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AN ILLUSTRATION OF THE VALIDITY OF
THE SIMULATION APPROACH FOR SOLVING
PROBLEMS FOR THE EFFECTIVENESS OF
CHEMICAL MUNITION SYSTEMS

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constrained version of the simulation. Then that portion of the simulation that had to be eliminated in order to test the analytic solution is tested separately. By inference, this method of verification can be utilized on much more complicated simulation models by subdividing them into simpler models each of which can be verified independently.

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SUMMARY

It is true that there are a vast number of techniques that can be used in computer simulation. For validation, each of these requires its own individual method of verification. The compounding of these techniques makes verification of the total simulation often approach the impossible. However, this does not eliminate simulation as a very valuable methodology for performing systems analysis. This paper has presented an example of a simulation that can be shown to produce accurate results by verifying its output in two stages. First an analytic solution is derived for a constrained version of the simulation. Then that portion of the simulation that had to be eliminated in order to test the analytic solution is tested separately. By inference, this method of verification can be utilized on much more complicated simulation models by subdividing them into simpler models each of which can be verified independently.

PREFACE

The work described in this report was conducted under Project 1W162116A084, Chemical Testing and Assessment Technology. This work was started in June 1970 and completed in November 1970.

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AN ILLUSTRATION OF THE VALIDITY OF THE SIMULATION APPROACH FOR SOLVING PROBLEMS FOR THE EFFECTIVENESS OF CHEMICAL MUNITION SYSTEMS

CHAPTER 1

I. INTRODUCTION.

It is obvious that the most accurate way of deriving the theoretical effectiveness of a chemical munition system is by analytic means. The complicity of the functioning of many such systems, however, negates the ability of developing purely mathematical models in closed form to represent these systems. The most widely used alternative to the mathematical model in recent times has been the simulation. Advances in computer technology and the related advent of computer techniques for approximating both mathematical and physical systems have provided the analyst with an excellent tool for evaluating the effectiveness of a chemical munition. Again, however, a problem has arisen. Unless empirical data are available to validate simulated results, decisions (which involve large resources of both money and manpower) must often be made solely from the outputs of unverifiable computer models. At the present time, this problem can not be totally resolved. One means of partially circumventing this problem, however, is by showing the validity of a total simulation of a complex munition system by subdividing the functioning of the system into subunits, each of which can be verified independently by either analytic or empirical means. Inferences concerning the validity of the total simulation model can then be made from the proven verification of these subunits. This paper develops one such computer simulation model for a weapons system that was analyzed at Edgewood Arsenal. It then proves the validity of the effectiveness outputs of the computer program by constraining the simulation to a case that can be proven analytically. The verification of the total simulation is then inferred from statistical procedures that are applied to that portion of the program which was eliminated in order to obtain a subunit to which an analytic solution existed.

It is hoped that this model will serve as an example to those who might doubt the ability of complicated simulation models to truly represent the systems they are designed to emulate. It must be noted that the validity of outputs from simulation-type programs is extremely dependent upon the parameters used as inputs. These inputs are often either empirical in nature or are outputs of other computer models. In either event, if imprecisions are indigenous to the inputs, inaccuracy will result in the output even if the most perfect model is used. This paper is concerned solely with the modeling of the functioning of a munition system and does not attempt to deal with the ability of the analyst to obtain accurate inputs

A. The Weapons System.

The XM99 riot-control rocket, developed by the Department of the Army to supplement helicopter armament, is used to temporarily disorganize and disorient an enemy. It consists of a warhead mounted on a 2.75-inch folding-fin aircraft rocket. Inside the warhead, 32 wedge-shaped XM100 canisters are arranged around a fuze-activated central burster. The rocket is designed to airburst at about 300 feet, thus scattering the submunitions which burn from 8 to 26 seconds after impact. While burning, these submunitions disperse o-chlorobenzylmalononitrile (commonly called CS), a white, crystalline, sensory irritant, which at medium concentrations produces immediate lachrimation and respiratory effects.¹

Before the development of the XM99 system advanced beyond the experimental stage, it was necessary to ascertain whether the rocket would be able to deliver the desired amount of chemical agent inside a target. Before additional monetary outlays were made for production and testing of the system, an effectiveness study was initiated to evaluate its accuracy. This study determined that a computer program capable of simulating the impact patterns produced by the rocket would be the best means of accomplishing this evaluation. The computer program developed simulates the workings of the XM99 system, provides the percentage of target impacts that the system is expected to produce, and includes in its output a diagram of the simulated impacts.

B. The Background.

In July 1969, in anticipation of a request from the Weapons Development and Engineering Laboratory, the Systems Analysis Office at Edgewood Arsenal undertook an effectiveness study of the XM99 system. The intent of this study was to determine the effect that variations in aircraft speed, target size, and aiming error would have on the accuracy of the system. To form conclusions concerning this effect, it would be necessary to find the average percentage of submunitions that the system would be able to impact inside a specified target using these variations in system conditions. This percentage was not obtainable from field trials because the testing of as many variations as would be necessary would require an excessive outlay of capital. It was evident that a theoretical means would have to be found for obtaining the desired percentage.

Initially, it was believed that the target impact percentage could be derived by either of two methods. An analytic solution for the percentage could be formulated using mathematical models to represent the different parts of the system, or the percentage could be obtained from impact patterns produced by a computer program capable of simulating the system under practical conditions. A subsequent analysis of the feasibility of deriving an analytic solution indicated that considerable difficulty would be encountered if that method were employed. From this analysis it was found that impacts from the same munition could not be regarded from a probability viewpoint as independent occurrences. The positions of these impacts were dependent upon the points at which the submunitions were dispersed (the munition function points). Using known technology, an analytic solution could be derived for the probability that any submunition would impact inside the target. Unless the impacts were independent occurrences, however, this probability did not equal the mean percentage of target impacts. This finding proved that the derivation of the analytic solution should not be used to obtain the desired percentage.

C. The Methodology.

The development of the simulation can be divided into writing of the computer program and its subsequent verification. The procedural steps which comprise the methodology of each of these parts will be developed and explained. These steps are summarized in the following paragraphs.

The first consideration to be made in the preparation of the computer program is the designation of its input and output. The input will consist of those parameters that are necessary to accurately describe a system that is to be simulated. It will include launcher speed, target size, aiming error, submunition effects pattern size, trial number, number of munitions per

trial, number of submunitions per munition, and the number of replications for each set of system conditions. The basic output will include the target impact diagram, a printout of the input, and a printout of the percentage of target impacts for each trial. After all trials have been simulated, the final mean percentage for the test will be computed and printed.

After the format for each part of this input and output has been designated, each phase of the weapons system must be represented by a precise mathematical model. There are three phases of the XM99 system for which these models must be formulated: the shifting of the aiming point on the target due to the movement of the launcher, the distribution of the munition function points, and the distribution of the submunitions dispersed by each munition. The mathematical models that will emulate these phases are models based upon probability distributions. In the program, they will be simulated by means of pseudo-random number generators that will produce the coordinates necessary in each of the phases according to that distribution to which each phase has been associated.

When the total mathematical model for the system has been assembled, a micro-flowchart will be prepared to outline the sequence of general steps that the program must contain. Each of its macro-steps will then be broken down into individual mathematical and logical operations. From these, a micro-flowchart will be prepared depicting the final logic and step sequence of the program. Each step from this flowchart will finally be programmed to produce the finished simulation.

The simulation will be coded into FORTRAN V. The version that will be used is that developed for the UNIVAC 1108 by the Sperry Rand Corporation.²

When the simulation has been completed, the resulting submunition patterns must accurately represent those produced by the simulated system. This will be verified by comparing the values of the percentage of target impacts obtained from the simulation with the same values obtained from the analytic solution. The analytic solution that will be derived for this purpose is the same one that was disregarded as the method for obtaining the percentage of target impacts. It can be used to verify the simulation if slight restrictions are placed upon the simulated system.

If the submunition impacts are assumed to be independent occurrences, it can be shown that the probability that any submunition impacts inside the target equals the average percentage of target impacts over repeated trials. If a system is simulated assuming that each munition disperses only one submunition, the resulting submunition impacts can be regarded as independent occurrences. The percentage of target impacts obtained from this restricted simulation should, therefore, equal the probability obtained from the analytic solution.

The derivation of the analytic solution will be the major step in the verification of the simulation. After the solution has been formulated, the values of the percentages obtained from it using variations in system conditions will be compared with the same values obtained from the simulation. Statistical tests will then be utilized to test for significant differences. The lack of such differences in the two sets of percentages will show that the program is accurately simulating both the movement of the target aiming point and the dispersion of the munition function points.

At this point, one portion of the simulation will remain to be verified. The relative equality of the two sets of percentages will not prove that the simulation accurately represents the dispersion of the submunitions by each munition. This portion of the program will have been

omitted when the assumption was placed upon the system that there could be only one submunition per rocket. The dispersion will be verified by testing the pseudo-random number generator, which is used to simulate the dispersion, for goodness of fit with respect to the numbers it produces. A Chi-square test will be used for this part of the verification. The results of these final tests will be the last proof that is needed to show that the simulation is producing patterns which accurately represent those produced by the actual system.

D. Additional Applications for the Simulation.

In addition to the analysis of the XM99 rocket, the simulation can be useful in the analysis of other weapons systems having the same general characteristics as the XM99 system: a moving firing mechanism which launches submunitions dispersing projectiles toward a stationary target. Single-round systems and systems with stationary launchers will become applicable for the program by merely adjusting the input parameters. This adaptability should prove useful because similar studies on different types of weapon systems will be required in the future.

In addition to determining the accuracy of a weapons system, the program will be used in other types of weapons effectiveness studies. Systems can be simulated to determine the number of rounds required to place a specified number of submunitions inside a target area. This information is necessary in studies comparing the effectiveness of different systems and in studies determining the number of munitions necessary to place a designated dosage of chemical agent over a target. The impact points produced by the pattern simulation can be used in future programs as the generation points for the chemical agent clouds released by a system. The diffusion and travel of these clouds over various system conditions would then be simulated, and analyses of the resulting dosages of chemical agent would furnish information concerning the casualty effects and the area coverage that the system can be expected to produce.

CHAPTER 2

II. THE ASSUMPTIONS.

The development of the simulation and the derivation of the analytic solution each require that the XM99 system be linked to a mathematical model. This model must be such that its components can be precisely emulated by programming methods to produce a computer program that accurately imitates the workings of the weapons system. As stated in the first chapter, the parts of the system that must be incorporated into the model include: the movement of the target aiming point caused by the motion of the launcher, the distribution of the munition function points, and the dispersion of the submunitions by each munition. Before each of these are dealt with, however, the system must be oriented to a frame of reference so that a clear description can be given as to what is being done.

