shell- Theory Analysis for Axi-symmetric Vibrations of Orthotropic Shells," Journal of Acous. Soc., Am. Vol. 39, No. 3, 1966, pp. 549-555.

41. Resnick, B.S., and Dujundji, J., "Effects of Orthotropicity, Boundary Conditions, and Eccentricity on the Vibrations of Cylindrical Shells," AFOSR Scientifc Rep. AFOSR 66-2821, ASRL TR 134-2, M.I.T. Aeroelastic and Structures Research Laboratory, Nov. 1966.

42. Junger, M.C., "Dynamic Behavior or Reinforced Cylindrical Shells, in a Vacuum and in Fluid," Journal of Appl.Mech.,ASME Vol.21,1954,pp.35-41.

43. Baron, M.L., "Circular-Symmetric Vibrations of Infinitely Long Cylindrical Shells with Equidistant Stiffeners," Journal of Appl. Mech. ASME, Vol.23, 1956, pp.316-318.

44. Hoppmann II, W.H., "Flexural Vibrations of Orthogonally Stiffened Cylindrical Shells," Proc. 9th Intern. Congr. of Appl.Mech. Bruxelles, 1956, pp. 225-237.

45. Hoppmann II, W.H., "Some Characteristics of the Flexural Vibrations of Orthogonally Stiffened Cylindrical Shells," Journal of Acous. Soc. Am. Vol. 30, No. 1, 1958, pp.77-82.

46. Penzes, L.E., " Effect of Boundary Conditions on Flexural Vibrations of Thin Orthogonally Stiffened Cylindrical Shells," Journal of Acous. Soc. Am. Vol. 42, No.4, 1967, pp. 901-903.

47. Miller, P.R., "Free Vibrations of a Stiffened Cylindrical Shell," A.R.C. Tech. Report, R & M. No.3154, London, 1960.

48. Bleich, H.H., "Approximate Determination of the Frequencies of Ring Stiffened Cylindrical Shells," Oesterreichisches Ingenieur Archiv, Vol.15, 1961, pp.6-25.

49. Baruch, M. and Singer, J., "Effect of Eccentricity of Stiffeners on the General Instability of Stiffened Cylindrical Shells under Hydrostatic Pressure," Journal Mech. Eng. Sci. Vol. 35, No. 1, 1963, pp. 23-27.

50. Baruch, M., "Equilibrium and Stability Equations for Stiffened Shells," Israel Journal of Tech. Vol. 2, No. 1, 1964, pp. 117-124.

51. Mikulas, M.M., Jr. and McElman, J.A., " On the Free Vibrations of Eccentrically Stiffened Cylindrical Shells and Flat Plates," NASA Tech. Notes, TN D-3010, Sept. 1965.

52. McElman, J.A., Mikulas, M.M., Jr. and Stein, M., "Static and Dynamic Effects of Eccentric Stiffening of Plates and Cylindrical Shells," AIAA Journal Vol. 4, No. 5, 1966, pp.887-894.

53. Sewall, J.L., and Naumann, E.C., " An Experimental and Analytical Vibration Study of Thin Cylindrical Shells With and Without Longitudinal Stiffeners," NASA Tech. Notes, NASA TN D-4705, Sept. 1968, 54 pp.

54. Baruch, M., "Equilibrium and Stability Equations for Discretely Stiffened Shells," Israel Journal of Tech. Vol. 3, No. 2, 1965, pp.138-146.

55. Block, D.L., "Influence of Discrete Ring Stiffeners and Prebuckling Deformations on the Buckling of Eccentrically Stiffened Orthotropic Cylinders." NASA TN D-4283, January,1968.

56. Singer, J., and Haftka, R., "Buckling of Discretely Ring Stiffened Cylindrical Shells," Israel Journal of Tech.,Vol.6,1968,pp. 125-137.

57. Parthan, S., and Johns, D.J., "Vibration and Flutter of Unstiffened and Orthogonally Stiffened Circular Cylindrical Shells," TT 7106, Dept. of Transport Tech. Loughborough Univ. of Tech. June 1971, Vol. I, II.

58. Parthan, S., and Johns, D.J.,"Effects of Inplane and Rotary Inertia on the Frequencies of Eccentrically Stiffened Cylindrical Shells," Proc. of the AIAA Structural Dynamics and Aeroelasticity Specialist Conference, New-Orleans, April 14-16, 1969, pp.253-262.

59. Parthan, S., and Johns, D.J., "Effects of Inplane and Rotary Inertia on the Frequencies of Eccentrically Stiffened Cylindrical Shells," AIAA Journal Vol. 8, No. 3, 1970, pp. 592-594.

60. Soder, K.E., Jr., "An Analysis of Free Vibration of Thin Cylindrical Shells with Rings and Stringers Treated as Discrete Elements which may be Nonsymmetric, Eccentric and Arbitrarily Spaced,"Univ. of Oklahoma, Ph.D., 1968, Eng. Aeronautical.

61. Galletly, G.D., "On the In-Vaco Vibrations of Simply Supported Ring-Stiffened Cylindrical Shells," Proc. 2nd, U.S. National Congress of Appl. Mech. June 1954, pp.225-231.

62. Egle, D.M., and Sewall, J.L., "An Analysis of Free Vibration of Orthogonally Stiffened Cylindrical Shells with Stiffeners Treated as Discrete Elements." AIAA Journal Vol. 6, No. 3, 1968, pp.518-526.

63. Bray, F.M., and Egle, D.M.,"An Experimental Investigation on the Free Vibration of Thin Cylindrical Shells with Discrete Longitudinal Stiffening,"Journal of Sound & Vibration, Vol. 12, No.2,1970,pp.153-164.

