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A STUDY OF PRINCIPAL COMPONENT ANALYSIS AS APPLIED TO SEPARATION OF MULTIPLE PLANE WAVES

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Teledyne Geotech

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Principal component analysis, eigenvalue-eigenvector decomposition of array spectral matrices, has been proposed by several authors as a technique for the separation of the signal components of mixed signals observed at an array since the eigenvectors of the spectral matrix often resemble signal delay vectors. This report is a critical evaluation of the method. The findings of the report are as follows.

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In the strict theoretical sense the eigenvectors of array spectral matrices derived from multiple plane waves cannot be considered as delay vectors of the individual plane wave components.

If some restrictive conditions are met, however, the eigenvectors tend to be similar to signal delay vectors. These conditions are that the power levels of the various plane waves be grossly unequal and/or that the individual plane wave delay vectors are orthogonal.

If these conditions are met the eigenvector decomposition is found to be superior at small arrays to both two-segment maximum likelihood processing and to FKCOMB with stripping.

A STUDY OF PRINCIPAL COMPONENT ANALYSIS AS APPLIED TO SEPARATION OF MULTIPLE PLANE WAVES

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

THEORY

Principal component analysis has been proposed by several authors as a tool for separating multiple signals (Goodman 1967, Choi 1967, Barnard 1968, Booker and Ong 1972). The technique is based on the observation that eigenvectors of the spectral matrix of signals recorded at an array are often found to be similar to signal delay vectors and that the eigenvalues of the matrix are proportional to the relative power of the various signal components. The signal delay vector is defined in equation 2 below, it is a complex vector such that the phase of each component is the phase shift of the signal at the corresponding array element. The absolute value of each component is the same. In the presence of random noise each eigenvector is still proportional to a signal delay vector but the eigenvalue loses its proportionality to the total signal powers. This report is a critical evaluation of the method as applied to the separation of plane wave signals.

Strictly speaking the method as applied to seismic arrays with a small number of elements (less than ten) stands on a very shaky foundation. Assuming that the signal observed consists of a superposition of n plane waves, where $n \leq m$, m being the number of array elements, the spectral matrix can be written (dependence of the quantities on frequency is implied) as the weighted outer product of the eigenvectors $\tilde{V}_i$

$$\Phi = \sum_{i=1}^{n} p_i \tilde{V}_i \tilde{V}_i^* + \rho I \qquad (1)$$

assuming that the waveforms are uncorrelated between signals, where p_i is the power level of the i'th signal, ρ is the random noise level, and $\tilde{V}_i$ is the column vector (~ above symbols denote complex vectors in the notation used.).

$$\tilde{V}_i = \frac{1}{m} \operatorname{col}[e^{i\tilde{k}_i\tilde{x}_1}, e^{i\tilde{k}_i\tilde{x}_2}, \ldots, e^{i\tilde{k}_i\tilde{x}_m}] \tag{2}$$

where $\tilde{k}_i$ is the wavenumber vector of the i'th signal, $\tilde{x}_i$ is the position vector of the i'th array element and the star denotes conjugate transpose. The same spectral matrix can also be written

$$\Phi = \sum_{i=1}^{m} \lambda_i \tilde{E}_i \tilde{E}_i^* \tag{3}$$

where λ_i and $\tilde{E}_i$ are the i'th eigenvalue and corresponding eigenvector of Φ. It is obvious that the two representations are not identical even if we put $\rho = 0$ and $\lambda_i = 0$ for $i > n$, since

$$\tilde{E}_i^* \tilde{E}_j = \delta_{ij} \tag{4a}$$

but in general

$$\tilde{V}_i^* \tilde{V}_j \neq \delta_{ij} \tag{4b}$$

thus only in special cases do the two representations coincide. The performance of the method improves if the number of array elements increases and the power levels of the various signals differ. In fact it has been shown that in the limiting case $m \to \infty$ the signals separate on the eigenvectors (Grenander and Szegö). Further discussion of the theory can be found in the Appendix. The mathematical background for the discussion can be found in any basic text on matrix theory, e.g. (Ayres, 1962).

Booker and Ong (1972) showed that the signals can be separated successfully with a 20-element linear array even if their amplitudes differ by several orders of magnitude. They also showed that the method is relatively stable with respect to variations in the sample and random noise level. In this report we re-examine the problem using a smaller array which is of greater practical interest than a linear 20-element array, and we also include cases when several signals of comparable magnitude are present.

The results are shown in the form of f-k plots. The eigenvectors can be beamed, treating the eigenvector components as signal Fourier delay values at each frequency to obtain the f-k plot of

$$S(\tilde{k}) = |\tilde{E}^*\tilde{B}|^2 \qquad (5)$$

where $\tilde{B}$ is the beamsteer vector of the form identical to V_i with $\tilde{k}$ instead of $\tilde{k}_i$.

Alternatively some kind of high resolution f-k spectra can be computed (Capon 1967, Lintz 1968, McCowan and Lintz 1968). To compute the commonly used maximum likelihood f-k spectra, a pseudo spectral matrix of the j'th eigenvector can be formed

$$\Phi = \tilde{E}_i\tilde{E}_i^* + \gamma I$$

and the maximum likelihood f-k spectrum

$$\tilde{S}(\tilde{k}) = \frac{1}{\tilde{B}^{*}\Phi^{-1}\tilde{B}} \qquad (6)$$

computed.

Since Φ^{-1} does not exist if $\gamma=0$, a small amount of independent random noise has to be added by adding γI to the diagonal elements of $\tilde{\Phi}$, where I is the unit matrix.

In this report the latter representation is used as a matter of convenience, since the maximum likelihood f-k spectrum is less dependent on the array response, and thus the sidelobes present in the $\tilde{k}$ response function of small arrays are less confusing (Barnard 1968).

EXPERIMENTS ON SYNTHETIC SPECTRAL MATRICES

The numerical experiments described below were performed on hypothetical L-shaped five-element array. This configuration was chosen as an example for its simplicity, and no particular advantage for this array is claimed. The array configuration is shown in Figure 1. Plane wave signals were superposed using equations (1) and (2), and 2% random noise power was added to the diagonal elements.

The results from the first example are shown in Figure 2. The signal power levels and azimuths are given in Table 1a. The f-k plots of the various eigenvectors in the order of decreasing eigenvalues (i.e. decreasing power levels) are denoted by letters A-E. The figures show that the first three signal components are clearly discernible on the f-k plots although the power levels of the first and third signals differ by two orders of magnitude, the fourth and fifth signal are, however, not resolved. The signals were assumed to have a phase velocity of 3.8 km/sec and a period of 20 seconds. The circle in the figure denotes the 3.8 km/sec velocity and the solid line from the center gives the azimuth of the successive signal components. The figures show that the first three signals are fairly well separated by the eigenvector-eigenvalue analysis.

Figure 3 further illustrates what is happening. This figure shows the polar representations of signal delay vectors and spectral eigenvectors in the complex plane. The first signal and the first eigenvector are

