AD-775 584

ON THE BALANCE EQUATIONS FOR A MIXTURE OF GRANULAR MATERIALS

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Prepared for:

Army Research Office-Durham

January 1974

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Contract No. DA-31-124-ARO-D-462

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MRC Technical Summary Report # 1390
January 1974

Received August 31, 1973

ia/

Madison, Wisconsin 53706

AR 70-31 Unclassified
Security Classification

AD 775 584

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Mathematics Research Center University of Wisconsin, Madison, Wis. 53706	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

ON THE BALANCE EQUATIONS FOR A MIXTURE OF GRANULAR MATERIALS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Summary Report: no specific reporting period.

5. AUTHOR(S) (First name, middle initial, last name)

S. L. Passman

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1974	18	4

8a. CONTRACT OR GRANT NO. Contract No. DA-31-124-ARO-D-462 b. PROJECT NO. None c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) #1390 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) None

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Army Research Office-Durham, N.C.

13. ABSTRACT

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DD FORM 1 NOV 65 1473

Unclassified
Security Classification

AGO 5696A

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ON THE BALANCE EQUATIONS FOR A MIXTURE OF GRANULAR MATERIALS

S. L. Passman

Introduction

The theory of motion of a granular material in three dimensions has been given by Goodman and Cowin [2]. It would be interesting, with certain applications in mind, to extend this theory to the case of an arbitrary finite number of different granular materials mixed with an arbitrary finite number of ordinary continua (in particular, fluids). As a preliminary to this end, I present balance laws for a mixture of an arbitrary finite number of granular materials.

1. Preliminaries

The number of symbols and the complexity of the calculations involved in the theory presented here is so great that any attempt to present either the axiomatic foundations or the physical explanations and motivations of the theory would enlarge the work greatly. I therefore refer the reader to the standard works on mixture theory and granular materials.[†] A list of symbols and a short set of preliminaries and standard equations is given in order to make the work partially self-contained.

A sequence of bodies $\mathcal{B}_\alpha$, $\alpha = 1, 2, \ldots, n$ is considered. A fixed reference configuration is chosen for each body, and in this configuration $\underset{\sim}{X}_\alpha$ is the place occupied by a particle of $\mathcal{B}_\alpha$. The motion of $\mathcal{B}_\alpha$ is the smooth mapping

$$\underset{\sim}{x} = \underset{\sim}{x}_\alpha(\underset{\sim}{X}_\alpha, t), \qquad t \in (-\infty, \infty). \tag{1.1}$$

[†] In particular, I follow Truesdell [1], and Goodman and Cowin [2].

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of $\underset{\alpha}{\mathcal{B}}$ onto a region of three-dimensional Euclidean space $\mathcal{E}$. In general a backward prime is used to denote a time derivative with $\underset{\alpha}{\mathbf{X}}$ held fixed. Thus the velocity and acceleration of constituent α

$$\underset{\alpha}{\grave{\mathbf{x}}} = \partial_t \underset{\alpha}{\mathbf{x}} (\underset{\alpha}{\mathbf{X}}, t) = \underset{\alpha}{\grave{\mathbf{x}}} (\mathbf{x}, t) , \tag{1.2}$$

$$\underset{\alpha}{\grave{\grave{\mathbf{x}}}} = \partial_t^2 \underset{\alpha}{\mathbf{x}} (\underset{\alpha}{\mathbf{X}}, t) = \underset{\alpha}{\grave{\grave{\mathbf{x}}}} (\mathbf{x}, t) ,$$

where the second forms follow by the assumed smoothness of (1.1). Often the term "peculiar______" will be used in place of "_________ of constituent α". Thus $\underset{\alpha}{\grave{\mathbf{x}}}$ is the peculiar velocity of the α-th constituent. It is assumed that, for each t, there is a region of $\mathcal{E}$ each point of which is occupied simultaneously by particles of each $\underset{\alpha}{\mathcal{B}}$. Henceforth, each formula written will be assumed to hold in subregions or at points of this region.

Assume that each $\underset{\alpha}{\mathcal{B}}$ has a mass, which is a measure on $\underset{\alpha}{\mathcal{B}}$, absolutely continuous with respect to volume on each configuration of $\underset{\alpha}{\mathcal{B}}$. Then a mass density $\underset{\alpha}{\rho}$ exists.

Other quantities defined on $\underset{\alpha}{\mathcal{B}}$ are:

$\underset{\alpha}{\mathbf{b}}$, body force,

$\underset{\alpha}{s}$, body heating ,

$\underset{\alpha}{\mathbf{T}}$, stress ,

$\underset{\alpha}{\mathbf{q}}$, heat flux ,

$\underset{\alpha}{\varepsilon}$, internal energy ,

$\underset{\alpha}{k}$, equilibrated inertia , (1.4)

$\underset{\alpha}{\nu}$, volume distribution

$\underset{a}{\mathbf{h}}$, equilibrated stress ,

$\underset{a}{\ell}$, equilibrated body force,

$\underset{a}{g}$, intrinsic body force,

$\underset{a}{k}$, equilibrated inertial force ,

$\underset{a}{K}$, inertial body force .

Growth terms are

$\underset{a}{\overset{+}{c}}$, growth of mass ,

$\underset{a}{\overset{+}{\mathbf{m}}}$, growth of linear momentum,

$\underset{a}{\overset{+}{\mathbf{M}}}$, growth of angular momentum ,

$\underset{a}{\overset{+}{e}}$, growth of energy ,

$\underset{a}{\overset{+}{v}}$, growth of equilibrated force,

$\underset{a}{\overset{+}{k}}$, growth of equilibrated inertia .

Here, lower-case lightface symbols denote scalars, lower-case boldface symbols denote vectors, (with the exception of $\mathbf{x}$), and upper-case boldface symbols denote (second-order) tensors, thought of as linear transformations on a vector space.

The mixture may also be thought of as a single body $\mathcal{B}$. Corresponding to each quantity in (1.4) for $\underset{a}{\mathcal{B}}$, there is a similarly named and symbolized quantity for the composite body $\mathcal{B}$; e.g. body force $\mathbf{b}$, body heating s .

Truesdell [1,3], lays down three "metaphysical principles" relating $\mathcal{B}$ to $\underset{a}{\mathcal{B}}$. Although it has been pointed out that there is some ambiguity in the interpretation of these principles, I quote them, then proceed to show that they are satisfied by this theory. I have some confidence that, for a particular theory at least, they may be stated as axioms to be satisfied by the balance laws.

1. All properties of the mixture must be mathematical consequences of properties of the constituents.

2. So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.

3. The motion of the mixture is governed by the same equations as is a single body.†

Here the "single body" considered in the third principle is a slight generalization of the "granular material" considered by Goodman and Cowin [2].

