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THE OMEGA-VALUE OF A TWO-BY-TWO MATRIX-DIFFERENTIAL GAME

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ABSTRACT

In this technical report, a general formula for the Ω-value of a 2×2 matrix-differential game is derived, using two different methods.

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1. Introduction.

Consider a two-person zero-sum game with matrix $[a_{ij}]$ which is being played for one unit of time. If in a time interval Δt player 1 plays row i and player 2 column j the payoff to player 1 is $a_{ij}\Delta t$. Payoffs are accumulated with time so that the total payoff is

$$\int_0^1 a_{ij}\, dt.$$

Of course, if instantaneous changes of rows or columns were allowed the integral would not be defined. John Danskin [1] has proposed to resolve this dilemma by approximating the game by n-stage games with perfect information. In the n-stage approximation, player 2 is allowed to change the column index only at times $t = \frac{k}{n}$; $k = 0,1,\ldots,n-1$, while player 1 is allowed to change the row index only when $t = \frac{k}{n} + \frac{\sigma}{n}$, where $0 \le \sigma \le 1$ is a fixed parameter. The payoff is then

$$\int_0^1 a_{ij}\, dt = \frac{1}{n} \sum_{k=1}^{n} \sigma a_{i_{k-1}j_k} + (1-\sigma)a_{i_k j_k}.$$

The row index i_0 is called the predecessor. We denote this game $G_n(\sigma;i_0)$ and call it the n-stage Ω-game with predecessor i_0. Its value

$$\Omega_n(\sigma;i_0) = \min_{j_1} \max_{i_1} \ldots \min_{j_n} \max_{i_n} \frac{1}{n} \sum_{k=1}^{n} \sigma a_{i_{k-1}j_k} + (1-\sigma)a_{i_k j_k} \tag{1.1}$$

John Danskin has proved (in a far more general context) that the sequence

$$\Omega_1(\sigma;i_0), \Omega_2(\sigma;i_0), \ldots \tag{1.2}$$

converges to a limit which does not depend on the predecessor i_0. This limit

$$\Omega(\sigma) = \lim_{n\to\infty} \Omega_n(\sigma;i_0)$$

is then called the Ω-value of a matrix-differential game $[a_{ij}]$.

The main purpose of this technical report is to establish the following formula for the Ω-value of a two-by-two game.

Let $A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ be a matrix, let $I = \{1,2\}$ be the index set, let $0 \le \sigma \le 1$. Denote $\bar{M} = \max_{i\in I} \max_{j\in I} \{a_{ij}\}$, $\underline{M} = \min_{i\in I} \min_{j\in I} \{a_{ij}\}$,

$\bar{V} = \min_{j\in I} \max_{i\in I} \{a_{ij}\}$, $\underline{V} = \max_{i\in I} \min_{j\in I} \{a_{ij}\}$, and

$$v(\sigma) = \frac{1}{2}[\sigma(\underline{M}+\underline{V}) + (1-\sigma)(\bar{M}+\bar{V})].$$

Then

$$\Omega(\sigma) = \begin{cases} \underline{V} & \text{if } v(\sigma) \le \underline{V}, \\ v(\sigma) & \text{if } \underline{V} < v(\sigma) < \bar{V}, \\ \bar{V} & \text{if } v(\sigma) \ge \bar{V}. \end{cases} \tag{1.3}$$

We provide two independent proofs of this formula, each using a different approach--Shubert's (Sec. 2) or Washburn's (Sec. 3). These two sections can be read independently. The notation is common. A brief comparison and discussion of the two approaches is contained in Sec. 4, together with some ideas about possible generalization to larger games.

2. Shubert's Approach.

From (1.1) it is easily seen that $\Omega_n(\sigma;\cdot)$ satisfy the recurrence relation

$$\Omega_{n+1}(\sigma;k) = \min_{j\in I}\max_{i\in I}\{\frac{1}{n+1}[\sigma a_{kj} + (1-\sigma)a_{ij}] + \frac{n}{n+1}\Omega_n(\sigma;i)\}, \quad k \in I, \qquad (2.1)$$

with initial condition $\Omega_0 \equiv 0$. This relation could be used to evaluate $\Omega(\sigma)$, which can be done on a computer provided numerical values for σ and the entries a_{ij} are given.

However, if we wish to obtain the entire function $\Omega(\cdot)$ and establish some of its properties, the recurrence relation (2.1) may be of little help. An analogy with similar problems suggests that perhaps $\Omega(\cdot)$ can also be found as a solution of a functional equation obtained from (2.2) by letting $n \to \infty$. Unfortunately, such a process would yield the equation

$$\Omega(\cdot;k) = \min_j \max_i \Omega(\cdot;i),$$

which is satisfied by any function such that $\Omega(\sigma;k) = \Omega(\sigma;\ell)$ for $k \neq \ell$.

In what follows we are going to show that the difficulty can nevertheless be circumvented by solving the equation

$$\Omega(\sigma;k) = \min_{j\in I}\max_{i\in I}\{\lambda[\sigma a_{kj}+(1-\sigma)a_{ij}] + (1-\lambda)\Omega(\sigma;i)\}, \quad k \in I, \qquad (2.2)$$

where $0 < \lambda \leq 1$ is a parameter.

We first prove that this equation has a unique solution, depending, of course, on the parameter λ. Next we show that as $\lambda \to 0$ this solution converges to the Ω-value. As a byproduct we obtain a new proof for the convergence of the sequence (1.2). Finally, by actually finding the solution of (2.2) for small $\lambda > 0$ and letting $\lambda \to 0$ we establish the formula (1.3) for the Ω-value.

