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Romano M. DeSantis, et al

Michigan University

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This report analyses the input-output behavior of feedback systems whose open loop map can be modeled by an operator, K, defined on a Hilbert space. In particular, attention is focused on the case where K is multipower, bounded, and strictly causal. The analysis utilizes Hilbert resolution space, the causality structure of K and contraction mapping techniques. The main objective is to clarify questions of the following type: (1) Given an input, y, is the output, x, well defined? (Existence, Uniqueness); (2) If x is well defined, what can be said about the mapping y - x? (Causality, Continuity); (3) Given

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Block 20. Abstract (Continued)

y how can x be computed? (Computational method). To illustrate the difficulty associated with questions of this type, it is shown that a system input with a finite energy may generate an infinite energy, possibly with a finite escape time, feedback system output. The set of inputs for which this does not happen, however, is unbounded, open set with interior. Moreover, on this set, the ougput is a continuous and causal function of both the input and the open loop system. From a practical point of view it follows that the output can always be computed by using a number of nonlinear computational algorithms already available. Applications are relevant to the area of stability, sensitivity and controllability of dynamical systems.

The contents of this document are also available in the French language report: "ANALYSE DES SYSTEMES A CONTREACTION AVEC CHAINE OUVERTE MULTIPUISSANCE" (Rapport technique de l'Ecole Polytechnique de Montréal, Montréal, P. Q., Canada).

THE UNIVERSITY OF MICHIGAN

SYSTEMS ENGINEERING LABORATORY

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ON THE ANALYSIS OF FEEDBACK SYSTEMS WITH A MULTIPOWER OPEN LOOP CHAIN

by

Dr. Romano Mario De Santis
Departement de Genie Electrique
Ecole Polytechnique
Montreal, Quebec

Dr. William A. Porter
Department of Electrical and Computer Engineering
The University of Michigan
Ann Arbor, Michigan

October, 1973

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0. Summary

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The main objective is to clarify questions of the following type: (1) Given an input, y, is the output, x, well defined? (Existence, Uniqueness); (2) If x is well defined, what can be said about the mapping $y \to x$? (Causality, Continuity); (3) Given y how can x be computed? (Computational method). To illustrate the difficulty associated with questions of this type, it is shown that a system input with a finite energy may generate an infinite energy, possibly with a finite escape time, feedback system output. The set of inputs for which this does not happen, however, is unbounded, open set with interior. Moreover, on this set, the output is a continuous and causal function of both the input and the open loop system. From a practical point of view it follows that the output can always be computed by using a number of nonlinear computational algorithms already available.

Applications are relevant to the area of stability, sensitivity and controllability of dynamical systems.

1. Introduction

In recent months it has become increasingly apparent that the concept of the resolution space is a fruitful, and in some cases essential, tool for the analysis of causality, stability, and sensitivity problems in dynamical systems. This study is an addendum to the literature on invertibility and causality questions of operators on Hilbert resolution spaces. The question that we would like to answer is the following. If K is a summation of multipower operators on a Hilbert resolution space: (1) when is I + K one-to-one?, (2) what is the range of I + K? and (3) If y = (I + K)x when is the map y → x causal?

Questions of this generality are of course elusive. However, by restricting attention to the causality structure of K and how it impinges on the three questions posed above some progress can be made. In particular the properties of strict causality and nonmemory play important roles in the development.

The question of invertibility for, I + K, has received considerable attention. In particular Browder [1], Dolph [2] and Minty [3], among others, have given sufficient invertibility conditions for the case in which K is a monotone operator. Similar results have been obtained by Petryshyn [4] and Shinbrot [5] who considered operators K with special compactness properties.

In regard to the causality of $(I + K)^{-1}$, the early work concentrated on linear stationary systems, (Foures and Segal [6] and Youla Castriota and Carlin [7]), and was based on the Paley and Wiener theorem [8]. More recently Sandberg [9] considered nonlinear time variant systems and exposed a connection between causality and energy concepts. Damborg [10], [11] has established a sufficient condition for the causality of $(I + K)^{-1}$ in terms of an expression involving "incremental truncated" phase gain and phase shift concepts. Saeks [12] has considered linear systems in Hilbert space and has established the causality of $(I + K)^{-1}$ when K is causal and satisfies an inner product type condition. In a similar context Porter [13] has shown that $(I + K)^{-1}$ is causal whenever K is causal and dissipative. Finally DeSantis [17][18] focused attention on those causal systems for which the future of the output is determined by the strict past of the input; such systems are said to be strictly causal. This approach leads in a natural way to the utilization of the Hilbert resolution space framework proposed in [12], [13], and [14] and is the format adopted in the present article.

The interrelation of strict causality with causal invertibility has precedent in the technical literature. In particular, some results of Gohberg and Krein [16], on the abstract theory of Volterra operators can be interpreted [18] as showing that if K is linear, completely continuous and strictly causal then, $I + K$, is invertible and $(I + K)^{-1}$ is causal. The scope and applicability of these results are seriously hampered however by the rather restrictive requirements of linearity and complete continuity. This article, together with [20], may be viewed as efforts to remove these restrictive requirements.

2. Mathematical Preliminaries

In this section we summarize the main definitions of multilinear operators and Hilbert Resolution Spaces. Consider first a Hilbert space, H, over the field, F, and its n-th power Cartesian product H^n, equipped with the usual rules for addition and scalar multiplication. Suppose that W is a function with domain H^n and range in H. Using the notations $(x_1, \ldots, x_n) \in H^n$ and $W[x_1 \ldots x_n]$ in the obvious manner we have:

Definition 1. A function $W: H^n \to H$ is said to be n-linear if

$$W[x^1, \ldots, x^i + \alpha y, \ldots, x^n] = W[x^1, \ldots, x^i, \ldots, x^n] + \alpha W[x^1, \ldots, y, \ldots, x^n]$$

for every $x^i, y \in H$, $i = 1, \ldots, n$ and all $\alpha \in F$. *

In the case n = 2 the terminology bilinear is used in lieu of n-linear. Similarly the terminology multilinear is used when n is arbitrary or unimportant to the discussion. By an abuse of terminology we shall refer to W as a multilinear operator on H.

To every n-linear operator $W: H^n \to H$ there is associated a function $\hat{W}: H \to H$ defined by

$$\hat{W}(x) = W[x, \ldots, x], \quad x \in H. \tag{1}$$

Definition 2. A function $\hat{W}: H \to H$ is called an n-power operator if there exists an n-linear operator $W: H^n \to H$ such that eqn. (1) holds.

*Note that a linear operator can be viewed as a special case of a n-linear operator.

Definition 3. A mapping $\tilde{W}: H \to H$ is multipower if it is given by an algebraic sum of n-power operators.

