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# SACLANT ASW RESEARCH CENTRE MEMORANDUM

## MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

by

BRIAN W. CONOLLY

1 SEPTEMBER 1973

NORTH ATLANTIC TREATY ORGANIZATION

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SACLANT ASW RESEARCH CENTRE
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TABLE OF CONTENTS

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ABSTRACT

Mathematical models and their analysis are the basis of many operational research investigations, whether for civilian or military applications. They provide a rationale which, when validated, yields insight into the structure of the processes at work, and the material for decision making. It has been found that the peculiar mathematical analyses that arise in military situations are less well documented than those that arise in civilian situations. This volume is, accordingly, the first contribution to a reading programme that beginners in military operational research might be interested to follow, and to which the writer feels it is a duty of practitioners to contribute.

In the first part a stochastic model is formulated to represent a brief campaign in which major naval units sojourn in a given area for the fulfilment of a certain mission, and are subjected to submarine attack by the enemy. The analysis is confined to the elements likely to be basic to a full-scale study of such an operation for military decision making. The point is also made that, while the problem seems complicated, it is nevertheless amenable to an analysis that has the capability of revealing the structural dependence of the quantities of interest on the parameters of the problem more clearly than would be the case if computer simulation were the only tool available.

The second part is also of stochastic nature and describes the use of probabilistic methods in an aspect of mine warfare. The model is capable of providing material for decision-making by both the offence and the defence. The problem is characteristic of harbour defence and is concerned with shallow waters.

In the third part the problems considered are geometrical in nature. They are divided into three categories: (a) Geometrical Probability: The statistics of distances between points on a line and in an area, with some consideration of their maxima and minima in the linear case. Applications to patrols and mining situations are mentioned; (b) Closure: Collision course; probability of hitting a target of finite length with a salvo of straight running torpedoes; some weapon homing systems – pursuit, pursuit with lead, line of sight guidance; probability of closing and detecting a submarine located at a remote datum; (c) Location: Principally the mathematics of some bearings-only methods.

Following the general introduction to the series there is a list identifying unclassified items from the SACLANTCEN Bibliography, and elsewhere, recommended for further reading.

# MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

## GENERAL INTRODUCTION TO SERIES (PARTS 1 - 3)

by

Brian W. Conolly

Increasing numbers of younger practitioners of military Operational Research will have followed an Operational Research course at the University level, but these courses, consisting usually of lectures in a number of disciplines commonly held to be recurrent themes in O.R. practice, with the addition of case studies, will rarely have had any military flavour. This raises the question of the desirability of offering preparatory courses in military Operational Research. Many workers of an older generation will assert that they came to the subject by accident and that certainly they had no preparation. They had to learn on the job. Anyway, any theory that occupies more space than the back of an envelope should be regarded with suspicion. There is no room for long-haired young mathematicians, thinking they know every thing, tossing off long-winded and incomprehensible theories. Half the time a fellow can not understand even the first line. And what about the man in the ship/tank/aircraft? What is he going to think when he finds out that defence budgets are being eroded by this sort of thing? What is needed is bigger bangs... Many of us have heard views of this kind expressed. Fortunately they are becoming increasingly rare and by and large they can be ignored. So, accepting that some preparation for the practice of Operational Research is desirable, what remains to consider seriously is whether additional orientation towards military Operational Research is advantageous.

The answer seems likely to be affirmative. Such preparation is given at least in the U.S., U.K., and in France. And since, in the cases known personally to the writer, public money is involved, little room for doubt is left as to the existence of a demand and the evidence of its satisfaction. A similar inference may be drawn from activity on the international plane. In NATO the Office of the Assistant Secretary General for Scientific Affairs has for more than ten years conducted a vigorous and distinguished educational programme with some military content through its Advisory Panel on Operational Research, and through current organizational changes there appears to be a broadening of activity with identification and isolation for special treatment of the military aspect. It seems then that it is the plain duty of military O.R. practitioners within NATO to support the Scientific Affairs Division whereever and whenever they can. An obligation thus devolves upon suitably qualified scientists within the two NATO laboratories where military O.R. is practised.

Fulfilment of the obligation can take many forms. Just doing the job well is part of it, but particularly taking care to write clear and well documented reports that can be widely diffused and used by students for collateral reading. In this connection special attention must be given to careful and lucid explanation of novel methods, or novel applications of well known methods. To do this adequately may mean writing two reports on a single project, one destined for the customer and the other for the student: in addition, opportunities should always be seized to give suitable work the wider publicity afforded by the open literature.

It is in the spirit of offering some student-oriented material that the present series, "Mathematics in Military Operational Research", has been initiated. The contributions have the flavour of "leaves from the analyst's notebook", and were intended to be a reading companion to a series of lectures. They are frankly didactic and not specially original either in content or presentation. Their distinction is solely in having genuinely been used personally by the author in the course of military Operational Research work. It is to be hoped that colleagues at SACLANTCEN may volunteer contributions to the series, which could ultimately form a valuable addition to a rather scanty military O.R. literature.

In conclusion it seems worth while to use this opportunity to identify SACLANTCEN unclassified Reports and Memos considered to be of use for collateral reading. These follow with comments, which like the views expressed here, are the author's own.

## SUGGESTED SACLANTCEN DOCUMENTS

### Technical Reports

T.R. 60 Conolly, B.W. (1966)
An Unrestricted Linear Random Walk with Negative Exponentially Distributed Step Lengths.
Comment: An account is given of the theory of a doubly infinite linear random walk in which step lengths have a negative exponential distribution and the direction of each step is not necessarily equiprobable. The problem of first passage time is also studied. The theory was developed in connection with a study of random linear anti-submarine patrols.
Also published in Annales de l'Institut Henri Poincaré, 2, 1965, 173-184.

T.R. 117 Bresson, M. (1968)
The ASW Role in the Logistic Support of Ground Forces.
Comment: On the basis of the observed slowing-down of land offensive, a model is developed to explain the logistical constraints that impose an upper limit on the tactical or strategic capabilities of ground forces. The model is then generalized to cover not only the space/time constraints within the land theatre itself, but also

those constraints that are imposed by the availability of supplies. This leads to the development of a global model showing the interactions between land and naval battles for conditions in which the main source of supply of large land units depends on the safe arrival of transport ships. The role of ASW in the logistic support of ground forces is thereby demonstrated.

T.R. 144 Conolly, B.W. (1969)
A probabilistic Theory of Antisubmarine Warfare Models Developed in Terms of Congestion Theory.
Comment: This report, which is methodological, develops a probabilistic theory that has direct application to both antisubmarine warfare and congestion models. The theory is expressed in congestion terminology because of the presumed wider knowledge and appeal of the field. This results in a simplified presentation of the general theory of infinite service facility systems with specific application to $M/Y/\infty$ and $X/M/\infty$, some of which have already been studied by Takacs and Khintchine. A new result is given for the output of the latter process. The analogy between certain infinite service facility systems and a single-server system with queue length dependent service is exploited to provide results for the latter process. A further new result for the busy period of such a process is quoted. The antisubmarine applications are to the formally similar models of the number of units present in a geographical area, and to the attrition of an enemy submarine force subjected to a steady threat from an anti-submarine barrier that geographical or other constraints compel it to transit.

T.R. 178 Diess, H.G. (1970)
A Game Theory Solution to an Aiming Problem.
Comment: This paper discusses the game theory solution to the following aiming problem:
An attacker receives information about the location of a target and launches a weapon. It is assumed that, at the moment it is supposed to be hit, the target may be anywhere within an annulus with radii $R_1$, $R_2$, which depend on the weapon delivery time and the target evasion manoeuvres. Using a polar coordinate system R, $\theta$, the assumption is made that the target is uniformly distributed in $\theta$, but chooses R between the limits $R_1$ and $R_2$ in order to maximize its chance of escape. The attacker will then distribute the weapon aimpoints uniformly in the angle $\theta$ (over its range $0, 2\pi$). For a single weapon $\theta = 0$ is chosen arbitrarily and the problem is reduced to the choice of the radial coordinate X for the aimpoint. The pay-off for this two-person game, where both players have continuous strategies, is expressed by the probability that the distance of the target from the impact point is less than the effective damage radius, e, of the weapon. Pure and mixed strategy solutions are discussed and conditions are derived from the normalized parameters $2e(R_2 - R_1)$ and $R_2(R_2 - R_1)$ that allow one to determine the type of strategies for a given set of values of the parameters e, $R_1$ and $R_2$.

T.R. 207 Fabry, C. (1972)
Entrance Time Distribution and Limiting Transition Probabilities for Continuous-Time Markov Chains, with Applications to Stochastic Combat Models.
Comment: The methods presented in this report make possible, in stochastic combat models, the determination of victory probabilities, distribution of the number of survivors and moments of the distribution of the combat duration. More generally, when applied to continuous-time Markov chains, the same methods provide the limiting transition probabilities and the moments of the distribution of the entrance time into the set of all absorbing states.
The computation time required to obtain numerically the quantities of practical interest is very small in comparison to the time that would be necessary to solve the huge system of differential equations describing a combat stochastically.

T.R. 217 Fabry, C. (1972)
On Taking Sequential Decisions in Changing Environment.
Comment: An attempt is made to extend Walds's sequential decision theory to the case where the state of the system being observed can change during the observation-decision process. Most of the results obtained concern systems that can have only two possible states and are based on a partial differential equation which describes approximately the evolution in time of the probability ratio (or likelihood ratio). In particular, the following elements can be deduced from that partial differential equation for a sequential decision procedure with constant thresholds: mean number of observations before a terminal decision is taken, probabilities of errors, and expected loss.

T.R. 220 Mjelde, K.M. (1972)
A Time Dependent Stochastic Model for a Combat between Submarines and Defended Convoys.
Comment: A situation where submarine attack merchant ship convoys is analysed stochastically. The convoys are defended by surface and air screens and by antisubmarine area defences. Submarines' encounters with convoys are described statistically and differential equations for the expected values and variance of the losses on both sides are given. They are valid for all times after the start of the battle and for both small and large numbers of submarines engaged in combat. Lower bounds of the variances are expressed in terms of mean values. Expected value equations that have been used in previous SACLANTCEN studies are given.

Technical Memos

T.M. 82 Knight, J.C. (1964)
Digital Computer Simulation of Tactical Situations.
Comment: This memorandum is a summary of work done in the field of simulation at the SACLANT ASW Research Centre during the period 1961-1963. The emphasis is on methodology rather than results. The techniques described are extremely simple, and should be easily adaptable to a wide variety of problems which cannot readily be tackled by analytical methods.
The memorandum is in two parts. Part I contains a general discussion of the simulation method adopted. Part II contains flow diagrams of the more important programmes and subroutines which have been writen.

T.M. 88 Conolly, B.W. (1964)
On the Digital Computer Simulation of a Tactical Game.
Comment: Of particular interest is the emphasis given to checking the mechanics of the simulation.

T.M. 95 Conolly, B.W. (1965)
On the Distribution of the Lead in a Chase.
Comment: More details on checking aspect of T.M. 88.
In a series of tactical studies a problem arose which can be described in the following terms.
A runner P moves with constant speed along a straight line. At negative, exponentially-distributed time intervals the position of P is communicated to a second runner S who then tries to reach that position. Interest centres on the lead that P has over S at the instants when information is given to S.
This memorandum is devoted to the derivation of the probability distribution of the lead. Some numerical tables are appended.

T.M. 139 Conolly, B.W. (1968)
Experiments with Pólya Processes.
Comment: Certain events are habitually observed in naval exercises. Their interest lies in their nuisance value, which it is required to eliminate. When considered as time series, the event sequences appear to cluster by comparison with a Poisson sequence. This report arises out of an investigation of the Pólya process as a possibly better statistical description of the observation.
The central problem was the estimation of the two parameters of Pólya processes, but only a brief mention is made of the problem of estimation in this report. Some theory is given and this is supplemented with numerical results of digital computer experiments intended to illustrate some startling behaviour that Pólya processes appear to exhibit, and which it is believed are not well known. Subsequent reflection leads the author to the view that the Pólya process is not the most adequate model for the phenomenon originally under consideration.

T.M. 155 Fabry, C. (1970)
Nested Bounds for Solutions of Differential Equations.
Comment: For a particular class of ordinary differential equations, an iterative procedure is described that gives a sequence of nested pairs of lower and upper bounds for the solution. Simple conditions are found under which the bounds converge to the solution. The method is used to study equations of Lanchester's type, which have applications to the deterministic description of situations in naval warfare.

SACLANTCEN Memos

S.M. 1 Mjelde, K.M.
Extensions of a Time-Dependent Stochastic Model for a Combat Between Submarines and Defended Convoys.
Comment: A time-dependent stochastic model for the losses of convoyed ships due to submarine attacks has been extended in three different ways to cover the following cases:

1. A non-zero time to complete an attack on a convoy.
2. Different kinds of weapons.
3. Different types of targets.

Several tactical options are included in each extension.

## OTHER SUGGESTED READING

Morse, Philip M. & Kimball, George E. 1951.
'Methods of Operations Research', New York: Wiley.

Zehna, Peter W., ed. 1971.
'Selected Methods and Models in Military Operations Research', Washington: Office of Naval Research — US Government Printing Office.

Saaty, T.L. 1959.
'Mathematical Methods in Operations Research', New York: McGraw-Hill.

# MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

## PART 1: A THEORY OF SUBMARINE OPERATIONS AGAINST A MAJOR, ESCORTED, SURFACE TARGET IN THE PRESENCE OF ARMED DECOYS

MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

# PART 1: A THEORY OF SUBMARINE OPERATIONS AGAINST A MAJOR, ESCORTED, SURFACE TARGET IN THE PRESENCE OF ARMED DECOYS

by

Brian W. Conolly

## ABSTRACT

This is the first in a brief series of expositions of the use of mathematical analysis in some military operational research problems. A stochastic model is formulated to represent a brief campaign in which major naval units sojourn in a given area for the fulfilment of a certain mission, and are subjected to submarine attack by the enemy. The analysis is confined to the elements likely to be basic to a full-scale study of such an operation for military decision making. The point is also made that, while the problem seems complicated, it is nevertheless amenable to an analysis that has the capability of revealing the structural dependence of the quantities of interest on the parameters of the problem more clearly than would be the case if computer simulation were the only tool available.

## INTRODUCTION

Mathematical models and their analysis are usually the basis of operational research studies, whether in civilian or military contexts. They provide a rationale which, when tested, yields insight into the structure of the processes at work, and the material upon which decisions can be made.

The application of operational research to military situations, has, however, been found to be less well documented than the civilian counterpart. We advance no explanations for this situation but, accepting it as a fact, we believe that, to remedy it, it would be useful to potential practitioners to document some accounts of models that have been developed and analysed in the military, and, in particular, naval connection. This is the reason for planning the short series of SACLANTCEN Memos of which this is the first.

We emphasize that the Memos are not case studies, but that in every instance they describe the kind of research work that has to be done in the first phases of operational research studies.

A shortened version of this paper was presented to the NATO Advisory Panel on Operational Research Conference on Modern Developments in Lanchester Theory held in Munich, 1967. The relevance both of the techniques and of the subject matter to modern antisubmarine warfare preserves the interest of the paper which, at the time of writing for the Munich Conference, emphasized the technical aspect. The reader will nevertheless guess that the problem itself, idealized though the setting may be, arose from a real life scenario. An added justification for offering the paper now is the continued and lamentable failure to appear of the proceedings of the Munich Conference.

An analysis of the following situation is provided. Two major naval powers are at war. In a relatively short duration (days rather than months) phase of the campaign one power deploys major surface units U (e.g. cruisers, battle-ships, aircraft carriers), one at a time, in a certain area to carry out a prescribed task.
The unit deployed is provided with close escorts for protection against the submarines, S, deployed by the enemy for the purpose of destroying U. In addition, the major surface-unit power deploys decoys scattered uniformly throughout the operational area. Their role is to simulate U and thereby to lure the S (assumed to rely on passive methods for detection) away from the real U, and then to attack and destroy as many S as possible. In this scenario the "U" force is granted air superiority and no direct account is taken of the use of aircraft by either side.

The task of a deployed U requires it to remain essentially stationary in the operational area which we shall treat as a rectangle of dimensions L km long and B km wide. The enemy, with N submarines available to put to sea, divides the area into N strips parallel to the long side and therefore each of width $W = B/N$ km. Each submarine is instructed to patrol a strip back and forth parallel to the long side and not to stray from that strip. Then if, on behalf of U, there are D decoys it follows from the assumption of uniformity that each strip contains $D/N = n$ decoys.

In what follows we shall model mathematically the S vs. U operations <u>in that submarine patrol strip containing U</u>. When this has been done it will not be difficult for the reader to devise a (simpler) theory to deal with the N-1 strips containing no U. Reference to U will henceforward be taken to mean its close escorts as well.

A crucial simplifying assumption made on theoretical grounds is that losses can be, and are, instantly replaced on both sides. The statements made by the theory about the statistics of the lifetime of a U, of an S, and of the losses on either side, are therefore distorted. This is of less concern because of the supposed short duration of this phase of the campaign. In addition the statistics are meaningful for planning purposes. For example, the continual instantaneous replacement of S means that U is subjected to a more sustained threat than is really the case. From U's point of view estimates of U's survivability are therefore pessimistic. The enemy may, of course, draw a similar conclusion concerning the fate of submarines. The theory can be thought of as supplying the basis for estimating upper limits of force requirements.

The major part of the paper deals with the situation in which submarines are instructed <u>not</u> to attack anything classified as a decoy. It was part of the original problem to see what difference would be made by the adoption of an aggressive anti-decoy policy, and so, at the end, this is considered and shown theoretically to be expressible as a simple extension of the non-aggressive case.

## 1. PROBABILITY OF AN ENCOUNTER

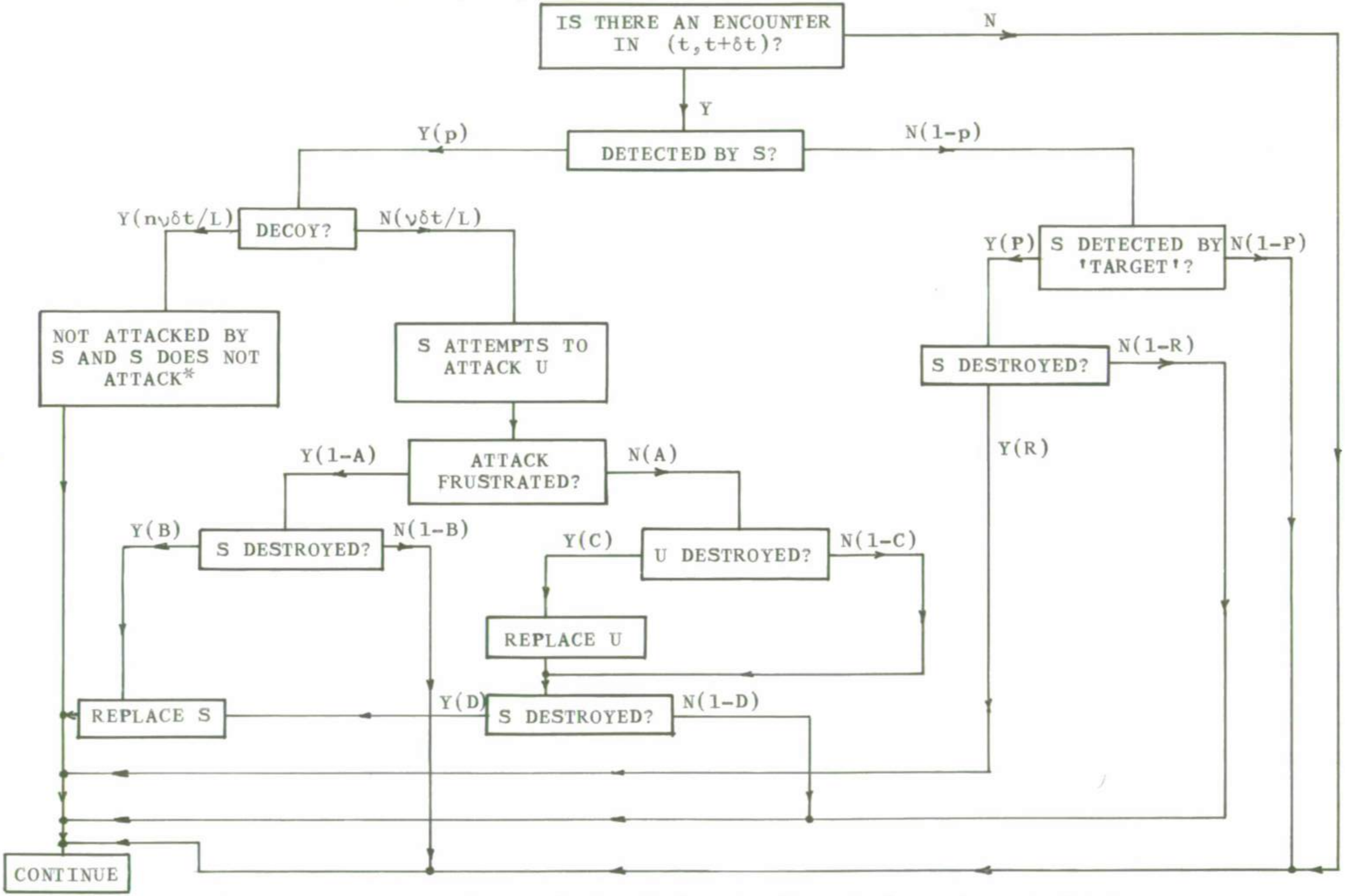
In the strip containing U there are n decoys, and so (n+1) objects providing potential targets for S. The density of targets is thus $\frac{(n+1)}{LW}$ targets per unit area. If S patrols at mean speed $\nu$ km/unit time, then the probability of detecting a target in an interval of length $\delta t$ time units will be assumed to be $W\nu\delta t(n+1)/LW$ i.e. $(n+1)\nu\delta t/L$. The probability of detecting U in $\delta t/L$ is then simply $\nu\delta t/L$; and of a decoy, $n\nu\delta t/L$.