We begin with some definitions. To simplify notation we temporarily suppress σ and abbreviate

$$h_{ij}^k = \sigma a_{kj} + (1-\sigma)a_{ij},$$

where $i \in I$, $j \in I$, $k \in I$.

Let $\underline{x} = (x_1,x_2) \in Re^2$ be a two component vector, let $0 \leq \lambda \leq 1$. Denoting

$$H_{ij}^k(\lambda,\underline{x}) = \lambda h_{ij}^k + (1-\lambda)x_i$$

$$i \in I, \quad k \in I, \quad j \in I,$$

we define a parametric family of mappings

$$T(\lambda,\underline{x}) = (T_1(\lambda;\underline{x}),T_2(\lambda,\underline{x}))$$

by (2.3)

$$T_k(\lambda,\underline{x}) = \min_{j \in I} \max_{i \in I} H_{ij}^k(\lambda,\underline{x}); \quad k \in I.$$

Let $\underline{x}_1,\underline{x}_2,\ldots$ be a sequence of vectors from Re^2 defined by

$$\underline{x}_1 \in Re^2, \quad \underline{x}_{n+1} = T(\tfrac{1}{n},\underline{x}_n), \quad n = 1,2,\ldots \tag{2.4}$$

Comparing with (2.1) we see that with $\underline{x}_1 = \underline{0}$

$$\underline{x}_{n+1} = (\Omega_n(\sigma;1),\Omega_n(\sigma;2)),$$

so that $\underline{x}_1,\underline{x}_2,\ldots$ is indeed the sequence defining the Ω-value of the Ω-game with matrix A.

Next, let $||\cdot||$ be the maximum norm on Re^2

$$||\underline{x}|| = \max\{|x_1|,|x_2|\},$$

let α be a constant such that

$$\max_{i\in I} \max_{j\in I} |a_{ij}| \leq \alpha.$$

Let

$$X = \{\underline{x} \in Re^2 : ||\underline{x}|| \leq \alpha\}$$

We are now ready to state the following theorem

Theorem 1: Let $0 \leq \lambda \leq 1$, let T be the mapping defined by (2.3) and restricted to the domain X, let $\underline{x}_1, \underline{x}_2, \ldots$ be the sequence (2.4). Then the following is true.

(1) For every $0 < \lambda \leq 1$. there is a unique fixed point

$$\underline{\tilde{x}}(\lambda) = T(\lambda, \underline{\tilde{x}}(\lambda)).$$

(2) The limit

$$\lim_{\lambda\to 0} \underline{\tilde{x}}(\lambda) = \underline{\xi} = (\xi_1, \xi_2)$$

exists and $\xi_1 = \xi_2$.

(3) The sequence $\underline{x}_1, \underline{x}_2, \ldots$ converges and

$$\lim_{n\to\infty} \underline{x}_n = \lim_{x\to 0} \underline{\tilde{x}}(\lambda).$$

Proof: The proof is divided into a sequence of eight lemmas. Lemma 3 proves statement (1), Lemma 7 statement (2) and Lemma 8 statement (3). We would like to point out that the proof may seem rather uneconomical for the two-by-two case (for instance, Lemma 3 can be proved without using Banach's theorem).

We, nevertheless, leave it in its present form with an eye on possible generalization.

Lemma 1: For any $0 \le \lambda \le 1$

$$\underline{x} \in X \Rightarrow T(\lambda,\underline{x}) \in X$$

Proof:

$$||T(\lambda,\underline{x})|| = \max\{|T_1(\lambda,\underline{x})|,|T_2(\lambda,\underline{x})|\}$$

$$\le \max_{k\in I}\max_{i\in I}\max_{j\in I} |\lambda h^k_{ij} + (1-\lambda)x_i|$$

$$\le \lambda\alpha + (1-\lambda)||\underline{x}|| \le \alpha \quad \text{if} \quad \underline{x} \in X.$$

Lemma 2: For any $0 \le \lambda \le 1$

$$||T(\lambda,x) - T(\lambda,\underline{x}')|| \le (1-\lambda)||\underline{x}-\underline{x}'||$$

Proof: If (f_1,f_2) and (g_1,g_2) are any two real vectors then

$$|\max\{f_1,f_2\} - \max\{g_1,g_2\} \le \max\{|f_1-g_1|,|f_2-g_2|\},$$

and

$$|\min\{f_1,f_2\} - \min\{g_1,g_2\}| \le \max\{|f_1-g_1|,|f_2-g_2|\}.$$

Using these two inequalities in the reversed order we obtain

$$||T(\lambda,\underline{x}) - T(\lambda,\underline{x}')|| = \max_{k\in I}|\min_{j\in I}\max_{i\in I} H^k_{ij}(\lambda,\underline{x}) - \min_{j\in I}\max_{i\in I} H^k_{ij}(\lambda,\underline{x}')|$$

$$\le \max_{k\in I}\max_{j\in I}|\max_{i\in I} H^k_{ij}(\lambda,\underline{x}) - \max_{i\in I} H^k_{ij}(\lambda,\underline{x}')|$$

$$\le \max_{k\in I}\max_{j\in I}\max_{i\in I}|H^k_{ij}(\lambda,\underline{x}) - H^k_{ij}(\lambda,\underline{x}')|$$

$$= \max_{i\in I} |(1-\lambda)(x_i-x_i')| = (1-\lambda)||\underline{x} - \underline{x}'||.$$

Lemma 3: For every $0 < \lambda \le 1$ there is a unique $\underline{\tilde{x}} \in X$ such that $T(\lambda,\underline{\tilde{x}}) = \underline{\tilde{x}}$.

