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EVALUATION OF CERTAIN PROBABILITIES ASSOCIATED WITH A CLASS OF MARKOV CHAINS

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Two formulae are derived for ratios of limiting probabilities for a class of finite homogeneous Markov chains. The class consists of chains obtained by a generalization of Bernoulli random walk with reflecting or absorbing barriers. These chains are closely related to problems of testing hypotheses with finite memory. The formulae are recursive in nature and hence much easier to use than classical methods.

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Two formulae are derived for ratios of limiting probabilities for a class of finite homogeneous Markov chains. The class consists of chains obtained by generalization of Bernoulli random walk with reflecting or absorbing barriers. These chains are closely related to problems of testing hypotheses with finite memory. The formulae are recursive in nature and hence much easier to use than classical methods.

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1. Introduction.

In this report we propose a method for recursive evaluation of certain probabilities associated with two classes of finite homogeneous Markov chains. These chains are next of kin to Bernoulli random walks with reflecting and absorbing barriers. They may be roughly characterized by the following four properties:

(1) Except for the barrier states, (one-step) transitions from each state can be made to exactly two other states.

(2) The states are divided into two subsets, S_a and S_b, plus an initial state in the absorbing case. States in S_a communicate with states in S_b only via a pair of states, one in each subset.

(3) Except for the absorbing states, the probability of transition among states in S_a and among states in S_b has only two values, p_a, $1 - p_a$ and p_b, $1 - p_b$ respectively.

(4) If states in each of the two subsets are ordered such that the transitions with probability p are all between adjacent states in one direction then the transitions with probability $1 - p$ are all in the opposite direction, but not necessarily to the adjacent state.

A glance at Figures 1 and 3 may help to reveal the structure of a typical member of those two classes.

We are interested in the ratio of limiting probabilities of a chain being in the subset S_a and the subset S_b. In sections 2 and 3 we present a recursive formulae for evaluating these probabilities. Subsequent examples show that the computation is considerably simpler than the classical method of solving systems of linear equations. Our method involves nothing but repeated substitution and is easy to perform and program even for a large number of states.

The need for studying these ratios arises in problems connected with finite automata with binary inputs and outputs driven by a Bernoulli sequence. These, in turn, appear in the so-called finite memory problems

(References [1] through [4]), which are currently receiving considerable attention in literature.

The reason for writing this report is twofold. First since the proofs of our formulae (Sections 4 and 5) are basically algebraic and thus rather long it is usually necessary to condense the proof when the formula is used as a lemma. Hence we wanted to have the proof documented in full detail for reference. Next, it is conceivable that Markov chains of the type studied here may be encountered in various stochastic models. Hence, the second purpose of this report is to provide an access to our results to other workers in the general area of stochastic modelling.

To this we would like to add that the two formulae can probably be generalized in several directions. For instance, inspection of the proofs indicate that the same method could still be used to establish similar formulae for a larger class of chains, namely without the property (3) above.

The part on ergodic chains (Sections 2 and 4) and the part on absorbing chains (Sections 3 and 5) can be read independently.

2. Ergodic Chains.

Let $\underline{r}_a = \{r_a(2), r_a(3), \ldots\}$ and $\underline{r}_b = \{r_b(2), r_b(3), \ldots\}$ be two sequences of positive integers such that $1 \le r_a(i) < i$, $1 \le r_b(i) < i$, $i = 2,3,\ldots$ With each such pair $(\underline{r}_a, \underline{r}_b)$ we associate a class

$$E(\underline{r}_a, \underline{r}_b) = \{M_{n,m} : n=1,2,\ldots\ ;\ m=1,2,\ldots\}$$

of finite ergodic Markcv chains. The chain $M_{n,m}$ has $n + m$ states which are divided into two subsets S_a and S_b with n and m states respectively.

We label the states in S_a by (i,a), $i = 1,\ldots,n$, and the states in S_b by (i,b), $i = 1,\ldots,m$. The transition probabilities are as follows:

$$P((i,a) \to (i+1,a)) = p_a, \quad i = 1,\ldots,n-1,$$

$$P((n,a) \to (m,b)) = p_a,$$

$$P((i,b) \to (i+1,b)) = p_b, \quad i = 1,\ldots,m-1,$$

$$P((m,b) \to (n,a)) = p_b,$$

$$P((i,a) \to (r_a(i),a)) = q_a, \quad i = 2,\ldots,n,$$

$$P((1,a) \to (1,a)) = q_a,$$

$$P((i,b) \to (r_b(i),b)) = q_b, \quad i = 2,\ldots,m,$$

$$P((1,b) \to (1,b)) = q_b.$$

Here $0 < p_a < 1$, $0 < p_b < 1$, $q_a = 1 - p_a$, $q_b = 1 - p_b$. All other transition probabilities are zero. The transition diagram is depicted in Figure 1.

