

AD-779 401

MECHANICS AND THERMODYNAMICS OF  
A MIXTURE OF A GRANULAR MATERIAL  
WITH A FLUID

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Prepared for:

Army Research Office-Durham

April 1974

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| DOCUMENT CONTROL DATA - R & D  |   |  |
|--|---|--|
| <i>(Security classification of this body of abstract and index or annotation must be entered when the overall report is classified)</i>  |   |  |
| 1. ORIGINATING ACTIVITY (Corporate number)<br>Mathematics Research Center<br>University of Wisconsin, Madison, Wis. 53706  |   | 29. REPORT SECURITY CLASSIFICATION<br>Unclassified |
|  |   | 30. GROUP<br>None                                  |
| 2. REPORT TITLE<br>MECHANICS AND THERMODYNAMICS OF A MIXTURE OF A GRANULAR MATERIAL WITH A FLUID   |   |  |
| 3. DESCRIPTIVE NOTES (Type of report and inclusive dates)<br>Summary Report: no specific reporting period.   |   |  |
| 4. AUTHOR(S) (First name, middle initial, last name)<br>S. L. Passman  |   |  |
| 5. REPORT DATE<br>April 1974   | 7A. TOTAL NO. OF PAGES<br>31  | 7B. NO. OF REFS<br>3                               |
| 6A. CONTRACT OR GRANT NO.<br>Contract No. DA-31-124-ARO-D-462  | 6B. ORIGINATOR'S REPORT NUMBER(S)<br>1391   |  |
| 6C. PROJECT NO.<br>None  | 6D. OTHER REPORT NUMBER(S) (Any other numbers that may be assigned this report)<br>None |  |
| 10. DISTRIBUTION STATEMENT<br>Distribution of this document is unlimited.  |   |  |
| 11. SUPPLEMENTARY NOTES<br>None  | 12. SPONSORING MILITARY ACTIVITY<br>Army Research Office-Durham, N. C.                  |  |
| 13. ABSTRACT<br><br>Constitutive equations are postulated for a mixture of a granular material and a fluid. Consequences of the entropy inequality, for both linear and nonlinear constitutive equations, are derived. |   |  |
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Contract No. DA-31-124-ARO-D-462

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MRC Technical Summary Report # 1391  
April 1974

Received August 31, 1973

Madison, Wisconsin 53706

UNIVERSITY OF WISCONSIN - MADISON  
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# MECHANICS AND THERMODYNAMICS OF A MIXTURE OF A GRANULAR MATERIAL WITH A FLUID

S. L. Passman

## Introduction

In a previous work [1] balance equations for a mixture of an arbitrary finite number of granular materials has been given; however, no constitutive theory has been developed. In this work I consider the special case of two materials, one a granular material as defined by Goodman and Cowin [2], the other a viscous fluid. A constitutive theory is postulated, and restrictions due to the entropy inequality are explored, following closely the analysis of Müller [3].

I use without further comment the equations and notations of [1]. Furthermore, sections and equations herein are numbered as if this work were a continuation of [1].

## 5. Further Analysis of the Entropy Inequality

Recall the equations for balance of energy and entropy for the mixture :

$$\begin{aligned} \rho \dot{e} = & \operatorname{tr}(\mathbb{T}^I \operatorname{grad} \underline{\dot{x}}) + \underline{h} \cdot \operatorname{grad} \dot{\nu} + \rho k : \dot{\underline{x}}^2 \\ & + \rho g \dot{\nu} + \operatorname{div} \underline{g} + \rho s , \end{aligned} \tag{5.1}$$

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$$\rho \dot{\eta} \geq \text{div } \underline{q} + \rho \sigma \theta, \quad (5.2)$$

where

$$\rho s = \sum_a \rho_a s_a + \sum_a \rho_a \left[ \frac{b_a}{\eta} \cdot \underline{u}_a + l(\dot{v}_a - \dot{v}) \right], \quad (5.3)$$

$$\rho \theta \sigma = \sum_a \rho_a \theta_a s_a. \quad (5.4)$$

Assume a common coldness for each component\*

$$\theta_a = \theta. \quad (5.5)$$

Then

$$\rho \sigma = \sum_a \rho_a s_a$$

and

$$\rho s = \rho \sigma + \sum_a \rho_a \left[ \frac{b_a}{\eta} \cdot \underline{u}_a + l(\dot{v}_a - \dot{v}) \right], \quad (5.7)$$

so that (5.1) becomes

$$\begin{aligned} \rho \dot{e} = & \text{tr}(\underline{T}^T \text{grad } \underline{x}) + \underline{h} \cdot \text{grad } \dot{v} + \rho k \dot{v}^2 + \rho g \dot{v} \\ & + \text{div } \underline{q} + \rho \sigma + \sum_a \rho_a \left[ \frac{b_a}{\eta} \cdot \underline{u}_a + l(\dot{v}_a - \dot{v}) \right]. \end{aligned} \quad (5.8)$$

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\* There is for the special case considered here a physical argument indicating that this assumption is too strong.

Eliminating  $\rho\sigma$  between (5.8) and (5.2) gives

$$\begin{aligned} \rho\dot{\eta} \geq & \operatorname{div} \underline{\phi} + \vartheta \{ \rho\dot{\varepsilon} - \operatorname{tr}(\underline{T}^T \operatorname{grad} \underline{x}) - \underline{h} \cdot \operatorname{grad} \dot{\nu} - \rho k \dot{\nu}^2 \\ & - \rho g \dot{\nu} - \operatorname{div} \underline{g} - \sum_a \rho \left[ \frac{\underline{b}}{\underline{a}} \cdot \frac{\underline{u}}{\underline{a}} + \frac{\dot{\ell}}{\underline{a}} (\dot{\nu} - \dot{\nu}) \right] \}. \end{aligned} \quad (5.9)$$

By the linear momentum balance for a constituent

$$\rho \frac{\underline{h}}{\underline{a}} \cdot \frac{\underline{u}}{\underline{a}} = \rho \frac{\dot{\underline{x}}}{\underline{a}} \cdot \frac{\underline{u}}{\underline{a}} + \rho \frac{\dot{\underline{x}}}{\underline{a}} \cdot \frac{\underline{u}}{\underline{a}} - \underline{u} \cdot \operatorname{div} \underline{T} - \rho \frac{\dot{\underline{m}}}{\underline{a}} \cdot \frac{\underline{u}}{\underline{a}}, \quad (5.10)$$

also by the balance of equilibrated force (2.21),

$$\begin{aligned} \rho \frac{\dot{\ell}}{\underline{a}} (\dot{\nu} - \dot{\nu}) = & \rho k \frac{\dot{\nu}}{\underline{a}} (\dot{\nu} - \dot{\nu}) + \rho k \frac{\dot{\nu}}{\underline{a}} (\dot{\nu} - \dot{\nu}) + \rho k \frac{\dot{\nu}}{\underline{a}} \frac{\dot{\nu}}{\underline{a}} (\dot{\nu} - \dot{\nu}) - (\dot{\nu} - \dot{\nu}) \operatorname{div} \frac{\underline{h}}{\underline{a}} \\ & - \rho g (\dot{\nu} - \dot{\nu}) - \rho \frac{\dot{\nu}}{\underline{a}} (\dot{\nu} - \dot{\nu}). \end{aligned} \quad (5.11)$$

Define

$$\hat{\underline{\phi}} = \underline{\phi} - \vartheta \underline{g} + \vartheta \sum_a \underline{T}_a^T \frac{\underline{u}}{\underline{a}} + \vartheta \sum_a \underline{h}_a (\dot{\nu} - \dot{\nu}), \quad (5.12)$$

s. that

$$\begin{aligned} \operatorname{div} \underline{\phi} = & \operatorname{div} \hat{\underline{\phi}} + \underline{g} \cdot \operatorname{grad} \vartheta + \vartheta \operatorname{div} \underline{g} \\ & - \sum_a \underline{T}_a^T \frac{\underline{u}}{\underline{a}} \cdot \operatorname{grad} \vartheta - \vartheta \sum_a \underline{u}_a \cdot \operatorname{div} \underline{T}_a - \vartheta \sum_a \operatorname{tr}(\underline{T}_a^T \operatorname{grad} \frac{\underline{u}}{\underline{a}}) \\ & - \sum_a \underline{h}_a (\dot{\nu} - \dot{\nu}) \cdot \operatorname{grad} \vartheta - \vartheta \sum_a (\dot{\nu} - \dot{\nu}) \operatorname{div} \frac{\underline{h}}{\underline{a}} - \vartheta \sum_a \underline{h}_a \cdot \operatorname{grad} (\dot{\nu} - \dot{\nu}). \end{aligned} \quad (5.13)$$