Proof: By Lemma 2 the map T is for any fixed $0 < \lambda \le 1$ a contraction and by Lemma 1 it maps a closed bounded set X into itself. Hence by Banach's fixed point theorem it must have a unique fixed point.

Lemma 4: For every $0 < \lambda \le 1$ there is $p \in I$ and $q \in I$ such that either

$$\tilde{x}_1(\lambda) = h^1_{1p}\ ,\ \tilde{x}_2(\lambda) = h^1_{1p} + \lambda(h^2_{1q}-h^1_{1p}), \tag{2.5}$$

or

$$\tilde{x}_1(\lambda) = h^2_{2q}\ ,\ \tilde{x}_2(\lambda) = h^2_{2q} + \lambda(h^1_{2p}-h^2_{2q}), \tag{2.6}$$

or

$$\tilde{x}_1(\lambda) = \frac{h^1_{2p} + h^2_{1q} - \lambda h^2_{1q}}{2-\lambda}\ ,$$

$$\tilde{x}_2(\lambda) = \frac{h^1_{2p} + h^2_{1q} - \lambda h^1_{2p}}{2-\lambda}\ , \tag{2.7}$$

or

$$\tilde{x}_1(\lambda) = h^1_{1p}\ ,\ \tilde{x}_2(\lambda) = h^2_{2q}. \tag{2.8}$$

Proof: Since $\underline{\tilde{x}} = (\tilde{x}_1, \tilde{x}_2)$ is a fixed point of the map $T(\lambda,\cdot)$ we must have

$$\tilde{x}_1 = h^1_{ip} + (1-\lambda)\tilde{x}_i,$$

$$\tilde{x}_2 = h^2_{jq} + (1-\lambda)\tilde{x}_j,$$

where $i \in I$, $j \in I$, $p \in I$, $q \in I$. Choosing values 1 or 2 for the index i and the index j we obtain four sets of pairs of linear equations:

$$\tilde{x}_1 = \lambda h^1_{1p} + (1-\lambda)\tilde{x}_1$$

$$\tilde{x}_2 = \lambda h^2_{1q} + (1-\lambda)\tilde{x}_1$$

$$\tilde{x}_1 = \lambda h^1_{2p} + (1-\lambda)\tilde{x}_2$$

$$\tilde{x}_2 = \lambda h^2_{2q} + (1-\lambda)\tilde{x}_2$$

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$$\tilde{x}_2 = \lambda h^2_{2q} + (1-\lambda)\tilde{x}_2$$

The solution of each set gives the corresponding expression (2.5)-(2.8).

Lemma 5: $\lim_{\lambda \to 0} |\tilde{x}_1(\lambda) - \tilde{x}_2(\lambda)| = 0.$

Proof: Assume the statement is false, i.e. there is $\varepsilon_0 > 0$ such that for every $\lambda_1 > 0$ there is $0 < \lambda < \lambda_1$ for which

$$|\tilde{x}_1(\lambda) - \tilde{x}_2(\lambda)| \geq \varepsilon_0.$$

By Lemma 4 this could only be if $\tilde{\underline{x}}(\lambda)$ is given by (2.8) i.e.

$$\tilde{x}_2(\lambda) = h^1_{1p} \, , \; \tilde{x}_2(\lambda) = h^2_{2q} \tag{2.9}$$

and

$$|h^1_{1p} - h^2_{2q}| \geq \varepsilon_0 . \tag{2.10}$$

Next since $\tilde{\underline{x}}(\lambda)$ is a fixed point

$$\tilde{x}_k(\lambda) = \min_{j \in I} \max_{i \in I} H^k_{ij}(\lambda, \tilde{\underline{x}}), \qquad k \in I,$$

and because of (2.9) this implies

$$H^1_{1p}(\lambda, \tilde{\underline{x}}) \geq H^1_{2p}(\lambda, \tilde{\underline{x}}),$$

and

$$H^2_{2q}(\lambda, \tilde{\underline{x}}) \geq H^2_{1q}(\lambda, \tilde{\underline{x}}).$$

Substituting from (2.9) we obtain

$$\lambda h^1_{1p} + (1-\lambda) h^1_{1p} \geq \lambda h^1_{2p} + (1-\lambda) h^2_{2q},$$

and

$$\lambda h^2_{2q} + (1-\lambda) h^2_{2q} \geq \lambda h^2_{1q} + (1-\lambda) h^1_{1p},$$

which is equivalent to

$$h^1_{1p} - h^2_{2q} \geq \lambda (h^1_{2p} - h^2_{2q}),$$

$$h^1_{1p} - h^2_{2q} \leq \lambda (h^1_{1p} - h^2_{2q}).$$

However, these two inequalities cannot hold for arbitrarily small $\lambda > 0$ unless $h^1_{1p} = h^2_{2q}$, which contradicts (2.10). The lemma is proved.

Lemma 6: Let $\underline{f}_j = (f_{1j},\ldots,f_{mj})$ and $\underline{g}_j = (g_{1j},\ldots,g_{mj})$, $j = 1,\ldots,n$ be given vectors. Let for every $0 < \lambda \leq \lambda_1$ at least one of the n inequalities $\lambda\underline{f}_j \leq \underline{g}_j$ be satisfied.