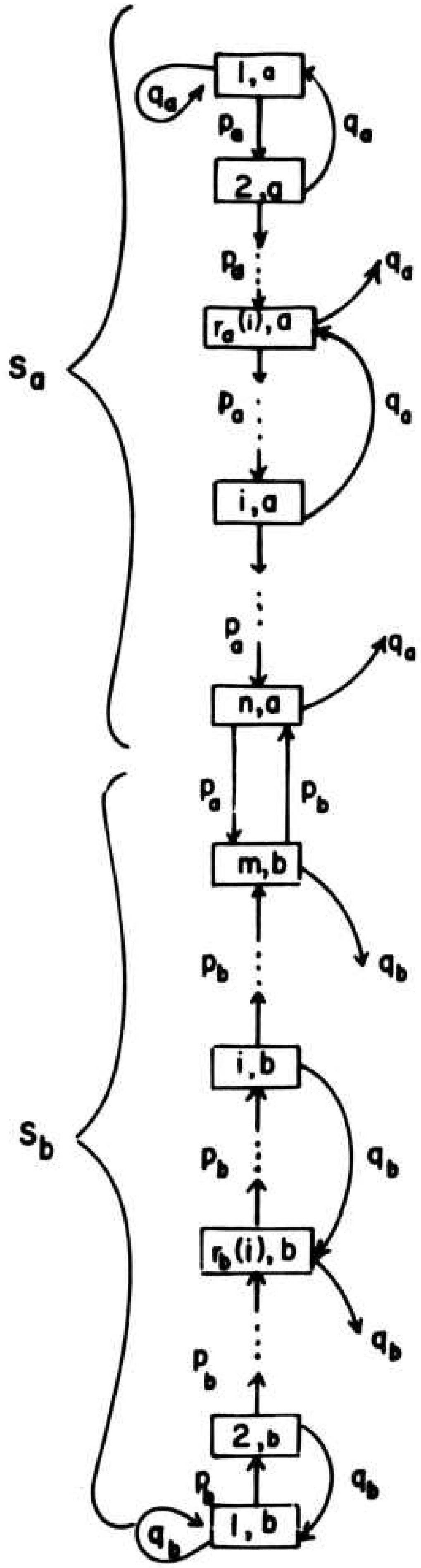

Figure 1

Proposition 1: Let $M_{n,m} \in E(\underline{r}_a, \underline{r}_b)$, let $\mu(s)$, $s \in S_a \cup S_b$ be its stationary distribution, let

$$\mu(S_a) = \sum_{s \in S_a} \mu(s) \quad \text{and} \quad \mu(S_b) = \sum_{s \in S_b} \mu(s)$$

be the stationary probabilities of the chain being in S_a and S_b respectively. Then

$$\frac{\mu(S_a)}{\mu(S_b)} = \frac{p_b^m}{p_a^n} \frac{A_n}{B_m}, \tag{2.1}$$

where A_n and B_m are polynomials in p_a and p_b respectively satisfying the recurrence relations

$$A_{n+1} = p_a^n + q_a \sum_{\ell=r_a(n+1)}^{n} A_\ell p_a^{n-\ell}, \quad A_1 = 1, \qquad n = 1,2,\ldots, \tag{2.2}$$

$$B_{m+1} = p_b^m + q_b \sum_{\ell=r_b(m+1)}^{m} B_\ell p_b^{m-\ell}, \quad B_1 = 1. \qquad m = 1,2,\ldots . \tag{2.3}$$

Hence, both A_n and B_m have integral coefficients and are of degree less than n and m, respectively.

(For the proof see Section 4.)

Example 1: Let $(\underline{r}_a, \underline{r}_b)$ be given by the following table

i	$r_a(i)$	$r_b(i)$
2	1	1
3	1	1
4	2	3
5	.	3
6	.	.

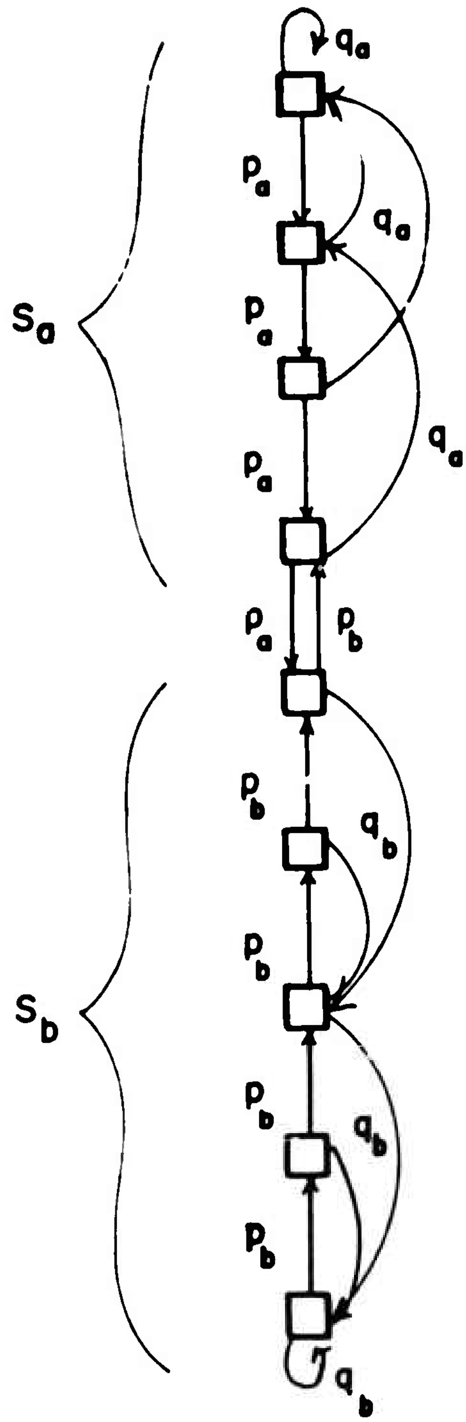

Figure 2

let $n = 4$, $m = 5$. The transition diagram of this 9-state chain is in Figure 3.

First evaluate A_4 and B_5. From (2.2) we have

$$A_4 = p_a^3 + q_a(A_2p_a^{3-2}+A_3p_a^{3-3}) ,$$

$$A_3 = p_a^2 + q_aA_1p_a^{2-1} + q_aA_2p_a^{2-2} ,$$

$$A_2 = p_a + q_aA_1p_a^{1-1} ,$$

$$A_1 = 1 ,$$

and substituting from the bottom to the top gives

$$A_1 = 1 ,$$

$$A_2 = p_a + q_a = 1 ,$$

$$A_3 = p_a^2 + q_ap_a + q_a ,$$

$$A_4 = p_a^3 + q_ap_a + q_ap_a^2 + q_a^2p_a + q_a^2 ,$$

or by substituting $q_a = 1 - p_a$

$$A_4 = p_a^3 - p_a^2 + 1.$$

Similarly from (2.3)