A. The Frame of Reference.

The target will be assumed to be centered at the origin of a Cartesian coordinate system. The x-axis will extend in its negative direction from the center of the target through the launcher. The positive direction of the x-axis away from the launcher will be referred to in this paper as the direction of the range. The direction which is perpendicular to the range will be designated the direction of deflection.

The target will be assumed to be rectangular. Its sides will lie parallel to the axes of the coordinate system. The length of the rectangle in the direction of the range will be referred to as LL. Its width in the direction of deflection will be WW.

Using this coordinate system, the assumptions can now be made for the formulation of the mathematical model. Each of the three parts of the system which must be incorporated into this model will be individually dealt with in the following paragraphs.

B. The Aiming Point.

While it is firing its munitions, the launcher travels a distance p toward the center of the target. This motion causes the abscissa of the aiming point, which lies inside the target, to also shift a distance p in the positive direction. To simulate this motion, the program will place the initial aiming point at a distance $p/2$ in front of the target center on the negative x-axis. After the first munition has been launched, the aiming point for the next munition will shift a distance beyond the first aiming point equal to that which the launcher has moved between the two firings. This procedure will be repeated for each subsequent munition launching. Since the launcher is considered to fire its munitions at equal time intervals and is assumed to be moving at a constant velocity, the subsequent aiming points will lie at equal intervals along the x-axis. When the final munition is released, the aiming point will lie on the axis, a distance $p/2$ beyond the center of the target.

In the simulation, the movement of the aiming point will be imitated by physically adding the incremental distances traveled by the launcher to each successive range function point coordinate. In the analytic solution, the coordinates of the aiming point will be considered to be distributed according to a uniform distribution. This distribution has a probability density function j such that

$$j = 1/p$$

where p is the total distance traveled by the launcher while firing its munitions. The interval over which this distribution will occur is the interval from $p/2$ to $-p/2$ on the x-axis.

C. The Munition Function Points.

The munition function points for each trial will be assumed to be distributed according to a normal distribution. In the computer program the function point coordinates will be generated by a pseudo-random number generator that will produce a coordinate when given the specified mean and standard deviation.³ The mean for the placement of each range coordinate will be the aiming point for the munition. It will be designated u_x . The mean for each deflection coordinate will be assumed to be zero. The standard deviations for the two normal distributions will be constant over each firing and will be referred to as s_x in the direction of the range and as s_y in the direction of deflection. These deviations will correspond to the aiming errors associated with the system in each direction.

The analytic solution requires that a mathematical formula be found to represent this dispersion. According to the preceding paragraph, it was assumed that the probability density for the function point coordinates in the direction of the range is the normal density function

$$\frac{1}{\sqrt{\pi} \sqrt{2} s_x} \text{EXP} \left(\frac{-(x-u_x)^2}{2s_x^2} \right)$$

where x is the random variable denoting these coordinates. Similarly, the probability density function for the random variable y , which denotes the coordinates of the function points in the direction of deflection, is the normal density

$$\frac{1}{\sqrt{\pi} \sqrt{2} s_y} \text{EXP} \left(\frac{-y^2}{2s_y^2} \right)$$

The independence of the variables x and y permits the joint probability density function of both variables to be expressed as the multiple of their respective densities⁴

$$\frac{1}{2\pi s_y s_x} \text{EXP} \left(\frac{-(x-u_x)^2}{2s_x^2} + \frac{-y^2}{2s_y^2} \right)$$

This is the expression that will be used in the analytic solution to emulate the dispersion of the function points.

D. The Submunitions.

At its function point, the munition disperses its submunitions over the target. These submunitions are assumed to impact within a rectangular effects area that is centered at the munition function point. The sides of the effects rectangle are parallel to the sides of the target and, thus to the coordinate axes. The length of the effects area in the direction of the range will be referred to as L , and its width in the direction of deflection as W .

The submunition impacts will be distributed within the rectangle according to a uniform distribution. The joint probability density function for the positioning of the submunitions is thus

$$\frac{1}{(W \cdot L)}$$

This formula does not take into account anything that has previously happened within the system. The mathematical combination of this formula with those of the other parts of the system, using principles of dependence and conditional probability, will be dealt with when the analytic solution is derived.

In the computer simulation, the submunition coordinates will be dispersed over the effects rectangle by means of a pseudo-random number generator. This generator produces numbers according to a uniform distribution when it is given the interval over which that distribution lies.³ The abscissa of each submunition impact will be generated on an interval lying in the direction of the range. This interval is centered at the munition function point and has a length equal to that of the effects rectangle in the x direction. This length has been previously

designated L. The ordinate will be generated on an interval lying in the direction of deflection. It is also centered at the function point and has a length equal to W, the length of the effects rectangle in the y direction.

E. Summary.

These are the basic assumptions that will be used in the formulation of the computer simulation program and the analytic solution. There are additional assumptions and alterations that must be made to the system in order to simplify its programming. These, however, will not affect the mathematical models which emulate the system; they will be explained as they occur in the development of the program.

CHAPTER 3

III. THE SIMULATION.

This chapter deals with the development of the computer program that will simulate the XM99 system. To aid in explaining the theory behind the steps that will be incorporated into the simulation, the program variables will be referred to in this chapter by the mnemonic names that will be used to describe them in the program. A table of these mnemonic names and their meanings appears later in this chapter. A copy of the finished program and its output appear in appendix A.

A. The Input.

There are several conditions associated with each trial that affect the final distribution of submunition impacts. The variables which represent these conditions must be consolidated into the program so that they will affect the computer results in a similar manner to the way the conditions affect the actual munition firing. These variables make up the major portion of the input to the program. They are listed alphabetically in table I according to their mnemonic names. Included in this table is a brief description of the meaning or use of each variable and a separate listing of its units. Also included are variables which have been added to the input during programming. These variables either activate the random number of generators used by the program or aid in converting the target area to a grid system.

B. The Output.

The first portion of the program output will display the conditions under which each system is being simulated. These conditions remain constant over each test, and thus only their initial printout is necessary. The remaining output will be printed for each trial. It will consist of the trial number, the pattern diagram, the regional totals, and the trial impact percentage.

The pattern diagram will display the submunition impacts which occur inside the target. It will differentiate between points at which one submunition has landed and points at which more than one landing has occurred. For points receiving multiple impacts, it will display the number of landings at each point.

Table I. Input Variables

Variable	Description	Units
DT	The distance between grid lines in both directions across the target	Meters
L	The length of the effects rectangle in the direction of the range	Meters
LL	The length of the target rectangle in the direction of the range	Meters
NMUN	The number of submunitions to be dispersed by each munition	
NOREP	The number of trials to be replicated per test using the same system conditions	
NUMIN	The number of munitions to be fired per trial	
SIGX	The standard deviation of the distribution of the function point coordinates in the direction of the range. This corresponds to the aiming error associated with the system.	Meters
SIGY	The standard deviation of the distribution of the function point coordinates in the direction of deflection. This corresponds to the aiming error associated with the system.	Meters
T	The time interval between the first munition firing and the last	Seconds
V	The velocity of the launcher in the direction of the range	Knots
W	The width of the effects rectangle in the direction of deflection	Meters
WW	The width of the target rectangle in the direction of deflection	Meters
X(1)	The positive integer primer for the normal pseudo-random number generator	
XM(1)	The positive integer primer for the uniform pseudo-random number generator	

For impacts outside the target, the exact position of each impact will not be printed. The relative locations of these impacts will be shown by dividing the area surrounding the target into zones. The total number of impacts that occur in each zone will then be listed to indicate the relative positions of submunitions missing the target.

In order to eliminate the enormous amount of output produced by a large number of trials, the printing of the impact diagrams and of other trial information, except for the initial printing of the input conditions, will be suppressible by removing from the card deck the four necessary printout instruction cards. This modification should decrease turn around time to a substantial degree as it will eliminate much of the time that is wasted by the slowness of peripheral output equipment. In this situation it is also inconvenient to compute by hand the average percentage of target impacts over all trials. The program will thus compute and print this final mean percentage after all simulated trials have been completed.

As mentioned previously, a sample of the program output appears in appendix A.

C. The Impact Zones.

The area outside the target is divided into zones into which the submunitions missing the target fall. These zones are formed by extending the sides of the target rectangle infinitely in all directions. This produces eight zones which are in figure 1.

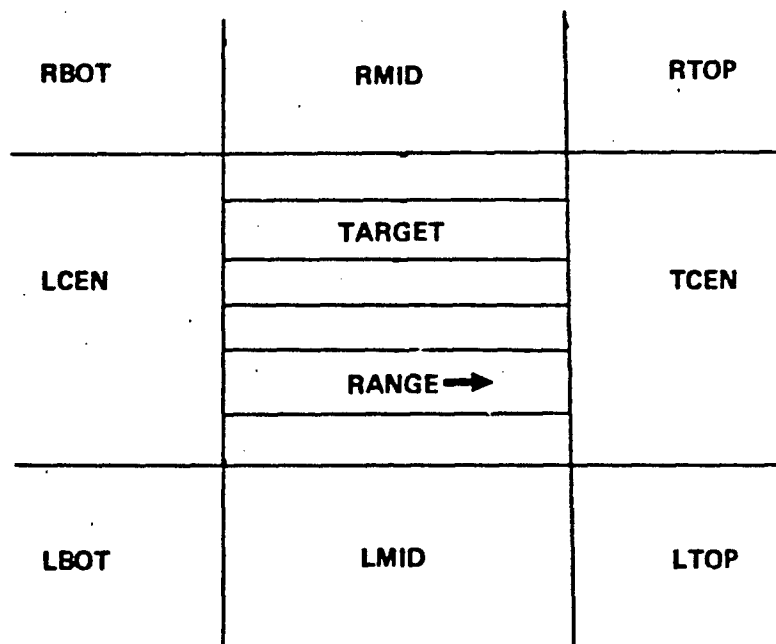


Figure 1. Diagram of the Impact Zones

D. The Programming Model.

As discussed in chapter 2, there are three phases of the workings of the XM99 system that must be emulated by mathematical and programming models. The first of these is

the effect of the movement of the launcher on the target aiming point. If the launcher moves a distance p during the time period in which it launches NUMIN munitions, the aiming point inside the target will shift along the x -axis from a distance $p/2$ in front of the target center to a distance $p/2$ in back of the center. This movement will occur in increments of $p/(NUMIN-1)$. Each shifted aiming point will subsequently serve as the mean for the normal distribution according to which the range coordinate of the munition function point is dispersed.

