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DYNAMICS OF ROTATING DEFORMABLE SOLIDS

G. L. Anderson

Watervliet Arsenal
Watervliet, New York

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WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

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13 ABSTRACT

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ABSTRACT

The kinematics of rotating deformable solids are developed, and a conservation law is postulated. The general equations of motion for a deformable solid rotating about a fixed axis are obtained from this conservation law, which then serves as the theoretical basis for the derivation of suitable beam theories for rotating beams and shafts subjected to conservative and non-conservative loads. The effects of internal and external damping as well as gyroscopic inertial forces are included in the formulation.

Cross-Reference Data

Elasticity

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Damping

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INTRODUCTION

During the last half century, the stability and vibrations of rotating elastic disks, beams and shafts, which find engineering application in the design of turbines, helicopter blades, and other types of rotary machinery, have been the subjects of numerous theoretical investigations [1-30].* In general, the partial differential equations which describe the deformations of the above-mentioned rotating systems are derived by *ad hoc* techniques rather than some systematic procedure based upon, say, Hamilton's principle, energy considerations, or some conservation law. It is this lack in the technical literature of such systematic procedures that has provided the impetus for the present investigation, particularly in light of the fact that recently there has been considerable interest (see, for example, references [31-33]) in the stability of deformable bodies loaded by complicated non-conservative forces. For conservative, rotating systems, one can employ for such purposes Hamilton's principle upon formulating a suitable expression for the kinetic energy. Further-

*This list of references is by no means complete.

Note: [1-30] See References pp. 92-95.

[31] ANDERSON, G. L., On the role of the adjoint problem in dissipative, nonconservative problems of elastic stability. Meccanica, 7, pp. 165-173 (1972).

[32] ANDERSON, G. L., Application of a variational method to dissipative, non-conservative problems of elastic stability. J. Sound Vib., 27, pp. 279-296 (1973).

[33] ANDERSON, G. L., On the stability of a deep beam subjected to non-conservative and dissipative forces. Watervliet Arsenal Technical Report R-WV-T-2-14-73, Watervliet, New York 12189 (1973).

more, a similar step can be accomplished even in the case of non-conservatively loaded systems, however in order to obtain a functional, viz., the Lagrangian function, it is necessary to introduce a set of field equations which is adjoint to that which describes the originally posed problem. It is of considerable interest and importance to note that for non-conservative systems one can avoid the necessity of introducing the adjoint problem by working with a generalized conservation law. Indeed, this technique has been used very effectively by Nemat-Nasser et al. [34-35] for non-rotating systems. In this report, the procedure suggested in references [34-35] will be extended so as to include the gyroscopic effect which is present in rotating systems.

In order to motivate the discussion which appears in subsequent sections, it is instructive to begin with an elementary example of a non-conservative system.

Consider the equation of motion of a damped spring-mass system, namely,

$$\ddot{x} + c\dot{x} + \omega^2 x = 0, \qquad \dot{x} = dx/dt. \tag{1.1}$$

Multiplication of equation (1.1) by $\dot{x}$ yields

$$\dot{x}\ddot{x} + c\dot{x}^2 + \omega^2 x\dot{x} = 0,$$

[34] NEMAT-NASSER, S. and TSAI, P. F., Effect of warping rigidity on stability of a bar under eccentric follower force. Int. J. Solids Structures, 5, pp. 271-279 (1969).

[35] LIN, K.-H., NEMAT-NASSER, S. and HERRMANN, G., Stability of a bar under eccentric follower force. Proc. ASCE, J. Engng. Mech. Div., 93, pp. 105-115 (1967).

which, in view of the identities

$$\dot{x}\ddot{x} = d(\tfrac{1}{2}\dot{x}^2)/dt, \qquad x\dot{x} = d(\tfrac{1}{2}x^2)/dt, \qquad \dot{x}^2 = \frac{d}{dt}\int_0^t \left(\frac{dx}{d\tau}\right)^2 d\tau,$$

can be expressed as

$$dH/dt = 0, \tag{1.2}$$

where

$$H = \frac{1}{2}\dot{x}^2 + c\int_0^t \left(\frac{dx}{d\tau}\right)^2 d\tau + \frac{1}{2}\omega^2 x^2. \tag{1.3}$$

Therefore,

$$H = \text{constant},$$

i.e., the quantity H is conserved throughout the motion.

Suppose next that these steps are reversed, i.e., equation (1.3) is substituted into equation (1.2). The result, obviously, is

$$(\ddot{x} + c\dot{x} + \omega^2 x)\dot{x} = 0.$$

But since $\dot{x} \neq 0$ for all t, it follows that

$$\ddot{x} + c\dot{x} + \omega^2 x = 0,$$

which is merely equation (1.1). The point to be made here is that the equation of motion (1.1) can be derived from the conservation law in equation (1.2).

The procedure illustrated in the preceding paragraph can, however, be fraught with peril when applied to gyroscopic systems. For example, Ziegler [36], p. 17, has discussed the pair of coupled equations

$$m \ddot{x}_1 + (c_1 - m \Omega^2)x_1 - 2m \Omega\dot{x}_2 = 0, \tag{1.4}$$

$$2m \Omega \dot{x}_1 + m \ddot{x}_2 + (c_2 - m \Omega^2)x_2 = 0, \tag{1.5}$$

which describe the motion of a disk of mass m mounted on a shaft rotating with constant angular velocity Ω. The constants c_1 and c_2 denote the stiffnesses of the system in the x_1- and x_2- directions, respectively. The gyroscopic effect is embodied in the Coriolis terms $-2m\Omega\dot{x}_2$ and $2m\Omega\dot{x}_1$.

If equations (1.4) and (1.5) are multiplied by $\dot{x}_1$ and $\dot{x}_2$, respectively, then

$$m \ddot{x}_1 \dot{x}_1 + (c_1 - m\Omega^2)x_1 \dot{x}_1 - 2m \Omega \dot{x}_1 \dot{x}_2 = 0,$$

$$m \ddot{x}_2 \dot{x}_2 + (c_2 - m \Omega^2)x_2 \dot{x}_2 + 2m \Omega \dot{x}_1 \dot{x}_2 = 0.$$

Addition of these equations yields

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[36] ZIEGLER, H., Principles of Structural Stability. Waltham, Massachusetts: Blaisdell Publishing Company (1968).

$$dH/dt = 0,$$

where now

$$H = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} (c_1 x_1^2 + c_2 x_2^2) - \frac{1}{2} m \Omega^2 (x_1^2 + x_2^2). \quad (1.6)$$

Clearly,

$$H = \text{constant}, \quad (1.7)$$

i.e., equation (1.7) is a conservation law. It must be observed here that the Coriolis terms do not appear in equation (1.6).

If the derivative of equation (1.6) is formed, the result is

$$[m\ddot{x}_1 + (c_1 - m\Omega^2)x_1]\dot{x}_1 + [m\ddot{x}_2 + (c_2 - m\Omega^2)x_2]\dot{x}_2 = 0 \quad (1.8)$$

A comparison of the square brackets in equation (1.8) with equations (1.4) and (1.5) reveals that the quantities in the square brackets do not vanish identically. If, however, the identity $2m\Omega(\dot{x}_1 \dot{x}_2 - \dot{x}_1 \dot{x}_2) = 0$ is inserted into equation (1.8), then equations (1.4) and (1.5) can be reclaimed. Apparently, there exists no general method for selecting the appropriate identity in situations of this type; one must simply know what it must be. At first glance, this situation would appear to eliminate the conservation law approach of deriving the equations of motion of non-conservative, rotating bodies. However, this technique can be salvaged, provided that an artifice can be introduced which, at least formally, prevents the cancellation of the Coriolis terms.

Consider the rotating system depicted in Figure 1. If $\underline{e}_1$, $\underline{e}_2$ denote the unit basis vectors in the x_1, x_2- directions, respectively, then the position and velocity vectors are

$$\underline{r} = x_1 \underline{e}_1 + x_2 \underline{e}_2, \qquad \dot{\underline{r}} = (\dot{x}_1 - \Omega x_2)\underline{e}_1 + (\dot{x}_2 + \Omega x_1)\underline{e}_2.$$

Therefore, the kinetic energy T of the mass m is

$$T = \frac{1}{2} m \, \dot{\underline{r}} \cdot \dot{\underline{r}} = \frac{1}{2} m[\dot{x}_1^2 + \dot{x}_2^2 + 2\Omega(x_1\dot{x}_2 - \dot{x}_1 x_2) + \Omega^2(x_1^2 + x_2^2)]. \tag{1.9}$$

Since the force vector is

$$\underline{F} = -(c_1 x_1 \underline{e}_1 + c_2 x_2 \underline{e}_2),$$

the potential energy is easily shown to be

$$V = \frac{1}{2} c_1 x_1^2 + \frac{1}{2} c_2 x_2^2. \tag{1.10}$$

In view of equations (1.9) and (1.10) suppose that one now make the following definitions:

$$T_K = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2),$$

$$T_G = 2m\,\Omega \int_0^t [(Dx_1)\dot{x}_2 - (Dx_2)\dot{x}_1]d\hat{t}, \tag{1.11}$$

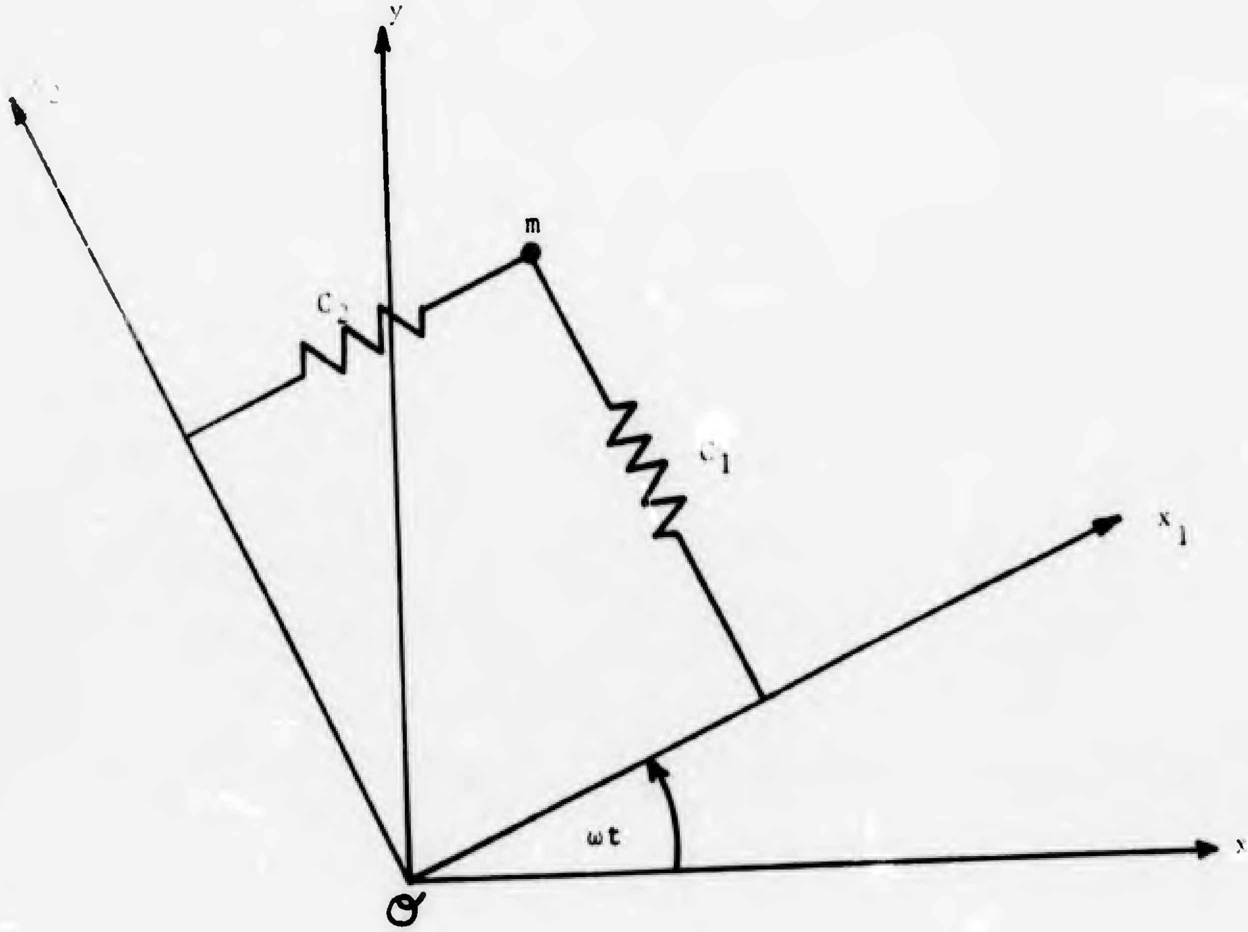

Figure 1. Model of a disk mounted on a rotating shaft.

$$V_T = \frac{1}{2}(c_1 x_1^2 + c_2 x_2^2) - \frac{1}{2} m \Omega^2 (x_1^2 + x_2^2),$$

where it is understood that $D = d/d\hat{t}$. Indeed, T_G is identically equal to zero, but the form of T_G as given in equation (1.11) will be retained. Next, it is postulated that

$$H \equiv T_K + T_G + V_T = \text{constant}, \tag{1.12}$$

and consequently

$$dH/dt = 0. \tag{1.13}$$

The quantities T_K, T_G, and V_T may be called the kinetic energy, the gyroscopic potential, and the potential energy, respectively, of the rotating system.

Substitution of equations (1.11) into equation (1.13) yields

$$[m\ddot{x}_1 + (c_1 - m\Omega^2)x_1 - 2m\,\Omega\,\dot{x}_2]\dot{x}_1 +$$

$$+ [m\ddot{x}_2 + (c_2 - m\,\Omega^2)x_2 + 2m\,\Omega\,\dot{x}_1]\dot{x}_2 = 0.$$

If the quantities in square brackets vanish, then equations (1.4) and (1.5) are reclaimed.

The technique introduced above can easily be extended to the case of a discrete system of n-degrees of freedom, for which the equation of motion is

$$m_{ij}\ddot{x}_j + g_{ij}\dot{x}_j + (k_{ij} - c_{ij})x_j = 0, \qquad (1.14)$$

$$(i,j = 1,2,\ldots,n;\ \text{sum on } j).$$

Here, the x_j's are generalized coordinates, m_{ij}, k_{ij}, and c_{ij} are, respectively, the symmetric mass, stiffness, and conservative force matrices, and g_{ij} is the antisymmetric gyroscopic force matrix. The following definitions are made:

$$T_K = \frac{1}{2} m_{ij}\dot{x}_i\dot{x}_j,$$

$$T_G = \int_0^t g_{ij}(Dx_j)\dot{x}_i\, d\hat{t},$$

$$V_T = \frac{1}{2}(k_{ij} - c_{ij})x_i x_j.$$

Equations (1.12) and (1.13) are again assumed to be valid. Proceeding as in the preceding paragraph, it is found that

$$[m_{ij}\ddot{x}_j + g_{ij}\dot{x}_j + (k_{ij} - c_{ij})x_j]\dot{x}_i = 0.$$

If the quantity in square brackets is required to vanish, then equation (1.14) is reclaimed.

Therefore, it has been demonstrated formally that the system of differential equations for a gyroscopic system of n-degrees of freedom can be derived from the conservation law in equation (1.12). This same procedure can be extended to the case of deformable rotating solids. These details will be discussed in Section 4.

2. KINEMATICS OF A ROTATING DEFORMABLE SOLID

It is generally recognized that it is possible in the study of problems of elastic stability to use a simplified version of the equations of the non-linear theory of elasticity, provided that attention is confined to the case of small strains but large displacements. In this report the concept of stability shall always refer to the stability of a certain form of equilibrium. This will be called the undisturbed form of equilibrium, and the associated state of stress will be called the initial state of stress. Of particular interest, however, will be the disturbed forms of motion which are close to the undisturbed form of equilibrium. In many cases an assessment of the stability of equilibrium can be made by assuming that the disturbances are sufficiently small and by effecting the analysis on the basis of linearized differential equations.

Consider a deformable body of volume τ which is bounded by a finite closed surface S. Let displacements v_i^* be prescribed on the portion S_u of S and tractions $\bar{p}_i$ be prescribed on the remaining portion S_σ, where $S = S_u + S_\sigma$. According to reference [31], the equation of motion and boundary conditions are

$$\bar{\sigma}_{ij,j} + (\bar{\sigma}_{jk}\,\bar{u}_{i,k})_{,j} + \bar{X}_i = \rho\, a_i \quad \text{in } \tau, \tag{2.1}$$

$$(\bar{\sigma}_{ij} + \bar{\sigma}_{jk}\,\bar{u}_{i,k})n_j = \bar{p}_i \quad \text{on } S_\sigma, \tag{2.2}$$

$$\bar{u}_i = v_i^* \quad \text{on } S_u. \tag{2.3}$$

where the summation convention is employed. Here $\bar{\sigma}_{ij}$ denotes the stress tensor, x_i the cartesian coordinates, $\bar{u}_i$ the components of the displacement vector, $\bar{X}_i$ the body force vector per unit volume, ρ the mass density, a_i the acceleration vector, and n_i the unit exterior normal vector to S.

