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# EXPLICIT SOLUTIONS IN CONVEX GOAL PROGRAMMING

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13. ABSTRACT

Goal programming has now become an important tool in areas such as public management science. There is therefore a need for examining ways of securing improved computational efficiency, as is done in this paper, instead of resting only on the linear programming equivalences that were set forth when the original goal programming article was published in Vol. 1, No. 2 of Management Science. Based on lemmata which permit reduction of various important classes of convex goal programming models to problems of full row rank-interval programming type, explicit solutions to convex goal programming problems are exhibited. Some of the equivalences herein established are also useful in their own right and for other classes of problems--e.g., interval programming, as well as advanced start procedures and other such computational matters.

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| Constrained regression | | | | | | |
| Constrained generalized hypermedians | | | | | | |
| Advanced starts | | | | | | |
| Algorithms | | | | | | |
| Explicit solution | | | | | | |
| Manpower planning | | | | | | |
| Organization design | | | | | | |
| Linear programming under uncertainty | | | | | | |
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# EXPLICIT SOLUTIONS IN CONVEX GOAL PROGRAMMING

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June 1974



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Background:

As material for possible historical (as well as motivational) interest, we may observe that Volume 1, No. 1 of Management Science contained an article entitled "The Stepping Stone Method for Explaining Linear Programming Calculations in Transportation Type Models." Volume 1, No. 2 contained an article entitled "Optimal Estimation of Executive Compensation by Linear Programming."[1] The purpose of these two papers was to help this new journal and the new society, TIMS, which published it get off to a start that would also provide a basis for further growth not only in themselves but also in a society (TIMS) which would, in turn, greatly enlarge the prospects for these and other related developments in the management sciences.

By emphasis at least, the first of these two articles was designed to appeal to immediate use and understanding. The emphasis in the second article was directed to longer range potentials for additional applications in the then new discipline of linear programming. Some of these applications included extensions to areas like "inequality constrained regression," "multi-dimensional objective optimizations" and their extensions to ordinal non-metric scaling, including non-Archimedean constructs.[2] New theory as well as new methodological possibilities were also naturally kept in mind, and exploited as their

1. See [2] and [3] in the bibliography appendix to this paper.

2. See [4].

potency for applications appeared to warrant. Thus, the transportation model in the first of these two articles was extended via a variety of new ideas which ranged from the development of the "poly - $\omega$ method"[3] to the use of model transformations and approximations. Ideas in the second article subsequently gave birth to attempts to exploit the use of linear programming models and methods to attack a variety of nonlinear problems.[4] They also gave rise to the goal programming ideas which have now begun to be used in a variety of novel contexts. One such use, the OCMM[5] series of models [12], has also involved a change in the state of modeling per se by joining the ideas of goal programming and Markov processes into a new approach for dealing with decision (Markov) processes.

As in these other developments, applications associated with the OCMM models have provided points which now make it possible to undertake further methodological and theoretical developments, as well as applications, that will provide still better bases for future extensions. This, in fact, is the point of the present paper, which will be followed by others, in which we shall undertake to join some of the preceding work (e.g., in model approximation and goal programming)

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3. See [4].

4. See [11], [13], [14].

5. OCMM (Office of Civilian Manpower Management, U. S. Navy) This series of models is described in [12].

not only with each other but also with presently disparate developments (e.g., in linear programming under uncertainty) while we simultaneously attempt to provide solution procedures (including good starts) that can supply operational significance for the new problem areas that might then be addressed.

## Introduction

Explicit solutions to linear programming or linear fractional programming problems presently exist only for special cases, the most general of which are the full row rank interval programming[6] and the full row rank linear fractional programming.[7]

In some cases these models are of direct interest. This naturally includes those cases where problems can be modelled to fit the theoretical and computational requirements. It also includes cases for which model (structure) approximation and parameterization methods can yield either exact or approximate solutions to more complicated problems.[8] Finally it includes cases in which advanced starts can be thus obtained including ones which greatly accelerate the attainment of solutions--as when, e.g., primal simplex technique applied to such starts have yielded great improvements in efficiency

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6. See [1].

7. See [5].

8. See [6].

relative to alternative solution procedures that might otherwise have been required.[9]

There are at present no comparable classes of explicit solutions to "goal programming" models. The increased usage now being accorded to this type of model, however, would seem to justify such developments in order to anticipate benefits for this class which are analogous to those described in the preceding paragraph.

The present paper is directed toward developing such solution procedures for the general case of piecewise linear, but separable, "goal functionals." We define a separable functional as a sum of one-variable goal functionals. A one-variable goal functional, $f_j(x_j)$, is a function that is monotone decreasing for $x_j \leq g_j$ --$g_j$ = the stipulated $j^{th}$ goal--and monotone increasing for $x_j > g_j$. Note, however, that <u>strict</u> monotonicity is <u>not</u> required. In fact for the important "goal interval" type of model, the functional has a constant minimum over a whole interval.

The following examples are included in this class:

(i) Absolute value functions, including those with asymmetric weights and multi-goal components.