Then there is a $\lambda_0 \in (0,\lambda_1]$ such that if the inequality $\lambda\underline{f}_{j_0} \leq \underline{g}_{j_0}$ is satisfied for $\lambda = \lambda_0$ it is satisfied for any $0 < \lambda \leq \lambda_0$.

Proof: Since all these inequalities are linear there is, for every j, a closed interval $J_j \subset [0,\lambda_1]$ such that the j-th inequality is satisfied if and only if $\lambda \in J_j$. Although some of those intervals may be vacuous the system $\{J_1,\ldots,J_m\}$ must cover $(0,\lambda_1]$ since by assumption at least one of the inequalities is satisfied for any $\lambda \in (0,\lambda_1]$. Hence, there must be at least one interval with nonempty interior and left endpoint zero, say $J_{j_0} = [0,\lambda_0]$, $\lambda_0 > 0$, and the statement follows.

Lemma 7: There exists $\lambda_0 > 0$, a vector $\underline{\xi} = (\xi_1,\xi_2)$, and a vector $\underline{b} = (b_1,b_2)$ such that either

$$\tilde{\underline{x}}(\lambda) = \underline{\xi} + \lambda\underline{b} \quad \text{for all} \quad 0 < \lambda \leq \lambda_0, \tag{2.11}$$

or

$$\tilde{\underline{x}}(\lambda) = \frac{\underline{\xi} + \lambda\underline{b}}{2 - \lambda} \quad \text{for all} \quad 0 < \lambda \leq \lambda_0. \tag{2.12}$$

In either case the vector $\underline{\xi}$ is such that

$$\xi_1 = \xi_2.$$

Proof: By Lemma 5 the fixed point $\tilde{\underline{x}}(\lambda)$ is given by one of the expressions (2.5)-(2.8). Since p and q generally depend on λ this leaves 16 possibilities.

Since $\tilde{\underline{x}}(\lambda)$ is a fixed point there is $p \in I$, $q \in I$, $r \in I$, $s \in I$, in general depending on λ, such that

$$\left.\begin{aligned} \tilde{x}_1 &= H^1_{pq}(\lambda,\tilde{\underline{x}}) = \min_{j\in I}\max_{i\in I} H^1_{ij}(\lambda,\tilde{\underline{x}}), \\ \text{and} \quad & \\ \tilde{x}_2 &= H^2_{rs}(\lambda,\tilde{\underline{x}}) = \min_{j\in I}\max_{i\in I} H^2_{ij}(\lambda,\tilde{\underline{x}}). \end{aligned}\right\} \qquad (2.13)$$

But $A = \min[\max(A,B), \max(C,D)]$ is equivalent to

$$A \geq B \quad \text{and} \quad A \leq C$$

or

$$A \geq B \quad \text{and} \quad A \leq D.$$

Hence, (2.13) is satisfied (with some specific p, q, r, s) if and only if at least one of the resulting four systems of four inequalities between pairs of the H's is satisfied. Now, each of these inequalities is of the form

$$\lambda h + (1-\lambda)\tilde{x} \geq \lambda h' + (1-\lambda)\tilde{x}', \qquad (2.14)$$

where h and h' each stands for one of the h^k_{ij}'s and $\tilde{x}$ and $\tilde{x}'$ each for one of the two components of $\tilde{\underline{x}}$.

By Lemma 6 there must be $\lambda_1 > 0$ such that for $0 < \lambda \leq \lambda_1$ $\tilde{\underline{x}}(\lambda)$ cannot be given by (2.8) unless $h^1_{1p} = h^2_{2q}$.

Hence, substituting for $\tilde{x}$ and $\tilde{x}'$ any one of the expressions (2.5)-(2.7) or the expression (2.8) with $h^1_{1p} = h^2_{2q}$, the inequality (2.14) becomes either

$$\lambda(h-h') + (1-\lambda)\lambda B \geq 0$$

or

$$(2-\lambda)\lambda(h-h') + (1-\lambda)\lambda B \geq 0$$

where B is a constant involving the h_{ij}^k's. But λ, being positive, can be canceled and the inequalities become linear in λ.

Thus we have a finite number of systems of four linear inequalities and since the fixed point $\tilde{\underline{x}}$ exists by Lemma 3 for every $0 < \lambda \le 1$ at least one of the systems must be satisfied for every $0 < \lambda \le \lambda_1$. Hence, by Lemma 6 there is a $\lambda_0 > 0$ such that if a particular system is satisfied for $\lambda = \lambda_0$ it remains satisfied for all $0 < \lambda \le \lambda_0$. Consequently, the indexes p, q, r, s, which identify the fixed point for $\lambda = \lambda_0$ will remain unchanged as λ decreases to zero. Hence $\tilde{\underline{x}}(\lambda)$ will for every $0 < \lambda \le \lambda_0$ be given by one of the expressions (2.5)-(2.7) with p and q no longer changing with λ and thus by Lemma 5 must have one of two forms (2.11) or (2.12) with $\xi_1 = \xi_2$. The lemma is proved.

<u>Lemma 8</u>: Let $\underline{x}_1, \underline{x}_2, \ldots$ be a sequence satisfying (2.4), let $\underline{\xi} = \lim_{\lambda \to 0} \underline{x}(\lambda)$. Then the sequence $\underline{x}_1, \underline{x}_2, \ldots$ converges and $\lim_{n\to\infty} \underline{x}_n = \underline{\xi}$.

