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MUTUAL COHERENCE FUNCTION OF A BEAM PROPAGATING IN A TURBULENT MEDIUM

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13. ABSTRACT

Using transport theory the mutual coherence function of a beam propagating in a turbulent medium has been studied in detail, including the effect of the beam size, focal length and strength of the turbulence. The most detailed results are obtained for the MCF when the fluctuations have a Gaussian spectrum, but simple closed-form asymptotic results have also been obtained when the spectrum is a von Karman.

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1. INTRODUCTION

The mutual coherence function (MCF) is important since it describes the loss of coherence of an initially-coherent beam propagating through turbulence, and can be shown[1] to determine the signal-to-noise ratio of an optical heterodyne system. The MCF for a plane wave is readily calculated for an arbitrary spectrum of the index of refraction fluctuations, but for a beam of finite lateral extent, numerical results have not been obtained for realistic fluctuation spectra, except over short distances. That is, numerical results for the MCF have been obtained using the method of smooth perturbations[2] but these results are valid[3] only over distances z, such that $C_n^2 k_o^{7/6} z^{11/6} \ll 1$, where k_o is the signal wavenumber and C_n^2 is the index structure constant. Formulations valid over arbitrary distances, z, have

(Received for Publication 6 September 1973)

1. Fried, D. (1967) Signal-to-noise ratio in optical heterodyne systems, Proc. IEEE 55:57-65.

2. Tatarski, V. (1971) The Effects of the Turbulent Atmosphere on Wave Propagation, U.S. Dept. of Commerce, Springfield, Virginia.

3. Barabanenkov, V., Kravtsov, Y., Rytov, S. and Tatarski, V. (1971) Status of the theory of propagation of waves in a randomly inhomogeneous medium, Soviet Phys. Usp. 13:551-680.

recently been developed using the Markov approximation[4] transport theory,[5-7] the Huygens-Fresnel principle[8] and the first ladder approximation to the Bethe-Salpeter equation.[9] It is shown in Section 2 that all these, apparently different, formulations lead to identical results for the MCF, and are valid at optical frequencies provided the attenuation over a wavelength is small. Unfortunately, the expression obtained by these methods for the MCF of a finite beam is quite complicated, and is not readily evaluated, except when the spectrum of the index fluctuations is assumed to be Gaussian. However, for atmospheric turbulence a Gaussian spectrum is not a good assumption, since the spectrum is given more nearly by the modified von Karman form. As we shall demonstrate in Section 3, it is possible to obtain a readily evaluated result for the MCF when the spectrum has a modified von Karman form by solving the transport equation in the diffusion approximation. Unfortunately, the result is not valid over all lateral distances, but does give valid results over nearly all lateral distances of practical interest at very large propagation distances.

2. RELATIONSHIP OF TRANSPORT THEORY AND OTHER METHODS

It was shown previously[5] that the radiative intensity I(x, n) at position $\underline{x}$ propagating in the direction of the unit vector n in a turbulent medium satisfies

$$\left(\hat{n}\cdot\nabla+\frac{1}{\ell}\right) I(\underline{x},\hat{n}) = \iint \sigma_g(\hat{n}-\hat{n}')\, I(\underline{x},\hat{n}')\, d\Omega' \tag{1}$$

where σ_g is the Born-scattering cross section per unit volume, $d\Omega'$ is the element of solid angle, $\ell^{-1} = \ell_t^{-1} + \ell_a^{-1}$, ℓ_t is the mean-free path for photon scatter by the

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4. Tatarski, V. (1969) Light Propagation in a medium with random process approximation, Soviet Phys. JETP 29:1133-1138.

5. Dolin, L. (1964) Scattering of a light beam in a turbulent plasma (in Russian) Radiofizika 7:380-391.

6. Fante, R. (1972) Propagation of Electromagnetic Waves Through Turbulent Media Using Transport Theory, AFCRL-72-0733.

7. Bugnolo, D. (1960) Transport equation for the spectral density of a multiple-scattered electromagnetic field, J. Appl. Phys. 31:1176-1182.

8. Yura, H. (1972) Mutual coherence function of a finite-cross-section optical beam propagating in a turbulent medium, Appl. Optics 11, 1399-1406.

9. Brown, W. (1971) Second moment of a wave in a random medium, J. Opt. Soc. Am. 61:1051-1059.

turbulence, and ℓ_a is the mean free path for absorption. For a narrow-angle beam propagating along the z-axis, Eq. (1) may be approximated by

$$\frac{\partial I}{\partial z} + \underline{n}_\perp \cdot \frac{\partial I}{\partial \underline{R}} + \frac{1}{\ell} I = \iint_{-\infty}^{\infty} d^2 n'_\perp \, \sigma_g(\underline{n}_\perp - n'_\perp) \, I(z, \underline{R}, n'_\perp) \tag{2}$$

where $\underline{n}_\perp$ is the component of $\hat{n}$ transverse to the z-axis, and $\underline{R}$ is the component of $\underline{x}$ transverse to the z-axis. If we now define the mutual coherence function

$$\Gamma(z, \underline{\rho}_1, \underline{\rho}_2) = \langle E(z, \underline{\rho}_1) E^*(z, \underline{\rho}_2) \rangle$$

where $\underline{\rho}_1$ and $\underline{\rho}_2$ are position vectors lying in the plane z, it can be shown that I and Γ are related via

$$\Gamma(z, \underline{R}, \underline{\rho}) = \iint_{-\infty}^{\infty} d^2 n_\perp \, e^{ik_o \underline{n}_\perp \cdot \underline{\rho}} \, I(z, \underline{R}, \underline{n}_\perp) \tag{3}$$

where $\underline{R} = 1/2\,(\underline{\rho}_1 + \underline{\rho}_2)$ and $\underline{\rho} = \underline{\rho}_2 - \underline{\rho}_1$. Upon using Eq. (2) in Eq. (3), we then obtain for Γ

$$2ik_o \frac{\partial \Gamma}{\partial z} + 2\,\nabla_\rho \cdot \nabla_R \Gamma + \frac{i\pi k_o^3}{2} \Gamma H(\rho) = 0 \tag{4}$$

where

$$H(\underline{\rho}) = 8 \iint_{-\infty}^{\infty} d^2\kappa \, \Phi(\kappa) [\, 1 - \cos \underline{\kappa} \cdot \underline{\rho} \,] \, . \tag{5}$$

In obtaining Eq. (4) we have used the relationship between the cross section σ_g and the two-dimensional spectral density $\Phi(\kappa)$, which is given by

$$\sigma_g(\underline{n}_\perp - \underline{n}'_\perp) = 2\pi k_o^4 \, \Phi[\, \underline{\kappa} = k_o(\underline{n}_\perp - \underline{n}'_\perp) \,] \quad .$$

Upon recalling the relationship between the two-dimensional spectral density $\Phi(\kappa)$ and the function $\phi_\epsilon(o, \kappa)$ defined by Tatarski[4], it can be seen that Eq. (4) is identical with Tatarski's result for the equation satisfied by Γ, which is based on the Markov approximation. Therefore, transport methods and the Markov approximation led to identical results for the mutual coherence function and the intensity distribution. It can be shown[2] that the general solution to Eq. (4) is given by

$$\Gamma(z,\underline{R},\underline{\rho}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2R' \iint_{-\infty}^{\infty} d^2\xi\, u_o\left(\underline{R}'+\frac{\rho}{2} - \frac{\xi z}{2k_o}\right)$$

$$\times\, u_o^*\left(R'-\frac{\rho}{2}+\frac{\xi z}{2k_o}\right) \exp\left\{+i\underline{\xi}\cdot(\underline{R}-\underline{R}') - \frac{\pi k_o^2}{4}\int_0^z dz' H\left[z', \underline{\rho}-\frac{\xi}{k_o}(z-z')\right]\right\} \tag{6}$$

where $u_o = E(0,\underline{\rho})$ is the initial field distribution in the plane $z = 0$. In the plane wave limit, Eq. (6) reduces to

$$\Gamma(z,\underline{\rho}) = |u_o|^2 \exp\left\{-\frac{\pi k_o^2}{4}\int_0^{z'} dz'\, H(z',\underline{\rho})\right\}. \tag{6a}$$

More recently, Lutomirski and Yura[10], apparently unaware of Tatarski's earlier work in this area, have developed a method for computing Γ based on the Huygens-Fresnel principle. We will now demonstrate that the expression obtained by Yura[8], using this method, is identical with Eq. (5). We first set $\underline{p} = \underline{\rho}$, $\underline{R} = \underline{P}$ and $\underline{r} = \underline{p} = \xi z/k_o$. Then Eq. (5) becomes

$$\Gamma(z,\underline{p},\underline{P}) = \left(\frac{k_o}{2\pi z}\right)^2 \iint_{-\infty}^{\infty} d^2R' \iint_{-\infty}^{\infty} d^2r\, u_o\,(\underline{R}'+\frac{r}{2})\, u_o^*\,(\underline{R}'-\frac{r}{2})$$

$$\times \exp\left\{i\frac{k_o}{z}(\underline{p}-\underline{r})\cdot(\underline{P}-\underline{R}') - \frac{\pi k_o^2}{4}\int_0^z H\left[z', \underline{p}-(\underline{p}-\underline{r})(1-\frac{z'}{z})\right] dz\right\}. \tag{7}$$