If the body is not rotating relative to an observer's fixed frame of reference, then $a_i = \ddot{\bar{u}}_i$, as is well known. However, if the body is rotating about some axis relative to an inertial frame of reference, then the expression for a_i becomes somewhat more complicated. Let the acceleration vector be denoted by

$$\ddot{\underline{r}} = a_i\, \underline{e}_i, \tag{2.4}$$

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[31] ANDERSON, G. L., On the role of the adjoint problem in dissipative, nonconservative problems of elastic stability. Meccanica, 7, pp. 165-173 (1972).

where the $\underline{e}_i$'s form an orthogonal set of basis vectors moving with the body. Suppose that the body rotates at a uniform angular velocity $\underline{\Omega}$ about some axis, and write

$$\underline{\Omega} = \Omega_i \underline{e}_i, \qquad (2.5)$$

it being understood that

$$\partial\Omega_i/\partial t = 0.$$

In general, it may be assumed that the axis of rotation lies outside the body τ. Let the origin $\mathcal{O}'$ of an inertial coordinate frame (xyz-coordinates in Figure 2), lie at some point on the axis of rotation, and let the moving coordinate frame, (x_1, x_2, x_3), be established at some convenient point in or near the moving solid, as shown in Figure 2. If

$$\underline{c} = c_i \underline{e}_i, \qquad |\underline{c}| = \text{constant}$$

is the position vector from $\mathcal{O}'$ to $\mathcal{O}$, then the position vector $\underline{r}$ from $\mathcal{O}'$ to a generic point P in the body may be expressed as

$$\underline{r} = \underline{c} + \underline{x} + \underline{\bar{u}}$$

$$= (c_i + x_i + \bar{u}_i)\underline{e}_i.$$

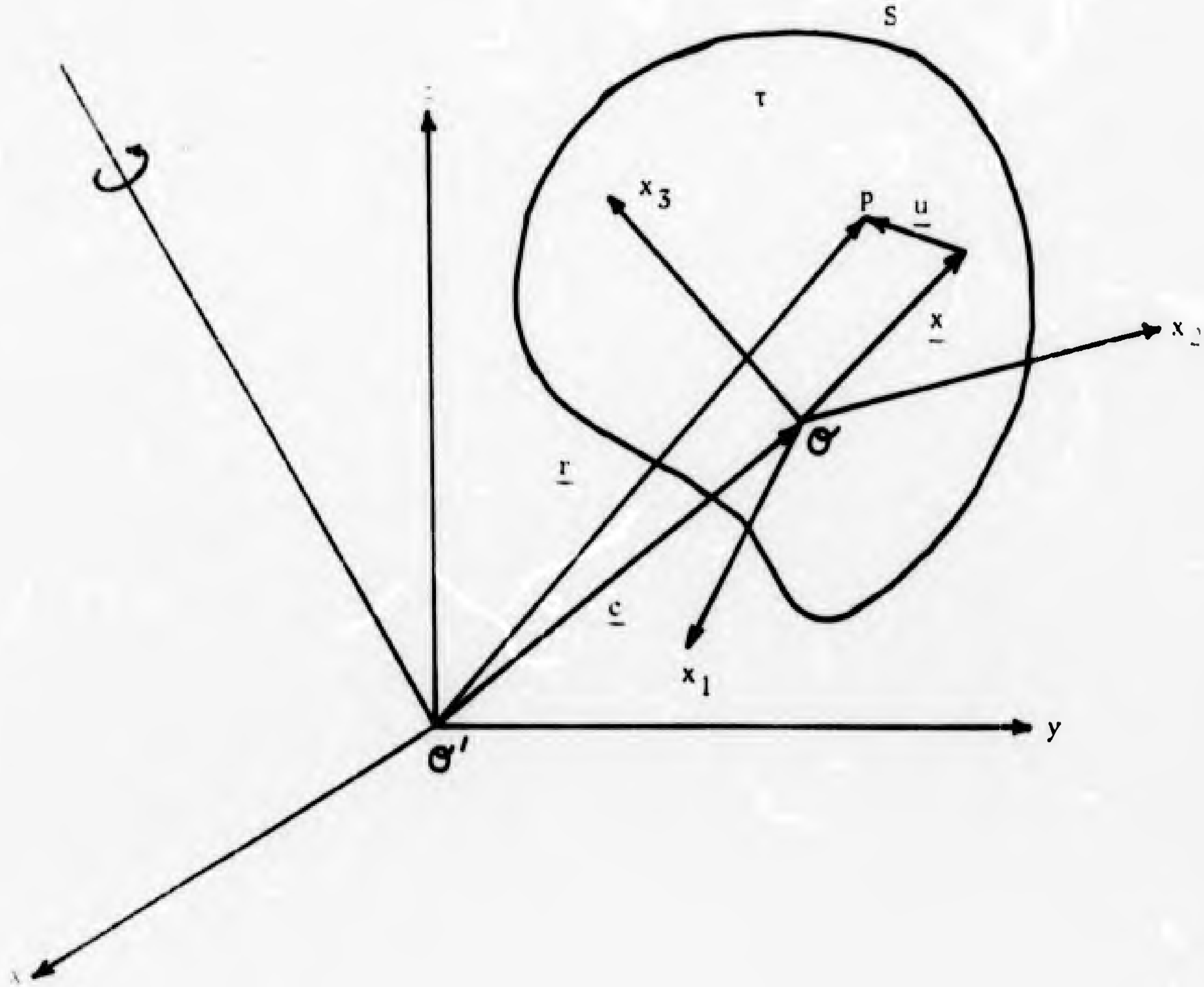

Figure 2. Coordinate systems for a rotating deformable solid.

Recalling from elementary mechanics that

$$\dot{\underline{e}}_i = \underline{\Omega} \times \underline{e}_i$$

$$= \epsilon_{ijk}\, \Omega_j\, \underline{e}_k, \qquad (2.6)$$

where ϵ_{ijk} is the third order alternating tensor, it follows that the velocity vector can be expressed as

$$\dot{\underline{r}} = \dot{\overline{u}}_i\, \underline{e}_i + (c_i + x_i + \overline{u}_i)\dot{\underline{e}}_i$$

$$= [\dot{\overline{u}}_i + \epsilon_{kji}\, \Omega_j (c_k + x_k + \overline{u}_k)]\underline{e}_i, \quad (\dot{\overline{u}}_i = \partial\overline{u}_i/\partial t). \qquad (2.7)$$

If the identity

$$\epsilon_{kji}\, \epsilon_{\ell mi} = \delta_{jm}\, \delta_{k\ell} - \delta_{km}\, \delta_{j\ell} \qquad (2.8)$$

is used, then it is easily shown that

$$\dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{\overline{u}}_i\, \dot{\overline{u}}_i + 2\, \epsilon_{kji}\, \Omega_j (c_k + x_k + \overline{u}_k)\dot{\overline{u}}_i -$$

$$- \Omega_j\, \Omega_k (c_k + x_k + \overline{u}_k)(c_j + x_j + \overline{u}_j) +$$

$$+ \Omega_j\, \Omega_j (c_k + x_k + \overline{u}_k)(c_k + x_k + \overline{u}_k).$$

Therefore, the kinetic energy of a rotating deformable body is

$$T = \int_\tau \frac{1}{2}\rho[\dot{\bar{u}}_i\,\dot{\bar{u}}_i + 2\,\varepsilon_{kji}\,\Omega_j(c_k + x_k + \bar{u}_k)\dot{\bar{u}}_i -$$

$$- \Omega_j\,\Omega_k(c_k + x_k + \bar{u}_k)(c_j + x_j + \bar{u}_j) +$$

$$+ \Omega_j\,\Omega_j(c_k + x_k + \bar{u}_k)(c_k + x_k + \bar{u}_k)]d\tau. \qquad (2.9)$$

Differentiating equation (2.7) one finds

$$\ddot{\underline{r}} = [\ddot{\bar{u}}_i + 2\,\varepsilon_{kji}\,\Omega_j\,\dot{\bar{u}}_k + \Omega_i\,\Omega_k(c_k + x_k + \bar{u}_k) -$$

$$- \Omega_j\,\Omega_j(c_i + x_i + \bar{u}_i)]\underline{e}_i. \qquad (2.10)$$

Hence, in view of equations (2.4) and (2.10), it is evident that the components a_i of the acceleration vector are

$$a_i = \ddot{\bar{u}}_i + 2\,\varepsilon_{kji}\,\Omega_j\,\dot{\bar{u}}_k + \Omega_i\,\Omega_k(c_k + x_k + \bar{u}_k) -$$

$$- \Omega_j\,\Omega_j(c_i + x_i + \bar{u}_i). \qquad (2.11)$$

Therefore, substitution of equation (2.11) into equation (2.1) yields the equation of motion in the following form:

$$\bar{\sigma}_{ij,j} + (\bar{\sigma}_{jk}\, \bar{u}_{i,k})_{,j} + \bar{X}_i = \rho[\ddot{\bar{u}}_i + 2\, \epsilon_{kji}\, \Omega_j\, \dot{\bar{u}}_k +$$

$$+ \Omega_i\, \Omega_k (c_k + x_k + \bar{u}_k) - \Omega_j\, \Omega_j (c_i + x_i + \bar{u}_i)]. \tag{2.12}$$

Suppose that for the state of undisturbed equilibrium $\sigma_{ij}(x)$ denotes the state of initial stress, $v_i(x)$ the corresponding displacements, X_i^* the body force field, and p_i^* the surface tractions applied on S_σ. Next, the body is caused to assume certain small deviations from the position of undisturbed equilibrium. Let the components of the characteristics of the disturbed motion be designated by $\tau_{ij}(x,t)$, $u_i(x,t)$, X_i, and p_i, such that

$$\bar{\sigma}_{ij}(x,t) = \sigma_{ij}(x) + \tau_{ij}(x,t), \qquad \bar{u}_i(x,t) = v_i(x) + u_i(x,t),$$

$$\bar{X}_i = X_i^* + X_i, \qquad \bar{p}_i = p_i^* + p_i. \tag{2.13}$$

Substitution of equation (2.13) into equation (2.12) yields

$$\sigma_{ij,j} + \tau_{ij,j} + (\sigma_{jk}\, v_{i,k} + \sigma_{jk}\, u_{i,k} + \tau_{jk}\, v_{i,k} + \tau_{jk}\, u_{i,k})_{,j} +$$

$$+ X_i^* + X_i = \rho[\ddot{u}_i + 2\, \epsilon_{kji}\, \Omega_j\, \dot{u}_k +$$

$$+ \Omega_i\, \Omega_k (c_k + x_k + v_k + u_k) - \Omega_j\, \Omega_j (c_i + x_i + v_i + u_i)].$$

$$\tag{2.14}$$

The assumption that the disturbances are small enables one to discard the terms $\sigma_{jk}\, v_{i,k}$, $\tau_{jk}\, v_{i,k}$, and $\tau_{jk}\, u_{i,k}$ from equation (2.14) since they are terms of the second order of smallness. Consequently, equation (2.14) becomes

$$\sigma_{ij,j} + \tau_{ij,j} + (\sigma_{jk}\, u_{i,k})_{,j} + X_i^* + X_i =$$

$$= \rho[\ddot{u}_i + 2\, \varepsilon_{kji}\, \Omega_j\, \dot{u}_k + \Omega_i\, \Omega_k (c_k + x_k + v_k + u_k) -$$

$$- \Omega_j\, \Omega_j (c_i + x_i + v_i + u_i)]. \tag{2.15}$$

If it is now required that

$$\sigma_{ij,j} + X_i^* = \rho[\Omega_i\, \Omega_k (c_k + x_k + v_k) - \Omega_j\, \Omega_j (c_i + x_i + v_i)], \tag{2.16}$$

then equation (2.15) reduces to

$$\tau_{ij,j} + (\sigma_{jk}\, u_{i,k})_{,j} + X_i = \rho[\ddot{u}_i + 2\, \varepsilon_{kji}\, \Omega_j\, \dot{u}_k +$$

$$+ \Omega_i\, \Omega_k\, u_k - \Omega_j\, \Omega_j\, u_i]. \tag{2.17}$$

The state of initial stress is described by equation (2.16), whereas the components of the disturbed motion must satisfy equation (2.17).

Proceeding in an analogous fashion with the boundary conditions in equations (2.2) - (2.3), one can show that

$$\sigma_{ij}\, n_j = p_i^* \text{ on } S_\sigma, \qquad v_i = v_i^* \text{ on } S_u \tag{2.18}$$

and

$$(\tau_{ij} + \sigma_{jk}\, u_{i,k})n_j = p_i \text{ on } S_\sigma, \quad u_i = 0 \text{ on } S_u \tag{2.19}$$

are the boundary conditions for the initial stress problem and the disturbed motion problem, respectively.

Therefore, in summary, the initial stress problem

$$\sigma_{ij,j} + X_i^* = \rho[\Omega_i\, \Omega_k(c_k + x_k + v_k) - \Omega_j\, \Omega_j(c_i + x_i + v_i)] \text{ in } \tau, \tag{2.20}$$

$$\sigma_{ij}\, n_j = p_i^* \text{ on } S_\sigma, \qquad v_i = v_i^* \text{ on } S_u \tag{2.21}$$

must be solved first for σ_{ij}, so that the disturbed motion problem

$$\tau_{ij,j} + (\sigma_{jk}\, u_{i,k})_{,j} + X_i = \rho[\ddot{v}_i + 2\, \epsilon_{kji}\, \Omega_j\, \dot{u}_k +$$

$$+ \Omega_i\, \Omega_k\, u_k - \Omega_j\, \Omega_j\, u_i] \text{ in } \tau, \tag{2.22}$$

$$(\tau_{ij} + \sigma_{jk}\, u_{i,k})n_j = p_i \text{ on } S_\sigma, \quad u_i = 0 \text{ on } S_u, \tag{2.23}$$

can subsequently be analyzed.

3. CONSTITUTIVE RELATIONS

The constitutive equations for a Kelvin-Voigt viscoelastic solid are assumed to be

$$\tau_{ij} = c_{ijk\ell}\,\varepsilon_{k\ell} + d_{ijk\ell}\,\dot{\varepsilon}_{k\ell}, \tag{3.1}$$

where the strains ε_{ij} are given by

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{3.2}$$

and the constants $c_{ijk\ell}$, $d_{ijk\ell}$ are the elastic and viscous moduli, respectively.

Suppose that the body force field may be expressed as

$$X_i = f_i(x) + a_{ij}(x)\,u_j - b_{ij}(x)\,\dot{u}_j \tag{3.3}$$

with $a_{ij} = a_{ji}$. This form of the body force permits the inclusion in the present formulation of certain displacement or velocity dependent body forces.

4. A CONSERVATION LAW

For rotating deformable bodies subjected to conservative and non-conservative loads, it is possible, as indicated in Section 1, to postulate a conservation law that can be exploited for the purpose of deriving in a systematic fashion beam, plate, and shell theories.

It is now hypothesized that for the class of boundary value problems described by equations (2.22) - (2.23) and (3.1) - (3.3) there exists a quantity H such that

$$H = \text{constant}, \tag{4.1}$$

where

$$H = T_K + T_G + V + D + W, \tag{4.2}$$

$$T_K = \int_\tau \frac{1}{2} \rho\, \dot{u}_i\, \dot{u}_i\, d\tau, \tag{4.3}$$

$$T_G = \int_o^t \int_\tau 2\rho\, \varepsilon_{kji}\, \Omega_j (\partial_t\, u_k) \dot{u}_i\, d\tau\, d\hat{t}, \quad (\partial_t = \partial/\partial t), \tag{4.4}$$

$$V = \int_\tau [\frac{1}{2} c_{ijk\ell}\, \varepsilon_{ij}\, \varepsilon_{k\ell} + \frac{1}{2} \rho\, \Omega_j\, \Omega_k\, u_k\, u_j - \frac{1}{2} \rho\, \Omega_j\, \Omega_j\, u_k\, u_k] d\tau, \tag{4.5}$$

$$D = \int_o^t \int_\tau (d_{ijk\ell}\, \dot{\varepsilon}_{ij}\, \dot{\varepsilon}_{k\ell} + b_{ij}\, \dot{u}_i\, \dot{u}_j) d\tau\, d\hat{t}, \tag{4.6}$$

$$W = \int_\tau [\frac{1}{2} \sigma_{jk}\, u_{i,k}\, u_{i,j} - f_i\, u_i - \frac{1}{2} a_{ij}\, u_i\, u_j] d\tau - \int_{S_\sigma} [r_i\, u_i + \int_o^t \alpha\, q_j\, u_{i,j}\, \dot{u}_i\, d\hat{t}] dS. \tag{4.7}$$

For the purpose of writing equation (4.7), it has been assumed that the traction p_i appearing in equation (2.23) can be decomposed as follows:

$$p_i = r_i(x) + \alpha(x)\, q_j\, u_{i,j}, \qquad q_j = \sigma_{ij}\, n_i, \tag{4.8}$$

where $r_i(x)$ denotes the conservative portion of the surface traction and $q_j\, u_{i,j}$ the portion of the surface traction due to non-conservative forces. For $\alpha = 0$, the loading is purely conservative, whereas for $\alpha = 1$ the tractions are of the follower type. If $\alpha \neq 0,1$, the tractions may be called sub-tangential or super-tangential forces, depending upon the value of α (see, for example, [37-41]).

The derivative with respect to time of equation (4.1) is obviously

$$dH/dt = 0. \tag{4.9}$$

Substituting equations (4.2) - (4.7) into equation (4.9), one can show,

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[37] NEMAT-NASSER, S. and HERRMANN, G., On the stability of equilibrium of continuous systems. Ingen.-Arch., 35, pp. 17-24 (1966).

[38] NEMAT-NASSER, S. On the stability of equilibrium of nonconservative continuous systems with slight damping. J. Appl. Mech., 34, pp. 344-348 (1967).

[39] NEMAT-NASSER, S. On local stability of a finitely deformed solid subjected to follow type loads. Quart. Appl. Math., 26, pp. 119-129 (1968).

[40] KORDAS, Z. and ZYCZKOWSKI, M. On the loss of stability of a rod under a super-tangential force. Arch. Mech. Stos., 15, pp. 8-31 (1963).

[41] KÖNIG, H. Die Knickkraft beim einseitig eingespannten Stab unter nicht-richtungstreuer Kraftwirkung. Der Stahlbau, 29, pp. 150-154 (1960).

upon integration by parts, that the conservation law leads to

$$\int_\tau [\rho \ddot{u}_i + 2\rho\, \epsilon_{kji}\, \Omega_j\, \dot{u}_k + \rho\, \Omega_i\, \Omega_j\, u_j - \rho\, \Omega_j\, \Omega_j\, u_i - \tau_{ij,j} +$$

$$+ b_{ij}\, \dot{u}_i - (\sigma_{jk}\, u_{i,k})_{,j} - f_i - a_{ij}\, u_j]\dot{u}_i\, d\tau +$$

$$+ \int_{S_u} [(\tau_{ij} + \sigma_{jk}\, u_{i,k})n_j]\dot{u}_i\, dS + \int_{S_\sigma} [(\tau_{ij} + \sigma_{jk}\, u_{i,k})n_j -$$

$$- r_i - \alpha\, q_j\, u_{i,j}]\dot{u}_i\, dS = 0,$$

which is entirely consistent with equations (2.22) and (2.23).