(ii) General convex piecewise linear functionals

(iii) Goal interval functionals

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9. See [8] and [10].

(iv) Hypermedian functionals and related functionals in extensions of ordinary goal programming.[10]

This list is not exhaustive, but it is nevertheless indicative of the many significant kinds of problems that are included for the development that we shall now provide as follows: First we shall state and prove a very general lemma which can be used to reduce and simplify such problems. This lemma will be presented in the next section--i.e., section 2--along with its application to a simple (but significant) class of cases. The lemma, we may note, is not restricted to the latter class of cases and is, indeed, established in far greater generality than required for any part of the present paper. In addition, we shall proceed to our results for obtaining explicit solutions via general separable convex piecewise linear functionals, which will be covered in section 3. In this section--i.e., section 3--we shall obtain a linear programming equivalent which will be extended in section 4 where procedures for obtaining explicit solutions will be delineated. Section 5 will then conclude this paper with a numerical example which will serve to illustrate some of the results achieved in the present paper and also indicate some of the possible extensions that will be covered in later papers in this series.

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10. See [7]

## 2. A General Reduction Lemma:

The following lemma is established in far greater generality than we shall require for this paper. Indeed, no properties other than monotone decrease up to the goal and monotone increase to its right are stipulated. Thus, we let $f = \sum_j f_j(x_j)$ where $f_j(x_j)$ is monotone decreasing for $x_j \le h_j$ and monotone increasing for $x_j \ge h_j$. Then we consider the following problem:

(1.1) Min. $f(x) = \sum_j f_j(x_j)$

subject to $\sum_j x_j \le b_j$

(1.2) $a_j \le x_j \le b_j$

With this in mind we now develop

**Lemma 1**: If x satisfies (1.2) and for some $j_o$, $x_{j_o} > h_{j_o}$, then $\bar{x}$ satisfies (1.2) and $f(\bar{x}) \le f(x)$, where

$$\bar{x} = (\ldots, \bar{x}_j, \ldots),$$

$$\bar{x}_j = \begin{cases} x_j, & j \ne j_o \\ \max(a_{j_o}, h_{j_o}), & j = j_o. \end{cases}$$

Proof: If $x_{j_o}$ is replaced by $x_{j_o} - \Delta$, $0 \le \Delta \le \min.(x_{j_o} - h_{j_o}, x_{j_o} - a_{j_o})$, then

$$f_{j_o}(x_{j_o} - \Delta) \le f_{j_o}(x_{j_o})$$

$$a_j \le x_{j_o} - \Delta \le b_j$$

and $\sum_{\hat{j}_o} x_j + x_{j_o} - \Delta \le b_o$, where $\hat{j}_o$ signifies that the summation omits the variable with this index.

Thus, (1.2) is still satisfied. Also $f_{j_o}(x_{j_o})$ is monotone increasing for $h_{j_o} \le x_{j_o}$. We therefore obtain a decrease (or at lease no increase) by setting $\Delta$ at its upper limit $\bar{\Delta}$ and define $\bar{x}_{j_o} = x_{j_o} - \bar{\Delta} = \max(h_{j_o}, a_{j_o}) \le x_{j_o}$. But $x_{j_o} > h_{j_o}$. Hence

$$f_{j_o}(\bar{x}_{j_o}) \le f_{j_o}(x_{j_o})$$

and

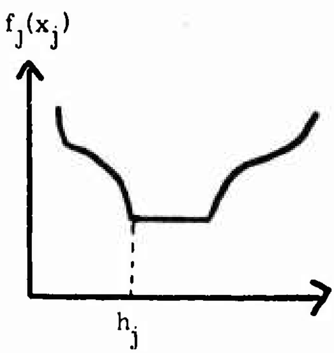
$$\sum_{\hat{j}} f_j(x_j) + f_{j_o}(\bar{x}_{j_o}) \le f(x).$$

I.e.,

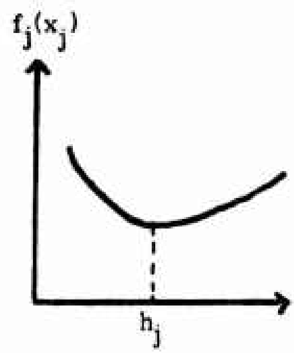
$$f(\bar{x}) \le f(x),$$

and the lemma is thus established.

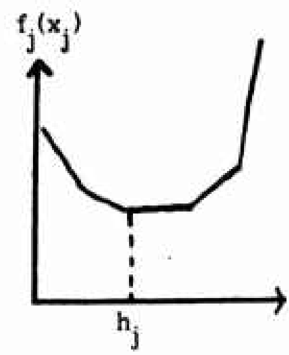
The lemma covers in particular nonlinear goal functionals of types such as those drawn in the following figures.



Non-Convex Functional
Figure 1a



Convex Functional
Figure 1b



Piecewise Linear
Functional
Figure 1c

Note, further, also that the simple constraint $\sum x_j \le b_o$ in the lemma may be replaced by any system of constraints such that decreasing any single variable preserves feasibility of the constraints. E. g., the system

$$\begin{aligned} 5x_1 + 2x_2 \phantom{+ 3x_3} + x_4 &\le 20 \\ x_2 + 3x_3 + 5x_4 &\le 50 \end{aligned}$$

would be a valid type of replacement for the single constraint of the lemma. See the illustration in Section 5.

