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MANPOWER PLANNING WITH UNCERTAIN REQUIREMENTS

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by

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ABSTRACT

This paper examines a longitudinal model of a manpower system in which the demands for effective manpower is determined by the state of a finite Markov chain, and there are delays in training an adequate supply of effective manpower. We present an operational method of calculating optimal appointment policies. This calculation can in turn be used to find the equilibrium operating rules for the system. The model is a useful device for measuring the impact of alternate assumptions about continuation rates, manpower utilization policies, the demands, and the transition probabilities in the demand process.

0. INTRODUCTION

This paper considers the problem of providing an adequate supply of effective manpower when demand for manpower is uncertain and there are time lags in the training of effective personnel. The model described in this paper is useful in three specific ways: (i) it can calculate optimal manpower input policy for alternate control objectives, (ii) it can be used to measure the effects of alternate training, retention, and utilization policies on system performance, and finally (iii) it can measure the sensitivity of optimal policies and system performance to various assumptions on the nature of the stochastic nature of the demand process.

In Section 1 we present a model for the flow of personnel. This model attempts to capture the essential features of the manpower system in a simple framework by utilizing the longitudinal stability of various manpower cohorts. The stochastic nature of the demand process is described in Section 2. At time t, the demand for trained manpower is determined by the state of a finite Markov chain. In Section 3 we present two alternate control objectives. The first objective is to minimize the expected, weighted, discounted square of the error between the effective supply and demand for manpower. The second objective is to minimize the expected discounted square error between the actual manpower stocks and ideal stocks for any demand state. In Section 4 we describe the form of the optimal policy and optimal value function. Numerical solutions are presented for the two control objectives. In Section 5 we outline several uses for the model. The first is the determination of the expected manpower stocks when the optimal appointment policy is used. Calculation of the equilibrium stocks can then be used to calculate the expected cost of the appointment policy along with some expected measure of system performance. These calculations allow us to measure the tradeoff between cost and performance for alternate manpower policies. In Section 5 we also consider the effect of errors in the specification of the stochastic law of motion. We find it is relatively easy to discover the effects of these errors. Finally in Section 5 we consider the effect of possible alterations in the stock of manpower, either by releasing people from the system or allowing personnel inflow that has a past accumulation of experience.

The appendix is devoted to the theoretical aspect of the model we find that the optimization procedure is a hybrid of the linear-quadratic optimal control problem ([1],[4],[7]) and the Markov decision problem. We are able to show that the optimal decision rules are linear and that the optimal policy functions are quadratic. In addition, under special assumptions on the problem data we can show that the optimal decision rules converge to an equilibrium optimal appointment policy.

The model was motivated by the problem of regulating the supply of naval aviators. We do not pretend to present an operational solution to that real problem, however, a numerical example, using the naval aviator problem as a setting, is used throughout the paper. The author is the sole source of the data for the numerical example.

The paper is based on a longitudinal manpower flow model; Oliver and Hopkins [5],[6], that has been used in a similar context by Grinold, Marshall, and Oliver, [2], and Grinold and Oliver, [3].

1. THE FLOW OF MANPOWER

This section describes the basic manpower flow process. For a more detailed description of a related deterministic model see [2] and [3].

The state of the manpower system is observed at discrete equally spaced time points $t = 0,1,2,3, \ldots, T$; where T is a planning horizon. The time interval $(t - 1,t]$ after time $t - 1$ and up to and including time t will be called *period* t .

At each time t individuals in the system are classified according to their *length of service* (LOS). The length of service is simply the number of occasions the individual has been classified. Thus individuals entering in period t will have length of service equal to one at time t , since they are in their first period of service at time t . Individuals entering in period $t - 1$ that are still present at time t will be in their second period of service. We assume that m is the maximum number of periods an individual can remain in the system.† It follows that individuals in the system at time t in their m^{th} period of service must leave the system in period $t + 1$.