In later sections, the conservation law in equation (4.1) will provide the theoretical basis for the derivation of the equations of motion for several problems involving rotating beams and conservative and non-conservative forces.

5. THE INITIAL STRESS PROBLEM

It should be observed that the components σ_{ij} of the initial stress tensor appear in equation (4.7). Consequently, before proceeding with the application of the conservation law in equation (4.1) to problems of rotating bars and shafts, it is worthwhile to devote some discussion at this point to certain types of initial stress fields which are very often encountered in the contemporary technical literature. In this section, therefore, attention will be focused upon the bar of rectangular cross section depicted in Figure 3. The beam has length ℓ and cross sectional area A. The coordinate axes have been selected so as to coincide with the centroidal axes as well as the principal axes of inertia of the cross section. In subsequent paragraphs, the integrals

$$A = \int_A dA = bh, \quad \int_A x_2 x_3 \, dA = 0, \quad \int_A x_s \, dA = 0, \quad s = 2,3,$$

$$I_{22} = \int_A x_2^2 \, dA = Ab^2/12, \qquad I_{33} = \int_A x_3^2 \, dA = Ah^2/12 \tag{5.1}$$

and the definitions

$$N(x_1) = \int_A \sigma_{11} \, dA, \qquad M_s(x_1) = \int_A x_s \sigma_{11} \, dA,$$

$$Q_s(x_1) = \int_A \sigma_{1s} \, dA, \qquad s = 2,3, \tag{5.2}$$

will be useful. The quantities N, M_s, and Q_s, $s = 2,3$, well known from beam theory, will be called the longitudinal force, the bending moments, and the shear forces, respectively.

For the initial stress problem embodied in equations (2.20) - (2.21), the equations of equilibrium may be expressed as

$$\sigma_{ij,j} + G_i = 0 \text{ in } \tau, \tag{5.3}$$

where

$$G_i = X_i^* - \rho \, \Omega_i \, \Omega_k (c_k + x_k + v_k) + \rho \, \Omega_j \, \Omega_j (c_i + x_i + v_i). \tag{5.4}$$

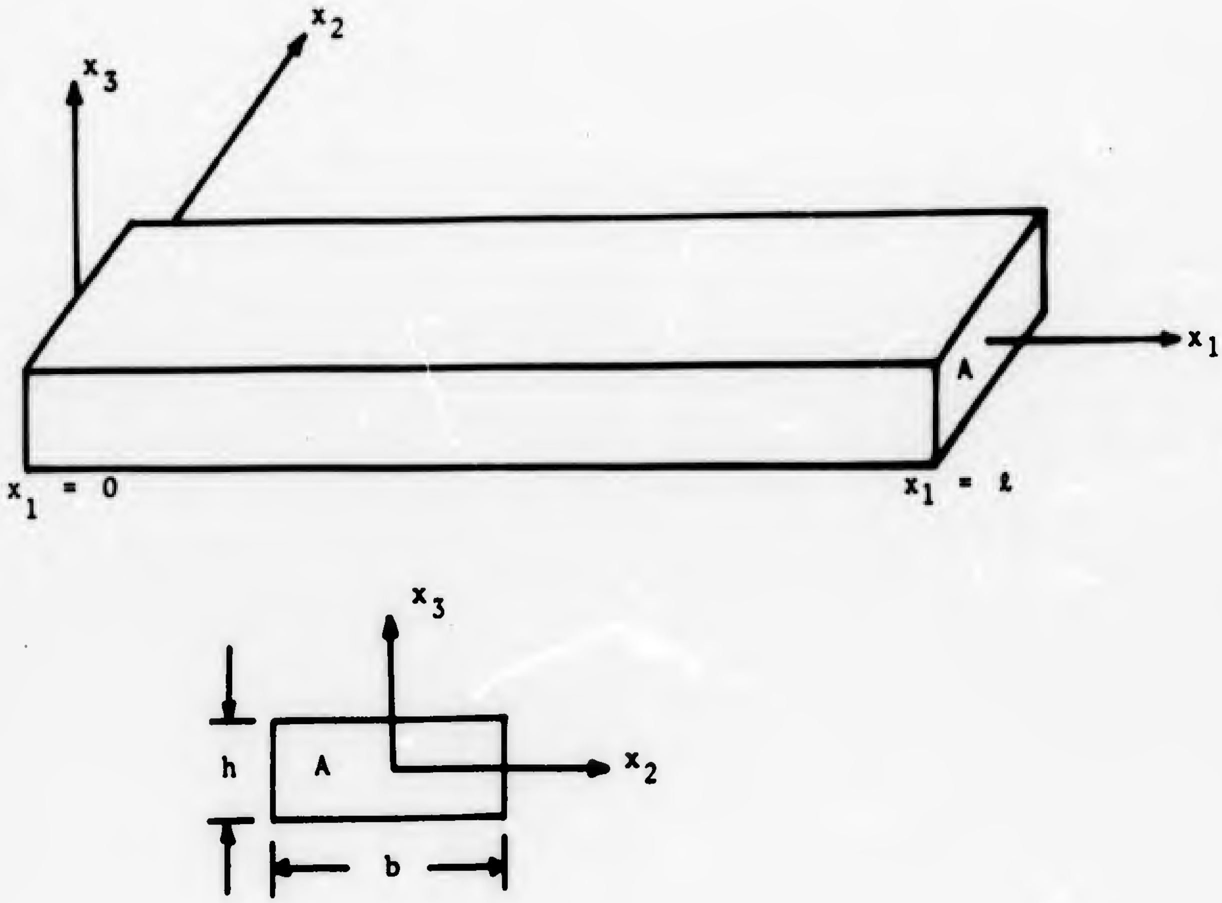

Figure 3. Coordinate system for a bar of rectangular cross section.

For the class of problems to be considered in this report, the initial stresses σ_{11}, σ_{12}, σ_{13}, and σ_{23} may be assumed to be of the form

$$\sigma_{11} = \frac{N(x_1)}{A} + \frac{x_2}{I_{22}} M_2(x_1) + \frac{x_3}{I_{33}} M_3(x_1), \quad \sigma_{23} = 0,$$

$$\sigma_{12} = \frac{Q_2(x_1)}{2I_{22}} [(b/2)^2 - x_2^2], \quad \sigma_{13} = \frac{Q_3(x_1)}{2\,I_{33}} [(h/2)^2 - x_3^2]. \tag{5.5}$$

From the forms assumed for σ_{1s}, $s = 2,3$, it is evident that the surface tractions σ_{13} on $x_3 = \pm\, h/2$ and σ_{12} on $x_2 = \pm\, b/2$ must vanish, whereas formulas for the stresses σ_{22} and σ_{33} will be derived below.

Substitution of equation (5.5) into equation (5.3) with $i = 1$ yields

$$\frac{1}{A} N_{,1} + \frac{x_2}{I_{22}} M_{2,1} + \frac{x_3}{I_{33}} M_{3,1} - \frac{x_2}{I_{22}} Q_2 - \frac{x_3}{I_{33}} Q_3 + G_1 = 0.$$

This equation is now multiplied by x_s^m, where $m = 0,1$ and $s = 2,3$, and the integral over the cross sectional area A of the result is

$$\frac{1}{A} N_{,1} \int_A x_s^m \, dA + \frac{M_{2,1}}{I_{22}} \int_A x_2 x_s^m \, dA + \frac{M_{3,1}}{I_{33}} \int_A x_3 x_s^m \, dA -$$

$$- \frac{Q_2}{I_{22}} \int_A x_2 x_s^m \, dA - \frac{Q_3}{I_{33}} \int_A x_3 x_s^m \, dA +$$

$$+ \int_A x_s^m G_1 \, dA = 0. \tag{5.6}$$

If the definitions

$$g_1(x_1) = \int_A G_1 \, dA, \qquad g_{1s}(x_1) = \int_A x_s G_1 \, dA, \quad s = 2,3, \tag{5.7}$$

are made, then one obtains from equation (5.6)

$$N_{,1} + g_1 = 0, \qquad 0 < x_1 < \ell, \tag{5.8}$$

$$M_{2,1} - Q_2 + g_{12} = 0, \qquad 0 < x_1 < \ell, \tag{5.9}$$

$$M_{3,1} - Q_3 + g_{13} = 0, \qquad 0 < x_1 < \ell, \tag{5.10}$$

for m = 0, m = 1 and s = 2 and m = 1 and s = 3, respectively.

Next, inserting equations (5.5) into equation (5.3) with i = 2 and i = 3, one finds

$$\begin{aligned} &\frac{Q_{2,1}}{2\, I_{22}} [(b/2)^2 - x_2^2] + \sigma_{22,2} + G_2 = 0, \\ &\frac{Q_{3,1}}{2\, I_{33}} [(h/2)^2 - x_3^2] + \sigma_{33,3} + G_3 = 0. \end{aligned} \tag{5.11}$$

If the integral over the area A is taken of equation (5.11), the result is found to be

$$Q_{s,1} + R_s + g_s = 0, \qquad s = 2,3, \tag{5.12}$$

where

$$R_2 = \int_{-h/2}^{h/2} [\sigma_{22}]_{x_2 = -b/2}^{x_2 = b/2} dx_3,$$

$$R_3 = \int_{-b/2}^{b/2} [\sigma_{33}]_{x_3 = -h/2}^{x_3 = h/2} dx_2, \qquad (5.13)$$

$$g_s = \int_A G_s \, dA$$

To find expressions for σ_{22} and σ_{33}, one writes equations (5.11) as

$$\sigma_{22,2} = - G_2 - \frac{Q_{2,1}}{2\, I_{22}} [(b/2)^2 - x_2^2],$$

$$\sigma_{33} = - G_3 - \frac{Q_{3,1}}{2\, I_{33}} [(h/2)^2 - x_3^2],$$

and performs the indicated integrations over the intervals $(-b/2, x_2)$ and $(-h/2, x_3)$, respectively,

$$\sigma_{22} = - \frac{Q_{2,1}}{24\, I_{22}} (b^3 + 3b^2 x_2 - 4x_2^3) - - \int_{-b/2}^{x_2} G_2(x_1, y, x_3)\, dy + \sigma_{22}\Big|_{x_2 = -b/2}, \qquad (5.14)$$

$$\sigma_{33} = -\frac{Q_{3,1}}{24\, I_{33}}(h^3 + 3h^2 x_3 - 4x_3^3) -$$

$$- \int_{-h/2}^{x_3} G_3(x_1, x_2, z)dz + \sigma_{33}\Big|_{x_3 = -h/2} . \tag{5.15}$$

In summary, then, to determine the initial stress field, one must first solve the ordinary differential equations

$$N_{,1} + g_1 = 0, \tag{5.16}$$

$$M_{s,1} - Q_s + g_{1s} = 0, \tag{5.17}$$

$$Q_{s,1} + R_s + g_s = 0, \tag{5.18}$$

for s = 2,3, on the interval $0 < x_1 < \ell$, subject to the appropriate boundary conditions. Finally, the initial stresses may be determined from equations (5.5), (5.14) - (5.15).

6. EXAMPLES OF INITIAL STRESS PROBLEMS

Beam in a Gravity Field

Consider a beam inclined at an angle α relative to the horizontal with the axis of the beam lying along the inclined x_1-axis as shown in Figure 4. The force of gravity acts vertically downward, i.e., in the negative y-direction. In this case the components of the body

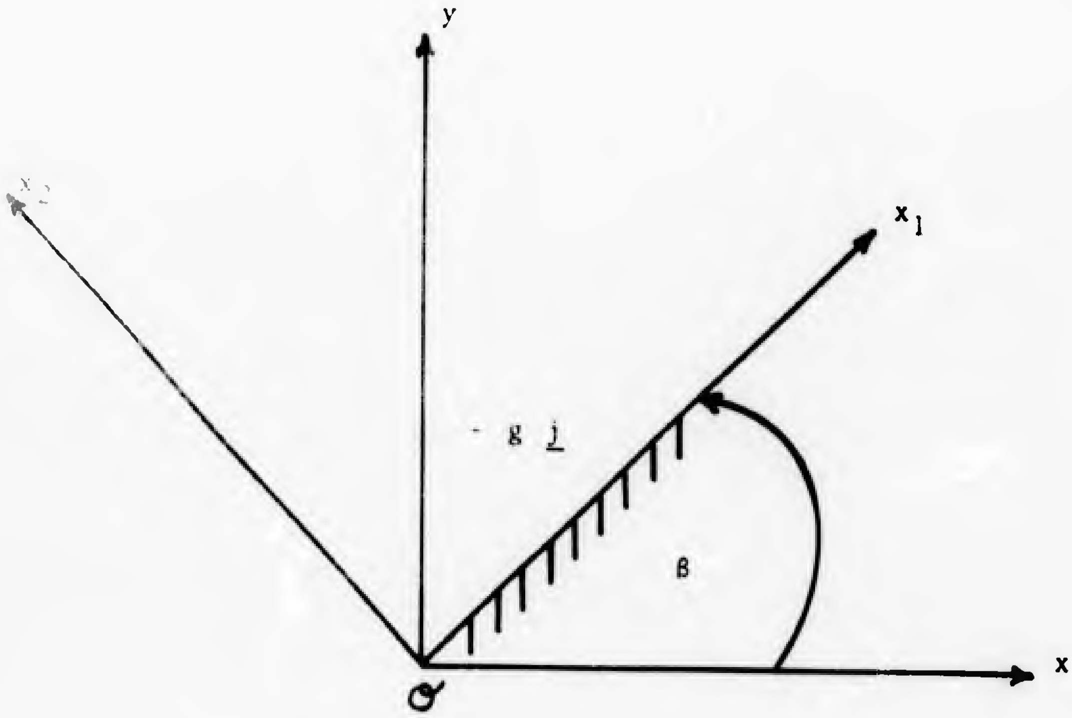

Figure 4. Coordinate system for an inclined bar.

force vector per unit volume are

$$G_1 = -\rho g \sin \beta, \qquad G_2 = 0, \qquad G_3 = -\rho g \cos \beta, \tag{6.1}$$

for a non-rotating beam ($\Omega_i = 0$). In this case, it may be assumed that $M_2 = Q_2 = 0$, so that equations (5.16) - (5.18) become

$$N_{,1} - \rho g A \sin \beta = 0, \quad Q_{3,1} - \rho g A \cos \beta = 0, \quad M_{3,1} - Q_3 = 0. \tag{6.2}$$

Because the end of the beam $x_1 = \ell$ is assumed to be traction free in the initial stress state, the boundary conditions associated with equation (6.2) must be

$$N(\ell) = M_3(\ell) = Q_3(\ell) = 0. \tag{6.3}$$

The solutions of equations (6.2) - (6.3) are easily verified to be

$$\begin{aligned} N(x_1) &= -\rho g A(\ell - x_1) \sin \beta, \\ Q_3(x_1) &= -\rho g A(\ell - x_1) \cos \beta, \\ M_3(x_1) &= \frac{1}{2} \rho g A(\ell - x_1)^2 \cos \beta. \end{aligned} \tag{6.4}$$

Therefore, one has

$$\sigma_{12} = \sigma_{23} = \sigma_{22} = 0, \tag{6.5}$$

whereas the stresses σ_{11}, σ_{13}, and σ_{33} may be obtained from equations (5.5) and (5.15).

An Eccentric Follower Force

Consider a beam of length ℓ loaded by a compressive force of magnitude P as shown in Figure 5. The force P is applied at a distance e from the centroid of the cross section at $x_1 = \ell$ and on the x_2-axis. Such a beam is said to be loaded eccentrically, and the stability of a cantilever under the action of an eccentric follower force has been discussed in references [34-35].

For the geometry and loading condition shown in Figure 5, it may be assumed that

$$M_3 = Q_3 = \sigma_{13} = \sigma_{33} = 0 \text{ and } \sigma_{22} = 0 \text{ on } x_2 = \pm b/2.$$

Hence, in the absence of body forces and for $\Omega_i = 0$, equations (5.16) - (5.18) reduce to

$$N_{,1} = Q_{2,1} = 0, \quad M_{2,1} - Q_2 = 0, \quad 0 < x_1 < \ell. \qquad (6.6)$$

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[34] NEMAT-NASSER, S. and TSAI, P. F. Effect of warping rigidity on stability of a bar under eccentric follower force. Int. J. Solids Structures, 5, pp. 271-279 (1969).

[35] LIN, K.-H., NEMAT-NASSER, S. and HERRMANN, G. Stability of a bar under eccentric follower force. Proc. ASCE, J. Engng. Mech. Div., 93, pp. 105-115 (1967).

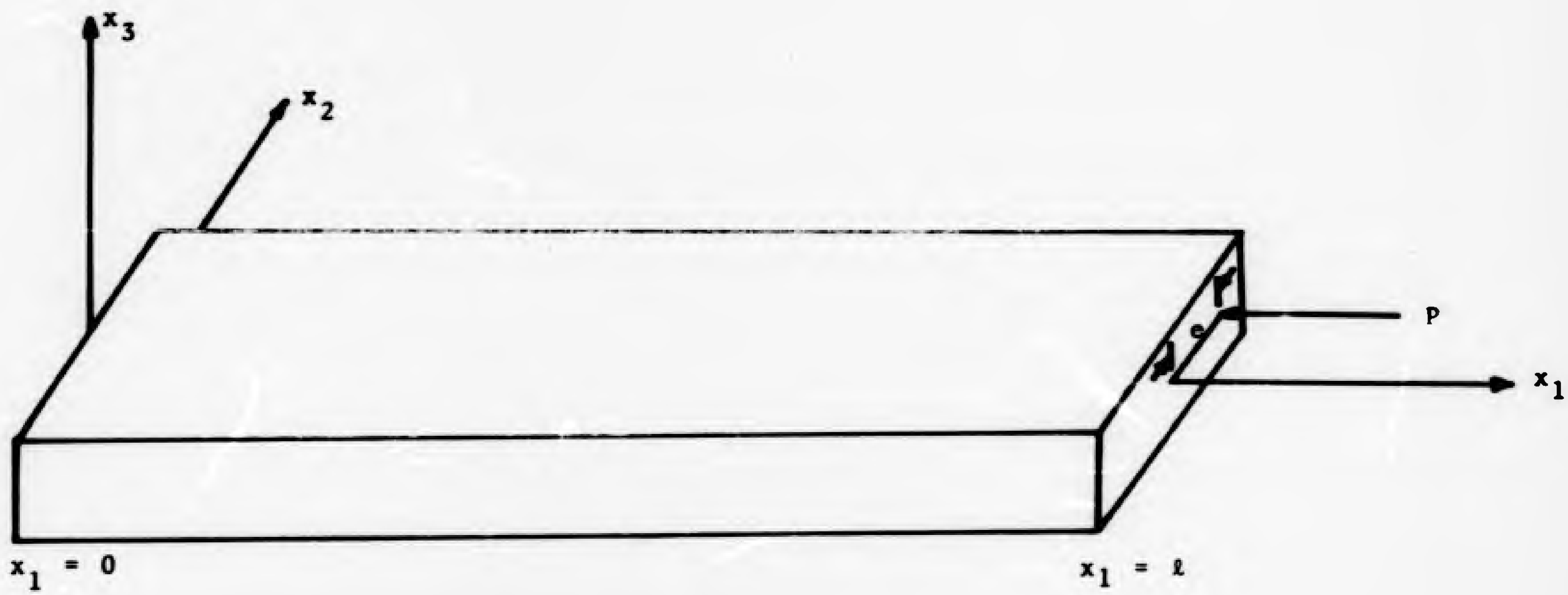

Figure 5. An eccentrically loaded bar.