The total mass density ρ for the body $\mathcal{B}$ is defined as the sum of the peculiar densities

$$\rho = \Sigma \rho_\alpha \quad , \tag{1.6}$$

where the symbol Σ , here and henceforth is an abbreviation for

$$\Sigma = \sum_{\alpha=1}^{n} \quad . \tag{1.7}$$

The concentration c_α of the α-th constituent is defined by

$$c_\alpha = \frac{\rho_\alpha}{\rho} \, . \tag{1.8}$$

†The principles are quoted from Truesdell [1, p. 83], where they are subsequently motivated. It may be noted that they are somewhat parallel in intent to the assertion of some molecular theorists that the properties of material bodies are determined by the properties of their molecules.

Let $\mathbf{p}$ be the position vector from some fixed point in $\mathcal{E}$ to $\mathbf{x}$. Let $\mathbf{a}$, $\mathbf{b}$ be vectors, let $\mathbf{a}\otimes\mathbf{b}$ denote their tensor product, and let $\mathbf{a}\wedge\mathbf{b}$ denote their outer product

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a}\,. \tag{1.9}$$

The densities of linear momentum and angular momentum, as well as equilibrated inertia and volume distribution momentum for $\mathcal{B}$ are defined by

$$\rho\dot{\mathbf{x}} = \Sigma\, \underset{a}{\rho}\,\underset{a}{\grave{\mathbf{x}}}\,, \tag{1.10}$$

$$\mathbf{p}\wedge\rho\dot{\mathbf{x}} = \Sigma\, \mathbf{p}\wedge\underset{a}{\rho}\,\underset{a}{\grave{\mathbf{x}}}\,,$$

and

$$\rho k = \Sigma\, \underset{a}{\rho}\,\underset{a}{k}\,, \tag{1.11}$$

$$\rho k\nu = \Sigma\, \underset{a}{\rho}\,\underset{a}{k}\,\underset{a}{\nu}\,.$$

Here $\dot{\mathbf{x}}$ is interpreted as velocity in $\mathcal{B}$.

2. Balance Equations

I postulate the following set of balance laws for each constituent of the mixture, for each part $\underset{a}{\mathcal{P}}$ of $\underset{a}{\mathcal{B}}$ for all time t:

$$\int \rho\,\underset{a}{\overset{+}{c}}\,dv = \Big(\int \underset{a}{\rho}\,dv\Big)^{\backprime}\,, \tag{2.1}$$

$$\int \rho\,\underset{a}{\overset{+}{\mathbf{m}}}\,dv = \Big(\int \underset{a}{\rho}\,\underset{a}{\grave{\mathbf{x}}}\,dv\Big)^{\backprime} - \oint \underset{a}{\mathbf{T}}\,\mathbf{n}\,dA - \int \underset{a}{\rho}\,\underset{a}{\mathbf{b}}\,dv\,, \tag{2.2†}$$

$$\int \rho\big(\underset{a}{\overset{+}{\mathbf{M}}} + \mathbf{p}\wedge\underset{a}{\overset{+}{\mathbf{m}}}\big)\,dv = \Big(\int \mathbf{p}\wedge\underset{a}{\rho}\,\underset{a}{\grave{\mathbf{x}}}\,dv\Big)^{\backprime} - \oint \mathbf{p}\wedge\underset{a}{\mathbf{T}}\,\mathbf{n}\,dA - \int \mathbf{p}\wedge\underset{a}{\rho}\,\underset{a}{\mathbf{b}}\,dv\,, \tag{2.3}$$

---

†There are certain refinements which I eschew in order to keep this work relatively reasonable in length. E.g. I assume the existence of a stress tensor rather than proving it by the classical argument of Cauchy.

$$\int \rho \overset{+}{\underset{a}{e}}\, dv = \Big(\int \underset{a}{\rho} (\underset{a}{\epsilon} + \tfrac{1}{2} \underset{a}{\dot{x}}^2 + \tfrac{1}{2} \underset{a}{k} \underset{a}{\dot{\nu}}^2) dv \Big)^{\cdot}$$

$$- \oint \underset{a}{\dot{x}} \cdot \underset{a}{T} n \, dA - \oint \underset{a}{q} \cdot n \, dA - \oint \underset{a}{h} \underset{a}{\dot{\nu}} \cdot \underset{a}{n} \, dA$$

$$- \int \underset{a}{\rho} (\underset{a}{s} + \underset{a}{\dot{x}} \cdot \underset{a}{b} + \underset{a}{\dot{\nu}} \underset{a}{\ell}) \, dv \,, \tag{2.4}$$

$$\int \rho \overset{+}{\underset{a}{v}}\, dv = \Big(\int \underset{a}{\rho} \underset{a}{k} \underset{a}{\dot{\nu}} dv\Big)^{\cdot} - \oint \underset{a}{h} \cdot n \, dA - \int \underset{a}{\rho} (\underset{a}{\ell} + \underset{a}{g}) dv \,, \tag{2.5}$$

$$\int \rho \overset{+}{\underset{a}{k}}\, dv = \Big(\int \underset{a}{\rho} \underset{a}{k}\, dv\Big)^{\cdot} - \oint \underset{a}{k} \cdot n \, dA - \int \underset{a}{\rho} \underset{a}{K}\, dv. \tag{2.6}$$

Here $\int dv$ denotes integration over the volume of an arbitrary part of $\underset{a}{\mathcal{B}}$ and $\oint dA$ denotes integration over its surface with unit outward normal n. The first four of these equations respectively express growth of mass, linear momentum, angular momentum and energy for each component of the mixture and, except for three terms in the equation for energy, are of standard form. The fifth equation expresses growth of equilibrated force. An early explicit recognition of a statement of this nature as a separate balance law is that of Ericksen [4]. Since that recognition, balance laws of this type have been widely accepted in the theory of continua with directors. The first explicit recognition of an equation of the form (2.6) as a separate balance law is apparently due to Goodman and Cowin [2].

A standard argument applied to (2.1) yields

$$\underset{a}{\rho} \overset{+}{\underset{a}{c}} = \underset{a}{\dot{\rho}} + \underset{a}{\rho}\, \mathrm{div}\, \underset{a}{\dot{x}} \equiv \partial_t \underset{a}{\rho} + \mathrm{div}\, (\underset{a}{\rho} \underset{a}{\dot{x}}) \,, \tag{2.7}$$

the local form of the balance of mass.