The actual firing of the munition is simulated with the use of a pseudo-random number generator. This generator produces a set of pseudo-random numbers which follow a normal distribution with specified mean and standard deviation.³ Two sets of NUMIN random numbers are generated. One set is used to produce the abscissas of the munition function points, and the other set, the ordinates. The following explanation describes the simulation of one munition and one of its submunitions. Any number of munitions and submunitions can be simulated by merely repeating the following procedure.

The normal pseudo-random number generator produces a number RX for each munition using a mean of $-p/2$. RX , when transformed, becomes the range coordinate X of the munition function point. The necessary transformation entails adding RX to the distance the launcher has moved prior to the firing of the munition and subsequent to the firing of the first munition. If the munition is the i th munition launched

$$X_i = RX_i + (i-1) [p/(NUMIN-1)].$$

This transformation produces the range coordinate of the munition function point. Because the mean of the normal distribution, according to which the deflection coordinates of the function points are dispersed, has been assumed to be zero, the deflection coordinate Y is equal to a second normal pseudo-random number which the generator has produced using a mean of zero.

The coordinates of the munition function point have now been determined. Centered at this point is a rectangular effects area over which the submunition impacts are uniformly dispersed. The means of simulating this dispersion is a pseudo-random number generator which produces two sets of NMUN numbers for each munition. These numbers are uniformly distributed on the interval $(0,1)$.³

For each submunition, the generator produces two numbers, RUX and RUY . They are transformed into the impact coordinates in the directions of the range and deflection, respectively. Because the effects pattern is centered at the munition function point, each generated number must first have .5 subtracted from it to produce a coordinate that can lie on either side of the effects pattern center. The numbers are then adjusted to the size of the effects rectangle by multiplying each by the length of the rectangle in its respective direction. The expression for the submunition impact range coordinate is

$$\text{Abscissa} = (RUX-.5)L + X.$$

Similarly, for the deflection coordinate

$$\text{Ordinate} = (RUY-.5)W + Y.$$

In these equations, X and Y are the respective coordinates of the munition function point.

The entire impact pattern is generated by simply repeating the above procedure for the desired number of munitions and submunitions.

E. The Program.

It is quite difficult to plot individually submunitions which impact very close to one another. To remedy this situation, the simulation was modified so that impacts occurring within a close distance of each other will be considered to occur at the same point. This modification entails forming the target rectangle into a grid. In the direction of the range, lines are drawn perpendicular to the x-axis across the target, a constant interval apart. The distance between the grid lines will be an input variable and will depend upon the size of the target. It will generally be less than 10 meters. The range coordinate of each impact point now assumes the value of the grid line nearest it. For example, if grid lines were drawn at 3 and 6 meters on the coordinate axis, and the range coordinate of an impact is 4.4, the coordinate would assume the value 3. This same procedure is used in the direction of deflection so that a checkerboard pattern is formed on which the impacts lie at the intersections of the horizontal and vertical grid lines. A nicely spaced pattern, which can be neatly printed, is the result.

In the output diagram, the impact points will be displayed in alphanumeric characters. This alleviates the confusion that would arise if 10 or more impacts should be located at one grid intersection. For example, if 11 impacts occurred at a single grid point, using numerals to represent the number of impacts at a point, confusion would exist as to whether there were two adjacent points with one impact each or one point with 11 impacts. This will be especially helpful in trials where a large target allows no blank spaces between intersections. It will permit large pattern diagrams to be printed on one computer sheet because each space on the sheet can represent one intersection on the target grid. Grid intersections at which no impacts occur will be displayed as blank spaces. Only the impacts inside the target will be displayed by the target diagram. Outside impacts will appear in their regional totals.

The next step in the development of the simulation program is the formulation of a macro-flowchart. This flowchart is presented in figure 2. The purpose of the flowchart is to outline the general sequence of steps that will occur within the program.

Each of the steps in the macro-flowchart is next broken down into mathematical and logical operations which can be directly coded into the computer language. A micro-flowchart is prepared to show the sequence of these operations. This flowchart is presented in figure 3. In this chart, the reading of input and the printing of output are not displayed in detail as they have previously been explained.

Table II lists all variables used in the simulation, along with a brief description of their use and meaning. The variables are presented in approximately the order in which they will appear in the program. As the input variables have already been presented in table I, they are omitted from the list.

The final coding of the program can be observed in appendix A along with a copy of the program output.

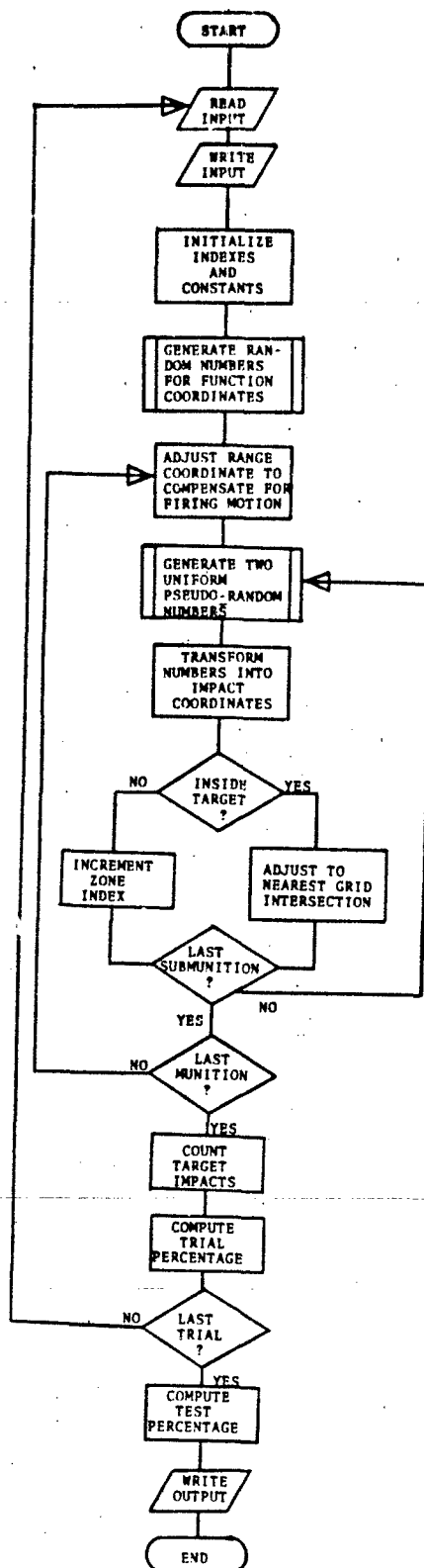


Figure 2. Macroflowchart of Program Logic

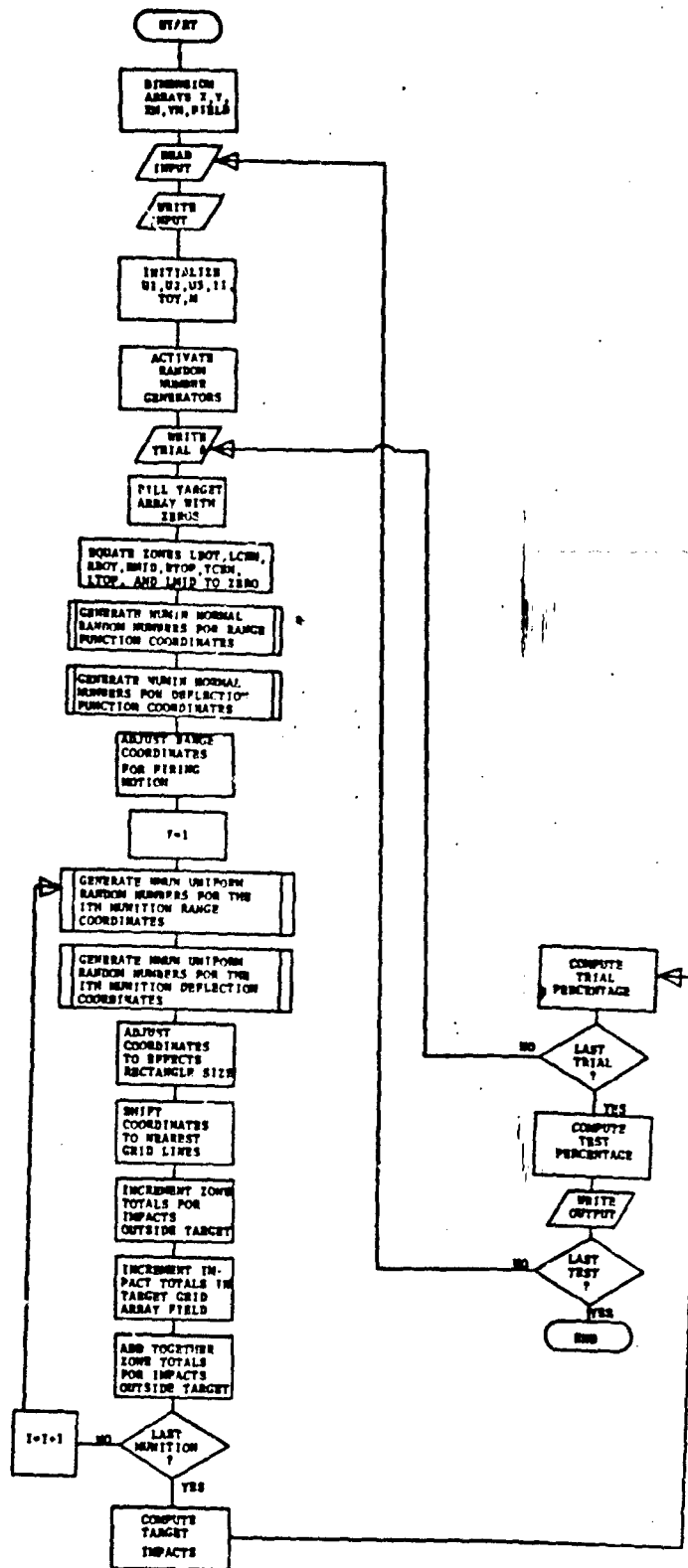


Figure 3. Mainflowchart of Program Logic

Table II. Program Variables

Variable	Description
TOT	The mean percentage of target impacts for the test
U1	The value of the mean used to normally generate the munition function point range coordinates
U2	The value of the mean used to normally generate the munition function point deflection coordinates
U3	The distance in meters that the launcher travels between successive munition firings
XM	An array used to store the range function point coordinates
YM	An array used to store the deflection function point coordinates
II	Integer variable used by the random number generators to make sure that the last number produced will be used as the next prime
M	Integer index which specifies the trial number
I,J,K,N	Integers used as array indexes
FIELD	Two-dimensional integer array whose elements represent the possible impact points inside the target
LBOT LCEN RBOT RMID RTOP TCEN LTOP LMID	Zones outside the target in which impacts can occur
NU	Integer index equal to NUMIN-1
IX,IY	Variables used to shift from coordinates to the grid system
NO,NI	Total impacts outside and inside the target
RP	The percentage of target impacts for the trial

CHAPTER 4

IV. THE ANALYTIC SOLUTION.

To show the validity of the simulated impact patterns, a comparison will now be made between the mean percentage of target impacts produced by repeated trials of the simulation and the same percentage derived analytically. The analytic percentage will be calculated by integrating over the target limits the joint probability density function for submunition impact position. The result of this integration will be the probability that any individual submunition lands inside the target. In the following paragraph, this probability is proven mathematically to be identical to the desired analytic percentage.