Also, the Helmholtz free energy of the mixture is

$$\psi = \epsilon - \frac{\eta}{\theta} \quad (5.14)$$

Eliminating  $\frac{b}{a}$  and  $t$  from (5.9), and using (5.13) and (5.14) gives, by use of (2.25)<sub>9</sub>,

$$\begin{aligned} \rho\eta \frac{\dot{\theta}}{\theta} - \rho\dot{\psi} \geq & \operatorname{div} \hat{\underline{e}} - \partial \left[ \sum_a \rho \hat{\underline{x}} \cdot \underline{u} + \sum_a \rho (k \hat{\underline{v}} + \hat{k} \underline{v})(\hat{\underline{v}} - \underline{v}) \right] \\ & + \left[ \underline{g} - \sum_a \underline{T}^T \underline{u} - \sum_a \underline{h}(\hat{\underline{v}} - \underline{v}) \right] \cdot \operatorname{grad} \theta \\ & + \partial \rho \sum_a \left( \frac{\dot{m}}{a} - \frac{\dot{c} \hat{\underline{x}}}{a} \right) \cdot \underline{u} + \partial \rho \sum_a \left( \frac{\dot{v}}{a} - \frac{\dot{c} k \underline{v}}{a} \right) (\hat{\underline{v}} - \underline{v}) \\ & - \partial \left[ \operatorname{tr}(\underline{T}^T \operatorname{grad} \hat{\underline{x}}) + \sum_a \operatorname{tr}(\underline{T}^T \operatorname{grad} \underline{u}) + \underline{h} \cdot \operatorname{grad} \hat{\underline{v}} + \sum_a \underline{h} \cdot \operatorname{grad}(\hat{\underline{v}} - \underline{v}) \right] \\ & - \rho k \hat{\underline{v}}^2 - \sum_a \rho g \hat{\underline{v}} \end{aligned} \quad (5.15)$$

Define

$$\underline{G} = \operatorname{grad} \hat{\underline{x}} \quad (5.16)$$

the velocity gradient, and

$$\underline{G}_a = \operatorname{grad} \frac{\hat{\underline{x}}}{a} \quad (5.17)$$

the peculiar velocity gradient of the  $a$ -th constituent. The symmetric and skew parts of these are



$$\underline{D} = \frac{1}{2} (\underline{G} + \underline{G}^T), \quad \underline{W} = \frac{1}{2} (\underline{G} - \underline{G}^T), \quad (5.18)$$

the stretching and spin, and

$$\underline{D}_a = \frac{1}{2} (\underline{G}_a + \underline{G}_a^T), \quad \underline{W}_a = \frac{1}{2} (\underline{G}_a - \underline{G}_a^T), \quad (5.19)$$

the peculiar stretching and peculiar spin of the  $a$ -th constituent.

By (2.11)

$$\text{grad } \underline{u}_a = \underline{G}_a - \underline{G}, \quad (5.20)$$

and by (1.6) and (1.10)

$$\rho \underline{G} = \sum_a \rho_a \underline{G}_a + \sum_a \underline{u}_a \otimes \text{grad } \rho_a, \quad (5.21)$$

so that

$$\rho \underline{D} = \sum_a \rho_a \underline{D}_a + \frac{1}{2} \sum_a (\underline{u}_a \otimes \text{grad } \rho_a + \text{grad } \rho_a \otimes \underline{u}_a), \quad (5.22)$$

and

$$\rho \underline{W} = \sum_a \rho_a \underline{W}_a + \frac{1}{2} \sum_a (\underline{u}_a \otimes \text{grad } \rho_a - \text{grad } \rho_a \otimes \underline{u}_a). \quad (5.22)$$

By (2.19) and (2.23)<sub>3</sub>  $\sum_a \underline{T}_a$  is symmetric, and by (2.25)<sub>1</sub>  $\underline{T}$  is symmetric. Then by (5.18) - (5.21),

$$\begin{aligned}
& \text{tr}(\underline{T}^T \text{grad } \underline{\dot{x}}) + \Sigma \text{tr}(\underline{T}_a^T \text{grad } \underline{u}_a) \\
&= \text{tr } \underline{T} \underline{D} - \text{tr}(\Sigma \underline{T}_a^T \underline{D}) + \Sigma \text{tr } \underline{T}_a^T \underline{G}_a \\
&= \text{tr}(\underline{T} - \Sigma \underline{T}_a^T) (\Sigma \frac{\rho}{p} \underline{D} + \frac{1}{p} \underline{u}_a \otimes \text{grad } \rho) + \Sigma \text{tr } \underline{T}_a^T \underline{G}_a
\end{aligned} \tag{5.23}$$

The gradient of (1.11)<sub>2</sub> is

$$(\text{grad } \rho k) \dot{v} + \rho k \text{grad } \dot{v} = \Sigma (\text{grad } \rho k) \dot{v} + \Sigma \rho k \text{grad } \dot{v} \tag{5.24}$$

However, by (1.11),

$$\text{grad } \rho k = \Sigma \text{grad } \rho k \tag{5.25}$$

Substituting (5.25) into (5.24). I obtain the result

$$\rho k \text{grad } \dot{v} = \Sigma \rho k \text{grad } \dot{v} + \Sigma [\text{grad}(\rho k)] (\dot{v} - \dot{v}) \tag{5.26}$$

Then

$$\begin{aligned}
& \underline{h} \cdot \text{grad } \dot{v} + \Sigma \underline{h}_a \cdot \text{grad}(\dot{v} - \dot{v}) \\
&= (\underline{h} - \Sigma \underline{h}_a) \cdot \text{grad } \dot{v} + \Sigma \underline{h}_a \cdot \text{grad } \dot{v} \\
&= (\underline{h} - \Sigma \underline{h}_a) \cdot \frac{1}{p} \rho k \text{grad } \dot{v} + \Sigma (\dot{v} - \dot{v}) \text{grad}(\rho k) \\
&\quad + \Sigma \underline{h}_a \cdot \text{grad } \dot{v}
\end{aligned} \tag{5.27}$$

Substituting (5.23) and (5.27) into (5.15), yields

$$\begin{aligned}
 \rho \eta \frac{\dot{\theta}}{\theta} - \rho \dot{\theta} \dot{\psi} &\geq \operatorname{div} \hat{\phi} - \partial \left[ \sum_a \rho \tilde{x}^a \cdot \frac{\underline{u}}{a} + \sum_a (\rho k \tilde{v}^a + \rho \dot{k} \tilde{v}^a) (\tilde{v}^a - \dot{v}^a) \right] \\
 &+ \left[ \underline{q} - \sum_a \underline{T}^T \frac{\underline{u}}{a} - \sum_a \underline{h} (\tilde{v}^a - \dot{v}^a) \right] \cdot \operatorname{grad} \theta \\
 &+ \dot{\rho} \sum_a (\underline{m}^a - \dot{c} \underline{x}^a) \cdot \frac{\underline{u}}{a} + \dot{\rho} \sum_a (\tilde{v}^a - \dot{c} k \tilde{v}^a) (\tilde{v}^a - \dot{v}^a) \\
 &- \frac{\partial}{\rho} \operatorname{tr} (\underline{T} - \sum_a \underline{T}^T) (\sum_a \rho \underline{D}^a + \sum_a \underline{u}^a \otimes \operatorname{grad} \rho) - \partial \sum_a \operatorname{tr} \frac{\underline{T}^T}{a} \underline{G}^a \\
 &- \frac{\partial}{\rho k} (\underline{h} - \sum_a \underline{h}^a) \cdot \left[ \sum_a \rho k^a \operatorname{grad} \tilde{v}^a + \sum_a (\tilde{v}^a - \dot{v}^a) (\operatorname{grad} \rho k^a) \right] - \partial \sum_a \underline{h}^a \cdot \operatorname{grad} \tilde{v}^a \\
 &- \dot{\rho} (\rho k \dot{v}^2 - \sum_a \rho g^a \dot{v}^a) .
 \end{aligned} \tag{5.28}$$

## 6. Constitutive Equations for a Special Two-Component Mixture

Although constitutive equations may be written for a general mixture and the restrictions imposed on them by the entropy inequality (5.15) may be found, the resulting algebraic manipulations are quite tedious. I consider here a special two-component mixture, one component being a granular material as defined by Goodman and Cowin [2], the other a fluid of complexity 1. In particular

$$\begin{aligned}
 k_1 &= 0, & \dot{k}_2 &= 0.
 \end{aligned} \tag{6.1}$$

Define

$$\bar{L} = \left\{ \rho_a, \rho_a, \text{grad } \rho_a, \dot{x}_a, \text{grad } \dot{x}_a, \vartheta, \text{grad } \vartheta, k_a, v_a, \text{grad } v_a, \dot{v}_a \right\}, \quad (6.2)$$

where  $\rho_a(\underline{x}) = \rho(\underline{x}, 0)$ , and  $a = 1, 2$ .