Let $\underset{a}{\Psi}$ be a scalar, vector, or tensor defined at points in $\underset{a}{\mathcal{B}}$. By a standard method, it may be shown that

$$\left(\int \underset{\alpha}{\rho}\,\underset{\alpha}{\Psi}\,dv\right)^{\cdot} = \int \left(\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{\Psi}} + \rho\,\underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{\Psi}\right)dv \,. \tag{2.8}$$

If $\underset{\alpha}{\Psi}$ is related to Ψ by

$$\rho\Psi = \Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\Psi}\,, \tag{2.9}$$

then again by a standard method it may be shown that

$$\rho\,\dot{\Psi} = \Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{\Psi}} - \operatorname{div}\Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\Psi}\,\underset{\alpha}{\mathbf{u}} + \Sigma\,\rho\,\underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{\Psi}\,, \tag{2.10}$$ †

where $\underset{\alpha}{\mathbf{u}}$ is the diffusion velocity of the α-th constituent

$$\underset{\alpha}{\mathbf{u}} = \underset{\alpha}{\grave{\mathbf{x}}} - \dot{\mathbf{x}}\,. \tag{2.11}$$

By (2.10), (1.10) and (1.11) become

$$\begin{aligned}
\rho\,\ddot{\mathbf{x}} &= \Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{\grave{\mathbf{x}}}} - \operatorname{div}\Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{\mathbf{x}}}\otimes\underset{\alpha}{\mathbf{u}} + \Sigma\,\rho\,\underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{\grave{\mathbf{x}}}\,,\\
\rho\,\dot{k} &= \Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{k}} - \operatorname{div}\Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{k}\,\underset{\alpha}{\mathbf{u}} + \Sigma\,\rho\,\underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{k}\,,\\
\rho k\,\ddot{\nu} + \rho\,\dot{k}\,\dot{\nu} &= \Sigma\,\underset{\alpha}{\rho}\left(\underset{\alpha}{k}\,\underset{\alpha}{\grave{\grave{\nu}}} + \underset{\alpha}{\grave{k}}\,\underset{\alpha}{\grave{\nu}}\right) - \operatorname{div}\Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{k}\,\underset{\alpha}{\grave{\nu}}\,\underset{\alpha}{\mathbf{u}} + \Sigma\,\rho\,\underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{k}\,\underset{\alpha}{\grave{\nu}}\,.
\end{aligned} \tag{2.12}$$

Using the lemmas (2.8) and (2.10), and noting that each of (2.2)-(2.6) hold for arbitrary parts of $\underset{\alpha}{\mathcal{P}}$, I obtain the local forms

$$\underset{\alpha}{\rho}\,\underset{\alpha}{\overset{+}{\mathbf{m}}} = \underset{\alpha}{\rho}\,\underset{\alpha}{\grave{\grave{\mathbf{x}}}} + \rho\,\underset{\alpha}{\grave{\mathbf{x}}}\,\underset{\alpha}{\overset{+}{c}} - \operatorname{div}\underset{\alpha}{\mathbf{T}} - \underset{\alpha}{\rho}\,\underset{\alpha}{\mathbf{b}}\,, \tag{2.13}$$

$$\begin{aligned}
\rho\,\underset{\alpha}{\overset{+}{\mathbf{M}}} + \mathbf{p}\wedge\rho\,\underset{\alpha}{\overset{+}{\mathbf{m}}} &= \mathbf{p}\wedge\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{\grave{\mathbf{x}}}} + \mathbf{p}\wedge\rho\,\underset{\alpha}{\grave{\mathbf{x}}}\,\underset{\alpha}{\overset{+}{c}} - \mathbf{p}\wedge\operatorname{div}\underset{\alpha}{\mathbf{T}}\\
&\quad + \underset{\alpha}{\mathbf{T}} - \underset{\alpha}{\mathbf{T}}^{T} - \mathbf{p}\wedge\underset{\alpha}{\rho}\,\underset{\alpha}{\mathbf{b}}\,,
\end{aligned} \tag{2.14}$$

---

† Here "div" denotes divergence with respect to $\mathbf{x}$. The divergence of a vector is the trace of its gradient. Gradient is denoted by "grad", trace by "tr".

$$\rho\overset{+}{e}_\alpha = \rho_\alpha\left(\grave{\epsilon}_\alpha + \grave{\mathbf{x}}_\alpha\cdot\grave{\grave{\mathbf{x}}}_\alpha + k_\alpha\grave{\nu}_\alpha\grave{\grave{\nu}}_\alpha + \tfrac{1}{2}\grave{k}_\alpha\grave{\nu}_\alpha^2\right)$$

$$+\rho\overset{+}{c}_\alpha\left(\epsilon_\alpha + \tfrac{1}{2}\grave{\mathbf{x}}_\alpha^2 + \tfrac{1}{2}k_\alpha\grave{\nu}_\alpha^2\right) - \grave{\mathbf{x}}_\alpha\cdot\operatorname{div}\mathbf{T}_\alpha - \operatorname{tr}(\mathbf{T}_\alpha^T\operatorname{grad}\grave{\mathbf{x}}_\alpha)$$

$$-\operatorname{div}\mathbf{q}_\alpha - \grave{\nu}_\alpha\operatorname{div}\mathbf{h}_\alpha - \mathbf{h}_\alpha\cdot\operatorname{grad}\grave{\nu}_\alpha$$

$$-\rho_\alpha\left(s_\alpha + \grave{\mathbf{x}}_\alpha\cdot\mathbf{b}_\alpha + \grave{\nu}_\alpha\ell_\alpha\right), \qquad (2.15)$$

$$\rho\overset{+}{v}_\alpha = \rho_\alpha k_\alpha\grave{\grave{\nu}}_\alpha + \rho_\alpha\grave{k}_\alpha\grave{\nu}_\alpha + \rho k_\alpha\grave{\nu}_\alpha\overset{+}{c}_\alpha - \operatorname{div}\mathbf{h}_\alpha - \rho_\alpha(\ell_\alpha + g_\alpha), \qquad (2.16)$$

$$\rho\overset{+}{k}_\alpha = \rho_\alpha\grave{k}_\alpha + \rho\overset{+}{c}_\alpha k_\alpha - \operatorname{div}\mathbf{k}_\alpha - \rho_\alpha K_\alpha. \qquad (2.17)$$