If each submunition impact is treated as the result of an independent repeated trial, the probability of impact incidence inside the target can be computed using a binomial distribution. There are two states of nature for each submunition firing: hitting the target and missing it. The probability of hitting the target is P , which will have been found analytically, and the probability of missing it is $1-P$. The total number of target impacts in n repeated trials will be simply the mean of a binomial distribution with n observations and probability P for an individual target impact. The value of this mean is $n \cdot P$.⁴ The mean percentage of submunitions that land inside the target area per trial is thus the total number of target hits over all trials divided by the number of trials

$$\text{mean percentage} = \frac{(n \cdot P)}{n} = P$$

A. The Density Function.

The joint probability density function for submunitions impact position will be designated $F(U,V)$. U is the random variable denoting the range coordinate of the submunition impact point, and V is the random variable denoting the coordinate in the direction of the deflection.

The coordinates of the submunition impacts are dependent upon the coordinates of the munition function point. $F(U,V)$ is thus a marginal probability density produced by integrating out the random variables for munition function point position from the total joint density

$$F(U,V) = \int \int f(U,V,X,Y) dX dY$$

In this expression, X and Y are random variables denoting the range and deflection coordinates, respectively, of the function point.

It has been assumed that X and Y are distributed normally with means u_x and zero, respectively. As u_x is itself a random variable on which X is dependent, the above expression for $F(U,V)$ must be extended to include u_x . Thus

$$F(U,V) = \int \int \int f(U,V,X,Y,u_x) dX dY du_x$$

Using a theorem of conditional probability,⁴ the expression for the density function f becomes

$$f(U,V,X,Y,u_x) = h(U,V/X,Y,u_x) \cdot g(X,Y,u_x)$$

In this equation, $g(X,Y,u_x)$ is the joint probability density function for the munition function point coordinates and the mean of X , and $h(U,V/X,Y,u_x)$ is the joint conditional density function for submunition impact position given the munition function point position and the mean of X .

Reapplying the same theorem to the joint density g

$$g(X,Y,u_x) = d(X,Y/u_x) \cdot j(u_x)$$

Here, $d(X,Y/u_x)$ is the joint conditional density function for the position of the munition function point given the mean of X , and $j(u_x)$ is the probability density function for the mean of X .

Because of the independence of the random variables X and Y , d can be expressed as the product of two simpler density functions

$$d(X,Y/u_x) = d(X/u_x) \cdot e(Y)$$

where $d(X/u_x)$ is the conditional density function of X given u_x , and $e(Y)$ is the density function for the random variable Y .

Assembling the components of the joint probability function yields the equation for f in its final form

$$f(U,V,X,Y,u_x) = h(U,V/X,Y,u_x) \cdot d(X/u_x) \cdot e(Y) \cdot j(u_x)$$

Each of the individual functions on the right side of this equation must now be represented by a mathematical model. These models are derived in the following paragraphs.

The submunition impacts have been assumed to be dispersed according to a uniform distribution within an effects rectangle of length L and width W . The mathematical formula for the joint conditional density function h is thus

$$\frac{1}{(L \cdot W)}$$

It has also been assumed that the coordinates of the munition function points are distributed normally with mean u_x in the direction of the range and mean zero in the direction of deflection. The conditional density function d is thus the normal density function with mean u_x and standard deviation s_x . Similarly, the probability density function e is the normal density function with mean zero and standard deviation s_y . These expressions are

$$d(X/u_x) = \frac{1}{\sqrt{\pi} \sqrt{2} s_x} \text{EXP} \left(\frac{-(X-u_x)^2}{2s_x^2} \right)$$

and

$$c(Y) = \frac{1}{\sqrt{\pi} \sqrt{2} s_y} \text{EXP} \left(\frac{-Y^2}{2s_y^2} \right)$$

Letting p represent the distance the firing mechanism moves during the time period in which it launches its munitions, the range coordinate of the mean of X also shifts a distance p . As it is assumed that the firing mechanism moves at constant velocity and launches its munitions at equal time intervals, the probability density function j for the random variable u_x is the uniform density function

$$1/p$$

The components of the mathematical formula for f can now be assembled

$$f(U,V,X,Y,u_x) = \frac{1}{pLW} \cdot \frac{1}{2s_x} \cdot \frac{1}{\pi s_y} \text{EXP} \left(\frac{-(X-u_x)^2}{2s_x^2} + \frac{-Y^2}{2s_y^2} \right)$$

This joint density function must next be integrated with respect to each of the five random variables. The result of this integration will be P , the desired percentage.

B. The Integration.

The joint density function f is first integrated with respect to u_x , the range coordinate of the aiming point. This integration will yield the joint marginal density of U , V , X , and Y . The limits for the integration are $p/2$ and $-p/2$ where p is the total distance traveled by the launcher during its firing time. As in the simulation model, the target is centered at the origin of a Cartesian coordinate system. This and subsequent integrations use the same coordinate system for the determination of limits.

The integration with respect to u_x is simplified by the fact that X is the only random variable dependent on u_x . The joint marginal density of U , V , X , and Y is thus

$$G(U,V,X,Y) = \int f(U,V,X,Y,u_x) du_x$$

which has been found to be

$$\frac{1}{pLW} \cdot \frac{1}{\sqrt{2\pi} s_y} \text{EXP} \left(\frac{-Y^2}{2s_y^2} \right) \int_{-p/2}^{p/2} \frac{1}{\sqrt{2\pi} s_x} \text{EXP} \left(\frac{-(X-u_x)^2}{2s_x^2} \right) du_x$$

If everything outside the integral is regarded as a constant and the transformations $c = X - u_x$ and $dc = -du_x$ are performed, the expression for G becomes

$$K \frac{1}{\sqrt{2\pi} s_x} \int_{p/2+X}^{-p/2+X} \exp\left(\frac{-c^2}{2s_x^2}\right) dc$$

Reversing the limits of integration and separating the integral into the difference of two integrals then yields

$$K \cdot \frac{1}{\sqrt{\pi} \sqrt{2} s_x} \left[\int_0^{X+p/2} \exp\left(\frac{-c^2}{2s_x^2}\right) dc - \int_0^{X-p/2} \exp\left(\frac{-c^2}{2s_x^2}\right) dc \right]$$

If a transformation is now made to G using the variable $A = c/(\sqrt{2} s_x)$

$$G(U,V,X,Y) = \frac{1 \cdot K^2}{\sqrt{\pi} K} \left[\int_0^{\frac{X+p/2}{\sqrt{2} s_x}} \exp(-A^2) dA - \int_0^{\frac{X-p/2}{\sqrt{2} s_x}} \exp(-A^2) dA \right]$$

This expression can be evaluated by using the error function evaluated at its upper limit. The error function is defined and will be referred to as

$$\text{ERF}\left(\frac{(X-p/2)}{\sqrt{2} s_x}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{X-p/2}{\sqrt{2} s_x}} \exp(-t^2) dt$$

The mathematical evaluation of the error function will be deferred until the final solution for P has been derived. Until that time, the above integral will simply be referred to as ERF ().

Using the error function, the expression for G takes on its final form

$$G(U,V,X,Y) = \frac{1}{2pLW} \cdot \frac{1}{\sqrt{\pi} \sqrt{2} s_y} \exp\left(\frac{-Y^2}{2s_y^2}\right) \cdot \left[\text{ERF}\left(\frac{X+p/2}{\sqrt{2} s_x}\right) - \text{ERF}\left(\frac{X-p/2}{\sqrt{2} s_x}\right) \right]$$

The next step in the derivation of P is the calculation of F(UV), the joint marginal density of U and V. F is derived by integrating G(U,V,X,Y) over the variables X and Y. X and Y are independent random variables. This integration thus reduces to the product of two separate integrations

$$\frac{1}{2pLW} \cdot \int \frac{1}{\sqrt{\pi} \sqrt{2} s_y} \exp\left(\frac{-Y^2}{2s_y^2}\right) dY \cdot \int \left[\text{ERF}\left(\frac{X+p/2}{s_x}\right) - \text{ERF}\left(\frac{X-p/2}{s_x}\right) \right] dX$$

These two integrals will be evaluated individually.

Because the munition function point is the center of the effects rectangle in which the submunitions are uniformly dispersed, the limits for the integration with respect to X are $U - L/2$ and $U + L/2$ where L is the length of the effects rectangle in the direction of the range. Similarly, the limits for the integration with respect to Y are $V - W/2$ and $V + W/2$ where W is the width of the rectangle in the direction of deflection.