<u>Proof</u>: Let $\varepsilon > 0$, let N be a positive integer such that

$$\frac{1}{N} \le \lambda_0 \tag{2.15}$$

and

$$\frac{1}{N+n} \sum_{k=0}^{n} \frac{1}{N+k} < \frac{\varepsilon}{K} \quad \text{for every } n = 0,1,\ldots, \tag{2.16}$$

where $K = \max\{||\underline{b}||, ||\underline{\xi}-2\underline{b}||(2-\lambda_0)^{-2}\}$ and $\lambda_0, \underline{b}$ are those of Lemma 7. Such an N exists for any $\varepsilon > 0$ since the harmonic series is Cesaro summable to zero.

Let n be a nonnegative integer. By Lemma 2

$$||\underline{x}_{N+n} - \tilde{\underline{x}}(\tfrac{1}{N+n})|| = ||T(\tfrac{1}{N+n}, \underline{x}_{N+n-1}) - T(\tfrac{1}{N+n}, \tilde{\underline{x}}(\tfrac{1}{N+n})||$$

$$\le \frac{N+n-1}{N+n} ||\underline{x}_{N+n-1} - \tilde{\underline{x}}(\tfrac{1}{N+n})|| \tag{2.17}$$

and by triangular inequality

$$||\underline{x}_{N+n-1} - \underline{\tilde{x}}(\frac{1}{N+n})|| \le ||\underline{x}_{N+n-1} - \underline{\tilde{x}}(\frac{1}{N+n-1})|| + ||\underline{\tilde{x}}(\frac{1}{N+n}) - \underline{x}(\frac{1}{N+n-1})|| \quad (2.18)$$

By Lemma 7 if $0 < \lambda \le \lambda_0$ and $0 < \lambda' \le \lambda_0$

$$||\underline{\tilde{x}}(\lambda) - \underline{\tilde{x}}(\lambda')|| \le |\lambda - \lambda'| \; ||\underline{b}||$$

or

$$||\underline{x}(\lambda) - \underline{x}(\lambda')|| \le |\lambda - \lambda'| \frac{||\underline{\xi} - 2\underline{b}||}{(2-\lambda_0)^2}$$

depending on which of the two expressions (2.11) and (2.12) applies.

Hence

$$||\underline{\tilde{x}}(\frac{1}{N+n}) - \underline{x}(\frac{1}{N+n-1})|| \le \frac{K}{(N+n)(N+n-1)} \quad (2.19)$$

Putting (2.17), (2.18) and (2.19) together and calling $||\underline{x}_{N+n} - \underline{\tilde{x}}(\frac{1}{N+n})|| = \Delta_n$ we have

$$\Delta_n \le \frac{N+n-1}{N+n} \Delta_{n-1} + \frac{K}{(N+n)^2}$$

Iterating we obtain eventually

$$\Delta_n \le \frac{N}{N+n} \Delta_0 + \frac{K}{N+n} \sum_{k=0}^{n} \frac{1}{k}$$

for every $n = 0,1,\ldots$ and hence by (2.15) and (2.16) and the obvious fact that $0 \le ||\Delta_n|| \le \alpha$, the radius of X, we have

$$\Delta_n < \varepsilon \quad \text{for every} \quad n = 0,1,\ldots .$$

Thus $\lim_{n\to\infty} \Delta_n < \varepsilon$ for any $\varepsilon > 0$ so that $\lim_{n\to\infty} ||\underline{x}_n - \underline{\tilde{x}}(\frac{1}{n})|| = 0$ and since by Lemma 7 $\lim_{n\to\infty} \underline{\tilde{x}}(\frac{1}{n}) = \underline{\xi}$ the proof of Lemma 8 and also of Theorem 1 is terminated.

Now we can proceed with the proof of the formula for the Ω-value.

Theorem 2: The formula (1.3) is true.

Proof: Since always $\underline{V} \le \bar{V}$ we distinguish two cases.

Case 1: $\underline{V} < \bar{V}$

With no loss of generality assume that the matrix A is such that

$$a_{11} \ge a_{22} \ge \max\{a_{12}, a_{21}\},$$

relabelling the entries if necessary. Then we have

$$\bar{M} = a_{11}, \quad \bar{V} = a_{22} \quad \text{and} \quad \underline{V} = \max\{a_{12}, a_{22}\}.$$

We will again use the notation

$$h_{ij}^k = \sigma a_{kj} + (1-\sigma) a_{ij} \quad \text{and} \quad H_{ij}^k(\lambda, \underline{x}) = \lambda h_{ij}^k + (1-\lambda) x_k.$$

Notice that in this notation

$$v(\sigma) = \frac{1}{2}(h_{22}^1 + h_{11}^2)$$

and that if $k = i$, $h_{ij}^k = a_{ij}$.