If we now let $z'/z = t$ and use Eq. (5), we see that Eq. (6) is identical with Eq. (25) of Yura.[8] Therefore, transport theory and the Huygens-Fresnel principle are equivalent in computing the second moment of the field. Finally, we note that Yura[11] has demonstrated that his results are identical with those obtained by Brown[9] using the Bethe-Salpeter equation in the ladder approximation. Therefore, the Markov approximation, transport theory, the Huygens-Fresnel principle and the Bethe-Salpeter equation in the ladder approximation all lead to identical results for the mutual coherence function of an optical beam in a turbulent medium. The restrictions under which the results obtained by all of these methods are valid appear to be identical, and are that the scale of the turbulence is much larger than

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10. Lutomirski, R. and Yura, H. (1971) Propagation of a finite optical beam in an inhomogeneous medium, Appl. Optics 10:1652-1658.

11. Yura, H. (1972) First and second moments of an optical wave propagating in a random medium, J. Opt. Soc. Am. 62, 889-892.

a wavelength, and that the attenuation over a wavelength is small. This requires that $k_o C_n^2 (\ell_o)^{5/3} << 1$, where $L_o = 2\pi \ell_o$ is the outer scale of the turbulence.

3. SOLUTION IN THE DIFFUSION APPROXIMATION

Equation (6) is an exact solution of Eq. (3) for the mutual coherence function. However, in many cases the integrations arising in Eq. (6) are quite difficult to evaluate, even using numerical methods. For this reason it is sometimes desirable to use the approximate solution of Eq. (4), based on solving Eq. (2) in the diffusion approximation, and then using Eq. (2) to calculate Γ. The solution for the radiative intensity $I(z, \underline{R}, \underline{n}_\perp)$ has been given previously.[6] The result obtained for isotropic turbulence is

$$I(z, \underline{R}, \underline{n}_\perp) = \iint_{-\infty}^{\infty} d^2p \iint_{-\infty}^{\infty} d^2k F(\underline{k}, \underline{p})\, e^{-i(\underline{k}_o \underline{R} + \underline{p} \cdot \underline{n}_\perp)} \tag{8}$$

where

$$F(\underline{k}, \underline{p}) = e^{-\epsilon z} F_o(\underline{k}, \underline{p} + \underline{k}z) \Bigg\{ 1 + (2\pi)^2 \int_o^z dz'\, e^{\epsilon(z-z')}$$
$$\times \exp\left[- \int_{z'}^{z} d\xi \lambda(\xi, \underline{k}, \underline{p} + \underline{k}z) \right] \hat{\sigma}_g(\underline{p} + \underline{k}[z - z']) \Bigg\} \tag{9}$$

and

$$\lambda(\xi, \underline{k}, \underline{p}) = \pi^2 |\underline{p} - \underline{k}\xi|^2 \int_o^\infty \kappa^3 \Phi(\kappa) d\kappa + \frac{1}{\ell_a}$$

$$\epsilon = \frac{1}{\ell_a} + (2\pi k_o)^2 \int_o^\infty \kappa\, \Phi(\kappa) d\kappa - \frac{1}{\ell_a} + \frac{1}{\ell_t}$$

$$F_o(\underline{k}, \underline{p}) = \left(\frac{1}{2\pi}\right)^4 \iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2n_\perp I(z=0, \underline{k}, \underline{n}_\perp) \exp\left[i(\underline{k} \cdot \underline{R} + \underline{p} \cdot \underline{n}_\perp) \right]$$

$$\hat{\sigma}_g(\underline{p}) = \left(\frac{k_o}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2\kappa\, \Phi(\underline{\kappa})\, e^{ik_o^{-1} \underline{p} \cdot \underline{\kappa}} .$$

Before proceeding with the calculation of the mutual coherence function, let us first consider the initial intensity distribution. For a focussed beam we can write

$$u_o(\underline{\rho}_1) = U_o \exp\left\{ -\frac{\pi\rho_1^2}{2S} + \frac{ik_o\rho_1^2}{2F} \right\} \tag{10}$$

where S is the effective beam area, and F is the radius of curvature of the wave-front. If $F = \infty$, Eq. (10) represents a collimated beam; if $F < 0$, the beam is focussed at $x = -F$; and if $F > 0$, the beam diverges from the point $x = -F$. The initial distribution of the radiative intensity $I(z, \underline{R}, \underline{n}_\perp)$ is given in terms of u_o by

$$I\,(z=0,\ \underline{R},\ \underline{n}_\perp) = \left(\frac{k_o}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d^2\rho\ e^{-ik_o(\underline{n}_\perp \cdot \underline{\rho})}\ u_o(\underline{R}+\tfrac{\underline{\rho}}{2})\ u_o^*(\underline{R}-\tfrac{\underline{\rho}}{2}) \ . \tag{11}$$

Upon substituting Eq. (10) into (11) we obtain for $I(0, \underline{R}, \underline{n}_\perp)$

$$I(0, \underline{R}, \underline{n}_\perp) = |U_o|^2 S\left(\frac{k_o}{\pi}\right)^2 \exp\left\{ -\frac{\pi R^2}{S} - \left(\frac{k_o^2 S}{\pi}\right) \left|\underline{n}_\perp + \frac{\underline{R}}{F}\right|^2 \right\} . \tag{12}$$

For the case of a collimated beam ($F \to \infty$), Eq. (12) reduces to

$$I(0, \underline{R}, \underline{n}_\perp) = |U_o|^2 S\left(\frac{k_o}{\pi}\right)^2 \exp\left\{ -\frac{\pi R^2}{S} - \left(\frac{k_o^2 S}{\pi}\right) n_\perp^2 \right\} \tag{13}$$

from which we see that the initial divergence angle of a collimated beam is $\delta\omega = (\pi/k_o^2 S)^{1/2}$.

Let us now use the results of Eqs. (7) through (12) in Eq. (3) to calculate the mutual coherence function $\Gamma(z, \underline{R}, \underline{\rho})$. For convenience we consider a finite beam centered about the z-axis and choose $\rho_1 = 0$, so that $\rho = \rho_2$ and $R = \rho/2$. Therefore, we will obtain an expression for the coherence at position ρ relative, to the center of the beam. The result is

$$\Gamma(z, \rho) = \frac{S}{4\pi} |U_o|^2 e^{-\epsilon z} \left\{ \frac{e^{-q\rho^2}}{\beta_1} + (2\pi k_o)^2 \int_o^z \frac{d\eta\, e^{\eta/\ell_t}\, e^{-s_4\rho^2}}{s_1} G(\rho,\eta) \right\} \tag{14}$$

where

$$G(\rho, \eta) = \int_o^\infty \kappa\, d\kappa\ J_o(s_6\kappa\rho)\ \exp(-s_5\kappa^2)\ \Phi(\kappa) \ .$$

The details of the derivation of Eq. (14) are contained in Appendix A. The quantities s_1, s_4, s_5 and s_6 appearing in Eq. (14) are functions of z and η, and are given by

$$s_1 = \alpha_1 + \alpha_2 z^2 + \alpha_3 z + \beta \frac{\eta^3}{3}$$

$$s_4 = k_o^2(\alpha_2 + \beta\eta) + \frac{1}{4s_1}\left[\frac{1}{2} - ik_o(2\alpha_2 z + \alpha_3 + \beta\eta^2)\right]^2 \quad .$$

$$s_5 = \frac{\eta^2}{4k_o^2 s_1}$$

$$s_6 = 1 - \frac{i\eta}{2s_1 k_o}\left[\frac{1}{2} - ik_o(2\alpha_2 z + \alpha_3 + \beta\eta^2)\right] \tag{15}$$

$$\alpha_1 = \frac{S}{4\pi} \quad , \quad \alpha_2 = \frac{\pi}{4k_o^2 S} + \frac{S}{4\pi F^2} \quad , \quad \alpha_3 = \frac{S}{2\pi F}$$

$$\beta = \pi^2 \int_0^\infty \kappa^3 \, \Phi(\kappa) d\kappa$$

and

$$q = k_o^2 \alpha_2 + \frac{\frac{1}{4} - k_o^2(2\alpha_2 z + \alpha_3)^2 - ik_o(2\alpha_2 z + \alpha_3)}{4\left[\alpha_1 + \alpha_2 z^2 + \alpha_3 z\right]} \quad .$$