Separated from the total integration of G, the expression for the integration with respect to Y is

$$\int_{V-W/2}^{V+W/2} \frac{1}{\sqrt{\pi} \sqrt{2} s_y} \exp\left(\frac{-Y^2}{2s_y^2}\right) dY$$

Making the substitution $A = Y/(\sqrt{2} s_y)$, this integral also reverts to a form of the error function

$$\frac{1}{2} \left[\operatorname{ERF}\left(\frac{V+W/2}{\sqrt{2} s_y}\right) - \operatorname{ERF}\left(\frac{V-W/2}{\sqrt{2} s_y}\right) \right]$$

The integral with respect to X is first separated into the difference of two integrals

$$\int_{U-L/2}^{U+L/2} \operatorname{ERF}\left(\frac{X+p/2}{\sqrt{2} s_x}\right) dX - \int_{U-L/2}^{U+L/2} \operatorname{ERF}\left(\frac{X-p/2}{\sqrt{2} s_x}\right) dX$$

When a transformation is made to each integral using the variables $Q = (X + p/2) / (\sqrt{2} s_x)$ and $R = (X - p/2) / (\sqrt{2} s_x)$, the expression for the integration becomes

$$\sqrt{2} s_x \left[\int_{\frac{U-L/2+p/2}{\sqrt{2} s_x}}^{\frac{U+L/2+p/2}{\sqrt{2} s_x}} \operatorname{ERF}(Q) dQ - \int_{\frac{U-L/2-p/2}{\sqrt{2} s_x}}^{\frac{U+L/2-p/2}{\sqrt{2} s_x}} \operatorname{ERF}(R) dR \right]$$

Using the relationship

$$\int_0^A \operatorname{ERF}(t) dt = A \operatorname{ERF}(A) - \frac{(1 - e^{-A^2})}{\sqrt{\pi}}$$

this sum is evaluated as

$$\begin{aligned}
 & \sqrt{2} s_x \left[\left(\frac{U + L/2 + p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U + L/2 + p/2}{\sqrt{2} s_x} \right) - \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(U + L/2 + p/2)^2}{2s_x^2} \right) \right] \right. \\
 & - \left(\frac{U - L/2 + p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U - L/2 + p/2}{\sqrt{2} s_x} \right) + \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(U - L/2 + p/2)^2}{2s_x^2} \right) \right] \\
 & - \left(\frac{U + L/2 - p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U + L/2 - p/2}{\sqrt{2} s_x} \right) + \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(U + L/2 - p/2)^2}{2s_x^2} \right) \right] \\
 & \left. + \left(\frac{U - L/2 - p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U - L/2 - p/2}{\sqrt{2} s_x} \right) - \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(U - L/2 - p/2)^2}{2s_x^2} \right) \right] \right]
 \end{aligned}$$

The final expression for the integration with respect to X is produced by adding together like terms

$$\begin{aligned}
 & \sqrt{2} s_x \left[\left(\frac{U + L/2 + p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U + L/2 + p/2}{\sqrt{2} s_x} \right) - \left(\frac{U - L/2 + p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U - L/2 + p/2}{\sqrt{2} s_x} \right) \right. \\
 & - \left(\frac{U + L/2 - p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U + L/2 - p/2}{\sqrt{2} s_x} \right) + \left(\frac{U - L/2 - p/2}{\sqrt{2} s_x} \right) \text{ERF} \left(\frac{U - L/2 - p/2}{\sqrt{2} s_x} \right) \\
 & + \frac{1}{\sqrt{\pi}} \left[\text{EXP} \left(\frac{-(U + L/2 + p/2)^2}{2s_x^2} \right) - \text{EXP} \left(\frac{-(U - L/2 + p/2)^2}{2s_x^2} \right) - \text{EXP} \left(\frac{-(U + L/2 - p/2)^2}{2s_x^2} \right) \right. \\
 & \left. \left. + \text{EXP} \left(\frac{-(U - L/2 - p/2)^2}{2s_x^2} \right) \right] \right]
 \end{aligned}$$

All components of the integration of G have been derived. Assembling them yields the final expression for F(U,V)

$$\begin{aligned} & \frac{\sqrt{2} s_x}{4pLW} \left[\operatorname{ERF} \left(\frac{V + W/2}{\sqrt{2} s_y} \right) - \operatorname{ERF} \left(\frac{V - W/2}{\sqrt{2} s_y} \right) \right] \left[\left(\frac{U + L/2 + p/2}{\sqrt{2} s_x} \right) \right. \\ & \cdot \operatorname{ERF} \left(\frac{U + L/2 + p/2}{\sqrt{2} s_x} \right) - \left(\frac{U - L/2 + p/2}{\sqrt{2} s_x} \right) \operatorname{ERF} \left(\frac{U - L/2 + p/2}{\sqrt{2} s_x} \right) - \left(\frac{U + L/2 - p/2}{\sqrt{2} s_x} \right) \\ & \cdot \operatorname{ERF} \left(\frac{U + L/2 - p/2}{\sqrt{2} s_x} \right) + \left(\frac{U - L/2 - p/2}{\sqrt{2} s_x} \right) \operatorname{ERF} \left(\frac{U - L/2 - p/2}{\sqrt{2} s_x} \right) \\ & + \frac{1}{\sqrt{\pi}} \left[\operatorname{EXP} \left(\frac{-(U + L/2 + p/2)^2}{2s_x^2} \right) - \operatorname{EXP} \left(\frac{-(U - L/2 + p/2)^2}{2s_x^2} \right) - \operatorname{EXP} \left(\frac{-(U + L/2 - p/2)^2}{2s_x^2} \right) \right. \\ & \left. \left. + \operatorname{EXP} \left(\frac{-(U - L/2 - p/2)^2}{2s_x^2} \right) \right] \right] \end{aligned}$$

The final step in the derivation of the analytic percentage is the integration of F(U,V). F(U,V) can most easily be integrated if it is expressed as the product of two separate integrals. This can be done because of the independence of the random variables U and V. The integration then takes the form

$$\iint F(U,V) dU dV = K \int g(V) dV \cdot \int h(U) dU$$

F is being integrated over the target which is centered at the origin. The limits of integration are thus half the target dimensions in the directions of the corresponding variables. The length of the target in the direction of the range is LL; therefore, the limits of integration with respect to the variable U are -LL/2 and LL/2. Similarly, the limits for V are -WW/2 and WW/2 where WW is the width of the target in the direction of deflection.

The integration of g(V) will be done first. From the previous integration

$$\int_{-WW/2}^{WW/2} g(V) dV = \int_{-WW/2}^{WW/2} \left[\text{ERF} \left(\frac{V+W/2}{\sqrt{2} s_y} \right) - \text{ERF} \left(\frac{V-W/2}{\sqrt{2} s_y} \right) \right] dV$$

The form of this integral is first altered so that it is expressed as the difference of two integrals. A transformation is then made to the arguments of the error functions using the variables

$$A = \frac{V+W/2}{\sqrt{2} s_y} \quad \text{and} \quad B = \frac{V-W/2}{\sqrt{2} s_y}$$

The integral then becomes

$$\sqrt{2} s_y \left[\int_{(W-WW)/(2\sqrt{2} s_y)}^{(WW+W)/(2\sqrt{2} s_y)} \text{ERF}(A) dA - \int_{(-WW-W)/(2\sqrt{2} s_y)}^{(WW-W)/(2\sqrt{2} s_y)} \text{ERF}(B) dB \right]$$

Using the previously stated method for integrating the error function, the integral of $g(V)$ is now expressed as

$$\begin{aligned} & \sqrt{2} s_y \left[\left(\frac{WW+W}{2\sqrt{2} s_y} \right) \text{ERF} \left(\frac{WW+W}{2\sqrt{2} s_y} \right) - \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(WW+W)^2}{8s_y^2} \right) \right] - \left(\frac{W-WW}{2\sqrt{2} s_y} \right) \right. \\ & \cdot \text{ERF} \left(\frac{W-WW}{2\sqrt{2} s_y} \right) + \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(W-WW)^2}{8s_y^2} \right) \right] - \left(\frac{WW-W}{2\sqrt{2} s_y} \right) \text{ERF} \left(\frac{WW-W}{2\sqrt{2} s_y} \right) \\ & + \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(WW-W)^2}{8s_y^2} \right) \right] + \left(\frac{-WW-W}{2\sqrt{2} s_y} \right) \text{ERF} \left(\frac{-WW-W}{2\sqrt{2} s_y} \right) \\ & \left. - \frac{1}{\sqrt{\pi}} \left[1 - \text{EXP} \left(\frac{-(WW-W)^2}{8s_y^2} \right) \right] \right] \end{aligned}$$

The final expression for the integral is achieved by combining terms. Further simplification is permitted by the relationship

$$\text{ERF}(-A) = -\text{ERF}(A)$$

The final equation is

$$\int g(V) dV = (WW + W) \operatorname{ERF} \left(\frac{WW + W}{2\sqrt{2}s_y} \right) - (W - WW) \operatorname{ERF} \left(\frac{WW - W}{2\sqrt{2}s_y} \right) \\ + \frac{2\sqrt{2}s_y}{\sqrt{\pi}} \left[\operatorname{EXP} \left(\frac{-(WW + W)^2}{8s_y^2} \right) - \operatorname{EXP} \left(\frac{-(W - WW)^2}{8s_y^2} \right) \right]$$

The second half of the integration of $F(U, V)$ entails the integration of $h(U)$. The expression for this integration is

$$\int_{-LL/2}^{LL/2} \left[\left(\frac{U + L/2 + p/2}{\sqrt{2}s_x} \right) \operatorname{ERF} \left(\frac{U + L/2 + p/2}{\sqrt{2}s_x} \right) - \left(\frac{U - L/2 + p/2}{\sqrt{2}s_x} \right) \operatorname{ERF} \left(\frac{U - L/2 + p/2}{\sqrt{2}s_x} \right) \right. \\ \left. - \left(\frac{U + L/2 - p/2}{\sqrt{2}s_x} \right) \operatorname{ERF} \left(\frac{U + L/2 - p/2}{\sqrt{2}s_x} \right) + \left(\frac{U - L/2 - p/2}{\sqrt{2}s_x} \right) \operatorname{ERF} \left(\frac{U - L/2 - p/2}{\sqrt{2}s_x} \right) \right. \\ \left. + \frac{1}{\sqrt{\pi}} \left[\operatorname{EXP} \left(\frac{-(U + L/2 - p/2)^2}{2s_x^2} \right) - \operatorname{EXP} \left(\frac{-(U - L/2 + p/2)^2}{2s_x^2} \right) - \operatorname{EXP} \left(\frac{-(U + L/2 - p/2)^2}{2s_x^2} \right) \right. \right. \\ \left. \left. + \operatorname{EXP} \left(\frac{-(U - L/2 - p/2)^2}{2s_x^2} \right) \right] \right] dU$$

The integral is first separated into the sum of eight integrals. When the following transformations are performed

$$A = \frac{U + L/2 + p/2}{\sqrt{2}s_x} \quad B = \frac{U + p/2 - L/2}{\sqrt{2}s_x} \quad C = \frac{U + L/2 - p/2}{\sqrt{2}s_x} \quad \text{and} \quad D = \frac{U - L/2 - p/2}{\sqrt{2}s_x}$$

the expression for the integration of $h(U)$ becomes

$$\int_{\frac{-LL+L+p}{2\sqrt{2}s_x}}^{\frac{LL+L+p}{2\sqrt{2}s_x}} \sqrt{2}s_x \text{AERF}(A) dA - \int_{\frac{-LL+p-L}{2\sqrt{2}s_x}}^{\frac{LL+p-L}{2\sqrt{2}s_x}} \sqrt{2}s_x \text{BERF}(B) dB$$

$$- \int_{\frac{-LL+L-p}{2\sqrt{2}s_x}}^{\frac{LL+L-p}{2\sqrt{2}s_x}} \sqrt{2}s_x \text{CERF}(C) dC + \int_{\frac{-LL-L-p}{2\sqrt{2}s_x}}^{\frac{LL-L-p}{2\sqrt{2}s_x}} \sqrt{2}s_x \text{DERF}(D) dD$$

$$+ \frac{\sqrt{2}s_x}{\sqrt{\pi}} \left[\int_{\frac{-LL+L+p}{2\sqrt{2}s_x}}^{\frac{LL+L+p}{2\sqrt{2}s_x}} \text{EXP}(-A^2) dA - \int_{\frac{-LL-L+p}{2\sqrt{2}s_x}}^{\frac{LL-L+p}{2\sqrt{2}s_x}} \text{EXP}(-B^2) dB \right.$$

$$\left. - \int_{\frac{-LL+L-p}{2\sqrt{2}s_x}}^{\frac{LL+L-p}{2\sqrt{2}s_x}} \text{EXP}(-C^2) dC + \int_{\frac{-LL-L-p}{2\sqrt{2}s_x}}^{\frac{LL-L-p}{2\sqrt{2}s_x}} \text{EXP}(-D^2) dD \right]$$