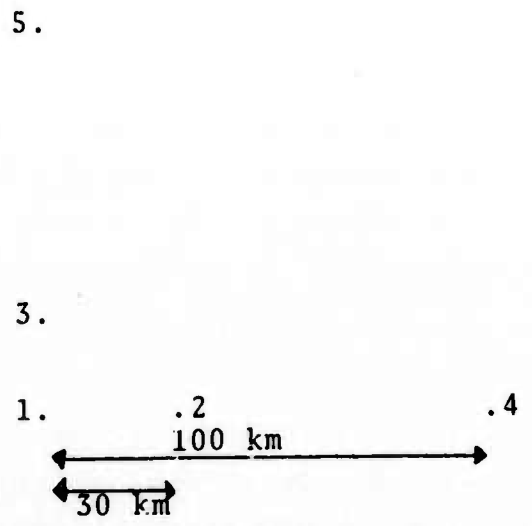


Figure 1. L shaped array assumed in synthesizing spectral matrices

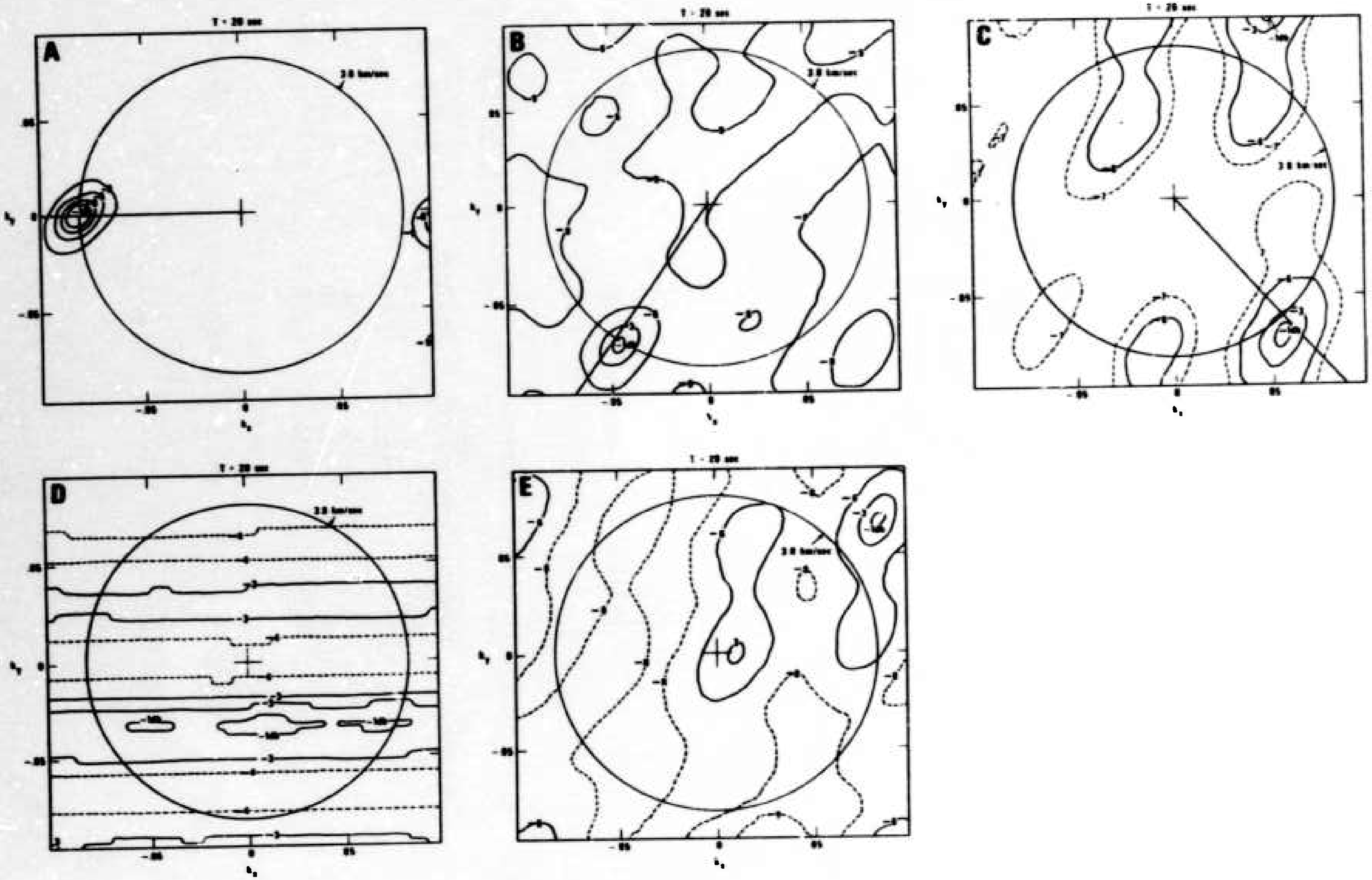


Figure 2 F-K plots obtained for the superposition of synthetic signals with parameters given in Table Ia.

TABLE I

Power Levels and Azimuths of Signals used in Simulation

(a)

Signal	Power	Azimuth (Degrees)	Figure
1	1.	270.	2A
2	.1	210.	2B
3	.01	150.	2C
4	.001	90.	2D
5	.0001	30.	2E

(b)

Signal	Power	Azimuth (Degrees)	Figure
1	1.	270.	4A
2	.1	240.	4B
3	.01	210.	4C
4	.001	180.	4D
5	.0001	150.	4E

(c)

Signal	Power	Azimuth (Degrees)	Figure
1	1.	270.	5A
2	1.	210.	5B
3	.5	150.	5C
4	.5	90.	5D
5	.2	30.	5E

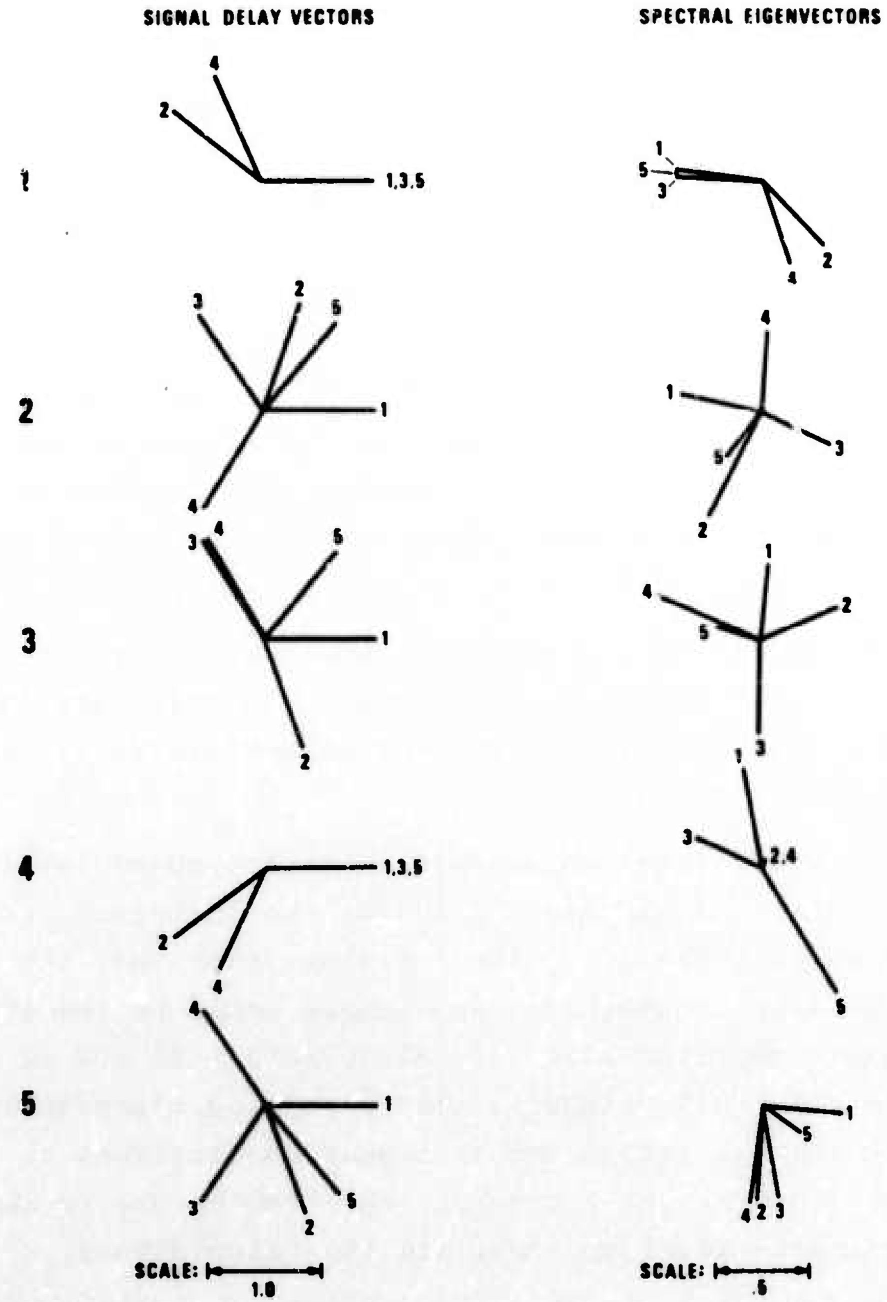


Figure 3. Comparison of the complex eigenvectors and signal delay vectors for the synthetic example with parameters given in Table Ia.

very similar (disregarding the rotation which corresponds to a simple phase shift). As the order of eigenvectors increases (in order of decreasing eigenvalues) the eigenvectors and signal vectors lose their resemblance. The eigenvectors corresponding to components which could not be resolved do not resemble any signal components at all. This did not result from any inaccuracies in the eigenvalue analysis routines; the original spectral matrix could be reconstructed in every case, but it is caused by the basic inadequacy of the method as outlined in the appendix. It is interesting that eigenvector #3 is already considerably different from the signal delay vector, yet it still produces a fair f-k spectra.