Equation (14) is the solution for the mutual coherence function in the diffusion approximation for an arbitrary spectrum $\Phi(\kappa)$. Because higher order terms are neglected* when the right-hand side of Eq. (2) is expanded in Taylor series to yield a diffusion term, we note that Eq. (14) is not valid for all values of ρ but only for values such that

$$\frac{k_o^2 \rho^2}{16} \frac{\overline{\gamma^4}}{\overline{\gamma^2}} << 1 \tag{16}$$

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*When $\sigma_g(|n_\perp - n'_\perp|)$ is sharply peaked about $n'_\perp = n_\perp$, we can expand $I(n'_\perp)$ in Eq. (2) in Taylor series about $n'_\perp = n_\perp$. We obtain

$$\iint_{-\infty}^{\infty} d^2 n'_\perp \sigma_g(n'_\perp - n_\perp) I(n'_\perp) \simeq I(n_\perp) \int_0^\infty \sigma_g(n)\, n dn + \left[\frac{\pi}{2} \int_0^\infty n^3 \sigma_g(n) dn\right] \nabla_{\hat{n}}^2 I$$

$$+ \left[\frac{\pi}{32} \int_0^\infty n^5 \sigma_g(n) dn\right] \nabla_{\hat{n}}^4 I + \ldots .$$

Therefore if we keep only the first two terms, as is done in the diffusion approximation, we must require that the last term above be much less than the preceding term. This leads to the requirement stated in Eq. (16).

where

$$\frac{\overline{\gamma^4}}{\overline{\gamma}^2} \sim \frac{\int_0^\infty \kappa^5 \Phi(\kappa)\, d\kappa}{k_o^2 \int_0^\infty \kappa^3 \Phi(\kappa) d\kappa} .$$

4. SIMPLIFICATION OF EQ. (14) FOR A GAUSSIAN SPECTRUM

Equation (14) is quite readily evaluated when $\Phi(\kappa)$ is Gaussian For example, let us choose

$$\Phi(\kappa) = \frac{1}{2\pi^2 \ell_t \overline{\gamma}^2 k_o^4} \exp\left(-\frac{\kappa^2}{k_o^2 \overline{\gamma}^2}\right) \tag{17}$$

where

$$\overline{\gamma}^2 = 4\ell_t \pi^2 \int_0^\infty \kappa^3 \Phi(\kappa)\, d\kappa = 4\ell_t \beta .$$

If Eq. (17) is substituted into the expression obtained previously for $G(\rho, \eta)$, we obtain

$$G(\rho, \eta) = \frac{1}{(2\pi)^2 \ell_t (1+k_o^2 \overline{\gamma}^2 s_5)} \exp\left[-\frac{(s_6 \rho)^2}{4\left(s_5 + \frac{1}{k_o^2 \overline{\gamma}^2}\right)}\right] .$$

Upon using this result in Eq. (14), we obtain

$$\Gamma(z,\rho) = \frac{S}{4\pi} |U_o|^2 e^{-\epsilon z} \left\{ \frac{e^{-q\rho^2}}{\beta_1} + \frac{1}{\ell_t} \int_0^z \frac{d\eta\, e^{\eta/\ell_t} e^{-\theta\rho^2}}{s_1 (1+k_o^2 \overline{\gamma}^2 s_5)} \right\} \tag{18}$$

where

$$\theta = s_4 + \frac{(s_6)^2}{4\left(s_5 + \frac{1}{k_o^2 \overline{\gamma}^2}\right)} .$$

As a check on the validity of Eq. (18), let us consider the plane wave limit. Letting $S \to \infty$, $F \to \infty$ gives

$$\Gamma(z,\rho) = |U_o|^2 \exp\left\{ -\frac{k_o^2 \rho^2 \bar{\gamma}^2 z}{4 \ell_t} \right\} \tag{19}$$

which is identical with the plane wave result, as seen from Appendix B, for values ρ such that

$$\frac{k_o^2 \rho^2 \bar{\gamma}^2}{4} < 1$$

as required by Eq. (16), for the validity of the diffusion approximation.

It is interesting to examine the result of Eq. (18) when $z >> \ell_t$. If we also assume that $\beta z >> \alpha_2$, $\beta z^2 >> \alpha_3$, and $\beta z^3 >> \alpha_1$ we obtain*

$$\Gamma(z \to \infty, \rho) = \frac{3 S \ell_t e^{-z/\ell_a}}{\pi \bar{\gamma}^2 z^3} \exp\left[-\frac{k_o^2 \bar{\gamma}^2 z \rho^2}{16 \ell_t} \right]. \tag{20}$$

From Eq. (20) we see that, for large z, the coherence length ρ_b is given by

$$\rho_b = \frac{4}{k_o \left(\frac{z}{\ell_t} \bar{\gamma}^2 \right)^{1/2}} = \frac{4}{k_o \left[(2\pi)^2 z \int_o^\infty \kappa^3 \Phi(\kappa) d\kappa \right]^{1/2}} . \tag{21}$$

Upon recalling that the plane wave coherence length $\rho_{p\omega}$ is

$$\rho_{pw} = \frac{2}{k_o \left[(2\pi)^2 z \int_o^\infty \kappa^3 \Phi(\kappa) d\kappa \right]^{1/2}} , \tag{22}$$

we see that the asymptotic coherence length of a finite beam is twice the coherence length for a plane wave. This result is quite reasonable physically, since at large distances the beam has spherical wave properties. At large distance, z, the coherence length of a spherical wave is approximately twice that of a plane wave.[12]

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* It can also be shown that Eq. (20) is valid even if the initial field distribution is highly asymmetric.

12. Lutomirski, R. and Yura, H. (1971) Wave structure function and mutual coherence function of an optical wave in a turbulent atmosphere, J. Opt. Soc. Am. 61:482-487.

5. NUMERICAL RESULTS FOR A GAUSSIAN SPECTRUM

Equation (18) has been programmed and evaluated numerically for numerous cases. Before discussing these results let us define

$$C_s = \frac{3S}{\pi \overline{\gamma}^2 \ell_t}$$

$$C_F = \frac{F}{\ell_t}$$

$$\lambda = k_o \ell_t \overline{\gamma}^2$$

$$\tau = z/\ell_t$$

$$R = k_o (\overline{\gamma}^2)^{1/2} \rho \ .$$

We recall that Eq. (18) is valid only when $k_o^2 \gamma^2 \rho^2 << 4$ or for $R << 2$. To illustrate this point, let us define $R_c = k_o \rho_c (\overline{\gamma}^2)^{1/2}$ as the normalized radial distance at which $|\Gamma(\tau, R)/\Gamma(\tau, 0)|$ is equal to e^{-1}. We first consider the plane wave limit, and compare in Figure 1 the exact results for R_c with those computed using Eq. (18). We note that for $R_c < 1.5$ the exact and approximate results agree quite well, but for large values of R_c the results are radically different. Therefore, in this report when we study the MCF of finite laser beams we will generally not present any cases for which $R_c > 1.5$, since Eq. (18) is not generally valid for these cases.

Figures 2, 3, and 4 show the normalized mutual coherence function for a finite width laser beam as a function of R, for various values of distance $\tau = z/\ell_t$ into the turbulent medium. From these results, and from a further study of Eq. 18, it can be seen that

$$\frac{\Gamma(\tau, R)}{\Gamma(\tau, 0)} \simeq \exp\left\{ -\left(\frac{R}{R_c}\right)^2 \right\} . \tag{23}$$

Plots of R_c as a function of τ for various appropriate values of C_s, C_F and λ are shown in Figures 5 through 7. For example, a 10.6-μm collimated laser beam of initial area $\pi/16$ cm^2 propagating in medium strength turbulence would have parameters $C_s = 79$, $C_F = -\infty$, and $\lambda = \pi \times 10^{-3}$. On Figure 5 we have also shown R_c for a plane wave and for a spherical wave. We note that for moderate values of τ, R_c (for the beam) generally lies between the plane wave and spherical wave values, while for large τ, R_c approaches the spherical wave value.