$$B_5 = p_b^4 + q_b(B_3p_b^{4-3}+B_4p_b^{4-4}) ,$$

$$B_4 = p_b^3 + q_bB_3p_b^{3-3} ,$$

$$B_3 = p_b^2 + q_b(B_1p_b^{2-1}+B_2p_b^{2-2}) ,$$

$$B_2 = p_b + q_b B_1 p_b^{1-1} \ ,$$

$$B_1 = 1,$$

and again substituting

$$B_1 = 1 \ ,$$

$$B_2 = p_b + q_b = 1 \ ,$$

$$B_3 = p_b^2 + q_b p_b + q_b \ ,$$

$$B_4 = p_b^3 + q_b p_b^2 + q_b^2 p_b + q_b^2 \ ,$$

$$B_5 = p_b^4 + q_b p_b^3 + q_b^2 p_b^2 + q_b^2 p_b$$
$$+ q_b p_b^3 + q_b^2 p_b^2 + q_b^3 p_b + q_b^3 \ ,$$

or substituting for $q_b = 1 - p_b$

$$B_5 = p_b^3 - p_b + 1.$$

Hence, from (2.1)

$$\frac{\mu(S_a)}{\mu(S_b)} = \frac{p_b^5}{p_a^4} \ \frac{p_a^3 - p_a^2 + 1}{p_b^3 - p_b + 1} \ .$$

3. <u>Absorbing Chains</u>.

Let $\underline{r}_a = \{r_a(1), r_a(2), \ldots\}$ and $\underline{r}_b = \{r_b(1), r_b(2), \ldots\}$ be two sequences of nonnegative integers such that

$$0 \le r_a(i) < i, \quad 0 \le r_b(i) < i, \qquad i = 1,2,\ldots$$

With each such a pair $(\underline{r}_a, \underline{r}_b)$ we associate a class

$$A(\underline{r}_a, \underline{r}_b) = \{M_{n,m} : n=1,2,\ldots ; m=1,2,\ldots\}$$

of finite absorbing Markov chains. The chain $M_{n,m}$ has $n + m + 1$ states, two of them absorbing and the rest transient. One state is always designated as an initial state while the remaining $n + m$ states are divided into two subsets S_a and S_b with n and m states respectively, each containing one of the two absorbing states.

We label the states in S_a by (i,a) $i = 1,\ldots,n$ with (n,a) absorbing, and the states in S_b by (i,b), $i = 1,\ldots,m$ with (m,b) absorbing. The initial state is labeled $(0,a)$ or $(0,b)$ or just 0 as needed.

The transition probabilities are as follows:

$$P((i,a) \to (i+1,a)) = p_a, \qquad i = 1,\ldots,n-1,$$

$$P((n,a) \to (n,a)) = 1,$$

$$P((i,b) \to (i+1,b)) = p_b, \qquad i = 1,\ldots,m-1,$$

$$P((m,b) \to (m,b)) = 1,$$

$$P((i,a) \to (r_a(i),a)) = q_a, \qquad i = 1,\ldots,n-1,$$

$$P((i,b) \to (r_b(i),b)) = q_b, \qquad i = 1,\ldots,m-1,$$

$$P(0 \to (1,a)) = \frac{p_a}{p_a + p_b},$$

$$P(0 \to (1,b)) = \frac{p_b}{p_a + p_b}.$$

Here $0 < p_a < 1$, $0 < p_b < 1$, $q_a = 1 - p_a$, $q_b = 1 - p_b$. All other transition probabilities are zero. The transition diagram is depicted in Figure 2.

<u>Proposition</u>: Let $M_{n,m} \in A(\underline{r}_a, \underline{r}_b)$, let $\pi(a)$ and $\pi(b)$ be the probabilities of absorption in the state (n,a) and (m,b) respectively, if the initial state is the state 0. Then

$$\frac{\pi(a)}{\pi(b)} = \frac{p_a^n}{p_b^m} \frac{B_m}{A_n}, \tag{3.1}$$

where A_n and B_m are polynomials in p_a and p_b respectively satisfying the recurrence relations

$$A_{n+1} = 1 - q_a \sum_{k=2}^{n} A_{r_a(k)} p_a^{k-r_a(k)}, \quad A_0 = 0, \qquad n = 1,2,\ldots, \tag{3.2}$$

$$B_{m+1} = 1 - q_b \sum_{k=2}^{m} B_{r_b(k)} p_b^{k-r_b(k)}, \quad B_0 = 0, \qquad m = 1,2,\ldots . \tag{3.3}$$

Hence both A_n and B_m have integral coefficients and are of degree less than n and m respectively.

(For the proof see Section 5.)

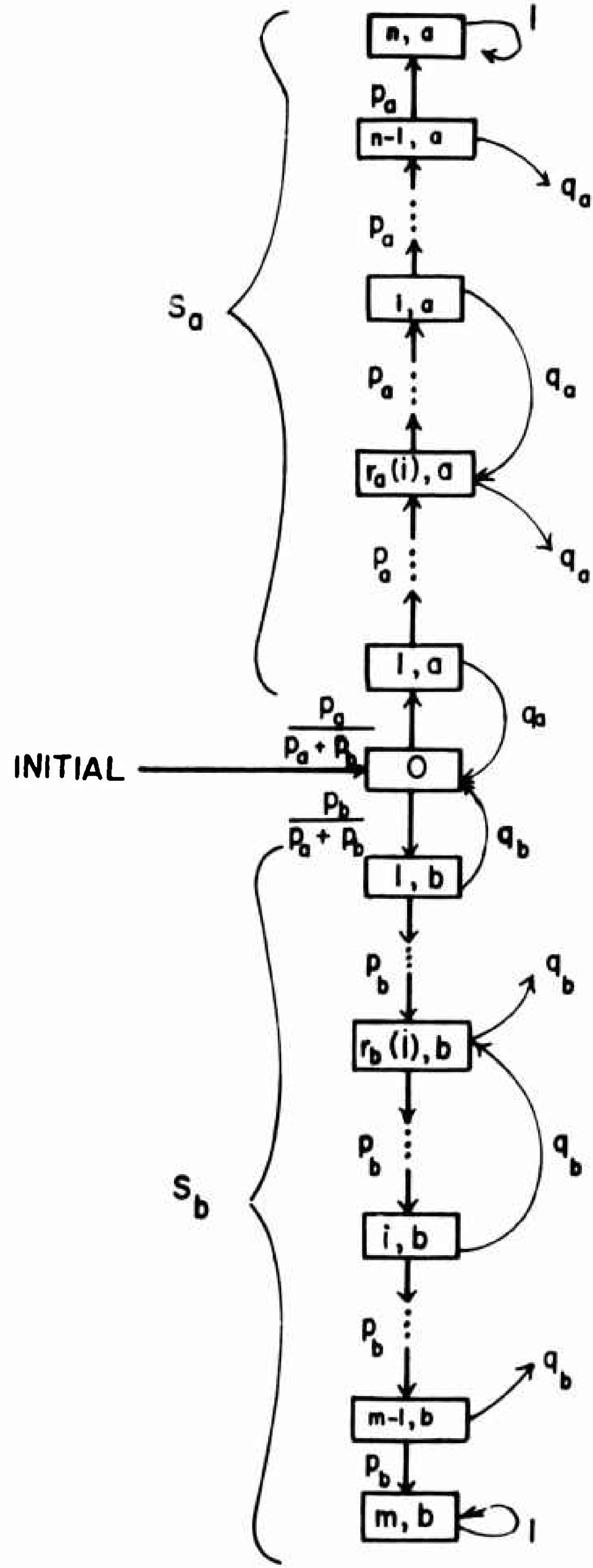
n, a
1
p_a
n-1, a
q_a
p_a
i, a
S_a
p_a
q_a
$r_a(i)$, a
p_a
q_a
1, a
q_a
$\frac{p_a}{p_a + p_b}$
INITIAL
O
$\frac{p_b}{p_a + p_b}$
q_b
1, b
p_b
q_b
$r_b(i)$, b
p_b
q_b
S_b
i, b
p_b
q_b
m-1, b
p_b
m, b
1