An n-linear operator W is said to be bounded if

$$\|W\| = \sup_{\Gamma} \|W[x^1, \ldots, x^n]\| < \infty$$

where $\Gamma = \{\|x^1\| = \|x^2\| = \|x^n\| = 1\}$. Similarly the n-power operator $\hat{W}$ is bounded if

$$\|\hat{W}\| = \sup_{\|x\|=1} \|\hat{W}(x)\| < \infty .$$

The numbers $\|W\|$ and $\|\hat{W}\|$ respectively are called the n-linear and n-power norms, clearly $\|\hat{W}\| \leq \|W\|$. A multipower operator is bounded if it is a sum of bounded n-power operators.

Let W be a bilinear operator on H. The permutation, W^*, and the mean, $\overline{W}$, of W are defined respectively by

$$W^*[x, y] = W[y, x]$$

$$\overline{W}[x, y] = \frac{1}{2}\{W[x, y] + W[y, x]\} .$$

If $W = W^* = \overline{W}$, the bilinear operator is said to be symmetric.

More generally we have

Definition 4. The n-linear operator W^{*p} is called a permutation of type p of the n-linear operator W if the following relation holds

$$W^{*p}[x^1, x^2, \ldots, x^n] = W[x^{\gamma 1}, x^{\gamma 2}, \ldots, x^{\gamma n}]$$

where $p = \{\gamma 1, \gamma 2, \ldots, \gamma n\}$ is a permutation of the ordered set $\{1, 2, \ldots, n\}$;

and $(x^1, x^2, \ldots, x^n)$ is any element of H^n.

Definition 5. $\overline{W}$ is said to be the mean of W if

$$\overline{W}[x^1, x^2, \ldots, x^n] = \frac{1}{n!}\sum_{j=1}^{n!} W^{*p_j}[x^1, x^2, \ldots, x^n]$$

where $\{p_j : j = 1; n!\}$ are all distinct permutations of $\{1, 2, \ldots, n\}$ and $(x^1, x^2, \ldots, x^n) \in H^n$.

Definition 6. W is said to be symmetric if

$$W[x^1, \ldots, x^n] = W^{*p}[x^1, \ldots, x^n] = \overline{W}[x^1, \ldots, x^n]$$

holds for any permutation p of $\{1, 2, \ldots, n\}$ and for arbitrary $(x^1, x^2, \ldots, x^n) \in H^n$.

In dealing with multipower operators it is obvious that W, W*, and $\overline{W}$ all generate the same $\hat{W}$.

Example 1. Let H consist of the Euclidean space R^2 equipped with the usual inner product. Consider the operator $W:[R^2]^2 \to R^2$ defined as follows: if $z = (z_1, z_2)$, $x^1 = (x_1^1, x_2^1)$, $x^2 = (x_1^2, x_2^2) \in R^2$, and $z = W[x^1, x^2]$, then

$$z_1 = \beta_{111}x_1^1x_1^2 + \beta_{112}x_1^1x_2^2 + \beta_{121}x_2^1x_1^2 + \beta_{122}x_2^1x_2^2$$

$$z_2 = \beta_{211}x_1^1x_1^2 + \beta_{212}x_1^1x_2^2 + \beta_{221}x_2^1x_1^2 + \beta_{222}x_2^1x_2^2,$$

where β_{ijk} is a real number. This operator is bilinear and bounded; it is symmetric if $\beta_{ijk} = \beta_{ikj}$ for every $i, j, k \in \{1, 2\}$.

Example 2. Let $H = L_2(\Lambda)$ be the Hilbert space of square integrable scalar functions defined on the interval $\Lambda = [0, \frac{\pi}{2}]$. Consider a scalar valued function $K(t, s_1, s_2)$ defined on Λ^3 and such that

$$\|K\| = \iiint_{\Lambda^3} |K(t, s_1, s_2)|^2 dt\, ds_1 ds_2 < \infty .$$

For any pair $x^1, x^2 \in L_2(\Lambda)$ the following integral

$$(\bullet) = \iint_{\Lambda^2} K(\bullet, s_1, s_2) x^1(s_1) x^2(s_2) ds_1 ds_2$$

defines a bilinear operator $L_2^2 \to L_2$. This operator is bounded because for each pair $x^1, x^2 \in L_2(\Lambda)$ one has

$$\|y\| \leq \|K\| \cdot \|x^1\| \cdot \|x^2\| .$$

It is symmetric if $K(t, s_1, s_2) = K(t, s_2, s_1)$.

Next the structure of Hilbert resolution spaces will be presented [12], [14]. [20]. Suppose that H is a Hilbert space, and ν a linearly ordered set with t_o and t_∞ respectively minimum and maximum elements. A family $\{P^t\} = \mathbb{R}$, $t \in \nu$, of orthogonal projections on H is a Resolution of the identity if it enjoys the following two properties:

Ri) $P^{t_o}H = 0$, $P^{t_\infty}H = H$, and $P^k H \supseteq P^\ell H$ whenever $k > \ell$;

Rii) if $\{P^i\}$ is a sequence of orthogonal projections in $\mathbb{R}$ and there exists an orthogonal projection P such that $\{P^i x\} \to Px$, for every $x \in H$, then $P \in \mathbb{R}$.

Definition 7. A Hilbert space, H, equipped with a Resolution of the identity, is called a Hilbert resolution space (in short: HRS) and is denoted by the symbol $[H, P^t]$.

Example 3. Let $H = R^2$ and consider the family of orthoprojectors $\{P^t\}$, $t \in \{0, 1, 2\}$ defined as follows:

If $x, y \in R^2$ and $y = P^t x$, then

$$y = \begin{cases} (0,0) \text{ if } t = 0 \\ (x_1, 0) \text{ if } t = 1 \\ (x_1, x_2) \text{ if } t = 2 . \end{cases}$$

The family $\{P^t\}$ is a resolution of the identity and the pair $[R^2, P^t]$ is a Hilbert resolution space.

Example 4. Let $H = L_2(\Lambda)$, and for $t \in \Lambda = [0, \frac{\pi}{2}]$ define P^t as follows: if $y, x \in L_2(\Lambda)$ and $y = P^t x$, then $y(\tau) = x(\tau)$ for $\tau < t$, $y(\tau) = 0$ otherwise. The family of orthoprojectors $\{P^t\}$ is a resolution of the identity and the pair $[L_2(\Lambda), P^t]$ is a Hilbert resolution space.

For an operator K on $[H, P^t]$ the notation $\int dPKdP$ will denote a derived operator computed in the following way. First, let $\Omega = \{t_0, t_1, \ldots, t_j, \ldots, t_n = t_\infty\}$ denote a partition of γ, that is $t_j < t_{j+1}$ all j. For convenience, let $P^j = P^{t_j}$ and form the incremental projection set $\Delta_j = P^j - P^{j-1}$, $j = 1, \ldots, n$. The set of all such partitions is partially ordered by containment, that is $\Omega_1 \geqq \Omega_2$ if all points of Ω_2 are in Ω_1.