Figure 4 shows a similar example with signals spaced at 30° intervals in azimuth. In this case too, three signals are resolved. The parameters of signals are given in Table 1b.

Figure 5 shows an example where the power levels of two pairs of signals, including the strongest, are comparable (Table 1c). The f-k plots show that the method fails completely: no signals could be resolved. The polar representations of eigenvectors do not resemble any of the signals. The resulting eigenvectors cannot even be recognized as linear combinations of signal vectors (see Appendix); this can be due to the random noise added to stabilize the calculations.

Summarizing, if the signal levels are different, the first eigenvector tends to line up on the most

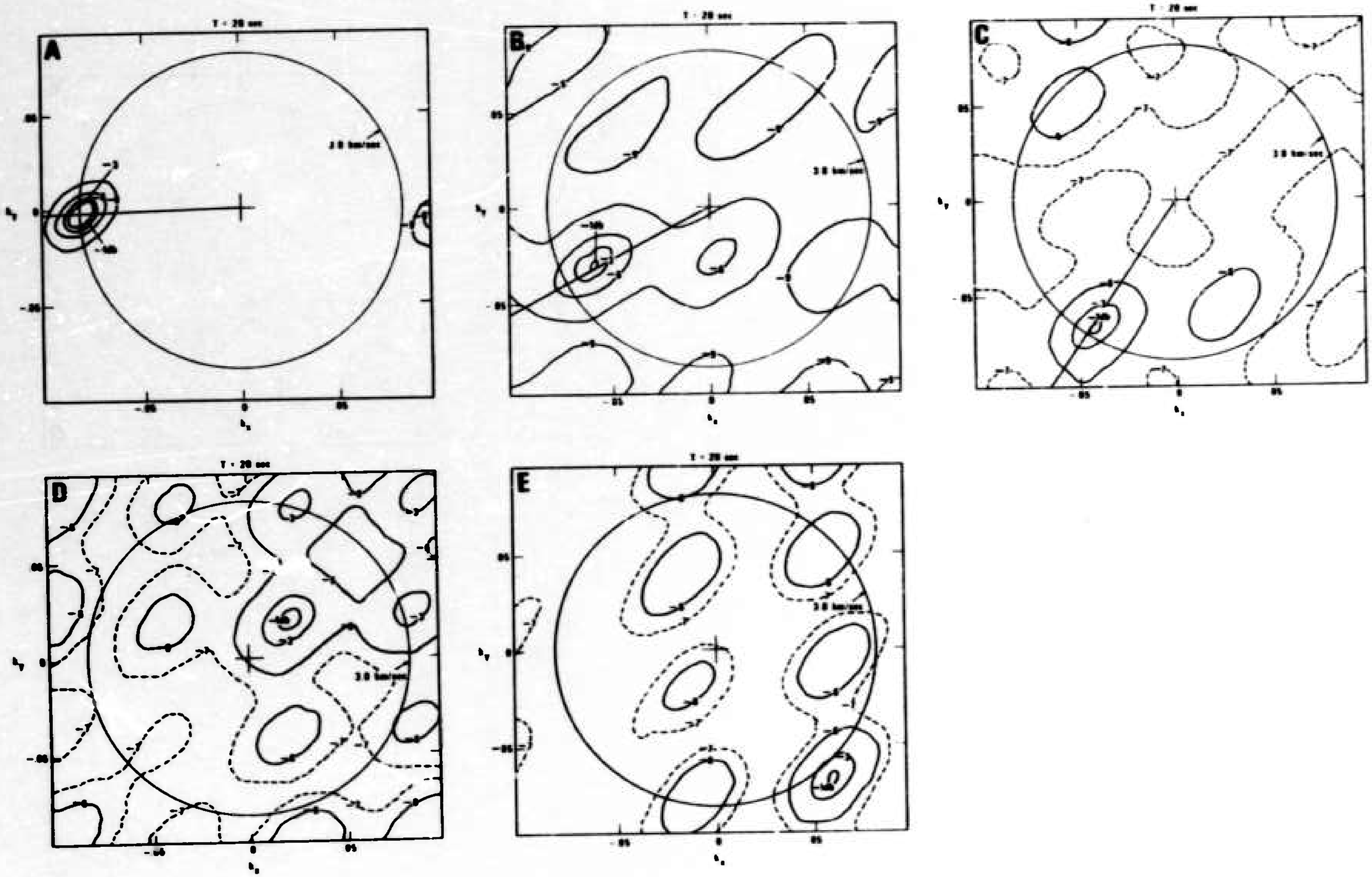


Figure 4 F-K plots obtained for the superposition of synthetic signals with parameters given in Table Ib.