## 2. DETECTION, CLASSIFICATION AND ATTACK

S is assumed to have a probability of detecting anything encountered. A time $t_c$ is allowed for classification after detection and it is assumed that correct classification is certain, an assumption easily modified. If the object is classified U, S immediately (and instantaneously) launches an attack, otherwise it does nothing. Thus, in this part, the role of the decoy is merely one of delay and possible attack on S. However, it is also assumed that S can not be attacked by any unit that it has classified before itself attacking that unit, implying that classification by S can be completed before S can itself be detected. But if a unit is not detected by S we allow the possibility of its attacking S. S may also be attacked and destroyed in the process of carrying out an attack.

Under the assumption that $\delta t$ is so small that only one type of unit can be encountered during $\delta t$ the situation and its possible consequences are shown in the flow diagram, Fig. 1. The expressions in brackets beside the branches are conditional probabilities of the various eventualities. They are now listed here for convenience:

| | |
|---|---|
| $(n+1)\nu\delta t/L$ | probability that S encounters a target (U or decoy) in time interval $(t, t+\delta t)$. |
| p | probability that S detects the target. |
| A | probability that S launches attack which is not frustrated; (1-A is probability of attempting to attack, but being frustrated before actual weapon launch). |
| B | probability that S is destroyed after a frustrated attack. |
| C | probability that U is destoyed in an attack by S. |
| D | probability that S is destroyed after attack on U. |
| P | probability that S is detected by a unit that it has not itself detected. |
| R | probability that S is destroyed by an unit that it has not itself detected. |




* We repeat that we are dealing first with the case where S does not attack decoys.

FIG. 1 FLOW DIAGRAM

## 3. THEORY

One purpose of this paper is to show that even when a situation appears to be complex there may be hope for the development of a 'paper theory', thereby possibly avoiding the tedium and pitfalls of a digital computer simulation and, in any case, providing a test bed for a simulation, if inevitable.

Without further ado we proceed to formulate equations for the following:

(a) $m_k(t)$ - the joint probability that S survives to age t, has destroyed k U's in that time, and, in addition, is disengaged. [By 'disengaged' it is meant that S is not engaged in classification, attack, or in being attacked. Thus, if an encounter were to occur during the interval $(t, t+\delta t)$ S in the disengaged condition would be free to pursue it.]

(b) $\ell_k(t)$ - the joint probability and density that S survives to age t and is destroyed in $(t, t+\delta t)$ having destroyed k U's in its life time.

(c) $f(t)$ - the probability density that U survives to age t and is destroyed in $(t, t+\delta t)$.

These functions provide all the statistical information we seek. $m_k(t)$ is the basis of $\ell_k(t)$ and $f(t)$. It satisfies a differential difference equation of Lanchester type, the attrition coefficients being, however, independent of k by virtue of the instant replacement hypothesis.

## 4. DETERMINATION OF $m_k(t)$

The fundamental differential difference equation for $m_k(t)$ has the form

$$\frac{dm_k(t)}{dt} + \alpha\, m_k(t) = \beta\, m_k(t-t_c) + \gamma\, m_{k-1}(t-t_c) \qquad \text{[Eq. 1]}$$

for $k \geq 1$, and lacks the $m_{k-1}$ term for $k = 0$. The initial conditions are $m_o = 1$, $m_k(0) = 0$ for $k \geq 1$. $\alpha, \beta, \gamma$ are all independent of t and will be given presently. Denoting the Laplace transform of $m_k(t)$ $\omega$, t . t by $M_k(z)$, then for all $k > 0$ we have

$$M_k(z) = e^{zt_c}\, \gamma^k / [(z+\alpha)\, e^{zt_c} - \beta]^{k+1} , \qquad \text{[Eq. 2]}$$

where $t_c$ is the time for S to classify.

Equation 1 is obtained in the usual way by formulating an equation for $m_k(t+\delta t)$ in terms of the probabilities of the various possible situations at $t$. In this formulation time is measured from the 'birth' of S (its entry into its strip). Thus 'time t' is equivalent to 'age t' for the particular S considered.

Now S is disengaged and alive at epoch $t+\delta t$ and has sunk k U's if either:

(a) it is free at epoch t, has already sunk k U's, and does not become engaged in $(t, t+\delta t)$

or

(b) it is engaged at epoch t and becomes disengaged during $(t, t+\delta t)$. At epoch t it may either

(i) have sunk k U's already, or

(ii) have sunk only (k-1) U's.

Let $\alpha\delta t$ be the probability that S becomes engaged in $(t,t+\delta t)$, given that it is disengaged at $t$, and is still alive at $t+\delta t$. Then S must

(a) encounter something in $(t, t+\delta t)$;

(b) detect it and therefore begin classification;

(c) not detect it and itself be detected, and therefore come under attack, but not itself be destroyed in the attack.

Then,

$$\alpha = \frac{(n+1)\nu}{L}(p+q\ PR) \qquad \text{[Eq. 3]}$$

where $q = 1-p$, and the contribution of (a) to $m_k(t+\delta t)$ is $m_k(t)(1-\alpha\delta t)$.

Let $\gamma\delta t$ be the probability that S encounters U in $(t-t_c, t-t_c+\delta t)$ and successfully completes its destruction in $(t,t+\delta t)$ while not itself being destroyed.

Then

$$\gamma = \frac{\nu P}{L} AC(1-D), \qquad \text{[Eq. 4]}$$

so that the contribution of (b) (ii) to $m_k(t+\delta t)$ is $m_{k-1}(t-t_c)\gamma\delta t$.

Finally, the contribution of (b) (i) depends on whether the encounter and detection in $(t-t_c, t-t_c+\delta t)$ (accounting for its engaged condition at t) was U or a decoy. If it was a decoy S simply becomes disengaged in $(t,t+\delta t)$, and because S detected the decoy there is no question of attack on S. If it was U the outcome must not be destruction of U. So either the attack must be frustrated, but S not destroyed, or the attack may be launched, but unsuccessfully, and S not subsequently destroyed.

The contribution to $m_k(t+\delta t)$ is $\beta\delta t\ m_k(t-t_c)$ where

$$\beta = \frac{n\nu p}{L} + \frac{\nu p}{L}\left[(1-A)\ (1-B) + A(1-C)\ (1-D)\right] \qquad [Eq.\ 5]$$

Figure 1 will be found helpful in showing clearly the construction of $\alpha,\beta$ and $\gamma$.

It follows that

$$m_k(t+\delta t) = m_k(t)(1-\alpha\delta t) + m_k(t-t_c)\,\beta\,\delta t + m_{k-1}(t-t_c)\gamma\delta t \qquad [Eq.\ 6]$$

from which [Eq. 1] follows immediately for $k \geq 1$. Obviously the last term does not enter when $k = 0$. Recalling that $m_o(0) = 1$ and noting that $m_k(t)$ must be identically zero for $0 \leqslant t \leqslant t_c$ we may apply easily Laplace transformation to [Eq. 1] to yield [Eq. 2]. In the general case where $t_c \neq 0$, [Eq. 2] may be inverted numerically to provide $m_k(t)$, thus yielding a statistical description of the life of an S and of the number of its victims.

## 5. DETERMINATION OF $\ell_k(t)$

It is recalled that $\ell_k(t)$ is the joint probability and density that S survives to age t and perishes in $(t,t+\delta t)$, having destroyed a total of k U's. $\ell_k(t)$ is constructed from $m_k(t)$ as follows:

$$\ell_k(t) = x\ m_k(t) + y\ m_k(t-t_c) + z\ m_{k-1}(t-t_c) \qquad [Eq.\ 7]$$

for $k \geq 1$, and without the last term for $k = 0$. x,y and z will be given below. The argument is as follows:

(a) S is disengaged at epoch t and has already destroyed k U's. Probability $m_k(t)$. In $(t, t+\delta t)$ it must perish. Its demise can not be the consequence of itself detecting a target, which would require time $t_c$ for classification. Hence it must itself be detected and destroyed. Altogether the contribution to $\ell_k(t)$ is $x\, m_k(t)$, where

$$x = \frac{(n+1)\,\nu}{L}\ q\ PR \qquad [Eq.\ 8]$$

(b) S is engaged at epoch t. Thus S must have encountered and detected a target in $(t-t_c,\ t-t_c+\delta t)$. This target must, however, be U for otherwise classification would reveal it to be a decoy which, by the rules, is not subject to attack by S, nor attacks S. If at $t-t_c$ S had already destroyed k U's then the attack on U in $(t, t+\delta t)$ must fail, but S must be destroyed, the contribution to $\ell_k(t)$ being $y\ m_k(t-t_c)$, where

$$y = \frac{\nu}{L}\ p\left[A(1-C)\,D + (1-A)\,\beta\right]. \qquad [Eq.\ 9]$$

But if $S$ had destroyed only $(k-1)$ U's at $t = t_c$ the attack on $U$ in $(t, t+\delta t)$ must succeed, but again $S$ must be destroyed. The contribution to $\ell_k(t)$ is then $z\, m_{k-1}(t-t_c)$ where

$$z = \frac{\nu}{L}\, p\, ACD \quad . \qquad \text{[Eq. 10]}$$

It can easily be confirmed that

$$x + y + z = \alpha - (\beta + \gamma) \quad . \qquad \text{[Eq. 11]}$$

Writing $L_k(u)$ for the Laplace transform of $\ell_k(t)$ we obtain

$$L_k(u) = x\, M_k(u) + y\, e^{-ut_c} M_k(u) + z\, e^{-ut_c} M_{k-1}(u) \qquad \text{[Eq. 12]}$$

for $k \geq 1$. It is convenient for the purpose of obtaining moments to introduce generating functions $M(s,u)$, $L(s,u)$. Then, from Eq. 2,

$$M(s,u) = \sum_{k \geq 0} s^k M_k(u) = [u + \alpha - (\beta + \gamma s)\, e^{-z\, t_c}]^{-1}; \qquad \text{[Eq. 13]}$$

and hence

$$L(s,u) = \sum_{k \geq 0} s^k L_k(u) = \frac{x + e^{-ut_c}(y + zs)}{u + \alpha - e^{-ut_c}(\beta + \gamma s)} \qquad \text{[Eq. 14]}$$

Equation 14 contains much valuable information:

(a) by putting $s = 1$ we can find the moments of the life time of an $S$ irrespective of how many U's destroyed.

(b) by putting $u = 0$, information about the number of U's destroyed during the lifetime of an $S$.

First, we note from Eq. 14 that $L(1,0) = 1$, as it should. This expresses the certainty that under the conditions of the model an $S$ will exist for a finite time and during its life will destroy 0, 1, 2... U's. Let $T$ be the life of an $S$. Then

$$E(T^n) = (-)^n \left[\frac{\partial^n L(1,u)}{\partial u^n}\right]_{u=0} \quad ; \qquad \text{[Eq. 15]}$$

and we obtain

$$E(T) = \frac{t_c + \dfrac{L}{(n+1)\nu p}}{\dfrac{q}{p}\; PR + \dfrac{AD + (1-A)B}{(n+1)}} \qquad \text{[Eq. 16]}$$

$$\mathrm{Var}(T) = \left[ t_c^2 \left[ 1 + \frac{p}{qPR(n+1)^2} \{AD+(1-A)B\} \{n+(1-A)(1-B)+A(1-D)\} \right] + \right.$$

$$\left. + \frac{p}{q(n+1)^2 PR} \left[ \frac{L}{\nu p} + t_c \{n+(1-A)(1-B)+A(1-D)\} \right]^2 \right] /$$

$$/ \left[ \frac{qPR}{p} \left[ 1 + \frac{p}{qPR(n+1)} \{ AD + (1-A)B \} \right]^2 \right] \qquad \text{[Eq. 17]}$$

This can in principle be extended as far as desired. The visibility of the dependence on the parameters of the problem is an important feature of the theory, and one that is not immediately evident from simulation.
One particular conclusion that follows immediately is that if the area be flooded with decoys, so that $n \to \infty$, then

$$E(T) \to \frac{pt_c}{qPR}$$

$$\mathrm{Var}(T) \to \frac{pt_c^2}{qPR} \left(1 + \frac{p}{qPR}\right) ,$$

both of which are independent of $L, r, A, B, D$. From this it might be concluded that the continued introduction of decoys tends to have diminishing returns, all other things being equal.

Next we examine the statistics of $k$, the number of U's destroyed in the life time of an $S$. The moments are obtained by differentiation of $L(s,0)$ with respect to $s$, subsequently putting $s = 1$.

Thus,

$$E(k) = \left[ \frac{\partial L(s,0)}{\partial s} \right]_{s=1} = \frac{z+\gamma}{x+y+z} \quad ; \qquad \text{[Eq. 18]}$$

$$\mathrm{Var}(k) = \frac{(z+\gamma)(x+y+\gamma)}{(x+y+z)^2} \quad . \qquad \text{[Eq. 19]}$$

Expressed explicitly in the parameters of the model we have, for example,

$$E(k) = \frac{p\,AC}{(n+1)q\,PR + p\{AD+(1-A)B\}} \qquad \text{[Eq. 18a]}$$

## 6. THE LIFE OF A U

It is easy to argue that if $T_u$ is the life of a U then

$$E(T_u) = \frac{E(T)}{E(k)} \quad . \qquad \text{[Eq. 20]}$$

However, to obtain all the moments of $T_u$ we consider the density $f(t)$ of the life of a U measured, of course, from its birth upon entering the area.

Now $T_u$ can be composed in any one of the following mutually exclusive ways:

(a) the time it takes the S in the strip at $t = 0$ to detect, classify, attack, and destroy U;

(b) the sum of the time taken by the first S to expire (with no further kills) and the time required by its successor to do (a);

(c) the sum of the times taken by the first two S's to expire without a further kill, and the time required by its successor to do (a), etc. Since the introduction of a new U means resetting time to zero we have

$$f(t) = \frac{\nu pAC}{L}\left[ m_o(t-t_c)+m_o(t-t_c) * \ell_o(t-t_c)+m_o(t-t_c) * \ell_o^{(2)}(t-t_c)+\cdots\right]$$

where $*$ means convolution and $g^{(n)}(t)$ is the convolution of $g(t)$ with itself n-times. If $F(u)$ is the Laplace transform of $f(t)$ we have

$$F(u) = \frac{\nu pAC\ M_o(u)e^{-ut_c}}{L[1-L_o(u)]} = \frac{\nu pAC\ e^{-ut_c}}{L\{u+\alpha-x-(\beta+y)e^{-ut_c}\}} \qquad \text{[Eq. 21]}$$

It is easily confirmed that $F(0) = 1$, as it should be. Also, in confirmation of Eq. 20,

$$E(T_c) = \frac{1+(\alpha-x)t_c}{\gamma+2} = \frac{L+(n+1)\nu pt_c}{\nu\, p\, AC} \quad , \qquad \text{[Eq. 22]}$$

while

$$\mathrm{Var}(T_c) = E^2(T_c) + t_c\left\{\frac{2+(\alpha-x)t_c}{\gamma+z}\right\} . \qquad \text{[Eq. 23]}$$

If the total numbers of decoys and S's deployed are D and N respectively it follows that

$$E(T_c) = \frac{L+\nu pt_c}{\nu p\, AC} + \frac{Dt_c}{NAC} \qquad \text{[Eq. 24]}$$

This shows that to increase the number of S's merely decreases the expected life of a U to a fixed value dependent only on L, $\nu$, p, $t_c$, A and C, while to increase decoys is to prolong the mean life of a U. This is hardly unexpected, but perhaps the parameter dependence is not immediately obvious.

## 7. CASE WHERE S ATTACKS DECOYS

The case is of interest where S attacks all decoys detected instead of leaving them alone. This can be dealt with by a minor extension of the foregoing theory.

For illustration we take a simple case. The extreme left hand branch of Fig. 1 following a detection by S looks as shown in Fig. 2.

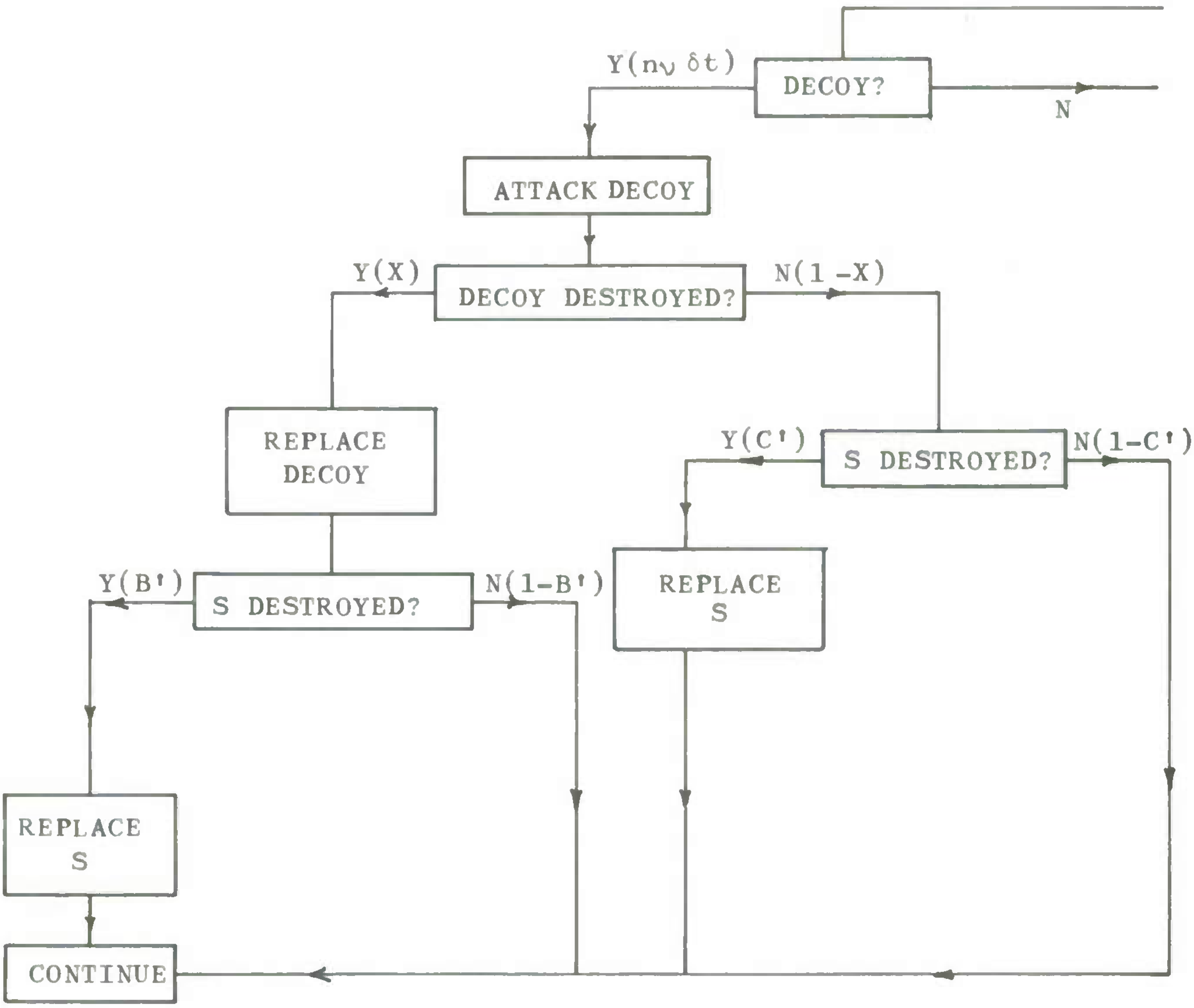



FIG. 2 PART OF FLOW DIAGRAM FOLLOWING A DETECTION BY S

Consider now the probability $m_k(t)$ as defined in Chap. 3 and the arguments by which it is constructed. These lead to the equation

$$m_k(t+\delta t) = m_k(t)(1-\alpha\delta t) + m_k(t-t_c)\beta' \delta t + m_{k-1}(t-t_c)\gamma \delta t ,$$

where

$$\alpha = (n+1)\nu(p+qPR)/L ,$$

$$\beta' = \frac{n\nu p}{L}[X(1-B') + (1-X)(1-C')] + \frac{\nu p}{L}[(1-A)(1-B)+A(1-C)(1-D)] ,$$

$$\gamma = \frac{\nu p}{L} AC(1-D) . \qquad [\text{Eq. } 25]$$

$\alpha$ and $\gamma$ are as before. $\beta'$ differs from the previous $\beta$. But formally $m_k(t)$ is as before and leads to Eq. 2 with $\beta'$ instead of $\beta$. [Flexibility has been preserved by introducing probabilities X, B', C' to which of course any plausible numerical values can be assigned.]