Case 1.1: $v(\sigma) \ge a_{22}$

Let

$$\left.\begin{aligned} x_1 &= \lambda h_{22}^1 + (1-\lambda) h_{22}^2 \\ x_2 &= h_{22}^2 \end{aligned}\right\} \qquad (2.20)$$

so that

$$H^1_{22}(\lambda,\underline{x}) = x_1 \quad \text{and} \quad H^2_{22}(\lambda,\underline{x}) = x_2 \tag{2.21}$$

Next

$$H^1_{22}(\lambda,\underline{x}) - H^1_{12}(\lambda,\underline{x}) = \lambda(h^1_{22}-h^1_{12}) + (1-\lambda)(x_2-x_1)$$

$$= \lambda[h^1_{22} - h^1_{12} + (1-\lambda)(h^2_{22}-h^1_{22})],$$

which is nonnegative in a positive neighborhood of zero if and only if $h^2_{22} - h^1_{12} \geq 0$. But $h^2_{22} = a_{22}$, $h^1_{12} = a_{12}$ so that indeed

$$H^1_{22}(\lambda,\underline{x}) - H^1_{12}(\lambda,\underline{x}) \geq 0 \quad \text{for small} \quad \lambda > 0.$$

Next

$$H^1_{22}(\lambda,\underline{x}) - H^1_{11}(\lambda,\underline{x}) = \lambda[h^1_{22} - h^2_{12} + (1-\lambda)(h^2_{22}-h^1_{22})] \leq 0$$

if and only if $h^2_{22} \leq h^1_{11}$, which is true since $h^2_{22} = a_{22}$ and $h^1_{11} = a_{11}$. Thus for small $\lambda > 0$

$$H^1_{22}(\lambda,\underline{x}) = \min_j \max_i H^1_{ij}(\lambda,\underline{x}) \tag{2.22}$$

Similarly

$$H^2_{22} - H^2_{12} \geq 0 \Leftrightarrow 2h^2_{22} - (h^2_{12}+h^1_{22}) \geq 0 \Leftrightarrow 2a_{22} \geq a_{22} + a_{12},$$

and

$$H^2_{22} - H^1_{11} \leq 0 \Leftrightarrow 2h^2_{22} - 2v(\sigma) \leq 0,$$

which is true since $h^2_{22} = a_{22}$. Hence, for small $\lambda > 0$

$$H^2_{22}(\lambda,\underline{x}) = \min_j \max_i H^2_{ij}(\lambda,\underline{x}),$$

which together with (2.21) and (2.22) means that for all $\lambda > 0$, in some neighborhood of zero (2.20) is a fixed point of the mapping (2.3). Consequently by Theorem 1 the Ω-value

$$\Omega(\sigma) = \lim_{\lambda \to 0} x_1 = h^2_{22} = a_{22}.$$

<u>Case 1.2</u>: $v(\sigma) \geq \underline{V}$

Let $p \in I$ and $q \in I$ be such that $\underline{V} = a_{pq}$. Notice that $p \neq q$ so that

$$v(\sigma) = \frac{1}{2}[h^p_{qq} + h^q_{pp}].$$

Let

$$x_p = h^p_{pq} \;, \quad x_q = \lambda h^q_{pp} + (1-\lambda)h^p_{pq} \tag{2.23}$$

so that

$$x_p = H^p_{pq}(\lambda,\underline{x}) \quad \text{and} \quad x_q = H^q_{pp}(\lambda,\underline{x}). \tag{2.24}$$

Similarly as in Case 1.1.

$$H^p_{pq} - H^p_{qq} \geq 0 \iff 2h^p_{pq} - 2v(\sigma) \geq 0,$$

which is true since $h^p_{pq} = a_{pq} = \underline{V}$. Next

$$H^p_{pq} - H^p_{pp} = \lambda(h^p_{pq} - h^p_{pp}) = \lambda(a_{pq} - a_{pp}) \leq 0,$$

so that

$$H^p_{pq} = \min_j \max_i H^p_{ij} \; . \tag{2.25}$$

Similarly

$$H^q_{pp} - H^q_{qp} \geq 0 \Leftrightarrow h^p_{pq} - h^q_{qp} \geq 0$$

since

$$h^p_{pq} = a_{pq} = \underline{V} \geq \underline{M} = a_{qp} = h^q_{qp},$$

and

$$H^q_{pp} - H^q_{qq} \leq 0 \Leftrightarrow h^p_{pq} - h^q_{qq} \leq 0$$

since

$$a_{pq} \leq a_{qq}.$$ Hence

$$H^q_{pp} = \min_j \max_i H^q_{ij} \qquad (2.26)$$

and (2.23)-(2.26) imply $\Omega(\sigma) = h^p_{pq} = \underline{V}$.

Case 1.3. $\underline{V} < v(\sigma) < \bar{V}$

Calling again $\underline{V} = a_{pq}$ we now take

$$x_p = \frac{1}{2-\lambda}(2v(\sigma) - \lambda h^q_{pp}),$$
$$x_q = \frac{1}{2-\lambda}(2v(\sigma) - \lambda h^p_{qq}). \qquad (2.27)$$

We have now

$$x_p = H^p_{qq}(\lambda,\underline{x}) \;,\; x_q = H^q_{pp}(\lambda,\underline{x}). \qquad (2.28)$$

Next

$$H^p_{qq} - H^p_{pq} = \lambda[h^p_{qq} - h^p_{pq} + \frac{1-\lambda}{2-\lambda}(h^q_{pp} - h^p_{qq})]$$

so that for small $\lambda > 0$

$$H^p_{qq} - H^p_{pq} \geq 0 \text{ if and only if } 2(h^p_{qq} - h^p_{pq}) + h^q_{pp} - h^p_{qq} \geq 0$$