64. Scruggs, R.M., Pierce, C.V., and Reese, J.R., "An Analytical and Experimental Study of the Vibration of Orthogonally Stiffened Cylindrical Shells," Journal Spacecraft, Vol. 6, No. 5, 1969, pp.603-609.

65. Wah, T., "Circular Symmetric Vibration of Ring-Stiffened Cylindrical Shells," Journal Soc. Indust.Appl.Math. Vol. 12,No.3, 1964, pp.649-662.

66. Wah, T., "Flexural Vibrations of Ring-Stiffened Cylindrical Shells," Journal of Sound and Vibration, Vol. 3, No. 3, 1966, pp.242-251.

67. Hu, W.C.L., Gormley, J.F., and Lindholm, U.S., "An Analytical and Experimental Study of Vibrations of Ring-Stiffened Cylindrical Shells," Tech. Report No. 9, Contract NASr-94(06) SwRI Project 02-1504, South-west Research Inst. June 1967.

68. Wah, T., and Hu, W.C., "Vibration Analysis of Stiffened Cylinders Including Inter-Ring Motion,"Journal of Acous.Soc.Am.Vol.43, No.5, 1968, pp.1005-1001.

69. Al-Najafi, A.N.J., and Warburton, G.B., "Free Vibration of Ring-Stiffened Cylindrical Shells", Journal of Sound and Vibration, Vol. 13, No. 1, 1970, pp.9-25.

70. McDonald,D., "A Problem in the Free Vibration of Stiffened Cylindrical Shells", AIAA Journal, Vol. 8, 1970, pp.252-258.

71. Schnell, W. and Heinrichsbauer, F.J.," The Determination of Free Vibrations of Longitudinally-Stiffened, Thin-Walled,Circular Cylindrical Shells," Jahrbuch 1963 der Wissenschaftliche Gesellschaft für Luft- und Raumfahrt (WALR) pp. 278-286,(Translation NASA TT-F-8856, April 1964).

72. McGarattan, R.J., and North, E.L., "Vibration Analysis of Shells using Discrete Mass Techniques," Journal of Eng. for Industry, Vol. 89, 1967, pp. 766-772.

73. Garnet, H., and Levy, A., "Free Vibrations of Reinforced Elastic Shells," Journal of Appl. Mech. Vol. 36, 1969, pp.835-844.

74. Harari, A. and Baron, M.L.,"Discrete Analysis for the Dynamic Response of Stiffened Shells," Paul Weidlinger, Consulting Engineer, Tech. Report No. 10, Office of Naval Research, Prepared under Contract Number N00014-70-C-0359 Task No. NR-064-464, March 1971.

75. Basdekas, N.L., "Response of Oddly-Stiffened Circular Cylindrical Shells," Journal of Sound and Vibration,Vol.17,No.2,1971,pp.187-206.

76. Michalopoulos, C.D., and Muster, D., "The In-Vaco Vibrations of a Simply Supported Ring-Stiffened, Mass-Loaded Cylindrical Shell," Proc. of the Symposium on the Theory of Shells, University of Texas, Houston, 1967, pp. 345-377.

77. Forsberg, K., "Exact Solution for Natual Frequencies of Ring-Stiffened Cylinders," AIAA/ASME 10th Structures, Structural Dynamics and Materials Conf. New-Orleans, Louisiana, Apr. 14-16, 1969, pp. 18-30.

78. Fung, Y.C., Sechler, E.E., and Kaplan, A.," On the Vibration of Thin Cylindrical Shells under Internal Pressure," Journal of Aero/Sci. Vol. 24, 1957, pp.650-660.

79. Koval, L.R., "On the Free Vibration of Thin Clamped Cylindrical Shells Subjected to Radial Pressure," Journal of the Aero/Sci.Vol.29, 1962, pp. 615-616.

80. Gottenberg, W.C., "Experimental Study of the Vibrations of a Circular Cylindrical Shell," Journal of Acous. Soc. Am. Vol. 32, No. 8, 1960, pp. 1002-1006.

81. Ebner, A.M., " Effects of Prestress on Vibration Behavior of Certain Shells of Revolution", AIAA Journal Vol. 9, No. 4, 1971, pp. 742-743.

82. Dym, C.L., "Vibrations of Pressurized Orthotropic Cylindrical Membranes," AIAA Journal, Vol. 8, No. 4, 1970, pp. 693-699.

83. Dym, C.L.,"Vibrations of Pressurized Orthotropic Shells," AIAA Journal Vol. 9, No. 6, 1971, pp. 1201-1203.

84. Ojalvo, I.U., and Newman, M.,"Natural Vibrations of a Stiffened Pressurized Cylinder with an Attached Mass," AIAA Journal Vol. 5, No. 6, 1967, pp. 1139-1146.

85. Greenspon, J.E., "Effect of External or Internal Static Pressure on the Natural Frequencies of Unstiffened, Cross-Stiffened, and Sandwich Cylindrical Shells," Journal of Acous.Soc. Am. Vol. 39, No. 2, 1966, pp.407-408.

86. Koval, L.R., and Cranch, E.T., " On the Free Vibrations of Thin Cylindrical Shells, Subjected to an Initial Static Torque," Proc. of the 4th U.S. National Congress of Appl. Mech., 1962, pp. 107-117.

87. Weingarten, V.I.,"Free Vibrations of a Thin Walled Cylindrical Shell Subjected to a Bending Moment," AIAA Journal Vol. 3, No. 1, 1965, pp.40-44.