These local equations may be reduced to simpler forms by substitution, yielding

$$\rho\overset{+}{c}_\alpha = \grave{\rho}_\alpha + \rho_\alpha\operatorname{div}\grave{\mathbf{x}}_\alpha, \qquad (2.7)$$

$$\rho\overset{+}{\mathbf{m}}_\alpha = \rho_\alpha\grave{\grave{\mathbf{x}}}_\alpha + \rho\grave{\mathbf{x}}_\alpha\overset{+}{c}_\alpha - \operatorname{div}\mathbf{T}_\alpha - \rho_\alpha\mathbf{b}_\alpha, \qquad (2.18)$$

$$\rho\overset{+}{\mathbf{M}}_\alpha = \mathbf{T}_\alpha - \mathbf{T}_\alpha^T, \qquad (2.19)$$

$$\rho\overset{+}{e}_\alpha = \rho_\alpha\grave{\epsilon}_\alpha + \rho\overset{+}{c}_\alpha\left(\epsilon_\alpha - \tfrac{1}{2}\grave{\mathbf{x}}_\alpha^2\right) + \rho\overset{+}{\mathbf{m}}_\alpha\cdot\grave{\mathbf{x}}_\alpha + \rho\overset{+}{v}_\alpha\grave{\nu}_\alpha - \rho\overset{+}{k}_\alpha\left(\tfrac{1}{2}\grave{\nu}_\alpha^2\right)$$

$$+\rho_\alpha g_\alpha\grave{\nu}_\alpha - \operatorname{tr}(\mathbf{T}_\alpha^T\operatorname{grad}\grave{\mathbf{x}}_\alpha) - \operatorname{div}\mathbf{q}_\alpha - \mathbf{h}_\alpha\cdot\operatorname{grad}\grave{\nu}_\alpha - \rho_\alpha s_\alpha$$

$$+(\rho_\alpha K_\alpha - \operatorname{div}\mathbf{k}_\alpha)\tfrac{1}{2}\grave{\nu}_\alpha^2, \qquad (2.20)$$

$$\rho\overset{+}{v}_\alpha = \rho_\alpha k_\alpha\grave{\grave{\nu}}_\alpha + \rho_\alpha\grave{k}_\alpha\grave{\nu}_\alpha + \rho k_\alpha\grave{\nu}_\alpha\overset{+}{c}_\alpha - \operatorname{div}\mathbf{h}_\alpha - \rho_\alpha(\ell_\alpha + g_\alpha), \qquad (2.21)$$

$$\rho\overset{+}{k}_\alpha = \rho_\alpha\grave{k}_\alpha + \rho\overset{+}{c}_\alpha k_\alpha - \operatorname{div}\mathbf{k}_\alpha - \rho_\alpha K_\alpha. \qquad (2.22)$$

Equations (2.7) and (2.18)-(2.22) appear to satisfy Truesdell's second metaphysical principle.

I assume that mass, linear momentum, angular momentum, energy, equilibrated force, and equilibrated inertia are each conserved for the mixture, that is

$$
\begin{aligned}
\Sigma \overset{+}{c}_{\alpha} &= 0 \, , \\
\Sigma \overset{+}{\mathbf{m}}_{\alpha} &= \mathbf{0} \, , \\
\Sigma \overset{+}{\mathbf{M}}_{\alpha} &= \mathbf{0} \, , \\
\Sigma \overset{+}{e}_{\alpha} &= 0 \, , \\
\Sigma \overset{+}{v}_{\alpha} &= 0 \, , \\
\Sigma \overset{+}{\mathbf{k}}_{\alpha} &= 0 \, .
\end{aligned}
\tag{2.23}
$$

I show that, through the use of (2.23), (2.7) and (2.13)-(2.17) satisfy Truesdell's third metaphysical principle .

By a standard argument, (1.6) and (2.7), $(2.23)_1$ becomes

$$\dot{\rho} + \rho \, \mathrm{div} \, \dot{\mathbf{x}} = 0 \, . \tag{2.24}$$

Define

$$
\begin{aligned}
\mathbf{T} &= \Sigma \mathbf{T}_{\alpha} - \Sigma \rho_{\alpha} \mathbf{u}_{\alpha} \otimes \mathbf{u}_{\alpha} \, , \\
\mathbf{q} &= \Sigma \mathbf{q}_{\alpha} + \Sigma [\mathbf{T}_{\alpha}^{T} \mathbf{u}_{\alpha} + \mathbf{h}_{\alpha} (\grave{\nu}_{\alpha} - \dot{\nu}) - \rho_{\alpha} (\epsilon_{\alpha} + \tfrac{1}{2} u_{\alpha}^{2} + k_{\alpha} \grave{\nu}_{\alpha} (\dot{\nu} - \tfrac{1}{2} \grave{\nu}_{\alpha}) \mathbf{u}_{\alpha}] \, , \\
\mathbf{h} &= \Sigma \mathbf{h}_{\alpha} - \Sigma \rho_{\alpha} k_{\alpha} \grave{\nu}_{\alpha} \mathbf{u}_{\alpha} \, , \\
\mathbf{k} &= \Sigma \mathbf{k}_{\alpha} - \Sigma \rho_{\alpha} k_{\alpha} \mathbf{u}_{\alpha} \, , \\
\rho \mathbf{b} &= \Sigma \rho_{\alpha} \mathbf{b}_{\alpha} \, ,
\end{aligned}
\tag{2.25}
$$

$$\rho s = \Sigma \rho_\alpha s_\alpha + \Sigma \rho_\alpha [\mathbf{b}_\alpha \cdot \mathbf{u}_\alpha + \ell_\alpha (\grave{\nu}_\alpha - \dot{\nu})] \; , \tag{2.25 cont.}$$

$$\rho \epsilon = \Sigma \rho_\alpha \epsilon_\alpha + \Sigma \tfrac{1}{2} \rho_\alpha [u_\alpha^2 + k_\alpha (\grave{\nu}_\alpha - \dot{\nu})^2]$$

$$\rho \ell = \Sigma \rho_\alpha \ell_\alpha \quad ,$$

$$\rho g = \Sigma \rho_\alpha g_\alpha \quad ,$$

$$\rho K = \Sigma \rho_\alpha K_\alpha \; .$$

These definitions, along with (1.10) and (1.11) appear to satisfy Truesdell's first metaphysical principle.

Summing (2.18) over all constituents, noting $(2.23)_2$, $(2.25)_{1,5}$ yields, by (1.6), $(1.10)_1$, and $(2.12)_1$,

$$\rho \ddot{\mathbf{x}} = \rho \mathbf{b} + \operatorname{div} \mathbf{T} \; . \tag{2.26}$$

Summing (2.19) over all constituents, noting $(2.23)_3$ and $(2.25)_1$ yields

$$\mathbf{T} = \mathbf{T}^T \; . \tag{2.27}$$

Equations (2.24), (2.26), and (2.27) are respectively the usual local forms of balance of mass, linear momentum, and moment of momentum for a single body.