Likewise $\ell_k(t)$ can be treated exactly as in Chap. 5 with x and z as before, and y replaced by

$$y' = y + \frac{n\nu p}{L}[X\beta' + (1-X)C']$$

The process of extracting moments is the same.

## 8. STATISTICS OF DECOY LOSSES

The number of decoys lost is also of interest, and similarly one could extend the theory to deal with the life of a decoy both in and out of the strip containing U.
We make just one more excursion and examine first $m_{j,k}(t)$, the joint probability that S by time t has destroyed j decoys and k U's, being itself disengaged at time t, this to be related (of course) to the strip containing U. Then:

$$m_{j,k}(t+\delta t) = (1-\alpha\delta t) m_{j,k}(t) + \beta'_1 \delta t\, m_{j-1,k}(t-t_c) +$$

$$+ \gamma\delta t\, m_{j,k-1}(t-t_c) + \beta'_2 \delta t\, m_{j,k}(t-t_c)$$

where $\alpha$ and $\gamma$ are as always, and

$$\beta'_1 = \frac{n\nu p}{L} X(1-B') ,$$

$$\beta'_2 = \frac{n\nu p}{L}(1-X)(1-C') + \frac{\nu p}{L}\{(1-A)(1-B)+A(1-C)(1-D)\} \qquad [\text{Eq. } 26]$$

The terms with j-1 and k-1 are not present when $j = 0$ and $k = 0$, respectively.

If we introduce generating functions of Laplace transforms

$$N_j(y,z) = \sum_{k \geq 0} y^k \, n^*{}_{jk}(t)$$

$$N(x,y,z) = \sum_{j \geq 0} x^j \, N_j(y,z)$$

it is easy to show that

$$N(x,\ y,\ z) = \frac{1}{z+\alpha - e^{-zt}c\ (\beta'_2+\beta'_1 x+\gamma y)'} \qquad \text{[Eq. 27]}$$

which holds the key to the moments.

## CONCLUSION

Thus one may proceed, complicating the model more and more, but without complicating the mathematics required to analyse it. It is emphasized that such an analysis as this is an essential preliminary to the systematic derivation of conclusions about structure, and the provision of military advice on the operation.

# MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

## PART 2: A MINEFIELD MODEL

MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

PART 2 : A MINEFIELD MODEL

by

Brian W. Conolly

ABSTRACT

The use of probabilistic methods for the development of a simple model of a minefield is described. The model is capable of providing material for decision-making by both the offence and the defence. The problem is characteristic of harbour defence and is concerned with shallow waters.

## INTRODUCTION

This Memorandum is one of a series presenting mathematical topics in military Operational Research studies. The series has been prepared primarily for instructional purposes. Attention is invited to the General Introduction to the series included in Vol. I, where a reading list, drawn mainly from SACLANTCEN publications, is provided.

The following describes a simple mathematical model for the sweeping and use by shipping of a shallow water channel through a minefield constantly replenished with ground mines by an enemy. The object is the development of a "discrete" time theory which will enable the main features of the process to be described and which could be extended to more complex situations if required. It should be noted that a "continuous time" treatment can also be formulated in terms of birth and death processes. Such a model can be of use to one adversary for the design of minefields, and to the other for the organization of minesweeping and minehunting.

## 1. ASSUMPTIONS

We assume that the enemy replenishes the field by making lays of mines at intervals. By suitable use of arming delays he can secure that mines ripen at a constant rate throughout the field. We assume further that the lays randomly scatter the mines throughout the field so that equal numbers tend to be dropped in equal areas.

Attention is now fixed on a channel through the field. For simplicity and definiteness we assume this is used by shipping leaving and entering harbours in the following way. First the channel is swept by a succession of $s$ sweepers, and these are followed by a succession of $t$ ships. A single sequence of $s$ sweepers and $t$ ships will be called a cycle, and cycles are assumed to follow one after the other.

We now assume that one mine arms, or ripens, in the channel at the end of every cycle so that the sweepers and ships of the $n^{th}$ cycle have to contend with one new mine as well as any residual left over from the $(n-1)^{th}$ cycle. In general we shall suppose that all mines are equipped with a ship count device such that $N$ impulses are needed before the mine explodes (on the $N^{th}$ and last). Interest centres on the partition of explosions between ships and sweepers and in spite of the fact that an explosion may wreck a ship before it has run the gauntlet of the whole channel we shall nevertheless assume that all the ships and sweepers do pass right through the channel. Explosions caused by ships can be interpreted as ship sinkings, and, though this will not be strictly correct it will be a satisfactory first approximation.

The above assumption that one mine ripens per cycle is a matter of convenience, and the actual number of explosions per cycle can be scaled up or down linearly according as the ripening rate in the area of the channel is greater or less than one per cycle. In investigations of the effect of mixed lays with different ship count settings the result for a single fixed setting $N$ is basic.

Finally, we suppose a "steady state". This means that the distribution in the channel of mines of different counts is identical at the end of any prescribed cycle with that at the end of the preceding cycle. It means moreover that the total number of explosions per cycle is constant, and the partition of explosions is always the same. Obviously, if the number of mines in the channel reaches a steady value, then as many explosions must occur in a cycle as fresh mines are added, viz. one.

## 2. FORMULATION

When the stream of $s$ sweepers passes down the channel each mine therein may receive 0, 1, 2, ... $s$ actuating impulses. Let the associated probabilities be $f_r (0 \leqslant r \leqslant s)$, with

$$\sum_{r=0}^{s} f_r = 1 \quad ;$$

and let the generating function of the $f_r$ be

$$F(x) = \sum_{r=0}^{s} f_r x^r \quad .$$

Similarly, let the probabilities of $r (0 \leqslant r \leqslant t)$ actuations by the ships be $g_r$, with generating function

$$G(x) = \sum_{r=0}^{t} g_r x^r$$

and

$$G(1) = 1 \ .$$

Let $(u_N, u_{N-1}, \ldots u_2, u_1)$ be the distribution of expected numbers of mines of counts N, N-1, ... 2, 1 left in the channel at the end of a cycle (i.e. after the last of the ships has passed). Then the distribution at the beginning of the next cycle is $(u'_N, u_{N-1}, \ldots u_2, u_1)$ where $u'_N = u_N+1$. Moreover, let the distribution of mines after the sweepers have passed be $(\nu_N, \nu_{N-1}, \ldots \nu_2, \nu_1)$.

We now formulate sets of difference equations for the $u$ and $\nu$, first in the simple case of $N = 1$. Thus

$$\nu_1 = u'_1 \, f_o$$

$$u_1 = \nu_1 \, g_o = u'_1 \, f_o \, g_o \quad .$$

Moreover since $u'_1 = 1 + u_1$,

$$u_1 = \frac{f_o \, g_o}{1 - f_o \, g_o} \quad .$$

If we interpret $\nu_o$ and $u_o$ as the number of explosions caused by sweepers and ships respectively, we have

$$\nu_o = u'_1 \, \bar{f}_1 \quad , \qquad u_o = \nu_1 \, \bar{g}_1$$

where

$$\bar{f}_1 = \sum_{k=1}^{s} f_k \quad , \qquad \bar{g}_1 = \sum_{k=1}^{t} g_k \quad .$$

Then

$$u_o + \nu_o = u'_1 \, [\bar{f}_1 + f_o \, \bar{g}_1]$$

$$= \frac{[1 - f_o + f_o \, (1 - g_o)]}{1 - f_o g_o} = 1$$

as expected, since a steady state has been assumed.

We now generalize. Let the newly ripening mine have count setting N. For definiteness it will be assumed that $t > s$, and we shall formulate the equations in two extreme cases:

(a) $N < s < + t \; (= M)$

(b) $s + t \; (= M) < N$.

Intermediate cases can be dealt with in similar fashion,

(a) $\underline{N < s < s + t}$

$$\nu_N = u'_N f_o \quad ,$$

$$\nu_{N-1} = u'_N f_1 + u_{N-1} f_o \quad ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \text{[Eq. 1]}$$

$$\nu_1 = u'_N f_{N-1} + u_{N-1} f_{N-2} + \ldots + u_1 f_o \quad ,$$

$$\nu_o = u'_N \bar{f}_N + u_{N-1} \bar{f}_{N-1} + \ldots + u_1 \bar{f}_1 \quad ;$$

$$u_N = \nu_N g_o \quad ,$$

$$u_{N-1} = \nu_N g_1 + \nu_{N-1} g_o \quad ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \text{[Eq. 2]}$$

$$u_1 = \nu_N g_{N-1} + \nu_{N-1} g_{N-2} + \ldots + \nu_1 g_o \quad ,$$

$$u_o = \nu_N \bar{g}_N + \nu_{N-1} \bar{g}_{N-1} + \ldots + \nu_1 \bar{g}_1 \quad .$$

(b) $\underline{s + t = M < N}$

$$\nu_N = u'_N f_o \quad ,$$

$$\nu_{N-1} = u'_N f_1 + u_{N-1} f_o \quad ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \text{[Eq. 3]}$$

$$\nu_{N-s} = u'_N f_s + u_{N-1} f_{s-1} + \ldots + u_{N-s} f_o \quad ,$$

$$\nu_{N-s-1} = u_{N-1} f_s + \ldots + u_{N-s-1} f_o \quad ,$$

$$\ldots$$

$$\nu_1 = u_{s+1} f_s + \ldots + u_1 f_o \quad ,$$

$$\nu_o = u_s \bar{f}_s + u_{s-1} \bar{f}_{s-1} + \ldots + u_1 \bar{f}_1 \quad ;$$

$$
\begin{aligned}
u_N &= \nu_N\, g_o \;, \\
u_{N-1} &= \nu_N\, g_1 + \nu_{N-1}\, g_o \;, \\
&\ldots\ldots\ldots \\
u_{N-t} &= \nu_N\, g_{N-t} + \ldots + \nu_{N-t}\, g_o \;, \\
&\ldots\ldots \\
u_1 &= \qquad \nu_{t+1}\, g_t + \ldots + \nu_1\, g_o \;, \\
u_o &= \qquad\qquad \nu_t\, \bar{g}_t + \nu_{t-1}\, \bar{g}_{t-1} + \ldots + \nu_1\, \bar{g}_1 \;.
\end{aligned}
\qquad [\text{Eq. } 4]
$$

Sets of equations of this type are very familiar to students of stochastic processes. A variety of ways to solve them exists; an excellent reference is Feller [Ref. 1].

Here we shall proceed introducing the generating function $H(x)$ defined to be the product $F(x)\ G(x)$ and therefore a polynomial of degree M. Writing.

$$H(x) = F(x)\ G(x) = h_o + h_1 x + h_2 x^2 + h_M\ x^M \quad ,$$

we see that Eqs. 1 & 2 can be telescoped into

$$
\begin{aligned}
u_N &= h_o\, u'_N \;, \\
u_{N-1} &= h_1\, u'_N + h_o\, u_{N-1} \;, \\
&\ldots\ldots\ldots \\
u_1 &= h_{N-1}\, u'_N + \ldots + h_o\, u_1 \;.
\end{aligned}
\qquad [\text{Eq. } 5]
$$

and Eqs. 3 & 4 into

$$
\begin{aligned}
u_N &= h_o\, u'_N \\
u_{N-1} &= h_1\, u'_N + h_o\, u_{N-1} \\
&\ldots\ldots\ldots \\
u_{N-M} &= h_M\, u'_N + h_{M-1}\, u_{N-1} + \ldots + h_o\, u_{N-M} \\
u_1 &= \qquad h_M\, u_{M+1} + h_{M-1}\, u_M + \ldots + h_o\, u_1 \;.
\end{aligned}
\qquad [\text{Eq. } 6]
$$

From these it follows, in both cases (a) and (b), and indeed for all possible combinations, that if we introduce a further generating function

$$A(x) = \sum_{k=0}^{\infty} a_k\, x^k$$

by the relation

$$A(x)\,[1 - H(x)] = 1$$

then, for $0 \leqslant k \leqslant N-1$

$$u_{N-k} = a_k \,. \qquad \text{[Eq. 7]}$$

This provides an algorithm for the recursive computation of $u_k$. $(1 \leqslant k \leqslant N)$, thus describing statistically the state of the swept channel in a steady state at the end (or beginning) of a cycle. Another way of putting it is to say that for $\underline{0 \leqslant k \leqslant N-1}$ coefficients $a_k$ are generating by the relation

$$A(x) = \frac{1}{1 - H(x)} \qquad \text{[Eq. 8]}$$

Similarly, if we write

$$b_k = \nu_{N-k}$$

and

$$B(x) = \sum_{k=0} b_k\, x^k$$

it is easy to show that for $0 \leqslant k \leqslant N-1$ $b_k$ is generated by the relation

$$B(x) = A(x)\, F(x) = \frac{F(x)}{1 - H(x)} \qquad \text{[Eq. 9]}$$

We now proceed to obtain explicit alternative formulae for $u_k$. Take first the case $N < s < s + t$. If in the last member of the set of difference equations [Eq. 5] we substitute $u_n = y^n$ for $1 \leqslant n \leqslant N-1$, and $u_N^4 = y^N$, the equation reduces to

$$y = y\, \tilde{H}(y) \quad ,$$

where

$$\tilde{H}(y) = h_o + h_1\, y + \ldots + h_{N-1}\, y^{N-1}. \qquad \text{[Eq. 10]}$$

The theory of difference equations then informs us that if

$$\tilde{H}(y) = 1 \quad , \qquad \text{[Eq. 11]}$$

an equation of the $(N-1)^{th}$ degree, has $N-1$ distinct roots,*

---

* Extension to the case of multiple roots is left to the reader.

say $\eta_1, \eta_2 \ldots \eta_{N-1}$ then

$$u_n = \sum_{i=1}^{N-1} A_i \eta_i^n \qquad \text{[Eq. 12]}$$

$$u'_N = \sum_{i=1}^{N-1} A_i \eta_i^N$$

where the (N-1) constants $A_i$ are independent of $x$.

These are determined by substitution in the N-1 remaining equations of the set [Eq. 5]. For example, we have

$$u_2 = h_{N-2} u'_N + h_{N-3} u_{N-1} + \ldots + h_1 u_3 + h_o u_2 \quad ,$$

and by substitution this gives

$$\sum_{i=1}^{N-1} A_i \eta_i^2 = \sum_{i=1}^{N-1} A_i [h_{N-2} \eta_i^N + h_{N-3} \eta_i^{N-1} + \ldots + h_o \eta_i^2]$$

$$= \sum_{i=1}^{N-1} A_i \eta_i^2 [\tilde{H}(\eta_i) - h_{N-1} \eta_i^{N-1}]$$

$$= \sum_{i=1}^{N-1} A_i \eta_i^2 [1 - h_{N-1} \eta_i^{N-1}]$$

Thus

$$\sum_{i=1}^{N-1} A_i \eta_i^{N+1} = 0 \quad .$$

Similarly, it can be shown that for $1 < r < N-2$

$$\sum_i A_i \eta_i^{N+r} = 0 \quad , \qquad \text{[Eq. 13]}$$

and

$$\sum_i A_i \eta_i^N = (1 - h_o)^{-1}$$

This set [Eq. 13] of equations determines $A_i$. We note that if we write

$$\tilde{J}(y) = \tilde{H}(y) - 1 = \prod_{i=1}^{N-1} (y - \eta_i) \qquad \text{[Eq. 14]}$$

then

$$A_i = [\eta_i^{N+1} \tilde{J}'(\eta_i)]^{-1} \qquad \text{[Eq. 15]}$$

where the dash denotes differentiation.

This may be seen by noting the following partial fraction representation, valid for $0 \leqslant r \leqslant N-2$.

$$\frac{y^r}{\tilde{J}(y)} = \sum_i \frac{\eta_i^r}{(y-\eta_i)\tilde{J}'(\eta_i)} \quad . \qquad [\text{Eq. } 16]$$

Then, if Eq. 15 holds,

$$\sum_i A_i \eta_i^{N+r} = \sum_i \frac{\eta_i^{r-1}}{\tilde{J}'(\eta_i)} \quad ,$$

and the righthand side is seen to be zero for $1 \leqslant r \leqslant N-2$ by putting $y = 0$ in Eq. 16. Also

$$\sum_i A_i \eta_i^N = \sum_i [\eta_i \tilde{J}'(\eta_i)]^{-1} = -[\tilde{J}(0)]^{-1} = (1-h_o)^{-1}.$$

Thus the set in Eq. 13 is satisfied and we have

$$u_{N-k} = \sum_i [\eta_i^{k+1} \tilde{J}'(\eta_i)]^{-1} \qquad [\text{Eq. } 17]$$

for $1 \leqslant k \leqslant N-1$, together with

$$u'_N = \sum_i [\eta_i \tilde{J}'(\eta_i)]^{-1}$$

The set [Eq. 17] does provide a practical way of calculating $u_k$. For if the N-1 roots $\eta_i$ are ordered in such a way that

$$|\eta_1| < |\eta_2| < \dots \qquad < |\eta_{N-1}| \quad ,$$

then since also $\tilde{J}(y)$ and its derivatives are increasing functions of $y$ it follows that the term of Eq. 17 corresponding to $i = 1$ is dominant.

It is left to the reader to show that the same result [Eq. 17] holds good also when $N > M(= s + t)$ provided that $\tilde{J}(y)$ is replaced by $J(y) = H(y) - 1$ and the $\eta_i$ are the $M$ (assumed simple) zeros of $J(y)$.

From the numerical point of view an alternative contour integral representation of $u_{N-k}$ is of interest. In the case $N < M$ we have seen that $a_k = u_{N-k}$ is generated for $0 < k < N-1$ by the generating function relation

$$A(x) = [1 - H(x)]^{-1} = -[J(x)]^{-1} .$$

Using Cauchy's theorem we then may write

$$a_k = -\frac{1}{2\pi i} \int_C \frac{dr}{z^{k+1} J(z)} \qquad \text{[Eq. 18]}$$

where $C$ is a closed contour encircling the origin in the z-plane and excluding the zeros of $J(z)$. The direct evaluation of contour integrals of this kind is nowadays a well-known process. The reader is left with the task of demonstrating the direct equivalence of Eq. 18 and Eq. 17, which may be carried out by using Eq. 16.

Before proceeding to practical numerical details it is salutary to show, as a check on the theory that indeed

$$u_o + \nu_o = 1 , \qquad \text{[Eq. 19]}$$

is a condition that must exist in a steady state deriving from the appearance of a single freshly arming mine at the beginning of each cycle. As an example we take the case $N < s < t$. Then by re-arrangement

$$\begin{aligned} u_o = {} & u'_N(\bar{g}_N f_o + \bar{g}_{N-1} f_1 + \ldots + \bar{g}_1 f_{N-1}) + \\ & + u_{N-1}(\bar{g}_{N-1} f_o + \bar{g}_{N-2} f_1 + \ldots + \bar{g}_1 f_{N-2}) + \\ & + \ldots \\ & + u_1 f_o \bar{g}_1 . \end{aligned}$$

Now it is easily seen that if

$$\bar{h}_r = \sum_{k=r}^{\max k} h_k ,$$

then

$$\bar{h}_o = 1 ,$$

$$\bar{h}_1 = f_o \bar{g}_1 + \bar{f}_1 ,$$

$$\bar{h}_N = f_o \bar{g}_N + f_1 \bar{g}_{N-1} + \ldots + f_{N-1} \bar{g}_1 + \bar{f}_N ,$$

so that

$$u_o = u'_N(\bar{h}_N - \bar{f}_N) + u_{N-1}(\bar{h}_{N-1} - \bar{f}_{N-1}) + \dots + u_1(\bar{h}_1 - f_1)$$

$$= \sum_{k=1}^{N-1} u_k \bar{h}_k + u'_N \bar{h}_N - \nu_o \quad ,$$

by Eq. 1.