88. Seggelke, P.,"Vibration Behavior of Axial-Symmetrically Loaded Thin-Walled Circular Cylinders under Different Boundary Conditions," Deutsche Luft-und Raumfahrt, Report 64-43, 1964, pp.31.

89. Herrmann, G., and Armenakas, A.E., "Dynamic Behavior of Cylindrical Shells under Initial Stress," Proc. of the 4th U.S. National Congress of Appl. Mech. 1962, pp.203-213.

90. Armenakas, A.E. and Herrmann, G.,"Vibrations of Infinitely Long Cylindrical Shells under Initial Stress," AIAA Journal Vol. 1, No. 1, 1963, pp. 100-106.

91. Armenakas, A.E., "Influence of Initial Stress on the Vibrations of Simply Supported Circular Cylindrical Shells," AIAA Journal Vol. 2, No. 9, 1964, pp. 1607-1612.

92. Herrmann, G., and Shaw, J., "Vibration of Thin Shells under Initial Stress," Journal of The Eng. Mech. Div. Proc. ASCE. Vol. 91, No. EM5, pp. 37-58.

93. Baker, E.H., and Herrmann, G., "Vibrations of Orthotropic Cylindrical Sandwich Shells under Initial Stress," AIAA Journal Vol. 4, No. 6, 1966, pp.1063-1070.

94. Cooper, P.A., "Vibration and Buckling of Prestressed Shells of Revolution," NASA TN D-3831, March 1967.

95. Cooper, P.A., and Robinson, M.P., "Effect of Static Stress and Edge Restraint on the Vibration of Nearly Cylindrical Shells with Circular Cross Section," NASA TN D-6133, April 1971.

96. Padovan, J., "Frequency and Buckling Eigenvalues of Anisotropic Cylinders Subjected to Nonuniform Lateral Prestress," Int. Journal Solids Structures, Vol. 7, 1971, pp. 1449-1466.

97. Singer, J., Baruch, M., and Harari, O.," On the Stability of Eccentrically Stiffened Cylindrical Shells under Axial Compression," Int. Journal Solids Structures, Vol. 3, 1967, pp.445-470.

98. Cooper, P.A., "Vibration of Stressed Shells of Double Curvature," Virginia Polytechnic Institute, Ph.D. 1968, Eng. Mech.

99. Bushnell, D., "Stress, Stability and Vibration of Complex Shells of Revolution: Analysis and User's Manual for BOSOR 3,"Report N-5J-69-1, SAMSO TR-69-375, Sept. 1969.

100. Bushnell, D., Almroth, N.O. and Sobel, L.H., "Buckling of Shells of Revolution with Various Wall Construction," Vol. 2 - Basic Equations and Method of Solution. NASA CR-1050, May 1968.

101. Rosen, A., and Singer, J., "Vibrations of Axially Loaded Stiffened Cylindrical Shells: Part II: Experimental Analysis," TAE Report No. 163, Technion-Israel Institute of Technology, Haifa, Israel. August, 1973.

102. Bleich, H.H. and Dimaggio, F., "Strain-Energy Expressions for Thin Cylindrical Shells," Journal of Appl. Mech. ASME, Vol. 20, 1953, pp.448-449.

TABLE I - GEOMETRICAL AND STRUCTURAL PROPERTIES OF MODELS FROM REF. [53].

PROPERTY	MODELS
Radius to shell middle surface	9.537 in (242.2mm)
Shell thickness $\bar{h}$	0.0256 in (0.650mm)
Stiffener width, c_1	0.100 in (2.540mm)
Stiffener height, d_1	0.2764 in (7.021mm)
Stiffener spacing, b_1	1.00 in (2.54 mm)
Cylinder length for various end conditions, L	
Free-free	25.125 in (63.82cm)
Clamped-free	24.625 (62.55cm)
Simply supported	24.00 in (60.96 cm)
Clamped-Clamped	24.00 in (60.96 cm)
Young's modulus for shell and stringers, $E = E_1$	10^7 psi ($68.95\ GN/m^2$)
Poisson's ratio for shell, ν	0.315
Shear modulus for stringers, G_1	$3.8\cdot10^6$ psi ($26.2\ GN/m^2$)
Mass density for shell and stringers $\rho = \rho_1$	$2.54\cdot10^{-4}$ $lb\text{-}sec^2$ ($2.7145\cdot10^3\ kg/m^3$)
Cross sectional area of stringer, $\bar{\bar{A}}_1$	0.029319 in^2 ($18.92\ mm^2$)
Distance from shell middle surface to stringer centroid, e_1	0.1439 in (3.655 mm)
Moment of inertia of stringer about shell middle surface, I_{o_1}	$0.836124\cdot10^{-3}$ in^4 ($0.034802\ cm^2$)