Assume that each of the quantities

$$\psi, \eta, \underline{q}, \hat{\underline{q}}, \underline{T}, \underline{m}, \underline{c}, \underline{h}, \underline{v}, g \quad (6.3)$$

depends on  $\bar{L}$ . The material defined by this constitutive assumption will be called a mixture of a granular medium with a fluid.

Constitutive equations are often assumed to be subject to certain restrictions, one of which is called the "principle of frame-indifference".

A change of frame is defined by

$$\underline{x}^* = \underline{x}_0^* + \underline{Q}(\underline{x} - \underline{x}_0), \quad \underline{Q}\underline{Q}^T = \underline{1},$$

$$\begin{aligned} \rho_a^* &= \rho_a, \\ v_a^* &= v_a, \\ k_a^* &= k_a, \end{aligned} \quad (6.4)$$

This is a generalization of the usual definition.

Define

$$\bar{L}^* = \left\{ \rho_0, \rho, Q \operatorname{grad} \rho, \dot{\underline{x}}_a^* + Q(\underline{x} - \underline{x}_0) + Q \dot{\underline{x}}_a, Q \operatorname{grad} \dot{\underline{x}}_a Q^T + \dot{Q} Q^T, \vartheta, \right. \\ \left. Q \operatorname{grad} \vartheta, k, v, Q \operatorname{grad} v, \dot{v} \right\}.$$

Consider scalar-, vector-, and tensor-valued functions  $s(\bar{L})$ ,  $\underline{w}(\bar{L})$ ,  $\underline{T}(\bar{L})$ . These functions are said to be frame-indifferent if

$$\begin{aligned} s(\bar{L}^*) &= s(\bar{L}), \\ \underline{w}(\bar{L}^*) &= Q \underline{w}(\bar{L}), \\ \underline{T}(\bar{L}^*) &= Q \underline{T}(\bar{L}) Q^T. \end{aligned} \tag{6.5}$$

I postulate that each of the quantities in (6.3) is frame-indifferent. Then by a familiar argument, dependence of these functions on  $\bar{L}$  reduces to dependence on  $L$ , where

$$L = \left\{ \rho_0, \rho, \operatorname{grad} \rho, \underline{v}, \underline{D}, \underline{\Omega}, \vartheta, \operatorname{grad} \vartheta, k, v, \operatorname{grad} v, \dot{v} \right\}, \tag{6.6}$$

the dependence on vectors is only through their inner products, and

$$\underline{\Omega} = \underline{W}_1 - \underline{W}_2, \tag{6.7}$$

$$\underline{v} = \dot{\underline{x}}_1 - \dot{\underline{x}}_2 \equiv \underline{u}_1 - \underline{u}_2. \tag{6.8}$$

Furthermore, representation theorems for (6.3) as functions of  $L$  are known, although quite complicated. In the special case where the dependence on the vector and tensor variables in  $L$  is linear,

$$s(L) = \bar{s}, \quad (6.9)$$

$$\underline{w}(L) = \sum_{\alpha\rho} w_{\alpha\rho} \text{grad } \rho + w_{D\alpha} \underline{v} + w_T \text{grad } \vartheta + \sum_{\alpha\nu} w_{\alpha\nu} \text{grad } \nu, \quad (6.10)$$

$$\overset{s}{\underline{T}}(L) = -p \underline{1} + \sum_{\alpha b} \xi_{\alpha b} (\text{tr } \underline{D}) \underline{1} + 2 \sum_{\alpha b} \eta_{\alpha b} \underline{D}, \quad (6.11)$$

$$\overset{a}{\underline{T}}(L) = -\mu \underline{\Omega}, \quad (6.12)$$

where  $\overset{s}{\underline{T}}$  and  $\overset{a}{\underline{T}}$  are the symmetric and skew parts of  $\underline{T}$ , the summations are over the two components, and the functions  $\bar{s}$ ,  $w_{\alpha\rho}$ ,  $w_{D\alpha}$ ,  $w_T$ ,  $w_{\alpha\nu}$ ,  $p$ ,  $\xi_{\alpha b}$ ,  $\eta_{\alpha b}$ ,  $\mu$  are functions of  $l$ , where

$$l = \left\{ \rho_{\alpha 0}, \rho, \vartheta, k, \nu, \dot{\nu} \right\}. \quad (6.13)$$

I impose the restriction that  $L$  and  $l$  do not in fact contain  $\rho_{\alpha 0}$ , so that the first constituent is in fact a fluid. In addition, I assume that

$$\overset{+}{c} = 0, \quad (6.14)$$

thus excluding certain physical phenomena, including chemical reactions.

By (2.19), (2.22)<sub>3</sub>, (5.19), (6.7) and some straightforward algebra, it may be shown that

$$\Sigma \operatorname{tr} \left( \underset{\alpha}{T} \underset{\alpha}{G} \right) = \operatorname{tr} \left( \underset{1}{T} \underset{1}{D} + \underset{2}{T} \underset{2}{D} + \underset{1}{T} \underset{1}{\Omega} \right). \quad (6.15)$$

Also, by (2.23)<sub>2</sub> and (6.8)

$$\Sigma \underset{\alpha}{m}^+ \cdot \underset{\alpha}{u} = \underset{1}{m}^+ \cdot \underset{1}{v}, \quad (6.16)$$

and by (2.23)<sub>5</sub>

$$\Sigma \underset{\alpha}{v}^+ (\overset{\sim}{v} - \underset{\alpha}{v}) = \underset{1}{v}^+ (\overset{\sim}{v} - \underset{2}{v}). \quad (6.17)$$

By (1.6), (1.10), (2.11) and (6.8),

$$\Sigma \underset{\alpha\alpha}{\rho \overset{\sim}{x}} \cdot \underset{\alpha}{u} = \frac{\rho_1 \rho_2}{\rho} \underset{1}{v} \cdot \underset{2}{\dot{v}}. \quad (6.18)$$

It is easily shown that

$$\overset{\sim}{x} \underset{\alpha} = \underset{\alpha}{x} + \underset{\alpha\alpha}{G} \underset{\alpha}{u}, \quad (6.19)$$

so

$$\begin{aligned} \Sigma \underset{\alpha\alpha}{\rho \overset{\sim}{x}} \cdot \underset{\alpha}{u} &= \Sigma \underset{\alpha\alpha}{\rho \overset{\sim}{x}} \cdot \underset{\alpha}{u} + \Sigma \underset{\alpha\alpha}{\rho u} \cdot \underset{\alpha\alpha}{G} \underset{\alpha}{u}, \\ &= \Sigma \underset{\alpha\alpha}{\rho \overset{\sim}{x}} \cdot \underset{\alpha}{u} + \Sigma \underset{\alpha\alpha}{\rho u} \cdot \underset{\alpha\alpha}{D} \underset{\alpha}{u}, \end{aligned} \quad (6.20)$$

or, by (6.18)

$$\Sigma \underset{\alpha\alpha}{\rho \overset{\sim}{x}} \cdot \underset{\alpha}{u} = \frac{\rho_1 \rho_2}{\rho} \underset{1}{v} \cdot \underset{2}{\dot{v}} + \Sigma \underset{\alpha\alpha}{\rho u} \cdot \underset{\alpha\alpha}{D} \underset{\alpha}{u}. \quad (6.21)$$