The stock of manpower at time t is described by the length of service distribution $s(t)$; $s(t)$ is a vector with m components, where $s_k(t)$ is the number of individuals at time t with LOS equal to k .

$$s(t) = [s_1(t),s_2(t), \ldots, s_m(t)]' \tag{1}$$

In each period t a number of new *accessions* $f(t)$ enter the manpower system. The change in the manpower system from time t to time $t + 1$ depends on three factors: the stock at time t , the accessions in period $t + 1$, and the *continuation rates* of the stock of manpower at time t . The continuation rate q_k for $k = 1,2, \ldots, m - 1$ is defined to be the fraction

†Caution: in [2] and [3] the length of service is defined to be the number of completed periods of service, and is then one less than the measure used here.

of manpower with LOS equal to k, that remains in the system an additional period. With this definition we have

$$(2) \qquad s_{k+1}(t+1) = \begin{cases} f(t+1) & \text{if } k = 0 \\ q_k s_k(t) & \text{if } k = 1, 2, \ldots, m-1 \end{cases} .$$

The continuation rates are intended to include several factors that influence the flow of manpower: the organization's retention, promotion, and retirement policy, natural forces such as mortality, and the behavior of personnel.

2. THE DEMAND AND SUPPLY OF EFFECTIVE MANPOWER

This section describes the stochastic demand process and the related deterministic supply process.

The demand Z_t for effective manpower is a random variable that can take on a finite number of values $z_1, z_2, \ldots, z_n$. The stochastic law of motion that governs the evolution of the process z_t is Markovian:

$$(1) \qquad P_r[Z_{t+1} = z_i \mid Z_t = z_j] = r_{ij}$$

We assume r_{ij} is independent of t, and that the uncertain demand z_t persists through period $t + 1$.

The example below is a six state system and the manpower category is naval aviators. State 1 of the Markov chain represents a calm peacetime period, while State 2 represents a higher condition of peacetime preparation. States 3, 4, 5, and 6 represent various stages of a conflict. Notice the conflict may end, by a return to State 1 or 2, may stay at the same stage or it may proceed to a higher stage. The transition probabilities are shown below along with the demand for effective manpower and the equilibrium distribution of the Markov chain.

Let $i(t)$ be the state of demand at time t : i.e. $Z_t = z_{i(t)}$. The state of the system at time t depends on two factors: the stock of manpower $s(t)$, and the state of demand, $i(t)$.

We shall now generalize the nation of continuation rates introduced in (I;2), by allowing the continuation rates to depend on the state of demand. Thus we have continuation rate q_{ik} and the law of motion for manpower is

$$(3) \qquad s_{k+1}(t+1) = \begin{cases} f(t+1) & \text{if } k = 0 \\ q_{ik} s_k(t) & \text{if } k = 1, 2, \ldots, m-1 \end{cases} .$$

TABLE 1

	1	2	3	4	5	6	z	Π
1	0.90	0.07	0.03				500	0.783
2	0.5	0.4	0.1				900	0.147
3	0.05	0.5	0.05	0.4			1200	0.040
4	0.1	0.35		0.2	0.35		1200	0.020
5	0.1	0.6			0.1	0.3	1200	0.008
6	0.1	0.7				0.2	1000	0.002

TABLE 2

		Length of Service						
		1	2	3	4	5	6	7
Demand State	1	0.9	0.9	0.95	0.95	0.95	0.1	0.9
	2	0.9	0.9	0.95	0.95	0.95	0.2	0.9
	3	0.95	0.95	0.9	0.9	0.8	0.8	0.9
	4	0.95	0.9	0.9	0.9	0.9	0.9	0.9
	5	0.95	0.8	0.9	0.8	0.8	0.9	0.9
	6	0.95	0.9	0.9	0.9	0.9	0.9	0.9

This allows for alternate retention, mortality, and behavior assumptions for each state of the demand process. In this paper we shall solve numerical examples with $n = 6$ states and $m = 8$ maximum years of service. The continuation rates are shown in Table 2.

We can utilize personnel in alternate ways depending on the demand environment. Let d_{ik} be the contribution to effectiveness of each individual in the system with length of service equal to k when the demand environment is i. The total effectiveness is then

$$(4) \qquad \sum_{k=1}^{m} d_{ik} s_k(t)$$

and the excess of the supply of effective manpower is given by

$$(5) \qquad \sum_{k=1}^{m} d_{ik} s_k(t) - z_i \ .$$

We shall allow the d_{ik} to be nonnegative. Consider the problem of supplying effective naval aviators. Individuals in the first period of service, are being trained to fly; thus they are not making any contribution to meeting the demand for naval aviators. In addition, it takes trained and qualified pilots to teach new recruits. Suppose one teacher is needed for every two pupils in state i. Then $d_{i1} = -.5$. Every two pilot trainees take up the services of one instructor who is therefore not eligible to meet the demand for operational pilots. The contributions to effectiveness for the problems solved in this paper are seen in Table 3.