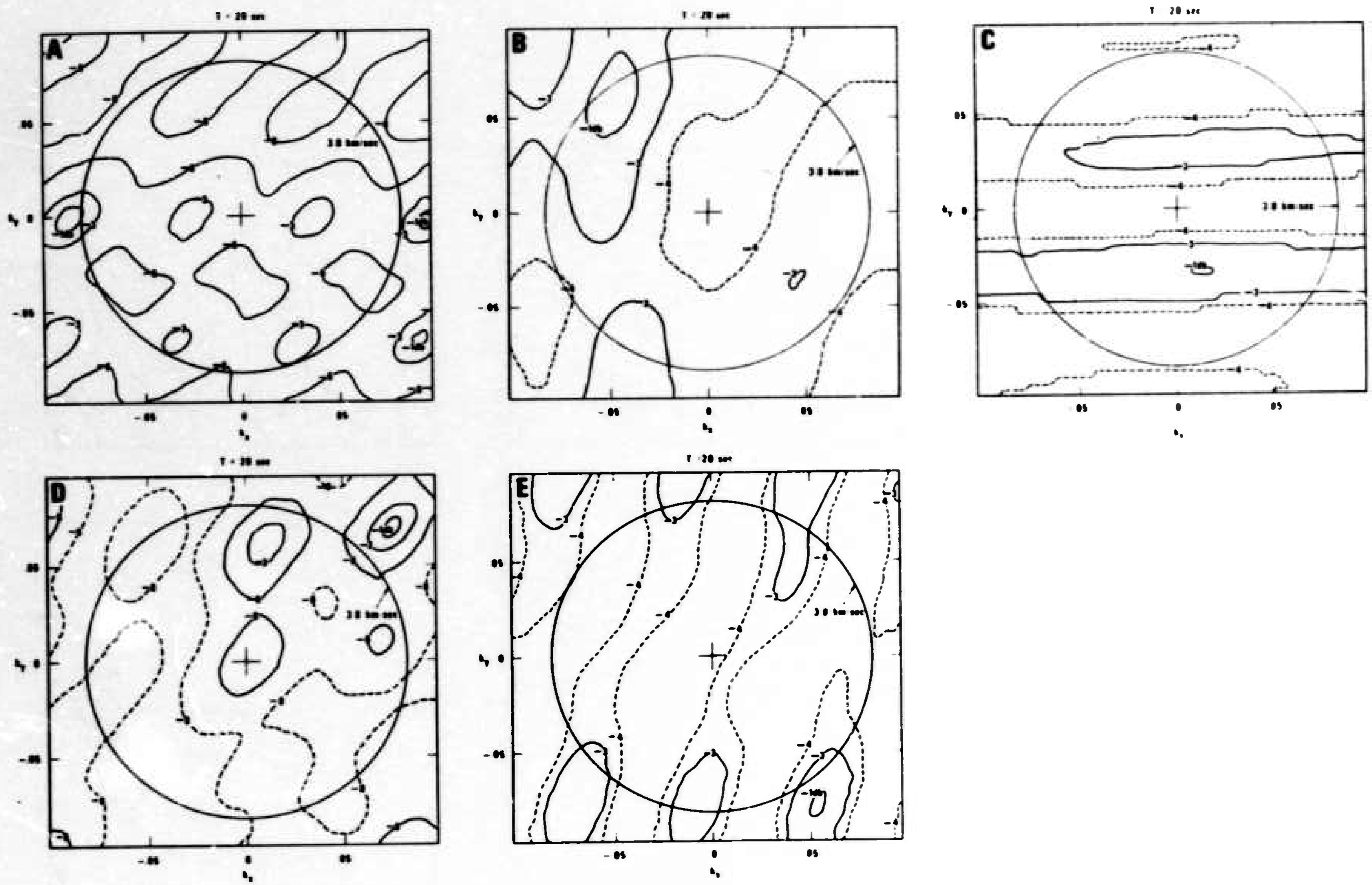


Figure 5   F-K plots obtained for the superposition of synthetic signals with parameters given in Table Ic.

powerful signal, the following eigenvectors on the signals following, in decreasing magnitude (if only one signal is present it will be, in fact, identical to the first eigenvector). This is expected, theoretically. The eigenvectors resemble the signals less and less as their relative power level decreases, because the first eigenvector lines up on the first signal, and the following eigenvectors have to be orthogonal to the previous ones. The second eigenvector, for example, must be similar to a projection of the second signal to a plane perpendicular to the vector E_i. Increasing the dimensionality of the problem (the number of array elements) helps because a projection can be more easily found which resembles the second signal if more dimensions are available. This intuitively explains why the method works better for large arrays, like the 20 element array used by Booker and Ong (1972). The failure of the method for signals of equal amplitude can be explained by the fact that the first eigenvector does not line up on any of the signals. Consideration of equations (4a,b) also explains why it did not work for the vertical arrays.
Although

$$\tilde{V}_i{}^*\tilde{V}_j \simeq \delta_{ij}$$

if many sensors are used along the vertical and the noise is composed of a superposition of normal modes, it is not necessarily true of a vertical array consisting of only a few elements. Further discussion of the theory is given in the Appendix.

EXPERIMENTS ON REAL DATA

For these experiments segments of composited seismograms constructed for another report (Lambert and Der, 1973) were used. These seismograms were obtained by adding long-period recordings of seismic surface waves from two events recorded at LASA channel by channel with predetermined amplitude ratios. Two consecutive segments, each 128 seconds long, of the data were Fourier transformed, all the cross products of the transforms calculated, and the products for the two segments added to obtain a spectral matrix. For stability in the following calculations the diagonal elements were increased about 2% by adding a multiple of the unit matrix. The above procedure was used for computing maximum likelihood f-k spectra in the report mentioned. A window of 256 seconds was used as a standard for computations with FKCOMB, since it was found optimum for detecting dispersed surface wave signals. The resulting spectral matrix, in general, differs greatly from the idealized spectral matrices derived by using equation (1). Because of the shortness of the window used, the spectral matrix has a considerable sample variance, and the two signals cannot be regarded as completely uncorrelated. The effects of cross-correlation terms in the spectral matrix have been considered by Woods and Lintz (1973). The epicentral data for the events used are given in Table 2.

Because it is hard to evaluate all these effects theoretically, an experiment is necessary to establish

TABLE 2

Epicentral Data for the Events

	Date	Orig. Time	Lat.	Long.	Depth km	(From LASA) Dist. km	Azm	m_b	M_s
Event 1 Philippine Islands Region	8 Jan 72	05:27:53	20.874N	120.213E	33	11155	316.5	6.2	6.3
Event 2 New Britain Region	28 Jul 71	01:10:24	5.142S	152.880E	33	11229	274.4	6.0	6.4
Event 3 New Hebrides Islands	24 Jan 72	03:55:42	13.040S	166.398E	28	10839	259.1	5.6	6.2
Event 4 Baja, California	14 Apr 71	11:38:42	27.695N	112.356W	33	2176	196.5	5.4	5.2
Event 5 Guatemala	22 Jan 72	13:08:50	14.029N	91.014W	102	3895	153.7	5.5	-

the usefulness of the technique for this non-ideal case. A five-element array consisting of the D ring and the center element of LASA was used. Figure 6a and b show the f-k spectra associated with the first two eigen-vectors of the spectral matrix composed of the recordings of two superposed events, one of which occurred in the New Britain area with the other in the Philippines. The first eigenvector yields a direction appropriate to the first event and the second indicates the presence of the second event. The amplitude ratio of the two events is approximately 3:1. Lambert and Der (1973) showed that this case was not resolved by maximum likelihood spectra or by stripping using the FKCOMB method (Smart and Flinn, 1971, Mack and Smart, 1972).

Figures 7a and b show the same technique applied to a superposition of an event from the New Hebrides region and the previously mentioned Philippine event, and Figures 8a and b show the decomposition of a seis-mogram made up of one event from Baja, California and one from Guatamala. The relative amplitude ratio is around 3:1 in all cases. In the last case the decomposition is ambiguous, just as is the case of FKPLOT and maximum likelihood f-k spectra (Lambert and Der, 1973). Figures 9 and 10 show the application of the method to the first two combinations of events analyzed above but with noise superposed such that the amplitude level of noise is one half that of the larger event (this puts the smaller event below the noise level). The noise sample used is the same one used in