On the face $x_1 = \ell$ of the beam, the force

$$\underline{F}_1 = - P\, \delta(x_2 - e)\, \delta(x_3) \underline{e}_1,$$

where $\delta(. \; . \; .)$ denotes the Dirac delta function, is acting, and the position vector $\underline{r}_1$ from the centroid of this face to the point of application of $\underline{F}_1$ is

$$\underline{r}_1 = e\, \underline{e}_2 .$$

Thus, the longitudinal force and bending moment, $N(x_1)$ and $M_2(x_1)$, respectively, acting over the end of the beam are given by

$$N(\ell)\underline{e}_1 = \int_A \underline{F}_1 \, dA = - P\, \underline{e}_1 \int_A \delta(x_2 - e)\, \delta(x_3) dA = - P\, \underline{e}_1,$$

$$M_2(\ell)\underline{e}_3 = \int_A \underline{r}_1 \times \underline{F}_1 \, dA = P\, e\, \underline{e}_3,$$

so that

$$N(\ell) = - P, \qquad M_2(\ell) = P\, e. \tag{6.7}$$

Because no shear forces are applied on $x_1 = \ell$, one concludes that

$$Q_2(\ell) = 0. \tag{6.8}$$

Consequently, the solutions of equations (6.6) subject to the boundary conditions in equations (6.7) - (6.8) are easily shown to be

$$N(x_1) = -P, \quad Q_2(x_1) = 0, \quad M_2(x_1) = Pe. \tag{6.9}$$

Therefore, by virtue of equations (5.5) and (5.14), the initial stresses are

$$\sigma_{11} = -\frac{P}{A} + x_2\frac{Pe}{I_{22}}, \quad \sigma_{12} = \sigma_{22} = 0. \tag{6.10}$$

If a second concentrated force of magnitude P is applied at the point $x_2 = -e$, $x_3 = 0$ on the face $x_1 = \ell$, then, in analogy with $\underline{r}_1$ and $\underline{F}_1$, the position and force vectors

$$\underline{r}_2 = -e\,\underline{e}_2, \quad \underline{F}_2 = -P\,\delta(x_2 + e)\delta(x_3)\underline{e}_1,$$

respectively, may be introduced. Following the steps outlined in the previous paragraph, it can be shown for the case of a symmetric pair of eccentric forces that

$$N(x_1) = -2P, \quad Q_2(x_1) = M_2(x_1) = 0.$$

Consequently, the initial stress state must be given by

$$\sigma_{11} = -\frac{2P}{A}, \quad \sigma_{12} = \sigma_{22} = 0. \tag{6.11}$$

The torsional stability of a cantilever subjected at its free end to a pair of symmetrically applied compressive follower forces was described in reference [42].

A Transverse Follower Force

For the beam loaded as shown in Figure 6, it may again be assumed that

$$M_3 = Q_3 = \sigma_{13} = \sigma_{33} = 0,$$

but now

$$\sigma_{22} = \begin{cases} -p(x_1, x_3) & \text{on } x_2 = b/2, \\ 0 & \text{on } x_2 = -b/2, \end{cases}$$

so that by equation (5.13),

$$R_2(x_1) = -\int_{-h/2}^{h/2} p(x_1, z)\,dz. \qquad (6.12)$$

In addition, it is assumed that $X_i^* = \Omega_i = 0$. In this case, equations (5.16)-(5.18) simplify to

$$N_{,1} = 0, \quad M_{2,1} - Q_2 = 0, \quad Q_{2,1} + R_2 = 0, \; 0 < x_1 < \ell. \qquad (6.13)$$

[42] NEMAT-NASSER, S. and HERRMANN, G. Torsional instability of cantilevered bars subjected to nonconservative loading. J. Appl. Mech., 33, pp. 102-104 (1966).

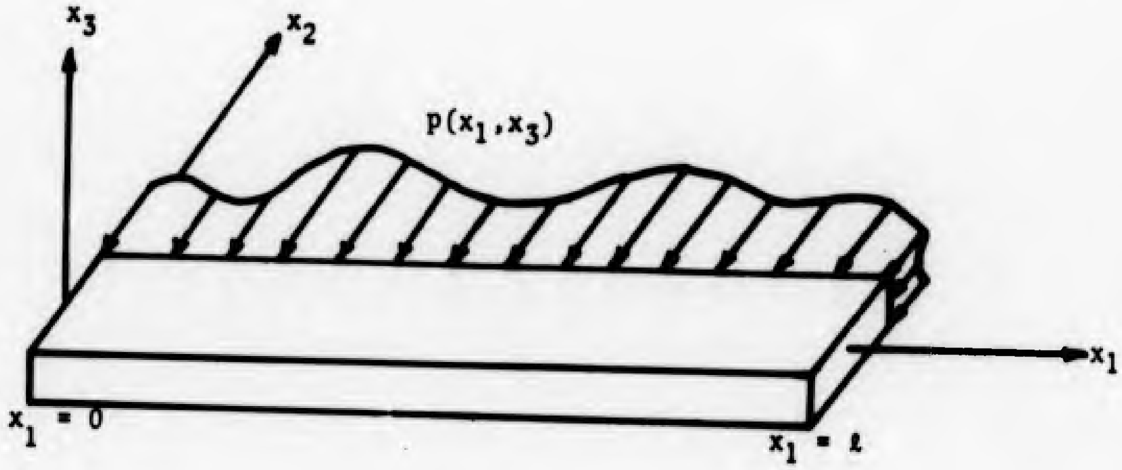

Figure 6. A transversely loaded bar.

Since the end $x_1 = \ell$ of the beam is traction free, the boundary conditions must be

$$N(\ell) = Q_2(\ell) = M_2(\ell) = 0. \tag{6.14}$$

The solutions of equations (6.13) subject to the boundary conditions in equation (6.14) are easily shown to be

$$N(x_1) = 0, \qquad Q_2(x_1) = \int_{x_1}^{\ell} R_2(y)dy,$$

$$M_2(x_1) = -\int_{x_1}^{\ell} (y-x_1)R_2(y)dy. \tag{6.15}$$

Hence, according to equations (5.5) and (5.14), the initial stress components are

$$\sigma_{11} = \frac{x_2}{I_{22}} M_2(x_1), \qquad \sigma_{12} = \frac{Q_2(x_1)}{2\ I_{22}} [(b/2)^2 - x_2^2],$$

$$\sigma_{22} = \frac{R_2(x_1)}{24\ I_{22}} (b^3 + 3b^2x_2 - 4x_2^3). \tag{6.16}$$

In the special case that

$$p(x_1, x_3) = P\ \delta(x_1 - c)\delta(x_3),$$

it follows from equation (6.12) that

$$R_2(x_1) = -P\,\delta(x_1 - c), \qquad (6.17)$$

and as a result equation (6.15) yields

$$Q_2(x_1) = -P\int_{x_1}^{\ell} \delta(y-c)dy = \begin{cases} -P \text{ if } x_1 < c \\ 0 \text{ if } c < x_1 \end{cases} \qquad (6.18)$$

and

$$M_2(x_1) = P\int_{x_1}^{\ell} (y - x_1)\delta(y-c)dy = \begin{cases} P(c-x_1) & \text{if } x_1 < c \\ 0 & \text{if } c < x_1. \end{cases}$$

(6.19)

Centrifugal Force

Consider a beam of length ℓ, the axis of which lies along the x_1-axis in the coordinate system shown in Figure 7. The $y_1\ y_2\ y_3$-coordinate system is fixed in space, and the $x_1\ x_2\ x_3$-coordinate frame rotates at uniform angular velocity Ω about the vertical y_3-axis. The x_1-axis makes a constant angle β, called the coning angle, with the $y_1\ y_2$-plane. The components of $\underline{\Omega}$ relative to the moving coordinate frame are

$$\Omega_1 = \Omega \sin\beta, \quad \Omega_2 = 0, \quad \Omega_3 = \Omega\cos\beta. \qquad (6.20)$$

In the absence of body forces X_i^*, the initial stresses in the rotating beam are due to the so-called centrifugal forces. If the displacements v_i are neglected in equation (5.4), one has

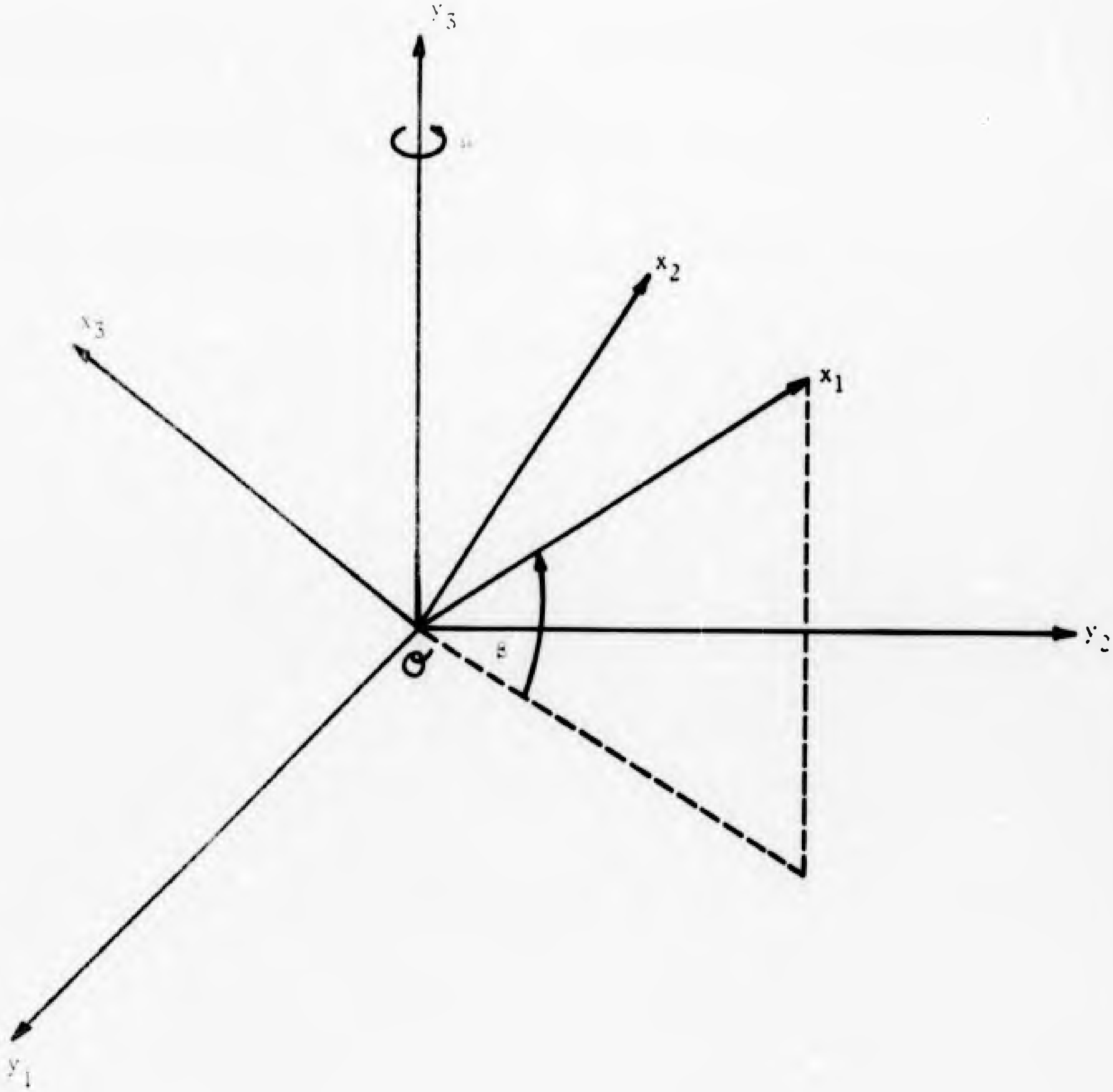

Figure 7. A rotating bar inclined at an angle β.

$$G_i = -\rho\,\Omega_i\,\Omega_k(c_k + x_k) + \rho\,\Omega_j\,\Omega_j(c_i + x_i). \tag{6.21}$$

If it is supposed further that the root of the beam is offset from the axis of rotation, as in Figure 2, by a distance c_1 (sometimes called the hub radius) measured along the x_1-axis, i.e., $c_1 > 0$, $c_2 = c_3 = 0$, then from equations (6.20) - (6.21) the components of the effective body force vector are found to be

$$G_1 = f_{11}(c_1 + x_1) + f_{13}\,x_3, \qquad G_2 = \rho\,\Omega^2\,x_2,$$

$$G_3 = f_{13}(c_1 + x_1) + f_{33}\,x_3 \tag{6.22}$$

where

$$f_{11} = \rho\,\Omega^2\cos^2\beta, \qquad f_{13} = -\rho\,\Omega^2\sin\beta\cos\beta,$$

$$f_{33} = \rho\,\Omega^2\sin^2\beta. \tag{6.23}$$

According to equations (5.7) and (5.13), it follows from equation (6.22) that

$$g_1 = f_{11}\,A(c_1 + x_1), \quad g_2 = 0, \quad g_3 = f_{13}\,A(c_1 + x_1),$$

$$R_2 = R_3 = 0, \quad g_{12} = 0, \quad g_{13} = f_{13}\,I_{33}.$$

In view of these equations, equations (5.16) - (5.18) become

$$N_{,1} + f_{11} A(c_1 + x_1) = 0, \quad M_{2,1} - Q_2 = 0,$$

$$M_{3,1} - Q_3 + f_{13} I_{33} = 0, \quad Q_{2,1} = 0, \tag{6.24}$$

$$Q_{3,1} + f_{13} A(c_1 + x_1) = 0.$$

The boundary conditions are

$$N(\ell) = Q_s(\ell) = M_s(\ell) = 0, \quad s = 2,3. \tag{6.25}$$

The integrals of equations (6.24) - (6.25) are

$$N(x_1) = f_{11} A[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)],$$

$$Q_2(x_1) = M_2(x_1) = 0,$$

$$Q_3(x_1) = f_{13} A[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)], \tag{6.26}$$

$$M_3(x_1) = f_{13}[I_{33} - A\ell(c_1 + \ell/2)](\ell - x_1) +$$

$$+ \frac{1}{2} f_{13} A[c_1(\ell^2 - x_1^2) + \frac{1}{3}(\ell^3 - x_1^3)].$$

Consequently, the stresses are

$$\sigma_{11} = \frac{N(x_1)}{A} + \frac{x_3}{I_{33}} M_3(x_1), \quad \sigma_{12} = \sigma_{23} = 0,$$

$$\sigma_{13} = \frac{Q_3(x_1)}{2\, I_{33}} [(h/2)^2 - x_3^2], \quad \sigma_{22} = \frac{1}{2} \rho\, \Omega^2 [(b/2)^2 - x_2^2], \tag{6.27}$$

$$\sigma_{33} = [2(f_{13}/h^2) x_3 (c_1 + x_1) + f_{33}/2][(h/2)^2 - x_3^2].$$

For $\beta = 0$, equations (6.26) reduce to

$$N(x_1) = \rho\, \Omega^2 A[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)], \tag{6.28}$$

$$Q_s(x_1) = M_s(x_1) = 0, \quad s = 2,3. \tag{6.29}$$

A second example, of considerable importance, of the flexural-torsional deformation of a rotating beam is that of a beam for which the angle β, as described above, is zero but whose cross section is inclined at an angle γ, called the angle of attack, relative to the horizontal plane (i.e., the $y_1\, y_2$-plane in Figure 8). If the beam rotates with constant angular velocity Ω about the vertical y_3-axis, then the components of $\underline{\Omega}$ relative to the moving $x_1\, x_2\, x_3$-frame of reference are

$$\Omega_1 = 0, \quad \Omega_2 = \Omega \sin\gamma, \quad \Omega_3 = \Omega \cos\gamma. \tag{6.30}$$

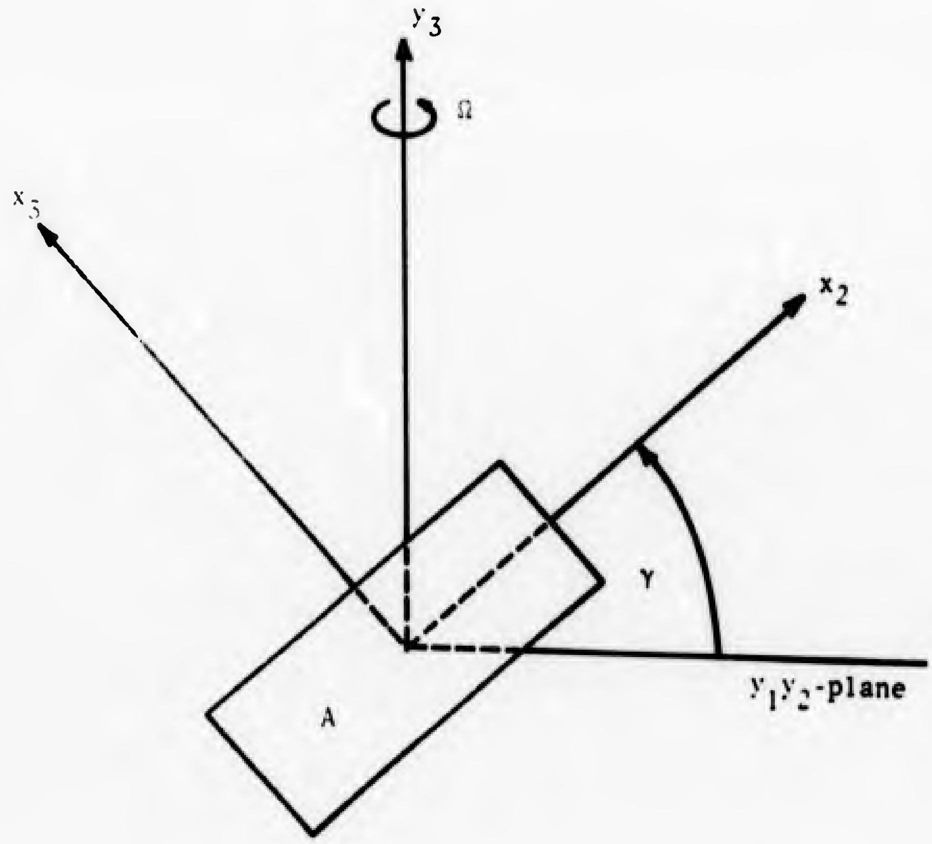

Figure 8. An inclined rotating cross section.