Figure 3

<u>Example 2</u>: Let $(\underline{r}_a,\underline{r}_b)$ be given by the following table

i	$r_a(i)$	$r_b(i)$
1	0	0
2	1	1
3	1	1
4	2	3
5	·	3

let $n = 4$, $m = 5$. The transition diagram of this 10-state chain is in Figure 4.

First evaluate A_4 and B_5. From (3.2) we have

$$A_4 = 1 - q_a(A_1p_a^{2-1}+A_1p_a^{3-1}) \ ,$$

$$A_1 = 1 - q_aA_0p_a^{2-0} \ ,$$

$$A_0 = 0 \ ,$$

and substituting from the bottom to the top

$$A_0 = 0 \ ,$$

$$A_1 = 1 \ ,$$

$$A_4 = 1 - q_ap_a - q_ap_a^2 \ ,$$

or substituting for $q_a = 1 - p_a$

$$A_4 = p_a^3 - p_a + 1.$$

Similarly from (3.3)

$$B_5 = 1 - q_b(B_1p_b^{2-1}+B_1p_b^{3-1}+B_3p_b^{4-1}) \ ,$$

$$B_3 = 1 - q_bB_1p_b^{2-1} \ ,$$

$$B_1 = 1,$$

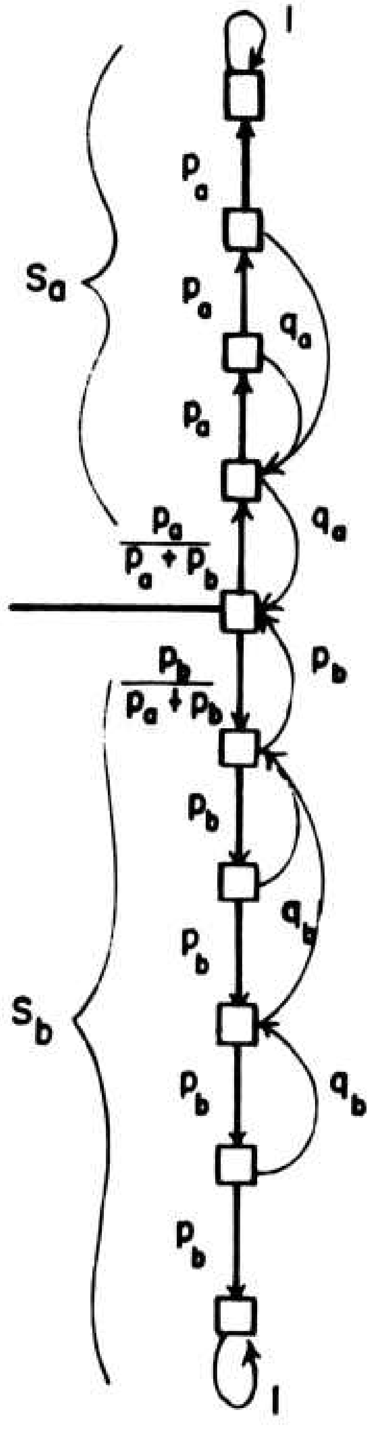
1
p_a
p_a
q_a
p_a
q_a
$\frac{p_a}{p_a + p_b}$
$\frac{p_b}{p_a + p_b}$
p_b
p_b
q_b
p_b
p_b
q_b
p_b
1
S_a
S_b

Figure 4

and again substituting

$$B_1 = 1 ,$$

$$B_3 = 1 - p_b q_b ,$$

$$B_5 = 1 - p_b q_b - p_b^2 q_b - p_b q_b + p_b^2 q_b^2 ,$$

or

$$B_5 = 2p_b^3 - p_b^2 - p_b + 1.$$

Hence, from (3.1)

$$\frac{\pi(a)}{\pi(b)} = \frac{p_a^4}{p_b^5} \frac{2p_b^3 - p_b^2 - p_b + 1}{p_a^3 - p_a + 1} .$$

4. Proof of Proposition 1.

Let P be the transition probability matrix for the chain $M_{n,m}$, where the first n rows and columns correspond to states $(1,a),\ldots,(n,a)$ and the following m rows and columns to states $(m,b),\ldots,(1,b)$.