Hence

$$u_o + \nu_o = \sum_{i=1}^{N-1} \frac{1}{\eta_i^{N+1} J'(\eta_i)} [\bar{h}_1 \eta_2 + \bar{h}_2 \eta_i^r + \dots + \bar{h}_N \eta_i^N]$$

But for

$$1 \leqslant k \leqslant N-1$$

$$\bar{h}_k = h_k + h_{k+1} + \dots + \bar{h}_N$$

so that the square bracket is

$$h_N \eta_i \frac{(1-\eta_i^N)}{1-\eta_i} + \frac{\tilde{H}(1) - \eta_i \tilde{H}(\eta_i)}{1-\eta_i} - \tilde{H}(1)$$

$$= \frac{\eta_i^{N+1} \tilde{J}(1)}{1-\eta_i} \quad ,$$

since

$$\tilde{H}(1) + h_N = 1$$

and

$$\tilde{H}(\eta_i) = 1 \quad .$$

Thus

$$u_o + \nu_o = \tilde{J}(1) \sum_{i=1}^{N-1} [(1-\eta_i) \tilde{J}'(\eta_i)]^{-1} = 1$$

by Eq. 16. This provides a check, and should be further checked for other critical ranges of N.

## 3. NUMERICAL EXAMPLE

It is instructive now to give a simple numerical example. We suppose that during each cycle $t = 3$ ships have to use the channel. We investigate the effect of making $s = 2$ sweeper passages.

Let

$$F(x) = (q + px)^2 ,$$

$$G(x) = (Q + Px)^3 ,$$

where

$$p = 1 - q = 0.75$$

$$Q = P = 0.5$$

Then it is simple to construct a table of $f_k$, $g_k$ and $h_k$ as follows:

TABLE 1

COEFFICIENTS IN FUNDAMENTAL GENERATING FUNCTIONS

| $k$ | $f_k$ | $\bar{f}_k$ | $g_k$ | $\bar{g}_k$ | $h_k$ |
|---|---|---|---|---|---|
| 0 | 0.0625 | 1 | 0.125 | 1 | 0.0078125 |
| 1 | 0.375 | 0.9375 | 0.375 | 0.875 | 0.0703125 |
| 2 | 0.5625 | 0.5625 | 0.375 | 0.500 | 0.234375 |
| 3 | - | - | 0.125 | 0.125 | 0.359375 |
| 4 | - | - | - | - | 0.2578125 |
| 5 | - | - | - | - | 0.0703125 |

Using the generating functions $A(x)$ and $B(x)$ defined by

$$A(x) = \frac{1}{1 - H(x)} ,$$

and

$$B(x) = \frac{F(x)}{1 - H(x)} ,$$

we then construct the following table of the first eleven coefficients $a_k$ and $b_k$.

## TABLE 2

### TABLE OF $a_k=u_{N-k}$, $b_k=\nu_{N-k}$ (all N)

| $k$ | $a_k$ | $b_k$ |
|---|---|---|
| 0 | 1.00787 4016 | 0.06299 2126 |
| 1 | 0.07142 4143 | 0.38241 6765 |
| 2 | 0.24314 2030 | 0.60890 9564 |
| 3 | 0.39915 9113 | 0.15630 1786 |
| 4 | 0.37348 0678 | 0.30979 4601 |
| 5 | 0.29880 7053 | 0.38325 7696 |
| 6 | 0.32221 6593 | 0.34227 4063 |
| 7 | 0.34964 4103 | 0.31076 2946 |
| 8 | 0.33445 4438 | 0.40380 8243 |
| 9 | 0.32711 2875 | 0.34753 8778 |
| 10 | 0.33372 9833 | 0.34867 3663 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\infty$ | 0.33333 3333 | 0.33333 3333 |

The purpose of this table is to supply values of

$$u'_N = a_o, \quad u_k = a_{N-k}$$

$$(1 \leq k \leq N-1) \quad .$$

$$\nu_N = b_o, \quad \nu_k = b_{N-k}$$

The fact that as $k \to \infty$ both $a_k$ and $b_k$ can be seen to oscillate with decreasing amplitude about limits is a result that can be deduced from Renewal Theory (see Ref. 1, for example). We have

$$a_k \to \frac{1}{\hat{h}} \quad ,$$

$$b_k \to \frac{F(1)}{\hat{h}} \quad ,$$

where

$$\hat{h} = \sum_k k\, h_k = \hat{s} + \hat{t} ,$$

and

$$\hat{s} = \sum_k k\, f_k ,$$

$$\hat{t} = \sum_k k\, g_k .$$

It is also not difficult to show that as $N \to \infty$

$$u_o \to \hat{t}/\hat{h} , \qquad \nu_o \to \hat{s}/\hat{h} .$$

This can also serve as a fair approximation when $N$ is large compared with $M$.

In this case

$$\hat{s} = sp = 1.5 ,$$

$$\hat{t} = tP = 1.5 .$$

We now utilise the Tables 1 to 3 to examine enemy policy of effecting the ripening of a mine of count $N\,(1 < N < 10)$ at the beginning of each cycle. Table 3 gives

$$\nu_o = u'_N \bar{f}_N + u_{N-1} \bar{f}_{N-1} + \ldots + u_1 \bar{f}_1$$

(subject to obvious limitations on the indices) and

$$u_o = 1 - \nu_o$$

for $1 < N < 10$.

TABLE 3

TABLE OF $u_o$ and $\nu_o$: $s=2;\ t=3;\ p=0.75;\ P=0.5$

| Mine Count | $\nu_o$ | $u_o$ |
|---|---|---|
| 1 | 0.9449 | 0.0551 |
| 2 | 0.6339 | 0.3661 |
| 3 | 0.2681 | 0.7319 |
| 4 | 0.5111 | 0.4889 |
| 5 | 0.5747 | 0.4253 |
| 6 | 0.4902 | 0.5098 |
| 7 | 0.4702 | 0.5298 |
| 8 | 0.5090 | 0.4910 |
| 9 | 0.5102 | 0.4898 |
| 10 | 0.4948 | 0.5052 |

It has been explained earlier that $\nu_o$ and $u_o$ are the steady-state proportions of mines exploded respectively by sweepers and ships during each cycle. To illustrate this concept in another way let us suppose that the enemy arranges that mine ripenings will be at a rate of 5 per cycle (all of count N). The assumption of a steady state means that during such a cycle there will be on average 5 mine explosions. Then $\nu_o$ is the fraction of these exploded by the sweepers and the fraction $u_o$ would be exploded by the ships (on the assumption of survival or substitution). Remembering that in the numerical example we have used two sweeper passages to protect three ship passages, if mines with count setting $N = 5$ were used the last Table 3 explains that on average about three mines would be destroyed by sweepers and two by ships. Not a very good outlook for the ships! Excellent for the opposition. Hence, in this situation one would obviously step up the minesweeping effort, equivalent to increasing s. However, it is not as simple as that. Indeed doubling s from 2 to 4 gives $\nu_o = 0.4651$ and $u_o = 0.5349$, even more to the disfavour of the ships! There is a possibly unexpected oscillatory relationship between $\nu_o$, $u_o$, N and s, which is to be seen in Table 3. A use for models of this kind is then that they can be of material assistance in the determination of a minesweeping policy. Conversely they are of help to the adversary. If the latter knows the broad lines of his enemy's policy (i.e. s, t, p and P) then he can calculate easily which count setting is worst for the ships. In our example it is plainly 3. In passing it may be remarked that choices of N and s constitute strategies in the sense of game theory, and our model also then provides a basis for payoff calculations.

The figures given in Table 4 corresponding to $s = 18$ sweeper passages, covering $t = 3$, as before, are instructive.

TABLE 4

TABLE OF $u_o$ and $\nu_o$: s=18; t=3; p=0.75; P=0.5

| N | $\nu_o$ | $u_o$ |
|---|---|---|
| 1 | 1.0000 | 0.0000 |
| 5 | 1.0000 | 0.0000 |
| 10 | 0.9856 | 0.0144 |
| 15 | 0.3857 | 0.6143 |
| 20 | 0.7962 | 0.2038 |

Protection for the ships is good and consistant at least up to count setting 10, but then strange things begin to happen.

## CONCLUSION

In conclusion we repeat that the foregoing is a sketch of a theory of possible utility in connexion with the tactics of warfare in shallow water using ground mines. It provides an example of a simple stochastic problem belonging to the random walk family, of interest in military operational research. The reader is reminded that because attention has been focussed on techniques a critical discussion of the assumptions made, both tacitly and overtly is lacking. The theory is not fully stochastic: it would indeed have been more satisfying to develop the actual probability distributions of mines of different count settings at the beginning of each cycle, deducing therefrom conditions for the existence of a steady state, and information about the rate of convergence to that steady state. These are essential elements in operational research, which the customer need not see but should feel can be produced upon demand. The research worker for professional reasons must not neglect them.

## REFERENCES

1. Feller, William. 1968. "An Introduction to Probability Theory and its Applications", Vol. 1, New York: Wiley.

# MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

## PART 3: SOME APPLICATIONS OF GEOMETRY

# MATHEMATICS IN MILITARY OPERATIONAL RESEARCH

## Part 3 : SOME APPLICATIONS OF GEOMETRY

by

Brian W. Conolly

ABSTRACT

The problems considered are of geometrical character. They are divided into three categories: (a) Geometrical Probability; The statistics of distances between points on a line and in an area, with some consideration of their maxima and minima in the linear case. Applications to patrols and mining situations are mentioned; (b) Closure: Collision course; probability of hitting a target of finite length with a salvo of straight running torpedoes; some weapon homing systems - pursuit, pursuit with lead, line of sight guidance; probability of closing and detecting a submarine located at a remote datum; (c) Location: Principally the mathematics of some bearings-only methods.

## 1. SOME PROBLEMS IN GEOMETRICAL PROBABILITY

### 1.1 Points on a Line

Military operational research creates many fascinating and not always easy problems in what is technically called geometrical probability. Buffon's needle problem is a familiar and unusually simple member of the genus. We start with an example. As part of a geographically fixed antisubmarine barrier a surface ship patrols at constant speed on a straight line of length a between two points A and B. Submarines, using their accustomed tactics of secrecy, attempt to traverse the barrier. In many cases it is reasonable to suppose that the projection Q on the surface of the crossing point can not be predicted, while the position P of the surface ship at the time when the submarine is at Q is equally likely to be anywhere from A to B inclusive. Clearly the distance PQ is of interest to the operational authorities and the analyst's job is to provide the statistics of PQ. It is a geometrical problem with essential probabilistic content.

We have asserted that P is uniformly distributed between A and B and the same assumption will be made for Q. Then if $PQ = \xi$ (always taken as positive number) we want basically the probability density function (p.d.f.)

$$f(X)\,dX = \Pr[X < \xi < X + dX] \,.$$

For generality we write $p(x)$ for the p.d.f. that P lies at distance $x$ from A.
Likewise $p(y)$ is the p.d.f. that Q lies at distance $y$ from A and the joint p.d.f. of P and Q is, on the assumption of independence, $p(x)\,p(y)$. Since PQ can be of length $\xi$ when either Q is to the right or left of P we have to distinguish the two cases:

(a) $y > x$ and $\xi = y - x$ ;

(b) $y < x$ and $\xi = x - y$ .

Thus, whatever $p(x)$ may be,

$$f(\xi) = \int_0^{a-\xi} p(x)\,p(x+\xi)\,dx + \int_\xi^a p(x)\,p(x-\xi)\,dx$$

$$= 2\int_0^{a-\xi} p(x)\,p(x+\xi)\,dx$$

which gives all the information needed about the separation PQ. In the case of uniformly distributed P and Q $p(x) = a^{-1}$, $a$ being the length of the patrol.

$$f(\xi) = \frac{2}{a^2}(a-\xi) \quad ,$$

(i.e. triangular),

$$E(\xi) = \frac{1}{3}a \quad ,$$

$$\mathrm{Var}(\xi) = \frac{1}{18}a^2 \ .$$

The mean distance is NOT $\frac{1}{2}$ a, as instinct might predict, but LESS.*

Another naval application is to mines in a channel. Two splashes were heard during the night passage of an aircraft. What are the statistics of their distance apart? With suitable assumptions the above theory provides an approximate answer.

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* Indeed it is an interesting exercise to look at the statistics of PQ when one of the points is fixed (e.g. fixed defence installation). In that case let the fixed point be P and AP = $X(<\frac{1}{2}a)$. Then for a uniform distribution of Q

$$f(\xi) = \frac{2}{a} \quad (0 < \xi \le X)$$

$$= \frac{1}{a} \quad (X < \xi \le a - X).$$

Then, for example $E(\xi) = (a^2 - 2aX + 2X^2)/2a$ which has value $\frac{1}{2}a$ when $P \equiv$ A or B, and $\frac{1}{4}a$ when P is midway between A and B.

A generalisation of interest would be:

Let $P_1, P_2, \ldots P_n$

be $n$ uniformly distributed points on the line AB and let $\xi_{mn}$ be the distance $P_m P_n$, always taken as a positive number.

Let

$$m = \min \xi_{mn}$$

$$M = \max \xi_{mn}$$

taken over all possible pairs, $m, n$,

What are the statistics of $m$ and $M$?
This problem will be addressed after the following cognate preliminary.

Consider the following problem:

Two points are selected from a uniform distribution on a line of length $a$. We already know that the density $f(\xi)$ of their separation is

$$f(\xi) = \frac{2}{a^2}(a - \xi)$$

with distribution function

$$P[\xi \leqslant X] = F(X) = \int_0^X f(\xi)d\xi = 1 - \left(1 - \frac{X}{a}\right)^2 .$$

Let the experiment be repeated independently $n$ times giving a set $\{\xi_k\}$ of $n$ distances. We would like to investigate the statistics of

$$M_n = \max_k \xi_k ,$$

$$m_n = \min_k \xi_k .$$

This is in the spirit of the statistics of records.
In the case of $M_n$ we plainly have

$$P[X < M_n < X + dX] = n\ f(X)\ F^{n-1}(X)\ dX$$
$$= d\ \{F^n(X)\}$$

and for $m_n$

$$P[X < m_n < X + dX] = n\ f(X)\ [1-F(X)]^{n-1}\ dX$$
$$= -d[1 - F(X)]^n ,$$

and the exact distributions are

$$P[M_n < X] = F^n(X) \quad ,$$

$$P[m_n < X] = 1 - [1 - F(X)]^n \; .$$

These results are independent of the form of $F(X)$. Technically then this is a distribution-free result.

For the record, in the particular case of uniformly distributed points, the means and variances are:

$$E(M_n) = a[1 - \frac{2^{2n}(n!)^2}{(2n+1)!}] \quad ;$$

$$Var(M_n) = a^2[\frac{1}{n+1} - \frac{(n!)^4 2^{4n}}{\{(2n+1)!\}^2}] \quad ;$$

$$E(m_n) = \frac{a}{2n+1} \quad ;$$

$$Var(m_n) = \frac{n\,a^2}{(n+1)\;(2n+1)^2} \; .$$

It can easily be checked that the first results are recovered when $n = 1$.

These results provide ALSO the answer to the problem posed earlier: what are the statistics of $m$ and $M$? Let us carry out a short experiment. We know that five mines have fallen in a channel. What are the statistics of their maximum and minimum separation? Let the coordinates of the mines be $x_1, x_2, x_3, x_4, x_5$ with respect to one end of the channel and let the inter-mine distances be $\xi_{12} = |x_1 - x_2|$, $\xi_{13} = |x_1 - x_3|$, ... $\xi_{45} = |x_4 - x_5|$, viz. ten of them, so that $n = 10$.
According to the previous paragraph

$$E(M_{10}) = 0.7297\ a$$

$$Var(M_{10}) = 0.01786\ a^2 \quad (\sigma_{M10} = 0.134\ a)$$

$$E(m_{10}) = 0.04762\ a$$

$$Var(m_{10}) = 0.002061\ a^2 \quad (\sigma_{m10} = 0.04540\ a)$$

The experiment is as follows. We draw $x_1, \ldots x_5$ from a table of uniform deviates in the range 00-99. Then we form the ten values $|x_m - x_n|$ and select the minimum and maximum. Then we repeat the experiment ten times (rather few) to obtain statistics of $m_{10}$ and $M_{10}$. The following will be self explanatory.

TABLE 1

| Drawing No. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| 1 | 17 | 45 | 77 | 79 | 31 |
| 2 | 46 | 34 | 63 | 85 | 87 |
| 3 | 94 | 21 | 32 | 92 | 93 |
| 4 | 35 | 45 | 64 | 53 | 93 |
| 5 | 54 | 41 | 4 | 56 | 9 |
| 6 | 70 | 58 | 28 | 49 | 54 |
| 7 | 3 | 27 | 48 | 97 | 41 |
| 8 | 8 | 64 | 71 | 62 | 76 |
| 9 | 12 | 37 | 85 | 36 | 32 |
| 10 | 68 | 97 | 36 | 84 | 30 |

TABLE 2

| Distances | Experiment no. | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\xi_{12}$ | 28 | 12 | 73 | 10 | 13 | 12 | 24 | 56 | 25 | 29 |
| $\xi_{13}$ | 60 | 17 | 62 | 29 | 50 | 42 | 45 | 63 | 73 | 32 |
| $\xi_{14}$ | 62 | 39 | 2 | 18 | 2 | 21 | 94 | 54 | 24 | 16 |
| $\xi_{15}$ | 14 | 41 | 1 | 58 | 45 | 16 | 38 | 68 | 20 | 38 |
| $\xi_{23}$ | 32 | 29 | 12 | 19 | 37 | 30 | 21 | 7 | 48 | 61 |
| $\xi_{24}$ | 34 | 51 | 71 | 8 | 15 | 9 | 70 | 2 | 1 | 13 |
| $\xi_{25}$ | 14 | 53 | 72 | 48 | 32 | 4 | 14 | 12 | 5 | 67 |
| $\xi_{34}$ | 2 | 22 | 60 | 11 | 52 | 21 | 49 | 9 | 49 | 48 |
| $\xi_{35}$ | 46 | 24 | 61 | 29 | 5 | 26 | 7 | 5 | 53 | 6 |
| $\xi_{45}$ | 48 | 2 | 1 | 40 | 47 | 5 | 56 | 14 | 4 | 54 |
| $m_{10}$ | 2 | 2 | 1 | 8 | 2 | 4 | 7 | 2 | 1 | 6 |
| $M_{10}$ | 62 | 53 | 73 | 58 | 52 | 42 | 94 | 68 | 73 | 67 |

The mean of the experimental $m_{10}$ and $M_{10}$ are 64.2 and 3.5 both within probabilistically acceptable limits of experimental error from the theoretical values of 73 and 4.8. It is a small experiment and has been conducted as an example to the operational research worker of how a simple experiment can give confidence in theoretical results. In this case we might feel it desirable to continue the experiments.

## 1.2 Points in a Rectangle

An extension to two dimensions of the fundamental problem of the previous paragraphs is indicated if one considers random antisubmarine patrols in an area instead of on a line. Now both P and Q are selected from uniform distributions over a rectangle of dimensions (a,b) and the problem is to find the probability distribution/density function of the always positively measured distance PQ. The diagram below illustrates the notation

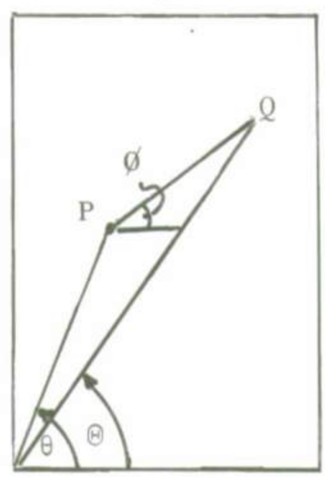



FIG. 1

The positions of P and Q are expressed in polar coordinates with respect to a prescribed corner of the fixed rectangle whose sides are a and b in length. By uniformity

$$d\ \Pr[P] \propto r\,dr\,d\theta$$

$$d\ \Pr[Q] \propto R\,dR\,d\Theta$$

Then if Q has polar coordinates $(X,\emptyset)$ with respect to P, and their selection is independent, we have

$$d\ \Pr[P,Q] \propto r\,dr\,d\theta\,X\,dX\,d\emptyset \quad ,$$

so that the p.d.f. $F(X)$ of $X$ is given by

$$F(X) \propto X \iiint r\,dr\,d\theta\,d\emptyset \quad ,$$

the integration being taken over all values of $\theta$ and $\emptyset$ compatible with the particular choice of X.