It is convenient to leave (2.20) until (2.21) and (2.22) are considered. Summing (2.22) over all constituents and noting $(2.23)_6$ yields

$$\Sigma \rho_\alpha \grave{k}_\alpha + \Sigma \rho \overset{+}{c}_\alpha k_\alpha - \operatorname{div} \Sigma \mathbf{k}_\alpha - \Sigma \rho_\alpha K_\alpha = 0 \; . \tag{2.28}$$

By $(2.12)_2$ this becomes

$$\rho \dot{k} + \operatorname{div} \Sigma \rho_\alpha k_\alpha \mathbf{u}_\alpha - \operatorname{div} \Sigma \mathbf{k}_\alpha - \Sigma \rho_\alpha K_\alpha = 0 \; . \tag{2.29}$$

By $(2.25)_{4,10}$ then

$$\rho \dot{k} - \operatorname{div} \mathbf{k} - \rho K = 0 \; . \tag{2.30}$$

In the particular case when $\underset{\sim}{k}$ is solenoidal and $K = 0$, (2.30) reduces to

$$\dot{k} = 0 . \tag{2.31}$$

This corresponds to the local equation for the balance of equilibrated inertia as given by Goodman and Cowin [2, equation (4.9)].

Summing (2.16) over all constituents and noting $(2.23)_5$ yields

$$\Sigma \underset{a}{\rho} \underset{a}{k} \underset{a}{\ddot{\nu}} + \Sigma \underset{a}{\rho} \underset{a}{\dot{k}} \underset{a}{\dot{\nu}} + \Sigma \underset{aaa}{\rho k \dot{\nu}} \overset{+}{c} - \operatorname{div} \Sigma \underset{a}{\underset{\sim}{h}} - \Sigma \underset{a}{\rho} (\underset{a}{\ell} + \underset{a}{g}) = 0 . \tag{2.32}$$

By $(2.12)_3$, (2.32) becomes

$$\rho k \ddot{\nu} + \rho \dot{k} \dot{\nu} + \operatorname{div} \Sigma \underset{a}{\rho} \underset{a}{k} \underset{a}{\dot{\nu}} \underset{a}{\underset{\sim}{u}} - \operatorname{div} \Sigma \underset{a}{\underset{\sim}{h}} - \Sigma \underset{a}{\rho} (\underset{a}{\ell} + \underset{a}{g}) = 0 , \tag{2.33}$$

which, by $(2.25)_{3,8,9}$ is

$$\rho k \ddot{\nu} + \rho k \dot{\nu} = \operatorname{div} \underset{\sim}{h} + \rho (\ell + g) . \tag{2.33}$$

This may also be written as

$$\rho k \ddot{\nu} + (\operatorname{div} \underset{\sim}{k} + \rho K) \dot{\nu} = \operatorname{div} \underset{\sim}{h} + \rho(\ell + g) , \tag{2.34}$$

where (2.30) has been used. In the case where $\underset{\sim}{k}$ is solenoidal and $K = 0$, (2.34) becomes

$$\rho k \ddot{\nu} = \operatorname{div} \underset{\sim}{h} + \rho(\ell + g) . \tag{2.35}$$

This corresponds to the local equation for the balance of equilibrated force as given by Goodman and Cowin [2, equation (4.10)].

The proof that (2.20) and $(2.23)_4$ yield the usual equation of balance of energy, although somewhat repetitive of the derivation for the non-polar case, is given in some detail here in order to emphasize the interaction of the new terms in the total internal energy, heat supply, and heat flux with the equilibrated stress and equilibrated inertia.

Note that by (1.6), (1.10) and (2.11)

$$\Sigma \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha = \Sigma \rho_\alpha \grave{\mathbf{x}}_\alpha \otimes \grave{\mathbf{x}}_\alpha - \rho \dot{\mathbf{x}} \otimes \dot{\mathbf{x}} \, . \tag{2.36}$$

As a corollary

$$\Sigma \rho_\alpha u_\alpha^2 = \Sigma \rho_\alpha \grave{x}_\alpha^2 - \rho \dot{x}^2 \, . \tag{2.37}$$

Consider the last term in $(2.25)_7$

$$\Sigma \rho_\alpha k_\alpha (\grave{\nu}_\alpha - \dot{\nu})^2 = \Sigma \rho_\alpha k_\alpha \grave{\nu}_\alpha^2 - 2(\Sigma \rho_\alpha k_\alpha \grave{\nu}_\alpha)\dot{\nu} + (\Sigma \rho_\alpha k_\alpha)\dot{\nu}^2 \, . \tag{2.38}$$

By $(1.11)_{1,2}$, this becomes

$$\Sigma \rho_\alpha k_\alpha (\grave{\nu}_\alpha - \dot{\nu})^2 = \Sigma \rho_\alpha k_\alpha \grave{\nu}_\alpha^2 - \rho k \dot{\nu}^2 \, . \tag{2.39}$$

Substituting (2.37) and (2.39) into $(2.25)_7$ yields

$$\rho(\epsilon + \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2} k \dot{\nu}^2) = \Sigma \rho_\alpha (\epsilon_\alpha + \tfrac{1}{2}\grave{x}_\alpha^2 + \tfrac{1}{2} k_\alpha \grave{\nu}_\alpha^2) \, . \tag{2.40}$$

The result (2.40) has the form (2.9), and the lemma (2.10) gives

$$\begin{aligned} \rho(\dot{\epsilon} + \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \tfrac{1}{2}\dot{k}\dot{\nu}^2 + k \dot{\nu} \ddot{\nu}) &= \Sigma \rho_\alpha (\grave{\epsilon}_\alpha + \grave{\mathbf{x}}_\alpha \cdot \grave{\grave{\mathbf{x}}}_\alpha + \tfrac{1}{2}\grave{k}_\alpha \grave{\nu}_\alpha^2 + k_\alpha \grave{\nu}_\alpha \grave{\grave{\nu}}_\alpha) \\ &\quad - \operatorname{div} \Sigma \rho_\alpha (\epsilon_\alpha + \tfrac{1}{2}\grave{x}_\alpha^2 + \tfrac{1}{2} k_\alpha \grave{\nu}_\alpha^2) \grave{\mathbf{x}}_\alpha \\ &\quad + \operatorname{div} \Sigma \rho_\alpha (\epsilon_\alpha + \tfrac{1}{2}\grave{x}_\alpha^2 + \tfrac{1}{2} k_\alpha \grave{\nu}_\alpha^2) \dot{\mathbf{x}} \\ &\quad + \Sigma \rho_\alpha \overset{+}{c}_\alpha (\epsilon_\alpha + \tfrac{1}{2}\grave{x}_\alpha^2 + \tfrac{1}{2} k_\alpha \grave{\nu}_\alpha^2) \, . \end{aligned} \tag{2.41}$$