Let $\underline{\mu} = (\mu(1,a),\ldots,\mu(n,a),\mu(m,b),\ldots,\mu(1,b))$ be the stationary distribution, so that

$$\underline{\mu}(I-P) = \underline{0}\ , \tag{4.1}$$

where I is the identity matrix. Now partition the matrix P into four submatrices

$$P = \left\| \begin{matrix} P_a & V_a \\ V_b & P_b \end{matrix} \right\| ,$$

where P_a is an $n \times n$ matrix

$$P_a = \begin{array}{c} \left\| \begin{matrix} q_a & p_a & & \text{All } 0\text{'s} & & & 0 \\ q_a & 0 & p_a & & & & 0 \\ & & \ddots & \ddots & & & \vdots \\ & q_a & & 0 & p_a & & 0 \\ & & & & \ddots & \ddots & \vdots \\ & & q_a & & 0 & p_a & 0 \\ & & & & & \ddots & p_a \\ & & q_a & & & & 0 \end{matrix} \right\| \\ \text{Column: } \quad 1 \qquad r_a(i) \qquad i \qquad n \end{array} \quad \begin{array}{c} \text{Row:} \\ 1 \\ 2 \\ \vdots \\ r_a(i) \\ \vdots \\ i \\ \vdots \\ n \end{array}$$

and P_b is an $m \times m$ matrix

$$\begin{array}{lcc}
 & & \text{Row:} \\
P_b = & \left\| \begin{array}{ccccccc}
0 & \ddots & & q_b & & & \\
p_b & \ddots & \ddots & & & & \\
\cdot & \ddots & \ddots & & & & \\
0 & p_b & 0 & \ddots & q_b & & \\
\vdots & & \ddots & \ddots & & & \\
0 & & & p_b & 0 & q_b & \\
\vdots & & & & \ddots & \ddots & \\
0 & & & & p_b & 0 & q_b \\
0 & & & & & p_b & q_b
\end{array} \right\| & \begin{array}{c} n \\ \\ \\ i \\ \\ r_a(i) \\ \\ 2 \\ 1 \end{array} \\
\text{Column:} & \begin{array}{ccccccc} n & & i & & r_b(i) & & 1 \end{array} &
\end{array}$$

Notice that each row of these matrixes contains exactly one entry q, namely the $(i,r(i))^{\underline{th}}$ one, and that the labelling of rows and columns of P_b begins at the lower right corner while the labelling of P_a is the usual one (beginning at the upper left corner).

The off-diagonal matrices V_a and V_b consist of all zeros except for the lower-left corner entry of V_a, which is p_a, and the upper-right corner entry of V_b, which is p_b.

With this partitioning the equation (4.1) decomposes into two equations

$$\underline{\mu}_a(I-P_a) = (0,\ldots,0,\mu(m,b)p_b), \tag{4.2}$$

$$\underline{\mu}_b(I-P_b) = (\mu(n,a)p_a,0,\ldots,0), \tag{4.3}$$

where

$$\underline{\mu}_a = (\mu(1,a),\ldots,\mu(n,a)) ,$$

and

$$\underline{\mu}_b = (\mu(m,b),\ldots,\mu(1,b)).$$

Consider the matrix equation (4.2) first. Solving for $\underline{\mu}_a$ gives

$$\underline{\mu}_a = (0,\ldots,0,\mu(m,b)p_b)(I-P_a)^{-1} ,$$

or denoting α_{ij} the $(i,j)^{\underline{th}}$ entry of the inverse $(I-P_a)^{-1}$

$$\mu(i,a) = \mu(m,b)p_b\alpha_{n,i}, \qquad j = 1,\ldots n. \tag{4.4}$$

By the well-known formula for matrix inversion

$$\alpha_{ij} = \frac{\left| I - P_a \right|_{(j,i)}}{\left| I - P_a \right|} ,$$

where $\left| I - P_a \right|$ is the determinant of $I - P_a$ and $\left| I - P_a \right|_{(k,\ell)}$ is the $(k,\ell)^{\underline{th}}$ cofactor of $\left| I - P_a \right|$. Hence

$$\mu(S_a) = \sum_{i=1}^{n} \mu(i,a) = \frac{\mu(m,b)p_b}{\left| I - P_a \right|} \sum_{i=1}^{n} \left| I - P_a \right|_{(i,n)}. \tag{4.5}$$

Next let A_n be the determinant of the $n \times n$ matrix obtained from $I - P_a$ by replacing the $n^{\underline{th}}$ column by a column of 1's,

$$A_n = \begin{vmatrix} p_a & -p_a & & & & & 1 \\ -q_a & 1 & -p_a & & & & 1 \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & -q_a & 1 & -p_a & & 1 \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -q_a & 1 & -p_a \quad 1 \\ & & & & & \ddots & \vdots \\ & & & & & -q_a & 1 \end{vmatrix} \tag{4.6}$$

Expanding A_n along this last column we obtain

$$A_n = \sum_{i=1}^{n} | I - P_a |_{(i,n)},$$

since the $(i,n)^{\underline{th}}$ cofactors of A_n and $| I - P_a |$ are identical. Thus, (4.5) can be written as

$$\mu(S_a) = \frac{\mu(m,b)p_b}{| I - P_a |} A_n .$$

Next since the only flow of probability between sets S_a and S_b is through states (n,a) and (m,b) we must have

$$\mu(m,b)p_b = \mu(n,a)p_a \tag{4.7}$$

in the stationary regime. Using (4.4) we obtain

$$\mu(n,a) = \mu(m,b)p_b \frac{| I - P_a |_{(n,n)}}{| I - P_a |} ,$$

and substituting from (4.7)

$$| I - P_a | = p_a | I - P_a |_{(n,n)} .$$

But

$$| I - P_a |_{(n,n)} = \begin{vmatrix} p_a & -p_a & & & & & 0 \\ -q_a & 1 & -p_a & & & & \cdot \\ & & \ddots & \ddots & & & \cdot \\ & pq_a & & 1 & -p_a & & \cdot \\ & & & & \ddots & \ddots & \cdot \\ & & -q_a & & & 1 & -p_a \quad 0 \\ & & & & & & -p_a \\ & & & & & & 1 \end{vmatrix} ,$$

which is same as the determinant of the $(n-1) \times (n-1)$ matrix $I - P_a$ obtained for the chain $M_{n-1,m} \in E(\underline{r}_a, \underline{r}_b)$. Employing temporarily the superscript (n) for the number of states in S_a we have a recurrence relation