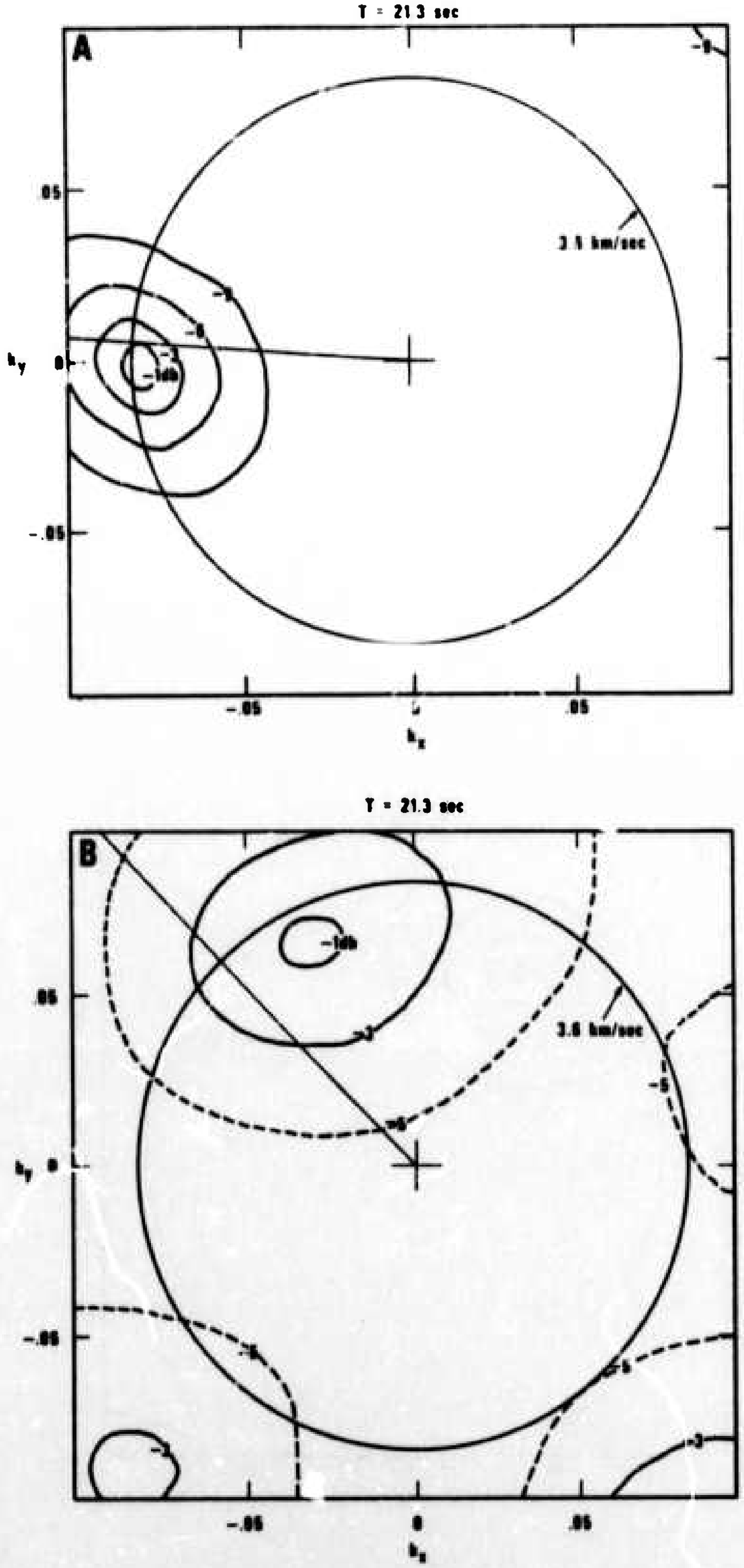


Figure 6. Decomposition of the composite of New Britain (A) and Philippines (B) events with amplitude ratio 3:1.

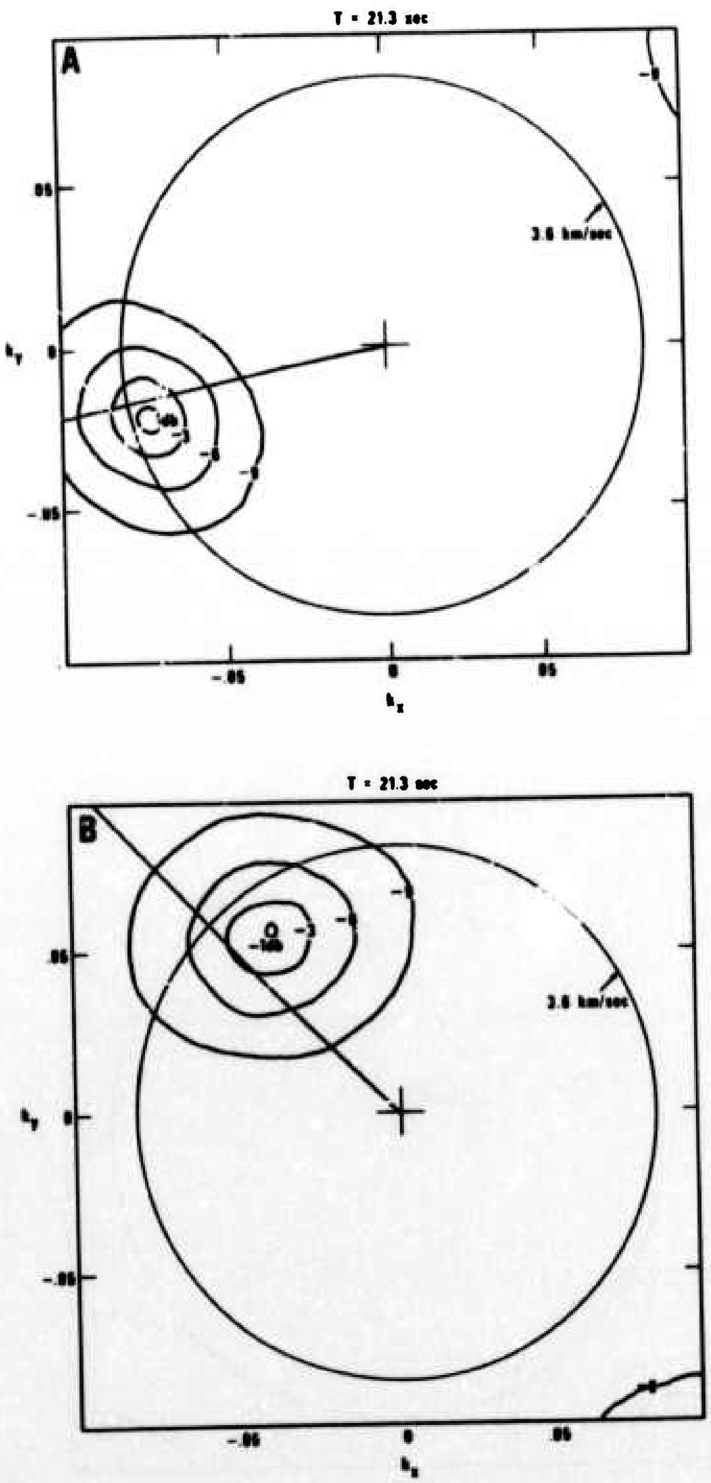


Figure 7. Decomposition of the composite of New Hebrides (A) and Philippines (B) events with amplitude ratio 3:1.

