AD-784 019

A NOTE ON EXCHANGES IN MATROID BASES

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Prepared for:

Naval Personnel Research and Development Laboratory
Office of Naval Research

July 1974

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Unclassified

Security Classification

AD 784019

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Center for Cybernetic Studies The University of Texas	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A Note on Exchanges in Matroid Bases

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

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6. REPORT DATE July 1974	7a. TOTAL NO. OF PAGES 11	7b. NO. OF REFS 2
8a. CONTRACT OR GRANT NO. N00123-74-C-2272 N00014-67-A-0126-0008; 0009 b. PROJECT NO. NR047-021 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Center for Cybernetic Studies Research Report CS 184 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research (Code 434) Washington, D.C.

13. ABSTRACT

In a matroid with bases B and B', a B-exchange is a pair of elements e, e', where B - e + e' is a base. A serial exchange of B into B' is a sequence of pairs e_i, e_i', for i = 1, ..., n, such that e_i, e_i', is a B_{i-1}-exchange, where $B_0 = B$, $B_i = B_{i-1} - e_i + e_i'$, and $B_n = B'$. This paper shows there is a one-to-one correspondence between elements of B and B' such that corresponding elements e, e' give B-exchanges; furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'. A symmetric exchange is a pair of elements e, e' such that e, e' is a B-exchange and e', e is a B'-exchange. Any element of B can be symmetrically exchanged with at least one element of B'. But in contrast to B-exchanges, it is not always possible to make a correspondence between B and B' so corresponding elements give symmetric exchanges.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Matroids						
Spanning Trees						
Telecommunications						

DD FORM 1 NOV 65 1473 (BACK)

1a

Research Report
C.S. 184

A NOTE ON EXCHANGES IN MATROID BASES

by

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July 1974

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This research was partly supported by Project No. NR 047-021, ONR Contracts N00014-67-A-0126-0008 and N00014-67-A-0126-0009 with the Center for Cybernetic Studies, The University of Texas and Contract N00123-74-C-2272 with the Naval Personnel Research & Development Laboratory, San Diego, California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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ABSTRACT

In a matroid with bases B and B', a B-exchange is a pair of elements e, e', where $B - e + e'$ is a base. A serial exchange of B into B' is a sequence of pairs e_i, e_i', for $i = 1, \ldots, n$, such that e_i, e_i' is a B_{i-1}-exchange, where $B_0 = B$, $B_i = B_{i-1} - e_i + e_i'$, and $B_n = B'$. This paper shows there is a one-to-one correspondence between elements of B and B' such that corresponding elements e, e' give B-exchanges; furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'. A symmetric exchange is a pair of elements e, e' such that e, e' is a B-exchange and e', e is a B'-exchange. Any element of B can be symmetrically exchanged with at least one element of B'. But in contrast to B-exchanges, it is not always possible to make a correspondence between B and B' so corresponding elements give symmetric exchanges.

Many network and linear programming problems are solved by repeatedly exchanging elements of a base. The pivot step in linear programming is a general example. The existence of such exchanges can be taken as a defining property of a matroid [2]. This note presents results concerning several types of matroid base exchanges.

First we define three types of exchanges. Let M be a matroid, with bases B and B'. For example, Figure shows the graphic matroid on four nodes. One base consists of the solid arcs 1, 2, 3; another base consists of the dotted arcs 4, 5, 6.

An ordered pair of elements e, e' is a <u>B-exchange</u> if $B - \{e\} + \{e'\}$ is a base. Table 1 shows the possible B-exchanges for each element in $B = \{1, 2, 3\}$.

A serial exchange of B into B' is a sequence of ordered pairs, e_1, e_1'; $e_2, e_2', \ldots, e_n, e_n'$, such that for all i in $1 \le i \le n$, a base is formed by the set

$$B_i = B - \{e_1, \ldots, e_i\} + \{e_1', \ldots, e_i'\}$$

Furthermore, $B_n = B'$. The definition implies each pair e_i, e_i' is a B_{i-1} exchange. Hence the sequence of exchanges can be executed serially. Figure 2 shows a serial exchange of the base {1, 2, 3} into {4, 5, 6}.

A <u>symmetric exchange</u> is an ordered pair of elements e, e' such that the sets $B - \{e\} + \{e'\}$ and $B' - \{e'\} + \{e\}$ are bases. Equivalently, the pair e, e' is a B-exchange and e', e is a B' exchange. Table 2 shows the possible symmetric exchanges for each element in $B = \{1, 2, 3\}$

To characterize these exchanges, we introduce notation for some well-known matroid concepts [2]. For a base B and an element $f \notin B$, <u>B(f)</u> denotes the unique circuit in the set B + f. In Figure 1, for base $B = \{1, 2, 3\}$, $B(5) = \{2, 3, 5\}$.

For a set of elements D, <u>sp(D)</u> denotes the span of D. This set is defined as the smallest superset of D such that for any element f, if sp(D) + f contains a circuit containing f, then $f \in sp(D)$. In Figure 1, sp ({ 4, 6}) = {2, 4, 6}.

<u>Lemma 1</u>: For elements $e \in B$, $e' \notin B$, these conditions are equivalent:

(i) e, e' is a B-exchange

(ii) $e \in B(e')$.

(iii) $e' \notin sp(B-e)$.