With $c_2 = c_3 = 0$, equations (6.30) and (6.21) provide

$$G_1 = \rho \, \Omega^2 (c_1 + x_1),$$

$$G_2 = \rho \, \Omega^2 \cos \gamma (x_2 \cos \gamma - x_3 \sin \gamma), \qquad (6.31)$$

$$G_3 = \rho \, \Omega^2 \sin \gamma (x_3 \sin \gamma - x_2 \cos \gamma).$$

Thus, from equations (5.7) and (5.13), it immediately follows that

$$g_1 = \rho \, A \, \Omega^2 (c_1 + x_1), \quad g_{1s} = g_s = R_s = 0,$$

$$s = 2,3. \qquad (6.32)$$

The result of substituting equation (6.32) into equations (5.16) - (5.18) is

$$N_{,1} + g_1 = 0, \quad M_{s,1} - Q_s = 0, \quad Q_{s,1} = 0. \qquad (6.33)$$

The solutions of equation (6.33) subject to the boundary conditions in equation (6.25) are

$$N(x_1) = \rho \, A \, \Omega^2 [c_1 (\ell - x_1) + \frac{1}{2} (\ell^2 - x_1^2)],$$

$$M_s = Q_s = 0. \qquad (6.34)$$

Finally, from equations (6.31), (6.34), (5.5), and (5.14) - (5.15), the initial state of stress is found to be

$$\sigma_{11} = N(x_1)/A, \quad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0,$$

$$\sigma_{22} = -\rho\,\Omega^2 \cos\gamma\,[\tfrac{1}{2}(x_2^2 - b^2/4)\cos\gamma -$$

$$- x_3(x_2 + b/2)\sin\gamma], \qquad (6.35)$$

$$\sigma_{33} = \rho\,\Omega^2 \sin\gamma\,[x_2(x_3 + h/2)\cos\gamma - \tfrac{1}{2}(x_3^2 - h^2/4)\sin\gamma].$$

7. THE TIMOSHENKO BEAM UNDER GRAVITY

As a first example of the applicability of the conservation law stated in equation (4.1), suppose that a non-rotating beam is inclined at an angle β relative to the horizontal and is subjected to the force of gravity as depicted in Figure 4. In addition, it will be assumed that a follower force of magnitude P is applied at the centroid of the face of the beam at $x_1 = \ell$. Thus, for

$$\Omega_i = f_i = a_{ij} = r_i = 0, \quad \alpha(x) = 1, \quad b_{12} = b_{21} = 0,$$

equations (4.3) - (4.7) become

$$T_K = \int_\tau \frac{1}{2} \rho \dot{u}_i \dot{u}_i \, d\tau, \quad T_G = 0,$$

$$V = \int_\tau \frac{1}{2} c_{ijk\ell} \varepsilon_{ij} \varepsilon_{k\ell} \, d\tau,$$

$$D = \int_0^t \int_\tau (d_{ijk\ell} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{k\ell} + b_{11} \dot{u}_1^2 + b_{22} \dot{u}_2^2) d\tau d\hat{t}, \tag{7.1}$$

$$W = \int_\tau \frac{1}{2} \sigma_{jk} u_{i,k} u_{i,j} \, d\tau - \int_0^t \int_{S_\sigma} q_j u_{i,j} \dot{u}_i \, dS \, d\hat{t}.$$

According to equations (6.4) and (6.9), with $e = 0$, i.e., vanishing eccentricity, the expressions for N, Q_3, and M_3 are

$$N(x_1) = -\rho g A(\ell - x_1) \sin \beta, \quad Q_3(x_1) = -\rho g A(\ell - x_1) \cos \beta,$$

$$M_3(x_1) = \frac{1}{2} \rho g A(\ell - x_1)^2 \cos \beta,$$

and

$$N(x_1) = -P, \quad Q_2(x_1) = 0, \quad M_2(x_1) = 0$$

for the case of gravity and the case of the axially applied compressive force P, respectively. But by the principle of superposition, the effective longitudinal and shear forces and bending moment must be

$$N(x_1) = -[P + \rho g A(\ell - x_1) \sin \beta],$$

$$Q_3(x_1) = -\rho g A(\ell - x_1) \cos \beta, \tag{7.2}$$

$$M_3(x_1) = \frac{1}{2} \rho g A(\ell - x_1)^2 \cos \beta.$$

From equations (5.15) - (5.16), the initial stresses are

$$
\begin{aligned}
\sigma_{11} &= \frac{N(x_1)}{A} + \frac{x_3}{I_{33}} M_3(x_1), \\
\sigma_{13} &= \frac{Q_3(x_1)}{2\, I_{33}} [(h/2)^2 - x_3^2], \qquad (7.3) \\
\sigma_{33} &= \frac{-\rho\, g\, A}{6\, I_{33}} x_3 [(h/2)^2 - x_3^2] \cos \beta.
\end{aligned}
$$

A Timoshenko type theory for an isotropic solid will be obtained for the beam geometry shown in Figure 3. The beam is assumed to be clamped at the end $x_1 = 0$ and free at the end $x_1 = \ell$. The displacements are assumed to be approximated by

$$u_1 = x_3\, \psi(x_1, t), \quad u_2 = 0, \quad u_3 = w(x_1, t). \qquad (7.4)$$

Thus, by virtue of equations (7.4) and (3.2), the strains ε_{11} and ε_{13} are

$$\varepsilon_{11} = x_3\, \psi_{,1}, \quad \varepsilon_{13} = \frac{1}{2} (\psi + w_{,1}). \qquad (7.5)$$

The strains ε_{22} and ε_{33}, however, are not computed in this manner; instead one assumes that $\tau_{22}^{(1)} = \tau_{33}^{(1)} = 0$, where for an isotropic material

$$\tau_{ij}^{(1)} = 2\, \mu\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk}\, \delta_{ij},$$

μ and λ being the Lamé constants. Thus,

$$\tau_{22}^{(1)} = (2\mu + \lambda)\,\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{33}) = 0,$$

$$\tau_{33}^{(1)} = (2\mu + \lambda)\,\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22}) = 0,$$

whence one finds

$$\varepsilon_{22} = \varepsilon_{33} = \frac{-\lambda}{2(\mu+\lambda)}\,\varepsilon_{11}.$$

Therefore, the expression for V in equation (7.1) becomes

$$V = \int_\tau \frac{1}{2}\,[E\,\varepsilon_{11}^2 + \mu\,\varepsilon_{13}^2]d\tau, \tag{7.6}$$

where E denotes Young's,modulus. It is customary to introduce a shear correction factor K, where $K^2 = \pi^2/12$, by replacing ε_{13} by $K\,\varepsilon_{13}$. Thus, equation (7.6) becomes

$$V = \int_\tau \frac{1}{2}\,[E\,\varepsilon_{11}^2 + K^2\,\mu\,\varepsilon_{13}^2]d\tau. \tag{7.7}$$

Analogously, the expression for D in equation (7.1) can be shown to be

$$D = \int_0^t \int_\tau [E^*\,\dot{\varepsilon}_{11}^2 + K^2\,\mu^*\,\dot{\varepsilon}_{13}^2]d\tau\,d\hat{t}, \tag{7.8}$$

where E^*, μ^* are the moduli of viscosity.

Therefore, in view of equations (5.1) and (7.3) - (7.5), equations (7.1), (7.7), and (7.8) become

$$T_K = \int_0^{\ell} \frac{1}{2} \rho [I_{33} \dot{\psi}^2 + A \dot{w}^2] dx_1,$$

$$T_G = 0,$$

$$V = \int_0^{\ell} \frac{1}{2} [E I_{33} \psi_{,1}^2 + K^2 \mu A (w_{,1} - \psi)^2] dx_1,$$

$$D = \int_0^t \int_0^{\ell} [E^* I_{33} \dot{\psi}_{,1}^2 + K^2 \mu^* A (\dot{w}_{,1} - \dot{\psi})^2 +$$

$$+ b_{11} I_{33} \dot{\psi}^2 + b_{22} A \dot{w}^2] dx_1 \, d\hat{t},$$

$$W = \int_0^{\ell} \frac{1}{2} [(I_{33}/A) N \psi_{,1}^2 + N w_{,1}^2] dx_1 +$$

$$+ \int_0^t P(I_{33} \psi_{,1} \dot{\psi} + A w_{,1} \dot{w}) \Big|_{x_1 = \ell} d\hat{t}.$$

The time derivatives of these quantities are, after integration by parts,

$$dT_K/dt = \int_0^{\ell} (\rho I_{33} \ddot{\psi} \dot{\psi} + \rho A \ddot{w} \dot{w}) dx_1, \quad dT_G/dt = 0,$$

$$dV/dt = \int_0^{\ell} [-E I_{33} \psi_{,11} \dot{\psi} - K^2 \mu A (w_{,1} - \psi) \dot{\psi} -$$

$$- K^2 \mu A (w_{,11} - \psi_{,1}) \dot{w}] dx_1 +$$

$$+ [E I_{33} \psi_{,1} \dot{\psi} + K^2 \mu A (w_{,1} - \psi) \dot{w}]_{x_1 = 0}^{x_1 = \ell},$$

$$dD/dt = \int_0^{\ell} \{-[E^* I_{33} \dot{\psi}_{,11} + K^2 \mu^* A(\dot{w}_{,1} - \dot{\psi}) - b_{11} I_{33} \dot{\psi}]\dot{\psi} -$$

$$- [K^2 \mu^* A(\dot{w}_{,11} - \dot{\psi}_{,1}) - b_{22} A\dot{w}]\dot{w}\}dx_1 +$$

$$+ [E^* I_{33} \dot{\psi}_{,1} \dot{\psi} + K^2 \mu^* A(\dot{w}_{,1} - \psi)\dot{w}]_{x_1=0}^{x_1=\ell} ,$$

$$dW/dt = \int_0^{\ell} [-(I_{33}/A)(N \psi_{,1})_{,1} \dot{\psi} - (Nw_{,1})_{,1} \dot{w}]dx_1 +$$

$$+ [(I_{33}/A)P \psi_{,1} \dot{\psi} + P w_{,1} \dot{w}]_{x_1=\ell} +$$

$$+ [(I_{33}/A)N \psi_{,1} \dot{\psi} + N w_{,1} \dot{w}]_{x_1=0}^{x_1=\ell} . \qquad (7.9)$$

Therefore, substitution of equation (7.9) into equations (4.2) and (4.9) yields

$$\int_0^{\ell} \{[\rho I_{33} \ddot{\psi} + b_{11} I_{33} \dot{\psi} - E I_{33} \psi_{,11} - E^* I_{33} \dot{\psi}_{,11} - K^2 \mu A(w_{,1} - \psi) -$$

$$- K^2 \mu^* A(\dot{w}_{,1} - \dot{\psi}) - (I_{33}/A)(N \psi_{,1})_{,1}]\dot{\psi} + [\rho A\ddot{w} + b_{22} A\dot{w} -$$

$$- K^2 \mu A(w_{,11} - \psi_{,1}) - K^2 \mu^* A(\dot{w}_{,11} - \dot{\psi}_{,1}) - (Nw_{,1})_{,1}]\dot{w}\}dx_1 +$$

$$+ [EI_{33} \psi_{,1} + E^* I_{33} \dot{\psi}_{,1} + (I_{33}/A)N \psi_{,1} +$$

$$+ P(I_{33}/A)\psi_{,1}]\dot{\psi}\Big|_{x_1=\ell} - [E I_{33} \psi_{,1} + E^* I_{33} \dot{\psi}_{,1} +$$

$$+ (I_{33}/A)N\,\psi_{,1}]\dot{\psi}\Big|_{x_1=0} + [K^2\mu A(w_{,1} - \psi) + K^2\mu^* A(\dot{w}_{,1} - \dot{\psi}) +$$

$$+ (N+P)w_{,1}]\dot{w}\Big|_{x_1=\ell} - [K^2\mu A(w_{,1} - \psi) + K^2\mu^* A(\dot{w}_{,1} - \dot{\psi}) +$$

$$+ Nw_{,1}]\dot{w}\Big|_{x_1=0} = 0. \qquad (7.10)$$

Now from equation (7.10) one obtains the equations of motion

$$E\,I_{33}\,\psi_{,11} + E^* I_{33}\,\dot{\psi}_{,11} + K^2\mu A(w_{,1} - \psi) + K^2\mu^* A(\dot{w}_{,1} - \dot{\psi}) -$$

$$- (I_{33}/A)(N\,\psi_{,1})_{,1} = \rho\, I_{33}\,\ddot{\psi} + b_{11}\, I_{33}\,\dot{\psi}, \qquad (7.11)$$

$$K^2\mu A(w_{,11} - \psi_{,1}) + K^2\mu A(\dot{w}_{,11} - \dot{\psi}_{,1}) + (Nw_{,1})_{,1} =$$

$$= \rho A\ddot{w} + b_{22}\, A\dot{w}, \qquad (7.12)$$

and the boundary conditions

$$w = \psi = 0 \quad \text{at } x_1 = 0, \qquad (7.13)$$

$$\left.\begin{aligned} E\,I_{33}\,\psi_{,1} + E^* I_{33}\,\dot{\psi}_{,1} + (I_{33}/A)(N + P)\psi_{,1} &= 0, \\ K^2\mu A(w_{,1} - \psi) + K^2\mu^* A(\dot{w}_{,1} - \dot{\psi}) + (N + P)w_{,1} &= 0, \end{aligned}\right\} \text{at } x_1 = \ell. \qquad (7.14)$$

To obtain from equations (7.11) - (7.14) the Euler-Bernoulli counterpart, one disregards all terms in equation (7.11) except

$$EI_{33}\,\psi_{,11} + E^{*}I_{33}\,\dot{\psi}_{,11} + K^{2}\mu A(w_{,1} - \psi) + K^{2}\mu^{*}A(\dot{w}_{,1} - \dot{\psi}) = 0,$$

whence

$$K^{2}\mu A(w_{,1} - \psi) + K^{2}\mu^{*}A(\dot{w}_{,1} - \dot{\psi}) = -EI_{33}\,\psi_{,11} - E^{*}I_{33}\,\dot{\psi}_{,11}. \tag{7.15}$$

Substitution of equation (7.15) into equation (7.12) yields

$$EI_{33}\,\psi_{,111} + E^{*}I_{33}\,\dot{\psi}_{,111} - (N\,w_{,1})_{,1} + \rho A\ddot{w} + b_{22}A\dot{w} = 0. \tag{7.16}$$

But setting $\psi = w_{,1}$, which implies that $\varepsilon_{13} = 0$, equation (7.16) assumes the form

$$EI_{33}\,w_{,1111} + E^{*}I_{33}\,\dot{w}_{,1111} - (N\,w_{,1})_{,1} + \rho A\ddot{w} + b_{22}A\dot{w} = 0. \tag{7.17}$$

Similarly, the boundary conditions are found to be

$$w = w_{,1} = 0 \quad \text{at} \quad x_1 = 0, \tag{7.18}$$

$$\left.\begin{aligned} &EI_{33}\,w_{,11} + E^{*}I_{33}\,\dot{w}_{,11} = 0, \\ &EI_{33}\,w_{,111} + E^{*}I_{33}\,\dot{w}_{,111} - (N+P)w_{,1} = 0 \end{aligned}\right\} \text{at } x_1 = \ell. \tag{7.19}$$

McGill [43] investigated equation (7.17) for $\alpha = \pi/2$ and $E^* = b_{22} = 0$, and Nemat-Nasser [44] studied equations (7.11) - (7.14) in the absence of the gravity field, i.e., $\beta = 0$. The case of $\beta = 0$ corresponds to the well known non-conservative stability problem of Beck (see, for example, reference [32]).

8. STABILITY OF A ROTATING SHAFT

Formulation

Shieh [23] discussed the stability of a rotating shaft subjected to conservative loads. In this section, the usefulness of the conservation law in equation (4.1) for the purpose of deriving the equations of motion of a rotating shaft subjected to non-conservative loads will be demonstrated. The effect of internal damping upon the value of the critical load will also be considered.

Consider a shaft of length ℓ rotating about its own axis with a

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[23] SHIEH, R. C. Energy and variational principles for generalized (gyroscopic) conservative problems. Int. J. Non-Linear Mech., 5, pp. 495-509 (1971).

[32] ANDERSON, G. L. Application of a variational method to dissipative, non-conservative problems of elastic stability. J. Sound Vib., 27, pp. 279-296 (1973).

[43] McGILL, D. J. Column instability under weight and follower loads. Proc. ASCE, J. Engng. Mech. Div., 97, pp. 629-635 (1971).