We proceed first to consider, however, the particular case

$$(2.1)\quad \min \sum_j \mu_j \, |x_j - g_j|$$

$$\text{subject to } \sum_j x_j \le b_o$$

$$(2.2)\quad a_j \le x_j \le b_j$$

where, without loss of generality, the constants $\mu_j$ are indexed so that $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_j \ge \ldots \ge \mu_n \ge 0$. Because of Lemma 1, we can impose the additional condition

$$(3.0)\quad x_j \le \max\,(g_j, a_j) \equiv b'_j.$$

Via (2.2), however, $a_j \le x_j$ so that if $g_j \le a_j$ we need only set $x_j = a_j$ and reduce the preceding problem to one in the remaining variables

$$(3.1)\quad \min \sum_j \mu_j \, | x_j - g_j |$$

subject to

$$\sum_j x_j \le b'_o$$

$$(3.2)\quad a_j \le x_j \le b'_j$$

where $b'_j$ is defined as in (3.0) and $a_j \le g_j \le b'_j$ with, again, $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_j \ge \ldots \ge \mu_n \ge 0$, by re numbering, if required, and $b'_o$ is $b_o$ reduced by the $x_j = a_j$ values. Abusing notation, we replace $n'$ by $n$ too.

Because of Lemma 1 and the definition of $b'_j$, however, we now have $x_j - g_j \le 0$ all $j$ and this implies

$$|x_j - g_j| = g_j - x_j$$

Thus we may now replace the preceding problem by

(4.1) $\max \sum_j \mu_j x_j$

subject to $\sum_j x_j \leq b_o'$

(4.2) $x_j \leq - a_j.$

This is an ordinary linear programming problem with a dual that may be written

(5.1) $\min v_o b_o' + \sum_j (v_j^+ b_j' - v_j^- a_j)$

subject to

(5.2) $v_o \quad + v_j^+ \quad - v_j^- \quad = \mu_j$

$$v_o,\ v_j^+,\ v_j^- \geq 0.$$

If $\sum_j a_j > b_o'$ then the constraints of (4.2) are evidently inconsistent since the $a_j$ are lower bounds which the corresponding $x_j$ must satisfy. If we assume that $\sum_j a_j \geq b_o'$ then the optimum solution of (4.2) can be immediately written:

$$x_j^* = b_j',\ j=1, \ldots, k-1$$

$$x_k^* = b_o' - \sum_{j=1}^{k-1} b_j - \sum_{j=k+1}^{n} a_j$$

$$x_j^* = a_j,\ j = k+1, \ldots, n$$

where k is the smallest positive integer such that

$$\sum_{j=1}^{k-1} b_j' \leq b_o' - \sum_{k+1}^{n} a_j$$

and

$$\sum_{j=1}^{k} b_j' > b_o' - \sum_{k+1}^{n} a_j$$

To see that these $x^*$ choices are optimal we observe that they satisfy the constraints in (4.2) and produce $\sum_{j=1}^{n} \mu_j x_j^* = \sum_{1}^{k-1} \mu_j b_j' + \mu_k (b_o' - \sum_{1}^{k-1} b_j' - \sum_{k+1}^{n} a_j) + \sum_{k+1}^{n} \mu_j a_j$ in the functional (4.1). Via regrouping[11] this becomes $\mu_k b_o' + \sum_{1}^{k-1} (\mu_j - \mu_k) b_j' + \sum_{k+1}^{n} (\mu_j - \mu_k) a_j$.

Applying the "regrouping principle" (see [ ]), we posit these coefficients as optimal dual variables and need only verify that they satisfy the dual constraints--*viz.*,

$$
\begin{aligned}
v_o^* &= \mu_k \\
v_j^{*+} &= \mu_j - \mu_k \geq 0, \quad j=1, \ldots, k-1 \\
v_j^{*-} &= 0, \quad j=1, \ldots, k-1 \\
v_k^{*+} &= v^{*-} = 0 \\
v_j^{*-} &= \mu_k - \mu_j \geq 0, \quad j=k+1, \ldots, n \\
v_j^{*+} &= 0, \quad j=k+1, \ldots, n.
\end{aligned}
$$

These values of $v_o^*$, $v_j^{*+}$, $v_j^{*-}$ are non-negative as required. They also satisfy the other dual constraints in (5.2) since, as may be seen, these $v^*$ choices give

$$
\begin{aligned}
\mu_k + \mu_j - \mu_k &= \mu_j & \quad j=1, \ldots, k-1 \\
\mu_k &= \mu_k & \\
\mu_k + \mu_j - \mu_k &= \mu_j, & \quad j=k+1, \ldots, n
\end{aligned}
$$

upon substitution in (5.2).

---

11. I. e., we are here using a variant of the "regrouping principle" first set forth in [9] as a means of obtaining explicit solutions via the dual problem. See also [4].