But this is equivalent to

$$2v(\sigma) - 2h^p_{pq} \geq 0,$$

which is true since $h^p_{pq} = a_{pq} = \underline{V}$. Similarly

$$H^p_{qq} - H^p_{pp} \le 0 \Leftrightarrow 2(h^p_{qq}-h^p_{pp}) + h^q_{pp} - h^p_{qq} \le 0 \Leftrightarrow 2v(\sigma) - 2h^p_{pp} \le 0$$

since

$$h^p_{pp} = a_{pp} \ge \bar{V} \quad \text{and} \quad v(\sigma) < \bar{V}.$$

Thus

$$H^p_{qq} = \min_j \max_i H^p_{ij} \tag{2.29}$$

Finally

$$H^q_{pp} - H^q_{qp} \ge 0 \Leftrightarrow 2(h^q_{pp}-h^q_{qp}) + h^p_{qq} - h^q_{pp} \ge 0 \Leftrightarrow 2v(\sigma) - 2a_{qp} \ge 0$$

since $a_{qp} = \underline{M}$, and

$$H^q_{pp} - H^q_{qq} \le 0 \Leftrightarrow 2(h^q_{pp}-h^q_{qq}) + h^p_{qq} - h^q_{pp} \le 0 \Leftrightarrow 2v(\sigma) - 2h^q_{qq} \le 0$$

since $v(\sigma) < \bar{V} \le a_{qq}$. Hence

$$H^q_{pp} = \min_j \max_i H^q_{ij}, \tag{2.30}$$

and (2.27)-(2.30) yield

$$\Omega(\sigma) = v(\sigma).$$

Case 2: $\underline{V} = \bar{V}$

Although we could use the same procedure as in Case 1 again we prefer a simpler argument. If $\underline{V} = \bar{V} = V$ then the matrix A has a saddle, say $a_{k\ell} = V$. Since every n-stage Ω-game $G_n(\sigma;k)$, being a game with perfect information, has a pure value we must have

$$\Omega_n(\sigma;k) = V \quad \text{for every} \quad n = 1,2,\ldots$$

But by Theorem 1 $\lim_{n\to\infty} \Omega_n(\sigma;i) = \Omega(\sigma)$ regardless of $i \in I$. Hence $\Omega(\sigma) = V$ for all $0 \le \sigma \le 1$ and the theorem is proved.

3. Washburn's Approach.

We will proceed to prove formula (1.3) by showing that Player I can get at least $\Omega^{I}(\sigma) + O(1/n)$, that Player II can prevent player I from getting more than $\Omega^{II}(\sigma) + O(1/n)$, and that $\Omega^{I}(\sigma) = \Omega^{II}(\sigma) = \Omega(\sigma)$. The two functions $\Omega^{I}(\sigma)$ and $\Omega^{II}(\sigma)$ will be defined within the theorem. For future reference, let the clockwise and counterclockwise averages be

$$v_c(\sigma) \equiv .5((a_{11}+a_{22})(1-\sigma) + (a_{12}+a_{21})\sigma),$$

and

$$v_{cc}(\sigma) \equiv .5((a_{11}+a_{22})\sigma + (a_{12}+a_{21})(1-\sigma)).$$

Theorem: Formula (1.3) is true.

Proof: We will show that players I and II can asymptotically confine themselves to four strategies each, those strategies being the four functions mapping $\{1,2\}$ onto itself. For example, the strategy $(2,1)$ for I means that I will always choose 2 if II has previously chosen 1, or 1 if II has previously chosen 2. We will use the symbol $s \underset{I}{\geq} x$ to mean that the strategy s for I assures that the average payoff to I will be at least $x + O(1/n)$, where n is the number of moves, regardless of the starting state. Similarly for II.

We proceed by showing that

$$(1,2) \underset{I}{\to} \min\{v_c(\sigma), a_{11}, a_{22}\}$$

$$(2,1) \underset{I}{\to} \min\{v_{cc}(\sigma), a_{12}, a_{21}\}$$

$$(1,1) \underset{I}{\to} \min\{a_{11}, a_{12}\}$$

$$(2,2) \underset{I}{\to} \min\{a_{21}, a_{22}\}$$

$$(1,2) \underset{II}{\to} \max\{v_{cc}(\sigma), a_{11}, a_{22}\}$$

$$(2,1) \underset{II}{\to} \max\{v_c(\sigma), a_{12}, a_{21}\}$$

$$(1,1) \underset{II}{\to} \max\{a_{11}, a_{21}\}$$

$$(2,2) \underset{II}{\to} \max\{a_{12}, a_{22}\} .$$

For example, consider the strategy (1,2) for I. After n stages, let $n_{ij}^{I}(n_{ij}^{II})$ be the number of visits to state (i,j) initiated by I(II). Then as long as I uses (1,2), regardless of the starting point,

$$n_{II}^{I} \geq n_{21}^{II} + n_{11}^{II} - 1$$

$$n_{22}^{I} \geq n_{12}^{II} + n_{22}^{II} - 1$$

$$|n_{21}^{II} - n_{12}^{II}| \leq 1$$

$$n_{21}^{I} = 0$$

$$n_{12}^{I} = 0$$

$$\sum_i \sum_j n_{ij}^{II} = n .$$

Since the average payoff to I is

$$v_{(1,2)}^{I} \equiv \frac{1}{n} \sum_i \sum_j (n_{ij}^{I}(1-\sigma) + n_{ij}^{II}\,\sigma) a_{ij},$$

a lower bound on $v^{I}_{(1,2)}$ can be obtained by solving a linear programming problem. Furthermore, if we define

$$p^{I}_{ij} \equiv n^{I}_{ij}/n \quad \text{and} \quad p^{II}_{ij} = n^{II}_{ij}/n,$$