Now if for every $\epsilon > 0$ there exists Ω_ϵ and a function, denoted by $\int dPKdP$, such that

$$\| \int dPKdP - \sum_1^n \Delta_j K \Delta_j \| < \epsilon$$

for all $\Omega \geqq \Omega_\epsilon$ then the series of sums converge and

$$\int dPKdP = \lim \sum \Delta_j K \Delta_j$$

where the limit is taken over refinement.

Example 5. Consider in $[L_2(\Lambda), P^t]$ the bipower operator induced by the bilinear mapping described in Example 2. For simplicity assume $|K(t, s_1, s_2)| \leq M < \infty$. Then for any integer $N \geq 2$, choose

$$\Omega_\epsilon = \{\xi_i^\epsilon\} = \{0, \pi/2N, 2\pi/2N, \cdots \pi/2\}.$$

Then obviously

$$\iiint_{\xi_i^\epsilon}^{\xi_{i+1}^\epsilon} K^2(t, s_1, s_2) dt\, ds_1 ds_2 < (\pi/2N)^3 M^2$$

Note that for every $\{\xi_j\} = \Omega \supseteq \Omega_\epsilon$ one must have

$$\| \sum_{j=1}^N \Delta_j W[\Delta_j, \Delta_j] \|^2 < N(\pi/2N)^3 M^2$$

It follows then that for the operator under consideration $\int dPWdP = 0$.

For purposes of familiarity we shall single out three specific properties that relate to $\int dPKdP$. First if K is n-power and bounded, then

$$\|\Delta_j K \Delta_j x\| \leqq \|M \Delta_j x\|^n \quad j \in \Omega,\ x \in H \qquad (2)$$

for some scalar M. If $\int dPKdP = 0$ then Ω can be refined such that M can be taken arbitrarily small.

3. Causality Concepts

An n-linear (in particular, linear) operator W on $[H, P^t]$ is said to be causal, (anticausal) if $P^tW = P^tWP^t$, $((I-P^t)W = (I-P^t)W(I-P^t))$. W is memoryless if it is simultaneously causal and anticausal. W is precausal if there exists a partition $\Omega \in \nu$ such that $W = \sum_{i=1}^{N} \Delta_i W P^{i-1}$. W is strictly causal if there exists a sequence of precausal operators $\{W_i\}$, such that $\{W_i\} \to W$, where the convergence is intended in the uniform n-linear operator norm. An n-power operator is causal (precausal, strictly causal) if it is generated by a causal, (precausal, strictly causal) n-linear operator.* Similarly, a multipower operator is causal (precausal, strictly causal) if it is the sum of causal (precausal, strictly causal) n-power components.

Example 1. Consider the bilinear operator $W:[R^2, P^t]^2 \to [R^2, P^t]$ described in example 2.1. This operator is:

causal if $\beta_{ijk} = 0$ $j > i$ or $k > i$;

memoryless if $\beta_{ijk} \neq 0$ only when $j = k = i$;

precausal if $\beta_{ijk} = 0$ whenever $j \geq i$ or $k \geq i$.

Example 2. The bilinear operator $[L_2(\Lambda), P^t]^2 \to [L_2(\Lambda), P^t]$ described in example 2.2 is strictly causal if $K(t, s_1, s_2) = 0$ whenever either s_1 or s_2 is bigger than t. It is precausal if there exists an $\epsilon > 0$

*Note that if W is precausal, then it is also strictly causal and causal; if W is strictly causal then it is causal but not necessarily precausal.

such that $K(t, \Delta_1, \Delta_2) = 0$ whenever either Δ_1 or Δ_2 is bigger than $t - \epsilon$. Memoryless properties would require that $K(t, \Delta_1, \Delta_2) = 0$ unless $\Delta_1 = \Delta_2 = t$ in which case a distribution type behavior of the kernal would be necessary to rescue the example from trivial result $K \equiv 0$.

Note that if W is causal then the relations

$$
\begin{aligned}
\Delta_1 Wx &= \Delta_1 W \Delta_1 x \\
&\vdots \\
\Delta_j Wx &= \Delta_j W[\Delta_j + \Delta_{j-1} + \dots + \Delta_1]x
\end{aligned}
\tag{3}
$$

hold for arbitrary $x \in H$ and any partition Ω. Similarly, if W is memoryless then

$$\Delta_j Wx = \Delta_j W \Delta_j x$$

holds for all $x \in H$ and arbitrary Δ_j.

Proposition 1. If n-power W is causal then for every $t \in \nu$ one has that

$$W(I - P^t) = (I - P^t)W(I - P^t).$$

Proposition 2. ([12], Coro. 4.11). W is memoryless if and only if

$$W = \int dPWdP.$$

Proposition 3. If W is strictly causal then $\int dPWdP = 0$.

Proposition 4. Suppose that T and W are respectively linear and multilinear causal bounded operators. If either T or W is precausal (strictly causal), then TW and WT are also precausal (strictly causal).

Proof. If T is precausal then one can write $T = \sum_{i=1}^{N} \Delta_i TP^{i-1}$. From here, using the causality of W, it follows

$$TW = \sum_{i=1}^{N} \Delta TP^{i-1} WP^{i-1} = \sum_{i=1}^{N} \Delta_i TWP^{i-1} .$$

Thus TW is precausal. Suppose now that T is strictly causal. Then there exists a sequence of precausal operators $\{T_i\}$ such that $\{T_i W\} \rightarrow TW$, where each $T_i W$ is precausal.

Proposition 5. The class of bounded n-linear causal (strictly causal) operators is a Banach algebra.

4. Multipower Operator Equations.

Given the Hilbert space, H, and an element $y \in H$, we will consider the equation

$$x = y + \tilde{W}x \qquad (4)$$

where $\tilde{W} = \sum_{n=1}^{N} W_n$, where W_n denotes an n-power operator. We shall need some causality structure and for this H becomes a Hilbert resolution space $[H, P^t]$. For simplicity and without any loss of generality we shall assume that our identity resolution has no gaps; that is for every $x \in H$ the partition Ω can be refined such that

$$\|\Delta_j x\| < \beta \quad \text{all } j \in \Omega$$

holds for every $\beta > 0$.

The problem of determining the existence and uniqueness of a solution, x, for equations of the above kind has been studied, among others, by Rall [21], McFarland [32], and Prenter [33]. Their results concern the case where the operator $\hat{W}$ and the assigned element y satisfy norm conditions of the type described in the following specialized version of the contraction mapping theorem.* (See Prenter [33].)