This completes the proof that our explicit solution $x^*$ is optimal. It also shows in full detail how we can use the lemma to obtain explicit solutions in this class of cases. The next section will extend this result to the case of general separable convex piecewise linear functionals.

## 3. The General Separable Convex Piecewise Linear Functional

The functional we are considering is separable--i.e., it is a sum of functions each involving only one variable. Hence, we will be able to apply Lemma 1 and perform a reduction on one variable at a time to obtain a linear programming format. The explicit solution will then be developed and exhibited in the section after this one.

The general (continuous) one variable case has a graph consisting of straight line segments with slopes that increase from left to right. See Figure 1c. As is well known classically, such a function can be written as

$$(6.1)\quad f^a(t) = \sum_j \mu_j \left| t - g_j \right| + pt + q$$

where the $g_j$, $\mu_j$, p and q are given constants with $\mu_j = 0$, all j.

In this form we state the following:

<u>Lemma 2</u>: A necessary and sufficient condition for $f^a(t)$ with $\mu_j \geq 0, \forall_j$, to be a goal functional is that $\sum_j \mu_j \geq |p|$.

Another representation of such a function is also possible, bearing in mind that we shall wish to minimize on f(t). Because it explicitly specifies the slopes involved in f(t) we shall specify it by $f^s(t)$. We shall now proceed to develop this second representation of f(t) and then relate it to the $f^a(t)$ representation that we have already given.

Without loss of generality, we may suppose that $g_j \le g_{j+1}$ for all j. Suppose also that $a \le t \le b$. We may then write $t = \sum_j t_j$, for some sequence of $t_j$ starting with $t_1$, wherein we restrict the $t_j$ by the following conditions:

$$t_1 \le g_1$$

$$0 \le t_j \le g_j - g_{j-1}, \qquad j=2,\ldots, n$$

$$g_n \le t_{n+1}$$

Since we are minimizing on f(t) for $a \le t \le b$, we will choose $t_j > 0$ only if $t_{j-1}^{*}$ (which is associated with the greater slope) has reached its upper bound. Thus, for minimization purposes, f(t) convex can be equivalently written in separated linear form as

$$(6.2)\quad f^s(t) = \sum_{j=1}^{n+1} k_j t_j + d_1$$

where $k_j$ is the slope of the line segment of the graph of f(t) which runs from $g_{j-1}$ to $g_j$.

For various important goal functionals we shall need to be able to pass from one representation of the function to the other. We proceed to develop the needed formulae.

Consider then

$$f^a(t) = \sum_{j=1}^{n} \mu_j \left| t - g_j \right| + pt + q$$

and

$$f^s(t) = \sum_{j=1}^{n+1} k_j t_j + d, \; k_j \leq k_{j+1}, \; \forall j.$$

The graph of $f^s(t)$ is shown in Figure 2. Now consider

$$t_1 \leq g_1$$

$$0 \leq t_j \leq g_j - g_{j-1}$$

$$0 \leq t_{n+1}$$

For $t \to \infty$: $\quad f^a(t) \sim (\sum \mu_j + p)\, t_{n+1}$

$$f^s(t) \sim k_{n+1}\, t_{n+1}$$

Hence

$$p + \sum_{1}^{n} \mu_j = k_{n+1} .$$

For $t \to -\infty$: $\quad f^a(t) \sim (-\sum_{1}^{n} \mu_j + p)t.$

$$f^s(t) \sim k_1\, t_1$$

Hence

$$p - \sum_{1}^{n} \mu_j = k_1$$

Therefore

$$p = 1/2\,(k_1 + k_{n+1}), \quad \sum_{1}^{n} \mu_j = 1/2\,(k_{n+1} - k_1)$$

Yet, for $g_{j-1} \leq t \leq g_j$, we have $t = g_{j-1} + t_j$, so

$$f^a(t) = \sum_{r=1}^{j-1} \mu_r (g_{j-1} + t_j - g_r) + \sum_{r=j}^{n} \mu_r (g_r - g_j - t_j) + p(g_{j-1} + t_j) + q$$

$$f^s(t) = \sum_{r=1}^{j-1} k_r(g_r - g_{r-1}) + k_j t_j + d_1 ,$$

where $g_o = 0$.

Thus

$$f^a(t) = (\sum_1^{j-1} \mu_r - \sum_j^n \mu_r + p)t_j + (\sum_1^{j-1} \mu_r - \sum_j^r \mu_r + p)\, g_{j-1} +$$

$$+ \sum_1^{j-1} \mu_r g_r + \sum_j^r \mu_r g_r + q$$

$$f^s(t) = k_j t_j + \sum_1^{j-1} k_r(g_r - g_{r-1}) + d_1$$

For $t = t_1 \leq g_1$:

$$f^a(t) = (p - \sum_1^n \mu_j)\, t_1 + \sum_1^n \mu_j + q$$

$$f^s(t) = k_1 t_1 + d_1$$

$$k_1 = p - \sum_1^n \mu_j \; ; \quad d_1 = q + \sum_1^n \mu_j$$

From before:

$$k_{n+1} = p + \sum_1^n \mu_j \qquad p = 1/2\,(k_1 + k_{n+1}), \quad \sum_1^n \mu_j = 1/2\,(k_{n+1} - k_1)$$