Note that

$$\grave{\mathbf{x}}_\alpha \cdot \operatorname{div} \mathbf{T}_\alpha + \operatorname{tr}(\mathbf{T}_\alpha^T \operatorname{grad} \grave{\mathbf{x}}_\alpha) = \operatorname{div} \mathbf{T}_\alpha^T \grave{\mathbf{x}}_\alpha \, . \tag{2.42}$$

By (2.11) and $(2.25)_1$,

$$\begin{aligned} \Sigma \operatorname{div} \mathbf{T}_\alpha^T \grave{\mathbf{x}}_\alpha &= \operatorname{div} \Sigma \mathbf{T}_\alpha^T \mathbf{u}_\alpha + \operatorname{div} \Sigma \mathbf{T}_\alpha^T \dot{\mathbf{x}} \, , \\ &= \operatorname{div} \Sigma \mathbf{T}_\alpha^T \mathbf{u}_\alpha + \operatorname{div} \mathbf{T}^T \dot{\mathbf{x}} + \operatorname{div} \Sigma \rho_\alpha (\mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) \dot{\mathbf{x}} \, . \end{aligned} \tag{2.43}$$

However, by (2.36)

$$\Sigma \rho_\alpha (\mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) \dot{\mathbf{x}} = \Sigma \rho_\alpha (\grave{\mathbf{x}}_\alpha \otimes \grave{\mathbf{x}}_\alpha) \dot{\mathbf{x}} - (\rho \dot{x}^2) \dot{\mathbf{x}} \, , \tag{2.44}$$

so that (2.43) becomes

$$\Sigma \grave{\mathbf{x}}_\alpha \cdot \operatorname{div} \mathbf{T}_\alpha + \Sigma \operatorname{tr}(\mathbf{T}_\alpha^T \operatorname{grad} \grave{\mathbf{x}}_\alpha) = \operatorname{div} \Sigma \mathbf{T}_\alpha^T \mathbf{u}_\alpha + \dot{\mathbf{x}} \cdot \operatorname{div} \mathbf{T}$$

$$+ \operatorname{tr}(\mathbf{T}^T \operatorname{grad} \dot{\mathbf{x}}) + \operatorname{div} \Sigma \rho_\alpha (\grave{\mathbf{x}}_\alpha \otimes \grave{\mathbf{x}}_\alpha)\dot{\mathbf{x}} - \operatorname{div}(\rho \dot{x}^2)\dot{\mathbf{x}} . \qquad (2.45)$$

By $(2.25)_2$

$$\Sigma \mathbf{q}_\alpha = \mathbf{q} - \Sigma[\mathbf{T}_\alpha^T \mathbf{u}_\alpha + \mathbf{h}_\alpha(\grave{\nu}_\alpha - \dot{\nu}) - \rho_\alpha(\epsilon_\alpha + \tfrac{1}{2}u_\alpha^2 - k_\alpha \grave{\nu}_\alpha(\dot{\nu} - \tfrac{1}{2}\grave{\nu}_\alpha)\mathbf{u}_\alpha] . \qquad (2.46)$$

Equations (1.6), (1.10) and (2.11) yield

$$\Sigma \rho_\alpha u_\alpha^2 \mathbf{u}_\alpha = \Sigma \rho_\alpha \grave{x}_\alpha^2 \grave{\mathbf{x}}_\alpha - 2\Sigma \rho_\alpha (\grave{\mathbf{x}}_\alpha \otimes \grave{\mathbf{x}}_\alpha)\dot{\mathbf{x}} - \Sigma(\rho_\alpha \grave{x}_\alpha^2)\dot{\mathbf{x}} + 2\rho \dot{x}^2 \dot{\mathbf{x}} . \qquad (2.47)$$

Also, by $(2.25)_7$ and (2.37)

$$\Sigma \rho_\alpha \epsilon_\alpha \mathbf{u}_\alpha = \Sigma \rho_\alpha \epsilon_\alpha \grave{\mathbf{x}}_\alpha - \Sigma \rho_\alpha \epsilon_\alpha \dot{\mathbf{x}}$$

$$= \Sigma \rho_\alpha \epsilon_\alpha \grave{\mathbf{x}}_\alpha - \dot{\mathbf{x}}(\rho \epsilon - \Sigma \tfrac{1}{2}\rho_\alpha u_\alpha^2 - \Sigma \tfrac{1}{2}\rho_\alpha k_\alpha(\grave{\nu}_\alpha - \dot{\nu})^2) ,$$

$$= \Sigma \rho_\alpha \epsilon_\alpha \grave{\mathbf{x}}_\alpha - \rho \epsilon \dot{\mathbf{x}} + \tfrac{1}{2}\Sigma \rho_\alpha \grave{x}_\alpha^2 \dot{\mathbf{x}} - \tfrac{1}{2}(\rho \dot{x}^2)\dot{\mathbf{x}} - \Sigma \tfrac{1}{2}\rho_\alpha k_\alpha(\grave{\nu}_\alpha - \dot{\nu})^2 . \qquad (2.48)$$

Expanding the last term in (2.48) and using $(1.11)_{1,2}$ gives

$$\Sigma \rho_\alpha \epsilon_\alpha \mathbf{u}_\alpha = \Sigma \rho_\alpha \epsilon_\alpha \grave{\mathbf{x}}_\alpha - \rho \epsilon \dot{\mathbf{x}} + \Sigma \tfrac{1}{2}\rho_\alpha \grave{x}_\alpha^2 \dot{\mathbf{x}} - \tfrac{1}{2}(\rho \dot{x}^2)\dot{\mathbf{x}}$$

$$+ \Sigma \tfrac{1}{2}\rho_\alpha k_\alpha \grave{\nu}_\alpha^2 \dot{\mathbf{x}} - \tfrac{1}{2}\rho k \dot{\nu}^2 \dot{\mathbf{x}} . \qquad (2.49)$$

By $(2.25)_3$

$$\Sigma \rho_\alpha k_\alpha \grave{\nu}_\alpha \mathbf{u}_\alpha = \Sigma \mathbf{h}_\alpha - \mathbf{h} . \qquad (2.50)$$

By (2.40)

$$\rho \epsilon \dot{\mathbf{x}} = \Sigma \rho_\alpha(\epsilon_\alpha + \tfrac{1}{2}\grave{x}_\alpha^2 + \tfrac{1}{2}k_\alpha \grave{\nu}_\alpha^2)\dot{\mathbf{x}} - \tfrac{1}{2}(\rho \dot{x}^2)\dot{\mathbf{x}} - \tfrac{1}{2}(\rho k \dot{\nu}^2)\dot{\mathbf{x}} . \qquad (2.51)$$