Now it is clear that for fixed values of X and of $\emptyset$ $(0 \leq \emptyset < \frac{\pi}{2})$, all the possible positions that P can assume lie in a rectangle whose sides are $(b - X\cos\emptyset)$ and $(a - X\sin\emptyset)$ respectively. Hence for fixed X and

$$\{\emptyset : 0 \leq \emptyset < \tfrac{\pi}{2}\}$$

$$\int\int r\,dr\,d\theta = (b - X\cos\phi)\,(a - X\sin\phi) \quad ;$$

while for

$$\{\phi : \frac{\pi}{2} \leq \phi < \pi\}$$

$$\int\int r\,dr\,d\theta = (b + X\cos\phi)\,(a - X\sin\phi) \quad ,$$

with corresponding expressions for the other quadrants.

However, considerations of symmetry show that to obtain $F(X)$ it is only necessary to integrate $\int\int r\,dr\,d\theta$ over the first quadrant of $\phi$ (having regard to the restrictions imposed by the value of $X$) and to multiply the result by 4. Hence

$$\frac{F(X)}{4X} \propto \int (b - X\cos\phi)\,(a - X\sin\phi)\,d\phi$$

taken over that part of the first quadrant of $\phi$ which the value of $X$ permits.

There are three cases:

(a) $\quad 0 \leq X \leq b \leq a \leq (a^2 + b^2)^{\frac{1}{2}}$

and in this case $\quad 0 \leq \phi \leq \frac{\pi}{2} \quad ;$

(b) $\quad 0 \leq b \leq X \leq a \leq (a^2 + b^2)^{\frac{1}{2}} \quad ,$

when $\arccos \frac{b}{x} \leq \phi < \frac{\pi}{2} \quad ;$

(c) $\quad 0 \leq b \leq a \leq X \leq (a^2 + b^2)^{\frac{1}{2}} \quad ,$

when $\arccos \frac{b}{x} \leq \phi \leq \arcsin \frac{a}{x} \leq \frac{\pi}{2}$ .

Corresponding to these three cases we have

$$\text{(a)} \qquad F_1(X) = \frac{1}{(4X\,a^2\,b^2} \left[\tfrac{1}{2}\,ab\,\pi - (a+b)\,X + \tfrac{1}{2}\,X^2\right]$$

$$\text{for} \quad 0 \leq X \leq b$$

$$= 0 \,, \qquad\qquad b \leq X \leq (a^2 + b^2)^{\frac{1}{2}} \,.$$

(b) $F_2(X) = \frac{1}{4\,Xa^2b^2}[\frac{1}{2}\,ab\,\pi - \frac{1}{2}\,b^2 - aX + a(X^2-b^2)^{\frac{1}{2}} - ab\,\mathrm{arc\,cos}\,\frac{b}{X}]$

$$\text{for} \quad b \leq X \leq a$$

$$= 0\,, \qquad 0 \leq X \leq b \quad \text{and} \quad a \leq X \leq (a^2+b^2)^{\frac{1}{2}}\,.$$

(c) $F_3(X) = \frac{1}{4Xa^2b^2}[-\frac{1}{2}(a^2+b^2) - \frac{1}{2}X^2 + a(X^2 - b^2)^{\frac{1}{2}} +$

$$+ b(X^2 - a^2)^{\frac{1}{2}} + ab(\mathrm{arc\,sin}\,\frac{a}{X} - \mathrm{arc\,cos}\,\frac{b}{X})]$$

$$\text{for} \quad a \leq X \leq (a^2+b^2)^{\frac{1}{2}}$$

$$= 0\,, \qquad 0 \leq X \leq a.$$

The $n^{th}$ moment $E(X^n)$ of $X$ is given by

$$E(X^n) = \int_0^b X^n F_1(X)\,dX + \int_b^a X^n F_2(X)\,dX + \int_a^{(a^2+b^2)^{\frac{1}{2}}} X^n F_3(X)\,dX\,.$$

In the case of $n = 1$ we have

$$E(X) = \frac{a^5 + b^5 - (a^2+b^2)^{\frac{5}{2}}}{15\,a^2b^2} + \frac{1}{3}(a^2+b^2)^{\frac{1}{2}} +$$

$$+ \frac{1}{6}\left\{\frac{b^2}{a}\,\mathrm{sh}^{-1}\,\frac{a}{b} + \frac{a^2}{b}\,\mathrm{sh}^{-1}\,\frac{b}{a}\right\}\,.$$

The following is a table of $E(X)/a$. We write $b = ka$ and hold $a$ fixed. Then

TABLE 3

| K | E(X)/a |
|---|---|
| 0.0 | 0.33 |
| 0.2 | 0.350 |
| 0.4 | 0.383 |
| 0.6 | 0.424 |
| 0.8 | 0.471 |
| 1.0 | 0.521 |

To continue beyond $k = 1$ one simply interchanges the rôle of $a$ and $b$. It is worthwhile to verify the value $E(X) = \frac{1}{3} a$ (as expected) in the degenerate linear case from the general formula.

One might as in the case of the line patrol investigate maximum and minimum separations. Variances of separations should also be considered. The integrations for these and higher moments are tedious, but all can be performed in a finite number of operations to yield a closed form.

## 1.3 Other Dimensional Problems

There is a variety of problems of this kind in two dimensions, and of course the challenge of higher dimensions is operationally meaningful. We mention the following two dimensional extensions leaving the interpretation and proofs to the reader.

### 1.3.1 Parallel Lines

There is a rectangle of dimensions $(a,b)$ with $a > b$. Two points $P$ and $Q$ are selected from uniform distributions on the two long sides. What are:

(a) the mean value of $PQ$?

(b) the mean square value of $PQ$?

The answers are:

(a) $$E(PQ) = \frac{1}{a^2}\left[\frac{1}{3}(a^2+b^2)^{\frac{3}{2}} - b^2(a^2+b^2)^{\frac{1}{2}} + b^2 a \, sh^{-1}\frac{a}{b} + \frac{2}{3}b^3\right] ;$$

(and it will be seen after a little work on the $sh^{-1}$ term that $E(PQ) \to \frac{1}{3} a$ as $b \to 0)$ ;

(b) $$E(PQ^2) = \frac{1}{6} \quad a^2 + b^2 .$$

One purpose of this calculation is to see whether $[E(PQ^2)]^{\frac{1}{2}}$ is a serviceable approximation to $E(PQ)$. The reader may wish to conjecture why one would be interested.

### 1.3.2 Distance between Point on Side of Rectangle and Point within Rectangle

This is the two-dimensional analogue of the problem of the mean distance between one end of a line and a random point upon it. Solution requires care and patience. Let $P$ be selected on the longer side of length $a$. Denote, as usual, the length of the shorter side by $b$. The probability density function $f(X)$ of the distance

PQ = X is then:

$$f(X) = \begin{cases} f_1(X) & 0 \le X \le b \ . \\ f_2(X) & b \le X < a \ , \\ f_3(X) & a \le X \le (a^2+b^2)^{\frac{1}{2}} \ , \end{cases}$$

with

$$f_1(X) = (a\pi - 2X)\, X/a^2 b \ ,$$

$$f_2(X) = 2X(\tfrac{1}{2}\, a\pi - a\cos^{-1}(b/X) - b)/a^2 b \ ,$$

$$f_3(X) = 2X\,\{ a\sin^{-1}(a/X) - a\cos^{-1}(b/X) + (X^2-a^2)^{\frac{1}{2}} - b \}/ab.$$

Also

$$E(X) = -\frac{(a^2+b^2)^{\frac{3}{2}}}{6a^2} + \frac{5}{12}(a^2+b^2)^{\frac{1}{2}} + \frac{b^3}{6a^2} + \frac{a^2}{12b}\,\mathrm{sh}^{-1}\frac{b}{a} + \frac{b^2}{3a}\,\mathrm{sh}^{-1}\frac{a}{b} \ .$$

## 2. SOME PROBLEMS OF CLOSURE

### 2.1 Basic Geometry

The simplest possible problem of closure is this. A submarine which, for the purpose of description may be regarded as stationary at a point S, fires a torpedo at a target T whose position, course, and speed v, are known at the moment of firing. The geometry is shown in Fig. 2.

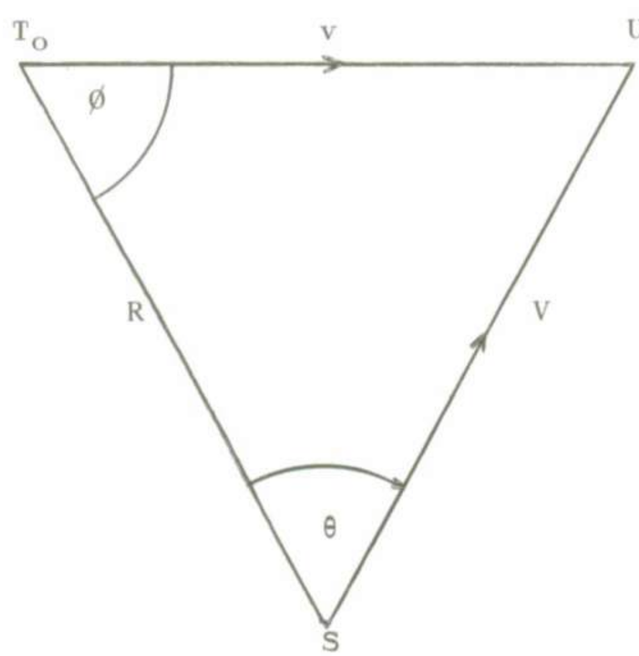



FIG. 2

$T_o$ is the position of the target at time of firing. The range on firing $S\,T_o$ may be assumed to be known to have value $R$(metres). This plus a bearing (not shown) establishes $T_o$. Conventionally the target's track is defined in terms of the initial bearing line by the angle $\phi$ on the target's bow. The speed of the target is $v$(m/s) and the speed of the torpedo is $V$. This is to be fired along SU, a track deflected from the initial bearing line ST by the deflection angle $\theta$, to hit the target at U. For the moment let us confine ourselves to point targets and accurately known values of $R$, $\phi$, $v$ and $V$. Then for a hit the time for the torpedo to run from S to U is equal to the time for the target to proceed from T to U. Thus, for a hit,

$$\frac{TU}{v} = \frac{SU}{V} \quad ,$$

whence the deflection angle $\theta$ is given by

$$\sin \theta = \frac{v}{V} \sin \phi \quad ,$$

and this solves the <u>fire control</u> problem. A course SU with such an angle $\theta$ is called a <u>collision course</u>.

Clearly if $\frac{v}{V} \sin \phi > 1$ there is no possible value (real) of $\theta$ that can be selected to yield a hit. This creates the notion of favourable and unfavourable regions for S. Thus define an angle $\phi_o$ such that

$$\phi_o = \arcsin \frac{V}{v} \quad , \qquad (V < v) \quad ,$$

$$= \frac{\pi}{2} \qquad (V > v) \quad .$$

Then a submarine that finds itself in the shaded region of the following diagram can not with its assumed weapon limitations possibly hit the target.

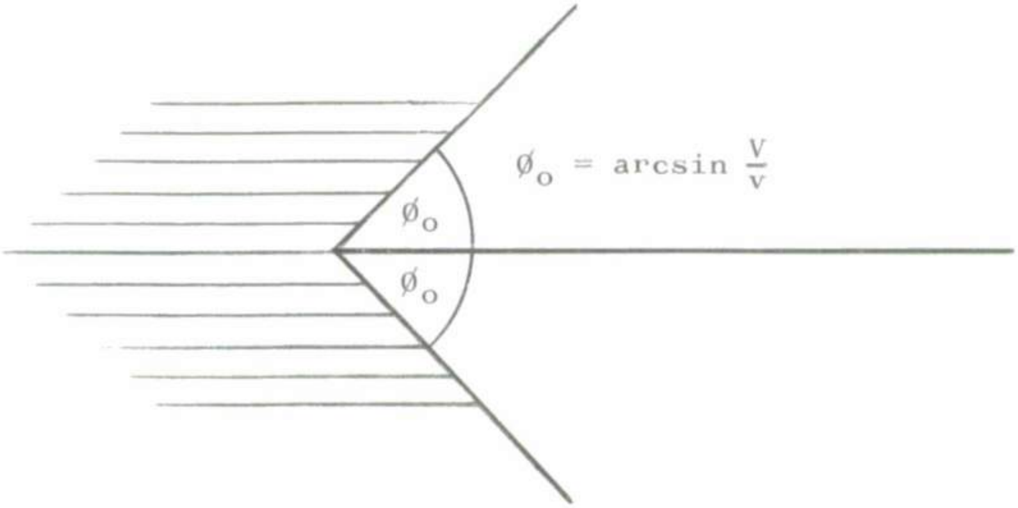



FIG. 3

This represents a limitation on the approach possibilities of the torpedo. The elementary setting we have chosen characterises a much more general situation. The problem of closure of the torpedo might equally well be the problem of submarine trying to reach an attacking position with respect to the target, usually requiring closure to a prescribed range from which a weapon can be fired and then reach the target. This would be a double application of the elementary problem. This raises the next limitation. Even when S is in a favourable position the weapon can not possibly arrive at U if the range SU exceeds the weapon's endurance. Thus, given the fire control solution — essentially the angle $\theta$, -- the geometry of the triangle is completely known and

$$SU = \frac{R \sin \emptyset}{\sin (\theta + \emptyset)} .$$

If this exceeds the weapon's endurance the submarine commander had better try to get closer.

It will be evident that in practice R, $\emptyset$ and v will all be known to the attacker imperfectly. As a consequence the calculated value of $\theta$ will be wrong. However the target is not really a point, so $\theta$ can be wrong but the target still hit. What is the probability that the target will be hit? We have returned to problems of geometrical probability. But the elementary nature of the model and the associated mathematics should not be permitted to deceive the reader who does not already know how complicated it is to use analysis to make probability statements.

Torpedoes may be of several varieties: straight runners; pattern runners; homers; combinations of these. The description "straight runners" means just that the weapon tries to follow a straight line track. Pattern runners are designed to follow curvilinear trajectories, partly to fool the enemy, partly to convert attack from an apparently unfavourable point, to one that is more favourable. Homing torpedoes run to a point somewhere near the target and then proceed to track it down by detecting and locking on some physical emanation from the target. Intermediate between homers and pattern runners are the guided varieties of weapon that may be controlled by the firing ship for most of their trajectory to the target, and then perhaps utilise terminal homing. All give rise to more or less simple mathematical descriptions and much more complicated error analysis on the basis of which probability of hitting statements may be made. In the next paragraph we shall look briefly at the case of salvos of straight runners, and then at some homing and guidance mathematics.

To increase the probability of hitting the target salvos (i.e. several) of weapons may be fired. The argument is that if a single weapon has probability p of hitting then the chance of one or more hits when n weapons are fired independently increases to $1 - (1 - p)^n$. This calculation is not completely applicable in real life since the weapons can rarely be regarded as being fired independently, but the principle of a probability increase holds. How does one go about making a probability of hitting statement? Take the case of a simple straight running torpedo. We have seen earlier that the torpedo must be aimed off. But since the target, a ship, is finite in length, and not a point, a range of values of $\theta$, not just a single value, favours a hit. In Fig. 4, the target $T_{-1}T_oT_1$ has length 2L and $T_o$ is the midship position. The range of possible hitting points corresponding to $T_{-1}T_o$ and $T_1$ is $U_{-1}U_o$ and $U_1$, and $U_{-1}U_o$ is in general different in length from $U_oU_1$. The permitted range of values of deflection angle $\theta$ for a hit (firing when the centre of the target is at $T_o$) is $(\theta_o - \delta_{-1},\ \theta_o + \delta_1)$.

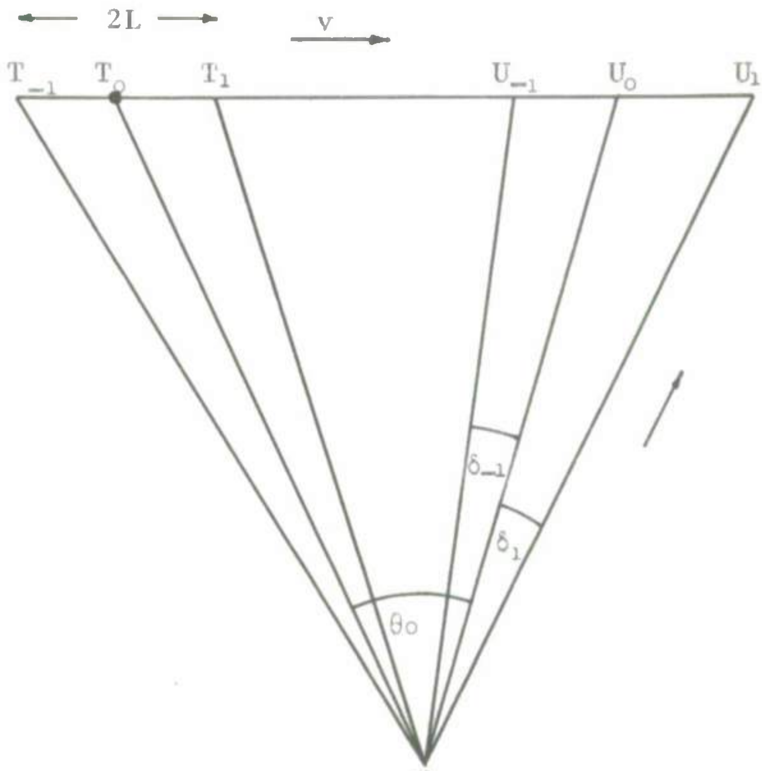


FIG. 4

To obtain $\theta_1$ and $\theta_{-1}$ the fire control computer would basically calculate:

(a) $\epsilon_{-1,1} = \arctan\left[\dfrac{\lambda \sin\theta_o}{1 \pm \lambda \cos\theta_o}\right]$ $(\lambda = L/R_o;\ R_o = ST_o)$

(b) $\phi_{-1,1} = \phi_o \mp \epsilon_{-1,1}$ ;

(c) $\theta_{-1,1} = \arcsin [\frac{v}{V} \sin \phi_{-1,1}]$ ;

(d) $\delta_{-1,1} = \pm \theta_o + \epsilon_{-1,1} \mp \theta_{-1,1}$ ;

where the first subscript is taken with the upper of any two sign choices available.

The probability of a hit depends upon the distribution of $\theta$ about $\theta_o$ and this is a more complex question than may at first sight appear since the errors that may affect $\theta$ derive at least from:

(a) tracking and location errors ($\phi$, v and $R_o$ may be in error);

(b) fire control errors (calculation in the computer, launching errors);

(c) running errors (the weapon may curve instead of running straight).

Naturally it is supposed that the target is in range.

## 2.2 Some Homing Mathematics

The most well known homing system leads to what is called a pursuit curve which we now describe in two dimensions. A rabbit R when at a point $R_o$ in a field perceives a dog D glaring menacingly at him from a point $D_o$ at distance d. The rabbit sets off at maximum speed V in a straight line and is pursued by the dog at his maximum speed v.
The initial positions are as shown in Fig. 5.

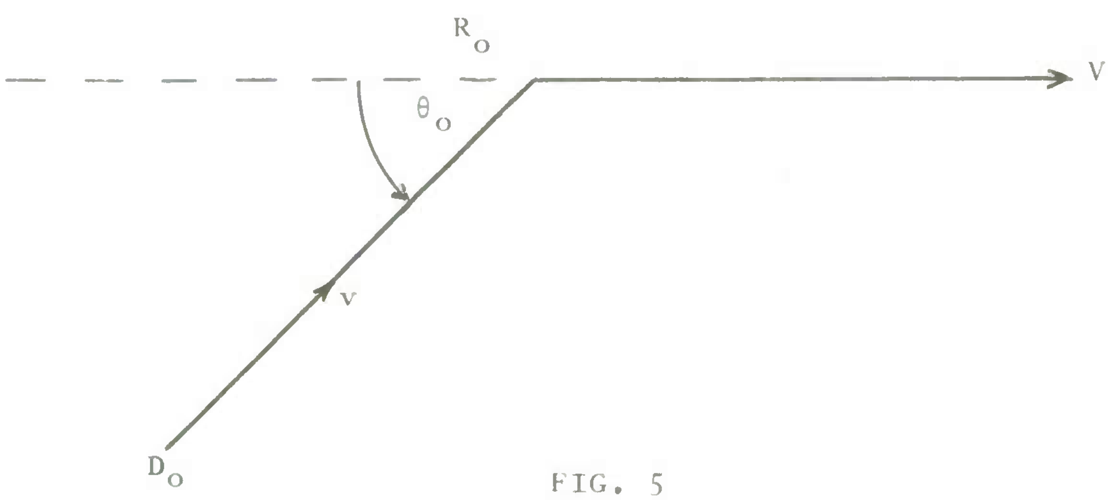



FIG. 5

The dog now chases the rabbit, always heading straight for him. To describe D's trajectory, the pursuit curve, it is convenient to work in target space, that is to "reduce R to rest" by applying a velocity equal and opposite to V to D. Then we have the polar coordinate system $(r, \theta)$ as shown in Fig. 6 with $r = RD$.