For $t = g_{j-1} + t_j$: $\quad k_j = p + \sum_1^{j-1} \mu_s - \sum_j^n \mu_s$

For $t = g_{j-2} + t_{j-1}$: $\quad k_{j-1} = p + \sum_1^{j-2} \mu_s - \sum_{j-1}^n \mu_s$

Hence $\quad k_j - k_{j-1} = 2\mu_{j-1}$

Thus, given $k_1, \ldots, k_{n+1}, g_1, \ldots, g_n; d_1$ in the $f^s(t)$ form, we can determine the $f^a(t)$ form via

$$p = 1/2\,(k_1 + k_{n+1})$$

$$g = d_1 - 1/2\,(k_{n+1} - k_1)$$

$$\mu_j = 1/2\,(k_{j+1} - k_j)$$

Conversely, given the $f^a(t)$ form with $\mu_1, \ldots, \mu_n$; p, q and $g_1, \ldots, g_n$, we can determine $d_1, k_1, \ldots, k_{n+1}$, for the $f^s(t)$ form via

$$d_1 = q + \sum_1^n \mu_j$$

$$k_1 = p - \sum_1^n \mu_j$$

$$k_j = p + \sum_1^{j-1} \mu_s - \sum_j^n \mu_s = k_{j-1} + 2\,\mu_{j-1}$$

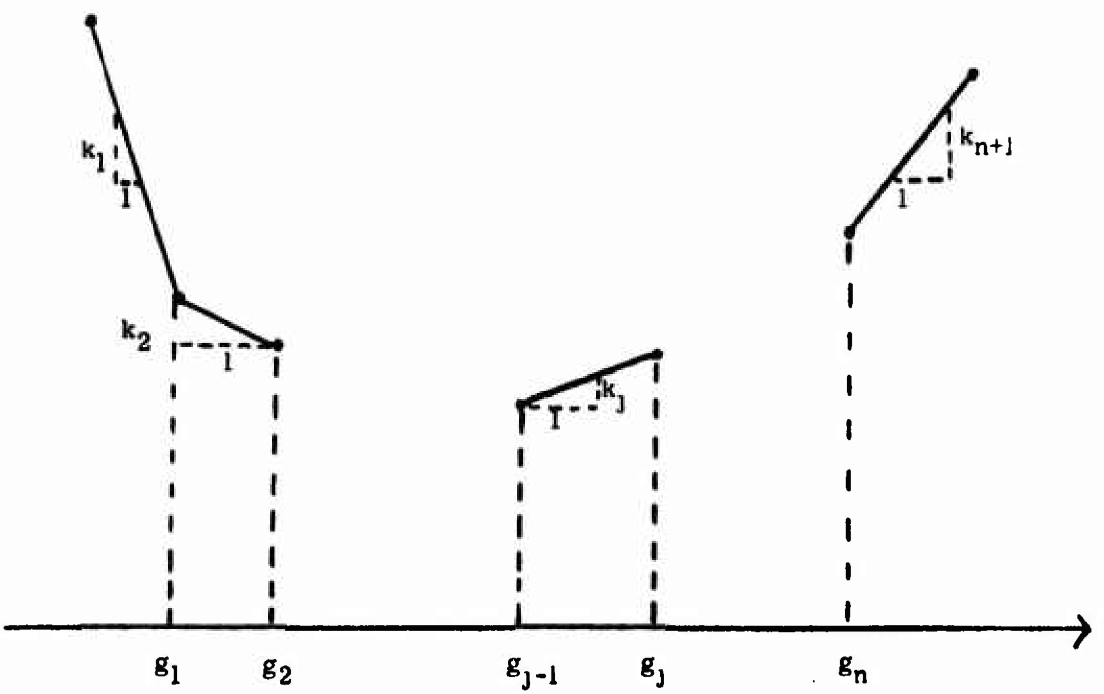



**Figure 2**

Because we are dealing with a piecewise linear and convex problem, we have been able to reduce the portion corresponding to one variable t to a separable linear form on the $t_j$ with individual interval constraints. Going back to our total functional (which is the sum of functions of one variable each) and splitting each variable whose functional is nonlinear, into a sum, as above, we have thereby reduced the solution of the general convex problem to that of a linear program. The solution of the latter will be exhibited in the following section.