Proposition 1. If $\tilde{W} = \sum_{n=1}^{N} W_n$ and $\sum_{n=1}^{N} n\|y\|^{n-1}\|W_n\| < 1$, then eqn. (4) has a well defined solution x. Moreover on its domain the mapping $y \to x$ is continuous and can be computed as the limit of the following series:

$$
\begin{aligned}
x_0 &= y \\
x_1 &= y + \tilde{W}x_0 \\
&\cdots\cdots\cdots\cdots \\
x_i &= y + \tilde{W}x_{i-1}\,.
\end{aligned}
\tag{5}
$$

Corollary 1. If in addition to the hypothesis of Proposition 1 one has that $\tilde{W}$ is causal, then the solution of eqn. (4) is a causal function of y.

The contraction mapping result does not rely on the causality structure. When $\tilde{W}$ does not satisfy the norm condition of Proposition 1 then, at this writing, the existing literature has nothing further to say about the solution properties of eqn. (4). It is easy to demonstrate, however, that when $\tilde{W}$ is precausal then one has the following.

Proposition 2. If $\tilde{W} = \sum_{n=1}^{N} W_n$ is bounded and precausal, then: (i) eqn. (4) defines a continuous and causal mapping $y \to x$; (ii) the mapping $y \to x$ can be computed using the contraction mapping iterations defined by eqns. (5).

The above result, together with our definition of a strictly causal operator as the uniform limit of a sequence of precausal operators, leads to the natural question about whether the precausality condition of Proposition 2

*We shall let W_n denote an n-power operator on H.

might be relaxed into a strict causality condition. To answer this question we will suppose that $\tilde{W}$ is strictly causal and will study existence, causality and continuity properties of the inverse operator $(I + \tilde{W})^{-1}$. In particular, in the following section we will discuss the case where $\tilde{W} = T$ is a linear operator (this case was studied in [17]). We will proceed then to the case where $\tilde{W} = W$ is bilinear (sections 6 and 7) and we will finally extend our results to the more general multilinear case.

5. The Linear Case

One of the main results in [20] was to the effect that if $\tilde{W} = T$ is a linear operator, then Proposition 3.2 can indeed be extended to the strictly causal case.

Proposition 1. ([17], Theorem 4.2). If T is a linear bounded and strictly causal operator then I + T is invertible, the series $\sum_{n=0}^{\infty} T^n$ converges in the uniform operator norm, and

$$(I + T)^{-1} = \sum_{n=0}^{\infty} (-T)^n \quad .$$

For the study of the multipower operator case it is useful to consider the following extension of Proposition 1.

Proposition 2. Suppose that T_o is a linear operator such that $(I + T_o)^{-1}$ exists and is causal and bounded. Then for every sequence of strictly causal linear operators, $\{T_i\}$, $(I + T_o + T_i)$ is invertible and $(I + T_o + T_i)^{-1}$ is causal and bounded. Moreover if $\{T_i\}$ converges to $\tilde{T}$, then $\{(I + T_o + T_i)^{-1}\}$ converges to $(I + T_o + \tilde{T})^{-1}$.

Proof. From the invertibility of $(I + T_o)$, one can write

$$I + T_o + T_i = [I + T_i(I + T_o)^{-1}](I + T_o).$$

To prove the first part of the proposition, it is then sufficient to show that $[I + T_i(I + T_o)^{-1}]$ is invertible and that its inverse is causal and bounded. This follows by observing that $T_i(I + T_o)^{-1}$ is strictly causal, (Proposition 3.3), and by applying Proposition 1. To prove the second part of the proposition, note that $\tilde{T}$ is strictly causal (Proposition 3.4). The result is then obvious from the identity

$$(I+T_o+T_i)^{-1}-(I+T_o+\tilde{T})^{-1} = (I+T_o+T_i)^{-1}(\tilde{T} - T_i)(I+T_o+\tilde{T})^{-1}.$$

Corollary 1. If $\{T_i\}$ is a convergent sequence of strictly causal linear operators, $\{T_i\} \to T_o$, then the sequence $\{(I + T_i)^{-1}\}$ is also convergent with $\{(I + T_i)^{-1}\} \to (I + T_o)^{-1}$.

Corollary 2. If W is strictly causal, then for every $y \in H$, the equation

$$x = u - W[y, u-x]$$

establishes a well defined continuous and causal mapping $u \to x$. This mapping is a continuous causal function of y and W.

6. The Bipower Case

To discuss solution properties of eqn. (4) in the case where $\hat{W}(x) = W[x, x]$ is bipower and strictly causal, we start by establishing existence and uniqueness of the solution of the homogeneous equation.

This result is then used first to obtain uniqueness in the nonhomogeneous case, (Proposition 2), and then causality of $(I + W)^{-1}$; (Proposition 3). Our subsequent results concern continuity properties of $(I + W)^{-1}$, (Propositions 4, 5 and 6).

Proposition 1. If $\hat{W}$ is bipower, bounded, causal and $\int dP\hat{W}dP = 0$. then eqn. (4) has a unique solution for $y = 0$ namely $x = 0$.

Proof. Suppose $x \in H$ such that $0 = x + \hat{W}(x)$. For any partition Ω the causality of $\hat{W}$ implies that

$$\Delta_1 x + \Delta_1 \hat{W} \Delta_1 x = 0$$

and, using eqn. (2),

$$\|\Delta_1 x\| = \|\Delta_1 \hat{W} \Delta_1 x\| \leqq \|M\Delta_1 x\|^2.$$

Refining the partition Ω until $M < 1$ and $\|\Delta_1 x\| < 1$ it is immediate that $\Delta_1 x = 0$. Continuing now we have also

$$\begin{aligned} \|\Delta_2 x\| &= \|\Delta_2 \hat{W}[\Delta_1 + \Delta_2] x\| \\ &= \|\Delta_2 \hat{W} \Delta_2 x\| \\ &\leqq \|M\Delta_2 x\|^2, \end{aligned}$$

which implies $\Delta_2 x = 0$. Repeating the procedure over all partition segments produces the $x = 0$ result.

<u>Proposition 2.</u> Let $\hat{W}$ be bipower, bounded and strictly causal. Then if $y = x_1 + \hat{W}(x_1) = x_2 + \hat{W}(x_2)$ then $x_1 = x_2$.