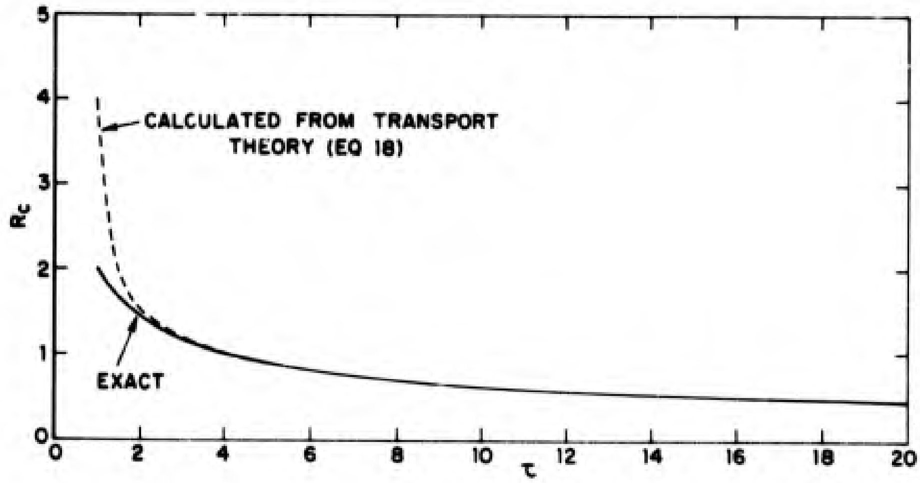
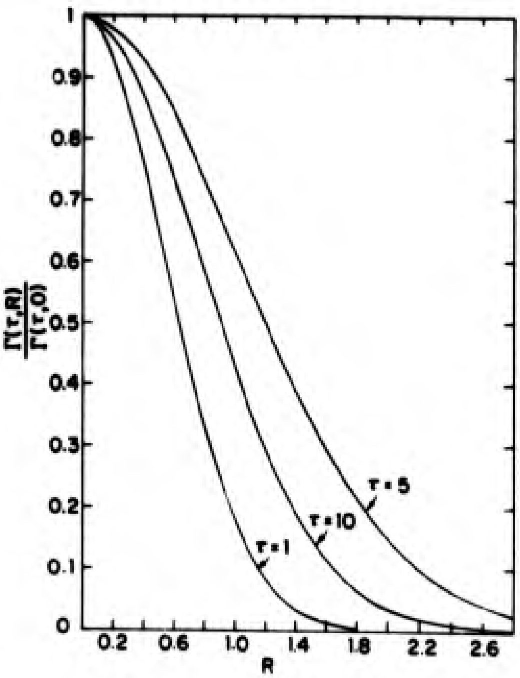


Figure 1. e^{-1} Coherence Length for a Plane Wave



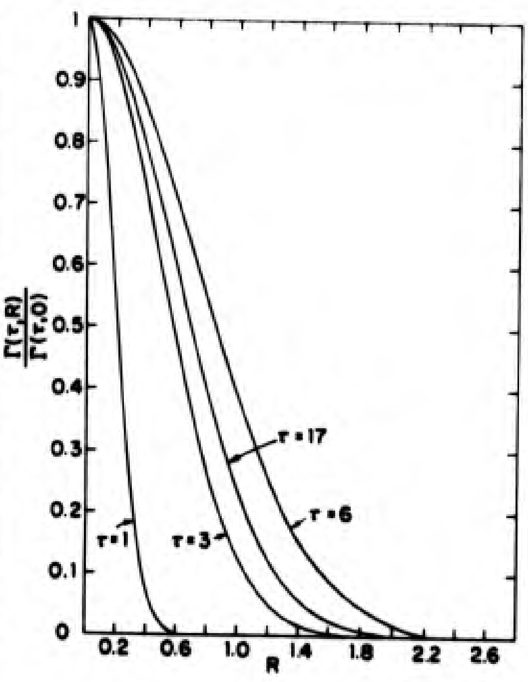


Figure 2. Normalized Mutual Coherence Function for a Beam With $C_s = 10$, $C_F = -\infty$, $\lambda = 0.001$

Figure 3. Normalized Mutual Coherence Function for a Beam With $C_s = 100$, $C_F = -\infty$, $\lambda = 0.001$

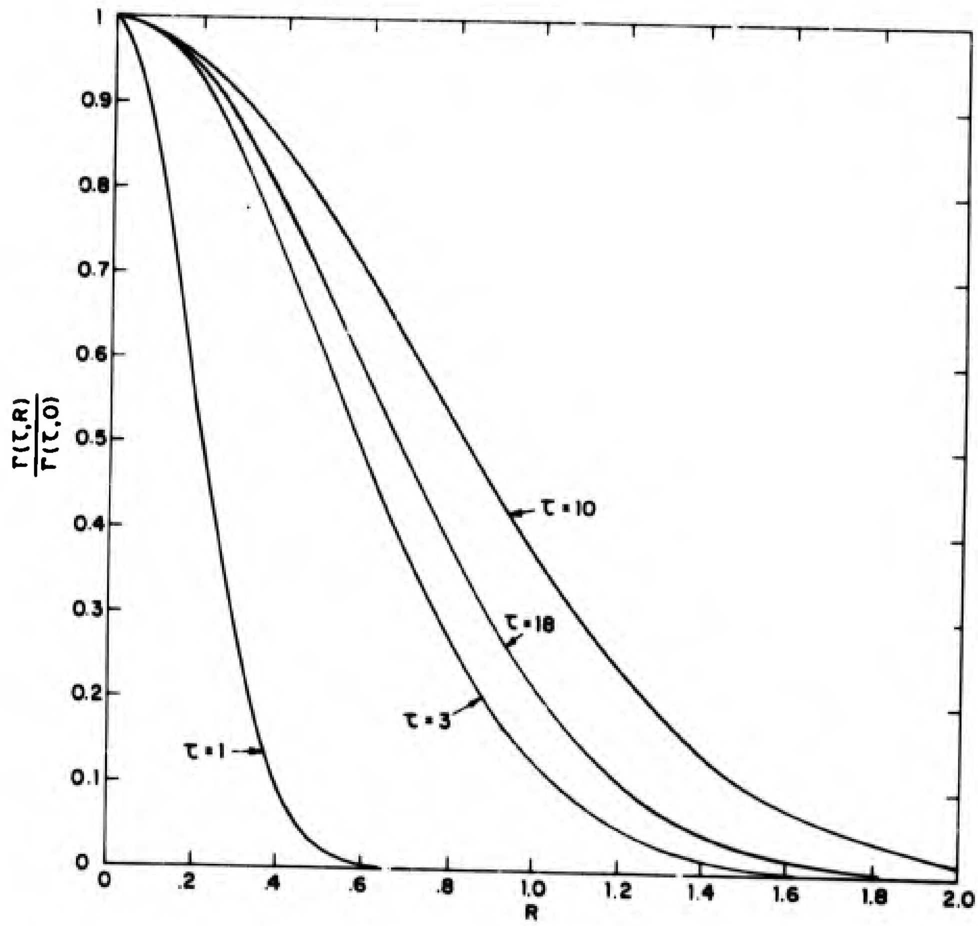


Figure 4. Normalized Mutual Coherence Function for a Beam With $C_s = 100$, $\lambda = 0.01$, $C_F = -\infty$

