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ON TIME TO FIRST FAILURE IN MULTI-
COMPONENT EXPONENTIAL RELIABILITY
SYSTEMS

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ABSTRACT

Consider an n component reliability system having the property that at any time each of its components is either up (i.e., working) or down (i.e., being repaired). Each component acts independently and we suppose that each time the i th component goes up it remains up for an exponentially distributed time having mean μ_i , and each time it goes down it remains down for an exponentially distributed time having mean v_i . We further suppose that whether or not the system itself is up at any time depends only on which components are up at that time. We are interested in the distribution of the time of first system failure when all components are initially up at time zero. In Section 2 we show that this distribution has the NBU (i.e., new better than used) property, and in Section 3 we make use of this and other results to obtain a lower bound to the mean time until first system failure.

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ON TIME TO FIRST FAILURE IN MULTICOMPONENT
EXPONENTIAL RELIABILITY SYSTEMS

by

Sheldon M. Ross

1. INTRODUCTION AND SUMMARY

Consider an n component reliability system having the property that at any time each of its components is either up (i.e., working) or down (i.e., being repaired). Each component acts independently and we suppose that each time the i th component goes up it remains up for an exponentially distributed time having mean μ_i , and each time it goes down it remains down for an exponentially distributed time having mean ν_i .

We suppose that whether or not the system itself is up at any time depends only on which components are up at that time. In particular, letting

$$X_i(t) = \begin{cases} 1 & \text{if the } i\text{th component is up at time } t \\ 0 & \text{otherwise} \end{cases}$$

and

$$X(t) = \begin{cases} 1 & \text{if the system is up at time } t \\ 0 & \text{otherwise} \end{cases}$$

we suppose that there exists a nondecreasing binary function ϕ such that

$$X(t) = \phi(X_1(t), \dots, X_n(t)) .$$

The function ϕ is called the structure function of the system.

The above system, with the added generality of arbitrary, rather than exponential, component uptime and downtime distributions was considered in Ross [5] where such quantities of interest as (i) the average system failure rate; (ii) the average length of system uptime; and (iii) the average length of system downtime,

were computed. In the present paper, we are interested in the distribution of the time of first system failure when all components are initially up at time zero. In Section 2 we show that this distribution has the NBU (i.e., new better than used) property, and in Section 3 we make use of this and other results to obtain a lower bound to the mean time until first system failure.

2. SOME PRELIMINARIES CONCERNING NBU DISTRIBUTIONS

We say that a lifetime distribution F is NBU (new better than used) if

$$\frac{\bar{F}(t+s)}{\bar{F}(t)} \leq \bar{F}(s) \quad \text{for all } s, t \geq 0.$$

In words, F is NBU if the probability that a t year old item survives an additional s years is less than or equal to the probability that a new item survives s years, for all s, t .

Proposition 1:

For an arbitrary structure ϕ . Assuming that all components are initially up, the time until the first system failure has an NBU distribution.

Proof:

We shall prove the proposition by first showing, by induction, that the time to first system failure will have a stochastically larger distribution if all components are initially up as opposed to $n - k$ of them being initially up and the other k initially down. To prove it for $k = 1$, suppose that components 2 through n are initially up and suppose that there are two identical components that can be used as component 1 --namely $1'$ which is initially up or $1''$ which is initially down. To show that the system will have a stochastically larger time to first failure if we use $1'$ as opposed to $1''$, let us condition on the first time τ such that

$$\phi(1, X_2(\tau), \dots, X_n(\tau)) = 1, \quad \phi(0, X_2(\tau), \dots, X_n(\tau)) = 0.$$