[44] NEMAT-NASSER, S. Instability of a cantilever under a follower force according to Timoshenko beam theory. J. Appl. Mech., 34, pp. 484-485 (1967).

constant angular velocity Ω, where

$$\underline{\Omega} = \Omega \underline{e}_1, \quad \Omega_1 = \Omega, \quad \Omega_2 = \Omega_3 = 0,$$

and loaded by a compressive follower force of magnitude P at the end $x_1 = \ell$ as shown in Figure 9.

As in Section 7 (neglecting the effect of gravity), the initial stress field is

$$\sigma_{11} = N(x_1)/A, \quad \sigma_{ij} = 0 \text{ for } i \neq 1 \text{ and } j \neq 1, \tag{8.1}$$

where

$$N(x_1) = -P. \tag{8.2}$$

In addition, if one assumes that the displacements of the beam may be approximated by

$$u_1 = -x_2 w_{1,1} - x_3 w_{2,1}, \quad u_2 = w_1, \quad u_3 = w_2,$$

then, upon neglecting the effect of rotatory inertia, one can derive from equations (8.1), (8.2), and (4.3) - (4.7)

$$T_K = \int_0^\ell \frac{1}{2} \rho A(\dot{w}_1^2 + \dot{w}_2^2) dx_1,$$

$$T_G = \int_0^t \int_0^\ell 2\rho\Omega A[-(\partial_t w_1)\dot{w}_2 + (\partial_t w_2)\dot{w}_1] dx_1 \, d\hat{t},$$

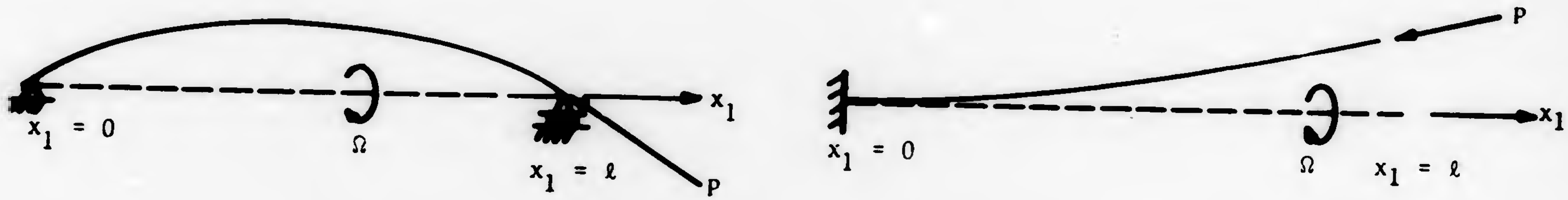

Figure 9. Rotating shafts subjected to a follower force.

$$V = \int_0^{\ell} \{\frac{1}{2} E[I_{22}(w_{1,11})^2 + I_{33}(w_{2,11})^2] - \frac{1}{2}\rho A\Omega^2(w_1^2 + w_2^2)\}dx_1,$$

(8.3)

$$D = \int_0^t \int_0^{\ell} \{E^*[I_{22}(\dot{w}_{1,11})^2 + I_{33}(\dot{w}_{2,11})^2] + b_{11}\dot{w}_1^2 + b_{22}\dot{w}_2^2\}dx_1,$$

$$W = \int_0^{\ell} -\frac{1}{2}P[(w_{1,1})^2 + (w_{2,1})^2]dx_1 + \int_0^t \alpha P(w_{1,1}\dot{w}_1 +$$

$$+ w_{2,1}\dot{w}_2)_{x_1=\ell}\, d\hat{t},$$

where an Euler-Bernoulli type of theory is being formulated. The quantities w_1 and w_2 are the transverse displacements of the axis of the beam in the x_2- and x_3-directions, respectively. Because no offset is assumed, $c_i = 0$. Just as in the preceding section, equation (8.3) may now be inserted into equations (4.2) and (4.9). Omitting the details of the algebraic manipulations, the resulting equations of motion are found to be

$$EI_{22}w_{1,1111} + E^*I_{22}\dot{w}_{1,1111} + Pw_{1,11} - \rho A\Omega^2 w_1 + \rho A\ddot{w}_1 +$$

$$+ b_{11}\dot{w}_1 + 2\rho\Omega A\dot{w}_2 = 0, \tag{8.4}$$

and

$$EI_{33}w_{2,1111} + E^*I_{33}\dot{w}_{2,1111} + Pw_{2,11} - \rho A\Omega^2 w_2 + \rho A\ddot{w}_2 +$$

$$+ b_{22}\dot{w}_2 - 2\rho\Omega A\dot{w}_1 = 0, \tag{8.5}$$

whereas the boundary conditions are

$$\text{either} \quad EI_{ss} w_{s,11} + E^* I_{ss} \dot{w}_{s,11} = 0 \quad \text{or} \quad \dot{w}_{s,1} = 0$$

$$(s = 1,2) \tag{8.6}$$

$$\text{either} \quad EI_{ss} w_{s,111} + E^* I_{ss} \dot{w}_{s,111} + Pw_{s,1} = 0 \quad \text{or} \quad \dot{w}_s = 0,$$

at $x_1 = 0$ and

$$\text{either} \quad EI_{ss} w_{s,11} + E^* I_{ss} \dot{w}_{s,11} = 0 \quad \text{or} \quad \dot{w}_{s,1} = 0,$$

$$\tag{8.7}$$

$$\text{either} \quad EI_{ss} w_{s,111} + E^* I_{ss} \dot{w}_{s,111} + P(1-\alpha)w_{s,1} = 0 \quad \text{or} \quad \dot{w}_s = 0,$$

at $x_1 = \ell$. It should be noted that equations (8.4) and (8.5) are coupled through the Coriolis terms $2\rho\Omega A\dot{w}_2$ and $-2\rho\Omega A\dot{w}_1$.

If w_2 is replaced by $-w_2$ in equation (8.5), then the resulting equation and equation (8.4) coincide with the equations discussed by Shieh [23], provided, of course, that the damping terms are neglected here.

Stability

Introducing the dimensionless variables

$$x_1 = \ell x, \qquad t = c\tau,$$

[23] SHIEH, R. C. Energy and variational principles for generalized (gyroscopic) conservative problems. Int. J. Non-Linear Mech., 5, pp. 495-509 (1971).

one can express equations (8.4) and (8.5) as

$$w_1'''' + \eta \dot{w}_1'''' + Q w_1'' - \omega^2 w_1 + \ddot{w}_1 + \beta_1 \dot{w}_1 + 2\omega \dot{w}_2 = 0, \quad (8.8)$$

$$r w_2'''' + r\eta \dot{w}_2'''' + Q w_2'' - \omega^2 w_2 + \ddot{w}_2 + \beta_2 \dot{w}_2 - 2\omega \dot{w}_1 = 0, \quad (8.9)$$

where $(\ldots)' = \partial(\ldots)/\partial x$ and $(\dot{\ldots}) = \partial(\ldots)/\partial\tau$. For a beam that is simply supported at both ends, the boundary conditions become

$$w_s = w_s'' + \eta \dot{w}_s'' = 0 \quad \text{at} \quad x = 0,1 \quad (s = 1,2). \quad (8.10)$$

In writing equations (8.8) and (8.9) the following definitions have been made:

$$c^2 = \rho A \ell^4/EI_{22}, \quad \eta = E^*/cE, \quad Q = P\ell^2/EI_{22},$$

$$\omega^2 = \rho A \Omega^2 \ell^4/EI_{22}, \quad \beta_1 = b_{11} \ell^4/cEI_{22}, \quad \beta_2 = b_{22}\ell^4/cEI_{22},$$

$$r = I_{33}/I_{22}.$$

By virtue of the form of the boundary conditions in equation (8.10), the solutions of equations (8.8) and (8.9) may be assumed in the form

$$w_s(x,\tau) = C_s \sin k\pi x \, e^{\lambda\tau}, \quad k = 1,2,3,\ldots, \quad s = 1,2. \quad (8.11)$$

Therefore, upon substitution of equation (8.11) into equations (8.8) and (8.9), the following system of algebraic equations is obtained:

$$\begin{aligned}
&\{\lambda^2 + [\beta_1 + \eta(k\pi)^4]\lambda + (k\pi)^4 - (k\pi)^2 Q - \omega^2\}C_1 + 2\omega\lambda\, C_2 = 0, \\
&-2\omega\lambda\, C_1 + \{\lambda^2 + [\beta_2 + r\eta(k\pi)^4]\lambda + r(k\pi)^4 - (k\pi)^2 Q - \omega^2\}C_2 = 0.
\end{aligned} \tag{8.12}$$

But if this system is to have a non-trivial solution, then the determinant of the coefficient matrix associated with equation (8.12) must vanish. Upon expansion of this determinant, one obtains

$$\lambda^4 + a_1 \lambda^3 + a_2\lambda^2 + a_3 \lambda + a_4 = 0, \tag{8.13}$$

where

$$\begin{aligned}
a_1 &= \beta_1 + \beta_2 + \eta(1 + r)(k\pi)^4, \\
a_2 &= (1+r)(k\pi)^4 - 2(k\pi)^2 Q + 2\omega^2 + \beta_1 \beta_2 + \eta(r\beta_1 + \beta_2)(k\pi)^4 + \\
&\qquad + r\eta^2(k\pi)^8, \\
a_3 &= [\beta_1 + \eta(k\pi)^4][r(k\pi)^4 - (k\pi)^2 Q - \omega^2] + \\
&\qquad + [\beta_2 + r\eta(k\pi)^4][(k\pi)^4 - (k\pi)^2 Q - \omega^2], \\
a_4 &= [(k\pi)^4 - (k\pi)^2 Q - \omega^2][r(k\pi)^4 - (k\pi)^2 Q - \omega^2].
\end{aligned} \tag{8.14}$$

The loss of stability by divergence can be investigated by setting $\lambda = 0$ in equation (8.13). This leads to $a_4 = 0$, or

$$\omega^2 = (k\pi)^4 - (k\pi)^2 Q \quad \text{and} \quad \omega^2 = r(k\pi)^4 - (k\pi)^2 Q. \tag{8.15}$$

To consider the possibility of loss of stability due to flutter, one may employ the Routh-Hurwitz criterion:

$$a_i > 0, \quad i = 1,2,3,4, \quad a_1 a_2 a_3 > a_3^2 + a_4 a_1^2 \tag{8.16}$$

as applied in reference [31].

A complete and exhaustive study of equation (8.16) is left for future study. The objective of the remainder of this section is to demonstrate that the effect of internal damping tends to destabilize the motion in much the same manner as that described in references [31-33]. In the discussion that follows, it will be assumed that $\beta_1 = \beta_2 = 0$, i.e., there is no external damping present. Under these circumstances, equations (8.14) simplify to

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[31] ANDERSON, G. L. On the role of the adjoint problem in dissipative, nonconservative problems of elastic stability. Meccanica, 7, pp. 165-173 (1972).

[32] ANDERSON, G. L. Application of a variational method to dissipative, non-conservative problems of elastic stability, J. Sound Vib., 27, pp. 279-296 (1973).

[33] ANDERSON, G. L. On the stability of a deep beam subjected to non-conservative and dissipative forces. Watervliet Arsenal Technical Report R-WV-T-2-14-73, Watervliet, New York 12189 (1973).

$$a_1 = (1+r)\eta(k\pi)^4, \quad a_2 = (1+r)(k\pi)^4 - 2(k\pi)^2Q + 2\omega^2 + r\eta^2(k\pi)^8,$$

$$a_3 = \eta(k\pi)^4[2r(k\pi)^4 - (1+r)(k\pi)^2Q - (1+r)\omega^2], \tag{8.17}$$

$$a_4 = [(k\pi)^4 - (k\pi)^2Q - \omega^2][r(k\pi)^4 - (k\pi)^2Q - \omega^2].$$

Substitution of these quantities into

$$a_1\, a_2\, a_3 = a_3^2 + a_4\, a_1^2$$

yields

$$\begin{aligned}(1+r)[(1+r)(k\pi)^4 - 2(k\pi)^2Q + 2\omega^2 + r\eta^2(k\pi)^8][2r(k\pi)^4 - (1+r)(k\pi)^2Q - \\ - (1+r)\omega^2] = [2r(k\pi)^4 - (1+r)(k\pi)^2Q - (1+r)\omega^2]^2 + \\ + (1+r)^2[(k\pi)^4 - (k\pi)^2Q - \omega^2][r(k\pi)^4 - (k\pi)^2Q - \omega^2].\end{aligned} \tag{8.18}$$

In the limit of vanishing internal damping, i.e., $\eta \to 0$, equation (8.18) reduces to

$$\begin{aligned}r(1+r)(k\pi)^4Q^2 - (k\pi)^2[(1+r)^3(k\pi)^4 - (1+r)(1+2r)(k\pi)^4 + 4(1+r)^2\omega^2]Q + \\ + 8r(1+r)(k\pi)^4\omega^2 - 4(1+r)^2\omega^4 - 8r^2(k\pi)^8 + r(1+r)^2(k\pi)^8 = 0.\end{aligned} \tag{8.19}$$

On the other hand, if it is assumed at the outset of the analysis that no damping is present, then $\eta = \beta_1 = \beta_2 = 0$, so that $a_1 = a_3 = 0$, and equation (8.13) reduces to

$$\lambda^4 + a_2 \lambda^2 + a_4 = 0, \tag{8.20}$$

where now

$$\begin{aligned} a_2 &= (1+r)(k\pi)^4 - 2(k\pi)^2 Q + 2\omega^2, \\ a_4 &= [(k\pi)^4 - (k\pi)^2 Q - \omega^2][r(k\pi)^4 - (k\pi)^2 Q - \omega^2]. \end{aligned} \tag{8.21}$$

In this case, the condition for the onset of instability is that equation (8.20) have multiple roots, i.e., the discriminant must vanish:

$$a_2^2 - 4a_4 = 0. \tag{8.22}$$

Substitution of equation (8.21) into equation (8.22) yields

$$\begin{aligned} (k\pi)^4 Q^2 - (k\pi)^2[(1+r)(k\pi)^4 + 6\omega^2]Q + \frac{1}{2}(1+r^2)(k\pi)^8 + \\ + 3(1+r)(k\pi)^4\omega^2 + \omega^4 = 0. \end{aligned} \tag{8.23}$$

Note that equations (8.19) and (8.23) are quite different. It is evident that internal damping can have a very pronounced effect upon the critical flutter load. It is of considerable interest to complete the stability analysis begun here and, in addition, to investigate the corresponding problem c the cantilevered rotating shaft.

9. DYNAMICS OF A WHIRLING BAR

Some attention [25], [27] and [45] has been devoted to the longitudinal vibrations and stability of a uniform bar of length ℓ rotating at constant angular velocity Ω about an axis perpendicular to the axis of the bar (see Figure 10). It was stated in these references that the equation of motion of such a bar is

$$E\, u_{,11} + \rho\, \Omega^2 (x_1 + u) = \rho\, \ddot{u}, \tag{9.1}$$

where E is Young's modulus and u is the longitudinal displacement of the bar. It is of interest to reconsider this problem, formulating the equation of motion by means of the general theory outlined in Section 4. It will be shown that a term that has the same order of magnitude as the term $\rho\Omega^2(x_1 + u)$ has not been included in equation (9.1) due to the fact

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[25] BRUNELLE, E. J. Stress redistribution and instability of rotating beams and disks. AIAA J., 9, pp. 758-759 (1971).

[27] BRUNELLE, E. J. The super flywheel: a second look. J. Engng. Mat. Tech., 95, pp. 63-65 (1973).

[45] BHUTA, P. G. and JONES, J. P. On axial vibrations of a whirling bar. J. Acoust. Soc. Amer., 35, pp. 217-221 (1963).

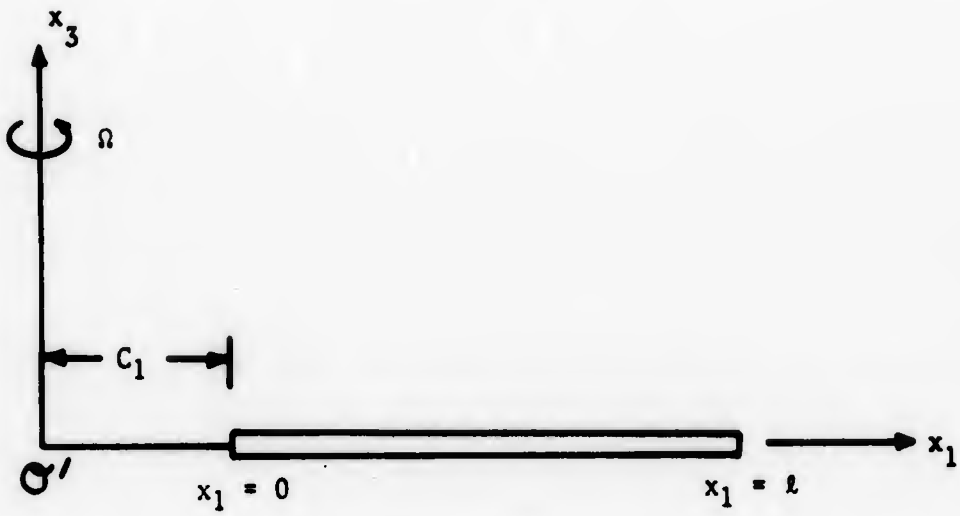

Figure 10. A whirling bar.

that the initial stresses in the bar were ignored.

According to equations (6.27), the initial stresses in the rotating bar are, for $\beta = 0$,

$$\sigma_{11} = N(x_1)/A, \quad \sigma_{13} = \sigma_{33} = 0, \tag{9.2}$$

where

$$N(x_1) = \rho \Omega^2 A[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)]. \tag{9.3}$$

The angular velocity vector has the components

$$\Omega_1 = \Omega_2 = 0, \quad \Omega_3 = \Omega.$$

Finally, the displacements are approximated by

$$u_1 = u(x_1, t), \quad u_2 = u_3 = 0. \tag{9.4}$$

If equations (9.2)-(9.4) are substituted into equations (4.3)-(4.7), one obtains

$$T_K = \int_0^\ell \frac{1}{2} \rho A \dot{u}^2 dx_1, \quad T_G = 0, \quad D = 0,$$

$$V = \int_0^\ell [\frac{1}{2} E A(u_{,1})^2 - \frac{1}{2} \rho A \Omega^2 u^2] dx_1,$$

$$W = \int_0^{\ell} \frac{1}{2} N(u_{,1})^2 dx_1,$$

for $\alpha(x) = 0$ and in the absence of internal and external damping. Application of the conservation law in equation (4.9) leads to the equation of motion

$$E\,A\,u_{,11} + (N\,u_{,1})_{,1} + \rho\,A\,\Omega^2\,u = \rho\,A\,\ddot{u}, \tag{9.5}$$

which replaces equation (9.1), whereas the boundary conditions are

$$\text{either} \quad (EA + N)u_{,1} = 0 \quad \text{or} \quad \dot{u} = 0 \text{ at } x_1 = 0, \ell. \tag{9.6}$$

The question now arises naturally as to whether the term $(N\,u_{,1})_{,1}$ can justifiably be neglected. This question can be settled through a consideration of the order of magnitude of each term in the partial differential equation in equation (9.5).