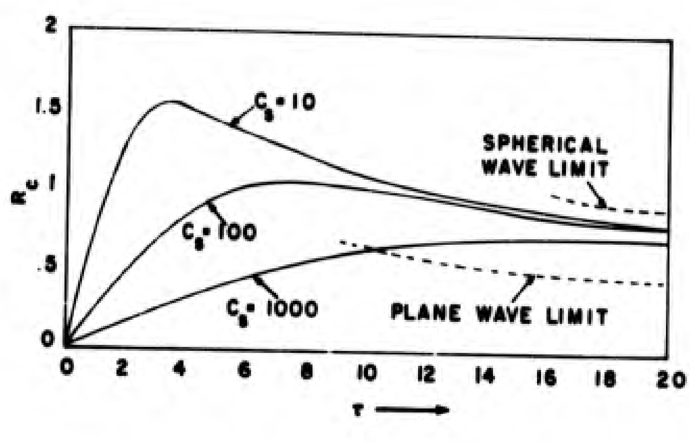


Figure 5. Coherence Length R_c for $\lambda = 0.001$, $C_F = -\infty$

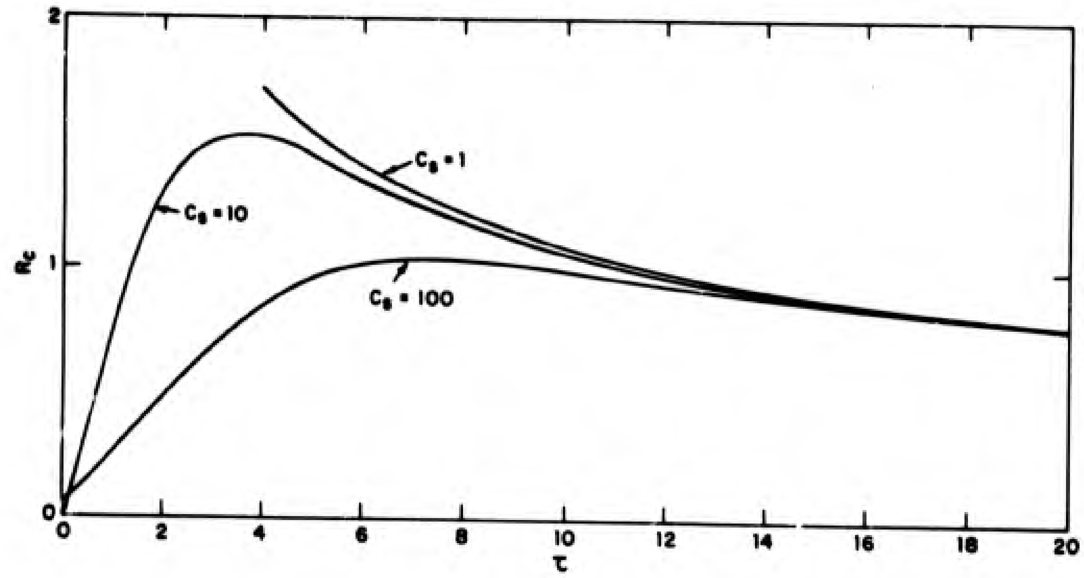


Figure 6. Coherence Length R_c for $\lambda = 0.01$, C_F -∞

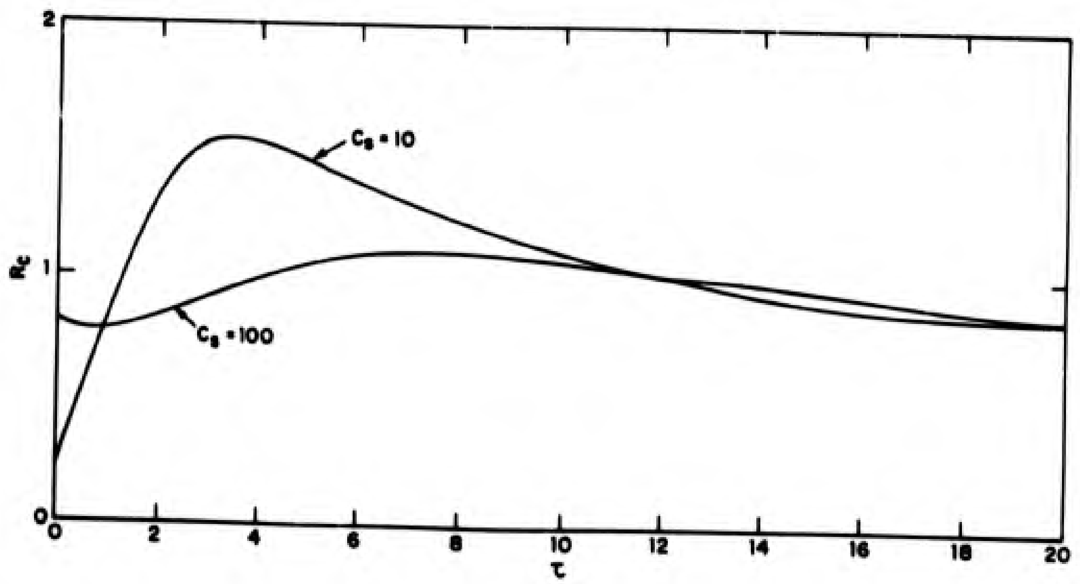


Figure 7. Coherence Length R_c for $\lambda = 0.1$, $C_F = -\infty$

It is also possible to study the effect of the focal length F on the normalized MCF. From Eq. (18) it is clear that for $z > F$ the MCF will be strongly dependent on F if

$$\frac{\pi F}{k_o S} << 1 \quad .$$

That is, the MCF is affected by changes in F if the beam focus lies in the Fresnel zone of an aperture of area S. This point has been verified numerically, but the results are omitted here. Of course for z sufficiently large, the MCF is again independent of F, as is expected from Eq. (20).

6. SOME RESULTS FOR A MODIFIED VON KARMAN SPECTRUM

When the spectrum of the fluctuations is von Karman, the diffusion approximation used in this report gives accurate results only for ρ of order of the inner scale of the turbulence. To see this consider Eq. (16), for the modified von Karman spectrum

$$\Phi(\kappa) = \frac{0.033\, C_n^2 \exp\left(-\kappa^2 \ell_o^2\right)}{\left[\kappa^2 + \mathcal{L}_o^{-2}\right]^{11/6}} \tag{24}$$

where C_n^2 is the index structure constant, $l_o = 2\pi\ell_o$ is the inner scale of the turbulence, and $L_o = 2\pi\mathcal{L}_o$ is the outer scale. For this spectrum Eq. (16) leads to the requirement that, for $\ell_o << \mathcal{L}_o$,

$$\frac{\rho^2}{96\,\ell_o^2} << 1 \tag{25}$$

so that the diffusion approximation is valid for ρ of order ℓ_o but not for ρ of order of the outer scale $\mathcal{L}_o$. This means that the diffusion approximation does not yield accurate results for the coherence length when z is small (since for small z the coherence length is of order $\mathcal{L}_o$) but does yield useful results for very large z, where the coherence length is of order ℓ_o. This situation is acceptable since, for small z, the commonly used method of smooth perturbations is valid anyway, and there is no need to employ the more exact theory which led to Eq. (6).

If we now use Eq. (24) in Eq. (14), we obtain (assuming $\ell_a \to \infty$)

$$\Gamma(\tau, R) = C_s |U_o|^2 e^{-\tau} \left\{ \frac{e^{-KR^2}}{T} + \frac{5\beta_o}{6} \int_o^{\tau} dy \frac{e^{y-\theta_o R^2}}{Q} \int_o^{\infty} \frac{dx\, J_o(s_6 R x^{1/2}) e^{-px}}{(1+\beta_o x)^{11/6}} \right\} \quad (26)$$

where

$R = k_o \rho (\overline{\gamma}^2)^{1/2}$, $\tau = z/\ell_t$, $\lambda = k_o \ell_t \overline{\gamma}^2$ and

$Q = T + y^3$

$T = C_s + \left(\frac{9}{\lambda^2 C_s} + \frac{C_s}{C_F^2} \right) \tau^2 + \frac{2C_s}{C_F} \tau$

$\theta_o = W + \frac{3}{\lambda^2 Q} \left[(1/2 - iV) \right]^2$

$W = M + \frac{y}{4}$

$M = \frac{3}{4\lambda^2 C_s} + \frac{C_s}{12 C_F^2}$

$V = g + \frac{\lambda}{4} \cdot y^2$

$g = 2\lambda \left(M\tau + \frac{C_s}{12 C_F} \right)$

$p = \alpha_o + \frac{3y^2}{Q}$

$\alpha_o = k_o^2 \ell_o^2 \overline{\gamma}^2$

$\beta_o = k_o^2 \ell_c^2 \overline{\gamma}^2$

$s_6 = 1 - \frac{i6y}{\lambda Q} \left[(1/2) - iV \right]$

$K = M + \frac{3}{\lambda^2 T} \left(\frac{1}{4} - g^2 - ig \right)$

and (see Appendix C) for a von Karman spectrum

$$\frac{1}{\ell_t} = 0.78\, C_n^2 k_o^2 (L_o)^{5/3} \qquad (27)$$

$$\overline{\gamma}^2 = \frac{2\pi^2 \ell_t (0.033 C_n^2)}{\ell_o^{1/3}} \left[5.58 - 7.2 (\ell_o/L_o)^{1/3}\right]. \qquad (28)$$

As stated before, Eq. (26) is stricly valid only for $\rho < 4\ell_o$. For $L_o/\ell_o = 1000$, we have $\overline{\gamma}^2 = 40.5 / (k_o L_o)^2$. Therefore, the dimensionless quantity R, which we use as a measure of transverse distance, can be rwritten as

$$R = k_o(\overline{\gamma}^2)^{1/2}\rho = 6.35 \frac{\rho}{\ell_o} \frac{\ell_o}{L_o} = 0.00635 \frac{\rho}{\ell_o} \quad .$$

In terms of R, the requirement that $\rho < 4\ell_o$ can be written as $R < 0.025$. Therefore, except for large τ, where the correlation length is of order ℓ_o, Eq. (26) does not yield useful results (except for the axial intensity).