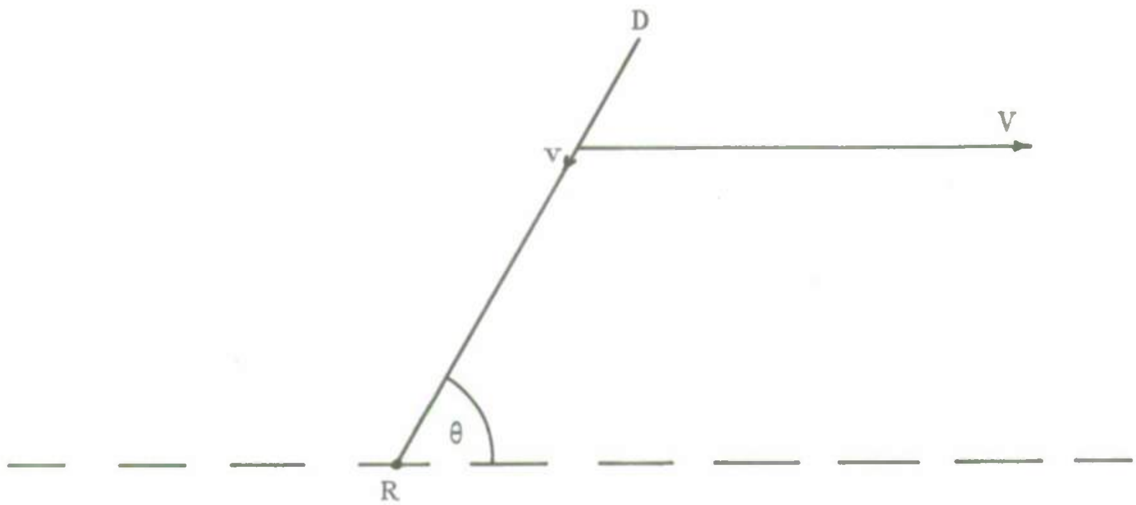



FIG. 6

In actual geographical space the rabbit would be running to the left along the dotted line. The equations describing D's relative motion are

$$\dot{r} = -v + V \cos\theta ,$$

$$r\dot{\theta} = -V \sin\theta ,$$

and the polar equation of the pursuit trajectory is accordingly

$$\frac{r}{r_o} = \left[\frac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta_o}\right]^{\lambda-1} \left[\frac{\cos\frac{1}{2}\theta_o}{\cos\frac{1}{2}\theta}\right]^{\lambda+1} .$$

The dog will catch the rabbit when $r = 0$. Let $0 < \theta_o < \frac{\pi}{2}$.

Then if $\lambda = v/V > 1$, as instinct tells us, this will occur when $\theta$ has reduced from $\theta_o$ to 0. (It is clear in this case from the second equation of motion that $\theta$ decreases as time $t$ increases.) With the value of $r$ given by this equation we can establish a relation between $\theta$ and $t$. One version is

$$\frac{Vt}{r_o} = \frac{1}{(\lambda^2-1)}\left[\lambda + \cos\theta_o - \left(\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\theta_o}\right)^{\lambda}\left(\frac{\sin\theta_o}{\sin\theta}\right)(\lambda + \cos\theta)\right].$$

The final value of time when/if the rabbit caught is more easily seen from the alternative:

$$\frac{Vt}{r_o} = \frac{1}{(\lambda^2 - 1)} \left[ \lambda + \cos \theta_o - \frac{\sin \theta_o}{(\tan \frac{1}{2} \theta_o)^\lambda} \left( \frac{\lambda + \cos\theta}{1 + \cos\theta} \right) (\tan \tfrac{1}{2} \theta)^{\lambda - 1} \right].$$

Then, when $\lambda > 1$ the final value $t_1$ of $t$ is

$$t_1 = \frac{r_o(\lambda + \cos \theta_o)}{V(\lambda^2 - 1)} .$$

Obviously for the dog to predict correctly what the rabbit will do and then to proceed on a straight collision course is more efficient. The time to intercept is then

$$t_1 = \frac{r_o}{\text{Components of D's relative speed along } D_oR_o}$$

$$= \frac{r_o}{v \cos D - V \cos \theta_o}$$

$$= \frac{r_o}{V (\cos D - \cos \theta_o)}$$

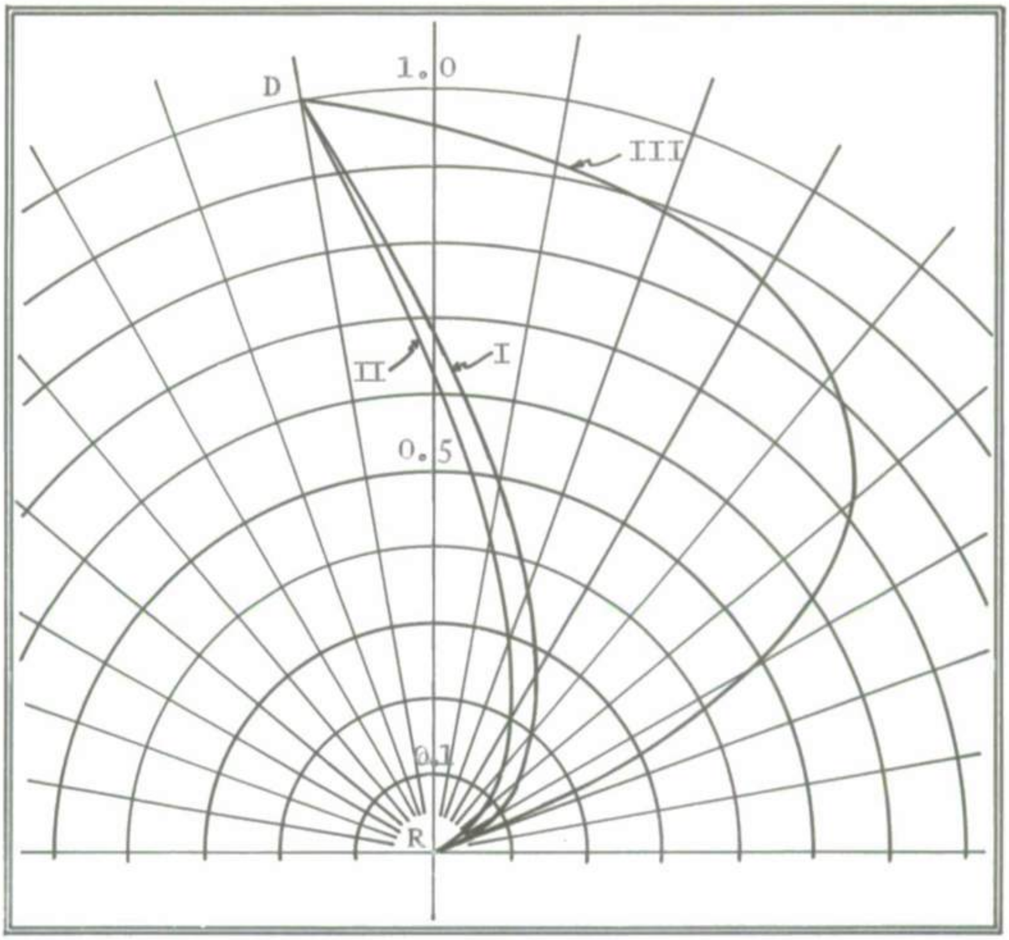
where D, the deflection angle, is given by

$$\sin D = \frac{1}{\lambda} \sin \theta_o.$$

Table 4 and curve I of Fig. 7 describe a pursuit curve for $r_o = 2000$ m, $v = 500$ m/min, $\lambda = 2$, $\theta_o = 100°$.

TABLE 4

| $\theta$ | t (min) | $r/r_o$ |
|---|---|---|
| 100 | 0 | 1.000 |
| 90 | 1.17 | 0.693 |
| 80 | 2.00 | 0.496 |
| 70 | 2.61 | 0.362 |
| 60 | 3.09 | 0.267 |
| 50 | 3.48 | 0.197 |
| 40 | 3.82 | 0.143 |
| 30 | 4.11 | 0.100 |
| 20 | 4.38 | 0.063 |
| 10 | 4.63 | 0.031 |
| 0 | 4.87 | 0.000 |




I Pursuit — no lead. II Pursuit — 5° lead angle.
III Line of sight guidance.

FIG. 7 Pursuit Curves in Target Space

The time to intercept on a collision course would be 4.18 minutes.

A variant of the basic pursuit homing system is "pursuit with lead". In this case (Fig. 8) the weapon's velocity vector makes a constant angle, $\phi$ say, with the line joining weapon to target in the sense of pointing ahead of it. ("Pursuit with lag" uses negative $\phi$).

In relative space

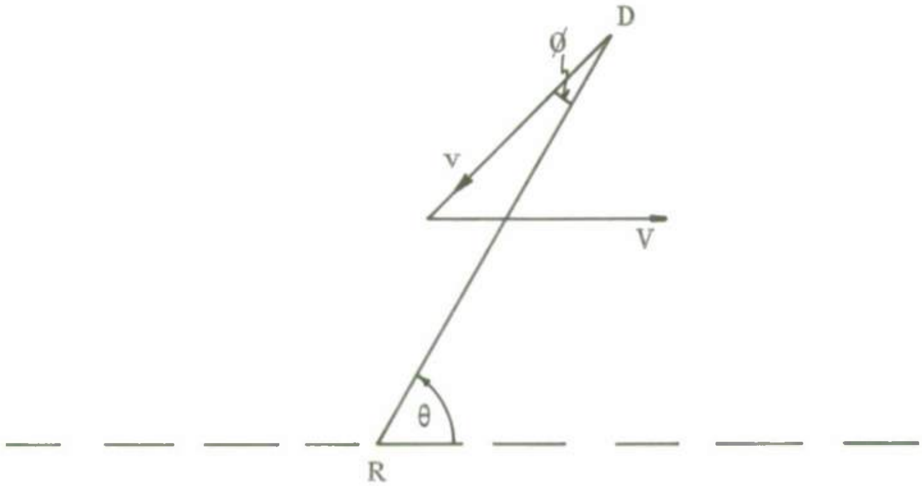



FIG. 8

the equations are

$$\dot{r} = - v \cos \phi + V \cos \theta$$

$$r\dot{\theta} = v \sin \phi - V \sin \theta$$

leading to

(a) when $b = \sin \phi > 1$;
$a = \lambda \cos \phi$;
$\lambda = \dfrac{v}{V}$;

$$\frac{r}{r_o} = \left[\frac{b - \sin \theta_o}{b - \sin \theta}\right] \exp\left[\frac{-2a}{(b^2 - 1)^{\frac{1}{2}}} \arctan \left\{\frac{(b^2-1) \sin \frac{1}{2} (\theta - \theta_o)}{b \cos \frac{1}{2}(\theta - \theta_o) - \sin \frac{1}{2} (\theta+\theta_o)}\right\}\right]$$

It is clear in this case that the weapon will never reach the target.

(b) $\lambda \sin \phi < 1$

It is now convenient to write

$$\lambda \sin \phi = \sin \psi ,$$

and we consider only the case where

$$\sin \psi < \sin \theta_o.$$

This means that $\dot{\theta}$ is initially negative and that $\theta$ <u>decreases</u> from its initial value $\theta_o$ until it reaches $\psi$ when a collision occurs.

We obtain

$$\frac{r}{r_o} = \left[\frac{\sin\frac{1}{2}(\theta-\psi)}{\sin\frac{1}{2}(\theta_o-\psi)}\right]^{\frac{\lambda\cos\emptyset}{\cos\psi}-1} \left[\frac{\cos\frac{1}{2}(\theta_o+\psi)}{\cos\frac{1}{2}(\theta+\psi)}\right]^{\frac{\cos\emptyset}{\cos\psi}+1}$$

which reduces satisfactorily to the equation obtained previously for pursuit without lead when we put $\emptyset = \psi = 0$. The time $t$ to reach angle $\theta$ is given by

$$t = \frac{r_o}{2V} \frac{[\cos\frac{1}{2}(\theta_o+\psi)]^{\mu+1}}{[\sin\frac{1}{2}(\theta_o-\psi)]^{\mu-1}} \int_{\theta_o}^{\theta} \frac{[\sin\frac{1}{2}(\theta-\psi)]^{\mu-2}}{[\cos\frac{1}{2}(\theta+\psi)]^{\mu+2}} \, d\theta \quad ,$$

where

$$\mu = \lambda \cos\emptyset/\cos\psi \; .$$

Integrating we obtain

$$t = \frac{r_o[\cos\frac{1}{2}(\theta_o+\psi)]^{\mu+1}}{V\cos^3\psi[\sin\frac{1}{2}(\theta_o-\psi)]^{\mu-1}} \left[\frac{(x_o^{\mu+1} - x^{\mu+1})}{\mu+1} + \frac{2\sin\psi(x_o^{\mu} - x^{\mu})}{\mu} + \frac{(x_o^{\mu-1} - x^{\mu-1})}{\mu-1}\right]$$

where

$$x = \frac{\sin\frac{1}{2}(\theta-\psi)}{\cos\frac{1}{2}(\theta+\psi)}$$

and $x_o$ is this with $\theta$ replaced by $\theta_o$.

The effect of a lead angle can be seen from Table 5 which may be compared with Table 4. All parameters are the same and the lead angle $\emptyset$ is $5^o$. The curve is marked II on Fig. 7 to provide a visual comparison.

TABLE 5

Pursuit with Lead Angle of $5^{o}$. Parameters as in Table 4

| $\theta^{o}$ | t(min) | $r/r_o$ |
|---|---|---|
| 100 | 0 | 1.000 |
| 90 | 1.75 | 0.642 |
| 80 | 2.27 | 0.428 |
| 70 | 2.59 | 0.292 |
| 60 | 3.40 | 0.201 |
| 50 | 3.72 | 0.137 |
| 40 | 3.96 | 0.089 |
| 30 | 4.27 | 0.053 |
| 20 | 4.47 | 0.024 |
| 10.04 | 4.66 | 0.000 |

It will be seen that in this case lead gets the weapon more quickly to the target than no lead. But perhaps the trajectory is more exacting for the weapon in terms of turning rates. The basic mathematics for a thorough investigation is here.

Analysis may be required to cope with the case of a manoeuvring target. Circular arcs are basic. If the target's path has constant radius $\rho$ and it travels with constant speed $V$ then the same relative space technique gives the differential equations

$$\dot{r} = V \cos\left(\theta - \frac{Vt}{\rho}\right) - v$$

$$r\dot{\theta} = - V \sin\left(\theta - \frac{Vt}{\rho}\right)$$

for straight pursuit. Numerical techniques are indicated for analysis.

Finally it is worth mentioning two other systems. The first may be described as "line of sight guidance". The system may be thought of as such that a fixed point $P$ of the weapon is kept on the bearing line joining parent ship (submarine) to current position of target. The weapon's velocity vector has a constant component in the direction of the bearing line, directed at the target. In a second system, related to what is sometimes called "proportional navigation", the lead angle $\emptyset$ varies with time and is corrected by servo-mechanisms which try to reduce it to zero. This system may be useful if there is poor fire control information.

The pursuit curve is the basis of most homing systems. A passive acoustic homing torpedo, for instance, always tries to point towards the target by adjusting its track according to where the noise is coming from. The trajectories themselves naturally become more elaborate if the target is disobliging enough not to follow a straight track. Also in some cases it is necessary to look at the three-dimensional form of the pursuit curves. The point of a mathematical analysis is, of course, to examine what demands are made on a weapon's speed and manoeuvrability by a certain homing or guidance system, and by plausible enemy evasive tactics. The mathematics thus enters both on the weapon design and employment side, and is an indispensable element in an investigation of pursuit and evasive tactics.

An elementary formulation of the problem for line of sight guidance is of interest as a final exercise in homing mathematics. In geographical space, as shown in Fig. 9, the firing ship S pursues a fixed trajectory with constant speed u and the target T does likewise with constant speed V. S controls the weapon W directly (wire), or remotely (radio), holding it always on the bearing line ST. The weapon has constant speed v at fixed angle $\chi$ to the bearing.

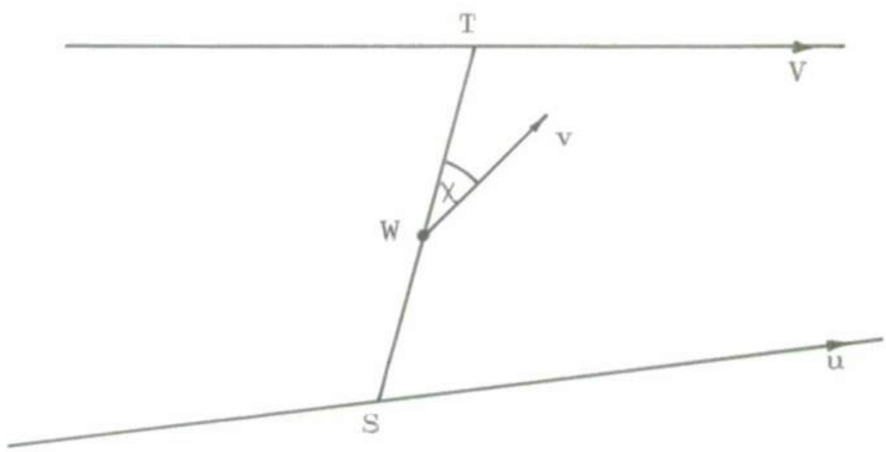



FIG. 9

In target space (target put to rest) we have Fig. 10.

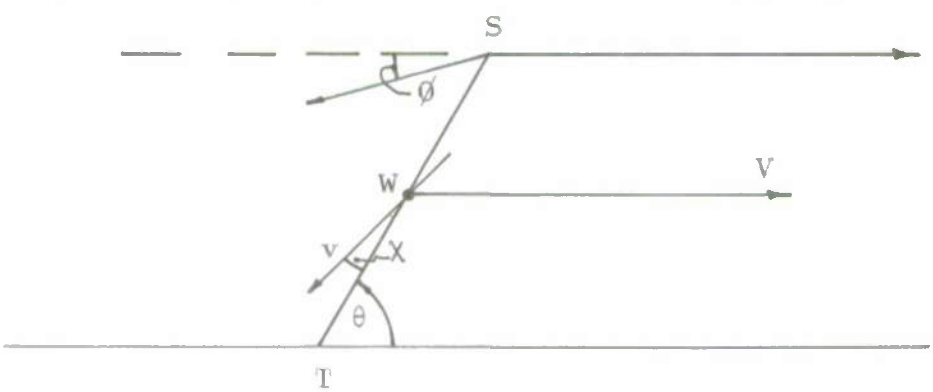



FIG. 10

Let $ST = R$. Then

$$\dot{R} = V \cos \theta - u \cos (\theta - \phi) = A \cos \theta - B \sin \theta \qquad \text{[Eq. 1]}$$

$$R\dot{\theta} = u \sin (\theta - \phi) - V \sin \theta = - A \sin \theta - B \cos \theta \qquad \text{[Eq. 2]}$$

where

$$A = V - u \cos \phi, \qquad B = u \sin \phi$$

These are the basic guidance equations and determine $\dot{\theta}$ which has to enter into the homing equations. Putting $TW = r$ we have, as usual,

$$\dot{r} = - v \cos \chi + V \cos \theta \, , \qquad \text{[Eq. 3]}$$

$$r\dot{\theta} = v \sin \chi - V \sin \theta \, . \qquad \text{[Eq. 4]}$$

Integration yields

$$\frac{R}{R_o} = \frac{A \sin \theta_o + B \cos \theta_o}{A \sin \theta + B \cos \theta} \qquad \text{[Eq. 5]}$$

and by substitution in Eq. 2

$$\dot{\theta} = - \frac{(A \sin \theta + B \cos \theta)^2}{R_o(A \sin \theta + B \cos \theta)} \, . \qquad \text{[Eq. 6]}$$

Finally, substitution in Eq. 4 gives

$$\frac{r}{r_o} = \frac{(V \sin \theta - v \sin \chi)\ (A \sin \theta_o + B \cos \theta_o)}{(A \sin \theta + B \cos \theta)^2} \qquad \text{[Eq. 7]}$$

with a hit if and when $\theta = \theta_h$ given by

$$\sin \theta_h = \frac{v \sin \chi}{V} \, . \qquad \text{[Eq. 8]}$$

There are, of course, considerations of signs of the factors of the numerator in order to have a practical geometry. Also the weapon is launched from S so that $r_o = R_o$. This means that for consistency,

$$V \sin \theta_o - v \sin \chi = A \sin \theta_o + B \cos \theta_o \qquad \text{[Eq. 9]}$$

must hold.