<u>Proof.</u> Suppose $x_1 \neq x_2$ in the proposition, then

$$x_1 - x_2 = \hat{W}(x_1) - \hat{W}(x_2)$$

$$= W[x_1, x_1] - W[x_2, x_2]$$

where W is symmetric. Using this symmetry

$$x_1 - x_2 = W[x_1 - x_2, x_1 - x_2] - 2W[x_1 - x_2, x_1] .$$

Using the notation $z = x_1 - x_2$ we have

$$(I + 2W[\bullet, x_1])z = W[z, z].$$

However the linear operator $2W[\bullet, x_1]$ is strictly causal and hence from Proposition 5.1, $I + 2W[\bullet, x_1]$ has a bounded causal inverse. Moreover applying Proposition 3.3 it follows that

$$\tilde{W}(\bullet) = (I + 2W[\bullet, x_1])^{-1}\hat{W}(\bullet)$$

satisfies the conditions of Proposition 1. Using the results of that lemma we have $z = 0 = x_1 - x_2$.

<u>Proposition 3.</u> If $\hat{W}$ is a bounded strictly causal operator, then on its domain the map $(I + \hat{W})^{-1}$ is causal.

<u>Proof.</u> Suppose that $y_1, x_1, y_2, x_2 \in H$ exist such that

$$x_i = y_i + \hat{W}(x_i).$$

For $t \in \nu$ suppose $P^t y_1 = P^t y_2$. Then from the above equations we must have that

$$P^t x_2 - P^t x_1 = P^t \hat{W}(x_2) - P^t \hat{W}(x_1)$$

and from the causality of $\hat{W}$

$$P^t x_2 - P^t x_1 = P^t \hat{W}(P^t x_2) - P^t \hat{W}(P^t x_1).$$

Invoking Proposition 2 it is easy to see that this latter equality can be verified if and only if $P^t x_2 = P^t x_1$. This implies $P^t(I + \hat{W})^{-1} = P^t(I + \hat{W})^{-1}P^t$.

Proposition 4. If bipower $\hat{W}$ is bounded and strictly causal then the domain of $(I + \hat{W})$ is an open unbounded set. Moreover, $(I + \hat{W})^{-1}$ on its domain is continuous.

Proof. Since W is homogeneous of order 2, the range of I + W is obviously unbounded. To prove that $(I + \hat{W})^{-1}$ is continuous, and that its domain is open, it is sufficient to show that if u and x are such that

$$x = u + W[x, x] \tag{6}$$

then there exists an $\epsilon > 0$ and an $M > 0$, such that given any $\Delta u \in H$, $\|\Delta u\| < \epsilon$, one can find a $\Delta x \in H$, $\|\Delta x\| \leq M\|\Delta u\|$, such that

$$x + \Delta x = u + \Delta u + W[x + \Delta x, x + \Delta x]. \tag{7}$$

Using the bilinearity of W and equation (6) we have

$$\Delta x = \Delta u + 2W[\Delta x, x] + W[\Delta x, \Delta x]. \tag{8}$$

This we rewrite in the form

$$\Delta x = (I - 2W[\cdot, x])^{-1}\Delta u + (I - 2W[\cdot, x])^{-1} W[\Delta x, \Delta x]$$

where, by Proposition 5.1, $(I - 2W[\cdot, x])^{-1}$ is a bounded causal linear mapping, and, by Proposition 3.3, $(I - 2W[\cdot, x])^{-1} W[\cdot, \cdot]$, is strictly causal.

Applying Proposition 4.1, it follows that if Δu is such that

$$\|(I - 2W[\cdot, x])^{-1}\Delta u\| \cdot \|(I - 2W[\cdot, x])^{-1} W[\cdot, \cdot]\| \leqq \frac{1}{4}$$

then eqn. (7) does indeed have a unique solution. This implies that if

$$4\|\Delta u\| \leqq \{\|(I - 2W[\cdot, x])^{-1}\| \cdot \|(I - 2W[\cdot, x])^{-1} W[\cdot, \cdot]\|\}^{-1} = 4\epsilon$$

then Δx satisfies eqn. (7) and $\|\Delta x\| \leqq M\|\Delta u\|$, where

$$M = 2\|(I - 2W[\cdot, x])^{-1}\|.$$

<u>Proposition 5.</u> If bilinear W is bounded and strictly causal and the pair $u, x \in H^2$ is such that

$$x = u + W[x, x],$$

then there exists an $\epsilon > 0$ such that for all bounded and strictly causal bilinear $\tilde{W}$ satisfying $\|\tilde{W} - W\| < \epsilon$, the equation

$$x' = u + \tilde{W}[x', x']$$

has a unique causal solution. Moreover this solution is a continuous function of $\tilde{W}$ and u.

<u>Proof.</u> Letting $\Delta W = \tilde{W} - W$ and $\Delta x = x' - x$, we start by proving that there exists an $\epsilon > 0$ such that if $\|\Delta W\| < \epsilon$, then the equation

$$x + \Delta x = u + (W + \Delta W)[x + \Delta x, x + \Delta x] \tag{9}$$

has a unique solution Δx. To do this, note that equation (9) is equivalent to

$$\Delta x = 2W[\Delta x, x] + W[\Delta x, \Delta x] + 2\Delta W[\Delta x, x] + \Delta W[\Delta x, \Delta x] + \Delta W[x, x]$$

that is

$$\Delta x = (I - 2(W + \Delta W)[\cdot, x])^{-1} \Delta W[x, x] + (I - 2(W + \Delta W)[\cdot, x])^{-1} (\Delta W + W)[\Delta x, \Delta x] \tag{10}$$

where the inverses exist by virtue of Proposition 5.1. From Proposition 4.1, we have then that eqn. (10) has a solution if

$$\|(I - 2(W + \Delta W)[\cdot, x])^{-1} \Delta W[x, x]\| \cdot \|(I - 2(W + \Delta W)[\cdot, x])^{-1} (W + \Delta W)[\cdot, \cdot]\| \leq \frac{1}{4}.$$

This implies that eqn.(10) has a solution if ΔW satisfies the following inequality

$$\|\Delta W[x, x]\| \leq \frac{1}{4} \frac{1}{\|(I - 2(W + \Delta W)[\cdot, x])^{-1}\|^2 \cdot \|W + \Delta W[\cdot, \cdot]\|} .$$

But, from Proposition 5.2, we can certainly find an $\epsilon > 0$ such that for $\|\Delta W\| < \epsilon$ the above inequality is indeed satisfied.

The continuity of Δx with respect to both u and ΔW follows from Proposition 1 which implies $\|\Delta x\| < \frac{1}{2}\|\Delta W[x,x]\|$ and Proposition 4 which says that x is a continuous function of u.

Proposition 6. Suppose that $\{W_i\}$ is a sequence of strictly causal bilinear operators such that $(I + W_i)^{-1}u$ exists for some $u \in H$. If $\{W_i\} \to W_o$, and $\{(I + W_i)^{-1}u\} \to x_o$, then $(I + W_o)^{-1}u$ also exists and is equal to x_o.