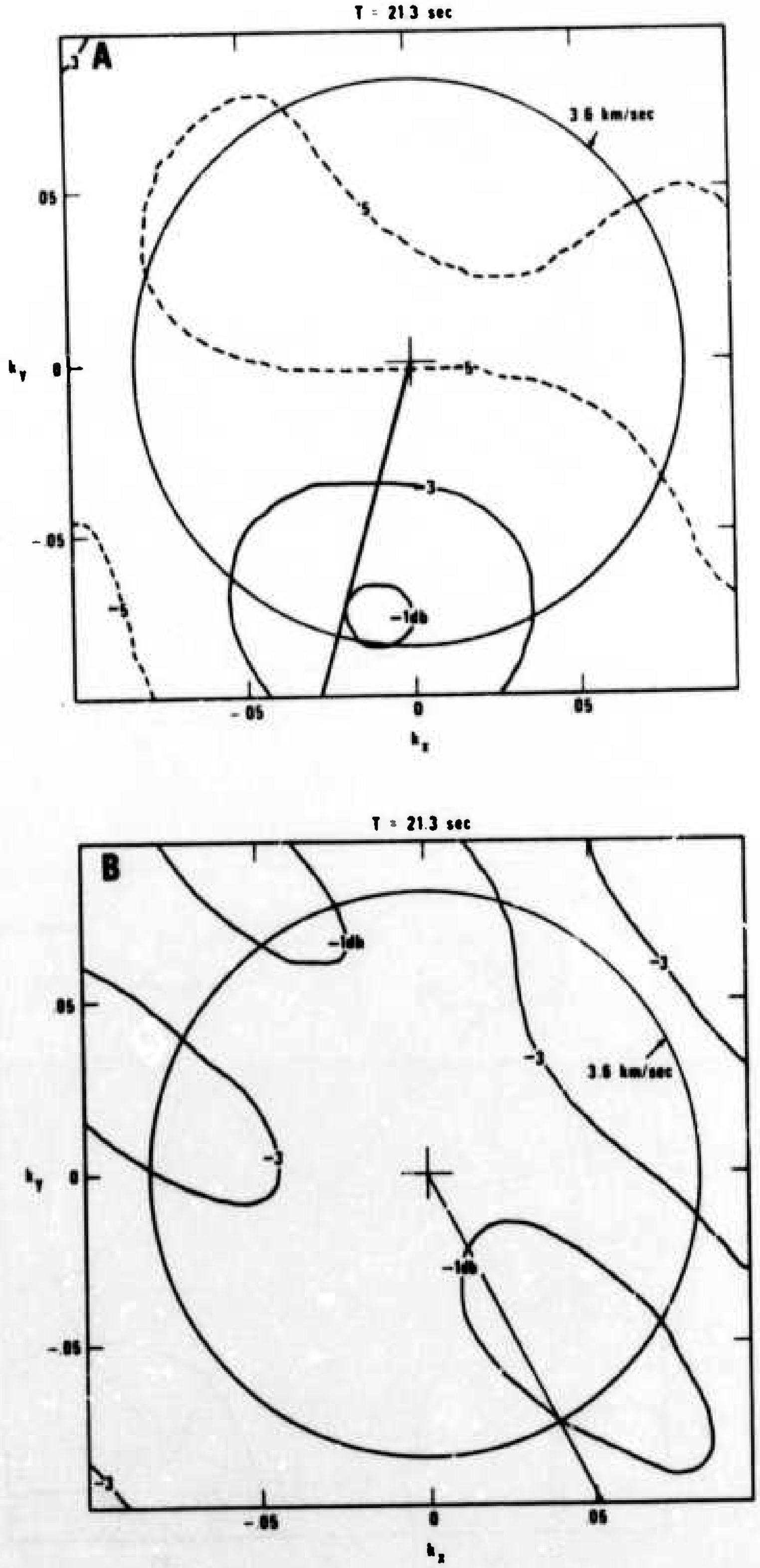
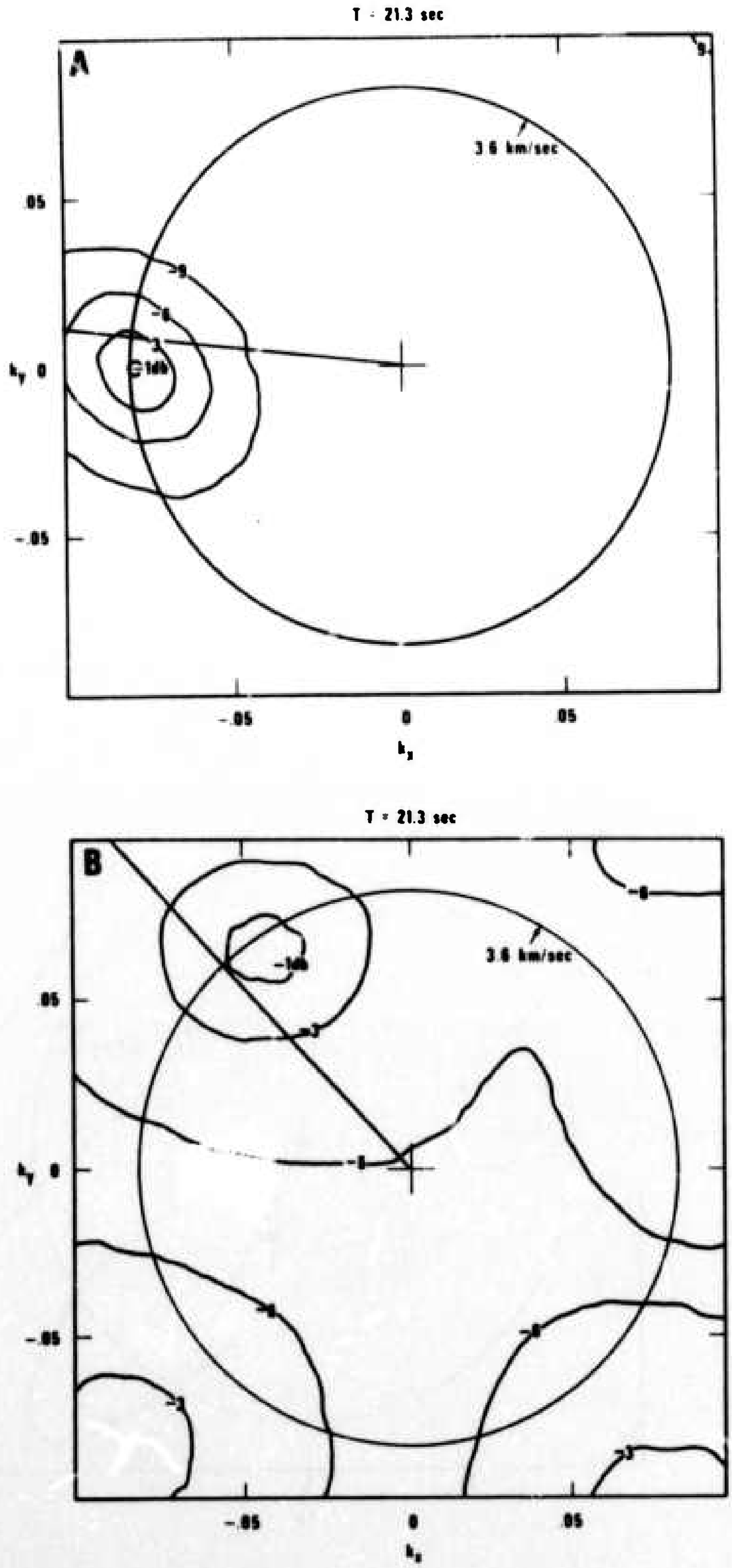


Figure 8. Decomposition of the composite of Baja California (A) and Guatamala (B) events with amplitude ratio 3:1.



Fgiure 9. Decomposition of the composite of New Britain (A) and Philippines (B) events. S/N ratio of larger event is approximately 2:1.

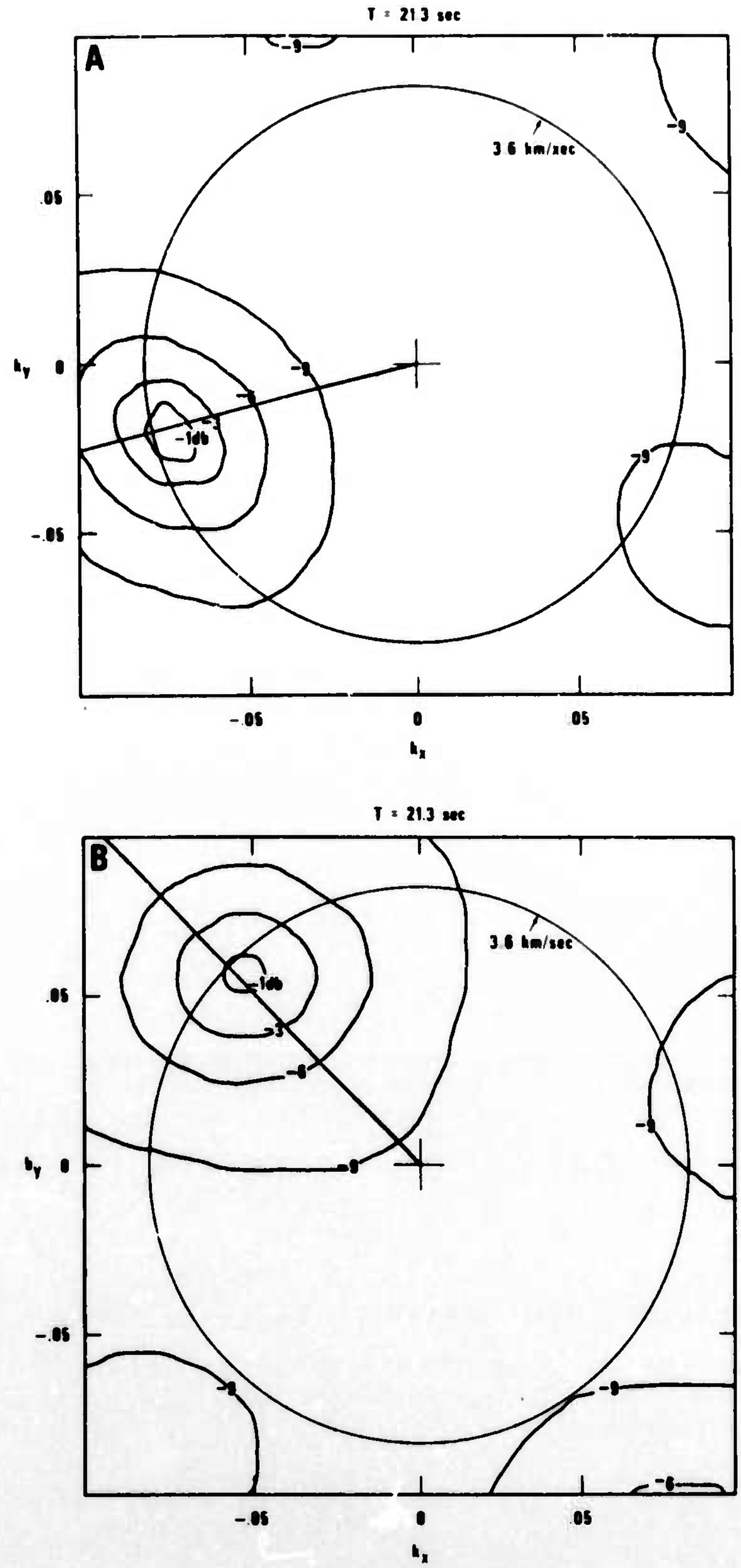


Figure 10. Decomposition of the composite of New Hebrides (A) and Philippines (B) events. S/N ratio of larger event is approximately 2:1.

the report by Lambert and Der (1974). The figures show that both events are well resolved. Application of the two segment maximum likelihood and FKCOMB methods resulted in no resolution or doubtful resolution respectively for the same combinations. In spite of the basic inadequacies of the method as outlined in the Appendix, the method can be used in practice for the separation of signals with unequal amplitudes, and seems to be superior in this case to both two-segment maximum likelihood and FKCOMB with stripping.