The implication of Eq. 9 is that

$$v \sin \chi = u \sin (\theta_o - \emptyset) \quad , \qquad \text{[Eq. 10]}$$

which places a restriction upon the parameters at the disposal of S. As an example consider the parameters of the example used previously, viz. $V = 250$ m/min, $v = 500$ m/min, $R = 2000$ m, $\theta_o = 100^o$. Taking $u = 100$ m/min, $\emptyset = 10^o$, we get from (Eq. 10) $\sin \chi = 0.2$ and $\chi = 11.5370^o$. Then $A = 151.5192$, $B = 17.3648$, and a check reveals that $\dot{R}_o$ is negative, as it should be for a hit, and that $\dot{\theta}_o$ is negative, so that $\theta$ decreases from its initial value of $100^o$ to $\theta_h$ which may be calculated from Eq. 8 and is $23.58^o$. The customary table for the homing curve and time required is given on Table 6. The curve is also plotted on Fig. 7 as curve III. It is visibly longer than the straight pursuit curves.

TABLE 6

| $\theta^o$ | $/R_o$ | t(min) |
|---|---|---|
| 100 | 1 | 0 |
| 95 | 0.976 | 1.15 |
| 90 | 0.955 | 2.29 |
| 85 | 0.938 | 3.40 |
| 80 | 0.922 | 4.49 |
| 60 | 0.870 | 9.19 |
| 40 | 0.724 | 15.65 |
| 35 | 0.620 | 17.92 |
| 30 | 0.443 | 20.70 |
| 25 | 0.130 | 24.22 |
| 24 | 0.041 | 25.04 |
| 23.58 | 0 | 25.40 |

The equation for calculating t as a function of $\theta$ is

$$t = \frac{R_o \cos \theta_o (\tan \theta_o - \tan \theta)}{B + A \tan \theta} \quad . \qquad \text{[Eq. 11]}$$

It will be seen that really a very long (ludicrously so) time is required under this system relatively to the time for straightforward pursuit. It is for the analyst to weigh the advantages of the system, find how to reduce its practical difficulties if there are tactical reasons for adopting such a system, and to make recommendations.

## 2.3 A Ship Closure Problem

To conclude this chapter on closure we discuss a typical and simple problem. It is supposed that the position of a submarine has been reported. It will be assumed that there is no uncertainty about this position. A group of ships with semi-swept path $W^*$ immediately begins to close the reported position from range $R$ with speed $V$. The submarine's speed $v$ is equally likely to have any value in the range $0 \leq v \leq v_0$, and its course is equally likely to be anything. What is the probability that the group sweeps over the submarine?

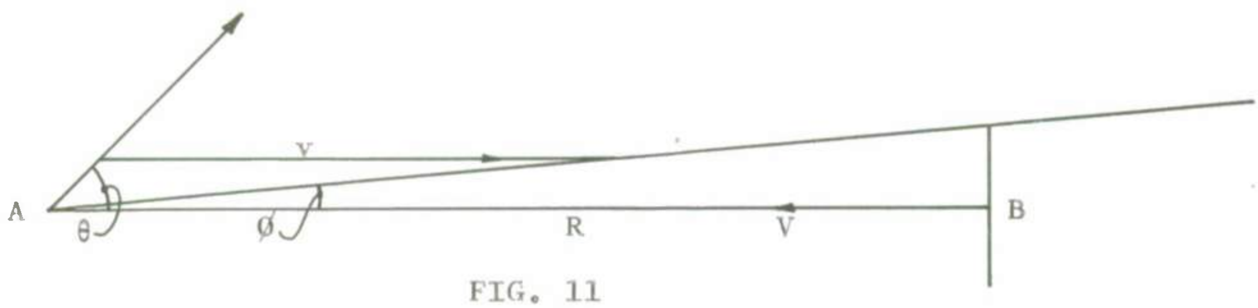



FIG. 11

The reported submarine position is A and the group starts from B along BA (BA = R) with speed V (Fig. 11). If the submarine is on a track making an angle $\theta$ with AB then it will just be detected if its track relative to the group makes an angle $\emptyset$ with AB such that

$$\tan \emptyset = \frac{W}{R} = r \text{ , say.} \qquad \text{[Eq. 12]}$$

* This means that detection of a submarine within a range W of the ship's position is 100% certain. Beyond W the probability is zero.

We have

$$\tan \phi = \frac{v \sin \theta}{V + v \cos \theta} \qquad [\text{Eq. } 13]$$

and so generally, for a detection

$$0 \le \frac{v \sin \theta}{V + v \cos \theta} \le r \quad , \qquad [\text{Eq. } 14]$$

where we restrict ourselves to $0 \le \theta \le \pi$ .

(There is symmetry about AB, so the first two quadrants supply all information.)

Now if $\theta = 0$, the submarine is moving towards B and clearly whatever speed $v$ which it makes good $(0 < v < \infty)$ will result in its being detected. A small increase in $\theta$ will result in a finite, but still large, range of values of $v$ which are favourable to the submarine's detection; as $\theta$ increases the corresponding range of $v$ decreases.

Clearly there is a critical value $\theta_1$ of $\theta$ to be associated with the speed $v_o$ for which all submarine speeds such that $0 < v \le v_o$ will result in detection. We may say that for values of $\theta$ such that $0 \le \theta \le \theta_1$ the submarine is bound to be detected since whatever speed it uses, even restricting itself to $0 < v \le v_o$ , it will be detected.

Also there will be a second critical value $\theta_2$ of $\theta$ which gives similar results - viz. for $\theta_2 < \theta < \pi$ the submarine is bound to be detected whatever speed it uses (and always supposing that $V > v_o$).

The joint probability that $\theta$ lies between $\theta$ and $\theta + d\theta$ and that $v$ lies between $v$ and $v + dv$ is

$$\frac{d\theta}{v_o} \quad \frac{d\theta}{\pi} \, . \qquad [\text{Eq. } 15]$$

The probability $P$ that the submarine is detected is therefore given by

$$P = \frac{\theta_1}{\pi} + A(\theta_1, \theta_2) + \frac{\pi - \theta_2}{\pi} \qquad [\text{Eq. } 16]$$

because of symmetry about AB, where the function $A(\theta_1, \theta_2)$ corresponds to detection for $\theta_1 \le \theta \le \theta_2$.

The critical values of $\theta$ are the solutions in $0 \leq \theta \leq \pi$ of

$$v_o \sin\theta = r(V + v_o \cos\theta)$$

With $t = \tan\frac{1}{2}\theta$, this equation gives

$$r\,t^2(V-v_o) - 2\,t\,v_o + r(V+v_o) = 0 \,.$$

Hence

$$t_{1,2} = \frac{v_o \pm [v_o^2(1+r^2) - r^2V^2]^{\frac{1}{2}}}{r(V-v_o)}$$

where $t_1 = \tan\frac{1}{2}\theta_1$ is obtained with the $-$ sign.

Now, choosing a $\theta$ in $(\theta_1, \theta_2)$ we see that, for a detection $0 \leq v \leq v_1$ where by Eq. 14

$$\frac{v_1 \sin\theta}{V + v_1\cos\theta} = r$$

i.e.

$$v_1 = \frac{Vr}{\sin\theta - r\cos\theta} \,.$$

The function $A(\theta_1, \theta_2)$ in Eq. 16 is thus given by

$$A(\theta_1, \theta_2) = \frac{1}{\pi\, v_o}\int_{\theta_1}^{\theta_2}\int_{o}^{v_1} dv\, d\theta = \frac{Vr}{\pi\, v_o}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\sin\theta - r\cos\theta}$$

$$= \frac{Vr}{\pi v_o}\, I(\theta_1, \theta_2) \,.$$

Writing $A^2 = 1 + \frac{1}{r^2}$ we have

$$I(\theta_1, \theta_2) = \frac{1}{r\,A}\,\ln\left[\frac{(t_2 + \frac{1}{r} - A)\,(t_1 + \frac{1}{r} + A)}{(t_1 + \frac{1}{r} - A)\,(t_2 + \frac{1}{r} + A)}\right],$$

where $t_2 = \dfrac{\lambda + r(\lambda^2A^2 - 1)^{\frac{1}{2}}}{r(1-\lambda)}$ , $t_1 = \dfrac{\lambda - r(\lambda^2A^2 - 1)^{\frac{1}{2}}}{r(1-\lambda)}$

$\lambda = v_o/V$.

Hence altogether the probability $P$ of the submarine being detected is

$$P = \frac{1}{\pi} \left[\pi + \theta_1 - \theta_2 + \frac{Vr}{v_o} I(\theta_1, \theta_2)\right] .$$

As an example take:

$v_o = 5$ knots; $\quad W = 2$ nautical miles;

$V = 20$ knots; $\quad \lambda = 0.25 : r = 0.1$;

$R = 20$ nautical miles; $\quad A^2 = \sqrt{101} : A = 10.05$

Noting that

$$t_{2,1} = \frac{\lambda \pm r\lambda A(1-\lambda^{\frac{1}{2}}A^2)^{\frac{1}{2}}}{r(1-\lambda)} = \frac{0.25 \pm 0.2304}{0.075} ,$$

$t_2 = 6.405 : \theta_2 = 2.832$ ;

$t_1 = 0.2613 : \theta_1 = 0.5112$ ;

$P = 0.659$ .

If $V$ is reduced to 18 knots $P$ is reduced to 0.618.

## 3. PROBLEMS OF LOCATION

The final chapter of this Part on geometrical problems is concerned only with the location of a target given that one knows one's own position. Problems of navigation, which include the latter, are both of great importance militarily, and mathematically quite rich, but to do them justice would require a further volume which this author does not plan at present.

Complete location of a target in two dimensions (all we deal with) with respect to a reference position and direction may be said to have taken place when the observer has determined four quantities :

(a) a position (two quantities) at some time;

(b) speed in magnitude and direction.

For example, if one obtained two accurate ranges and bearings separated by a fixed time interval then (a) and (b) are known. Since ranges and bearings, if used, are subject to error two alone of each will rarely be allowed to suffice. A number would be taken. Mutual inconsistencies will be observed and these may be attributed to errors, to target manoeuvres, or to both. Then arises the question of reconciliation, and the taking of a decision as to what the target is doing and then, usually, what it is likely to be doing in the immediate future. Thus one confronts a set of problems of smoothing and prediction, too vast to be more than mentioned here.

We shall focus attention on the mathematics associated with the location of a target using only bearings, and no range measurements. This is a situation that arises when an observer does not wish to disclose his own whereabouts and so merely listens to, or watches, passively, the target's movements, noting bearings. In principle four bearings are sufficient, but again the problem of inconsistencies arises. Before tackling that one must look at the basic mathematics.

In geographical space, with a target T on a constant course, constant speed, trajectory, the basic situation is as depicted in Fig. 12.

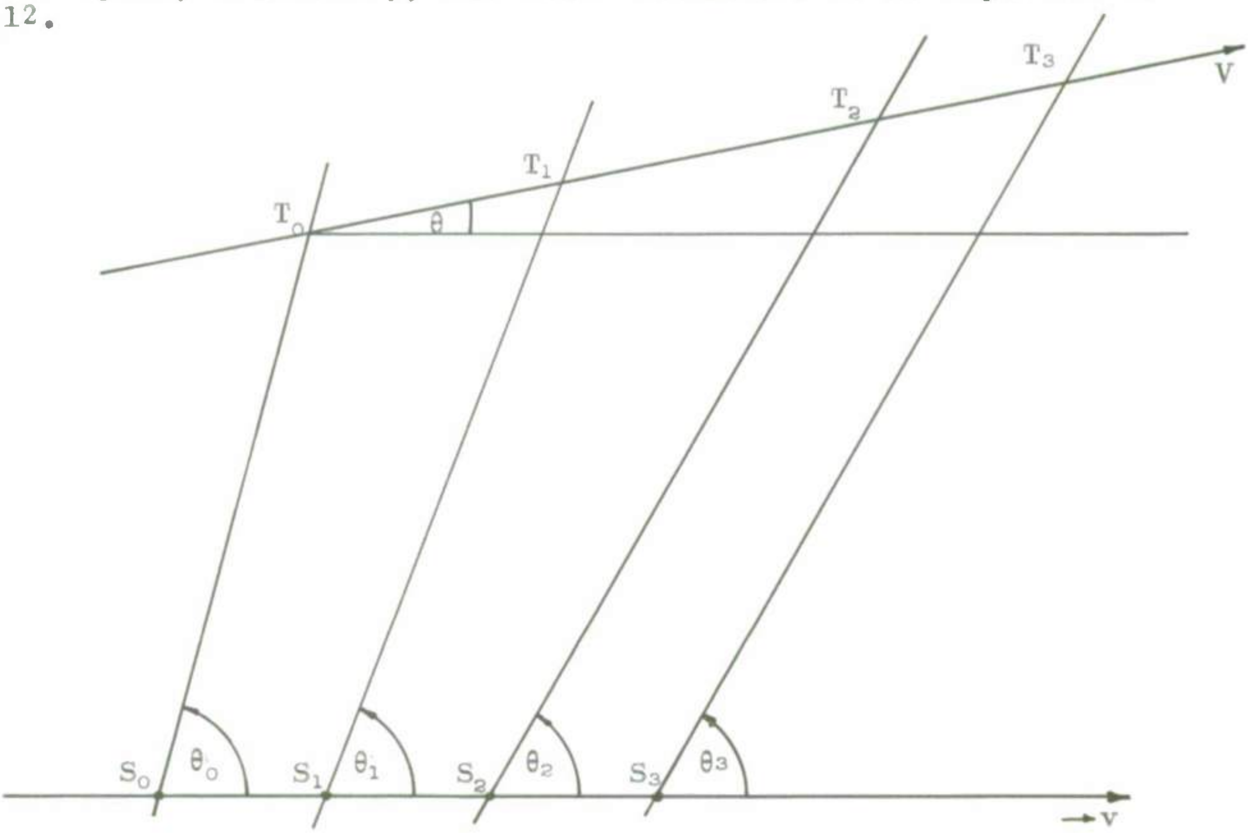



FIG. 12

The bearings are $\theta_0, \theta_1, \theta_2, \theta_3, \ldots$ taken at times $t_0, t_1, t_2, t_3, \ldots$ from ship positions $S_0, S_1, S_2, S_3, \ldots$ The ship is known to have constant speed $v$ in the direction shown. The problem is to establish a target position $T_0$, say, together with $V$ and $\phi$.

If we look at the problem in ship space, then at time $t$ we have the situation shown in Fig. 13,

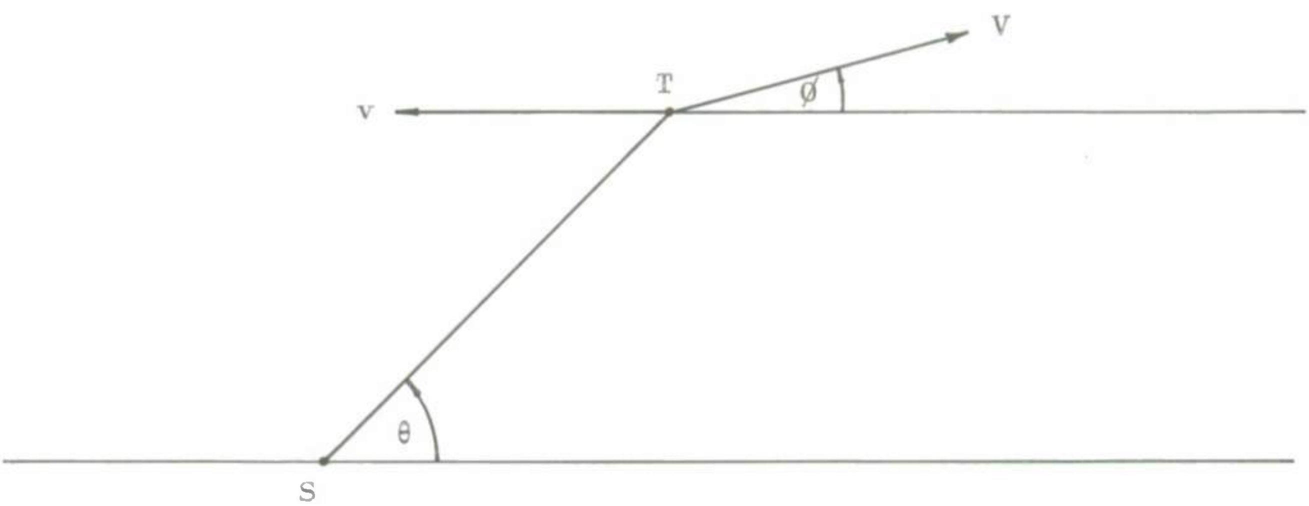



FIG. 13

so that in polar coords $(r, \theta)$ with $ST = r$,

$$\dot{r} = V \cos (\theta - \emptyset) - v \cos \theta \ , \qquad \text{[Eq. 17]}$$

$$r\dot{\theta} = v \sin \theta - V \sin (\theta - \emptyset) \ . \qquad \text{[Eq. 18]}$$

These equations resemble Eqs. 3 & 4.

Writing

$$A = V \cos \emptyset - v \ , \qquad \text{[Eq. 19]}$$

$$B = V \sin \emptyset$$

we then have

$$\frac{r}{r_o} = \frac{B \cos \theta_o - A \sin \theta_o}{B \cos \theta - A \sin \theta} \qquad \text{[Eq. 20]}$$

and

$$\dot{\theta} = \frac{(B \cos \theta - A \sin \theta)^2}{r_o(B \cos \theta_o - A \sin \theta_o)} \qquad \text{[Eq. 21]}$$

so that

$$r^2\dot{\theta} = r_o(B \cos \theta_o - A \sin \theta_o) = \text{constant} \ . \qquad \text{[Eq. 22]}$$

An interesting practical conclusion that can be drawn from Eq. 22 is that, provided the target remains at constant speed on constant course, the range $r$ at time $t$ is proportional to the inverse square root of the bearing rate $\dot{\theta}$ at the same time. This means that if estimates are formed of $\dot{\theta}$, for example by plotting the relative bearing $\theta$ as a function of time, smoothing, and reading off the slope of the $\theta$ versus time curve, one can construct a table of the type of Table 7,

TABLE 7

| $t$ | $\theta$ | $K \dot{\theta}^{-\frac{1}{2}}$ |
|---|---|---|
| $t_1$ | $\theta_1$ | $K \dot{\theta}_1^{-\frac{1}{2}} \propto r_1$ |
| $t_2$ | $\theta_2$ | $K \dot{\theta}_2^{-\frac{1}{2}} \propto r_2$ |
| $t_3$ | $\theta_3$ | $K \dot{\theta}_3^{-\frac{1}{2}} \propto r_3$ |
| ....... | ....... | ....... |

and if the entries in the third column are plotted on polar paper against $\theta$, for a suitable value of $K$ (not difficult to select on the basis of operational experience) then the line joining these points is, within the limits of accuracy, parallel to the target's estimated track relative to the observer.
To determine completly the target's location then needs only the identification of which of the infinite family of parallels really is the target's relative track, for then measurement of the intercepts gives the absolute value of the relative speed from which true speed and direction can be constructed by a vector diagram, while clearly any of the points on the identified relative track can be used to determine a target location in geographical space at a given time.