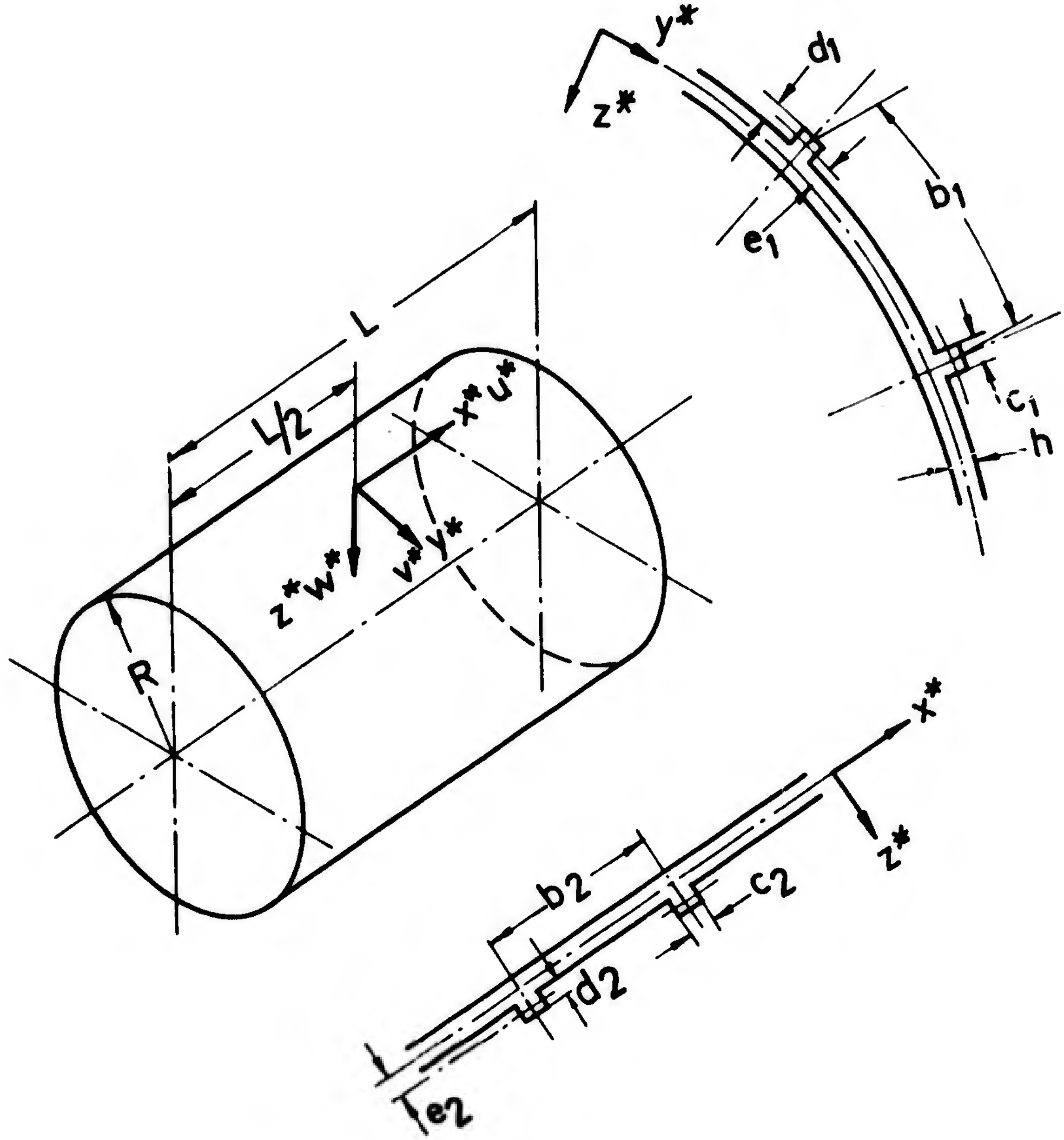

FIG. 1 NOTATION

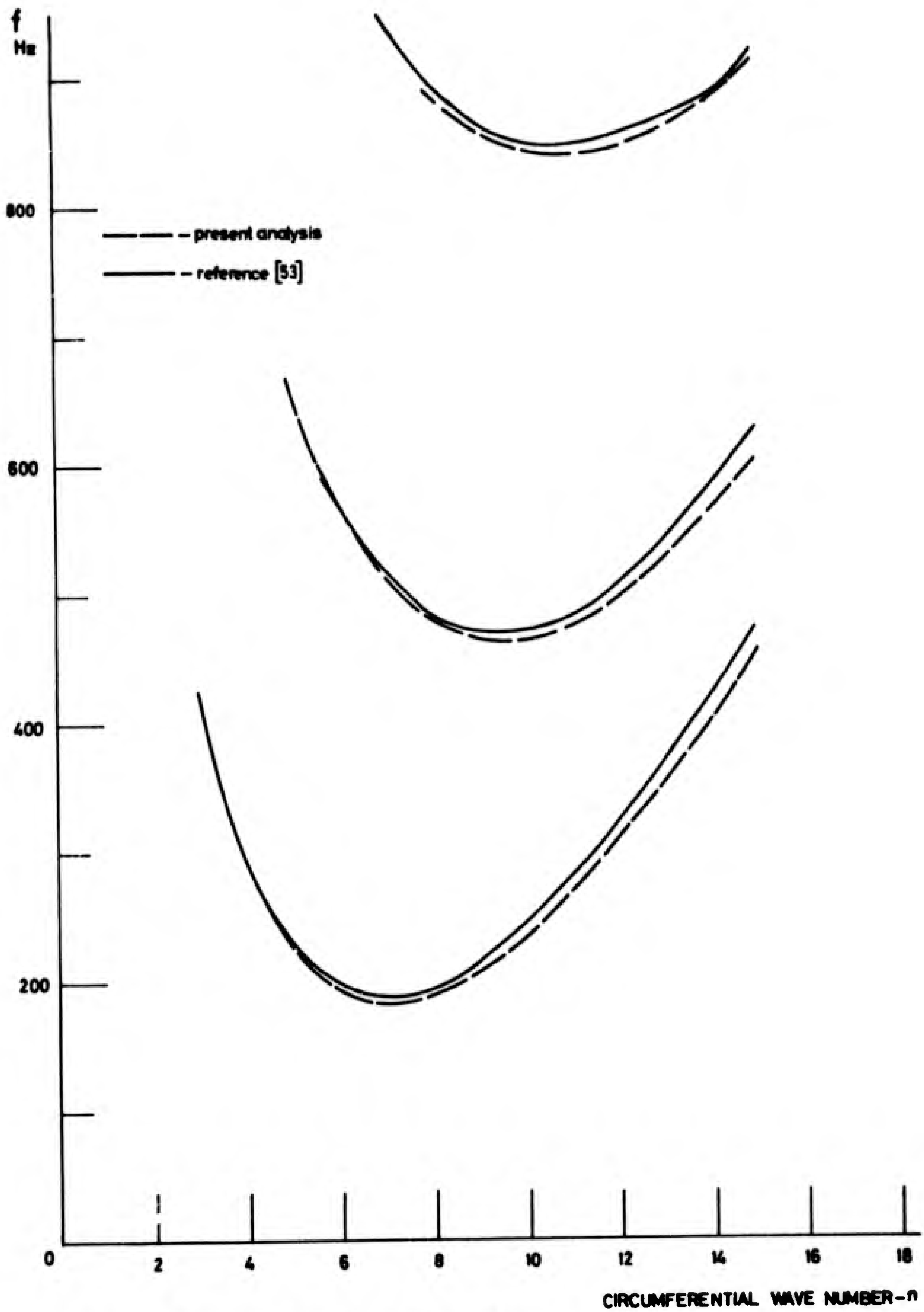

FIG. 2 THEORETICAL FREQUENCIES OF SIMPLY SUPPORTED EXTERNAL STRINGER-STIFFENED CYLINDRICAL