If it assumed that

$$\mathcal{O}(u_{,1}) \sim U/\ell,$$

then

$$\mathcal{O}(E\,u_{,11}) \sim E\,U/\ell^2, \tag{9.7}$$

$$\mathcal{O}[(N\,u_{,1})_{,1}/A] \sim \rho\,\Omega^2\,U, \tag{9.8}$$

$$\mathcal{O}[\rho\, \Omega^2\, u] \sim \rho\, \Omega^2\, U. \tag{9.9}$$

It was assumed in references [25], [27] and [45] that the terms appearing in equations (9.7) and (9.9) were of the same order of magnitude. But the term in equation (9.8) is of the same order of magnitude as that in equation (9.9). Hence, the term $(N\, u_{,1})_{,1}$ must be retained in equation (9.5), and equation (9.1) must be regarded as incomplete. Finally, it should be pointed out that equation (9.5) is a special case of an equation derived by Tomar [20] for the longitudinal vibration of a pre-twisted slender bar in a centrifugal force field.

For a fixed-free bar, the boundary conditions are, in view of equation (9.6),

$$u(o,t) = u_{,1}(\ell,\, t) = 0. \tag{9.10}$$

If

$$u(x_1,\, t) = y(x_1) \cos \omega t$$

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[20] TOMAR, J. S. Coupled torsional and longitudinal vibrations of a pretwisted slender beam in a centrifugal force field. Proc. Nat. Inst. Sci. India, 35A, pp. 779-787 (1969).

[25] BRUNELLE, E. J. Stress redistribution and instability of rotating beams and disks. AIAA J., 9, pp. 758-759 (1971).

[27] BRUNELLE, E. J. The super flywheel: a second look. J. Engng. Mat. Tech., 95, pp. 63-65 (1973).

[45] BHUTA, P. G. and JONES, J. P. On axial vibrations of a whirling bar. J. Acoust. Soc. Amer., 35, pp. 217-221 (1963).

is inserted into equations (9.5) and (9.10), one obtains

$$E\, y_{,11} + \rho\, \Omega^2\{[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)]y_{,1}\}_{,1} + \rho(\Omega^2 + \omega^2)y = 0, \tag{9.11}$$

$$y(o) = y_{,1}(\ell) = 0. \tag{9.12}$$

If the dimensionless parameters

$$x = x_1/\ell, \quad \gamma = c_1/\ell, \quad \lambda^2 = (\Omega\ell)^2/(E/\rho), \quad \zeta^2 = (\omega\ell)^2/(E/\rho),$$

are introduced, then equations (9.11)-(9.12) can be expressed as

$$y'' + \lambda^2\{[\gamma(1-x) + \frac{1}{2}(1-x^2)]y'\}' + (\lambda^2 + \zeta^2)y = 0, \tag{9.13}$$

$$y(o) = y'(1) = 0, \tag{9.14}$$

where $y'(x) = dy/dx$.

Rayleigh's quotient for the boundary value problem embodied in equations (9.13) and (9.14) may in the familiar fashion be shown to be

$$\zeta^2 = \frac{\int_0^1 \{(y')^2 + \lambda^2[\gamma(1-x) + \frac{1}{2}(1-x^2)](y')^2 - \lambda^2 y^2\}dx}{\int_0^1 y^2\, dx} \tag{9.15}$$

Equation (9.15) will now be used to compute an approximate value for the lowest natural frequency of free longitudinal oscillations of a rotating bar.

The coordinate function $y_1(x) = \sin(\pi x/2)$ satisfies both boundary conditions in equation (9.14). Then because

$$\int_0^1 (y_1')^2\,dx = \pi^2/8, \qquad \int_0^1 y_1^2\,dx = 1/2,$$

$$\int_0^1 (1-x)(y_1')^2 dx = (\pi^2 + 4)/16, \quad \int_0^1 (1-x^2)(y_1')^2 dx = (\pi^2+3)/12,$$

equation (9.15) leads to

$$\zeta_1^2 = \frac{\pi^2}{4} + \frac{\lambda^2}{24}[3\gamma(\pi^2 + 4) + 2(\pi^2 - 9)], \tag{9.16}$$

and for $\gamma = 0$, i.e., zero hub radius,

$$\zeta_1^2 = \frac{\pi^2}{4} + \frac{\lambda^2}{12}(\pi^2 - 9). \tag{9.17}$$

The simple formulas in equations (9.16)-(9.17) indicate that the natural frequency parameter ζ_1 increases as the rotational velocity of the bar is increased. On the other hand, if one had started with equation (9.1), in which the initial stress in the bar due to the centrifugal effect has been neglected, the (exact) counterpart of equation (9.17) would be found to be

$$\zeta_1^2 = (\pi/2)^2 - \lambda^2. \tag{9.18}$$

But, according to equation (9.18), the natural frequency of oscillation will vanish when $\lambda = \pi/2$. This condition corresponds to an instability of the divergence type, which was the principal conclusion reported in reference [27]. In contrast to this, however, according to equation (9.17), the value of ζ_1 cannot vanish. Therefore, at least on the basis of a one term approximation, instability due to longitudinal divergence appears to be impossible.

10. FLEXURAL-TORSIONAL VIBRATIONS OF A ROTATING BEAM

In this section the rotating beam depicted in Figure 7 will be considered. It is assumed that body forces other than those due to the rotation of the system can be neglected. Under the supposition that the rotating beam undergoes flexural deformations in the $x_1 x_2$- and $x_1 x_3$-planes and torsional deformation about its axis, the displacement field is approximated as follows:

$$
\begin{aligned}
u_1 &= -x_3 w_{,1}(x_1, t) - x_2 v_{,1}(x_1,t) + \theta(x_2, x_3)\phi_{,1}(x_1, t), \\
u_2 &= v(x_1, t) - x_3 \phi (x_1, t), \qquad (10.1) \\
u_3 &= w(x_1, t) + x_2 \phi (x_1, t),
\end{aligned}
$$

where v and w designate the transverse deflections of the beam in the x_2- and x_3- directions, respectively, and ϕ the angle of rotation of the beam about the x_1-axis. The quantity $\theta(x_2, x_3)$ is known as the warping

[27] BRUNELLE, E. J. The super flywheel: a second look. J. Engng. Mat. Tech., 95, pp. 63-65 (1973).

function. It is understood that the effects of rotatory inertia and transverse shear deformation will be neglected in the subsequent formulation.

The components of the angular velocity vector $\underline{\Omega}$ are

$$\Omega_1 = \Omega \sin \beta, \qquad \Omega_2 = 0, \qquad \Omega_3 = \Omega \cos \beta, \tag{10.2}$$

while the initial stress field is given in equation (6.27).

Therefore, substitution of equations (10.1), (10.2), and (6.27) into equations (4.3)-(4.7) leads to

$$T_K = \int_0^\ell \frac{1}{2} \rho [A \dot{v}^2 + A \dot{w}^2 + I_{23} \dot{\phi}^2] dx_1,$$

$$T_G = \int_0^t \int_0^\ell 2\rho A \Omega_1 [-(\partial_t v)\dot{w} + (\partial_t w)\dot{v}] dx_1 \, d\hat{t},$$

$$V = \int_0^\ell [\frac{1}{2} EI_{33}(w_{,11})^2 + \frac{1}{2} EI_{22}(v_{,11})^2 + \frac{1}{2} E\Gamma(\phi_{,11})^2 +$$

$$+ \frac{1}{2} \mu C(\phi_{,1})^2 - \frac{1}{2} \rho \Omega^2 A \sin^2\beta \, w^2 - \frac{1}{2} \rho \Omega^2 A v^2 -$$

$$- \frac{1}{2} \rho \Omega^2 (I_{22} \sin^2\beta + I_{33})\phi^2] dx_1, \tag{10.3}$$

$$D = \int_0^t \int_0^\ell [E^* I_{33}(\dot{w}_{,11})^2 + E^* I_{22}(\dot{v}_{,11})^2 + E^* (\dot{\phi}_{,11})^2 +$$

$$+ \mu^* C(\dot{\phi}_{,1})^2] dx_1 \, d\hat{t},$$

$$W = \int_0^{\ell} [\frac{1}{2} N(v_{,1})^2 + \frac{1}{2} N(w_{,1})^2 - M_3 v_{,1} \phi_{,1} - Q_3 v_{,1} \phi +$$

$$+ \frac{1}{2} \rho \Omega^2 (I_{22} + I_{33} \sin^2 \beta)\phi^2 + \frac{1}{2} (I_{23}/A) N(\phi_{,1})^2] dx_1,$$

where $I_{23} = I_{22} + I_{33}$ and

$$\Gamma = \int_A \theta^2 \, dA, \quad C = \int_A [(\theta_{,2})^2 + (\theta_{,3})^2 + x_2^2 + x_3^2] dA,$$

with N, M_3, and Q_3 being given in equation (6.26). In the development of equations (10.3), equation (5.1) as well as

$$\int_A x_3 \theta_{,2} \, dA = \int_A x_2 \theta_{,3} \, dA = \int_A x_3^3 \, dA = \int_A x_2 x_3 \, dA =$$

$$= \int_A x_2^2 x_3 \, dA = 0$$

have been used.

Therefore, based upon equations (4.2), (4.9), and (10.3) the equations of motion are found to be

$$EI_{33} w_{,1111} + E^* I_{33} \dot{w}_{,1111} - (N w_{,1})_{,1} - \rho A \Omega^2 w \sin^2 \beta + \rho A \ddot{w} -$$

$$- 2\rho A \Omega \dot{v} \sin \beta = 0, \tag{10.4}$$

$$EI_{22} v_{,1111} + E^* I_{22} \dot{v}_{,1111} - (N v_{,1})_{,1} - \rho A \Omega^2 v + \rho A \ddot{v} + 2\rho A \Omega \dot{w} \sin \beta +$$

$$+ (M_3 \phi)_{,11} + \rho I_{33} \Omega^2 \phi_{,1} \sin^2 \beta = 0, \tag{10.5}$$

$$E\Gamma \phi_{,1111} + E^* \Gamma \dot{\phi}_{,1111} - \mu C \phi_{,11} - \mu^* C \dot{\phi}_{,11} - (I_{23}/A)(N \phi_{,1})_{,1} +$$

$$+ \rho \Omega^2 (I_{22} - I_{33}) \phi \cos^2 \beta + \rho I_{23} \ddot{\phi} + M_3 v_{,11} -$$

$$- \rho I_{33} \Omega^2 v_{,1} \sin^2 \beta = 0, \tag{10.6}$$

with the boundary conditions

$$\left.\begin{aligned} &\text{either } EI_{33} w_{,11} + E^* I_{33} \dot{w}_{,11} = 0, \text{ or } \dot{w}_{,1} = 0, \\ &\text{either } EI_{33} w_{,111} + E^* I_{33} \dot{w}_{,111} - N w_{,1} = 0 \text{ or } \dot{w} = 0, \end{aligned}\right\} \tag{10.7}$$

$$\left.\begin{aligned} &\text{either } EI_{22} v_{,11} + E^* I_{22} \dot{v}_{,11} = 0 \text{ or } \dot{v}_{,1} = 0, \\ &\text{either } EI_{22} v_{,111} + E^* I_{22} \dot{v}_{,111} - N v_{,1} + (M_3 \phi)_{,1} + \\ &\qquad + \rho I_{33} \Omega^2 \phi \sin^2 \beta = 0 \text{ or } \dot{v} = 0, \end{aligned}\right\} \tag{10.8}$$

$$\left.\begin{aligned} &\text{either } E\Gamma \phi_{,11} + E^* \Gamma \dot{\phi}_{,11} = 0 \text{ or } \dot{\phi}_{,1} = 0, \\ &\text{either } E\Gamma \phi_{,111} + E^* \Gamma \dot{\phi}_{,111} - \mu C \phi_{,1} - \mu^* C \dot{\phi}_{,1} + M_3 v_{,1} - \\ &\qquad - (I_{23}/A) N \phi_{,1} = 0 \text{ or } \dot{\phi} = 0, \end{aligned}\right\} \tag{10.9}$$

where the identities

$$(M_3 \phi_{,1})_{,1} + (Q_3 \phi)_{,1} = (M_3 \phi)_{,11} + \rho I_{33} \Omega^2 \phi_{,1} \sin^2 \beta,$$

$$(M_3 v_{,1})_{,1} - Q_3 v_{,1} = M_3 v_{,11} - \rho I_{33} \Omega^2 v_{,1} \sin^2 \beta,$$

$$M_3 \phi_{,1} + Q_3 \phi = (M_3 \phi)_{,1} + \rho I_{33} \Omega^2 \phi \sin^2 \beta,$$

derived with the aid of equations (6.24) have been used.

For the rotating beam depicted in Figure 8, the components of the angular velocity vector are given in equation (6.30) and the corresponding initial state of stress is given in equation (6.35). Again for the displacements appearing in equation (10.1), the expressions in equations (4.3)-(4.7) assume the following forms:

$$T_K = \int_0^\ell \frac{1}{2} \rho [A \dot{v}^2 + A \dot{w}^2 + I_{23} \dot{\phi}^2] dx_1,$$

$$T_G = 0,$$

$$V = \int_0^\ell [\frac{1}{2} EI_{33}(w_{,11})^2 + \frac{1}{2} EI_{22}(v_{,11})^2 + \frac{1}{2} E\Gamma(\phi_{,11})^2 +$$

$$+ \frac{1}{2} \mu C(\phi_{,1})^2 - \frac{1}{2} \rho A \Omega^2 v^2 \cos^2 \gamma - \frac{1}{2} \rho A \Omega^2 w^2 \sin^2 \gamma +$$

$$+ \rho A \Omega^2 w v \sin \gamma \cos \gamma - \frac{1}{2} \rho \Omega^2 (I_{33} \cos^2 \gamma + I_{22} \sin^2 \gamma) \phi^2] dx_1,$$

$$D = \int_0^t \int_0^\ell [E^* I_{33}(\dot{w}_{,11})^2 + E^* I_{22}(\dot{v}_{,11})^2 + E^* \Gamma(\dot{\phi}_{,11}) +$$

$$+ \mu^* C(\dot{\phi}_{,1})^2] dx_1 \; d\hat{t},$$

$$W = \int_0^\ell \frac{1}{2} [N(v_{,1})^2 + N(w_{,1})^2 + (I_{23}/A) N(\phi_{,1})^2 + \rho\, \Omega^2 (I_{22} \cos^2 \gamma +$$

$$+ I_{33} \sin^2 \gamma)\phi^2] dx_1 .$$

Proceeding as in the previous examples, the equations of motion and the boundary conditions are found to be

$$EI_{33}\, w_{,1111} + E^* I_{33}\, \dot{w}_{,1111} - (N\, w_{,1})_{,1} - \rho\, A\, \Omega^2 \sin \gamma [w \sin \gamma -$$

$$- v \cos \gamma] + \rho\, A\, \ddot{w} = 0, \qquad (10.10)$$

$$EI_{22}\, v_{,1111} + E^* I_{22}\, \dot{v}_{,1111} - (N\, v_{,1})_{,1} + \rho\, A\, \Omega^2 \cos \gamma [w \sin \gamma -$$

$$-v \cos \gamma] + \rho\, A\, \ddot{v} = 0, \qquad (10.11)$$

$$E\Gamma\, \phi_{,1111} + E^* \Gamma\, \dot{\phi}_{,1111} - \mu\, C\, \phi_{,11} - \mu^* C\, \dot{\phi}_{,11} - (I_{23}/A)(N\, \phi_{,1})_{,1} +$$

$$+ \rho\, \Omega^2 (I_{22} - I_{33})\phi \cos 2\gamma + \rho\, I_{23}\, \ddot{\phi} = 0, \qquad (10.12)$$

and

$$\left.\begin{array}{l} \text{either } EI_{33}\, w_{,11} + E^{*}I_{33}\, \dot{w}_{,11} = 0 \text{ or } \dot{w}_{,1} = 0, \\ \\ \text{either } EI_{33}\, w_{,111} + E^{*}I_{33}\, \dot{w}_{,111} - N\, w_{,1} = 0 \text{ or } \dot{w} = 0, \end{array}\right\} \quad (10.13)$$

$$\left.\begin{array}{l} \text{either } EI_{22}\, v_{,11} + E^{*}I_{22}\, \dot{v}_{,11} = 0 \text{ or } \dot{v}_{,1} = 0, \\ \\ \text{either } EI_{22}\, v_{,111} + E^{*}I_{22}\, \dot{v}_{,111} - N\, v_{,1} = 0 \text{ or } \dot{v} = 0, \end{array}\right\} \quad (10.14)$$

$$\left.\begin{array}{l} \text{either } E\Gamma\, \phi_{,11} + E^{*}\Gamma\, \dot{\phi}_{,11} = 0 \text{ or } \dot{\phi}_{,1} = 0, \\ \\ \text{either } E\Gamma\, \phi_{,111} + E^{*}\Gamma\, \dot{\phi}_{,111} - \mu\, C\, \phi_{,1} - \mu^{*}C\, \dot{\phi}_{,1} - \\ \\ \qquad - (I_{23}/A)N\, \phi_{,1} = 0 \text{ or } \dot{\phi} = 0. \end{array}\right\} \quad (10.15)$$

These equations are consistent with those derived by Santini [14] by another method.