TABLE 3

Utilization

Demand	Length of Service 1	2	3	4	5	6	7	8
1	¯0.5	¯0.5	1	1	0.8	0.8	0.2	0.2
2	¯0.5	¯0.5	1	1	1	0.8	0.3	0.2
3	¯0.5	¯0.25	1	1	1	1	0.8	0.8
4	¯0.5	¯0.3	1	1	1	1	0.8	0.8
5	¯0.5	¯0.3	1	1	1	1	0.8	0.8
6	¯0.5	¯0.3	1	1	1	1	0.8	0.8

3. THE CONTROL OBJECTIVE

This section formulates two quadratic objective functions for controlling the flow of manpower through the system. We selected quadratic objectives for two reasons. First, they lead to a model for which results can be readily calculated. Second, the objectives are a reasonable measure of the departure of actual system performance from an ideal performance.

The simplest control objective is to minimize the expected weighted squared error between the supply and demand for effective manpower. Let w_j for $j = 1,2, \ldots, n$ be positive weights that sum to 1. If $i(t) = i$, and $s(t)$, then the expected error at time $t + 1$, as a function of $f = f(t + 1)$, is

$$E^i[s(t),f(t + 1)] = \sum_{j=1}^{n} r_{ij} w_j \left(z_j - f(t + 1) - \sum_{k=1}^{m-1} q_{jk} s_k(t)\right)^2 \tag{1}$$

Our second control objective is to provide a smooth flow of manpower through the system and to come as close as possible to meeting the demand for effective manpower. If the manpower system was always in state j, then an ideal distribution would satisfy

$$\sum_{k=1}^{m} d_{jk} s_k = z_j , \tag{2}$$

and

$$s_{k+1} = \begin{Bmatrix} f & \text{if} \quad k = 0 \\ q_{jk} s_k & \text{for} \quad k = 1,2, \ldots, m - 1 \end{Bmatrix} .$$

Note that (2) can be rewritten as

$$s_{k+1} = \begin{Bmatrix} f & \text{if} \quad k = 0 \\ \left[\prod_{\ell}^{k} q_{j\ell}\right] f & \text{for} \quad k = 1,2, \ldots, m - 1 \end{Bmatrix} . \tag{3}$$

For $k = 2,3, \ldots, m$ define

$$(4) \qquad P_{jk} = \prod_{\ell=1}^{k-1} q_{jk}$$

and $P_{j1} = 1$.

Combining (4), (3), and (2) we obtain

$$(5) \qquad s_{jk} = (z_j P_{jk}) / \sum_{k=1}^{m} P_{jk} d_{jk}$$

The discrepancy between actual and ideal is measured by

$$(6) \qquad \sum_{k=1}^{m} (s_k(t + 1) - s_{jk})^2$$

Using the law of motion (II;3) we can rewrite (7) as

$$(8) \qquad E^{ij}[s(t),f(t + 1)] = (f(t + 1) - s_{j1})^2 + \sum_{k=1}^{m-1} (q_{ik} s_k(t) - s_{jk+1})^2$$

This is a measure of the discrepancy at time $t + 1$ conditioned on four facts; manpower $s(t)$ at time t , demand i at time t , accessions $f(t + 1)$ in period $t + 1$, and demand j at time $t + 1$. The function is quadratic, strictly convex in $f(t + 1)$ and convex in $s(t)$. The expected discrepancy is

$$(9) \qquad E^{i}[s(t),f(t + 1)] = \sum_{j=1}^{n} r_{ij} E^{ij}[s(t),f(t + 1)]$$

is also quadratic convex in f (strictly) and s .

We note that it is possible to weigh the functions $E^{w}[s(t),f(t + 1)]$ in order to place greater or less importance on the expected discrepancy from state j .