4.1 Explicit Solution

Because of the equivalence we have developed in the last section between functions in $f^a(t)$ and $f^s(t)$ format, we can evidently always write the general convex piecewise linear separable functional (with a finite number of pieces) as a linear functional subject to internal constraints on the "incremental" variables. We shall therefore take our candidate for explicit solution in the reduced (equivalent) form:

$$\text{(8.1)} \qquad \begin{aligned} &\min k^{(1)T} x^{(1)} + k^{(2)T} x^{(2)} \\ &d^{(1)T} x^{(1)} + d^{(2)T} x^{(2)} \leq b_0 \\ &a^{(1)} \leq I x^{(1)} \leq b^{(1)} \\ &a^{(2)} \leq A x^{(2)} \leq b^{(2)} \end{aligned}$$

in which we now use vector-matrix notation with

$$k^{(1)T} = (k_1^{(1)}, \ldots, k_{n_1}^{(1)})$$

$$k^{(2)T} = (k_1^{(2)}, \ldots, k_{n_2}^{(2)})$$

$$x^{(1)T} = (x_1^{(1)}, \ldots, x_{n_1}^{(1)})$$

$$x^{(2)T} = (x_1^{(2)}, \ldots, x_{n_2}^{(2)})$$

$A = (a_{ij})$ is of full row rank

Note that if A is of full row rank, so is a matrix $\begin{pmatrix} I & D \\ O & A \end{pmatrix}$. Thus the above, which we have written to separate out that part of the constraints and functional which corresponds to the non-linear goals, can be rendered in the form

$$\begin{aligned} &\min k^T x \\ (8.2) \qquad &d^T x \le b_o \\ &\bar{a} \le \bar{A} x \le \bar{b} \end{aligned}$$

where $\bar{A}$ is of full row rank. We can make the transformation $y = \bar{A}x$, or $x = \bar{A}^{\#} y + (I - \bar{A}^{\#}\bar{A}) z$, where $A^{\#}$ is a generalized inverse of $\bar{A}$ (e.g., a right inverse) and z is an arbitrary vector.

The problem may now be rewritten

$$\min\ k^T\bar{A}^{\#}y + k^T(I-A^{\#}A)z \quad \text{with}$$

$$(9.1) \qquad d^T\bar{A}^{\#}y + d^T(I-\bar{A}^{\#}\bar{A})\,z \le b_o$$

$$\bar{a} \le y \qquad\qquad \le \bar{b}$$

Now let

$$n_j = \begin{cases} d^T\bar{A}_j^{\#}y_j, & \text{if } d^T\bar{A}_j^{\#} \ne 0 \\ y_j, & \text{if } d^T\bar{A}_j^{\#} = 0 \end{cases}$$

$$s_k = \begin{cases} d^T(I-\bar{A}^{\#}\bar{A})_k\, z_k, & \text{if } d^T(I-\bar{A}^{\#}\bar{A})_k \ne 0 \\ z_k, & \text{if } (I-\bar{A}^{\#}\bar{A})_k = 0 \end{cases}$$

Then the problem may be written

$$(9.2) \quad \min\ \hat{k}^T n + \hat{\hat{k}}^T s$$

$$\text{e.t. } \hat{e}^T n + \hat{\hat{e}}^T s \le b_o$$

$$\hat{a} \le n \qquad \le \hat{b}$$

where

$$\hat{e}_j = \begin{cases} 1, & \text{if } d^T\bar{A}_j^{\#} \ne 0 \\ 0, & \text{if } d^T\bar{A}_j^{\#} = 0 \end{cases}$$

$$(9.3) \qquad \hat{\hat{e}}_k = \begin{cases} 1, & \text{if } d^T(I-\bar{A}^{\#}\bar{A})_k \ne 0 \\ 0, & \text{if } (I-\bar{A}^{\#}\bar{A})_k = 0 \end{cases}$$

$$\hat{k}_j = \begin{cases} k^T\bar{A}_j^{\#} / d^T\bar{A}_j^{\#}, & \text{if } d^T\bar{A}_j^{\#} \ne 0 \\ k^T\bar{A}_j^{\#}, & \text{if } d^T\bar{A}_j^{\#} = 0 \end{cases}$$

$$\hat{k}_r = \begin{cases} k^T(I-\bar{A}^{\#}\bar{A})_r / d^T(I-\bar{A}^{\#}A)_r \text{ if } d^T(I-\bar{A}^{\#}\bar{A})_r \neq 0 \\ k^T(I-\bar{A}^{\#}\bar{A})_r \text{ if } d^T(I-\bar{A}^{\#}\bar{A})_r = 0 \end{cases}$$

$$\hat{a}_j = \begin{cases} d^T\bar{A}_j^{\#}a_j \text{, if } d^T\bar{A}_j^{\#} > 0 \\ a_j \text{, if } d^T\bar{A}_j^{\#} = 0 \\ d^T\bar{A}_j^{\#}b_j \text{, if } d^T\bar{A}_j^{\#} < 0 \end{cases}$$

$$\hat{b}_j = \begin{cases} d^T\bar{A}_j^{\#}b_j \text{, if } d^T\bar{A}_j^{\#} > 0 \\ b_j \text{, if } d^T\bar{A}_j^{\#} = 0 \\ d^T\bar{A}_j^{\#}a_j \text{, if } d^T\bar{A}_j^{\#} < 0 \end{cases}$$