SHELL (PROPERTIES IN TABLE 1)

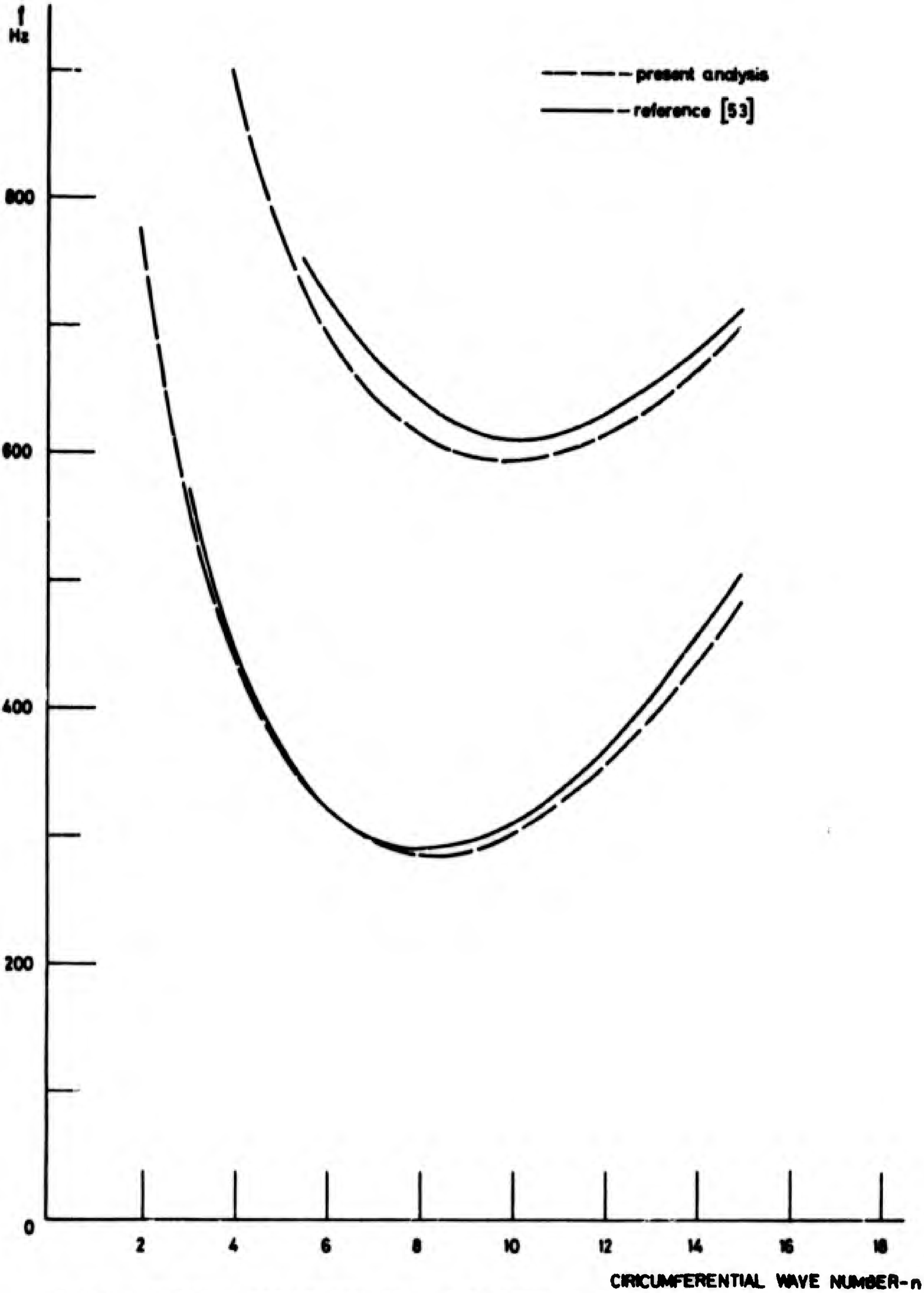

FIG. 3 THEORETICAL FREQUENCIES OF CLAMPED EXTERNAL STRINGER-STIFFENED CYLINDRICAL SHELL (PROPERTIES IN TABLE-1)