The first four integrals in this expression are evaluated using integration by parts. A model for this method is

$$\int x dt = xt - \int t dx$$

The relationship of this model and the first integral in the expression for the integration of $h(U)$ is such that the $\text{ERF}(A)$ is associated with the s and the A with the dt . A similar association is used in each of the other three integrals being evaluated. The expression for these four integrals now becomes

$$\begin{aligned} & \sqrt{2} s_x \left[\frac{A^2 \text{ERF}(A)}{2} \right]_{\frac{-LL+L+p}{2\sqrt{2}s_x}}^{\frac{LL+L+p}{2\sqrt{2}s_x}} - \frac{2}{\sqrt{\pi}} \int_{\frac{-LL+L+p}{2\sqrt{2}s_x}}^{\frac{LL+L+p}{2\sqrt{2}s_x}} \frac{A^2 \text{EXP}(-A^2)}{2} dA \\ & - \frac{B^2 \text{ERF}(B)}{2} \left[\right]_{\frac{-LL-L+p}{2\sqrt{2}s_x}}^{\frac{LL-L+p}{2\sqrt{2}s_x}} + \frac{2}{\sqrt{\pi}} \int_{\frac{-LL-L+p}{2\sqrt{2}s_x}}^{\frac{LL-L+p}{2\sqrt{2}s_x}} \frac{B^2 \text{EXP}(-B^2)}{2} dB \\ & - \frac{C^2 \text{ERF}(C)}{2} \left[\right]_{\frac{-LL+L-p}{2\sqrt{2}s_x}}^{\frac{LL+L-p}{2\sqrt{2}s_x}} + \frac{2}{\sqrt{\pi}} \int_{\frac{-LL+L-p}{2\sqrt{2}s_x}}^{\frac{LL+L-p}{2\sqrt{2}s_x}} \frac{C^2 \text{EXP}(-C^2)}{2} dC \\ & + \frac{D^2 \text{ERF}(D)}{2} \left[\right]_{\frac{-LL-L-p}{2\sqrt{2}s_x}}^{\frac{LL-L-p}{2\sqrt{2}s_x}} - \frac{2}{\sqrt{\pi}} \int_{\frac{-LL-L-p}{2\sqrt{2}s_x}}^{\frac{LL-L-p}{2\sqrt{2}s_x}} \frac{D^2 \text{EXP}(-D^2)}{2} dD \end{aligned}$$

In this expression, for each term in the form

$$-\frac{2}{\sqrt{\pi}} \int_B^A \frac{M^2}{2} \text{EXP}(-M^2) dM$$

there exists a term in the form

$$-\frac{2}{\sqrt{\pi}} \int_{-A}^{-B} \frac{N^2}{2} \text{EXP}(-N^2) dN$$

which is equal in value. Thus, the two terms add and the integration of the terms not containing the error function reduces to

$$-\frac{2}{\sqrt{\pi}} \int_{\frac{-LL+L+p}{2\sqrt{2}s_x}}^{\frac{LL+L+p}{2\sqrt{2}s_x}} A^2 \text{EXP}(-A^2) dA + \frac{2}{\sqrt{\pi}} \int_{\frac{-LL-L+p}{2\sqrt{2}s_x}}^{\frac{LL-L+p}{2\sqrt{2}s_x}} A^2 \text{EXP}(-A^2) dA$$

This expression is transformed using the variable $F = A^2$

$$-\frac{1}{\sqrt{\pi}} \int_{\frac{(-LL+L+p)^2}{8s_x^2}}^{\frac{(LL+L+p)^2}{8s_x^2}} F^{\frac{1}{2}} \text{EXP}(-F) dF + \frac{1}{\sqrt{\pi}} \int_{\frac{(-LL-L+p)^2}{8s_x^2}}^{\frac{(LL-L+p)^2}{8s_x^2}} F^{\frac{1}{2}} \text{EXP}(-F) dF$$

This two-term sum can be evaluated when it is recognized that the integrals are a form of the probability integral of the Chi-square distribution. This probability integral is expressed in the form

$$\text{Pr}(X^2/v) = \frac{1}{2^{(v/2)} \Gamma(v/2)} \int_0^{X^2} \frac{t^{v/2-1}}{e^{t/2}} dt$$

where v is the number of degrees of freedom.⁶ When a transformation is made to $\text{Pr}(x^2/v)$ using the variable $R = t/2$, the probability integral becomes

$$\text{Pr}(x^2/v) = \frac{1}{\Gamma(v/2)} \int_0^{\frac{x^2}{2}} \frac{R^{\frac{v}{2}-1}}{R^2} \text{EXP}(-R) dR$$

Using this relationship, the expression for the portion of $h(U)$ that is being evaluated becomes

$$\frac{\Gamma(3/2)}{\sqrt{\pi}} \left[-\text{Pr}\left(\frac{(LL+L+p)^2}{4s_x^2}/3\right) + \text{Pr}\left(\frac{(-LL+L+p)^2}{4s_x^2}/3\right) + \text{Pr}\left(\frac{(LL-L+p)^2}{4s_x^2}/3\right) - \text{Pr}\left(\frac{(-LL-L+p)^2}{4s_x^2}/3\right) \right]$$

The transformation $F = A^2$ introduces a complication into the integration. It deletes the fact that A can be negative. To compensate for this omission, the sign of each Chi-square term must be changed if the value of A contained in its argument is negative. For example, if $(-LL + L + p)$ is negative, the sign in front of the Chi-square term having that expression in its argument will change from plus to minus. To show this condition in the final expressions, both a plus and a minus will be placed in front of the Chi-square terms to show that their sign depends upon the sign of the A portion of their argument.

The values of the Chi-square probabilities are obtainable in most books of mathematical tables. Their exact evaluation will be deferred until the final solution is derived and then will be used only when comparing the analytically derived values against similar simulated ones. The Chi-square probability functions will be referred to throughout the remainder of this paper as $\text{Pr}(\cdot)$.

When the limits of the remaining terms in the integration by parts are substituted in and like terms are collected, the final expression for the first portion of the integration of $h(U)$ results

$$\sqrt{2} s_x \left[\left(\frac{LL+L+p}{2\sqrt{2} s_x} \right)^2 \text{ERF} \left(\frac{LL+L+p}{2\sqrt{2} s_x} \right) - \left(\frac{-LL+L+p}{2\sqrt{2} s_x} \right)^2 \text{ERF} \left(\frac{-LL+L+p}{2\sqrt{2} s_x} \right) \right. \\ \left. - \left(\frac{LL-L+p}{2\sqrt{2} s_x} \right)^2 \text{ERF} \left(\frac{LL-L+p}{2\sqrt{2} s_x} \right) + \left(\frac{-LL-L+p}{2\sqrt{2} s_x} \right)^2 \text{ERF} \left(\frac{-LL-L+p}{2\sqrt{2} s_x} \right) \right]$$

$$+ \frac{\Gamma(3/2)}{\sqrt{\pi}} \left[\pm \Pr \left(\frac{(LL + L + p)^2}{4s_x^2} / 3 \right) \pm \Pr \left(\frac{(-LL + L + p)^2}{4s_x^2} / 3 \right) \right. \\ \left. \pm \Pr \left(\frac{(LL - L + p)^2}{4s_x^2} / 3 \right) \pm \Pr \left(\frac{(-LL - L + p)^2}{4s_x^2} / 3 \right) \right]$$

The remaining four terms in the integration of $h(U)$ can be evaluated using the relationship

$$\frac{\sqrt{\pi}}{2} \cdot \text{ERF}(A) = \int_0^A \text{EXP}(-t^2) dt$$

They become

$$\frac{s_x}{\sqrt{2}} \left[\text{ERF} \left(\frac{LL + L + p}{2\sqrt{2}s_x} \right) - \text{ERF} \left(\frac{-LL + L + p}{2\sqrt{2}s_x} \right) - \text{ERF} \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right) + \text{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) \right. \\ \left. - \text{ERF} \left(\frac{LL + L - p}{2\sqrt{2}s_x} \right) + \text{ERF} \left(\frac{-LL + L - p}{2\sqrt{2}s_x} \right) + \text{ERF} \left(\frac{LL - L - p}{2\sqrt{2}s_x} \right) - \text{ERF} \left(\frac{-LL - L - p}{2\sqrt{2}s_x} \right) \right]$$

Combining terms reduces this expression to its final form

$$\sqrt{2}s_x \left[\text{ERF} \left(\frac{LL + L + p}{2\sqrt{2}s_x} \right) - \text{ERF} \left(\frac{-LL + L + p}{2\sqrt{2}s_x} \right) - \text{ERF} \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right) + \text{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) \right]$$

The two halves of the integration are now assembled to yield the final expression for $h(U)$

$$\sqrt{2}s_x \left[\left(\frac{LL + L + p}{2\sqrt{2}s_x} \right)^2 \text{ERF} \left(\frac{LL + L + p}{2\sqrt{2}s_x} \right) - \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right)^2 \text{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) \right. \\ \left. - \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right)^2 \text{ERF} \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right) + \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right)^2 \text{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) \right]$$

$$\begin{aligned}
& + \frac{\Gamma(3/2)}{\sqrt{\pi}} \left[\pm \Pr \left(\frac{(LL + L + p)^2}{4s_x^2} / 3 \right) \pm \Pr \left(\frac{(-LL + L + p)^2}{4s_x^2} / 3 \right) \right. \\
& \quad \left. \pm \Pr \left(\frac{(LL - L + p)^2}{4s_x^2} / 3 \right) \pm \Pr \left(\frac{(-LL - L + p)^2}{4s_x^2} / 3 \right) \right] \\
& + \sqrt{2} s_x \left[\operatorname{ERF} \left(\frac{LL + L + p}{2\sqrt{2}s_x} \right) - \operatorname{ERF} \left(\frac{-LL + L + p}{2\sqrt{2}s_x} \right) - \operatorname{ERF} \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right) + \operatorname{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) \right]
\end{aligned}$$