Combining the components of n and s into a single vector $\xi^T = (\xi_1, \ldots, \xi_n)$ in the order of decrease of their functional coefficients $\hat{k}_j$, $\hat{\hat{k}}_r$, and adjoining $-\infty$, $+\infty$ lower and upper bounds for the $\xi_r$ which are $S_r$, s, the problem may be written in the form

$$\text{(10.1)} \quad \min \ \tilde{k}^T\xi$$
$$\text{with } \tilde{e}^T\xi \leq b_o$$
$$\tilde{a} \leq \xi \leq \tilde{b}$$

where $\tilde{k}_1 \leq \tilde{k}_2 \leq \ldots \leq \tilde{k}_n$, the $\tilde{a}_j$ correspond to $\hat{a}_j$'s or $-\infty$, the $\tilde{b}_j$ correspond to $\hat{b}_j$'s or $+\infty$, the $\tilde{e}_j$ are the appropriate $\hat{e}_j$ or $\hat{\hat{e}}_k$. For consistency we must have $\sum_j^n \tilde{e}_j \tilde{a}_j \leq b$. Setting $\bar{\xi}_j = \tilde{a}_j + \bar{\xi}_j$, the problem may be rewritten,

$$\max. \quad \sum_j (-\tilde{k}_j)\,\bar{\xi}_j + \sum_j (-\tilde{k}_j)\,\tilde{a}_j$$

(10.2) with $$\sum_j \tilde{e}_j\,\bar{\xi}_j \le b_o - \sum_j \tilde{e}_j\,\tilde{a}_j \equiv b_o'$$

$$\bar{\xi}_j \le \tilde{b}_j - \tilde{a}_j \equiv \tilde{b}_j'$$

$$\bar{\xi}_j \ge 0.$$

This is a linear programming problem. It therefore has the following dual

$$\min. \quad w_o b_o' + \sum_j w_j\,\tilde{b}_j'$$

(10.3) with $$w_o\,\tilde{e}_j + w_j \ge -\tilde{k}_j$$

$$w_o,\ w_j \ge 0.$$

We now introduce

(11) $$J_o \equiv \left\{j:\ -\tilde{k}_j \ge 0 \text{ and } \tilde{e}_j = 0\right\}$$

$$J_1 \equiv \left\{j:\ -\tilde{k}_j \ge 0 \text{ and } \tilde{e}_j = 1\right\}$$

$$r = \max j \text{ such that } -\tilde{k}_j \ge 0,\ \tilde{e}_j = 1,\ \sum_1^{r-} \tilde{e}_s\,\tilde{b}_s' \le b_o'$$

and shall show that an optimal solution to the primal problem is

$$\bar{\xi}_j^* = \tilde{b}_j',\quad j \in J_o$$

(12) $$\bar{\xi}_j^* = \tilde{b}_j',\quad j \in J_1$$

$$\bar{\xi}_j^* = b_o - \sum_{j \in J_1} \tilde{e}_j\,\tilde{b}_j' = b_1' - \sum_{j \in J_1 \cup J_o} \tilde{e}_j\,\tilde{b}_j'$$

$$\bar{\xi}_j^* = 0,\quad j \in \{r\} \cup J_o \cup J_1$$

where $\{r\}$ is the singleton set with element r. Then

$$(13)\qquad \sum_j -\tilde{k}_j \bar{\xi}_j^* = \sum_{J_o}(-\tilde{k}_j)\tilde{b}_j' + \sum_{J_1}(-\tilde{k}_j)\tilde{b}_j' + (-\tilde{k}_r)\,(b_o' - \sum_{J_o \cup J_1} \tilde{b}_j')$$

$$= (-\tilde{k}_r)\,b_o' + \sum_{J_o}\left[(-\tilde{k}_j) - \tilde{e}_j(-\tilde{k}_r)\right]\tilde{b}_j' + \text{terms under } J_1.$$

We now employ the regrouping principle and posit the following values for the dual variables: $w_o^* = -\tilde{k}_r$, $w_j^* = -\tilde{k}_j - e_j(-\tilde{k}_r)$, $j \in J_o \cup J_1$ $w_j^* = 0$ otherwise. Noting that these are all non-negative, we check that these $w^*$ satisfy the other dual constraints--viz.,

$$(14.1)\qquad w_o^*\tilde{e}_j + w_j^* = (-\tilde{k}_r)\,\tilde{e}_j + [-\tilde{k}_j - \tilde{e}_j(-\tilde{k}_r)] = -\tilde{k}_j \text{ for } j \in J_o \cup J_1$$

Thus the $J_o \cup J_1$ dual constraints are satisfied. Next,

$$(14.2)\qquad w_o^*\tilde{e}_j + w_j^* = -(\tilde{k}_r)\,\tilde{e}_j \text{ for } j \notin J_o \cup J_1$$

We need to show here that $-(\tilde{k}_r)\,\tilde{e}_j \geq -\tilde{k}_j$. If $-\tilde{k}_j \geq 0$ then $\tilde{e}_j = 1$ and $j > r$ so that $-\tilde{k}_r \geq -\tilde{k}_j$. If $-\tilde{k}_j < 0$, then since $(-\tilde{k}_r)\tilde{e}_j \geq 0$ we have $(-\tilde{k}_r)\tilde{e}_j \geq \tilde{k}_j$. Thus all dual conditions are satisfied.