For either objective in a T stage problem, where $s(t)$, and $f(t)$ are the observed values of the manpower stocks and accessions for $t = 1,2, \ldots, T$, and $i(t)$ is the state of demand, the total discounted error is

$$(10) \qquad \sum_{t=1}^{T} \delta^{t-1} E^{i(t)}[s(t),f(t)]$$

where $0 < \delta < 1$ is a discount factor. The objective is to minimize the expected value of (10) over all possible appointment sequences.

4. THE OPTIMAL APPOINTMENT POLICY

This section presents two examples of optimal appointment policies: one for each of the objectives presented in Section 3.

A theoretical description of the optimization procedure is presented in the appendix. We shall use three important results from the appendix:

(i) For any finite planning horizon T, and any demand state i, there is an optimal appointment policy that is a linear function of the manpower stock s. For our model

$$f = b_0^{i,T} + \sum_{k=1}^{m} b_k^{i,T} s_k \ .$$

(ii) For any finite planning horizon T, and any demand state i, the minimum expected error is a convex quadratic function of the manpower stocks s:

$$s'P^{i,T}s + q^{i,T}s + r^{i,T}$$

(iii) As the planning horizon T becomes large, the values of $b^{i,T}$, $P^{i,T}$, $q^{i,T}$, $r^{i,T}$, converge to b^i, P^i, q^i, r^i.

In the examples we have solved to date, the convergence has been quite rapid. For the first example below a 1% change was observed between a 20 and 21 period horizon. In the second example convergence was nearly perfect in 5 periods.

The first solution below uses the objective (3:1) with discount factor $\delta = 0.95$ and weights

$$w_j = \begin{Bmatrix} 1/18 & \text{if} & j = 1,2 \\ 4/18 & \text{if} & j = 3,4,5,6 \end{Bmatrix} .$$

The optimal appointment policy is shown in Table 1.

For the second control objective, (3:9), there is a considerable simplification in the optimal decision rule: $f = b_0^i + \sum_{k=1}^{m} b^i{}_k s_k$. In the second case $b_k^i = 0$ for $k \geq 1$. Thus $f = b_o^i$ independently of the current manpower stock s.

The optimal input as a function of i is shown in Table 2. In this calculation we set $\alpha = 0.95$ and did not weigh the various terms in the sum (3:9).

TABLE 1

i	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
1	667	¯.95	¯.17	¯.14	.25	.30	0.0	.10	0
2	604	¯.95	¯.16	¯.12	.27	.31	.03	.16	0
3	285	¯.97	.04	.08	.37	.35	.29	.39	0
4	105	¯.91	.04	.21	.48	.47	.40	.46	0
5	345	¯.95	0.00	.06	.36	.36	.28	.32	0
6	475	¯.98	¯.08	¯.03	.34	.36	.22	.25	0

TABLE 2

i	b_0
1	304
2	332
3	370
4	374
5	357
6	354

5. COST PERFORMANCE TRADEOFFS

This section examines the use of the manpower planning model presented in earlier sections to measure the tradeoff between cost and system performance. We shall consider this tradeoff in the specific context of the naval aviation problem. In particular, we shall assume that costs are determined by the distribution, $s(t)$, of manpower stocks.

When an optimal linear appointment policy has been calculated, then it is possible to express the evolution of the manpower stocks in a linear fashion

$$(1) \qquad s_{k+1}(t+1) = \begin{cases} b_{io} + \sum_{j=1}^{m} b_{ij} s_j(t) & \text{if } k = 0 \\ q_{ik} s_k(t) & \text{if } k = 1, 2, \ldots, m-1 \end{cases}$$

This can be written in matrix form

$$(2) \qquad s(t+1) = A^i s(t) + h^i \ .$$

The equilibrium expected values of s as a function of the demand state j can be calculated.

Let ψ_{ji} be the probability that the previous demand state was i given that the current demand state is j. These are the inverse transition probabilities for the Markov chain. When the Markov chain is in equilibrium, a simple application of Bayes's Law gives

$$(3) \qquad \psi_{ji} = \frac{\pi_i r_{ij}}{\pi_j}$$

where $\pi = (\pi_1, \ldots, \pi_n)$ is the stationary distribution vector of the Markov chain.

Now let s^j for $j = 1, 2, \ldots, n$ be the expected manpower stock in demand state i when an optimal linear appointment policy is used.