By a set of steps similar to those leading to (6.21), it may be shown that, for a two-constituent mixture of granular materials

$$\begin{aligned} \Sigma_{aaa} (\rho k \dot{\tilde{v}} + \rho k \dot{\hat{v}}) (\dot{\tilde{v}} - \dot{\hat{v}}) \\ = \left[ \frac{\rho_1 \rho_2 k k}{\rho k} (\dot{\tilde{v}}_1 - \dot{\tilde{v}}_2) + \frac{\rho_1 \rho_2}{\rho k} (k k \dot{\tilde{v}}_{121} - k k \dot{\tilde{v}}_{122}) (\dot{\tilde{v}} - \dot{\hat{v}}) \right]. \end{aligned} \quad (6.22)$$

However, in the special case  $k = 0$ , by (1.11)

$$\dot{\tilde{v}} = \dot{\hat{v}}, \quad (6.23)$$

and

$$\Sigma_{aaa} (\rho k \dot{\tilde{v}} + \rho k \dot{\hat{v}}) (\dot{\tilde{v}} - \dot{\hat{v}}) = 0. \quad (6.24)$$

By (6.14) - (6.17), (6.21) and (6.24), (5.28) becomes

$$\begin{aligned} \rho \eta \frac{\dot{\vartheta}}{\vartheta} - \rho \dot{\vartheta} \dot{\psi} \geq \operatorname{div} \hat{\phi} - \vartheta \left[ \frac{\rho_1 \rho_2}{\rho} \underline{v} \cdot \dot{\underline{v}} + \Sigma_{aa} \rho \underline{u} \cdot \underline{D} \underline{u} \right] \\ + \left[ \underline{g} - \Sigma_{aa} \underline{T}^T \underline{u} - \underline{h}(\dot{\tilde{v}} - \dot{\hat{v}}) \right] \cdot \operatorname{grad} \vartheta + \vartheta \rho \underline{m}^+ \cdot \underline{v} + \vartheta \rho \underline{v}^+ (\dot{\tilde{v}} - \dot{\hat{v}}) \\ - \frac{\vartheta}{\rho} \operatorname{tr}(\underline{T} - \Sigma \underline{T}) (\Sigma \rho \underline{D} + \Sigma \underline{u} \otimes \operatorname{grad} \rho) \\ - \vartheta \operatorname{tr} \left( \underline{T}^s \underline{D} + \underline{T}^s \underline{D} + \underline{T}^a \underline{\Omega}^T \right) \\ - \vartheta (\underline{h} - \Sigma \underline{h}) \cdot \operatorname{grad} \dot{\tilde{v}} - \vartheta \Sigma \underline{h} \cdot \operatorname{grad} \dot{\tilde{v}} \\ - \vartheta (\rho k \dot{\tilde{v}}^2 - \Sigma \rho g \dot{\tilde{v}}). \end{aligned} \quad (6.25)$$



A result derived in a fashion similar to (6.19) is

$$\dot{\rho} = \dot{\rho} + \underline{u} \cdot \text{grad } \rho . \quad (6.26)$$

By the continuity equation (2.7)<sub>1</sub> and (6.14), it follows from (6.26) that

$$\dot{\rho} = -\underline{u} \cdot \text{grad } \rho - \rho \text{ tr } \underline{D} . \quad (6.27)$$

From a familiar commutation formula,

$$\frac{d}{dt} (\text{grad } \rho) = \text{grad } \dot{\rho} - \underline{G}^T \text{grad } \rho . \quad (6.28)$$

Taking the gradient of (6.27) and using (5.20) and (6.28), yields

$$\begin{aligned} \frac{d}{dt} (\text{grad } \rho) &= +\underline{G}^T \text{grad } \rho - (\text{grad grad } \rho) \underline{u} \\ &- (\text{tr } \underline{D}) \text{grad } \rho - \rho \text{ grad}(\text{tr } \underline{D}) . \end{aligned} \quad (6.29)$$

I note the following result:

$$\dot{v} = \dot{v} - \underline{u} \cdot \text{grad } v . \quad (6.30)$$

By (6.27), (6.29) and (6.30) then

$$\begin{aligned}
\dot{\psi} = & -\sum_a \frac{\partial \psi}{\partial \rho} (\underline{u}_a \cdot \text{grad}_a \rho + \rho \text{tr } \underline{D}) \\
& - \sum_a \frac{\partial \psi}{\partial (\text{grad}_a \rho)} \cdot [ \underline{G}^T \text{grad}_a \rho + (\text{grad}_a \text{grad}_a \rho) \underline{u}_a \\
& \quad + (\text{tr } \underline{D}) \text{grad}_a \rho + \rho \text{grad}_a (\text{tr } \underline{D}) ] \\
& + \frac{\partial \psi}{\partial \underline{v}} \cdot \dot{\underline{v}} + \sum_a \text{tr} \frac{\partial \psi}{\partial \underline{D}} \underline{D}_a + \text{tr} \frac{\partial \psi}{\partial \underline{\Omega}} \dot{\underline{\Omega}} + \frac{\partial \psi}{\partial \vartheta} \dot{\vartheta} \\
& + \frac{\partial \psi}{\partial (\text{grad } \vartheta)} \cdot \frac{d}{dt} \text{grad } \vartheta + \frac{\partial \psi}{\partial k} \frac{\dot{k}}{2} \\
& + \sum_a \frac{\partial \psi}{\partial v} (\underline{v}_a - \underline{u}_a \cdot \text{grad}_a v) \\
& + \sum_a \frac{\partial \psi}{\partial (\text{grad}_a v)} \cdot \frac{d}{dt} (\text{grad}_a v) + \sum_a \frac{\partial \psi}{\partial v} \cdot \frac{d}{dt} (\text{grad}_a \dot{v}) .
\end{aligned} \tag{6.31}$$

It is obvious that the computation of  $\text{div } \hat{\phi}$  will involve gradients of second-order tensors, which are third-order tensors. Such quantities are in general ill-adapted to the system of notation used here. I choose a system of orthogonal Cartesian coordinates, and establish the notation used in terms of components referred to these coordinates.

Let  $\underline{a}$  be a vector and  $\underline{q}$  be a tensor. Read the symbol " $\sim$ " as "has Cartesian components", so that

$$\underline{a} \sim a_i ,$$

$$\underline{q} \sim q_{;k} .$$

Then by definition

$$\text{grad } \underline{a} \sim a_{jk,l},$$

where the comma indicates partial differentiation, and

$$\frac{\partial \hat{\phi}}{\partial \underline{a}} \sim \frac{\partial \hat{\phi}_1}{\partial a_{jk}}.$$

Define

$$\text{tr} \left( \frac{\partial \hat{\phi}}{\partial \underline{a}} \text{grad } \underline{a} \right) \sim \frac{\partial \hat{\phi}_1}{\partial a_{jk}} a_{jk,i}.$$

Note the identity

$$\Omega_{ij,k} = (D_{1ki,j} - D_{1kj,i}) - (D_{2ki,j} - D_{2kj,i}), \quad (6.34)$$

which will be abbreviated as

$$\text{grad } \underline{\Omega} = (\text{grad } \underline{D}_1)^X - (\text{grad } \underline{D}_2)^X. \quad (6.35)$$

Differentiating (6.30), gives

$$\text{grad } \dot{v} = \frac{d}{dt} \text{grad } v + (\text{grad grad } v)_{\underline{a} \underline{a}} \quad (6.36)$$

Computing  $\text{div } \hat{\phi}$  is now a straightforward task. By (6.35) and (6.36),

$$\begin{aligned}
\operatorname{div} \hat{\phi} &= \frac{\partial \hat{\phi}}{\partial \rho} \cdot \operatorname{grad} \rho + \sum_a \frac{\partial \hat{\phi}}{\partial \rho_a} \cdot \operatorname{grad} \rho_a \\
&+ \sum_a \operatorname{tr} \frac{\partial \hat{\phi}}{\partial (\operatorname{grad} \rho_a)} \operatorname{grad} \operatorname{grad} \rho_a + \operatorname{tr} \frac{\partial \hat{\phi}}{\partial \gamma} \operatorname{grad} \gamma \\
&+ \sum_a \operatorname{tr} \frac{\partial \hat{\phi}}{\partial \underline{D}_a} \operatorname{grad} \underline{D}_a + \operatorname{tr} \frac{\partial \hat{\phi}}{\partial \underline{\Omega}_1} (\operatorname{grad} \underline{D}_1) - \operatorname{tr} \frac{\partial \hat{\phi}}{\partial \underline{\Omega}_2} (\operatorname{grad} \underline{D}_2) \\
&+ \frac{\partial \hat{\phi}}{\partial \vartheta} \cdot \operatorname{grad} \vartheta + \operatorname{tr} \frac{\partial \hat{\phi}}{\partial (\operatorname{grad} \vartheta)} \operatorname{grad} \operatorname{grad} \vartheta \\
&+ \frac{\partial \hat{\phi}}{\partial k} \cdot \operatorname{grad} k + \sum_a \frac{\partial \hat{\phi}}{\partial v_a} \cdot \operatorname{grad} v_a + \sum_a \operatorname{tr} \frac{\partial \hat{\phi}}{\partial (\operatorname{grad} v_a)} \operatorname{grad} \operatorname{grad} v_a \\
&+ \sum_a \frac{\partial \hat{\phi}}{\partial \dot{\gamma}_a} \cdot \left[ \frac{d}{dt} \operatorname{grad} v_a + (\operatorname{grad} \operatorname{grad} v_a) \underline{u}_a \right].
\end{aligned} \tag{6.37}$$