<u>Proof:</u> An immediate consequence of the definitions.

<u>Corollary 1</u>: For elements $e \in B - B'$, $e' \in B' - B$, these conditions are equivalent:

(i) e, e' is a symmetric exchange.

(ii) $e' \in B'(e) - sp(B - e)$

It is apparent from the lemma that any element $e \in B$ gives a B-exchange with at least one element of B'. We show the same is true for symmetric exchanges.

<u>Theorem 1</u>: For any element $e \in B$, there is an element $e' \in B$ such that e, e' is a symmetric exchange.

<u>Proof</u>: Consider any element $e \in B$. If $e \in B'$, then clearly e, e' is a symmetric exchange. So assume $e \notin B'$.

Since B is a base, element $e \notin sp(B-e)$. Thus the circuit B' (e) is not contained in sp(B - e) + e, that is,

$$B'(e) = e \not\subseteq sp(B - e).$$

Now corollary 1 shows there is a symmetric exchange for e, completing the proof.

In Figure 1, we can pair the elements of $B = \{1, 2, 3\}$ and $\{4, 5, 6\}$ so each pair gives a B-exchange: 1, 6; 2, 5; 3, 4. Figure 2 shows these pairs, in the given sequence, are a serial exchange of $\{1, 2, 3\}$ into $\{4, 5, 6\}$. Now we show such a pairing can be made in general.

Theorem 2: There is a one-to-one correspondence between elements of B and B', such that corresponding elements e, e' give a B-exchange. Furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'.

Proof: Denote the bases by

$$B = \{e_1, e_2, \ldots, e_n\}, \quad B' = \{e'_1, e'_2, \ldots, e'_n\}.$$

We assert indices can be chosen in B so for all i in $1 \leq i \leq n$, the pair e_i, e'_i is a B-exchange; furthermore, a base is formed by the set

$$B'_i = \{e_1, e_2, \ldots, e_i, \; e'_{i+1}, e'_{i+2}, \ldots, e'_n\}.$$

Note the assertion implies the theorem. For the pairs e_i e'_i give a correspondence of B-exchanges, and the sequence e_n e'_n; e_{n-1}; ...; e_1, e'_1 is a serial exchange of B into B'. The assertion is proved by induction on i. The initial step, $i = 0$, is obvious, since $B'_0 = B$ is a base. For the inductive step, suppose B'_i is a base. We prove the assertion for $i + 1$, as follows. Element e'_{i+1} of base B'_i gives a symmetric exchange with some element of base B. This element cannot be e_j, for j in $1 \leq j \leq i$, since $e_j \in B'_i$. With proper choice of indices, we can assume e_{i+1}, e'_{i+1} is a symmetric exchange. Thus e_{i+1}, e'_{i+1} is a B-exchange, and B'_{i+1} is a base. This completes the induction.

The proof of Theorem 2 gives a constructive procedure for finding the one-to-one correspondence of B-exchanges. The theorem itself, specialized to graphic matroids, is useful in finding minimum weight spanning trees with specified degree at one node [1].

It is natural to try to generalize Theorem 2 to symmetric exchanges. However Table 2 shows it is not always possible to pair the elements of two bases so each pair is a symmetric exchange.

4

Figure 1. Bases {1, 2, 3} and {4, 5, 6} in a graphic matroid.

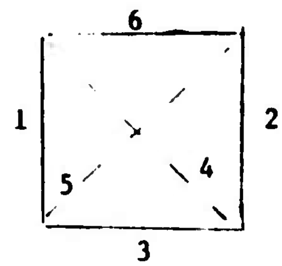


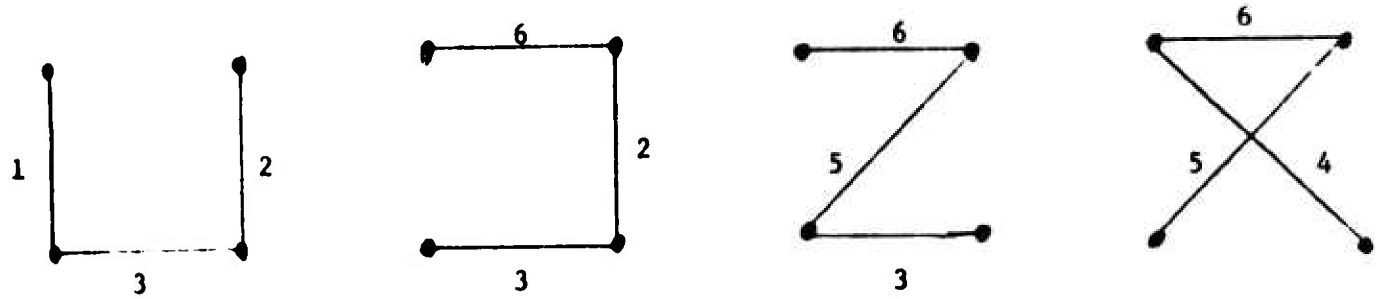
Table 1. B-exchanges for e, B = {1, 2, 3}.

e	e'
1	4, 6
2	5, 6
3	4, 5, 6

Table 2. Symmetric exchanges for e.

e	e'
1	6
2	6
3	4, 5, 6

Figure 2. Serial exchange of {1, 2, 3} into {4, 5, 6}.



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