The s^j must satisfy

(4) $$s^j = \sum_{i=1}^{n} \psi_{ji}(A^i s^i + h^i)$$

for $j = 1, 2, \ldots, n$.

This system of $n \times m$ equations with $n \times m$ unknowns can be solved for the equilibrium expected values of s^j . For the first objective (3:1), and the appointment policy in Table 1 of Section 4, we obtain the equilibrium expected values following in Table 1.

For the second objective (3:9) the expected equilibrium stock levels are shown in Table 2.

With these figures, suitable cost data and the vector π , it is easy to calculate the expected cost per period of operating this manpower system using the optimal linear appointment policy. Let c_{ik} be the cost of an individual in the k^{th} period of service. The equilibrium expected cost per period is then

(5) $$\sum_{i=1}^{n} \pi_i \sum_{k=1}^{m} c_{ik} s_k^i \ .$$

We can also construct a measure of system performance. Suppose we choose the expected weighted squared discrepancy between the demand and the steady state expected supply level. This would be

(6) $$\sum_{i=1}^{n} \pi_i w_i \left(z_i - \sum_{k=1}^{m} d_{ik} s_k^i\right)^2$$

With a measure of expected system performance and a measure of expected cost, we compare alternate manpower policies. For example, it is possible to change the demands z_j and recalculate a new optimal appointment strategy. We can also change the continuation rates q_{ik} , or the utilization policy d_{ik} ; note these changes would probably produce a corresponding change in the costs c_{ik} . It is also possible to change the objective function by either

TABLE 1

i	1	2	3	4	5	6	7	8
1	371	332	298	283	268	254	29	30
2	350	325	296	279	262	243	73	57
3	362	328	297	282	267	251	42	41
4	285	329	308	270	251	216	199	66
5	376	279	289	276	240	223	195	179
6	328	348	241	252	222	194	195	175

TABLE 2

i	1	2	3	4	5	6	7	8
1	307	279	252	241	229	217	25	26
2	330	295	262	243	226	209	63	49
3	318	285	257	243	229	215	36	35
4	371	312	273	234	217	186	171	57
5	373	353	281	246	209	192	168	153
6	357	351	289	254	203	171	169	151

using alternate weights or an entirely different objective function. Using our model, and trying alternate manpower policies we can discover, and explore the cost-performance tradeoffs that exist.

There is a second type of sensitivity analysis. If the probabilities in the Markov matrix are changed, this amounts to an alternate assumption about the stochastic nature of the demand process. By recalculating for several different values of the transition probabilities we can measure the impact of alternate assumptions concerning the demand process. A related question is to measure the impact of an incorrect assumption about the demand process. This leads to an easier calculation. Suppose we assume the transition probabilities are r_{ij} , when they actually are $\hat{r}_{ij}$ with equilibrium distribution vector $\hat{\pi}$. In this case the appointment policy will not change. However, the calculations (3)-(6) would have to use the values $\hat{r}_{ij}$ and $\hat{\pi}_i$. In this way we can measure the effect of incorrectly specifying the probabilities.

In some cases it is possible to alter the manpower stock in an ad-hoc fashion. In the language of the naval aviator problem, we can either bring in reserves or allow people in the system to leave before their obligated service is completed. If possibilities of this sort exist, then some of the information we have calculated can help in designing an alteration of the manpower stock. Recall that the minimum expected error starting in demand state i with stock s is

$$sP^i s + q^i s + r^i \quad (7)$$

The increase in this optimal expected error as a function of an increase in s_k is simply

$$2 \sum_{j=1}^{m} s_j P^j_{jk} + q^i_k .$$

This tells how an increase or decrease in s_k will effect the minimum expected error.

In addition, if the current demand state is i, then the current manpower stock s should be contrasted with its equilibrium value s^i. It is in the interest of system stability to change s closer to its equilibrium value.

It is even possible to calculate an optimal value for s when the demand state is i. The optimum is simply $-1/2(P^i)^{-1}(q^i)'$, the s that minimizes (7). The optimal values are not always realistic, as we indicate below. However, they do indicate a direction in which favorable changes can be made.

For the objective (3:1) we find the following optimal distributions shown in Table 3. Note any value of s_{18} is optimal.