Every component of the integration of $F(U, V)$ has been derived. The integral had been in the form

$$\int \int F(U, V) dU dV = K \int g(V) dV \int h(U) dU$$

Substituting the constant and the two integrals into the left side of the equation yields the final expression for the analytic solution P

$$\begin{aligned}
& \frac{s_x^2}{2pLW} \left[(WW + W) \operatorname{ERF} \left(\frac{WW + W}{2\sqrt{2}s_y} \right) - (W - WW) \operatorname{ERF} \left(\frac{W - WW}{2\sqrt{2}s_y} \right) \right. \\
& \quad \left. + \frac{2\sqrt{2}s_y}{\sqrt{\pi}} \left(\exp \left(\frac{-(WW + W)^2}{8s_y^2} \right) - \exp \left(\frac{-(W - WW)^2}{8s_y^2} \right) \right) \right] \left[\left(\frac{LL + L + p}{2\sqrt{2}s_x} \right)^2 \right. \\
& \quad \operatorname{ERF} \left(\frac{LL + L + p}{2\sqrt{2}s_x} \right) - \left(\frac{-LL + L + p}{2\sqrt{2}s_x} \right)^2 \operatorname{ERF} \left(\frac{-LL + L + p}{2\sqrt{2}s_x} \right) - \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right)^2 \operatorname{ERF} \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right) \\
& \quad \left. + \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right)^2 \operatorname{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) + \frac{\Gamma(3/2)}{\sqrt{\pi}} \left[\pm \Pr \left(\frac{(LL + L + p)^2}{4s_x^2} / 3 \right) \right. \right.
\end{aligned}$$

$$\pm \Pr \left(\frac{(LL - L + p)^2}{4s_x^2} / 3 \right) \pm \Pr \left(\frac{(-LL + L + p)^2}{4s_x^2} / 3 \right) \pm \Pr \left(\frac{(-LL - L + p)^2}{4s_x^2} / 3 \right) \Bigg]$$

$$+ \operatorname{ERF} \left(\frac{LL + L + p}{2\sqrt{2}s_x} \right) - \operatorname{ERF} \left(\frac{-LL + L + p}{2\sqrt{2}s_x} \right) - \operatorname{ERF} \left(\frac{LL - L + p}{2\sqrt{2}s_x} \right) + \operatorname{ERF} \left(\frac{-LL - L + p}{2\sqrt{2}s_x} \right) \Bigg]$$

C. Summary.

The probability that a submunition impacts inside a target area has been derived analytically. It has been shown mathematically that this probability equals the mean percentage of target impacts during any set of trials. The purpose of deriving the analytic solution was to verify the accuracy of the simulation. To accomplish this, a comparison must now be made between the results obtained from the substitution of similar parameters into each solution. A proof to substantiate a portion of the theory upon which the analytic solution was derived is presented in appendix B. This proof tests the validity of the solution at one of its boundary conditions.

CHAPTER 5

V. SUMMARY AND CONCLUSIONS.

A. Comparisons.

The comparison will now be made between percentages derived from the analytic solution and those obtained from the simulation. Each simulated mean percentage is the result of 1000 replicated trials with the simulation. Twenty-six different cases were compared using realistic extremes of system conditions. The compared values are listed in table III.

For each pair of percentages, the null hypothesis being tested is that the simulated percentage u equals the analytic percentage u_0 . A large sample test can be used when dealing with a population that is not necessarily normal, but which has a finite variance. The variance of the population is approximated by s , the sample variance⁴

$$s = \sum_{i=1}^{1000} \frac{(u_i - \bar{u})^2}{n-1}$$

The null hypothesis will be tested against the composite alternative $u \neq u_0$ using the best critical region of size α , $|z| > z_{\alpha/2}$ where

$$z = \frac{(u - u_0) \sqrt{1000}}{\sqrt{s}}$$

Table III. Simulated Trials and the Resulting Analytic and Simulated Percentages

Trial No.	W	L	WW	LL	P	S _x	S _y	Analytic %	Simulated %	Difference
1	50	50	100	100	0	20	20	0.9198	0.9184	0.0014
2	50	50	100	100	50	20	20	.8832	.8814	.0018
3	50	50	100	100	200	20	20	.4782	.4683	.0099
4	50	50	100	100	200	70	20	.3862	.3832	.0030
5	50	50	100	100	200	20	70	.2570	.2508	.0062
6	50	50	100	100	200	100	100	.1249	.1245	.0004
7	50	50	200	200	0	20	20	1.0000	1.0000	.0000
8	50	50	200	200	50	20	20	.9998	.9998	.0000
9	50	50	200	200	100	50	50	.8576	.8549	.0027
10	50	50	200	200	50	150	20	.4911	.4888	.0023
11	50	50	200	200	50	20	150	.4930	.4931	.0001
12	50	50	200	200	50	50	50	.8849	.8838	.0011
13	50	50	200	200	50	200	200	.1456	.1468	.0012
14	100	100	100	100	0	20	20	.7060	.7054	.0006
15	100	100	100	100	50	20	20	.6738	.6747	.0009
16	100	100	100	100	200	20	20	.4118	.4022	.0096
17	100	100	100	100	50	100	20	.3072	.3087	.0015
18	100	100	100	100	50	20	100	.2957	.2967	.0010
19	100	100	100	100	50	100	100	.1348	.1368	.0020
20	200	200	50	50	0	20	20	.0626	.0625	.0001
21	200	200	50	50	50	20	20	.0625	.0638	.0013
22	200	200	50	50	200	20	20	.0563	.0548	.0015
23	200	200	50	50	50	70	20	.0519	.0505	.0014
24	200	200	50	50	50	20	70	.0529	.0524	.0005
25	200	200	50	50	50	100	100	.0285	.0289	.0004
26	20	80	100	40	50	20	20	.4212	.4218	.0006

For this test, α , the probability of committing a type I error, will equal .05. The resulting $z_{\alpha/2}$ value is 1.96.⁴

The z values for the 26 trials are presented in table IV. The fact that all but four of these values are less than 1.96 means that the null hypothesis can not be rejected for 22 of the 26 cases.

An analysis of the four cases for which the null hypothesis is rejected finds that each of the four are characterized by a very large p. This implies that inaccuracy could be caused by a fault in the uniform pseudo-random number generator. In each of the four trials, the distance the launcher travels while firing is much greater than the target length in the direction of the range. If the generator is producing numbers within the interval (0,1) which have a mean much greater than .5, the large p could cause an excessive number of submunitions to impact around the periphery of the target rectangles and thus land in front of or beyond the target. Thus, hypothesis is further supported by the fact that all four cases possessed a simulated percentage that was smaller than the respective analytic one.

A subsequent analysis was made on the uniform generator. Two thousand numbers were generated using the same primer that was used in the simulation. The resulting pseudo-random numbers had a mean of .5112 and a variance of .08421. When the large sample test was run on these numbers using a two-sided alternative, the resulting z value was 1.713. This value is not high enough to reject the null hypothesis that the mean equals .5, but it is sufficiently close to 1.96 (the z value for $\alpha = .05$) so that an excess of coordinates could occur outside the target when p is very large with respect to the target dimension in the direction of the range.

It appears that it is more than coincidence that the largest differences between pairs of percentages occur at these same four trials. There are two other trials (4 and 13) for which p is large; however, they both also possess large s_x 's. This large s_x would cause fewer submunitions to land inside the target, thus reducing the obvious effect of the large p. The z values for these cases were still greater than 1.

The purpose of the effectiveness study was to ascertain the percentage of submunitions that the XM99 system would be able to impact inside a specified target using variations in system conditions. Since not one difference between compared percentages was greater than 1%, it can be concluded that the simulation accurately emulates the dispersion of the munition function points and the movement of the launcher.

To further substantiate the preceding conclusion, a second test will be applied to the pairs of percentages. The sign test is a statistical test commonly used to compare pairs of observations. For the test to be applicable, two conditions must exist. Each of the two observations of a given pair must be made under similar conditions, and different pairs must be observed under different conditions. The test is based upon the signs of the differences between each two observations. Differences that are zero are excluded and the sample size is reduced. The null hypothesis will be rejected when the number of positive and negative signs differ significantly from equality.⁷

Table IV. Statistical Test Values for Trial Comparison

Trial No.	Sign test	Variance	Z value
1	+	0.00193	1.022
2	+	.00265	1.062
3	+	.00379	5.003
4	+	.00605	1.057
5	+	.00421	2.529
6	+	.00314	.203
7	Deleted	.00000	.000
8	Deleted	.00000	.000
9	+	.00321	1.558
10	+	.00642	.904
11	-	.00620	.060
12	+	.00273	.684
13	-	.00345	.732
14	+	.00559	.271
15	-	.00577	.375
16	+	.00516	4.248
17	-	.00585	.232
18	-	.00583	.435
19	-	.00284	1.139
20	+	.00160	.032
21	-	.00144	1.083
22	+	.00002	10.889
23	+	.00127	1.242
24	-	.00123	.009
25	-	.00073	.456
26	-	.00546	.265

Setting a significance level of 25%, the critical value for this difference between numbers of signs is 8 for 24 observations (two cases were deleted because of zero differences). Table IV shows that there are 10 positive and 14 negative differences in the 24 trials. Since the number of times the less frequent sign occurs is greater than the critical value, the null hypothesis can not be rejected.

B. The Dispersion of the Submunitions.

One remaining portion of the simulation remains to be verified. When the restriction was placed upon the system that only one submunition would be emitted per rocket, the uniform dispersion of submunitions over the effects patterns became so reduced that an estimate of its accuracy could not be seen. The uniform pseudo-random number generator has already been tested using a large sample test. The z value obtained (1.713) was within the critical area with $\alpha = .05$. Thus, the hypothesis which stated that the mean of the generated numbers equals .5 could not be rejected.

The generator will now be studied further to show that its random numbers are evenly distributed throughout their interval, here (0,1). The uniform distribution property will be investigated by dividing the interval into 10 subintervals and then counting how many random numbers are produced within each subinterval.⁸ A Chi-square test is used to test the hypothesis that each subinterval contains equally many numbers. Using 10,000 random numbers, the value of the Chi-square for this generator was 11.442. Compared with the table values, using 9 degrees of freedom, it can be seen that this value will be exceeded 25% of the time.⁹ The null hypothesis, which states that an equal number of numbers fall into each interval, thus can not be rejected.