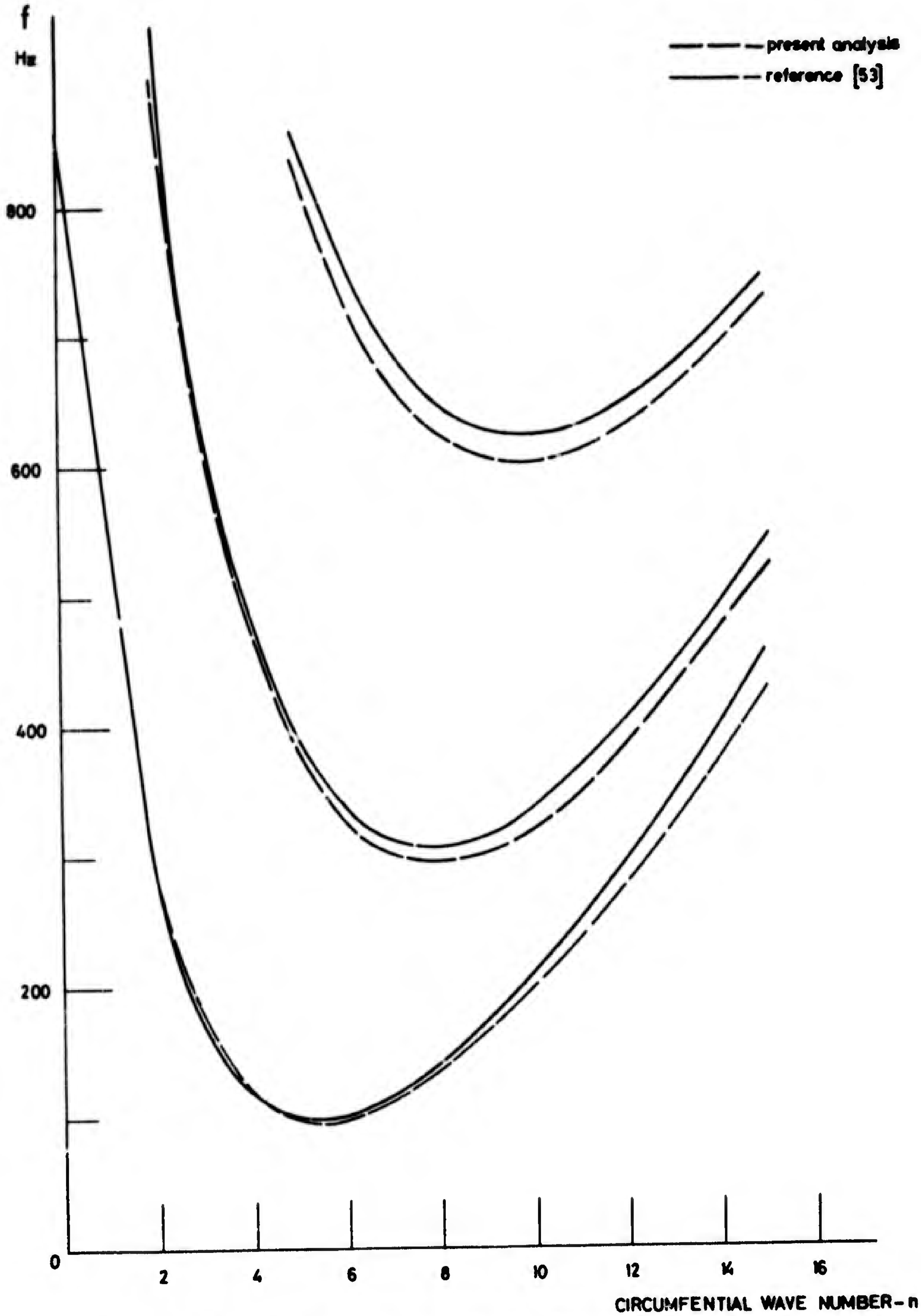

FIG. 4 THEORETICAL FREQUENCIES OF FREE-CLAMPED EXTERNAL STRINGER-STIFFENED CYLINDRICAL SHELL (PROPERTIES IN TABLE 1)