SUMMARY

In the strict theoretical sense the principal component decomposition does not yield components which are meaningful in terms of plane waves. In practice, however, it gives plausible looking f-k spectra if the power levels of the signals differ considerably. In this case of great practical interest the method works and is superior to two segment maximum likelihood and to FKCOMB.

The method does not yield useful results if several signals of comparable amplitudes are present, unless the delay vectors of the two signals are orthogonal. However, simple stripping or maximum likelihood f-k spectra work well in these cases, and could be used in parallel.

Principal component decomposition would appear to be the method of choice for routinely solving the masking problem at small arrays. Once detection is accomplished the interactive mixed signal processor of Blandford et al. (1973), or some other maximum likelihood approach may be used for signal estimation. These methods may, of course, be used initially if the direction of approach is known, e.g. LR from know epicenters, or looking for a test at a known test site in the coda of a large earthquake.

The next step would appear to be routine analysis of many naturally mixed or masked events by the methods discussed above in order to see how they perform in routine operation.

REFERENCES

Ayres, F., 1962, Matrices, Schaum Publishing Company, New York.

Barnard, T. E., 1968. Analytical studies of techniques for the computation of high resolution wavenumber spectra. Advanced Array Research, Special Report #9, Texas Instruments, Inc.

Blandford, R. R., T. Cohen, and J. Lambert, 1973, An iterative approximation of the mixed signal processor, SDAC-TR-73-7, Teledyne Geotech, Alexandria, Virginia.

Booker, A. H. and Ong. C., 1972, Resolution and stability of wavenumber spectral estimates, Extended Array Evaluation Program, Special Report #2, Texas Instruments, Inc.

Capon, J., 1969, High resolution frequency-wavenumber spectrum analysis. Proc. IEEE, 57, 1408-1418.

Choi, S. C., 1967, Principal component analysis of seismic data and direction of the principal component for seismic record. Seismic Data Laboratory Report #181, Teledyne Geotech, Alexandria, Virginia.

Goodman, N. R., 1967, Eigenvalues and eigenvectors of spectral density matrices. Seismic Data Laboratory Report #179, Teledyne Geotech, Alexandria, Virginia.

REFERENCES (Continued)

Grenander, U. and Szegö, G., 1958, Toeplitz forms and their applications. University of California Press, Berkeley.

Lambert, J. W., and Der, Z. A., 1973, Comparison of two segment maximum likelihood frequency wavenumber spectra with the fast beamed frequency wavenumber spectra (FKPLOT). SDAC Report (in press), Teledyne Geotech, Alexandria, Virginia.

Lintz, P. R., 1968. An analysis of a technique for the generation of high resolution wavenumber spectra. SDL Report #218, Teledyne Geotech, Alexandria, Virginia.

Mack, H. & Smart, E., 1972. Frequency domain processing of digital microbarograph data, J. Geophys. Res., 77, 488-490.

McCowan, D. W. and Lintz, P. R., 1968. High resolution frequency wavenumber spectra, Seismic Data Laboratory Report #206, Teledyne Geotech, Alexandria, Virginia.

Smart, E. and Flinn, E., 1971. Fast frequency-wavenumber analysis and Fisher signal detection in real-time infrasonic array data processing, Geophys. J. R. Astr. Soc., 26, 279-284.

Woods, J. W. and Lintz, P. R., 1973. Plane waves at small arrays, Geophysics, v. 53, 1023-1041.

APPENDIX

APPENDIX

Eigenvectors and eigenvalues of array spectral matrices generated by multiple plane waves.

The spectral matrix of the signals observed at various array elements, can be written as the outer product

$$\Phi = \sum_{L=1}^{n} \tilde{V}_i \tilde{V}_i^* \qquad \text{(A1)}$$

where the complex signal vectors are of the form (and the star denotes complex conjugate)

$$\tilde{V}_i = \frac{a_i}{m} \operatorname{col}[e^{i\tilde{k}_i\tilde{x}_1}, e^{i\tilde{k}_i\tilde{x}_2}, \ldots e^{i\tilde{k}_i\tilde{x}_m}] \qquad \text{(A2)}$$

where m is the number of array elements, a_i is the amplitude of the i'th signal, $\tilde{x}_j$ is the position vector of the j'th array element and $\tilde{k}_i$ is the wavenumber vector of the i'th signal. It is assumed in the above representation of the spectral matrix that the signals do not correlate and thus the cross spectra of various signals are zero. This temporal independence of signals should not be confused with the independence of the delay vectors V_i in the following discussion, which is related to the spatial characteristics of the array.

The matrix in the above form is singular for the case k<m; for the case m>n it is not singular but in this case the signals cannot be separated.

For simplicity let us consider the two-signal case. Denoting the two signal vectors by U and V

$$\Phi = \tilde{V}\tilde{V}^* + \tilde{U}\tilde{U}^* \qquad \text{(A3)}$$

eigenvectors of this matrix must satisfy the equation

$$\Phi E = \lambda E. \qquad \text{(A4)}$$

Vectors which are orthogonal to <u>both</u> U and V are eigenvectors of Φ and the corresponding eigenvalues are zero. There are m-2 such vectors which are mutually orthogonal. The remaining two eigenvectors are linear combinations of the vectors U and V and the corresponding eigenvalues are nonzero. This can be seen by rewriting (A4) as

$$(\tilde{V}\tilde{V}^* + \tilde{U}\tilde{U}^*)\,\tilde{E} = \lambda\tilde{E}$$

$$\tilde{V}\tilde{V}^*\tilde{E} + \tilde{U}\tilde{U}^*\tilde{E} = \lambda\tilde{E}$$

but V*E and U*E are complex scalars and assuming that λ is nonzero

$$\tilde{E} = \frac{\tilde{V}^*\tilde{E}}{\lambda}\,V + \frac{\tilde{U}^*\tilde{E}}{\lambda}\,\tilde{U} = \alpha\tilde{V} + \beta\tilde{U}.$$

Substituting this, the linear combination back into (A4)

$$(\tilde{V}\tilde{V}^* + \tilde{U}\tilde{U}^*)\,(\alpha\tilde{V} + \beta\tilde{U}) = \Lambda(\alpha\tilde{V} + \beta\tilde{U})$$

$$= \alpha\tilde{V}\tilde{V}^*\tilde{V} + \alpha\tilde{U}\tilde{U}^*\tilde{V} + \beta\tilde{V}\tilde{V}^*\tilde{U} + \beta\tilde{U}\tilde{U}^*\tilde{U}$$

$$= \tilde{V}\,(\alpha\tilde{V}^*\tilde{V} + \beta\tilde{V}^*\tilde{U}) + \tilde{U}\,(\alpha\tilde{U}^*\tilde{V} + \beta\tilde{U}^*\tilde{U}) = \Lambda\tilde{E}.$$