Proof. Denote $\{(I - W_i)^{-1}u\} = \{x_i\}$ and observe that

$$\|x_o - u - W_o[x_o, x_o]\| = \|x_o - x_i + W_i[x_i, x_i] - W_o[x_o, x_o]\|$$

$$\leq \|x_o - x_i\| + \|W_i[x_i, x_i] - W_i[x_o, x_o]\| + \|W_o[x_o, x_o] - W_i[x_o, x_o]\|.$$

If $\{x_i\} \to x_o$, and $W_i \to W_o$, then the second member of the inequality is as small as desired and we can then conclude that

$$x_o = u + W_o[x_o, x_o]$$

The results contained in Propositions 1-6, can be summarized by the following two theorems.

Theorem 1. Let bipower $\hat{W}$ be a bounded and strictly causal operator. Then: (i) $I + \hat{W}$ is one-to-one; (ii) the domain of $(I + \hat{W})^{-1}$ is an unbounded open set; (iii) $(I + \hat{W})^{-1}$ is on its domain causal and continuous.

Theorem 2. Let $\{\hat{W}_i\}$ be a sequence of precausal bipower operators such that $\{\hat{W}_i\} \to \hat{W}$ and $\{(I + \hat{W}_i)^{-1}y\} \to x_o$. Then: (i) $(I + \hat{W})^{-1}y$ is a well defined element in H and $(I + \hat{W})^{-1}y = x_o$; the element $(I + \hat{W})^{-1}y$ is a continuous function of both y and $\hat{W}$.

Corollary 1. If $\hat{W}$ is strictly causal, then a necessary and sufficient condition for the equation

$$x = u + \hat{W}x$$

to have a solution x_o is that for every $\epsilon > 0$ there exists a partition Ω_ϵ such that for every refinement $\Omega > \Omega_\epsilon$ one has

$$\left\| (I + \sum_{i=1}^{n} \Delta_i \hat{W} P^{i-1})^{-1} u - x_o \right\| < \epsilon .$$

Corollary 2. If $\hat{W}$ is strictly causal, then the equation

$$x = u + \hat{W}x$$

has a solution, x_o, if and only if there exists a convergent sequence of precausal approximants $\{\hat{W}_i\} \to \hat{W}$ such that $\{(I + \hat{W}_i)^{-1} u\} \to x_o$.

Corollary 3. Let $\hat{W}$ be strictly causal and suppose that $\{u_i\}$ is a sequence of elements in H such that $(I + \hat{W})^{-1} u_i$ exists for each i. If $\{u_i\} \to u_o$, and $(I + \hat{W})^{-1} u_o$ also exists, then

$$(I + \hat{W})^{-1} u_o = \lim_i (I + \hat{W})^{-1} u_i .$$

7. Results on the Range

We consider the range of the mapping defined by eqn. (4) in the special case where $\hat{W}$ is bipower and strictly causal. To begin with we start by showing that contrary to what might be expected from Proposition 4.2, the operator $I + \hat{W}$ is not necessarily onto. This is accomplished by the following (counter) example.

Example 1. Consider the following equation, $\Lambda = [0, \frac{\pi}{2}]$

$$x(t) = y(t) + [\int_0^t x(s)\, ds]^2, \quad t \in \Lambda \tag{11}$$

where y is an assigned scalar valued function in $L_2(\Lambda)$ and x is the unknown of the equation. Note that the bipower term is generated by a bilinear strictly causal operator of the type described in Example 3. 2. Observe also that for $y(t) = -k^2$ the solution x is absolutely continuous and the related differential equation that models our example is the Riccati equation

$$\dot{g}(t) = k^2 + g^2(t), \qquad g(0) = 0.$$

For this equation there exists a unique solution, namely $g(t) = k \tan(kt)$. It follows that the unique solution to eqn. (11) is given by $x(t) = \dot{g}(t) = k^2/\cos^2(kt)$. Note that for $K = 1$ this solution does not belong to $L_2(\Lambda)$ and therefore $y(t) = -1$ is not in the range of $I + W$.

Our attention turns now to necessary and sufficient conditions to guarantee the onto property.

Proposition 1. Suppose that $\hat{W}$ is strictly causal. If $y \in H$ is not in the range of $I + \hat{W}$, then there exists an element $t \in \nu$ such that for $s < t$ $P^s y$ belongs to the range of $P^s(I + \hat{W})$ and $\lim_{s \to t} \|x_s\| = \infty$, where

$$x_s = \|P^s(I + \hat{W})^{-1} P^s y\|.$$

Proof. Let $t_0 = \sup\{s : P^s y \text{ is in the range of } P^s(I + \hat{W})\}$. Clearly, from Proposition 4. 1, we have $t_0 \neq 0$. Suppose now that $\lim_{s \to t_0} \|x_s\| < \infty$.

Observing that for every $s_n > s_1$ one has

$$x_{s_2} - x_{s_1} = (P^{s_2} - P^{s_1})x_{s_2}$$

there would then exist a well defined element, $P^{t_o}x \in H$, such that $\lim_{s \to t_o} x_s = P^{t_o}x$. Moreover, $P^{t_o}x$ would have the property that

$$P^{t_o}x = P^{t_o}(I + \hat{W})^{-1}P^{t_o}y.$$

This latter equation would imply that $P^{t_o}y$ is in the range of $P^{t_o}(I + \hat{W})$. This implication, however, cannot hold becuase if $P^{t_o}y$ were in the range of $P^{t_o}(I + \hat{W})$, then applying Proposition 6.4 we would have that, for some $t > t_o$, $P^t y$ also is in the range of $P^t(I + \hat{W})$. A contradiction to the sup property of t_o would then be obtained.

Proposition 2. If $\hat{W}$ is a strictly causal bipower operator and there exists a sequence of precausal operators $\{\hat{W}_i\} \to \hat{W}$ such that, for every $y \in H$, the sequence $\{(I + \hat{W}_i)^{-1}y\}$ is bounded, then $I + \hat{W}$ is onto.

Proof. Suppose that y is not in the range of $I + \hat{W}$. Then by Proposition 1 there would exist a $t_o \in \nu$ such that for $s < t_o$ the element $P^s(I + \hat{W})^{-1}P^s y = x_s$ is well defined and $\lim_{s \to t} \|x_s\| = \infty$. Denoting $x_{is} = P^s(I + \hat{W}_i)^{-1}P^s y$, by Proposition 6.5, it would follow that

$$\lim_{\substack{i \to \infty \\ s \to t_o}} \|x_{is}\| = \infty.$$

One would then obtain a contradiction to the hypothesis that the sequence $\{(I + \hat{W}_i)^{-1}y\}$ is bounded.

Proposition 3. If $\hat{W}$ is strictly causal, then the equation $x = y + \hat{W}x$ has a solution $x \in H$ if and only if there exists a sequence of precausal operators $\{\hat{W}_i\} \to \hat{W}$ such that $\{(I + \hat{W}_i)^{-1}y\}$ is bounded.