It is possible to develop a simple closed form result for $\Gamma(\tau, R)$ when $\tau >> 1$. The result is

$$\Gamma(\tau >> 1, R) \simeq \frac{C_s |U_o|^2}{T_o + \tau^3} \exp\left[-\theta_o(y = \tau)R^2\right]. \qquad (29)$$

The derivation of this result is given in Appendix D. We can note from Eq. (28) that if $\tau >> C_s^{1/3}$, $\tau >> 9/\lambda^2 C_s + C_s/C_F^2$, $\tau >> \lambda^{-1/2}$, and $\tau >> (2C_s/C_F)^{1/2}$, then Eq. (29) can be approximated by the simpler form

$$\Gamma(\tau >> 1, R) \simeq \frac{C_s |U_o|^2}{\tau^3} e^{-\frac{\tau R^2}{16}} \quad . \qquad (30)$$

If we now use Eq. (28) for $\overline{\gamma}^2$, assuming $L_o/\ell_o = 10^{-3}$, Eq. (30) becomes

$$\Gamma(z \to \infty, R) = \frac{C_s |U_o|^2}{\tau^3} \exp\left(-\frac{0.197\, k_o^2 \rho^2 C_n^2 z}{\ell_o^{1/3}}\right) \qquad (31)$$

from which we see that, for sufficiently large z, the rate at which the coherence decays is independent of the outer scale of the turbulence and depends only on the inner scale.

As a further check on the validity of Eq. (26) for $R << 1$, we have studied it in detail in Appendix E for the case when $\rho = 0$ and $k_o^{7/6} C_n^2 z^{11/6} < 1$. There it is

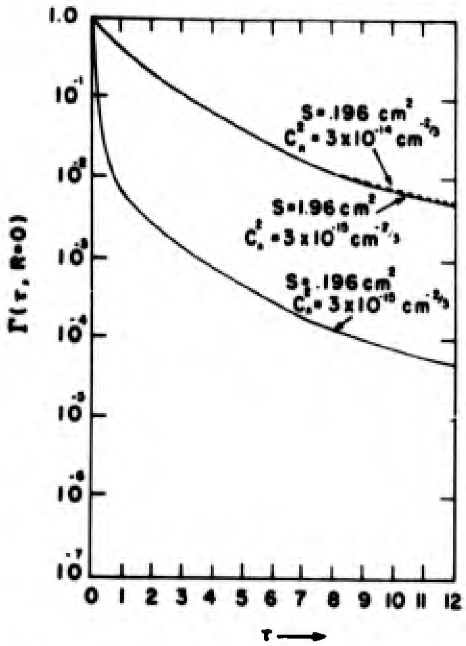


Figure 8. Axial Intensity for a Laser Beam Propagating in Turbulence Having a von Karman Spectrum, Assuming $k_o = 6280\ cm^{-1}$ and $F = -\infty$

shown that it reduces to the previous result of Ishimaru[13] in this limit. Some results computed from Eq. (26) for the axial intensity are shown in Figure 8.

Finally, we comment that it is also possible to develop an asymptotic ($\tau \gg 1$) expression for the intensity $I(\tau, R)$. The result is

$$I(\tau, R) = \frac{C_s |U_o|^2}{T_o + \tau^3} \exp\left[- \frac{3R^2}{\lambda^2 (T_o + \tau^3)} \right] . \qquad (32)$$

The proof proceeds in the same fashion as in Appendix D.

In a forthcoming report we will evaluate Eq. (6) directly to calculate the mutual coherence function for all values of ρ when the spectrum is von Karman.

13. Ishimaru, A. (1969) Fluctuations of a beam wave propagating through a locally inhomogeneous medium, Radio Science 4:295-305.

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Appendix A

When u_o is given by Eq. (10) and $I(0, \underline{R}, \underline{n}_\perp)$ by Eq. (12), it can be shown that the function F_o defined in Eq. (9) is

$$F_o(\underline{k}, \underline{p}) = A\, e^{-(\alpha_1 k^2 + \alpha_2 p^2 + \alpha_3 \underline{k} \cdot \underline{p})} \tag{A1}$$

where

$$A = \frac{S}{(2\pi)^4} |U_o|^2$$

$$\alpha_1 = \frac{S}{4\pi}$$

$$\alpha_2 = \left(\frac{\pi}{4k_o^2 S} + \frac{S}{4\pi F^2}\right)$$

$$\alpha_3 = \frac{S}{2\pi F} .$$

Upon using this result along with the relationship between $\hat{\sigma}_g$ and $\Phi(\kappa)$ in Eq. (8), we obtain

$$F(\underline{k}, \underline{p}) = A\, e^{-\epsilon z}\, e^{-(\beta_1 k^2 + \beta_2 p^2 + \beta_3 \underline{k} \cdot \underline{p})}\, J(\underline{k}, \underline{p}) \tag{A2}$$

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where

$$J(\underline{k}, \underline{p}) = 1 + 2\pi k_o^2 \iint_{-\infty}^{\infty} d^2\kappa\,\Phi(\kappa) e^{ik_o^{-1}\underline{\kappa}\cdot\underline{p}} \int_o^z d\eta\, e^{\ell_t^{-1}\eta}\, e^{ik_o^{-1}\eta\underline{\kappa}\cdot\underline{k}}$$

$$\times\, e^{-\beta\left(p^2\eta + \eta^2 \underline{p}\cdot\underline{k} + \frac{\eta^3}{3} k^2\right)} \qquad \text{(A3)}$$

$$\beta_1 = \alpha_1 + \alpha_2 z^2 + \alpha_3 z$$

$$\beta_2 = \alpha_2$$

$$\beta_3 = 2\alpha_2 z + \alpha_3$$

$$\beta = \pi^2 \int_o^{\infty} \kappa^3\, \Phi(\kappa)\, d\kappa\, .$$

Next define $s_1 = \beta_1 + \beta\eta^3/3$, $s_2 = k_o^2(\beta_2 + \beta\eta)$ and $s_3 = k_o(\beta_3 + \beta\eta^2)$. Then we can rewrite Eq. (A2) as

$$F(\underline{k}, \underline{p} = k_o\underline{\rho}) = F^o(\underline{k}, \underline{p} = k_o\underline{\rho}) + F^s(\underline{k}, \underline{p} = k_o\underline{\rho}) \qquad \text{(A4)}$$

where

$$F^o = A\, e^{-\epsilon z}\, e^{-(\beta_1 k^2 + \beta_2 k_o^2 \rho^2 + \beta_3 k_o \underline{k}\cdot\underline{\rho})} \qquad \text{(A5)}$$

$$F^s = 2\pi k_o^2 A e^{-\epsilon z} \iint_{-\infty}^{\infty} d^2\kappa\, \Phi(\kappa)\, e^{i\underline{\kappa}\cdot\underline{\rho}} \int_o^z d\eta \exp\left\{\frac{\eta}{\ell_t} + i k_o^{-1} \eta\, \underline{\kappa}\cdot\underline{k} - (s_1 k^2 + s_2 \rho^2 + s_3 \underline{k}\cdot\underline{\rho})\right\}. \qquad \text{(A6)}$$

Next, upon using Eq. (7) in Eq. (3) we recall that the mutual coherence function $\Gamma(z, \underline{R}, \underline{\rho})$ and $F(\underline{k}, \underline{p} = k_o\underline{\rho})$ are related via

$$\Gamma(z, \underline{R}, \underline{\rho}) = (2\pi)^2 \iint_{-\infty}^{\infty} d^2k\, F(\underline{k}, \underline{p} = k_o\underline{\rho})\, e^{-i\underline{k}\cdot\underline{R}}\, . \qquad \text{(A7)}$$

If Eqs. (A4) through (A6) are substituted into Eq. (A7), there results upon performing the integrations on d^2k:

$$\Gamma(z, \underline{R}, \underline{\rho}) = \Gamma^o(z, \underline{R}, \underline{\rho}) + \Gamma^s(z, \underline{R}, \underline{\rho}) \qquad \text{(A8)}$$

where

$$\Gamma^{o} = \frac{4\pi^3 A e^{-\epsilon z}}{\beta_1} \exp\left\{-\beta_2 k_o^2 \rho^2 - \frac{1}{4\beta_1}(\underline{R} - i\beta_3 k_o \underline{\rho})^2\right\} \qquad \text{(A9)}$$

$$\Gamma^{s} = 8\pi^4 A k_o^2 e^{-\epsilon z} \iint_{-\infty}^{\infty} d^2\kappa\, \Phi(\kappa) e^{i \underline{k}\cdot\underline{\rho}} \int_0^z \frac{d\eta}{s_1} \exp\left\{\frac{\eta}{l_t} - s_2 \rho^2 - \frac{v^2}{4s_1}\right\} \qquad \text{(A10)}$$

$$\underline{v} = \underline{R} - i s_3 \underline{\rho} - k_o^{-1} \eta\, \underline{\kappa}.$$