$$| I^{(n)} - P_a^{(n)} | = p_a | I^{(n-1)} - P_a^{(n-1)} | ,$$

and since $| I^{(1)} - P_a^{(1)} | = p_a$ we obtain

$$| I^{(n)} - P_a^{(n)} | = p_a^n .$$

Thus

$$\mu(S_a) = \frac{A_n}{p_a^n} \mu(m,b) p_b .$$

Now going back to (4.3) and repeating all the steps above we obtain a similar expression

$$\mu(S_b) = \frac{B_m}{p_b^m}\,\mu(n,a)p_a ,$$

where B_m is the determinant of order m

$$B_m = \begin{vmatrix}
1 & & & & -q_b & & & & & & \\
\cdot & \ddots & & & & \ddots & & & & & \\
\cdot & & \ddots & & & & & & & & \\
1 & & -p_b & 1 & & & -q_b & & & & \\
\cdot & & & \ddots & \ddots & & & & & & \\
\cdot & & & & & \ddots & & & & & \\
1 & & & & -p_b & 1 & & -q_b & & & \\
\cdot & & & & & \ddots & \ddots & & & & \\
\cdot & & & & & & & \ddots & & & \\
1 & & & & & & & -p_b & 1 & -q_b & \\
1 & & & & & & & & -p_b & p_b &
\end{vmatrix}$$

Hence, using again (4.7) we have

$$\frac{\mu(S_a)}{\mu(S_b)} = \frac{p_b^m}{p_a^n}\,\frac{A_n}{B_m} ,$$

and it remains to prove that A_n and B_m satisfy the recurrence relations (2.2), (2.3).

We begin with the determinant A_n. To evaluate this determinant let

$$I_1 = \{i = 2,\ldots,n : r_a(i) = 1\} .$$

Notice that I_1 is the set of exactly those row indices i for which the $(i,1)^{\underline{th}}$ entries in (4.6) are $-q_a$. Now multiply the first row in (4.6) by q_a/p_a and add it to all rows such that $i \in I_1$. The determinant becomes

$$
A_n = \begin{vmatrix}
p & -p & & & & & & t_{12}^{(n)} \\
0 & p & -p & & & & & \cdot \\
\cdot & -q & 1 & -p & & & & \cdot \\
\cdot & & \ddots & \ddots & \ddots & & & \cdot \\
\cdot & & & 1 & -p & & & \cdot \\
\cdot & & & \ddots & \ddots & \ddots & & \cdot \\
\cdot & & -q & & 1 & -p & & \cdot \\
\cdot & & & & & \ddots & \ddots & \cdot \\
0 & & & & & & & t_{12}^{(n)}
\end{vmatrix},
$$

where we temporarily dropped the subscript a to ease the notation. The entries in the last column are given by

$$
t_{2i}^{(n)} = \begin{cases} 1 + q/p & \text{if } i \in I_1, \\ 1 & \text{if } i \notin I_1. \end{cases}
$$

Expanding this determinant along the first column we have

$$
D_2^{(n)} = pD_1^{(n)},
$$

where $D_1^{(n)} = A_n$ is the original determinant (4.6) and

$$
D_2^{(n)} = \begin{vmatrix}
p & -p & & & & & t_{22}^{(n)} \\
-q & 1 & -p & & & & \cdot \\
& \ddots & \ddots & \ddots & & & \cdot \\
& & 1 & -p & & & \cdot \\
& & \ddots & \ddots & \ddots & & \cdot \\
& -q & & 1 & -p & & \cdot \\
& & & & \ddots & \ddots & \cdot \\
& & & & & & t_{2n}^{(n)}
\end{vmatrix}
$$

is of order $n - 1$. Notice that the entries in the first column in $D_2^{(n)}$ are $-q$ only for row indices $i = 3,\ldots,n$ such that either $r(i) = 1$ or $r(i) = 2$. Hence, calling $I_2 = \{i = 3,\ldots,n : r(i) < 3\}$, multiplying the first row in $D_2^{(n)}$ by p/q and adding to rows with $i \in I_2$ this determinant becomes

$$\begin{vmatrix} p & -p & & & & & t_{32}^{(n)} \\ 0 & p & -p & & & & \vdots \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & 1 & -p & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & -q & & & & 1 \quad -p & \vdots \\ \vdots & & & & & \ddots & \vdots \\ 0 & & & & & & t_{2n}^{(n)} \end{vmatrix} .$$

The entries in the last column are

$$t_{3i}^{(n)} = \begin{cases} t_{2i}^{(n)} + \dfrac{p}{q}\, t_{22}^{(n)} & \text{if } i \in I_2, \\ t_{2i}^{(n)} & \text{if } i \notin I_2. \end{cases}$$

Expanding again along the first column we have

$$D_2^{(n)} = pD_3^{(n)},$$

where

$$D_3^{(n)} = \begin{vmatrix} p & -p & & & & t_{33}^{(n)} \\ & \ddots & \ddots & & & \vdots \\ & & 1 & -p & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & -q & & 1 & -p & \vdots \\ & & & & \ddots & \vdots \\ & & & & & t_{3n}^{(n)} \end{vmatrix} .$$