Note that, by (2.12)<sub>2</sub>, (6.1) and (6.14)

$$\rho \dot{k} = -\operatorname{div}(\rho k \underline{u}), \tag{6.38}$$

so that

$$\rho \dot{k} = -k \operatorname{div}(\rho \underline{u}) - \rho \underline{u} \cdot \operatorname{grad} k. \tag{6.39}$$

By (6.23) and (6.39),

$$\rho \dot{k} \dot{v}^2 = -k \dot{v}^2 \operatorname{div}(\rho \underline{u}) - \rho \dot{v}^2 \underline{u} \cdot \operatorname{grad} k. \tag{6.40}$$

Also

$$\dot{k} = -\underline{u} \cdot \operatorname{grad} k. \tag{6.41}$$

Substituting (6.33), (6.40) and (6.41) into (6.25) then yields

$$\begin{aligned}
& \rho \eta \frac{\partial}{\partial t} - \rho \eta \left[ -\sum_a \frac{\partial \psi}{\partial p_a} (\bar{u}_a \cdot \text{grad } \rho + \rho \text{ tr } \underline{D}) - \sum_a \frac{\partial \psi}{\partial (\text{grad } \rho)} \cdot \left[ \sum_a \text{grad } \rho + (\text{grad grad } \rho) \bar{u}_a + (\text{tr } \underline{D}) \text{grad } \rho + \rho \text{ grad}(\text{tr } \underline{D}) \right] \right. \\
& + \frac{\partial \psi}{\partial \bar{y}} \cdot \bar{y} + \sum_a \text{tr} \frac{\partial \psi}{\partial \underline{D}_a} \underline{D}_a + \text{tr} \frac{\partial \psi}{\partial \underline{\Omega}} \underline{\Omega} + \frac{\partial \psi}{\partial \vartheta} \vartheta + \frac{\partial \psi}{\partial (\text{grad } \vartheta)} \cdot \frac{d}{dt} (\text{grad } \vartheta) - \frac{\partial \psi}{\partial k} \bar{u}_2 \cdot \text{grad } k \\
& + \sum_a \frac{\partial \psi}{\partial v_a} (\bar{v}_a \cdot \text{grad } v) + \sum_a \frac{\partial \psi}{\partial (\text{grad } v)} \cdot \frac{d}{dt} \text{grad } v + \sum_a \frac{\partial \psi}{\partial \dot{v}} \frac{dv}{dt} \left. \right] \geq \\
& \frac{\partial \hat{\phi}}{\partial \rho_0} \cdot \text{grad } \rho_0 + \sum_a \frac{\partial \hat{\phi}}{\partial \rho_a} \cdot \text{grad } \rho_a + \sum_a \text{tr} \frac{\partial \hat{\phi}}{\partial (\text{grad } \rho)} \text{grad grad } \rho_a + \text{tr} \frac{\partial \hat{\phi}}{\partial \dot{v}} \text{grad } \dot{v} \\
& + \sum_a \text{tr} \frac{\partial \hat{\phi}}{\partial \underline{D}_a} \text{grad } \underline{D}_a + \text{tr} \frac{\partial \hat{\phi}}{\partial \underline{\Omega}} (\text{grad } \underline{D})^X - \text{tr} \frac{\partial \hat{\phi}}{\partial \underline{\Omega}} (\text{grad } \underline{D})^X + \frac{\partial \hat{\phi}}{\partial \vartheta} \cdot \text{grad } \vartheta + \text{tr} \frac{\partial \hat{\phi}}{\partial (\text{grad } \vartheta)} \text{grad}(\text{grad } \vartheta) \\
& + \frac{\partial \hat{\phi}}{\partial k} \cdot \text{grad } k + \sum_a \frac{\partial \hat{\phi}}{\partial v_a} \cdot \text{grad } v_a + \sum_a \text{tr} \frac{\partial \hat{\phi}}{\partial (\text{grad } v)} \text{grad grad } v_a + \sum_a \frac{\partial \hat{\phi}}{\partial \dot{v}} \cdot \left[ \frac{d}{dt} \text{grad } v + (\text{grad grad } v) \bar{u}_a \right] \\
& - \vartheta \left[ \frac{\rho_1 \rho_2}{\rho} \bar{v} \cdot \bar{y} + \sum_{aa} \rho_{aa} \cdot \underline{D} \bar{u}_a \right] \\
& + \left[ \underline{g} - \sum_a \underline{I}_a^T \bar{u}_a - \underline{h}(\bar{v} - \dot{v}) \right] \cdot \text{grad } \vartheta + \vartheta \rho \underline{m} \cdot \bar{y} + \vartheta \rho \dot{v} (\bar{v} - \dot{v}) \\
& - \frac{\vartheta}{\rho} \text{tr}(\underline{I} - \sum_a \underline{I}_a) (\sum_a \rho_a \underline{D}_a + \sum_a \bar{u}_a \otimes \text{grad } \rho_a) - \vartheta \text{tr}(\underline{I} \underline{D} + \underline{I} \underline{\Omega} + \underline{I} \underline{\Omega}^T) \\
& - \vartheta (\underline{h} = \sum_a \underline{h}_a) \cdot \left[ \frac{d}{dt} \text{grad } v + (\text{grad grad } v) \bar{u}_a \right] - \vartheta \sum_a \underline{h}_a \cdot \left[ \frac{d}{dt} \text{grad } v + (\text{grad grad } v) \bar{u}_a \right] + \\
& + \vartheta \rho \bar{v} \cdot \bar{u}_2 \cdot \text{grad } k + \vartheta k \bar{v} \cdot \bar{u}_2 \cdot \text{div}(\rho \bar{u}_2) + \vartheta \sum_{aaa} \rho_{aaa}
\end{aligned}$$

(6.42)

A thermodynamic process is defined in a fashion similar to that for a conventional continuum. Each balance law with the exception of the one for mass contains one term which may be adjusted arbitrarily, and there exists at least one thermodynamic process in which  $\dot{\underline{v}}_a$ ,  $\dot{\underline{D}}_a$ ,  $\text{grad } \underline{D}_a$ ,  $\dot{\underline{\Omega}}_a$ ,  $\text{grad } \rho_a$ ,  $\text{grad grad } \rho_a$ ,  $\dot{\vartheta}_a$ ,  $\frac{d}{dt} \text{grad } \vartheta_a$ ,  $\text{grad grad } \vartheta_a$ ,  $\frac{d}{dt} \underline{v}_a$ ,  $\frac{d}{dt} \text{grad } \underline{v}_a$ ,  $\text{grad grad } \underline{v}_a$  and  $\text{grad } \underline{k}_2$  may be chosen arbitrarily and independently of any other term in the inequality. This implies the following results\*:

$$\begin{aligned} \text{a. } \psi & \text{ is independent of } \underline{D}_a, \underline{\Omega}_a, \text{grad } \vartheta_a, \underline{v}_a, \\ \text{b. } \frac{\partial \psi}{\partial \underline{v}} & = \frac{\rho_a \rho_a}{\rho} \underline{v}_a, \end{aligned} \quad (6.43)$$

so that

$$\psi = \psi_I(\rho_a, \rho_a, \text{grad } \rho_a, \vartheta_a, \underline{k}_2, \underline{v}_a, \text{grad } \underline{v}_a) + \frac{1}{2} \sum \frac{\rho_a}{\rho} \underline{u}_a \cdot \underline{u}_a. \quad (6.44)$$