We next specialize to the case when $\rho_1 = 0$ so that $R = \rho_2/2 = \rho/2$. In this case we obtain, upon writing $d^2k = k dk d\theta$ and performing the θ-integration

$$\Gamma^{o} = \frac{4\pi^3 A}{\beta_1} \exp\left\{-\epsilon z - q\rho^2\right\} \qquad \text{(A11)}$$

$$\Gamma^{s} = 16\pi^5 k_o^2 A\, e^{-\epsilon z} \int_0^z \frac{d\eta}{s_1} \exp\left\{\frac{\eta}{l_t} - s_4 \rho^2\right\} \int_0^{\infty} \kappa d\kappa\, J_o(s_6 \kappa\rho)\, e^{-s_5 \kappa^2}\, \Phi(\kappa) \qquad \text{(A12)}$$

where

$$q = \beta_2 k_o^2 + \frac{1}{4\beta_1}\left(\frac{1}{4} - \beta_3^2 k_o^2 - i\beta_3 k_o\right)$$

$$s_4 = s_2 + \frac{1}{4s_1}\left(\frac{1}{2} - i s_3\right)^2$$

$$s_5 = \eta^2 / 4k_o^2 s_1$$

$$s_6 = 1 - \frac{i\eta}{2k_o s_1}\left(\frac{1}{2} - i s_3\right)$$

Appendix B

The exact MCF for a plane wave can be demonstrated to be (Tatarski[1])

$$\Gamma(z,\rho) = \exp\left\{-(2\pi k_o^2)z \int_o^\infty \kappa\Phi(\kappa)\left[1 - J_o(\kappa\rho)\right] dk\right\}. \qquad \text{(B1)}$$

For a Gaussian spectrum,

$$\Phi(\kappa) = \frac{1}{2\pi^2 k_o^4 \bar{\gamma}^2 \ell_t}\, e^{-\frac{\kappa^2}{k_o^2 \bar{\gamma}^2}} .$$

Eq. (B1) becomes

$$\Gamma(z,\rho) = \exp\left\{-\tau\left[1 - \exp\left(-\frac{\rho^2\bar{\gamma}^2 k_o^2}{4}\right)\right]\right\}. \qquad \text{(B2)}$$

In the limit when $k_o^2\,\bar{\gamma}^2\rho^2 << 4$, Eq. (B2) can be approximated by

$$\Gamma(z,\rho) \simeq \exp\left\{-\frac{\tau\rho^2 k_o^2\bar{\gamma}^2}{4}\right\} \qquad \text{(B3)}$$

which is identical with the result of Eq. (19), in the limit when $S \to \infty$.

1. Tatarski, V. (1971) The Effects of the Turbulent Atmosphere on Wave Propagation, U.S. Dept. of Commerce, Springfield, Virginia

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Appendix C

In this Appendix we show the derivation of Eq. (27) for $\overline{\gamma}^2$, for the case of a von Karman spectrum. If Eq. (17) is substituted into the expression for $\overline{\gamma}^2$ and we let $\kappa^2 = x$, we obtain

$$\frac{\overline{\gamma}^2}{\ell_t} = \frac{(2\pi)^2}{2} C_n^2 \,(0.033) \int_0^\infty \frac{x\, dx\, e^{-\ell_0^2 x}}{(x + L_0^{-2})^{11/6}} \,. \tag{C1}$$

Now letting $u = x + L_0^{-2}$, we get

$$\frac{\overline{\gamma}^2}{\ell_t} = 2\pi^2 (0.033\, C_n^2) \exp\left(\frac{\ell_0^2}{L_0^2}\right) \left[\int_{L_0^{-2}}^\infty \frac{du\, e^{-\ell_0^2 u}}{u^{5/6}} - (L_0)^{-2} \int_{L_0^{-2}}^\infty \frac{e^{-\ell_0^2 u}\, du}{u^{11/6}} \right] . \tag{C2}$$

Assuming $\ell_0 / L_0 << 1$, and recalling the definition of the incomplete gamma function, $\Gamma(a, x)$, we see that Eq. (C2) can be rewritten as

$$\frac{\overline{\gamma}^2}{\ell_t} \simeq \frac{2\pi^2 (0.033\, C_n^2)}{(\ell_0)^{1/3}} \left[\Gamma\left(\frac{1}{6}, \frac{\ell_0^2}{L_0^2}\right) - \left(\frac{\ell_0}{L_0}\right)^2 \Gamma\left(-5/6, \frac{\ell_0^2}{L_0^2}\right) \right] . \tag{C3}$$

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For $x \ll 1$, $\Gamma(a, x)$ may be expanded as

$$\Gamma(a, x) = \Gamma(a) - x^a \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\,(a+n)} . \tag{C4}$$

Upon using this result in Eq. (C3), we obtain

$$\frac{\overline{\gamma}^2}{\ell_t} \simeq \frac{2\pi^2(0.033C_n^2)}{(\ell_o)^{1/3}} \left[\Gamma\left(\frac{1}{6}\right) - \left(\frac{\ell_o}{L_o}\right)^2 \Gamma\left(-\frac{5}{6}\right) - \left(\frac{36}{5}\right)\left(\frac{\ell_o}{L_o}\right)^{1/3} \right] . \tag{C5}$$

We now recall that $\Gamma(1/6) = 5.58$ and $\Gamma(-5/6) = -6.69$. Using these values in Eq. (C5), and neglecting higher order terms in (ℓ_o/L_o) then gives

$$\frac{\overline{\gamma}^2}{\ell_t} \simeq \frac{2\pi^2(0.033C_n^2)}{(\ell_o)^{1/3}} \left[5.58 - 7.2\,(\ell_o/L_o)^{1/3} \right] . \tag{C6}$$

When Eq. (26) is substituted for ℓ_t, we then get the result of Eq. (27) for $\overline{\gamma}^2$.

Appendix D

In this Appendix we derive an asymptotic expression for the MCF of a beam propagating in a turbulent medium having a von Karman spectrum. By studying Eq. (26), we can readily see that for $\tau \gg 1$ the first term is negligible in comparison with the second. Therefore, if we neglect the first term, substitute Eq. (25) for $\Phi(\kappa)$, and then write $u = \beta_o x$ we obtain

$$\Gamma(\tau \gg 1,\ R) \simeq \frac{5}{6} C_s e^{-\tau} \int_o^{\tau} \frac{dy\, e^y e^{-\theta_o R^2}}{T_o + y^3} \int_o^{\infty} \frac{du\, J_o\left(\frac{s_6 R}{\sqrt{\beta_o}} u\right) e^{-\frac{p}{\beta_o} u}}{(1+u)^{11/6}}$$

Now

$$\frac{p}{\beta_o} = \frac{\alpha_o}{\beta_o} + \frac{3y^2}{\beta_o(T_o + y^3)} = (\ell_o/L_o)^2 + \frac{3y^2}{\beta_o(T_o + y^3)} \ll 1 .$$

Therefore since the main contribution to the integral over u comes from u of order unity, it is clear that for $\tau \gg 1$ we may approximate $\exp(-pu/\beta_o)$ by unity. Next we recognize that since $R = o(1)$ and $(\beta_o)^{-1/2} \ll 1$, if we can show that $|s_6| = o(1)$, then we can also approximate $J_o(\ldots)$ by unity. Therefore let us write out s_6 setting $C_F = \infty$ to simplify our considerations. We have

$$s_6 = 1 - \frac{i3y}{\lambda C_s + \frac{9}{\lambda C_s}\tau^2 + \lambda y^3} - \frac{9y\tau}{\lambda^2 C_s^2 + 9\tau^2 + \lambda^2 C_s y^3} - \frac{\frac{3}{2}y^3}{C_s + \frac{9}{\lambda^2 C_s}\tau^2 + \tau^3} \, . \tag{D2}$$

Let us first consider the case when $C_s << 1$. Then the second term is, at most, of order $(\lambda C_s/3\tau) << 1$. The third and last terms are at most, of order unity. Therefore, for $\lambda C_s << 1$, $s_6 = o(1)$. For $\lambda C_s >> 1$ we can insure that all the terms in Eq. (D2) are less than or of order unity by requiring that $\tau > \lambda^{-1/2}$. Therefore, since $|s_6|$ will generally be of order unity for $\tau >> 1$, it is permissible to approximate $J_o\left(\frac{s_6 R}{\sqrt{\beta_o}} u\right)$ by unity.