The distributions for demand states $i = 1,2$ are unreasonable, probably close to the low weight attached to the objective for those states. The distribution for States 3 and 4 is quite good. It indicates that a large increase in the three year group would have a favorable impact, as well as a large increase in the six year group.

The optimal distributions for the second objective (3:9) are more reasonable.

Those values shown in Table 4 should be contrasted with those of the equilibrium distribution in Table 2.

TABLE 3

i	1	2	3	4	5	6	7	8
1	288	32	474	367	¯16	7290	104	0
2	331	¯61	481	554	131	2720	47	0
3	302	159	514	401	268	437	39	0
4	363	254	415	350	306	271	150	0
5	242	244	583	398	137	369	¯102	0
6	184	169	682	392	¯6	412	¯201	0

TABLE 4

i	1	2	3	4	5	6	7	8
1	303	272	243	228	213	296	27	0
2	334	303	274	263	249	256	54	0
3	360	332	328	295	280	143	131	0
4	379	355	341	299	217	130	143	0
5	350	377	323	342	274	91	98	0
6	342	330	318	305	266	80	83	0

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APPENDIX

DISCRETE TIME, LINEAR-QUADRATIC, DYNAMIC PROGRAMS WITH MARKOVIAN PARAMETERS

This appendix presents, in a more general form, the theoretical results used in the paper. We employ a notation close to that of the text.

The state of the system with $T + 1$ periods remaining is (s,i). Given action f, the state of the system with T periods remaining will be $(D^i(s,f),j)$ with probability r_{ij}, where, for each i, $D^i(s,f)$ is a linear function of s and f.

The error associated with choosing f in state (s,i) is $E^i(s,f)$, where for each i $E^i(s,f)$ is quadratic and convex in s and strictly convex in f. Future errors are discounted at rate δ.

For $i = 1,2, \ldots, n$ define $V_T(s,i)$ to be the minimum expected discounted error for a T period problem that starts with manpower stocks s, and demand state i.

For any choice of f, we have

$$(1) \qquad V_{T+1}(s,i) \leq E^i(s,f) + \delta \sum_{j=1}^{n} r_{ij} V_T(D^i(s,f),j)$$

By the usual principal of optimality arguments we can establish that the functions $V_{T+1}(s,i)$ must satisfy the following functional equation.

$$(2) \qquad V_{T+1}(s,i) = \min_f \left[E^i(s,f) + \delta \sum_{j=1}^{n} r_{ij} V_T(D^i(s,f),j) \right]$$

where $V_o(s,i) = 0$ for each s and i.

Now assume, that $V_T(s,i)$ is a convex quadratic function of s for each i. Thus we can write $V_T(s,i) = s'P^{i,T}s + q^{i,T}s + r^{i,T}$ where $P^{i,T}$ is positive semi-definite, and symmetric. It is easy to show that $V_T(D^i(s,f),j)$ is a convex quadratic function of (s,f), and therefore that

$$E^i(s,f) + \delta \sum_{j=1}^{n} r_{ij} V_T(D^i(s,f),j) \tag{3}$$

is quadratic, convex in s, and strictly convex in f.

When the quadratic function (3) is differentiated with respect to f, then a linear equation results, and the unique f that minimizes (3) as a function of s is in the form

$$f_{T+1}(s,i) = B^{i,T+1}(s) \tag{4}$$

where for each i and $T+1$, $B^{i,T+1}(s)$ is linear in s. When the optimal policy (4) is substituted into (3) we obtain a convex quadratic expression for $V_{T+1}(s,i)$.

Thus for any finite horizon T the optimal value and policy functions are, respectively, linear and quadratic. We now turn to the question of convergence of policies as the planning horizon increases.

Let $V_T(s,i) = s'P^{i,T}s + q^{i,T}s + r^{i,T}$ be the optimal value function starting in state i for an ℓ period problem.

When the discount factor δ is strictly less than 1, then the sequence $V^T(s,i)$ will remain bounded if there exists a policy $f(s,i)$ such that this policy leads to a bounded sequence $s(t)$ for any initial $s(o)$ and any evolution of the demand sequence $i(t)$. In the model described in this paper, the policy $f(s,i) = 0$ will insure $s(t) = 0$ for $t > m$. The general question of convergence of $V^T(s,i)$ and $B^T(s,i)$ is difficult, and is currently the object of further investigation by the author.