Furthermore,  $\psi_I$  depends on  $\text{grad } \rho_a$  and  $\text{grad } \underline{v}_a$  only through their inner products  $\text{grad } \rho_a \cdot \text{grad } \rho_a$ ,  $\text{grad } \rho_a \cdot \text{grad } \underline{v}_a$ ,  $\text{grad } \underline{v}_a \cdot \text{grad } \underline{v}_a$ .

$$\text{c. } \frac{\partial \psi}{\partial \vartheta} = \frac{\eta}{\vartheta}. \quad (6.45)$$

$$\text{d. } \frac{\partial \hat{\phi}}{\partial \underline{k}} = \rho \frac{\partial \underline{u}}{\partial \underline{k}} \left( \frac{\partial \psi}{\partial \underline{k}} - \frac{\underline{v}_a^2}{2} \right). \quad (6.46)$$

$$\text{e. } \left( \frac{\partial \hat{\phi}}{\partial (\text{grad } \vartheta)} \right)^s = \underline{0}, \quad (6.47)$$

\* These results reduce to those of Müller [3] in the case where both continua are ordinary fluids. I have corrected a minor printer's error in result b.

where, as before, the superscript <sup>s</sup> denotes the symmetric part of the indicated tensor.

$$f. \left( \frac{\partial}{\partial(\text{grad } \rho)} [\hat{\phi} - \rho \partial \psi_{I \underline{a}}] \right)^s = 0. \quad (6.48)$$

Here (6.44) has been used.

g. Again, I revert to Cartesian components.

$$\left( \rho \rho_1^{\partial} \frac{\partial \psi_I}{\partial \rho_{1,k}} \delta_{ij} - \frac{\partial \hat{\phi}_j}{\partial \Omega_{ki}} + \frac{\partial \hat{\phi}_k}{\partial \Omega_{ji}} - \frac{\partial \hat{\phi}_k}{\partial D_{1ij}} \right)^s_{ij} = 0, \quad (6.49)$$

$$\left( \rho \rho_2^{\partial} \frac{\partial \psi_I}{\partial \rho_{2,k}} \delta_{ij} + \frac{\partial \hat{\phi}_j}{\partial \Omega_{ki}} - \frac{\partial \hat{\phi}_k}{\partial \Omega_{ji}} - \frac{\partial \hat{\phi}_k}{\partial D_{2ij}} \right)^s_{ij} = 0, \quad (6.50)$$

where the superscript <sup>s</sup><sub>ij</sub> denotes the symmetric part with respect to i and j.

$$h. \frac{\partial \underline{h}_1}{\partial \underline{v}} = \frac{\partial \hat{\phi}}{\partial \underline{v}} + \partial \rho \frac{\partial \psi_I}{\partial(\text{grad } \underline{v})}, \quad (6.51)$$

$$\frac{\partial \underline{h}_2}{\partial \underline{v}} = \frac{\partial \hat{\phi}}{\partial \underline{v}} + \partial \rho \frac{\partial \psi_I}{\partial(\text{grad } \underline{v})} + \partial \rho k \hat{\nu} \underline{u}. \quad (6.52)$$

$$i. \left( \frac{\partial}{\partial(\text{grad } \rho)} [\hat{\phi} - \rho \partial \psi_{I \underline{a}}] \right)^s = 0, \quad (6.53)$$

where in deriving (6.53), (6.51) and (6.52) have been used.

j.  $\hat{\phi}$  is independent of  $\text{grad } \dot{\rho}_2$ .

I note that

$$\text{grad } \underline{y} = \underline{D}_1 - \underline{D}_2 + \underline{\Omega} . \quad (6.54)$$

Furthermore

$$\begin{aligned} \text{div } \rho \underline{u} &= \rho \text{ div}(\underline{x} - \underline{\dot{x}}) + \underline{u} \cdot \text{grad } \rho , \\ &= \rho \text{ tr } \underline{D}_2 - \rho \text{ tr } \underline{D}_1 + \underline{u} \cdot \text{grad } \rho . \end{aligned} \quad (6.55)$$

However, by (5.22)

$$\rho \text{ tr } \underline{D} = \rho_1 \text{ tr } \underline{D}_1 + \rho_2 \text{ tr } \underline{D}_2 + \underline{u} \cdot \text{grad } \rho_1 + \underline{u} \cdot \text{grad } \rho_2 . \quad (6.56)$$

Substituting (6.56) into (6.55) and taking (1.6) into account, I

obtain

$$\begin{aligned} \text{div } \rho \underline{u} &= \frac{\rho_1 \rho_2}{\rho} (\text{tr } \underline{D}_2 - \text{tr } \underline{D}_1) - \frac{\rho_2}{\rho_1} \underline{u} \cdot \text{grad } \rho_1 \\ &\quad + \frac{\rho_1}{\rho_2} \underline{u} \cdot \text{grad } \rho_2 . \end{aligned} \quad (6.57)$$

Substitution of (6.54), and (6.57) into (6.42) yields the residual inequality



$$\begin{aligned}
& \rho \vartheta \frac{\partial \psi}{\partial \rho} \frac{\underline{u}}{1} \cdot \text{grad } \rho - \frac{\partial \hat{\phi}}{\partial \rho} \cdot \text{grad } \rho + \frac{\vartheta}{\rho} \text{tr}(\underline{T} - \Sigma \underline{T}) \frac{\underline{u}}{1} \otimes \text{grad } \rho - \\
& \quad - \vartheta k \nu^2 \frac{\rho}{22} \frac{\underline{u}}{1} \cdot \text{grad } \rho + \\
& + \rho \vartheta \frac{\partial \psi}{\partial \rho} \frac{\underline{u}}{2} \cdot \text{grad } \rho - \frac{\partial \hat{\phi}}{\partial \rho} \cdot \text{grad } \rho + \frac{\vartheta}{\rho} \text{tr}(\underline{T} - \Sigma \underline{T}) \frac{\underline{u}}{2} \otimes \text{grad } \rho + \\
& \quad + \vartheta k \nu^2 \frac{\rho}{22} \frac{\underline{u}}{2} \cdot \text{grad } \rho + \\
& + \rho \vartheta \frac{\partial \psi}{\partial \rho} \text{tr } \underline{D} + \rho \vartheta \frac{\partial \psi}{\partial(\text{grad } \rho)} \cdot \underline{D} \text{ grad } \rho + \rho \vartheta \frac{\partial \psi}{\partial(\text{grad } \rho)} \cdot \text{grad } \rho (\text{tr } \underline{D}) - \\
& \quad - \text{tr} \frac{\partial \hat{\phi}}{\partial \underline{v}} \cdot \underline{D} + \vartheta \rho \underline{u} \cdot \underline{D} \underline{u} + \vartheta \frac{\rho}{\rho} \text{tr}(\underline{T} - \Sigma \underline{T}) \underline{D} + \vartheta \text{tr} \frac{\underline{s}}{\underline{1}} \underline{D} - \\
& \quad - \vartheta k \nu^2 \frac{\rho \rho}{22} \frac{1}{\rho} \text{tr } \underline{D} + \tag{6.58} \\
& + \rho \vartheta \frac{\partial \psi}{\partial \rho} \text{tr } \underline{D} + \rho \vartheta \frac{\partial \psi}{\partial(\text{grad } \rho)} \cdot \underline{D} \text{ grad } \rho + \rho \vartheta \frac{\partial \psi}{\partial(\text{grad } \rho)} \cdot \text{grad } \rho (\text{tr } \underline{D}) + \\
& \quad + \text{tr} \frac{\partial \hat{\phi}}{\partial \underline{v}} \underline{D} + \vartheta \rho \underline{u} \cdot \underline{D} \underline{u} + \vartheta \frac{\rho}{\rho} \text{tr}(\underline{T} - \Sigma \underline{T}) \underline{D} + \vartheta \text{tr} \frac{\underline{s}}{\underline{2}} \underline{D} + \vartheta k \nu^2 \frac{\rho_1 \rho_2}{22} \text{tr } \underline{D} - \\
& \quad - \text{tr} \left[ \left( \frac{\partial \hat{\phi}}{\partial \underline{v}} + \vartheta \underline{T} \right) \underline{\Omega} \right] + \left[ - \frac{\partial \hat{\phi}}{\partial \vartheta} - \underline{g} + \Sigma \frac{\underline{T}}{\underline{a}} \frac{\underline{u}}{\underline{a}} + \frac{\underline{h}(\underline{v} - \underline{v})}{\underline{1}} \right] \cdot \text{grad } \vartheta - \\
& \quad - \vartheta \rho \frac{\underline{m}}{\underline{1}} \cdot \underline{v} - \vartheta \rho \underline{v}(\underline{v} - \underline{v}) \\
& - \Sigma \left( \frac{\partial \psi}{\partial \underline{v}} \frac{\underline{u}}{\underline{a}} + \frac{\partial \hat{\phi}}{\partial \underline{v}} \right) \cdot \text{grad } \underline{v} - \Sigma \left( \rho \vartheta \frac{\partial \psi}{\partial \underline{v}} + \vartheta \rho \underline{g} \right) \underline{v} \geq 0 .
\end{aligned}$$