Proceeding in this fashion we obtain a sequence of determinants

$$D_1^{(n)}, D_2^{(n)}, \ldots, D_n^{(n)}, \tag{4.8}$$

where $D_k^{(n)}$ is of order $n - k + 1$. The entries $t_{ki}^{(n)}$ in the last column of $D_k^{(n)}$ satisfy the recurrence relation

$$t_{k+1,i}^{(n)} = \begin{cases} t_{ki}^{(n)} + \frac{p}{q}\, t_{kk}^{(n)} & \text{if } i \in I_k, \\ t_{ki}^{(n)} & \text{if } i \notin I_k, \end{cases} \tag{4.9}$$

$$k = 1,\ldots,n; \quad i = k,\ldots,n, \quad t_{1,i}^{(n)} = 1,$$

where

$$I_k = \{i = k+1,\ldots,n : r(i) \le k\}.$$

Further

$$D_{k+1}^{(n)} = pD_k^{(n)},$$

so that

$$D_1^{(n)} = p^{n-1}D_n^{(n)} = p^{n-1}t_{nn}^{(n)}. \tag{4.10}$$

Consider now the determinant $A^{(n+1)}$ of order $n + 1$ obtained by using the same sequence $\underline{r}$,

$$A^{(n+1)} = \left| \begin{array}{cccccccc} p & -p & & & & & & 1 \\ -q & 1 & -p & & & & & 1 \\ & & \ddots & & & & & \cdot \\ & & & \ddots & & & & \cdot \\ & -q & & 1 & -p & & & 1 \\ & & & & \ddots & & & \cdot \\ & & & & & \ddots & & \cdot \\ & & -q & & & 1 & -p & 1 \\ & & & & & & \ddots & \cdot \\ & & & & & & & \cdot \\ & & & -q & & & 1 & 1 \\ & & & & -q & & & 1 \end{array} \right| .$$

Applying the above procedure to $D_1^{(n+1)} = A^{(n+1)}$ we obtain a sequence

$$D_1^{(n+1)},\ldots,D_{n+1}^{(n+1)}, \tag{4.11}$$

where the determinants $D_k^{(n+1)}$ again satisfy (4.9) with n replaced by $n + 1$. Arrange now the last columns of the sequences (4.8) and (4.11) into triangular arrays as follows:

$$T^{(n)} = \left\{ \begin{array}{lllll} t_{11}^{(n)} & & & & \\ t_{12}^{(n)}, & t_{22}^{(n)} & & & \\ \cdot & \cdot & \cdot & \cdot & \\ t_{1n}^{(n)}, & t_{2n}^{(n)} & \cdot & \cdot & \cdot \quad t_{nn}^{(n)} \end{array} \right.$$

$$T^{(n+1)} = \left\{ \begin{array}{llllll} t_{11}^{(n+1)} & & & & & \\ t_{12}^{(n+1)}, & t_{22}^{(n+1)} & & & & \\ \cdot & \cdot & \cdot & & & \\ t_{1n}^{(n+1)}, & t_{2n}^{(n+1)} & \cdot & \cdot & \cdot & t_{nn}^{(n+1)} \\ t_{1,n+1}^{(n+1)}, & t_{2,n+1}^{(n+1)} & \cdot & \cdot & \cdot & t_{n,n+1}^{(n+1)}, t_{n+1,n+1}^{(n+1)} \end{array} \right.$$

Since for $i \leq n$ by the definition of sets $I_k^{(n)}$

$$i \in I_k^{(n)} \text{ if and only if } i \in I_k^{(n+1)}$$

the first n rows of $T^{(n)}$ and $T^{(n+1)}$ are identical, i.e.

$$t_{ki}^{(n)} = t_{ki}^{(n+1)}, \qquad i = k,\ldots,n\ ;\quad k = 1,\ldots,n. \tag{4.12}$$

Next by (4.9)

$$t_{k,n+1}^{(n+1)} = \begin{cases} 1 & \text{if } k = 1,\ldots,r(n+1), \\ 1 + \frac{q}{p} \sum\limits_{\ell=r(n+1)}^{k-1} t_{\ell,\ell}^{(n+1)} & \text{if } k = r(n+1)+1,\ldots,n+1. \end{cases}$$

In particular for $k = n + 1$ since $r(n+1) < n + 1$

$$t_{n+1,n+1}^{(n+1)} = 1 + \frac{q}{p} \sum_{\ell=r(n+1)}^{n} t_{\ell\ell}^{(n+1)}. \tag{4.13}$$

But by (4.12) for $\ell < n + 1$

$$t_{\ell\ell}^{(n+1)} = t_{\ell\ell}^{(n)} = \ldots = t_{\ell\ell}^{(\ell)},$$

so that (4.13) becomes

$$t_{n+1,n+1}^{(n+1)} = 1 + \frac{q}{p} \sum_{\ell=p(n+1)}^{n} t_{\ell\ell}^{(\ell)}.$$

Hence, by (4.10)

$$\frac{D_1^{(n+1)}}{p^n} = 1 + \frac{q}{p} \sum_{\ell=p(n+1)}^{n} \frac{D_1^{(\ell)}}{p^{\ell-1}}$$

or calling again $D_1^{(n)} = A_n$ we have

$$A_{n+1} = p_a^n + q_a \sum_{\ell=r(n+1)}^{n} A_\ell p^{n-\ell}, \qquad n = 1,2,\ldots,$$

where clearly $A_1 = 1$.

The recurrence relation for B_m is established in exactly the same fashion.

Noticing the obvious fact that the polynomials A_n and B_m must have integral coefficients completes the proof of Proposition 1.

5. Proof of Proposition 2.

Notice first that with 0 being the initial state any subsequent visit to this state is a recurrent event. Call this event E_0. Next call E_a the event which occurs if the chain after leaving the state 0 reaches the absorbing state (n,a) without any further visit to state 0. Similarly, define E_b for the absorbing state (m,b). Now clearly

$$P(E_0) > 0,$$

and since the absorbtion in (n,a) occurs if and only if we have either

$$E_a \quad \text{or} \quad R_0E_a \quad \text{or} \quad R_0R_0E_a \quad \text{etc.}$$

$$\pi(a) = \frac{P(E_a)}{1 - P(E_0)} ,$$

and similarly

$$\pi(b) = \frac{P(E_b)}{1 - P(E_0)} ,$$

so that,

$$\frac{\pi(a)}{\pi(b)} = \frac{P(E_a)}{P(E_b)} . \tag{5.1}$$

Next

$$P(E_a) = \frac{p_a}{p_a + p_b} P(E_a^1), \tag{5.2}$$

where E_a^1 is the event which occurs if and only if the chain after leaving the state (1,a) reaches the absorbing state (n,a) without ever visiting the state 0.