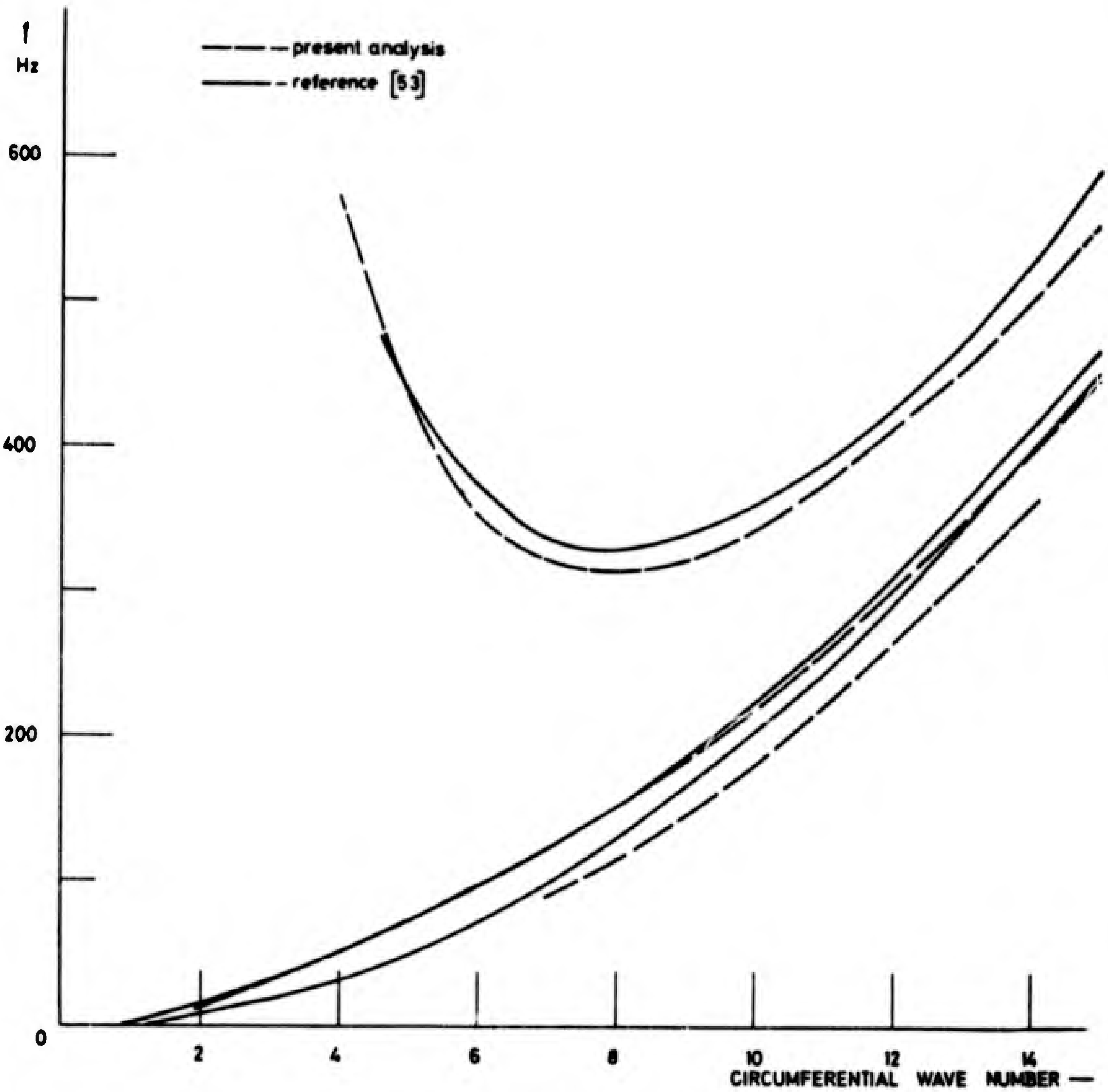

FIG. 5 THEORETICAL FREQUENCIES OF FREE FREE EXTERNAL STRINGER – STIFFENED CYLINDRICAL SHELL (PROPERTIES IN TABLE 1)