The consequences of results a. - f. may be investigated by standard methods and constitute generalizations of known results (see, e.g., [3]) for a mixture of two constituents to the case where one of the components is a granular material. In particular a. and b. indicate that the Helmholtz free energy, in addition to being independent of peculiar stretchings  $\underline{D}$ , the relative spin  $\underline{\Omega}$ , and the coldness gradient,  $\text{grad } \hat{\theta}$ , is independent of the peculiar volume distribution velocities  $\underline{\dot{v}}$ . Furthermore its dependence on diffusion velocities is made explicit by (6.44). Result c. is a familiar relation from thermodynamics, but result d. is new. Results h. generalize analogous results of Goodman and Cowin indicating, as might be expected, a contribution to peculiar equilibrated stress due to diffusion. Result g. constitutes a set of restrictions on the forms of the entropy flux and Helmholtz free energy. The restrictions imposed by the familiar relations e. and f., and the new relation i. are easily made explicit by a method due to Müller. Solving these equations, I obtain, in Cartesian tensor notation,

$$\hat{\phi}_i - \frac{1}{2} \rho \psi_{,i} v_i = (\Lambda_{ijkl} \rho_{,j} v_{,k} + \Lambda_{ijk} \rho_{,j} + \Gamma_{ijk} v_{,j} + \Lambda_{ij}) \hat{\theta}_{,j} \quad (6.59)$$

$$+ \Delta_{ijk} \rho_{,j} v_{,k} + \Lambda_{ij} v_{,i} + \Gamma_{ij} \rho_{,j} + \Lambda_i,$$

$$\begin{aligned} \hat{\phi}_i + \partial \rho \psi_{I i}^v = & (\Lambda_{2 i j k l} \rho_{1, j}^v \nu_{2, k} + \Lambda_{2 i j k} \rho_{2, j} + \Gamma_{2 i j k} \nu_{2, j} + \Lambda_{2 i j}) \partial_{, j} \\ & + \Delta_{2 i j k} \rho_{2, j}^v \nu_{2, k} + \Lambda_{2 i j} \nu_{2, j} + \Gamma_{2 i j} \rho_{2, j} + \Delta_{2 i}. \end{aligned} \quad (6.60)$$

Here the coefficients  $\Lambda$ ,  $\Gamma$ ,  $\Delta$  are skew in all indices, are independent of  $\text{grad } \partial$ , and a subscript 1 or 2 indicates independence of  $\text{grad } \rho_1$  and  $\text{grad } \nu$  or  $\text{grad } \rho$  and  $\text{grad } \nu$ , respectively.

It follows that

$$\begin{aligned} \rho \partial \psi_{I i}^v = & (\Delta_{2 i j k} \rho_{2, j}^v \nu_{2, k} + \Lambda_{2 i j} \nu_{2, i} + \Gamma_{2 i j} \rho_{2, j} + \Lambda_{2 i}) \\ & - (\Delta_{1 i j k} \rho_{1, j}^v \nu_{1, k} + \Lambda_{1 i j} \nu_{1, i} + \Gamma_{1 i j} \rho_{1, j} + \Lambda_{1 i}). \end{aligned} \quad (6.61)$$

Equations (6.59) - (6.61) may be used to derive a more explicit form for  $\hat{\phi}$  by straightforward addition and substitution. Furthermore, the functions  $\Lambda$ ,  $\Delta$ ,  $\Gamma$  may be recognized as combinations of the derivatives of  $\rho \partial \psi_{I i}^v$  with respect to  $\text{grad } \rho$ ,  $\text{grad } \nu$  and the values of these derivatives with one or more of the parameters vanishing. Finally, it may be recalled that  $\psi_I$  and  $\hat{\phi}$  depend on  $\text{grad } \rho$ ,  $\text{grad } \nu$  and  $\text{grad } \partial$  only through their inner products, thus yielding an alternative form for  $\psi_I$ . All of these results involve straightforward calculations and yield somewhat complicated results. Since they are not needed in their full generality in the following sections, they are not recorded here.

## 7. Equilibrium

Let

$$X_A = \left\{ \text{grad } \rho, \frac{D}{a}, \underline{\Omega}, \text{grad } \vartheta, \underline{v}, \text{grad } \nu, \underline{\dot{\nu}} \right\}. \quad (7.1)$$

If  $X_A = 0$ , the mixture is said to be in equilibrium. Let the entropy production  $\sigma$  be the left-hand side of (6.58), so that inequality is

$$\sigma \geq 0. \quad (7.2)$$

Then  $\sigma$  has a minimum at equilibrium. Necessary conditions for this are

$$\frac{\partial \sigma}{\partial X_A} = 0 \text{ at } X_A = 0, \quad (7.3)$$

$$\left\| \frac{\partial^2 \sigma}{\partial X_A \partial X_B} \right\| \text{ is non-negative definite at } X_A = 0. \quad (7.4)$$

By (2.25)<sub>1</sub> and (6.59) - (6.61), consequences of (7.3) are

$$\frac{\partial \hat{\phi}}{\partial \rho} = 0, \quad (7.5)$$

$$\frac{\partial \hat{T}}{\partial \rho} = \partial \rho \frac{\partial \psi}{\partial \rho} \underline{1} - \frac{\partial \hat{\phi}}{\partial \nu} \underline{1} \equiv -\partial p \underline{1}, \quad (7.6)$$

$$\frac{\partial \hat{T}}{\partial \rho} = \partial \rho \frac{\partial \psi}{\partial \rho} \underline{1} + \frac{\partial \hat{\phi}}{\partial \nu} \underline{1} \equiv -\partial p \underline{1}, \quad (7.7)$$

$$\frac{\partial \hat{T}}{\partial \rho} = 0, \quad (7.8)$$

$$\underline{g} = 0, \quad (7.9)$$

$$\frac{\dagger}{1} \underline{\underline{m}} = \underline{\underline{0}}, \quad (7.10)$$

$$\frac{\partial \hat{\phi}}{\partial v} = 0, \quad (7.11)$$

$$\rho g_{11} = -\rho \frac{\partial \psi}{\partial v} - \rho \underline{\underline{v}}_1, \quad (7.12)$$

$$\rho g_{22} = -\rho \frac{\partial \psi}{\partial v} + \rho \underline{\underline{v}}_1, \quad (7.13)$$

where all quantities are evaluated at equilibrium.

Equations (7.5) - (7.10) are familiar results from the thermostatics of a mixture of two ordinary fluids, and have been commented upon extensively by Müller [3]. Unlike the single component granular material of Goodman and Cowin, a mixture of a granular material and a fluid is incapable of sustaining a shear stress in equilibrium. The results (7.12) and (7.13) are analogous to those obtained in elasticity theory, stating that the intrinsic body forces are derived from the Helmholtz free energy and the growth of equilibrated force.

## 8. A Linear Theory

Consider an expansion about equilibrium, linear in  $X_A$  as defined by (7.1) of the quantities  $\psi$ ,  $\hat{\phi}$ ,  $\underline{\underline{T}}_a$ ,  $\underline{\underline{m}}_1$ ,  $\underline{\underline{h}}_a$ ,  $\underline{\underline{v}}_1$ ,  $\underline{\underline{g}}_a$ .