Passing back from $\bar{\xi}$ to $\xi$, our optimal solution for (10.1), is therefore

$$\xi_j^* = \tilde{b}_j,\ j \in J_o \cup J_1$$

$$(15)\qquad \xi_r^* = b_o - \sum_{J_o \cup J_1} \tilde{e}_j\,\tilde{b}_j - \sum_{j \notin J_o \cup J_1} \tilde{e}_j\,\tilde{a}_j$$

$$\xi_j^* = \tilde{a}_j,\ j \notin J_o \cup J_1$$

## 4.2 Further Computational Reduction:

For simplicity of exposition the general separable convex piecewise linear functional was transformed directly into linear interval programming format employing additional variables, one for each of the linear segments. Our fundamental lemma, however, makes it possible to reduce (sometimes very substantially) the number of segments which need to be considered and also reduces the number of variables and constraints which need to be introduced. Because of this lemma, we can restrict an original variable in the goal functional to be bounded above by the first $g_J$ at which the goal functional minimum over $a \leq t \leq b$ is attained.

The use of this technique, together with other reductions that are possible for special goal programming functionals, will be presented in additional reports in this series together with applications involving various utilizations of advance starts. As a start toward that end we have supplied a numerical illustration which illustrates some of the preceding developments.

## 5. Numerical Illustration:

We now conclude the present paper with a numerical example which illustrates the remarks we have just made. For this purpose we adapt an example from the paper which initiated this series of studies in manpower planning.

Applying the procedure of model approximation sketched in the next to last paragraph on page II-8 of [ ] to a curtailment of that example we have as our problem

$$\min. \ |N_1(1) - 30| + |N_2(1) - 200| + |N_1(2) - 70| + |N_2(2) - 300|$$

subject to

$$(16) \quad \begin{array}{rlllll} 0 \le & N_1(1) & & & & \le \infty \\ 0 \le & & N_2(1) & & & \le \infty \\ & 15N_1(1) & +13N_2(1) & & & \le 3,000 \\ 44 \le & & & N_1(2) & & \le \infty \\ 143 \le & & & & N_2(2) & \le \infty \\ & & & 15N_1(2) & 13N_2(2) & \le 4,000 \end{array}$$

Here we have confined ourselves to a model involving only two types of manpower in a 2-period plan. Thus, the variable $N_i(t)$ refers to the amount of the $i^{th}$ type of manpower, $i = 1, 2$, scheduled for recruitment in period $t = 1, 2$. The admissible values for these variables are in interval form except for the two budgetary constraints which are applicable at the indicated salary levels of 15 and 13(thousand dollars/annum) in each period. Essentially then this is all in the form of the expressions given in our General Lemma (see section 2, above) with the goals of 30, 200, 70 and 300 stipulated in the absolute value functional for (16). By our Lemma 1, the above example may be transformed into

$$
\begin{array}{lllllll}
 & \text{maximize } N_1(1) + N_2(1) + N_1(2) + N_2(2) & & & & & \\
 & \text{subject to} & & & & & \\
 & 0 \le N_1(1) & & & & \le 30 \\
 & 0 \le & N_2(1) & & & \le 200 \\
(17) & & 15N_1(1) + 13N_2(1) & & & \le 3{,}000 \\
 & 44 \le & & N_1(2) & & \le 70 \\
 & 143 \le & & & N_2(2) & \le 300 \\
 & & & 15N_1(2) + 13N_2(2) & & \le 4{,}000
\end{array}
$$

We now have only an ordinary linear programming problem that involves no more variables than the nonlinear problem (16). Furthermore, (17) may be split into the following two separate subproblems.

$$
\begin{array}{llll}
 & \text{maximize } N_1(1) + N_2(1) \text{ subject to} & & \\
 & 0 \le N_1(1) & & \le 30 \\
(18.1) & 0 \le & N_2(1) & \le 200 \\
 & 15N_1(1) + 13N_2(1) & & \le 3{,}000
\end{array}
$$

and

$$
\begin{array}{lll}
\text{maximize } N_1(2) + N_2(2) \text{ subject to} & & \\
44 \le N_1(2) & & \le 70 \\
143 \le & N_2(2) & \le 300 \\
15N_1(2) + 13N_2(2) & & \le 4{,}000
\end{array}
$$

Without further ado we can then write our optimal solution for

(17), and hence (16), via (18.1) and (18.2) This gives

$$N_1^*(1) = 400/15, \quad N_2^*(1) = 200$$

$$N_1^*(2) = 44 \qquad , \; N_2^*(2) = 143 + \frac{1481}{13} = 257.$$

Evidently our lemma has produced a simplification so that achieving a solution to the nonlinear problem (16) involved practically nothing more than merely a solution by inspection of two much smaller linear problems. This suggests further possible ways in which the ideas contained in our lemma may be extended to other classes of problems. Such developments are best delayed, however, until they can be treated in their own right via other papers in this series.

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