Consider now a subchain M_n^a obtained from the chain $M_{n,m}$ by making the state 0 an absorbing state and deleting states (1,b) though (m,b). The transition probability matrix for this subchain is the $(n+1) \times (n+1)$ matrix

$$P_a = \begin{Vmatrix} 1 & 0 & & & & & 0 \\ q_a & 0 & p_a & & \text{All 0's} & & \\ & & & \ddots & & & \\ & q_a & & 0 & p_a & & \\ & & & & \ddots & & \\ & q_a & & & 0 & p_a & \\ & & & & & 0 & p_a \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{Vmatrix} .$$

If this subchain is started at the state (1,a) then $P(E_a^1)$ is equal to the probability of absorption in (n,a) for this subchain. Using the well-known result from the algebraic theory of Markov chain (cf. [5], Theorem 3.3.7) we have

$$P(E_a^1) = p_a \alpha_{1,n-1}, \tag{5.3}$$

where $[\alpha_{ij}] = (I-Q_a)^{-1}$ and Q_a is the $(n-1) \times (n-1)$ matrix of transition probabilities between transient states of M_n^a, i.e.

$$Q_a = \left\|\begin{matrix} 0 & p_a & & & \text{All } 0\text{'s} \\ & \ddots & \ddots & & \\ q_a & & 0 & p_a & \\ & q_a & & \ddots & \ddots \\ & & & 0 & p_a \\ & q_a & & & 0 \end{matrix}\right\| .$$

By the formula for matrix inversion

$$\alpha_{1,n-1} = \frac{|\, I - Q_a|_{(n-1,1)}}{|\, I - Q_a|} , \tag{5.4}$$

where $|\, I - Q_a|$ is the determinant of $I - Q_a$ and $|\, I - Q_a|_{(n-1,1)}$ is its $(n-1,1)^{\underline{st}}$ cofactor. Now

$$|I-Q_a|_{(n-1,1)} = (-1)^n \begin{vmatrix} -p_a & & & \text{All } 0\text{'s} \\ & \ddots & & \\ -q_a & 1 & -p_a & \\ & & \ddots & \ddots \\ -q_a & & 1 & -p_a \\ -q_a & & & 1 \quad -p_a \end{vmatrix} = p_a^{n-2} ,$$

and calling $A_n = |\, I - Q_a\,|$ we obtain from (5.2), (5.3) and (5.4)

$$P(E_a) = \frac{p_a^n}{p_a + p_b} \frac{1}{A_n} , \tag{5.5}$$

Following the same procedure for the event E_b we obtain

$$P(E_b) = \frac{p_b^m}{p_a + p_b} \frac{1}{B_m}, \tag{5.6}$$

where $B_m = | I - Q_b |$ and Q_b is the $(m-1) \times (m-1)$ matrix of transition probabilities between transient states $(1,b),\ldots,(m-1,b)$. Hence, (5.1), (5.5) and (5.6) yields (3.1) and it remains to establish the recurrence relations for A_n and B_m.

To do this we evaluate the determinant

$$A_n = \begin{vmatrix} 1 & -p & & & \text{All } 0\text{'s} \\ & \ddots & \ddots & & \\ -q & & 1 & -p & \\ & & \ddots & \ddots & \\ & -q & & 1 & -p \\ & & & \ddots & \ddots \\ & -q & & & 1 \end{vmatrix},$$

where we dropped the subscript a to ease the notation. Notice that A_n is of order $n - 1$ and that each row but the first has either exactly one subdiagonal entry equal to $-q$, namely the $(i,r(i))^{\underline{th}}$, or all subdiagonal entries are zero. The former case occurs if $r(i) > 0$ while the latter if $r(i) = 0$.

Next consider the determinant A_{n+1} of order n obtain for the chain $M_{n+1,m}$ with the same $(\underline{r}_a,\underline{r}_b)$. Then

$$A_{n+1} = \left|\begin{array}{c|c} & 0 \\ & \cdot \\ A_n & \cdot \\ & \cdot \\ & 0 \\ & -p \\ \hline -q & 1 \end{array}\right| .$$

Now if $r_a(n) = 0$ then there is no $-q$ in the last row and hence

$$A_{n+1} = A_n. \tag{5.7}$$

If $r_a(n) > 0$ then expanding A_{n+1} along the last column gives

$$A_{n+1} = A_n + pD_n, \tag{5.8}$$

where

$$D_n = \left|\begin{array}{c|c} & 0 \\ & \cdot \\ & \cdot \\ A_{n-1} & \cdot \\ & 0 \\ & -p \\ \hline -q & 0 \end{array}\right|$$

is of order $n - 1$. Notice that the entry $-q$ in the last row moved one step to the right. Expanding D_n again along the last column gives

$$D_n = pD_{n-1},$$

where D_{n-1} is of order $n - 2$

$$D_{n-1} = \left|\begin{array}{c|c} & 0 \\ & \vdots \\ A_{n-2} & 0 \\ & -p \\ \hline -q & 0 \end{array}\right| .$$

Now repeating this the entry $-q$ eventually (after $n - r_a(n)$ steps) reaches the diagonal and we have

$$D_{r_a(n)+1} = \left|\begin{array}{c|c} & 0 \\ & \vdots \\ A_{r_a(n)} & 0 \\ & p \\ \hline 0 \; . \; . \; . \; . \; 0 & -q \end{array}\right| ,$$

and expanding this determinant along the last row yields

$$D_{r_a(n)+1} = -qA_{r_a(n)} .$$

Substituting back into (5.8) we obtain

$$A_{n+1} = A_n - qp^{n-r(n)} A_{r_a(n)}, \tag{5.9}$$

which holds for any $n = 1,2,\ldots$ such that

$$r_a(n) > 0.$$

To include the case $r_a(n) = 0$ define $A_0 = 0$. Then (5.9) reduces to (5.7). Finally, repeatedly substituting for A_n in (5.9) and using the obvious fact $A_2 = 1$ we obtain the recurrence relation (3.2). Notice that (3.2) holds also for $n = 1$ since $r_a(1) = 0$ always.

The relation (3.3) for B_m is established in exactly the same fashion from $|I - Q_b|$.

Noticing the obvious fact that the polynomials A_n and B_m must have integral coefficients completes the proof of Proposition 2.

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