Now, given that $\tau = t$, the probability that the first system failure will occur before (or at) time x is the same irregardless of whether $1'$ or $1''$ is used when $x < t$. On the other hand, if $\tau = t$ and the system has not failed prior to

time t , then the probability that it fails at time t is equal to the probability that component 1 is down at t . However, given that $\tau = t$, it follows from the theory of two-state Markov processes that the probability that the first component is up at time τ is greater if it is initially up as opposed to being

initially down (the probabilities being $\frac{\mu_1}{\mu_1 + v_1} + \frac{v_1}{\mu_1 + v_1} e^{-(\mu_1 + v_1)t}$ and $\frac{\mu_1}{\mu_1 + v_1} - \frac{v_1}{\mu_1 + v_1} e^{-(\mu_1 + v_1)t}$). Therefore, given $\tau = t$, the system is more likely

to experience its first failure by time t if $1''$ is used. Furthermore, by the lack of memory of the exponential distribution, it follows that if the system has not failed by time τ (and thus component 1 is up at time τ) then the distribution of the additional time until system failure is the same irregardless of which component-- $1'$ or $1''$ --was used as component 1. Hence, given τ , the system will have a stochastically larger time to first failure if $1'$ --as opposed to $1''$ --is used. Taking the expectation of τ (i.e., unconditioning), then yields the result when $k = 1$.

The general case is now easily established for if k of the components are initially down then by concentrating on one of these components the same argument as the one presented for the case $k = 1$ shows that it would have been better (in a stochastic sense) if this component was initially up. But by the induction hypothesis all components initially up is better than having any set of $k - 1$ initially down and thus the induction is complete.

The proposition now follows for if the system is up at time t we see, by conditioning on the number of components up at time t , that the system has less chance of surviving an additional s years than does a system that has all of its components up (i.e., a new system).||

Remark:

If all components are not initially up, then Proposition 1 is no longer valid. For instance, consider a parallel system of two components (i.e., the system is down when both components are down) which have identical exponential uptime and downtime distributions, and suppose that one of the components is up at time zero and the other is down. Then if the system has not yet failed by time t it follows that the remaining time to failure is a mixture of two distributions one of which is the distribution of the time from zero to the first failure while the other one is (strictly) stochastically larger. (The first distribution comes into effect if only one of the components is up at time t and the other if both are up at time t .) Hence, the distribution of additional time to failure would be stochastically larger than the distribution of time to first failure. In fact, it follows from results presented by Keilson [3] that the distribution of time to first failure is in this example, a mixture of exponential distributions, which not only is not an NBU distribution but is a DFR (decreasing failure rate) distribution (see [5] for definition.)

A second result about NBU distributions that we shall use is embodied in the following proposition.

Proposition 2:

Let T_1, T_2, \dots, T_m be nonnegative independent random variables, each having an NBU distribution, and let $\mu_i = E[T_i]$. Then

$$E[\text{Min}(T_1, \dots, T_m)] \geq \left[\sum_{i=1}^m 1/\mu_i \right]^{-1}.$$

Proof:

Let X be an NBU distribution having distribution F and mean μ_x and let

Y be an arbitrary nonnegative random variable having distribution G and mean μ_y , and assume that X and Y are independent. Now,

$$E[\text{Min}(X, Y)] = \int_0^{\infty} E[\text{Min}(X, t)] dG(t) .$$

Also,

$$\begin{aligned} E[\text{Min}(X, t)] &= \int_0^t x dF(x) + t(1 - F(t)) \\ &= \mu_x - \int_t^{\infty} x dF(x) + t(1 - F(t)) \\ &= \mu_x - E[X \mid X > t](1 - F(t)) + t(1 - F(t)) \\ &\geq \mu_x - (t + \mu_x)(1 - F(t)) + t(1 - F(t)) \\ &= F(t)\mu_x \end{aligned}$$

when we have used the fact that F NBU implies that $E[X \mid X > t] \leq t + \mu_x$.

Hence,

$$\begin{aligned} (1) \quad E[\text{Min}(X, Y)] &\geq \mu_x \int_0^{\infty} F(t) dG(t) \\ &= \mu_x P\{X \leq Y\} . \end{aligned}$$

Now let T_i , $i = 1, \dots, m$ be independent NBU random variables with means μ_i , $i = 1, \dots, m$. From Equation (1) we obtain, for each i ,

$$\begin{aligned} E[\text{Min}(T_1, \dots, T_m)] &= E[\text{Min}(T_i, \text{Min}(T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_m))] \\ &\geq \mu_i P\{T_i = \text{Min}(T_1, \dots, T_m)\} . \end{aligned}$$

Hence, dividing the above by μ_1 and then summing over i yields

$$\left(\sum_{i=1}^m 1/\mu_i \right) E[\text{Min}(T_1, \dots, T_m)] \geq \sum_{i=1}^m P\{T_i = \text{Min}(T_1, \dots, T_m)\} \\ = 1$$

and the proof is complete. ||

Remark:

Proposition 2 states that the mean life of a series system of independent NBU components is greater than or equal to the mean life of a series system of independent exponential components having the same set of means. (A reverse inequality can be proven, in the same way, for parallel systems.) This result had previously been proven (see [4]) under the stronger (than NBU) assumption that each component lifetime is IFRA, where we say that the lifetime distribution F is IFRA (increasing failure rate on the average) if $\frac{-\log(1 - F(t))}{t}$ is nondecreasing in t . Further we see from the proof of this proposition that the condition that the T_i are NBU can be reduced to the condition that $E[T_i | T_i > t] \leq t + E[T_i]$ for all t . This weaker than NBU assumption is referred to in the literature as assuming that T_i is NBUE (new better than used in expectation).[†]

[†]After formulating and proving Proposition 2, the author discovered that the result is contained as an exercise in a forthcoming text on reliability theory by R. Barlow and F. Proschan. However, their indicated method of proof is different from the one employed in the present paper.

3. MEAN TIME TO FIRST SYSTEM FAILURE

One way of characterizing the structure function ϕ of the reliability system is in terms of its minimal cut sets. We say that the subset C of the set of components is a cut set if the system is necessarily down when all components in this subset are down. In addition, we say that it is a minimal cut set if it does not (properly) contain any other cut sets. Let us denote by C_1, \dots, C_s , the minimal cut sets of the structure ϕ . Since

$$\phi(x_1, \dots, x_n) = \text{Min} (\alpha_1, \dots, \alpha_s)$$

where

$$\alpha_i = \begin{cases} 0 & \text{if all components of } C_i \text{ are down} \\ 1 & \text{otherwise} \end{cases}$$

it follows that knowing the minimal cut sets is equivalent to knowing the structure ϕ .

If, in our reliability system, we let T_i denote the first time that all components in the i th cut set are down, then T , the time to first system failure, is given by

$$T = \text{Min} (T_1, \dots, T_s).$$

As the T_i so defined will not, in general, be independent (they will be independent only if no two minimal cut sets overlap) we cannot immediately apply Propositions 1 and 2 to obtain a lower bound for $E(T)$ in terms of the $E[T_i]$. However, it has been shown by Esary and Proschan [2], using the concept of association of random variables, that if the component uptime and downtime distributions are exponential then

$$P\{T > t\} \geq \prod_{i=1}^s P\{T_i > t\}.$$

Using this result, now yields

Proposition 3:

Assuming that each component is initially up, then

$$E[T] \geq \left(\sum_{i=1}^s 1/E[T_i] \right)^{-1}.$$

Proof:

$$\begin{aligned} E[T] &= \int_0^{\infty} P\{T > t\} dt \\ &\geq \int_0^{\infty} \prod P\{T_i > t\} dt \quad \text{by Esary and Proschan [2]} \\ &\geq \left[\sum_{i=1}^s 1/E(T_i) \right]^{-1} \quad \text{by Proposition 2.} \end{aligned}$$

Thus, it remains to determine $E(T_i)$. Now T_i is the first time that all components of the i th minimal cut set C_i are down, and so considering only those components in C_i we see that determining $E(T_i)$ is equivalent to determining the mean life of a parallel system when all components are initially up. So let us consider the parallel system of the (say) r components of the minimal cut set C and suppose that these r components have mean exponential uptimes μ_1, \dots, μ_r and mean exponential downtimes v_1, \dots, v_r . Now it has been shown by Brown [1] that the mean time until all components of C are down is equal to

$$\sum_{k=1}^r \sum_{i_1 < i_2 < \dots < i_k} \left[\prod_{j=1}^k \frac{\mu_{i_j}}{v_{i_j}} - (-1)^k \right] \cdot \frac{1}{\sum_{j=1}^k \left(\frac{1}{\mu_{i_j}} + \frac{1}{v_{i_j}} \right)}.$$

Thus, we can use Brown's formula above to compute $E[T_i]$ for each minimal cut set and then use Proposition 3 to obtain a lower bound on the mean time to first system failure. These computations can be easily effected by use of a computer.

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