If the above approximations are made, the u integration in Eq. (D1) can be performed to yield

$$\Gamma(\tau >> 1, R) \simeq C_s e^{-\tau} \int_o^\tau \frac{dy\, e^y\, e^{-\theta_o(y)R^2}}{T_o + y^3} \, . \tag{D3}$$

For large τ, it is clear that nearly all the contribution to the integral in Eq. (D3) comes from y near τ. Therefore, it is permissible to expand the function $(T_o + y^3)^{-1} \exp(-\theta_o R^2)$ about $y = \tau$ (alternately, we may integrate by parts). If this is done, we obtain from Eq. (D3)

$$\Gamma(\tau >> 1, R) \simeq \frac{C_s e^{-\theta_o(\tau)R^2}}{T_o + \tau^3} + \text{higher order terms.} \tag{D4}$$

In the limit when τ is sufficiently large that $\tau >> C_s^{1/3}$,

$\tau >> \left(\frac{9}{\lambda^2 C_s} + \frac{C_s}{C_F^2}\right)$, $\tau >> \lambda^{-1/2}$, and $\tau >> (2C_s/C_F)^{1/2}$, Eq. (D4) can be approximated further by

$$\Gamma(\tau \to \infty, R) \; \frac{C_s}{\tau^3} e^{-\frac{\tau R^2}{16}} \, . \tag{D5}$$

Appendix E

Ishimaru[1] has previously studied the properties of a beam propagating in turbulence. Since he has used the Rytov approximation, his results are valid only for $C_n^2 k_o^{7/6} z^{11/6} << 1$. We will now demonstrate that in this regime our results agree with those of Ishimaru. Let us consider Eq. (26) for R = 0. We get

$$\Gamma(\tau, 0) = C_s e^{-\tau} |U_o|^2 \left\{ \frac{1}{T_o} + \frac{5\beta_o}{6} \int_o^{\tau} \frac{dy\, e^y}{T_o + y^3} \int_o^{\infty} \frac{dx\, e^{-px}}{[1+\beta_o x]^{11/6}} \right\} \qquad (E1)$$

where T_o, β_o, etc., are defined in Appendix A, and $p = \alpha_o + 3y^2/(T_o + y^3)$. To compare our results with those of Ishimaru, we first consider a collimated beam ($C_F \to \infty$) and assume that $\tau = z/\ell_t$ is small compared with T_o. In this case Eq. (E1) becomes

$$\Gamma(\tau, 0) = \frac{e^{-\tau}|U_o|^2}{\left(1 + \frac{9\tau^2}{\lambda^2 C_s^2}\right)} \left\{ 1 + \frac{5\beta_o}{6} \int_o^{\tau} dy\, e^y \int_o^{\infty} \frac{dx\, e^{-px}}{[1+\beta_o x]^{11/6}} \right\} . \qquad (E2)$$

1. Ishimaru, A. (1969) Fluctuations of a beam wave propagating through a locally inhomogeneous medium, Radio Science 4:295-305.

The inner integral in Eq. (E2) can be expressed in terms of the incomplete gamma function Γ (a, x), and Eq. (E2) then becomes:

$$\Gamma(\tau, 0) = \frac{|U_o|^2 e^{-\tau}}{\left(1 + \frac{9\tau^2}{\lambda^2 C_s^2}\right)} \left\{ 1 + \frac{5}{6} \int_0^\tau dy \exp\left(y + \frac{p}{\beta_o}\right) \Gamma\left(-5/6, \frac{p}{\beta_o}\right)\left(\frac{p}{\beta_o}\right)^{5/6} \right\}. \quad \text{(E3)}$$

We next assume that $p/\beta_o << 1$. This implies that both $\alpha_o/\beta_o = (\ell_o/L_o)^2 << 1$ and $3\tau^2/T_o\beta_o \simeq (z/k_o L_o^2)(z/k_o S) << 1$. The first condition is clearly satisfied since $\ell_o << L_o$, while in the Fresnel zone of the eddies the second is also generally satisfied. Using these assumptions we may therefore approximate $\exp(p/\beta_o) \simeq 1$, and

$$\Gamma\left(-\frac{5}{6}, \frac{p}{\beta_o}\right) \simeq \Gamma\left(-\frac{5}{6}\right) + \frac{6}{5}\left(\frac{p}{\beta_o}\right)^{-5/6} + \ldots. \quad \text{(E4)}$$

Upon using Eq. (E4) in Eq. (E3), we obtain

$$\Gamma(\tau, 0) = \frac{|U_o|^2 e^{-\tau}}{\left(1 + \frac{9\tau^2}{\lambda^2 C_s^2}\right)} \left\{ 1 + \frac{5}{6}\Gamma\left(\frac{-5}{6}\right) \int_0^\tau dy\, e^y \left(\frac{p}{\beta_o}\right)^{5/6} + \int_0^\tau dy\, e^y \right\}. \quad \text{(E5)}$$

The last integral in Eq. (E5) is readily performed, to yield

$$\Gamma(\tau, 0) = \frac{|U_o|^2}{\left(1 + \frac{9\tau^2}{\lambda^2 C_s^2}\right)} \left\{ 1 + \frac{5}{6}\Gamma\left(-\frac{5}{6}\right) e^{-\tau} \int_0^\tau dy\, e^y \left(\frac{p}{\beta_o}\right)^{5/6} \right\}. \quad \text{(E6)}$$

Over distances large compared with the inner scale of the turbulence, we may use the approximation

$$\frac{p}{\beta_o} \simeq \frac{3y^2}{T_o\beta_o}$$

and the integral on the right-hand side of Eq. (B6) can then be approximated to give

$$\Gamma(\tau, 0) = \frac{|U_o|^2}{\left(1 + \frac{9\tau^2}{\lambda^2 C_s^2}\right)} \left\{1 - \frac{5}{16}\Gamma\left(-\frac{5}{6}\right)\left(\frac{3}{\beta_o T_o}\right)^{5/6} \tau^{16/6} + \ldots\right\} . \quad \text{(E7)}$$

To compare with Ishimaru's results we next recall that the quantity α, defined by Ishimaru is, in our notation, $\alpha = 3/\lambda \ell_t C_s$. Therefore:

$$\left(\frac{3}{\beta_o T_o}\right)^{5/6} \tau^{16/6} = \left(\frac{3}{\beta_o C_s}\right)^{5/6}\left(\frac{1}{1+\alpha^2 z^2}\right)^{5/6}\left(\frac{\lambda C_s \alpha}{3}\right)^{5/6}\left(\frac{1}{\ell_t}\right)^{11/6} z^{16/6}$$

$$= \left(\frac{\lambda}{\beta_o}\right)^{5/6}\left(\frac{1}{\ell_t}\right)^{11/6} (\alpha z)^{5/6} z^{11/6}\left(\frac{1}{1+\alpha^2 z^2}\right)^{5/6}$$

$$\simeq 0.78\, k_o^{7/6} C_n^2 \left(\frac{1}{1+\alpha^2 z^2}\right)^{5/6} (\alpha z)^{5/6} z^{11/6} . \quad \text{(E8)}$$

Using Eq. (E8) in Eq. (E7) then gives

$$\Gamma(\tau, 0) = \frac{|U_o|^2}{(1+\alpha^2 z^2)} \left\{1 + 0.24\, k_o^{7/6} C_n^2 z^{11/6} \left(\frac{\alpha z}{1+\alpha^2 z^2}\right)^{5/6} \Gamma(-5/6)\right\} . \quad \text{(E9)}$$

Except for a slight difference in the coefficient of the second term in Eq. (E9), our limiting case is identical with Eq. (13) of Ishimaru. That is, Ishimaru obtains $5\pi^2(0.033)/8 = 0.205$, while we obtain 0.24. The difference arises because we have approximated the exact transport integral by the diffusion approximation.

It is interesting to compare Ishimaru's result with the result of Eq. (26). This is done in Figure E1, for the case when $C_n^2 = 3 \times 10^{-14} \text{cm}^{-2/3}$, $k_o = 6280 \text{ cm}^{-1}$, $S = 0.196 \text{ cm}^2$, and $F = -\infty$. We note that the two results agree quite well, but diverge rapidly when τ is sufficiently large that the second term in Eq. (E9) is comparable with the first.

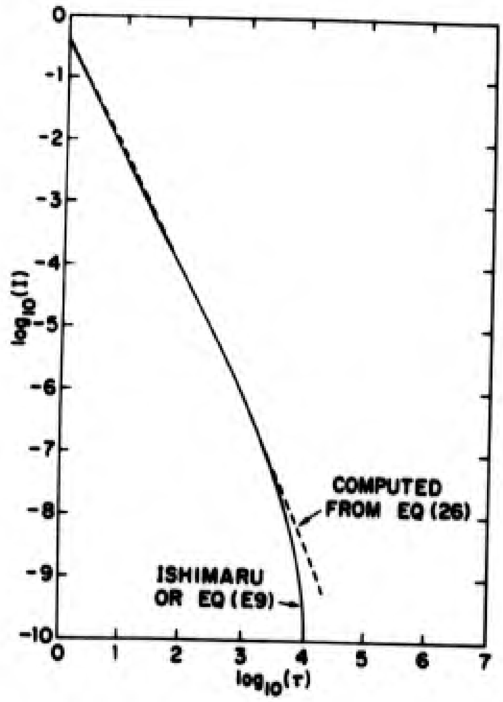

Figure E1. Comparison of Axial Intensity for a von Karman Spectrum Assuming k_o = 6280 cm^{-1}, $C_n^2 = 3 \times 10^{-14}$ cm$^{-2/3}$, S = 0.196 cm^2, and F = $-\infty$