The above results can be summarized in the following theorem.

Theorem 3. If $\hat{W}$ is a bounded and strictly causal bipower operator, then a necessary and sufficient condition for existence, causality and continuity of $(I + \hat{W})^{-1}$ is that there exists a convergent sequence of precausal operators, $\{\hat{W}_i\} \to \hat{W}$, such that, for every $y \in H$ the sequence $\{(I + W_i)^{-1}y\}$ is bounded.

The next result illustrates a different type of sufficient condition for $I + \hat{W}$ to be onto. In the following $T_x = I - 2W[x, \bullet]$ for every $x \in H$ is a bounded linear operator on H. The adjoint of T_x is denoted in the conventional style T_x^*.

Theorem 4. There exists a sequence $\{x_n\}$ such that

(1) $\|x_n - \hat{W}(x_n) - y\|$ is monotone decreasing

(2) $T_{x_n}^* (x_n - \hat{W}(x_n) - y) \to 0$

(3) $\|x_{n+1} - x_n\| \to 0$.

Corollary. If $T_{x_n}^*$ is uniformily bounded below on a set $\{x : \|x - \hat{W}(x) - y\| < a\}$ where $\inf[a - \|x_n - \hat{W}(x_n) - y\|] > 0$ then the map $x \to x - \hat{W}(x)$ is onto.

Remark. If W is strictly causal then from lemma 5 it follows that for every $x \epsilon H$, T_x is 1:1, onto, and is bounded below. These properties are inherited by T_x^*. However, it is not to the authors' knowledge true that a parameterized family of such operators is necessarily uniformily bounded below.

To prove the theorem consider the functional on H determined by

$$f(x) = \|x - \hat{W}(x) - y\|^2.$$

Expanding via the inner product it is rather straight forward but tedious to determine the first Frechet and second Gateau derivatives of F, namely

$$F^1(x;h) = 2\langle x - \hat{W}(x) - y, (I - 2W[x, \bullet])h\rangle$$

$$\nabla_x F = (I - 2W[x, \bullet])^*(x - \hat{W}(x) - y) = T_x^* a(x)$$

and

$$F^{11}(x;k, h) = 2\langle T_x k, T_x h\rangle - 4\langle a(x), W[k, h]\rangle$$

where $a(x) = x - \hat{W}(x) - y$. The theorem follows immediately then from Goldstein [22], page 125.

8. Extension to the Multipower Case.

The discussion about the solution properties of the multipower equation has thus far been confined to the special case where the operator $\hat{W}$ is bipower. It is to be noted, however, that most of our proofs have been based on Propositions 4.1 and 4.2 which are valid when $\hat{W}$ is given by the sum of n-power operators. It is then natural to ask whether our results might

<u>Remark.</u> If W is strictly causal then from lemma 5 it follows that for every $x \in H$, T_x is 1:1, onto, and is bounded below. These properties are inherited by T_x^*. However, it is not to the authors' knowledge true that a parameterized family of such operators is necessarily uniformily bounded below.

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and

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Theorem 1'. Let $\hat{W}$ be a bounded and strictly causal multipower operator. Then: (i) $I + \hat{W}$ is one to one; (ii) the domain of $(I + \hat{W})^{-1}$ is an unbounded open set; (iii) $(I + \hat{W})^{-1}$ is, on its domain, causal and continuous.

Theorem 2'. If $\{\hat{W}_i\}$ is a sequence of precausal multipower operators such that $\{\hat{W}_i\} \to \hat{W}_0$ and $\{(I + \hat{W}_i)^{-1} y\} \to x_0$, then $(I + \hat{W}_0)^{-1} y$ is a well defined element in H and is equal to x_0.

Theorem 3'. If $\hat{W}$ is a bounded and strictly causal multipower operator, then a necessary and sufficient condition for existence, causality, and continuity of $(I - \hat{W})^{-1}$ is that there exists a convergent sequence of precausal operators, $\{\hat{W}_i\} \to \hat{W}$, such that for every $y \in H$, the sequence $\{(I + \hat{W}_i)^{-1} y\}$ is bounded.

A somewhat more delicate generalization of our results is also available. To see this, suppose that F and G are mappings on $[H, P^t]$ such that

$$m \|\Delta_j (x_1 - x_2)\|^{\alpha} \leqq \|\Delta_j F \Delta_j (x_1 - x_2)\| \qquad (12)$$

and

$$\|\Delta_j G \Delta_j (x_1 - x_2)\| \leqq M \|\Delta_j (x_1 - x_2)\|^n \qquad (13)$$

hold for any pair $x_1, x_2 \in H$ and every suitably refined partition Ω.

Theorem 4. Let F be memoryless and satisfy eqn. (12) and let G be causal and satisfy eqn. (13). If $n > \alpha$ or $n = \alpha$ and $M < m$, then the operator $F + G$ is one to one and on its domain $(F + G)^{-1}$ is causal.

Proof. The proof of causality is identical to that given in connection with Proposition 3. The one-to-one property of F + G is based on the following two propositions.

Proposition 1. If F is memoryless and satisfies eqn. (12) and G is causal and satisfies eqn. (13) and if $n > \alpha$ or $n = \alpha$ and $M < m$, then

$$0 = F(x) - G(x)$$

has the unique solution $x = 0$.

In the proof of Proposition 2 we shall use a result which is easily verified and is contained in the

Proposition 2. If f, g are functions on R satisfying $(mr)^\alpha \leqq f(r)$, and $g(r) \leqq (Mr)^\beta$ for $m, M, \alpha, \beta > 0$ and $0 \leqq r \in R$. If $f(r) = g(r)$ for some $0 \leqq r$:

i) if $\beta > \alpha$ then $r = 0$ or $r \geqq r_0$

ii) if $\beta = \alpha$ and $M < m$ then $r = 0$

where $r_0 = (m^\alpha / M^\beta)^{\frac{1}{\beta - \alpha}}$.

Proof. (Proposition 1) If $F(x) = G(x)$ then using the causality properties of F, G we have

$$\Delta_1 F \Delta_1 x = \Delta_1 G \Delta_1 x$$

and invoking eqn.(12) and eqn. (13)

$$\|m\Delta_1 x\|^{\alpha} \leqq \|M\Delta_1 x\|^{n}.$$

Now since $\{P^t\}$ has no gaps we refine Ω until $\|\Delta_1 x\| < r_0$ the scalar of Proposition 1. Using that proposition then we have $\Delta_1 x = 0$. It remains only to note that

$$\Delta_2 F \Delta_2 x = \Delta_2 G[\Delta_1 + \Delta_2] x$$
$$= \Delta_2 G \Delta_2 x$$

and hence the above argument again applies and continuing over Ω we arrive at $x = 0$.