and neglect terms that are $O(1/n)$, the program is

$$\min \sum \sum [p^{I}_{ij}(1-\sigma) + p^{II}_{ij}\, \sigma] a_{ij} \equiv v_{(1,2)}$$

subject to

$$p^{I}_{11} = p^{II}_{21} + p^{II}_{11}$$

$$p^{I}_{22} = p^{II}_{12} + p^{II}_{22}$$

$$p^{II}_{21} = p^{II}_{12}$$

$$p^{I}_{21} = p^{I}_{12} = 0$$

$$\sum \sum p^{II}_{ij} = 1 \ .$$

There are six equations in eight variables. There is an optimal solution in which at least two of the variables, in addition to p^{I}_{21} and p^{I}_{12}, are 0. If these two variables are p^{II}_{11} and p^{II}_{22}, then $p^{I}_{11} = p^{II}_{21} = p^{II}_{12} = p^{I}_{22} = .5$ and $v^{I}_{(1,2)}$ is $v_c(\sigma)$. If $p^{I}_{11} = 0$, $v^{I}_{(1,2)} = a_{22}$, and if $p^{I}_{22} = 0$, $v^{I}_{(1,2)} = a_{11}$. If $p^{II}_{21} = 0$, $v^{I}_{(1,2)}$ is some mixture of a_{11} and a_{22}. In all cases, $(1,2) \underset{I}{\to} \min\{v_c(\sigma), a_{11}, a_{22}\}$. Essentially, II has his choice of either going around in a clockwise circle or else acepting one of the two payoffs that I keeps trying to obtain. The rest of the $\to$ statements can be obtained similarly.

Taking account of all four of I's (II's) strategies, we see that I (II) can guarantee that the payoff will be at least (at most)

$\Omega^{I}(\sigma) + O(1/n)$ and $\Omega^{II}(\sigma) + O(1/n)$, where

$$\Omega^{I}(\sigma) = \max\{\min(v_{cc}(\sigma), a_{12}, a_{21}), \min(v_{c}(\sigma), a_{11}, a_{22}), \underline{V}\}$$

$$\Omega^{II}(\sigma) \equiv \min\{\max(v_{cc}(\sigma), a_{11}, a_{22}), \max\{v_{c}(\sigma), a_{12}, a_{21}\}, \bar{V}\}$$

The fact that $\Omega^{I}(\sigma) = \Omega^{II}(\sigma) = \Omega(\sigma)$ can be shown by exhaustion of cases. Since $\Omega^{I}(\sigma) \geq \underline{V}$ and $\Omega^{II}(\sigma) \leq \bar{V}$, the proof is trivial if $\underline{V} = \bar{V}$. If there is no saddle point, suppose a_{11} or a_{22} is the largest element $\bar{M}$. It follows that the term involving $v_{cc}(\sigma)$ may be deleted from both $\Omega^{I}(\sigma)$ and $\Omega^{II}(\sigma)$, that the second largest element $\bar{V}$ is on the same diagonal, and also that $v(\sigma) = v_{c}(\sigma)$. The result is that

$$\Omega^{I}(\sigma) = \max\{\min\{v(\sigma), \bar{V}\}, \underline{V}\}$$

$$\Omega^{II}(\sigma) = \min\{\max\{v(\sigma), \underline{V}\}, \bar{V}\}$$

Both of these expressions are equivalent to 1.3. A similar proof holds if a_{12} or a_{21} is the largest element.

QED

4. Comparison of Techniques and Prognosis for Larger Games.

The 2×2 game would be much easier to solve if it could be discovered that the function $\Omega(\sigma)$ satisfies some simple functional equation that has a unique solution. This does not seem to be the case. Shubert deals with this problem by inventing a situation that is almost as good. He introduces a function $\Omega(\sigma,\lambda,k)(x_k(\lambda)$ in his notation) which does satisfy such a functional equation, and then shows that $\lim_{\lambda \to 0} \Omega(\sigma,\lambda,k) = \Omega(\sigma)$ regardless of the index k. He next shows there are only a finite number of candidates for the solution of the equation, one of which must apply for all λ smaller than some positive λ_0. Given the matrix A, he finds the solution, and consequently $\Omega(\sigma)$. This procedure is probably generalizable to larger games. The difficulty will be that the number of candidate solutions will grow very fast with the size of the matrix. It is possible, however, that a systematic procedure can be found to determine the correct solution without an exhaustive search.

Washburn uses a time-honored technique for solving two-person zero-sum games; he guesses the optimal strategies and shows that they are in equilibrium. The generalization of his procedure when I has m strategies and II has n strategies would be

1) Let I (II) confine himself to all functions mapping $\{1,\ldots,n(m)\}$ onto $\{1,\ldots,m(n)\}$. There are $m^n(n^m)$ such strategies.
2) For each such strategy, find the asmyptotic bound that it guarantees for the player who is using it by writing a Linear Program with $2MN$ variables.
3) Show that the greatest lower bound equals the least upper bound (a theorem is required).

Given the theorem in 3), one could find the Ω-value of an $m \times n$ game for one particular value of σ by solving $\min\{m^n, n^m\}$ Linear Programs with $2mn$ variables each.

It is apparent that either method for finding the Ω-value of a large game would require a vast amount of computational effort, even if all the required theorems can be proved. The central problem for matrix-differential games would therefore seem to be the invention of a practical procedure for finding solutions; it would be particularly valuable if the procedure were able to find solutions for all σ. By "practical," we mean a procedure where computational effort increases less than exponentially with m and n.

REFERENCES

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