Substituting (2.47) -(2.51) into (2.46) gives

$$\Sigma \mathbf{q}_\alpha = \mathbf{q} - \Sigma \mathbf{T}_\alpha^T \mathbf{u}_\alpha - \Sigma \mathbf{h}_\alpha \grave{\nu}_\alpha + \mathbf{h}\dot{\nu} + \Sigma \rho_\alpha \epsilon_\alpha \grave{\mathbf{x}}_\alpha - \rho k \dot{\nu}^2 \dot{\mathbf{x}}$$

$$+ \Sigma \tfrac{1}{2}\rho_\alpha k_\alpha \grave{\nu}_\alpha^2 \mathbf{u}_\alpha + \Sigma \tfrac{1}{2}(\rho_\alpha \grave{x}_\alpha^2)\grave{\mathbf{x}}_\alpha - \Sigma \rho_\alpha(\grave{\mathbf{x}}_\alpha \otimes \grave{\mathbf{x}}_\alpha)\dot{\mathbf{x}}$$

$$- \tfrac{1}{2}\Sigma(\rho_\alpha \grave{x}_\alpha^2)\dot{\mathbf{x}} . \qquad (2.52)$$

By $(2.25)_{5,8}$, $(2.25)_6$ becomes

$$\sum_\alpha \rho_\alpha s_\alpha = \rho s - \sum_\alpha \rho_\alpha \underset{\sim}{b}_\alpha \cdot \grave{\underset{\sim}{x}}_\alpha + \dot{\underset{\sim}{x}} \cdot \rho \underset{\sim}{b} - \sum_\alpha \rho_\alpha \ell_\alpha \grave{\nu}_\alpha + \dot{\nu} \rho \ell . \tag{2.53}$$

The equation (2.20) is now reduced by summing over all constituents, noting $(2.23)_4$, and substituting values obtained from (2.41), (2.45), (2.52) and (2.53). The resulting equation is

$$\begin{aligned} &\rho(\dot{\epsilon} + \dot{\underset{\sim}{x}} \cdot \ddot{\underset{\sim}{x}} + \tfrac{1}{2} \dot{k} \dot{\nu}^2 + k \dot{\nu} \ddot{\nu}) - \operatorname{tr}(\underset{\sim}{T}^T \operatorname{grad} \dot{\underset{\sim}{x}}) \\ &- \operatorname{div} \underset{\sim}{q} - \operatorname{div} \underset{\sim}{h} \dot{\nu} - \rho(s + \dot{\underset{\sim}{x}} \cdot \underset{\sim}{b} + \dot{\nu} \ell) = 0 . \end{aligned} \tag{2.54}$$

Substituting (2.26) and (2.32) into (2.54) yields

$$\rho \dot{\epsilon} = \operatorname{tr}(\underset{\sim}{T}^T \operatorname{grad} \dot{\underset{\sim}{x}}) + \underset{\sim}{h} \cdot \operatorname{grad} \dot{\nu} + \rho \dot{k} \dot{\nu}^2 + \rho g \dot{\nu} + \operatorname{div} \underset{\sim}{q} + \rho s. \tag{2.55}$$

This is the equation of balance of energy for the mixture. In the case when $\underset{\sim}{k}$ is solenoidal and $K = 0$, by (2.31) it becomes

$$\rho \dot{\epsilon} = \operatorname{tr}(\underset{\sim}{T}^T \operatorname{grad} \dot{\underset{\sim}{x}}) + \underset{\sim}{h} \cdot \operatorname{grad} \dot{\nu} + \rho g \dot{\nu} + \operatorname{div} \underset{\sim}{q} + \rho s , \tag{2.56}$$

which agrees with the energy equation of Goodman and Cowin [2, equation (4.11)]. In the case $\dot{\nu} = 0$, it further reduces to the classical equation for the balance of energy.

3. The Entropy Inequality

In addition to the balance laws analyzed in the preceding section, I introduce a postulate of growth of entropy, analogous to the "second law of thermodynamics." I assume, for each constituent the existence of a coldness $\underset{\alpha}{\vartheta}$, assumed to be strictly positive and interpreted as the reciprocal of the absolute temperature, an entropy $\underset{\alpha}{\eta}$, and an entropy growth $\underset{\alpha}{\overset{+}{\eta}}$. I assume a balance law for each constituent of the following form:

$$\int \rho \underset{\alpha}{\overset{+}{\eta}}\, dv = \left(\int \underset{\alpha}{\rho}\, \underset{\alpha}{\eta}\, dv\right) - \oint \underset{\alpha}{\vartheta}\, \underset{\alpha}{\mathbf{q}} \cdot \mathbf{n}\, dA - \int \underset{\alpha}{\vartheta}\, \underset{\alpha}{\rho}\, \underset{\alpha}{s}\, dv\,. \qquad (3.1)^{\dagger}$$

By (2.8) the local form of this equation is

$$\rho \underset{\alpha}{\overset{+}{\eta}} = \underset{\alpha}{\rho}\, \underset{\alpha}{\grave{\eta}} + \rho \underset{\alpha}{\overset{+}{c}}\, \underset{\alpha}{\eta} - \operatorname{div}\,(\underset{\alpha}{\vartheta}\, \underset{\alpha}{\mathbf{q}}) - \underset{\alpha}{\vartheta}\, \underset{\alpha}{\rho}\, \underset{\alpha}{s}\,. \qquad (3.2)$$

I introduce the axiom of dissipation

$$\Sigma \underset{\alpha}{\overset{+}{\eta}} \geq 0\,. \qquad (3.3)$$

This inequality is analogous to the conservation laws (2.23).

I define the total entropy η, entropy flux $\boldsymbol{\phi}$, and entropy supply σ multiplied by the coldness ϑ for the mixture by

$$\rho\, \eta = \Sigma \underset{\alpha}{\rho}\, \underset{\alpha}{\eta}\,, \qquad (3.5)$$

$$\boldsymbol{\phi} = \Sigma (\underset{\alpha}{\vartheta}\, \underset{\alpha}{\mathbf{q}} - \underset{\alpha}{\rho}\, \underset{\alpha}{\eta}\, \underset{\alpha}{\mathbf{u}})\,, \qquad (3.6)$$

$$\rho\, \vartheta\, \sigma = \Sigma \underset{\alpha}{\rho}\, \underset{\alpha}{\vartheta}\, \underset{\alpha}{s}\,. \qquad (3.7)^{\ddagger}$$

I proceed to show that the entropy inequality for the mixture is similar in form to the entropy inequality for a single continuum of the type considered.