For zero coning angle and zero angle of attack, equations (10.4)-(10.6) and (10.10)-(10.12), respectively, reduce to

$$EI_{33}\, w_{,1111} + E^{*}I_{33}\, \dot{w}_{,1111} - (N\, w_{,1})_{,1} + \rho\, A\, \ddot{w} = 0, \quad (10.16)$$

$$EI_{22}\, v_{,1111} + E^{*}I_{22}\, \dot{v}_{,1111} - (N\, v_{,1})_{,1} - \rho\, A\, \Omega^{2} v + \rho\, A\, \ddot{v} = 0, \quad (10.17)$$

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[14] SANTINI, P. Problemi dinamici dei rotori. L'Aerotecnica, <u>5</u>, pp. 199-215 (1963).

$$E\Gamma\,\phi_{,1111} + E^{*}\Gamma\,\dot{\phi}_{,1111} - \mu\,C\,\phi_{,11} - \mu^{*}\,C\,\dot{\phi}_{,11} - (I_{23}/A)(N\,\phi_{,1})_{,1} +$$

$$+ \rho\,\Omega^{2}(I_{22} - I_{33})\,\phi + \rho\,I_{23}\,\ddot{\phi} = 0, \tag{10.18}$$

whereas the boundary conditions in equations (10.7)-(10.9) and (10.13)-(10.15) become

$$\left.\begin{array}{l} \text{either } EI_{33}\,w_{,11} + E^{*}I_{33}\,\dot{w}_{,11} = 0 \text{ or } \dot{w}_{,1} = 0, \\ \text{either } EI_{33}\,w_{,111} + E^{*}I_{33}\,\dot{w}_{,111} - N\,w_{,1} = 0 \quad \dot{w} = 0, \end{array}\right\} \tag{10.19}$$

$$\left.\begin{array}{l} \text{either } EI_{22}\,v_{,11} + E^{*}I_{22}\,\dot{v}_{,11} = 0 \text{ or } \dot{v}_{,1} = 0, \\ \text{either } EI_{22}\,v_{,111} + E^{*}I_{22}\,\dot{v}_{,111} - N\,v_{,1} = 0 \text{ or } \dot{v} = 0, \end{array}\right\} \tag{10.20}$$

$$\left.\begin{array}{l} \text{either } E\Gamma\,\phi_{,11} + E^{*}\Gamma\,\dot{\phi}_{,11} = 0 \text{ or } \dot{\phi}_{,1} = 0, \\ \text{either } E\Gamma\,\phi_{,111} + E^{*}\Gamma\,\dot{\phi}_{,111} - \mu\,C\,\phi_{,1} - \mu^{*}\,C\,\dot{\phi}_{,1} - \\ \qquad - (I_{23}/A)\,N\,\phi_{,1} = 0 \text{ or } \dot{\phi} = 0. \end{array}\right\} \tag{10.21}$$

11. THE ROTATING TIMOSHENKO BEAM

For the Timoshenko beam theory, the displacements are again those assumed in equation (7.4). Furthermore, it is assumed here that the coning angle and the angle of attack are zero, i.e., $\beta = \gamma = 0$, and that the bar rotates about the vertical y_3-axis. Thus

$$\Omega_1 = \Omega_2 = 0, \quad \Omega_3 = \Omega.$$

The initial state of stress is given by

$$\begin{aligned} \sigma_{11} &= N(x_1)/A, \quad \sigma_{12} = \sigma_{23} = \sigma_{13} = \sigma_{33} = 0, \\ \sigma_{22} &= \frac{1}{2}\rho\, \Omega^2[(b/2)^2 - x_2^2], \end{aligned} \qquad (11.1)$$

where $N(x_1)$ is found in equation (6.28).

In the absence of damping and other external loads, equations (4.3)-(4.7) assume the following forms:

$$\begin{aligned} T_K &= \int_0^\ell \frac{1}{2}\rho[I_{33}\dot{\psi}^2 + A\dot{w}^2]dx_1, \quad T_G = D = 0, \\ V &= \int_0^\ell \frac{1}{2}[EI_{33}(\psi_{,1})^2 + K^2\mu A(w_{,1} - \psi)^2 - \rho I_{33}\Omega^2\psi^2]dx_1, \\ W &= \int_0^\ell \frac{1}{2}[(I_{33}/A)N(\psi_{,1})^2 + N(w_{,1})^2dx_1, \end{aligned} \qquad (11.2)$$

where $K^2 = \pi^2/12$ is the transverse shear correction factor introduced in Section 7. The result of applying the conservation law of Section 4 in conjunction with equation (11.2) may be shown to lead to the following equations of motion

$$K^2 \mu A(w_{,11} - \psi_{,1}) + (N w_{,1})_{,1} = \rho A \ddot{w}, \tag{11.3}$$

$$EI_{33} \psi_{,11} + K^2 \mu A(w_{,1} - \psi) + (I_{33}/A)(N \psi_{,1})_{,1} +$$

$$+ \rho \Omega^2 I_{33} \psi = \rho I_{33} \ddot{\psi}. \tag{11.4}$$

Simultaneously, the boundary conditions are found to be

$$\begin{aligned} &\text{either} \quad (E + N/A) I_{33} \psi_{,1} = 0 \quad \text{or} \quad \dot{\psi} = 0, \\ &\text{either} \quad K^2 \mu A(w_{,1} - \psi) + N w_{,1} = 0 \quad \text{or} \quad \dot{w} = 0, \end{aligned} \tag{11.5}$$

at $x_1 = 0, \ell$.

Equations (11.3)-(11.5) are the Timoshenko counterparts of equations (10.16) and (10.19).

12. AN ECCENTRIC FOLLOWER FORCE

According to equation (6.10), the initial stress field for an eccentrically applied compressive follower force of magnitude P is

$$\sigma_{11}^{(1)} = \frac{N^{(1)}}{A} + \frac{x_2}{I_{22}} M_2^{(1)},$$

$$\sigma_{12}^{(1)} = \sigma_{13}^{(1)} = \sigma_{22}^{(1)} = \sigma_{23}^{(1)} = \sigma_{33}^{(1)} = 0, \qquad (12.1)$$

$$N^{(1)} = -P, \qquad M_2^{(1)} = P\, e,$$

whereas that for a centrifugal force with $\Omega_1 = \Omega_2 = 0$, $\Omega_3 = \Omega$, and $\beta = \gamma = 0$ (see Section 10) is, according to equation (6.27),

$$\sigma_{11}^{(2)} = N^{(2)}/A, \qquad \sigma_{22}^{(2)} = \frac{1}{2}\rho\, \Omega^2[(b/2)^2 - x_2^2],$$

$$\sigma_{12}^{(2)} = \sigma_{13}^{(2)} = \sigma_{23}^{(2)} = \sigma_{33}^{(2)} = 0, \qquad (12.2)$$

where, by equation (6.28),

$$N^{(2)} = \rho\, A\, \Omega^2[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)]. \qquad (12.3)$$

Because the theory being applied here is linear, the principle of superposition can be used, so that in view of equations (12.1)-(12.3) the initial state of stress due to the combined effects of the eccentrically applied load and the centrifugal force lead to the following initial state of stress:

$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}, \tag{12.4}$$

i.e.,

$$\left.\begin{aligned} &\sigma_{11} = \frac{N}{A} + \frac{x_2}{I_{22}} M_2, \quad \sigma_{22} = \frac{1}{2}\rho\,\Omega^2[(b/2)^2 - x_2^2], \\ &\sigma_{12} = \sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \end{aligned}\right\} \tag{12.5}$$

where

$$N = N^{(1)} + N^{(2)} = -P + \rho\,\Omega^2 A[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)],$$

$$M_2 = M_2^{(1)} = P\,e.$$

If the displacement field is assumed to be

$$\begin{aligned} u_1 &= -x_3\,w_{,1}(x_1, t) + \theta(x_2, x_3)\,\phi_{,1}(x_1, t), \\ u_2 &= -x_3\,\phi(x_1, t), \\ u_3 &= w(x_1, t) + x_2\,\phi(x_1, t), \end{aligned} \tag{12.6}$$

and if the effect of rotatory inertia is ignored, then, by virtue of equations (12.5) and (12.6), it can be shown that

$$T_K = \int_0^{\ell} \frac{1}{2} \rho [A \dot{w}^2 + I_{23} \dot{\phi}^2] dx_1,$$

$$T_G = 0,$$

$$V = \int_0^{\ell} [\frac{1}{2} EI_{33}(w,_{11})^2 + \frac{1}{2} E\Gamma(\phi,_{11})^2 + \frac{1}{2} \mu C(\phi,_1)^2 - \\ - \frac{1}{2} \rho \Omega^2 I_{33} \phi^2] dx_1,$$

$$D = \int_0^t \int_0^{\ell} [E^* I_{33}(\dot{w},_{11})^2 + E^* \Gamma(\dot{\phi},_{11})^2 + \mu^* C(\dot{\phi},_1)^2] dx_1 \, d\hat{t},$$

$$W = \int_0^{\ell} [\frac{1}{2} N(w,_1)^2 + \frac{1}{2} (I_{23}/A) N(\phi,_1)^2 + M_2 w,_1 \phi,_1 + \\ + \frac{1}{2} \rho I_{22} \Omega^2 \phi^2] dx_1 - \int_0^t \{[(I_{23}/A) N \phi,_1 + \\ + M_2 w,_1] \dot{\phi} + [N w,_1 + M_2 \phi,_1] \dot{w}\}_{x_1 = \ell} d\hat{t}.$$

Applying the conservation law of Section 4, one finds

$$EI_{33} w,_{1111} + E^* I_{33} \dot{w},_{1111} - (N w,_1),_1 + \rho A \ddot{w} - M_2 \phi,_{11} = 0, \tag{12.7}$$

$$E\Gamma\,\phi_{,1111} + E^{*}\Gamma\,\dot{\phi}_{,1111} - \mu C\,\phi_{,11} - \mu^{*} C\,\dot{\phi}_{,11} - (I_{23}/A)(N\,\phi_{,1})_{,1} +$$

$$+ \rho\,\Omega^{2}(I_{22} - I_{33})\phi + \rho\, I_{23}\,\ddot{\phi} - M_{2}\, w_{,11} = 0, \qquad (12.8)$$

and the associated boundary conditions

$$\dot{w} = \dot{w}_{,1} = \dot{\phi} = \dot{\phi}_{,1} = 0 \quad \text{at} \quad x_1 = 0 \qquad (12.9)$$

and

$$\left.\begin{aligned} EI_{33}\, w_{,11} + E^{*} I_{33}\, \dot{w}_{,11} &= 0, \\ EI_{33}\, w_{,111} + E^{*} I_{33}\, \dot{w}_{,111} &= 0, \end{aligned}\right\} \qquad (12.10)$$

$$\left.\begin{aligned} E\Gamma\,\phi_{,11} + E^{*}\Gamma\,\dot{\phi}_{,11} &= 0, \\ E\Gamma\,\phi_{,111} + E^{*}\Gamma\,\dot{\phi}_{,111} - \mu C\phi_{,1} - \mu^{*} C\,\dot{\phi}_{,1} &= 0, \end{aligned}\right\} \qquad (12.11)$$

at $x_1 = \ell$ for a rotating cantilever If $\Omega = 0$ in equations (12.7)-(12.8), then the resulting differential equations are identical to those in reference [34].

Finally, if one considers a pair of symmetrically applied follower forces on the end $x_1 = \ell$ of the beam - see equation (6.11), then

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[34] NEMAT-NASSER, S. and TSAI, P. F. Effect of warping rigidity on stability of a bar under eccentric follower force. Int. J. Solids Structures, 5, pp. 271-279 (1969).

$$N^{(1)} = -2P, \qquad M_2^{(1)} = 0,$$

so that

$$N = -2P + \rho A \Omega^2[c_1(\ell - x_1) + \frac{1}{2}(\ell^2 - x_1^2)], \qquad M_2 = 0.$$

Repeating the procedure outlined above, one finds that equations (12.7)-(12.8) become replaced by

$$EI_{33} w_{,1111} + E^*I_{33} \dot{w}_{,1111} - (N w_{,1})_{,1} + \rho A \ddot{w} = 0, \qquad (12.12)$$

$$E\Gamma \phi_{,1111} + E^*\Gamma \dot{\phi}_{,1111} - \mu C \phi_{,11} - \mu^* C \dot{\phi}_{,11} -$$

$$- (I_{23}/A)(N \phi_{,1})_{,1} + \rho \Omega^2(I_{22} - I_{33})\phi + \rho I_{23}\ddot{\phi} = 0, \qquad (12.13)$$

with the boundary conditions still being those in equations (12.9)-(12.11). It should be noted that equations (12.12) and (12.13) are uncoupled. Furthermore, in the special case of $\Omega = 0$, these equations have been studied in references [32] and [42], respectively.

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[32] ANDERSON, G. L. Application of a variational method to dissipative, non-conservative problems of elastic stability. J. Sound Vib., 27, pp. 279-296 (1973).

[42] NEMAT-NASSER, S. and HERRMANN, G. Torsional instability of cantilevered bars subjected to nonconservative loading. J. Appl. Mech., 33, pp. 102-104 (1966).

13. A TRANSVERSE FOLLOWER FORCE APPLIED TO A ROTATING CANTILEVER

Consider a beam of length ℓ rotating at a uniform rate, $\underline{\Omega} = \Omega\, \underline{e}_3$, in a horizontal plane. Suppose, in addition, that the beam is loaded in a transverse sense along its length as shown in Figure 6. In view of equation (6.16), the state of initial stress is given by

$$\sigma_{11}^{(1)} = \frac{x_2}{I_{22}} M_2(x_1), \qquad \sigma_{12}^{(1)} = \frac{Q_2(x_1)}{2\, I_{22}} [(b/2)^2 - x_2^2],$$

$$\sigma_{22}^{(1)} = \frac{R_2(x_1)}{24\, I_{22}} (b^3 + 3b^2 x_2 - 4x_2^3), \tag{13.1}$$

$$\sigma_{13}^{(1)} = \sigma_{23}^{(1)} = \sigma_{33}^{(1)} = 0.$$

The initial state of stress due to the centrifugal force is that given in equations (12.2)-(12.3). Then using equation (12.4) with equations (12.2)-(12.3) and (13.1), one obtains

$$\sigma_{11} = \frac{N}{A} + \frac{x_2}{I_{22}} M_2, \qquad \sigma_{12} = \frac{Q_2}{2\, I_{22}} [(b/2)^2 - x_2^2],$$

$$\sigma_{22} = \frac{R_2}{24\, I_{22}} (b^3 + 3b^2 x_2 - 4x_2^3) + \frac{1}{2} \rho\, \Omega^2 [(b/2)^2 - x_2^2], \tag{13.2}$$

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0,$$

where N is given in equation (12.3).

For the displacement field assumed in equation (12.6) and for the initial stress field given in equation (13.2), it follows from equations (4.3)-(4.7) that

$$T_K = \int_0^\ell \frac{1}{2} \rho [A \dot{w}^2 + I_{23} \dot{\phi}^2] dx_1,$$

$$T_G = 0,$$

$$V = \int_0^\ell \frac{1}{2} [EI_{33}(w_{,11})^2 + E\Gamma(\phi_{,11})^2 + \mu C(\phi_{,1})^2 - \rho I_{33} \Omega^2 \phi^2] dx_1,$$

$$D = \int_0^t \int_0^\ell [E^* I_{33}(\dot{w}_{,11})^2 + E^*\Gamma(\dot{\phi}_{,11})^2 + \mu^* C(\dot{\phi}_{,1})^2] dx_1 \, d\hat{t},$$

$$W = \int_0^\ell [\frac{1}{2} N(w_{,1})^2 + \frac{1}{2}(I_{23}/A) N (\phi_{,1})^2 + M_2 w_{,1} \phi_{,1} + Q_2 \phi w_{,1} +$$

$$+ \frac{1}{4} b R_2 \phi^2 + \frac{1}{2} \rho I_{22} \Omega^2 \phi^2] dx_1 -$$

$$- \int_0^t \int_0^\ell R_2 (\phi \dot{w} + \frac{1}{2} b \phi \dot{\phi}) dx_1 \, d\hat{t}.$$

Consequently, the equations of motion are found to be

$$EI_{33} w_{,1111} + E^* I_{33} \dot{w}_{,1111} - (N w_{,1})_{,1} + \rho A \ddot{w} - (M_2 \phi)_{,11} -$$

$$- R_2 \phi = 0, \qquad (13.3)$$

$$EI' \phi_{,1111} + E^*\Gamma \dot{\phi}_{,1111} - \mu C \phi_{,11} - \mu C^* \dot{\phi}_{,11} - (I_{23}/A)(N \phi_{,1})_{,1} +$$

$$+ \rho \Omega^2 (I_{22} - I_{33})\phi + \rho I_{23} \ddot{\phi} - M_2 w_{,11} = 0, \qquad (13.4)$$

subject to the boundary conditions

$$\dot{w} = \dot{w}_{,1} = \dot{\phi} = \dot{\phi}_{,1} = 0 \quad \text{at} \quad x_1 = 0, \qquad (13.5)$$

and

$$\left.\begin{aligned} &EI_{33} w_{,11} + E^*I_{33} \dot{w}_{,11} = 0 \\ &EI_{33} w_{,111} + E^*I_{33} \dot{w}_{,111} - N w_{,1} - (M_2 \phi)_{,1} = 0, \end{aligned}\right\} \qquad (13.6)$$

$$\left.\begin{aligned} &E\Gamma \phi_{,11} + E^*\Gamma \dot{\phi}_{,11} = 0, \\ &E\Gamma \phi_{,111} + E^*\Gamma \dot{\phi}_{,111} - \mu C \phi_{,1} - \mu^* C \dot{\phi}_{,1} - (I_{23}/A)N \phi_{,1} - \\ &\qquad - M_2 w_{,1} = 0, \end{aligned}\right\} \qquad (13.7)$$

where the identities, based upon equation (6.13),

$$M_2 \phi_{,1} + Q_2 \phi = (M_2 \phi)_{,1},$$

$$(M_2 \phi_{,1})_{,1} + (Q_2 \phi)_{,1} = (M_2 \phi)_{,11},$$

$$M_{2,1}\, w_{,1} + M_2\, w_{,11} - Q_2\, w_{,1} = M_2\, w_{,11},$$

have been used.

A stability problem very closely related to that embodied in equations (13.3)-(13.7) with Ω set equal to zero has been discussed in references [33] and [46].

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