So

$$\Lambda\alpha = \alpha\tilde{V}^*\tilde{V} + \beta\tilde{V}^*\tilde{U}$$

$$\Lambda\beta = \alpha\tilde{U}^*\tilde{V} + \beta\tilde{U}^*\tilde{U}$$

or

$$\alpha(\tilde{V}^*\tilde{V}-\Lambda) + \beta\tilde{V}^*\tilde{U} = 0$$

$$\alpha\tilde{U}^*\tilde{V} + \beta(\tilde{U}^*\tilde{U}-\Lambda) = 0. \tag{A5}$$

Nontrivial solutions α and β exist only if

$$\begin{vmatrix} \tilde{V}^*\tilde{V}-\Lambda & \tilde{V}^*\tilde{U} \\ \tilde{U}^*\tilde{V} & \tilde{U}^*\tilde{U}-\Lambda \end{vmatrix} = 0. \tag{A6}$$

The roots of this equation are the eigenvalues. Substituting back into the various equations yields the ratios of α and β. An additional constraint on E is that

$$\tilde{E}^*\tilde{E} = 1.$$

Clearly if $\tilde{U}^*\tilde{V} = 0$

$$\Lambda_1 = \tilde{V}^*\tilde{V}$$

and

$$\Lambda_2 = \tilde{U}^*\tilde{U}$$

and the two eigenvectors become

$$\tilde{E}_1 = \beta\tilde{U} \text{ and } \tilde{E}_2 = \alpha\tilde{V}$$

$$|\beta| \neq 0 \qquad |\alpha| \neq 0$$

if $\tilde{U}^*\tilde{V}$ is small relatively to $\tilde{U}^*\tilde{U}$ and $\tilde{V}^*\tilde{V}$ the eigenvectors are still approximately lined up with the component signal vectors. The same is true if $\tilde{V}^*\tilde{V} >> \tilde{U}^*\tilde{U}$ or $\tilde{U}^*\tilde{U} >> \tilde{V}^*\tilde{V}$ even if $\tilde{U}^*\tilde{V}$ is not small. This can be easily proven from the characteristic equation

$$\Lambda^2-\Lambda\ (\tilde{V}^*\tilde{V} + \tilde{U}^*\tilde{U}) + \tilde{V}^*\tilde{V}\ \tilde{U}^*\tilde{U} - \tilde{V}^*\tilde{U}\ \tilde{U}^*\tilde{V} = 0$$

assuming the extreme case $|\tilde{U}|\ |\tilde{V}| = |\tilde{U}^*\tilde{V}|$

$$\Lambda^2-\Lambda(\tilde{V}^*\tilde{V} + \tilde{U}^*\tilde{U}) = 0$$

if

$$\tilde{V}^*\tilde{V} >> \tilde{U}^*\tilde{U}.$$

One of the eigenvalues becomes $\Lambda_1 \sim \tilde{V}^*\tilde{V}$ substituting into (A5) from which follows that $|\alpha| \neq 0$ $\beta \sim 0$ $\tilde{E}_1 \sim \alpha\tilde{V}$.

The products of the complex vectors U and V can be written as

$$UV^* = \frac{|U|\ |V|}{m^2} \sum_{i=1}^{m} e^{i\tilde{k}_u\tilde{x}_i} e^{-i\tilde{k}_v\tilde{x}_i}$$

$$= \frac{|U|\ |V|}{m^2} \sum_{i=1}^{m} e^{i\tilde{x}_i(\tilde{k}_u-\tilde{k}_v)}$$

$$= \frac{|U|\ |V|}{m^2} \sum_{i=1}^{m} e^{i\tilde{x}_i\Delta\tilde{k}}$$

$$\sim F(\Delta k)\,|\tilde{U}|\ |\tilde{V}|$$

where F is the array response, in this case with the argument $\Delta\tilde{K}$. If $\Delta\tilde{K}$ is such that $F(\Delta\tilde{K}) = 0$ that is, the array response is at a zero, the various components separate. If, on the other hand, $F(\Delta\tilde{K}) \neq 0$, especially if $F(\Delta\tilde{K}) \sim 1$, (one signal is in a large sidelobe of the array response when the array is beamed at the other signal) the separation cannot be achieved unless the various signals are considerably different in magnitude making $|\tilde{U}||\tilde{V}|$ small relative to the squared amplitude of the larger signal.

The discussion above clearly indicates that principal value analysis is not suitable for the separation of plane wave signals in general, and it works only if some very restrictive conditions are met.

Obviously the same theory applies to separation of signals at tapered (weighted) arrays or deep well arrays. For the weighted array case the exponentials in (A2) are multiplied by the sensor weights; in the deep well array case, assuming that the signal consists a superposition of normal surface wave modes V_i, the absolute values of elements in vector V_i are the relative amplitudes of a certain surface wave mode at the various sensors and the phase is either 0 or π depending on the phase reversals of that mode with depth. In general $V_i{}^*V_j \neq 0$ unless a large number of sensors is used to successfully approximate $\int_0^\infty V_i(Z)\cdot V_j(Z)dZ = 0$, where $V_i(Z)$ and $V_j(Z)$ are the amplitudes of various modes as functions of depth Z, which indeed are orthogonal in the sense given by the integral. Separation can be achieved with a few sensors only if their positions or weights are carefully chosen (as functions of frequency) knowing the particle motion-depth relationships of all the modes present, to make $\tilde{V}_i{*}\tilde{V}_j \neq 0$ for all mode combinations and periods.

In the case of correlated signals, that is, if the cross power spectra of the signals are not zero, the spectral matrix of the array takes the form (Woods and Lintz 1973) in the two signal case:

$$\Phi = \tilde{V}{*}\tilde{V} + \tilde{U}{*}\tilde{U} + \rho_{uv} (\tilde{U}{*}\tilde{V} + \tilde{V}{*}\tilde{U})$$

where ρ_{uv} is the coherence between the two plane waves. It is easy, following steps similar to those described above to derive equations for eigenvalues and eigenvectors. As shown above, the eigenvectors corresponding to nonzero eigenvalues will be linear combinations of the vectors U and V, which are not plane waves in the general case.

The finite sample length used to derive the spectral matrices poses the same problems as outlined by Woods and Lintz (1973) and Smart (1971). First of all the cross spectral terms of signals are never zero, if they are reduced by smoothing the spectra, leakage of energy leads to erroneous phase velocities and destroys the plane wave character of the signals.

The above discussion can be easily extended to the case of many signals.

Addition of noise to the signals also changes the results. Assuming that the noise is independent of the signals, the spectral matrix of the array inputs become

$$\Phi = \underbrace{\tilde{V}\,\tilde{V}^* + \tilde{U}\,\tilde{U}^* + \ldots}_{\text{Signals}} + \underbrace{N}_{\text{Noise}}$$

The noise spectral matrix is, like all properly derived spectral matrices, positive definite and Hermitian. A Hermitian matrix is unitarily similar to a diagonal real matrix. That is,

$$N = T\,\overline{\Lambda}\,T^*$$

where T is a unitary matrix and $\overline{\Lambda}$ is the diagonal matrix containing the eigenvalues of N in the diagonal elements. Putting

$$M = \overline{\Lambda}^{1/2}T$$

N becomes

$$N = M\,M^*.$$

But this also can be written as

$$N = (M_1, M_2, \ldots M_m)\begin{pmatrix} M_1^* \\ M_2^* \\ \cdot \\ \cdot \\ \cdot \\ M_m^* \end{pmatrix} = M_1M_1^* + M_2M_2^* + \ldots + M_mM_m^*$$

where M* are column matrices of M. Thus

$$\Phi = \tilde{V}\tilde{V}^* + \tilde{U}\tilde{U}^* + \ldots + \sum_{1}^{m} M_i M_i^*.$$

The M_i behave exactly as coherent signals and there are, in general, m of them. The derivations for the spectral eigenvectors are similar to those in the earlier part of this Appendix. The eigenvectors will be sums of the signal vectors vectors U, V, and the noise vectors M_i, but there can be no more than m nonzero eigenvalues (and independent eigenvectors) of Φ. Besides, the m vectors M_i span the whole dimensional complex space, thus depending on the noise; the eigenvectors of the total spectral matrix can assume almost any direction.