We would also like to assure that successive random numbers in the sequence are statistically independent, that is, uncorrelated. A test for this is made by again dividing the interval (0,1) into 10 equal subintervals. A count of the number of times a random number in the i th subinterval is followed by one in the j th subinterval is kept in T_{ij} where T is 10×10 matrix. The Chi-square test is again used to test the hypothesis that all T_{ij} are equal.⁸ The Chi-square value computed using this test is 100.299 using 99 degrees of freedom. Approximating this value to a z value in the normal table yields .12766 which is well under any appropriate testing value.⁴

C. Conclusions.

The submunition impact patterns produced by the XM99 riot-control rocket have been successfully simulated with the use of mathematical models and programming methods. The computer program resulting from this effort has been found to be capable of predicting the accuracy of the rocket over varying system conditions. The exactness of the patterns has been shown to be dependent upon these conditions; however, in all cases tested, the resulting patterns have given a good prediction of the average percentage of target impacts the system will be able to produce when compared to the percentages obtained from the analytic model. This simulation program will thus become a useful tool for analyzing the effectiveness of weapons systems which possess characteristics lying within the scope of its application.

In summary, this simulation reliably emulates the effectiveness of the XM99 system, at least with respect to the way it theoretically operates. It is true that there are a vast number of techniques that can be used in computer simulation. For validation, each of these requires its own individual method of verification. The compounding of these techniques makes verification of the total simulation often approach the impossible. However, this does not eliminate simulation as a very valuable methodology for performing systems analysis. This paper has presented an example of a simulation that can be shown to produce accurate results by verifying its output in two stages. First, an analytic solution is derived for a constrained version of the simulation. Then that portion of the simulation that had to be eliminated in order to test the analytic solution is tested separately. By inference, this method of verification can be utilized on much more complicated simulation models by subdividing them into simpler models each of which can be verified independently.

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APPENDIX A

THE SIMULATION PROGRAM AND ITS OUTPUT

```

C      SIMULATION OF THE EFFECTS PATTERN OF THE XM99 ROCKET
      INTEGER RBOT,RMID,RTOP,TCEN,LL,WW
      REAL L
      INTEGER FIELD (300,300)
      DIMENSION X(200),Y(200),XM(200),YM(200)
      READ (5,1001) X(1),XM(1)
1001   FORMAT(2F10.0)
      WRITE (6,3001) X(1),XM(1)
3001   FORMAT(1X,5HX(1)=,F10.0,5X,6HXM(1)=,F10.0)
      READ (5,1000) V,SIGX,W,SIGY,L
1000   FORMAT(5F7.3)
      WRITE (6,3000) V,SIGX,W,SIGY,L
3000   FORMAT(1H1,2HV=,F7.3,5X,5HSIGX=,F7.3,5X,2HW=,F7.3,5X,
      15HSIGY=,F7.3,5X,2HL=,F7.3)
      READ (5,1002) NUMIN,NMUN,LL,WW,INC,NOREP,T
1002   FORMAT(6I5,F5.3)
      WRITE (6,3002) NUMIN,NMUN,LL,WW,INC,NOREP,T
3002   FORMAT(1X,6HNUMIN=,I5,5X,5HNMUN=,I5,5X,3HLL=,I5,5X,3HWW=,
      1I5,5X,4HINC=,I5,5X,6HNOREP=,I5,5X,2HT=,F5.3)
      TOT=0
      U1=FLOAT(LL/2)-9.266*V*T/36
      U2=FLOAT(WW/2)
      U3=9.266*V*T/(18.*(FLOAT(NUMIN-1)))
      II=0
      CALL RANDN(X,3,U1,SIGX,II)
      CALL RANDU(XM,3,II)
      II=6
      DO 679 M=1,NOREP
      WRITE (6,683) M
683   FORMAT(1H1,5HTRIAL,I3)
      DO 676 K=1,WW,INC
      DO 15 J=1,LL,INC
15    FIELD(J,K)=1H
676   CONTINUE
      LBOT=0
      LCEN=0
      RBOT=0
      RMID=0
      RTOP=0
      TCEN=0
      LTOP=0
      LMID=0
      CALL RANDN(X,NUMIN,U1,SIGX,II)
      CALL RANDN(Y,NUMIN,U2,SIGY,II)
      NU=NUMIN-1
      DO 20 I=1,NU

```

```

20      X(I+1)=X(I+1) I*U3
      DO 30 I=1,NUMIN
      CALL RANDU(XM,NMUN,II)
      CALL RANDU(YM,NMUN,II)
      DO 40 N=1,NMUN
      XM(N)=(XM(N)-.5)*L+X(I)
      YM(N)=(YM(N)-.5)*W+Y(I)
      IF (XM(N).GE.0.) IX=IFIX((XM(N)+FLOAT(INC)/2.-1.)/FLOAT(INC))
      IF (XM(N).LT.0.) IX=IFIX((XM(N)-FLOAT(INC)/2.-1.)/FLOAT(INC))
      XM(N)=FLOAT(IX)*FLOAT(INC)+1.
      IF (YM(N).GE.0.) IY=IFIX((YM(N)+FLOAT(INC)/2.-1.)/FLOAT(INC))
      IF (YM(N).LT.0.) IY=IFIX((YM(N)-FLOAT(INC)/2.-1.)/FLOAT(INC))
40      YM(N)=FLOAT(IY)*FLOAT(INC)+1.
      DO 16 N=1,NMUN
      J=IFIX(XM(N))
      K=IFIX(YM(N))
      IF (J.GT.LL.OR.J.LT.1.OR.K.GT.WW.OR.K.LT.1) GO TO 21
      GO TO 17
21      IF (J.GT.LL.AND.K.GT.WW) RTOP=RTOP+1
      IF (J.GT.LL.AND.K.LT.1) LTOP=LTOP+1
      IF (J.LT.1.AND.K.LT.1) LBOT=LBOT+1
      IF (J.LT.1.AND.K.GT.WW) RBOT=RBOT+1
      IF (K.GE.1.AND.K.LE.WW) GO TO 80
      GO TO 81
80      IF (J.LT.1) LCEN=LCEN+1
      IF (J.GT.LL) TCEN=TCEN+1
      GO TO 16
81      IF (J.GT.1.AND.J.LE.LL) TO TO 82
      GO TO 16
82      IF (K.LT.1) LMID=LMID+1
      IF (K.GT.WW) RMID=RMID+1
      GO TO 16
17      FIELD(J,K)=FIELD(J,K)+6H(@@@@@@
16      CONTINUE
30      CONTINUE
      NO=LBOT+LCEN+RBOT+RMID+RTOP+TCEN+LTOP+LMID
      NI=NMUN*NUMIN-NO
      RP=FLOAT(NI)/FLOAT(NMUN*NUMIN)
      WRITE (6,2019) RP
2019    FORMAT(1H0,10X,28HSUBMUNITIONS IN TARGET AREA=,F7.5)
      WRITE (6,124) LBOT,RBOT,LTOP,RTOP,TCEN,LCEN,LMID,RMID
124     FORMAT(1X,5HLBOT=,14,5X,5HRBOT=,14,5X,5HLTOP=,14,5X,
      15HRTOP=,14,5X,5HTCEN=,14,5X,5HLCEN=,14,5X,5HRMID=,14,
      25X,5HLMID=,14)
      WRITE (6,123)
123     FORMAT(1H1,30X,29HRANGE OF MUNITION GOING RIGHT)
      DO 117 K=1,WW,INC
      WRITE (6,120) (FIELD(J,K),J=1,LL,INC)
120     FORMAT(1X,100A1)

```

```

117  CONTINUE
      TOT=TOT+RP
679  CONTINUE
      TOT=TOT/FLOAT(NOREP)
      WRITE (6,234) TOT
234  FORMAT(1X,17HMEAN PERCENTAGE= ,F7.5)
      GO TO 1
      END

```

X(1)= 13289. XM(1)= 72931.

V=32.375 SIGX=20.000 W=50.000 SIGY=20.000 L=50.000
 NUMIN=38 NMUN=32 LL=100 WW=100 INC=3 NOREP=25 T=3.000

TRIAL 1
 SUBMUNITION IN TARGET AREA= .90461

LBOT=3 RBOT=0 LTOP=0 RTOP=0 TCEN=43 BCEN=40 RMID=9 LMID=21

RANGE OF MUNITION GOING RIGHT

```

      C  AA  A
AABA AA  A  A  A  A  A
  ABA  BA  A  A  AAB
BAB AAA  A  AAA  B  AA
AAB AABA C  A  A
  BAABA AAAA AA
    BAABAA ACA A  AA
AAB ABC BA AA AB AAA AA
AAA AAB ABA CBE BAB BB A BB
AB  AB BB A AC AAB CB AAA A
AABA BACADA BCBACBA BA BAAB
A  C  AAB  CA  AA  AAA A
    AA  AAA AABBA A  A
      A  A  AB  A  B
A  ABBA  BBA  A  A

```

APPENDIX B

THEORETICAL TEST FOR THE ANALYTIC SOLUTION

An interesting method of testing the theory of the analytic solution is to check its limiting value as p approaches zero. Physically, this means that the firing mechanism is stationary, and that the aiming point does not move. Because only the probability density function associated with the range coordinate X of the munition function is dependent upon p , it will suffice to show that as p approaches zero, the limit of

$$\int_{-p/2}^{p/2} \frac{1}{\sqrt{\pi} p s_x} \text{EXP} \left(\frac{-(X-u_x)^2}{2s_x^2} \right) du_x$$

is the normal probability density function

$$\frac{1}{\sqrt{\pi} \sqrt{2} s_x} \text{EXP} \left(\frac{-X^2}{2s_x^2} \right)$$

with mean equal to zero and standard deviation equal to s_x .

Using L'Hospital's rule for indeterminate limits and the rules of calculus for calculating the derivatives of integrals,

$$\begin{aligned} \lim_{p \rightarrow 0} & \frac{\left[\int_0^{p/2} \text{EXP} \left(\frac{-(X-u_x)^2}{2s_x^2} \right) du_x - \int_0^{p/2} \text{EXP} \left(\frac{-(X-u_x)^2}{2s_x^2} \right) du_x \right]}{\sqrt{\pi} \sqrt{2} p s_x} \\ &= \frac{\text{EXP} \left(\frac{-(X-p/2)^2}{2s_x^2} \right) \text{EXP} \left(\frac{-(X+p/2)^2}{2s_x^2} \right)}{2\sqrt{2}\sqrt{\pi} s_x} \end{aligned}$$

Substituting zero for p into this expression and collecting terms produces the desired result

$$\frac{1}{\sqrt{\pi} \sqrt{2} s_x} \text{EXP} \left(\frac{-X^2}{2s_x^2} \right)$$

The theory behind this portion of the analytic solution is thus mathematically sound.