It will be noticed that continuation of the measurements of bearings and the plotting of points, however long continued, adds no further information. Hence the observer must change his tactics. Let us suppose this to be merely a change of speed. Then he must plot his new track relative to his old one carefully on the polar diagram. In principle one further observation is sufficient. For all the observer needs to do is to mark a new bearing line and find where it intersects the bearing line that would have corresponded to the time of the new observation had he not moved, this latter bearing line being simple to construct. That parallel through the point of intersection is the one sought to provide all supplementary information. Naturally it is preferable to prolong the new set of measurements. An experienced observer, it is to be noted, can deduce

a great deal about even a manoeuvring target from plots developed in the manner described. He can also often improve his locating ability by appropriate manoeuvres according to the behaviour of the target.
For example, a submarine target may move quite slowly so that the bearing rate is small and in that case particularly susceptible to error. It is clear that there are observation positions from which the bearing rate will be near its maximum for given tactical conditions, and such positions are for the observer to find and occupy. Finally, as noted above, the prolongation of the time of observation is naturally beneficial towards the formation of more reliable estimates, but on the other hand, provides the target with more opportunity to manoeuvre and escape.

The reader who is well versed in what used to be called geometrical conics will not be surprised, if indeed he did not immediately perceive, that the basic geometry is that of the parabola. We shall now review this briefly by well-known, but pedestrian, cartesian methods. The reader is advised that the methods of projective geometry are more elegant and expeditious.

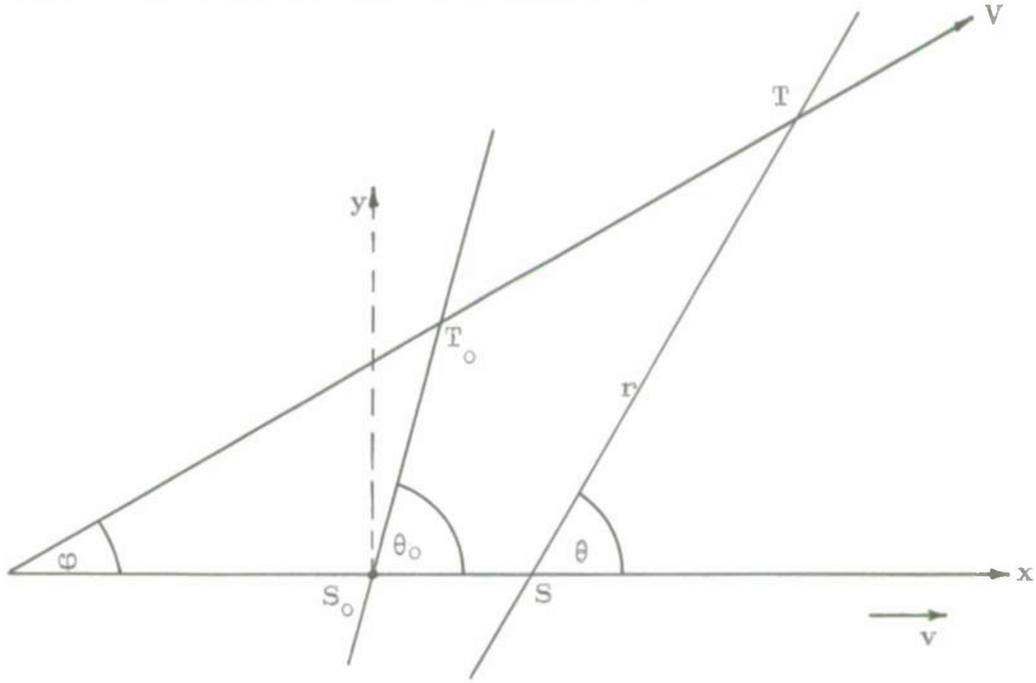



FIG. 14

Consider Fig. 14, let the origin of rectangular cartesian coordinates be $S_o$, as shown. $S_oT_o$ (bearing $\theta_o$) is the initial bearing line corresponding to time $t = 0$. $T_o$, the initial position of the target has coordinates $(x_o, y_o)$. After time $t$ we suppose the observer to be at $S$ and the target at $T$, bearing $\theta$. The points $S$ and $T$ have coordinates $(vt, 0)$,

$$(x_o + Vt \cos \phi, \; y_o + Vt \sin \phi) \quad , \qquad \text{[Eq. 23]}$$

respectively, where $V$ and $\phi$ define the target's speed vector.

Also

$$y_o = x_o \tan \theta_o \quad . \qquad \text{[Eq. 24]}$$

Observation gives $\theta_o$, while $x_o$, $y_o$ $V$ and $\phi$ have to be found.

The characteristic bearing line ST has equation

$$\frac{x - v\,t}{t(v - V\cos\phi) - x_o} = \frac{y}{-y_o - Vt\,\sin\phi} \,,$$

or  [Eq. 25]

$$\frac{x - v\,t}{At + x_o} = \frac{y}{y_o + B\,t} \quad .$$

This may be written as a quadratic in time $t$ thus:

$$Bvt^2 + t(Ay - Bx + vy_o) + (x_o y + y_o x) = 0 \qquad \text{[Eq. 26]}$$

The envelope of this line for varying $t$ is the parabola:

$$(Ay - Bx + vy_o)^2 = 4\ Bv\ (x_o y + y_o x) \quad . \qquad \text{[Eq. 27]}$$

To find the axis and tangent at the origin we write the parabola in the form

$$(Ay - Bx + C)^2 = 4\,L\,(By + Ax + D) \qquad \text{[Eq. 28]}$$

and determine $C$, $L$ and $D$ so that Eq. 28 is identical with Eq. 27. This gives

$$C = vy_o - \frac{2Bv(Ax_o - By_o)}{R^2} \;;$$

$$L = \frac{Bv(Ay_o + Bx_o)}{R^2} \;; \qquad \text{[Eq. 29]}$$

$$D = \frac{v}{(Ay_o + Bx_o)}\left[\frac{B(Ax_o - By_o)^2}{R^2} - y_o(Ax_o - By_o)\right] \;;$$

where $R^2 = A^2 + B^2$.

The axis is the line

$$Ay - Bx + C = 0;$$

The tangent at the vertex is

$$By + Ax + D = 0;$$

The latus rectum has total length 4L.
The vertex is the point

$$\left[\frac{BC-AD}{R^2}, \quad \frac{-(AC+BD)}{R^2}\right] .$$

The focus is

$$\left[\frac{BC-AD}{R^2} + \frac{2LA}{R}, \quad \frac{-(AC+BD)}{R^2} + \frac{2LB}{R}\right] .$$

These expressions can be simplified, and the student should do it. In particular it can be seen that the slope of the axis of the parabola, B/A, is parallel to the slope of the target's speed vector relative to the observer.

Any line that has the property that the ratio of its intercepts by the bearing lines is equal to the ratio of the times between the bearings could be generated by a target with appropriate choice of speed and initial position. To the well-versed student it will be obvious that such possible target tracks are also tangents to a parabola, and indeed to the same parabola as that enveloped by the bearing lines. We refer now to Fig. 15.

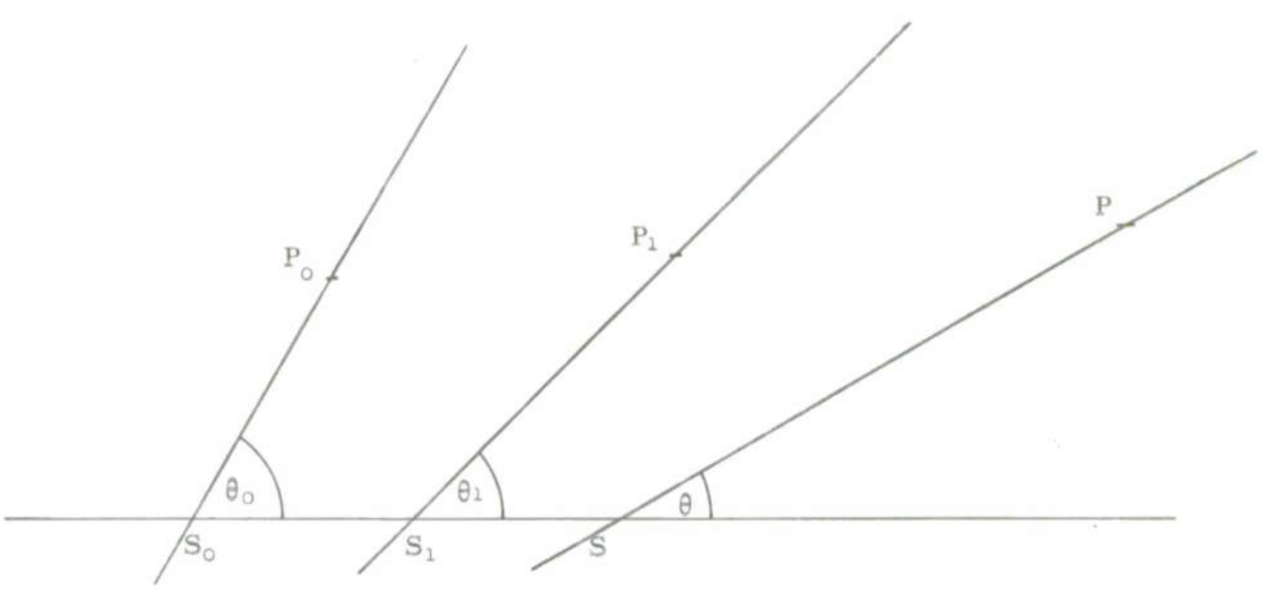



FIG. 15

Let the initial bearing be $\theta_o$, taken from $S_o(0,0)$. Let the next bearing be $\theta_1$ taken from $S_1$, where $S_oS_1 = vt_1$ and, of course, $v$ and $t_1$ are known. Now for any point $P_o$ selected on the first bearing line any other point $P_1$ may be chosen on the second bearing line, and clearly $P_oP_1$ could be a target track with speed $P_oP_1/t_1$ and direction given by the inclination of $P_oP_1$. The line becomes a unique possible target track corresponding to the choices $P_o$ and $P_1$ iff for any time $t > t_1$, a point $P$ is defined on the continuation of $P_oP_1$ such that $P_oP = \frac{P_oP_1}{t_1} t$ and $P$ lies on the appropriate bearing line.

With the usual notation $P_o(x_o,y_o)$, $P_1(x_1, y_1)$, $P(x,y)$ we have for any line through $P_o$ with inclination $\phi$,

$$x - x_o = s \cos \phi, \quad y - y_o = s \sin \phi \ . \qquad \text{[Eq. 30]}$$

with $y_o = x_o \tan \theta_o$,
where $s$ is the distance measured along the line from $P_o$ to $P$.

The next fixed bearing line has equation

$$x - vt_1 = r \cos \theta_1, \quad y = r \sin \theta_1, \qquad \text{[Eq. 31]}$$

where $r$ is the distance measured along the line from $S_1$ to $(x,y)$; with fixed $v$, $t$ and $\theta_1$. $r$ corresponding to $P_1$ is given by the intersection with the line through $P_o$. Thus

$$\left.\begin{aligned} x_o + s \cos \phi &= vt_1 + r \cos \theta_1 \\ y_o + s \sin \phi &= r \sin \theta_1 . \end{aligned}\right\} \qquad \text{[Eq. 32]}$$

Given $\phi$, everything is known except $r$ and $s$. These may easily be found. Thus

$$r = (vt_1 \sin \phi - x_o \sin \phi + y_o \cos \phi)/\sin(\theta_1 - \phi); \qquad \text{[Eq. 33]}$$

$$s = (vt_1 \sin \theta_1 - x_o \sin \theta_1 + y_o \cos \theta_1)/\sin(\theta_1 - \theta); \qquad \text{[Eq. 34]}$$

and the possible target speed corresponding to this choice of $\phi$ is $s/t_1$.

Any point $P(x_p,y_p)$ on this line corresponding to time $t$ is now given by

$$x_p = x_o + \frac{st}{t_1} \cos \phi, \quad y_p = y_o + \frac{st}{t_1} \sin \phi . \qquad \text{[Eq. 35]}$$

But the line $P_oP_1P$ will only qualify as a possible target track if the line PS joining P to the current observer position S(vt,0) has the observed slope/bearing $\theta$. The requisite condition on $\emptyset$, given $P_o$, is then that

$$\frac{y_o + \frac{st}{t_1}\sin\emptyset}{x_o + \frac{st}{t_1}\cos\emptyset - vt} = \tan\theta . \qquad \text{[Eq. 36]}$$

Equations 34 and 36 determine s and $\emptyset$ (in principle) for the observed and known values and the arbitrary choice of $x_o$ ($y_o$ follows from $y_o = x_o \tan\theta_o$, $\theta_o$ being given).

We seek the envelope of the possible target tracks corresponding to the variable parameter $x_o$, i.e. the envelope of the line

$$y - x_o \tan\theta_o = (x - x_o)\tan\emptyset \qquad \text{[Eq. 37]}$$

where

$$\tan\emptyset = \frac{\alpha\sin\theta_1 - \alpha_1\sin\theta}{\alpha\cos\theta_1 - \alpha_1\cos\theta} ,$$

$$\alpha t = (x_o - vt)\sin\theta - y_o\cos\theta , \qquad \text{[Eq. 38]}$$

$$\alpha_1 t_1 = (x_o - vt_1)\sin\theta_1 - y_o\cos\theta_1 .$$

Then Eq. 37 becomes

$$(y - x_o\tan\theta_o)(\alpha\cos\theta_1 - \alpha_1\cos\theta) = (x - x_o)(\alpha\sin\theta_1 - \alpha_1\sin\theta).$$

[Eq. 39]

We now differentiate with respect to $x_o$, find that $x_o$ for which the derivative vanishes and substitute in Eq. 39. This gives as envelope of possible tracks the parabola

$$(\ell y - nx + m\tan\theta_o)^2 = 4\,my\,(\ell\tan\theta_o - n) , \qquad \text{[Eq. 40]}$$

where

$$\ell = \frac{\cos\theta_1\sin(\theta-\theta_o)}{t} - \frac{\cos\theta\sin(\theta_1-\theta_o)}{t_1} ;$$

$$m = v \cos \theta_o \sin (\theta - \theta_1) \quad ; \qquad \text{[Eq. 41]}$$

$$n = \frac{\sin \theta_1 \sin(\theta - \theta_o)}{t} - \frac{\sin \theta \sin(\theta_1 \; \theta_o)}{t_1} .$$

There remains the problem of showing that Eqs. 28 and 40 are the same parabola. The basic consideration is this: Eq. 28 envisages <u>fixed</u> V and Ø with varying t; Eq. 40, on the other hand, envisages fixed $t_1$, t, $\theta_1$ and $\theta$, with varying $x_o$. The fundamental problem of conversion is thus to find V and Ø in terms of $t_1$, t, $\theta_1$ and $\theta$. This may be done by using Eqs. 34 & 36, recalling that $V = s/t_1$. Proceeding directly to A and B we find

$$A = \frac{x_o\{t_1 \cos \theta_1 \sin(\theta-\theta_o) - t \cos \theta \sin(\theta_1-\theta_o)\}}{t\, t_1 \cos \theta_o \sin(\theta_1-\theta)} = \frac{\ell x_o}{\cos \theta_o \sin(\theta_1-\theta)} ;$$

$$B = \frac{x_o\{t_1 \sin \theta_1 \sin(\theta-\theta_o) - t \sin \theta \sin(\theta_1-\theta_o)\}}{t\, t_1 \cos \theta_o \sin(\theta_1-\theta)} = \frac{n x_o}{\cos \theta_o \sin(\theta_1-\theta)} .$$

Then, for example, the slope B/A of the axis of the parabola of Eq. 28 is equal to n/1, the slope of the parabola of Eq. 40. A little further substitution provides the proof that the parabolae are identical. A visual indication is provided by the Fig. 16. Arbitrary target and observer tracks have been drawn. $S_o$ and $T_o$ are the initial positions. The parabola can be clearly seen emerging as the envelope of the bearing lines. By continuing the bearing lines "backwards" from $S_o$ one can see the approach to the observers' track, and thereafter to the actual targets' track, as possible target tracks, all members of the same tangent family to the fixed parabola. As a final exercise the student is recommended to develop expressions for $x_o$, $y_o$, V and Ø in terms of $(v, \theta_o, \theta_1, \theta_2, t_1, t_2)$ and $(v', \theta_3, t_3)$ after a change of speed.*

---

* For example three bearings $\theta_o$, $\theta_1$ and $\theta_2$ corresponding to times $t_o t_1$ and $t_2$ give $\tan Ø = (R_2 t_1 \tan \theta_1 - R_1 t_2 \tan \theta_2)/(R_2 t_1 - R_1 t_2)$

where $R_n = y_o - x_o \tan \theta_n + v t_n \tan \theta_n$ ;

and

$$V = \frac{[R_2^2 t_1^2 \cos^2 \theta_2 + R_1^2 t_2^2 \cos^2 \theta_1 - 2R_1R_2t_1t_2 \cos\theta_1 \cos\theta_2 \cos(\theta_2-\theta_1)]^{\frac{1}{2}}}{t_1 t_2 \sin(\theta_2-\theta_1)} .$$

The attempt to carry out a paper error analysis will be instructive. More useful will be to program the calculations and to investigate numerically the error distributions consequent upon measurement errors. This is one of the fruitful applications of digital computers to practical analysis.

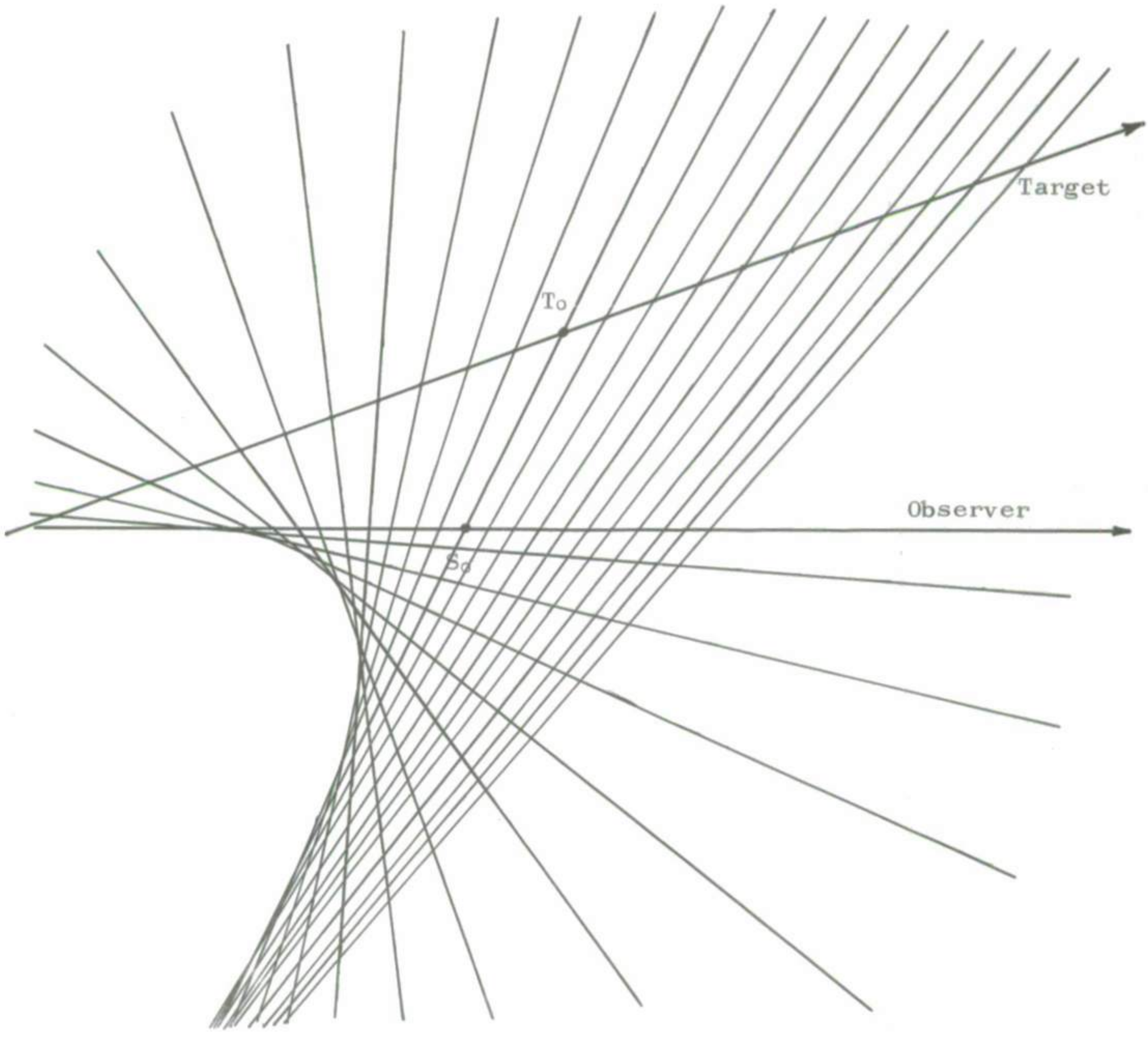



FIG. 16

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