9. Application.

Consider the operator K: $L_2[0,\infty) \to L_2[0,\infty)$ which is defined according to the following rule: if $u, x \in L_2[0,\infty)$ and $u = Kx$, then

$$u(t) = \sum_{n=1}^{N} \int_0^t \cdots \int_0^t K(t, s_1, \ldots, s_n) x(s_1) \ldots x(s_n)\, ds_1 \ldots ds_n \quad (14)$$

where

$$\int_0^\infty \cdots \int_0^\infty K^2(t, s_1, \ldots, s_n)\, dt\, ds_1 \ldots ds_n < \infty, \quad n = 1, 2, \ldots, N. \quad (15)$$

This operator has been proposed as a mathematical model to analyze the behavior of physical systems such as nuclear reactors [27], magnetic levitation [28], communication channels [29], biological systems, [24], [30], etc. The potential for representing such a variety of systems is that an operator of this form provides a good approximation for much more general nonlinear operators.

This fact is in part summarized in the following lemma.

Lemma [33]. Let C[E] be the set of continuous and causal operators on $L_2[0, T]$ restricted to a compact set E of $L_2[0, T]$ and let the topology in C[E] be determined by the uniform operator norm. Then the family of operators defined by eqns. (14) and (15) is dense in C[E].

Motivated in part by the above results, the properties of the operator K have recently been studied in connection with problems of synthesis [24], identification [25], [26] and Optimal Control [27]. A system configuration of great importance in all these problems is represented in Figure 1, and a first order of business is to analyze the input-output behavior of this type of feedback system. This is equivalent to studying the solution properties of the following equation

$$x(t) = y(t) + u(t) \tag{16}$$

where u(t) is given by eqn. (14).

Observing that the operator K is bounded multipower and strictly causal, the results of the present development can be readily utilized for the better purpose. In particular, Theorem 1' tells us that if eqn. (16) has a solution, x, then this solution is unique and it is a causal function of y. This, of course, does not imply that for every finite energy input, y, the corresponding solution x has necessarily a finite energy nor a finite

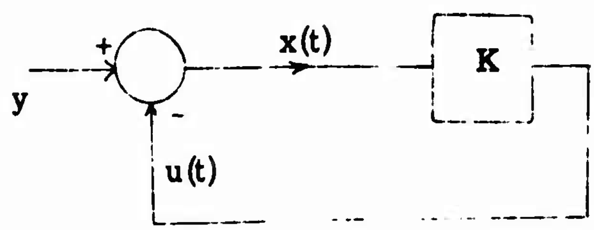

Figure 1. A basic feedback system with multipower open loop chain.

escape time (see counter-example 7.1). What it is possible to say, however, is that the domain of y for which these properties hold is an unbounded open set in $L_2[0,\infty)$. Moreover, if y is in this domain, the solution x is a continuous function of both y and K (theorem 2').

These considerations are relevant not only to our qualitative understanding of the input-output properties of the feedback system, but they do also have practical implications in terms of computational methods for solving eqn. (16).

To illustrate the last assertion let us specifically consider the context developed through examples 2 and 5 of section 2 and example 2 of section 3. We might start by choosing a sequence of positive real numbers $\{\xi_i\} \to 0$ and computing the sequence $\{x_i\}$ of solutions to the equation

$$x(t) = y(t) - \int_0^{t-\xi_i} \cdots \int_0^{t-\xi_i} K(t, s_1, \ldots, s_n) x(s_1) \ldots x(s_n) ds_1 \ldots ds_n$$

Observing that the multipower operator in this equation is bounded and precausal, this step can be carried by using the contraction mapping iterations considered in Proposition 4.1. At this point either one of the following two possibilities can occur. The sequence $\{x_i\}$ may turn out to be unbounded: in this case one can conclude that eqn. (16) does not have a finite energy solution. Alternatively, $\{x_i\}$ is bounded: in this case it must also be convergent with its limit providing the desired solution of eqn.(16).(Theorem 3').

The following example illustrates a large class of practical feedback control systems for which all of the above considerations are applicable.

Example 1. Suppose that in the scalar feedback system represented in Figure 2, the systems T_i, $i = 1, 2$, can be modeled as follows: if $x_i, z_i \in L_2[0,\infty)$ and $z_i = T_i x_i$, then

$$z_i(t) = \int_0^t K_i(t,s)x_i(s)ds$$

where $K_i(t,s)$ are such that

$$\int_0^\infty \int_0^\infty |K_i(t,s)|^2 \, ds\, dt < \infty, \qquad i = 1, 2.$$

Let N be a memoryless polynomial type nonlinearity, that is a nonlinearity such that if $z = Nx$, then

$$z(t) = a_1 x(t) + \ldots + a_n x^n(t)$$

The open loop chain of this feedback system is a bounded, multipower and strictly causal operator.

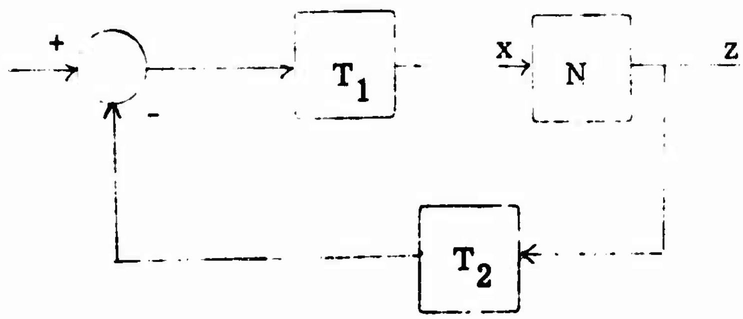

Figure 2. A practical multipower servo system.

10. Conclusions.

In this article we have investigated a number of connections between some special causality properties of a system, K, and questions related to the existence and causality of $(I + K)^{-1}$. This has been done by assuming that K is given by the sum of a finite number of multipower operators defined on a Hilbert resolution space. In analogy with a previous development, [20], which was addressed to the case where K is weakly additive, an important role has been played by the concept of strict causality.

Our results can be briefly summarized as follows. If K is bounded and strictly causal, then I + K is one-to-one and its inverse, when it exists, is causal and continuous (Theorems 1 and 2). This is in line with Theorems 4.1 - 2 in [20]. In contrast to the main conclusion of that study, however, when K is multipower the strict causality assumption is no longer sufficient to insure invertibility of I + K; in particular, as illustrated in Example 1, I + K may fail to be onto. To further investigate the structure of the range of I + K it is then natural to look for some appropriate additional conditions on K. In this regard, Theorem 3 gives a bounded above type necessary and sufficient conditions. Theorem 4 gives a bounded below type sufficient conditions.

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