By results a. and b. in Section 6

$$\psi = \psi(\rho_0, \rho, \text{grad } \rho, \vartheta, k, \nu, \text{grad } \nu). \quad (8.1)$$

However, by the assumption of frame-indifference,  $\psi$  depends on  $\text{grad } \rho$  and  $\text{grad } v$  through their inner products, which are nonlinear. Thus

$$\psi = \psi(\rho_0, \rho, \rho, \vartheta, k, v, v). \quad (8.2)$$

By (6.10), linear representations for  $q$ ,  $\overset{+}{m}_1$ , and  $\hat{\phi}$  are

$$q = -\kappa_T \text{grad } \vartheta - \sum_{\alpha} \kappa_{\alpha \rho} \text{grad } \rho_{\alpha} - \kappa_{Dv} v - \sum_{\alpha} \kappa_{\alpha v} \text{grad } v, \quad (8.3)$$

$$\overset{+}{m}_1 = -m_T \text{grad } \vartheta - \sum_{\alpha} m_{\alpha \rho} \text{grad } \rho_{\alpha} - m_{Dv} v - \sum_{\alpha} m_{\alpha v} \text{grad } v, \quad (8.4)$$

$$\hat{\phi} = -K_T \text{grad } \vartheta - \sum_{\alpha} K_{\alpha \rho} \text{grad } \rho_{\alpha} - K_{Dv} v - \sum_{\alpha} K_{\alpha v} \text{grad } v, \quad (8.4)$$

where the coefficients are functions of  $\rho_0, \rho, \vartheta, k, v, \dot{v}$ . The representation (8.5) for  $\hat{\phi}$  is further restricted by (6.47), (6.48), (6.53) and (8.2).

Straightforward computation leads to the conclusion that  $K_T, K_{\alpha \rho}, K_{\alpha v}$  vanish, giving

$$\hat{\phi} = -K_{Dv} v. \quad (8.6)$$

In this case (6.49) and (6.50) are satisfied identically. It is seen that, unlike the special case of a single granular material of Goodman and Cowin [2], the entropy flux does not have the classical form of the heat flux multiplied by the coldness. It is in addition affected by the diffusion velocity, with coefficient depending on density, volume distribution and its velocity, coldness, and equilibrated inertia.

There is, furthermore, a representation for equilibrated stress

$\hat{h}_a$  derived from (6.10). However,  $\hat{h}_a$  are obtained from  $\hat{\xi}_a$  as given by (8.6) and  $\hat{v}_a$  as given by (8.2). An easy computation yields

$$\frac{\partial \hat{h}_1}{\partial v} = -\frac{\partial K_D}{\partial v} \lambda, \quad (8.7)$$

$$\frac{\partial \hat{h}_2}{\partial v} = -\left(\frac{\partial K_D}{\partial v} + \frac{\rho \rho}{\rho} \frac{\partial v}{\partial v}\right) \lambda. \quad (8.8)$$

Thus the equilibrated stress vanishes when the diffusion velocity vanishes.

By (6.11) and (6.12), representations for  $\hat{T}_a$ ,  $\hat{v}_a$  and  $\hat{g}_a$  are

$$\hat{T}_a = -\mu \Omega, \quad (8.9)$$

$$\hat{T}_a - \hat{T}_a^0 = \Sigma(\lambda \hat{v} + \xi \text{tr } \hat{D})_a + 2\Sigma \eta \hat{D}_a, \quad (8.10)$$

$$\hat{v}_a - \hat{v}_a^0 = \Sigma(\nu \hat{v} + \mu \text{tr } \hat{D})_a, \quad (8.11)$$

$$\hat{g}_a - \hat{g}_a^0 = -\Sigma(\zeta \hat{v} + \delta \text{tr } \hat{D})_a, \quad (8.12)$$

where superscript  $^0$  denotes values in equilibrium.

Inserting (8.3) - (8.12) into (6.58), and noting (7.2) - (7.13), I obtain the following restrictions on the coefficients in the linear theory:

$$K_D = -\delta_{pp} \frac{\partial \psi}{\partial p} - \delta p,$$

$$K_D = \delta_{pp} \frac{\partial \psi}{\partial p} + \delta p,$$

$$\kappa_T \geq 0,$$

$$\kappa_{1^p} = 0, \quad \kappa_{2^p} = 0,$$

$$\delta_{pp} m_{1^p} = -\frac{\partial K_D}{\partial p} - \delta p \frac{\partial \psi}{\partial p},$$

$$\delta_{pp} m_{2^p} = -\frac{\partial K_D}{\partial p} + \delta p \frac{\partial \psi}{\partial p}, \quad (8.13)$$

$$\xi_{11} \geq 0, \quad \xi_{22} \geq 0, \quad \xi_{11} \xi_{22} - \frac{1}{4} (\xi_{12} + \xi_{21})^2 \geq 0,$$

$$\eta_{11} \geq 0, \quad \eta_{22} \geq 0, \quad \eta_{11} \eta_{22} - \frac{1}{4} (\eta_{12} + \eta_{21})^2 \geq 0,$$

$$\mu \geq 0,$$

$$2\delta m_D \geq 0,$$

$$2\kappa_T(2\delta m_D) \geq \left( \frac{\partial K_D}{\partial \theta} + \kappa_D + \delta_{pp} m_T + \frac{\partial}{\partial p} \left( \frac{\partial K_D}{\partial \theta} + \kappa_D + \delta_{pp} m_T + \frac{\partial}{\partial p} \right) \right).$$

Finally, the determinant



|   |   |  |  |
|---|---|--|--|
| $\begin{matrix} 2\xi \\ 11 \end{matrix}$  | $\begin{matrix} \xi + \xi \\ 12 \quad 21 \end{matrix}$                          | $\begin{matrix} \lambda + \delta - \rho\mu \\ 11 \quad 11 \quad 1 \end{matrix}$        | $\begin{matrix} \lambda + \delta + \rho\mu \\ 12 \quad 21 \quad 1 \end{matrix}$        |
| $\begin{matrix} \xi + \xi \\ 12 \quad 21 \end{matrix}$                          | $\begin{matrix} 2\xi \\ 22 \end{matrix}$  | $\begin{matrix} \lambda + \delta - \rho\mu \\ 21 \quad 12 \quad 2 \end{matrix}$        | $\begin{matrix} \lambda + \delta + \rho\mu \\ 22 \quad 22 \quad 2 \end{matrix}$        |
| $\begin{matrix} \lambda + \delta - \rho\mu \\ 11 \quad 11 \quad 1 \end{matrix}$ | $\begin{matrix} \lambda + \delta - \rho\mu \\ 21 \quad 12 \quad 2 \end{matrix}$ | $\begin{matrix} 2(\zeta - \rho\nu) \\ 11 \quad 1 \end{matrix}$                         | $\begin{matrix} \zeta + \zeta + \rho(u-v) \\ 12 \quad 21 \quad 2 \quad 1 \end{matrix}$ |
| $\begin{matrix} \lambda + \delta + \rho\mu \\ 12 \quad 1 \end{matrix}$          | $\begin{matrix} \lambda + \delta + \rho\mu \\ 22 \quad 22 \quad 2 \end{matrix}$ | $\begin{matrix} \zeta + \zeta + \rho(u-v) \\ 12 \quad 21 \quad 2 \quad 1 \end{matrix}$ | $\begin{matrix} 2(\zeta + \rho\nu) \\ 22 \quad 2 \end{matrix}$                         |

(8.14)

must be positive semi-definite. The restrictions (8.13) are classical restrictions for a mixture of two fluids as given by Müller. The inequalities (8.14) include (8.13)<sub>7</sub> but are otherwise new to this theory.

The form of the entropy flux  $\underline{\phi}$  in the linear theory can be found.

By (8.13)<sub>1,2</sub>

$$K_D = \vartheta \underline{v} \left[ \rho \rho \left( \frac{\partial \psi}{\partial \rho} \right)_{12} - \frac{\partial \psi}{\partial \rho} \right]_1 + \left( \frac{2}{\rho} p_1 - \frac{2}{\rho} p_2 \right) \quad (8.15)$$

Substituting (6.23), (7.6) and (8.6) into (5.12), and comparing with (8.15) substituted into (8.6) gives

$$\underline{\phi} = \vartheta \underline{q} - \vartheta \underline{v} \left[ \rho \rho \left( \frac{\partial \psi}{\partial \rho} \right)_{12} - \frac{\partial \psi}{\partial \rho} \right]_1 - \frac{\partial K_D}{\partial v} \left( \dot{v}_1 - \dot{v}_2 \right) \quad (8.16)$$

This generalizes Müller's (7.21).

## ACKNOWLEDGEMENT

My thanks go to the Mathematics Research Center and the Soil Science Department of the University of Wisconsin for support of this work.

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