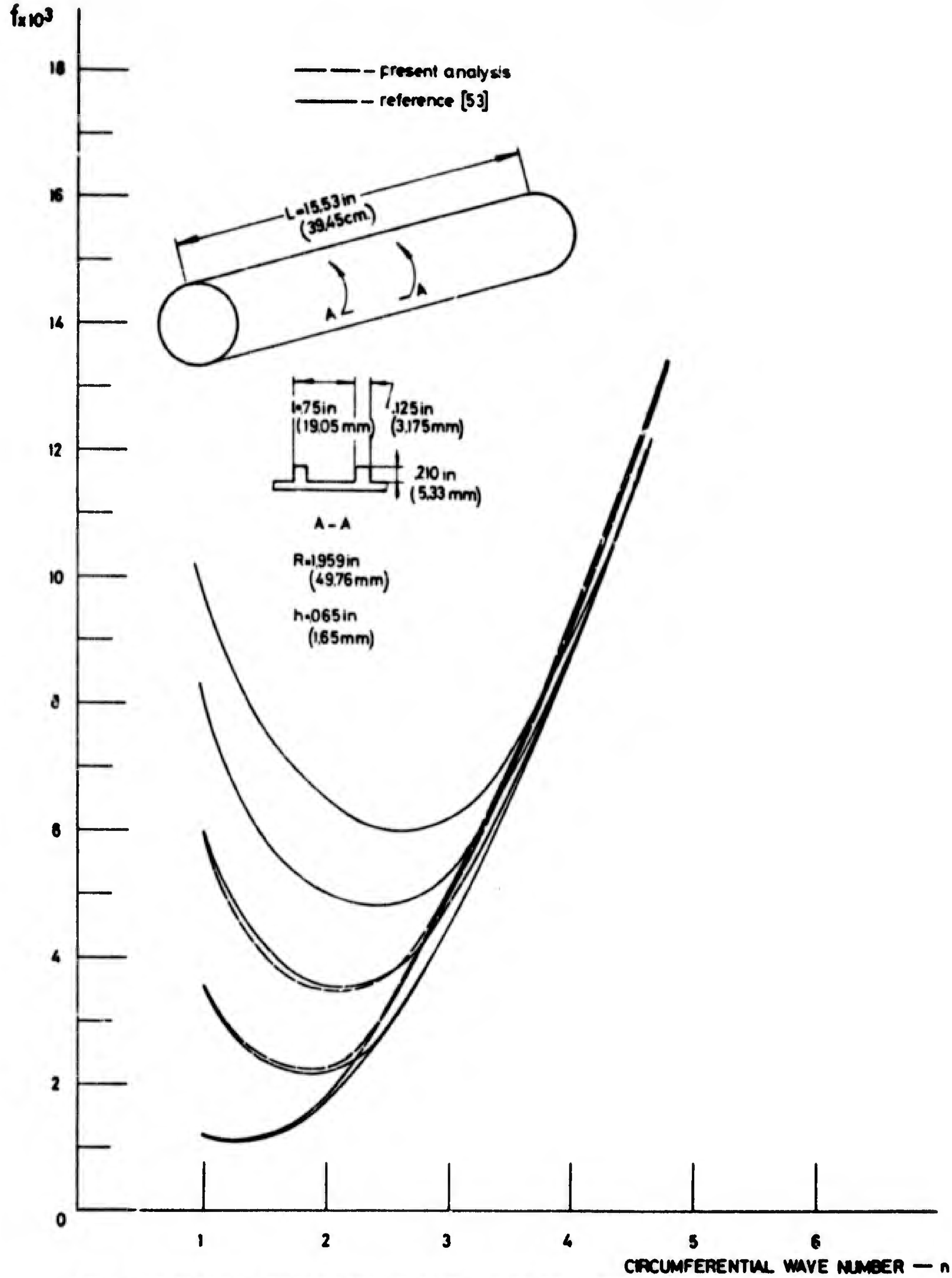

FIG. 6 THEORETICAL FREQUENCIES OF SIMPLY SUPPORTED EXTERNAL RING - STIFFENED CYLINDRICAL SHELL (PROPERTIES IN TABLE 1)

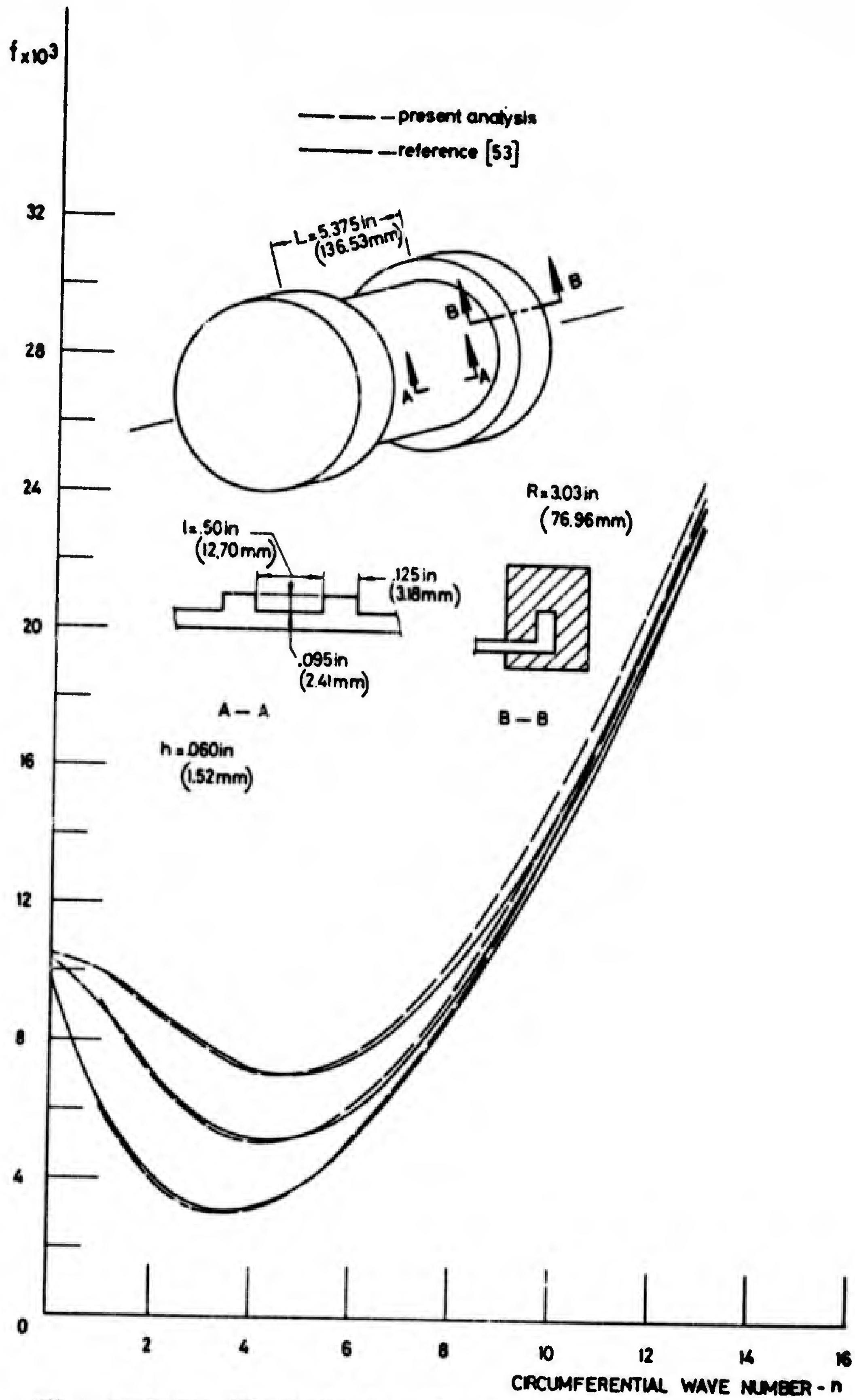

FIG. 7 THEORETICAL FREQUENCIES OF CLAMPED EXTERNAL RING-STIFFENED CYLINDRICAL SHELL (PROPERTIES IN TABLE 1)