---

† Goodman and Cowin [2] assume, for a single constituent, a similar inequality, but with a more general form for the entropy "flux" (the surface integral in (3.1)). They than proceed to show that, within a certain constitutive class, the entropy flux has the form assumed in (3.1). As assumption of that nature, if made here, would make some of the calculations in this section trivial.

‡ It should be noted that this equation does not uniquely specify σ or ϑ.

Equation (3.5) is of the form (2.9). Therefore by (2.10) ,

$$\rho\dot{\eta} = \Sigma\rho_\alpha\dot{\eta}_\alpha - \operatorname{div}\Sigma\rho_\alpha\eta_\alpha\underset{\sim}{u}_\alpha + \Sigma\rho_\alpha\overset{+}{c}_\alpha\eta_\alpha \quad . \qquad ($$

By (3.2) this becomes

$$\rho\Sigma\overset{+}{\eta}_\alpha = \rho\dot{\eta} + \operatorname{div}\Sigma(\rho_\alpha\eta_\alpha\underset{\sim}{u}_\alpha - \theta_\alpha\underset{\sim}{q}_\alpha) - \Sigma\theta_\alpha\rho_\alpha s_\alpha \ , \qquad (3.9)$$

or by (3.6) and (3.7) ,

$$\rho\Sigma\overset{+}{\eta}_\alpha = \rho\dot{\eta} - \operatorname{div}\underset{\sim}{\phi} - \rho s\sigma \ . \qquad (3.10)$$

The dissipation axiom (3.3) then yields

$$\rho\dot{\eta} \geq \operatorname{div}\underset{\sim}{\phi} + \rho s\sigma \ . \qquad (3.11)$$

This equation is the same as that of Goodman and Cowin [2, equation (4.12)].

It is often convenient to state the axiom of dissipation in another form, called the "reduced dissipation inequality." I substitute (2.20) into (3.2), obtaining

$$\begin{aligned}
\rho\overset{+}{\eta}_\alpha = {} & \rho_\alpha\grave{\eta}_\alpha + \rho_\alpha\overset{+}{c}_\alpha\eta_\alpha - \underset{\sim}{q}_\alpha\cdot\operatorname{grad}\theta_\alpha \\
& - \theta_\alpha[\rho_\alpha\grave{\epsilon}_\alpha + \rho_\alpha\overset{+}{c}_\alpha(\epsilon_\alpha - \tfrac{1}{2}\grave{x}_\alpha^2) + \rho_\alpha\overset{+}{\underset{\sim}{m}}_\alpha\cdot\grave{\underset{\sim}{x}}_\alpha - \rho_\alpha\overset{+}{e}_\alpha \\
& + \rho_\alpha g_\alpha\grave{\nu}_\alpha + \tfrac{1}{2}\grave{\nu}_\alpha^2(\rho_\alpha K_\alpha - \operatorname{div}\underset{\sim}{k}_\alpha - \rho_\alpha\overset{+}{k}_\alpha) \\
& - \operatorname{tr}(\underset{\sim}{T}_\alpha^T\operatorname{grad}\grave{\underset{\sim}{x}}_\alpha) - \underset{\sim}{h}_\alpha\cdot\operatorname{grad}\grave{\nu}_\alpha] \ .
\end{aligned} \qquad (3.12)$$

Define the Helmholtz free energy [illegible]

[illegible] (3.13)

so that

[illegible] (3.14)

Equations (3.13) and (3.14), when substituted into (3.12), yield

$$
\begin{aligned}
\rho_\alpha \overset{+}{\eta}_\alpha = {} & \rho_\alpha \vartheta_\alpha \left[\overset{+}{e}_\alpha - \underset{\sim}{m}_\alpha \cdot \grave{\underset{\sim}{x}}_\alpha - c_\alpha \left(\psi_\alpha - \tfrac{1}{2} \grave{x}_\alpha^2\right)\right] \\
& + \rho \left(\frac{\eta_\alpha}{\vartheta_\alpha} \grave{\vartheta}_\alpha - \vartheta_\alpha \grave{\psi}_\alpha \right) \\
& - \underset{\sim}{q}_\alpha \cdot \operatorname{grad} \vartheta_\alpha + \vartheta_\alpha \operatorname{tr}\left(\underset{\sim}{T}_\alpha^T \operatorname{grad} \grave{\underset{\sim}{x}}_\alpha\right) + \vartheta_\alpha \underset{\sim}{h}_\alpha \cdot \operatorname{grad} \grave{\nu}_\alpha \\
& - \vartheta_\alpha \left[\tfrac{1}{2} \grave{\nu}_\alpha^2 \left(\rho_\alpha K_\alpha - \operatorname{div} \underset{\sim}{k}_\alpha - \rho_\alpha \overset{+}{\ell}_\alpha\right) + \rho_\alpha g_\alpha \grave{\nu}_\alpha \right] .
\end{aligned}
\tag{3.15}
$$

Summing (3.15) over all constituents and taking note of (3.3), I obtain the reduced dissipation inequality

$$
\begin{aligned}
& \Sigma \rho_\alpha \vartheta_\alpha \left[\overset{+}{e}_\alpha - \underset{\sim}{m}_\alpha \cdot \grave{\underset{\sim}{x}}_\alpha - \overset{+}{c}_\alpha \left(\psi_\alpha - \tfrac{1}{2} \grave{x}_\alpha^2\right)\right] \\
& \quad + \Sigma \rho_\alpha \left(\frac{\eta_\alpha}{\vartheta_\alpha} \grave{\vartheta}_\alpha - \vartheta_\alpha \grave{\psi}_\alpha \right) \\
& - \Sigma \underset{\sim}{q}_\alpha \cdot \operatorname{grad} \vartheta_\alpha + \Sigma \vartheta_\alpha \operatorname{tr}\left(\underset{\sim}{T}_\alpha^T \operatorname{grad} \grave{\underset{\sim}{x}}_\alpha\right) + \Sigma \vartheta_\alpha \underset{\sim}{h}_\alpha \cdot \operatorname{grad} \grave{\nu}_\alpha \\
& - \Sigma \vartheta_\alpha \left[\tfrac{1}{2} \grave{\nu}_\alpha^2 \left(\rho_\alpha K_\alpha - \operatorname{div} \underset{\sim}{k}_\alpha - \rho_\alpha \overset{+}{\ell}_\alpha\right) + \rho_\alpha g_\alpha \grave{\nu}_\alpha \right] \geq 0 .
\end{aligned}
\tag{3.16}
$$

This reduces to the result of Goodman and Cowin [2, equation (4.15)] in the case of one constituent.

Acknowledgement

My thanks to the Mathematics Research Center and the Soil Science Department of the University of Wisconsin for support of this work.

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